Lecture Notes: Linear Programming II

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1 Preliminary

A *linear program*(LP) is a set of n *variables* and m *linear constraints*. The LP specifies an objective function which wants to maximize or minimize a linear combination of the decision variable and their coefficients subject to the m constraints. To formulate precisely:

$$\text{max: } \sum_{i=1}^{n} c_i x_i \tag{1}$$

$$\text{s.t } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, ..., m \tag{2}$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, ..., n \tag{3}$$

An LP can be solved with the *SIMPLEX* algorithm. SIMPLEX starts out with finding the *slack formulation* of the LP. To convert LP into slack form first n *slack variables* are introduced as follows.

$$\text{max: } \sum_{i=1}^{n} c_i x_i \tag{4}$$

$$\text{s.t } \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} \leq b_i \text{ for } i = 1, 2, ..., m \tag{5}$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, ..., n+m \tag{6}$$

Then slack formulation is achieved by isolating the slack variables in the constraints and changing the maximization to a equality with an unknown constant z. In this formulation every variable in the z-row is called the *non-basic* variables and the isolated variables in the constraints are called the *basic* variables. The formulation is as follows:

$$z = \sum_{i=1}^{n} c_i x_i \tag{7}$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \text{ for } i = 1, 2, ..., m \tag{8}$$

Then SIMPLEX iteratively pivots non-basic variable and a basic variables effectively changing positions in the formulation. This process is done such that the value of z increases or stays the same after the pivot. The variable chosen to leave the non-basic variable is based on picking only variables with positive coefficients. The variable chosen to enter the basic variables is based on which is the most constraining. When it is not possible to pivot in any variable without decreasing z the optimal solution has been found. The solution to the original problem is the set of variables, where every non-basic variable is 0 and the basic variables are equal to the corresponding constant.

2 Termination

Linear programs terminate given a sound formulation. There are two cases of termination, either all coefficients of the objective function are negative or SIMPLEX detects that the LP is *unbounded*.

There are a finite number of basic solutions. In a basic solution there is m basic variables which can be picked from the $n + m$ variables in the slack formulation. There is $\binom{n+m}{m} = \frac{(n+m)!}{n! * m!}$ different ways these basic variables can be picked and each basic solution has one objective value. Therefore if we continuously pick new basic variables which increases the objective function we will end up with a solution eventually.

2.1 Unbounded

In the case that the LP is unbounded the objective function will have one of its variables' coefficient positive and therefore available for pivoting. If x_i has been chosen to leave the non-basic variables, but all of the coefficients of x_i in the constraints are positive, then x_i is effectively allowed to increase infinitely since any of the basic variables' values will also increase and therefore does not violate the non-negativity constraints. In this situation none of the constraints are therefore binding and the LP is unbounded hence no optimal solution exists. An example of an unbounded LP

$$z = 0 + 2x_1 + 9x_2 \tag{9}$$

$$x_3 = 5 + 3x_1 - 2x_2 \tag{10}$$

$$x_4 = 2 - 6x_1 - 2x_2 \tag{11}$$

Notice that as soon as *one* of the non-basic variables are positive and it also appears as positive in just one or more of the constraints, SIMPLEX can terminate since the LP must be unbounded.

2.2 Degeneracy

Until now we assumed that SIMPLEX was able to choose a pivot which increased the objective function, but for some LP's it is possible to construct k pivots from a basic solution s_0 where each pivot conserves the value of the objective function. The main concern appears when the kth pivot results in a basic solution s_k where $s_k = s_0$. This concept is called cycling and happens when specific rules are used to decide the pivoting. For example if the entering variable is decided based on picking the non-basic variable with the largest positive coefficient and the ties for most binding constraint are broken based on smallest subscript, cycles can occur.

Since LP has a finite number of states we can conclude that if SIMPLEX does not terminate then it must be cycling, otherwise it would imply that there is an infinite number of basic solutions.

To avoid cycling two different methods for picking the pivoting variables are presented. The perturbation method which uses an ordered set of m significantly decreasing epsilons from 1 to 0. That is

$$1 >> e_1 >> e_2 >> ... >> e_{m-1} >> e_m > 0 \tag{12}$$

Each of these epsilons are then added to the constraints to make sure that it is impossible to have two basic solutions with the same objective value. Therefore the slack formulation is now as follows:

$$z = \sum_{i=1}^{n} c_i x_i \tag{13}$$

$$x_{n+i} = e_i + b_i - \sum_{j=1}^{n} a_{ij} x_j \text{ for } i = 1, 2, ..., m \tag{14}$$

In practise the epsilons are not materialized but are based on symbols to circumvent the otherwise expensive calculation of all the epsilons.

The second method for avoiding cycling is to use small indices as indicator for chosen the pivot. That is for any two non-basic variables x_i and x_j, $i \leq j$ with positive coefficients x_i is picked for entering the basic variables. To leave the basic variables the most binding constraint is picking but in the case of ties, the smallest indices rule applies again. This simple rule has been proved to prevent cycles in SIMPLEX by R.G. Bland (Hence its other name "Bland's rule") in 1977 and is more often used in practise.

3 Auxiliary LP

Sometimes the first basic solution is infeasible, such a situation occurs when a constraint requires a variable to be greater than zero, that is one of the constraints are in the following form $x_i \geq b$ for $b \geq 0$. This makes the first basic solution where all the non-basic variables are 0 infeasible.

To find the first feasible solution the *auxiliary* LP form(L_{aux}) is introduced. L_{aux} is both feasible and bounded and the optimal value of L_{aux} will both indicate that the original LP is feasible and its first feasible basic solution. For an LP in the following standard form:

$$\text{max:} \quad \sum_{i=1}^{n} c_i x_i \tag{15}$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, ..., m \tag{16}$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, ..., n \tag{17}$$

The auxiliary LP has the following form:

$$\text{max:} \quad -x_0 \tag{18}$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \text{ for } i = 1, 2, ..., m \tag{19}$$

$$x_j \geq 0 \qquad \text{for } j = 0, 1, 2, ..., n + m \tag{20}$$

Here a new variable x_0 is introduced both in the objective function where it must be maximized but also in each constraint as a subtraction. We solve this new LP by SIMPLEX until an optimal solution is found.

There are two possible results to the L_{aux}, either $x_0 = 0$ or $x_0 \geq 0$. First we proof that if the original LP L is feasible then the optimal objective value of L_{aux} is 0. For a feasible solution S of L let $s_0 = 0$ and add it to S which is now a feasible solution to L_{aux} with objective value 0. Since $x_0 \geq 0$, $x_0 = 0$ is the optimal value for maximizing $-x_0$. For an optimal solution S to L_{aux} with $x_0 = 0$ then x_0 does not influence L in any way and therefore S is still an optimal solution for L.

In the other case where $x_0 \geq 0$ L must be infeasible. We assume that L is infeasible, then the optimal objective value of L_{aux} is not 0 and since x_0 is positive and therefore is part of the feasible solution $(s0, 0, 0, ..., 0)$, the objective value must be negative.