Designing a motion model, an introduction

Sensor fusion & nonlinear filtering

Lars Hammarstrand

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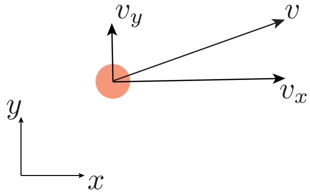
- **Translational kinematics.**
  Random walks, constant velocity and constant acceleration models are important examples. Here we often view objects as points objects.



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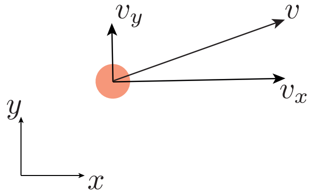
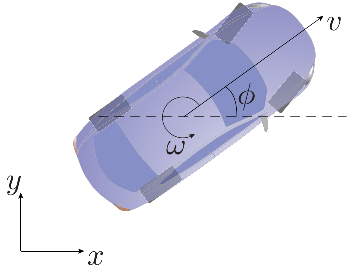


- **Rotational kinematics.**
  Useful when we wish to orient an object in $2D$ or in $3D$. The orientation may also be connected to the translation of the object.



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  - Motions are often conveniently described using differential equations.

    $\rightsquigarrow$ how can we discretize such models?
  - We may have a reasonable description of the noise in the time domain

    $\rightsquigarrow$ how can we describe the noise covariance in the discrete time model?

SELF-ASSESSMENT – ORIENTATION ESTIMATION

Why may it be of interest to include the orientation of, say, a car in a state vector:

- In some cases, we may be interested in estimating the orientation of the object.
- It is not possible to obtain a reasonable estimate of the position of the car (better than 10 meter accuracy) unless we know its orientation.
- Information about the orientation may yield better predictions of the future positions of the car.

SELF-ASSESSMENT – DISCRETIZATION

Why is it often useful to discretize a time continuous model?

- The computers that we use will introduce quantization error when they discretise our signals and it is therefore better that we do this ourselves.
- The discretized version provides a simpler model that we can use for prediction in our filters.
- Normally we obtain measurements at discrete time instants and in most situations it is sufficient to compute the posterior density at these time instants. For these reasons, it is sufficient to find the predicted density at the time when we receive the next measurement.

Discretization of continuous time systems

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DISCRETIZATION OVERVIEW

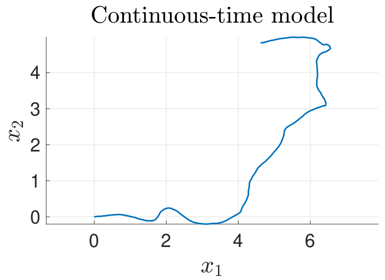
- Given a continuous-time motion model

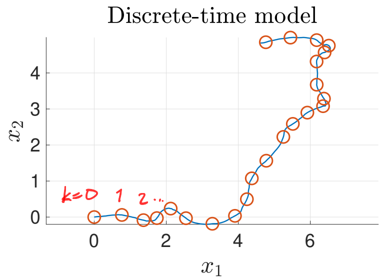
$$\dot{\mathbf{x}}(t) = \tilde{a}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$$

we would like to find a discrete-time motion model

$$\mathbf{x}_k = a(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}$$

where the discrete sequence is sampled from the continuous one, $\mathbf{x}_k = \mathbf{x}(kT)$.





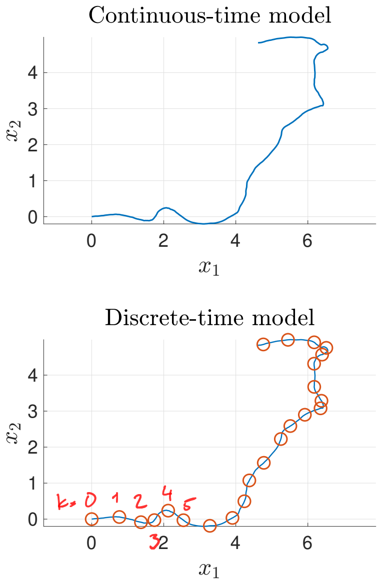
DISCRETIZATION OVERVIEW

$p(x_k|x_{k-1})$ x_k x_{k-1}

Current objective

- Express distribution of $\mathbf{x}(t+T)$ given $\mathbf{x}(t)$ for different continuous time motion models.

- Two important tools:
    1. The Euler discretization method.
    2. Analytical solution of linear systems.



THE EULER METHOD

- Suppose we have a differential equation (DE)

$$\dot{\mathbf{x}}(t) = \tilde{a}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t) \qquad (1)$$

Euler method

- A simple method to approximate the solution to a DE by using

$$\dot{\mathbf{x}}(t) \approx \frac{\mathbf{x}(t+T) - \mathbf{x}(t)}{T} \qquad (2)$$

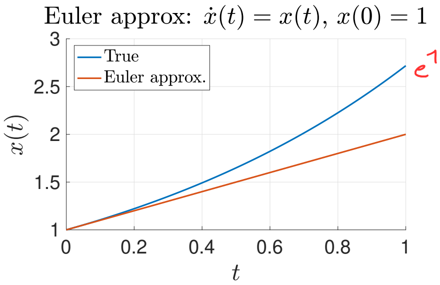
to find $\mathbf{x}(t+T)$.

Solution

$x(t+T) \approx x(t) + T\dot{x}(t)$

$= x(t) + T\tilde{a}(x(t)) + T\tilde{q}(t)$

$\Rightarrow x(1) = x(0) + 1 \cdot 1 = 1 + 1 = 2$



ANALYTICAL SOLUTION OF LINEAR SYSTEMS

- Suppose we have a linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{3}$$

  where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ is a constant vector.
- Solving for $\mathbf{x}(t + T)$:

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Note: $\frac{d}{dt} e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)$

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  2. Multiply both sides with $e^{-\mathbf{A}t}$: $\underbrace{e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)}_{\frac{d}{dt} e^{-At} x(t)} = e^{-\mathbf{A}t}\mathbf{b}$

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  3. Integrate both sides from t to $t+T$:

     $\Rightarrow e^{-\mathbf{A}(t+T)}\mathbf{x}(t+T) - e^{-\mathbf{A}t}\mathbf{x}(t) = \int_t^{t+T} e^{-\mathbf{A}\tau}\mathbf{b}\, d\tau$

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  4. Multiply with $e^{\mathbf{A}(t+T)}$:

     $$\Rightarrow \mathbf{x}(t+T) = e^{\mathbf{A}T}\mathbf{x}(t) + \int_0^T e^{\mathbf{A}\tau}\, d\tau\, \mathbf{b}$$

ANALYTICAL SOLUTION OF LINEAR SYSTEMS

- For the above linear system we found that

$$\mathbf{x}(t+T) = \exp(\mathbf{A}T)\mathbf{x}(t) + \int_0^T \exp(\mathbf{A}\tau)\, d\tau \mathbf{b},$$

  but what is $\int_0^T \exp(\mathbf{A}\tau)\, d\tau$?

- By definition $\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$

$$\Rightarrow \int_0^T \exp(\mathbf{A}\tau)\, d\tau = \mathbf{I}T + \frac{\mathbf{A}T^2}{2} + \frac{\mathbf{A}^2 T^3}{3!} + \dots$$

An analytical solution for linear systems

- If $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$ then

$$\mathbf{x}(t+T) = \exp(\mathbf{A}T)\mathbf{x}(t) + \left(\mathbf{I}T + \frac{\mathbf{A}T^2}{2} + \frac{\mathbf{A}^2 T^3}{3!} + \dots\right)\mathbf{b}$$

SELF ASSESSMENT

Both the above methods have strengths and weaknesses. Some of these are correctly described below.

Check all statements that apply.

- The Euler method is easy to apply also for nonlinear systems.
- The analytical solution is an exact solution also when T is large.
- The Euler method is more accurate when T is small than when T is large.

Discretizing linear models: the transition matrix

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FINDING THE TRANSITION MATRIX

- Suppose $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$ where $\tilde{\mathbf{A}}$ is a constant matrix.
- *How should we select* $\mathbf{A}_{k-1}$ *in* $\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$?

The Euler method

$$\mathbf{x}(t+T) \approx \mathbf{x}(t) + T(\tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t))$$

$$\Rightarrow \quad \mathbf{A}_{k-1} = \mathbf{I} + T\tilde{\mathbf{A}}$$

Exact solution for linear systems

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \int_t^{t+T} \exp(\mathbf{A}(t+T-\tau))\tilde{\mathbf{q}}(\tau)\, d\tau$$

$$\Rightarrow \quad \mathbf{A}_{k-1} = \exp(T\tilde{\mathbf{A}})$$

THE CONTINUOUS-TIME CONSTANT VELOCITY MODEL

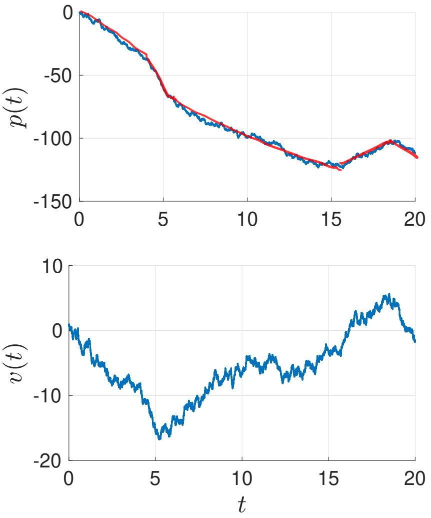
The constant velocity (CV) model

- Suppose we have a state vector $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) \end{bmatrix}^T$ where $p(t)$ is the position and $v(t)$ the velocity in one dimension.

- The continuous time constant velocity (CV) model for this state vector is:

$$\dot{x}(t) = \begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \tilde{\mathbf{q}}(t)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} q_c(t)$$



THE DISCRETE-TIME CONSTANT VELOCITY MODEL – TRANSITION MATRIX

- How should we select $\mathbf{A}_{k-1}$ in

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$

for the continuous-time CV model

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)?$$

$$\begin{bmatrix} P_k \\ V_k \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{k-1} \\ V_{k-1} \end{bmatrix} + q_{k-1}$$

Euler method

$$\begin{aligned} \mathbf{A}_{k-1} &= \mathbf{I} + T\tilde{\mathbf{A}} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Exact solution

$$\begin{aligned} \mathbf{A}_{k-1} &= \exp\left(\tilde{\mathbf{A}}T\right) \\ &= \mathbf{I} + \tilde{\mathbf{A}}T + \tilde{\mathbf{A}}^2 T^2/2 + \cdots = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

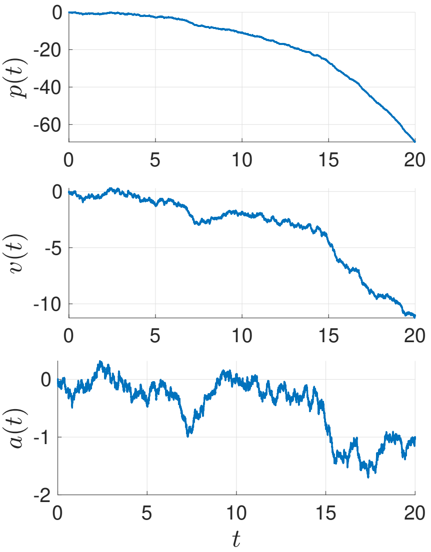
THE CONTINUOUS-TIME CONSTANT ACCELERATION MODEL

The constant acceleration (CA) model

- Suppose $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) & a(t) \end{bmatrix}^T$ where $p(t)$, $v(t)$ and $a(t)$ are position, velocity and acceleration in one dimension.
- The continuous time constant acceleration (CA) model for this state vector is:

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)$$

$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q_c(t) \right.$



THE DISCRETE-TIME CONSTANT VELOCITY MODEL – TRANSITION MATRIX

- How should we select $\mathbf{A}_{k-1}$ in

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$

for the continuous-time CA model

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)?$$

Euler method

$$\mathbf{A}_{k-1} = \mathbf{I} + T\tilde{\mathbf{A}} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$

Exact solution

$$\mathbf{A}_{k-1} = \exp\left(\tilde{\mathbf{A}}T\right) = \mathbf{I} + \tilde{\mathbf{A}}T + \tilde{\mathbf{A}}^2 T^2/2 + \ldots$$

$$= \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$

CV AND CA IN HIGHER DIMENSIONS

- In higher dimensions: assume motions in different dimensions are independent.
- The results using the exact discretization:

Constant velocity	Constant acceleration
$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{A}_{k-1} = \begin{bmatrix} \mathbf{I}_n & T\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}$	$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \mathbf{a} \end{bmatrix}, \quad \mathbf{A}_{k-1} = \begin{bmatrix} \mathbf{I}_n & T\mathbf{I}_n & T^2/2\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{I}_n & T\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}.$

$$\tilde{A} = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}$$

SELF ASSESSMENT

- Suppose we have a state vector $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) & \phi(t) \end{bmatrix}^T$ where $p(t)$ and $v(t)$ are position and velocity but $\phi(t)$ is an orientation angle in $2D$. It may then be reasonable to assume: $\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)$.

  Check the correct answer:

  - We would get

    $$\text{Euler: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{Analytical: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$

  - We would get

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Selecting the discrete time motion noise covariance

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CONTINUOUS-TIME MOTION NOISE

- Consider a linear continuous-time DE, $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, where $\tilde{\mathbf{q}}(t)$ is the motion noise.
- We usually assume that $\tilde{\mathbf{q}}(t)$ is a white Gaussian noise process:

$$\begin{cases} \mathbb{E}\{\tilde{\mathbf{q}}(t)\} = 0 & \text{i.e., zero mean} \\ \mathrm{Cov}\{\tilde{\mathbf{q}}(\tau_1), \tilde{\mathbf{q}}(\tau_2)\} = \delta(\tau_1 - \tau_2)\tilde{\mathbf{Q}} & \text{i.e., uncorrelated} \end{cases}$$

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- A simple example: the Wiener process $\mathbf{w}(t) = \int_0^t \tilde{\mathbf{q}}(\tau)\, d\tau$, which has zero mean and covariance

$$\begin{aligned} \mathbb{E}\{\mathbf{w}(t)\mathbf{w}(t)^T\} &= \mathbb{E}\left\{\int_0^t \tilde{\mathbf{q}}(\tau_1)\, d\tau_1 \int_0^t \tilde{\mathbf{q}}(\tau_2)^T\, d\tau_2\right\} \\ &= \int_0^t \int_0^t \underbrace{\mathbb{E}\left\{\tilde{\mathbf{q}}(\tau_1)\tilde{\mathbf{q}}(\tau_2)^T\right\}}_{\delta(\tau_1 - \tau_2)\tilde{\mathbf{Q}}} d\tau_2\, d\tau_1 = \int_0^t 1\, d\tau_1 \tilde{\mathbf{Q}} = t\tilde{\mathbf{Q}} \end{aligned}$$

DISCRETE-TIME MOTION NOISE

- Consider a linear continuous-time DE

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t),$$

  where $\tilde{\mathbf{q}}(t)$ is a white Gaussian noise process.

- We seek a discrete time motion model

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \quad \mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1}),$$

  but how can we select $\mathbf{Q}_{k-1}$?

- Note:
  - $\mathbf{Q}_{k-1} = \mathrm{Cov}\{\mathbf{q}_{k-1}\} = \mathrm{Cov}\{\mathbf{x}_k \big| \mathbf{x}_{k-1}\} = \mathrm{Cov}\{\mathbf{x}(t+T) \big| \mathbf{x}(t)\}.$

EXACT SOLUTION FOR LINEAR SYSTEMS

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, then

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \underbrace{\int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{q}}(\tau)\,d\tau}_{q_{k-1}}.$$

$$x_k = A_{k-1}\, x_{k-1} +$$

- The discrete time noise covariance is therefore:

$$\begin{aligned}
\mathbf{Q}_{k-1} &= \mathrm{Cov}\{\mathbf{x}(t+T) \big| \mathbf{x}(t)\} \\
&= \mathrm{Cov}\left\{\int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{q}}(\tau)\,d\tau\right\} = \dots \\
&= \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{Q}}\exp\left(\tilde{\mathbf{A}}^T\tau\right)\,d\tau
\end{aligned}$$

MODIFIED EULER METHOD

- Given a DE $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$
- We can view the Euler method as the approximation

$$\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t) \quad \text{for all } \tau \in [t, t+T]$$

$$\Rightarrow \ \mathbf{x}(t+T) = \mathbf{x}(t) + T(\tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t))$$

Modified Euler

- In the modified Euler method, we use $\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(\tau)$,

$$\begin{aligned} \Rightarrow \quad \mathbf{x}(t+T) &= \mathbf{x}(t) + \int_t^{t+T} \dot{\mathbf{x}}(\tau)\, d\tau \\ &= \mathbf{x}(t) + \underbrace{\int_t^{t+T} \tilde{\mathbf{a}}(\mathbf{x}(t))\, d\tau}_{T\tilde{\mathbf{a}}(\mathbf{x}(t))} + \int_t^{t+T} \tilde{\mathbf{q}}(\tau)\, d\tau. \end{aligned}$$

$$\Rightarrow \mathrm{Cov}\{\mathbf{x}(t+T) \,|\, \mathbf{x}(t)\} \approx T\tilde{\mathbf{Q}}.$$

TWO METHODS TO SELECT $\mathbf{Q}_{K-1}$

- We have proposed two methods to select the noise covariance matrix.

For linear continuous time systems

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, we can use

$$\mathbf{Q}_{k-1} = \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right) \tilde{\mathbf{Q}} \exp\left(\tilde{\mathbf{A}}^T\tau\right) \, d\tau$$

For nonlinear continuous time systems

- For linear and nonlinear systems $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$, we can use

$$\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}}$$

THE CONSTANT VELOCITY MODEL

- In many cases, the motion noise is zero on some of the state variables.
- Using a matrix $\mathbf{\Gamma}$, we can then express the motion noise as

$$\tilde{\mathbf{q}}(t) = \mathbf{\Gamma}\mathbf{q}_c(t)$$

where $\mathbf{q}_c(t)$ is the motion noise in some dimensions.

$$\Rightarrow \quad \tilde{\mathbf{Q}} = \mathbf{\Gamma}\mathbf{Q}_c\mathbf{\Gamma}^T$$

- The constant velocity model is a good example:

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{\Gamma}} q_c(t) \Rightarrow \quad \tilde{\mathbf{Q}} = \mathbf{\Gamma} Q_c \mathbf{\Gamma}^T = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}$$

SELF ASSESSMENT, PART 1

Things we know about CV and the modified Euler method:

- For linear and nonlinear systems $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$, we can use $\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}}$.
- The constant velocity model can be described as:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} q_c(t) \Rightarrow \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}$$

- The modified Euler method thus suggests that we use

  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & T^2 \end{bmatrix} Q_c$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q_c T$

  Check the correct statement.

SELF ASSESSMENT, PART 2

Summary of CV and the exact solution for linear systems:

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, we can use $\mathbf{Q}_{k-1} = \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right) \tilde{\mathbf{Q}} \exp\left(\tilde{\mathbf{A}}^T\tau\right)\, d\tau$
- We know that for the constant velocity model we have:

$$\Rightarrow \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}, \exp(\tilde{\mathbf{A}}\tau) = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \exp(\tilde{\mathbf{A}}^T\tau) = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}.$$

The exact solution for linear systems is

- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^3 & T^2 \\ 0 & T \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^2 & T \\ T & 1 \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} Q_c$

Check the correct statement.

Measurement models

Sensor fusion & nonlinear filtering

Lars Hammarstrand

MEASUREMENT MODELS

- A measurement model relates the measurement, $\mathbf{y}_k$, to the state vector, $\mathbf{x}_k$.
- Our models can often be expressed as

$$\mathbf{y}_k = h_k(\mathbf{x}_k) + \mathbf{r}_k, \qquad \mathbf{r}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k).$$

  or, more generally,

$$p(\mathbf{y}_k | \mathbf{x}_k).$$

- The list of useful and important sensors is long:
  radar, laser scanners, GNSS (e.g., GPS), accelerometers, gyroscopes, cameras, etc.

EXAMPLES OF MEASUREMENT MODELS

Global navigation and satellite system (GNSS)

State: $\mathbf{x}_k = \begin{bmatrix} p_k^1 & p_k^2 & v_k^1 & v_k^2 \end{bmatrix}^T$

Observation: noisy position in 2D

$$\mathbf{y}_k = \begin{bmatrix} p_k^1 \\ p_k^2 \end{bmatrix} + \begin{bmatrix} r_k^1 \\ r_k^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{H_k} \mathbf{x}_k + \mathbf{r}_k$$

Gyroscope (yaw-rate sensor)

State: $\mathbf{x}_k = \begin{bmatrix} p_k^1 & p_k^2 & v_k & \phi_k & \omega_k \end{bmatrix}^T$

Observation: noisy observation of yaw rate

$$\mathbf{y}_k = \omega_k + r_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k + r_k$$

EXAMPLES OF MEASUREMENT MODELS

Radar sensor

State: $\mathbf{x}_k = \begin{bmatrix} p_k^1 & p_k^2 & v_k^1 & v_k^2 \end{bmatrix}^T$

Observation: noisy observation of distance and angle

$$\mathbf{y}_k = \begin{bmatrix} \sqrt{(p_k^1)^2 + (p_k^2)^2} \\ \arctan\left(\frac{p_k^2}{p_k^1}\right) \end{bmatrix} + \begin{bmatrix} r_k^1 \\ r_k^2 \end{bmatrix}$$

Wheel speed encoders

State: $\mathbf{x}_k = \begin{bmatrix} p_k^1 & p_k^2 & v_k^1 & v_k^2 \end{bmatrix}^T$

Observation: noisy observation of speed

$$y_k = f_c\left(\sqrt{(v_k^1)^2 + (v_k^2)^2} + r_k\right)$$

SENSOR CALIBRATION AND BIAS FILTERING

- Suppose our sensor has an offset, or a bias, $\mathbf{s}$ such that we observe

$$\mathbf{y}_k = h_k(\mathbf{x}_k) + \mathbf{s} + \mathbf{r}_k, \qquad \mathbf{r}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k).$$

  instead of just $\mathbf{y}_k = h_k(\mathbf{x}_k) + \mathbf{r}_k$.
- If $\mathbf{s}$ is constant, it can usually be estimated from a set of training data.
- For low-quality sensors it is common that the bias drifts significantly over time.
- Common solution: include $\mathbf{s}_k$ in the state vector and describe its motion as a random walk

$$\mathbf{s}_k = \mathbf{s}_{k-1} + \mathbf{q}^s_{k-1}.$$

  $\rightsquigarrow$ Our filter now jointly estimates the kinematic states and the bias.

SELF ASSESSMENT

In many cases, the sensor model has to be adjusted to the geometry of the particular problem at hand. Here is a simple example of that type.

Suppose that we have a radar sensor positioned at (p_s^1, p_s^2) that observes range (distance), y_k^r. Suppose again that we have a state vector $\mathbf{x}_k = \begin{bmatrix} p_k^1 & p_k^2 & v_k^1 & v_k^2 \end{bmatrix}^T$.

Select one of the following possible measurement models:

- $y_k = \sqrt{(p_k^1 - p_s^1)^2 + (p_k^2 - p_s^2)^2} + r_k$
- $y_k = \sqrt{(p_k^1)^2 + (p_k^2)^2} + r_k$
- $y_k = \sqrt{(p_s^1)^2 + (p_s^2)^2} + r_k$
- $y_k = \arctan(p_k^2 / p_k^1) + r_k$

Nonlinear motion models

Sensor fusion & nonlinear filtering

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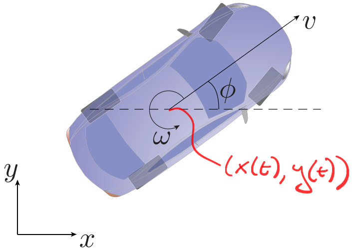
A COORDINATED TURN MODEL

Continuous-time coordinated turn model

- Assumptions:
  - Heading $\phi(t)$ is described by a CV model.
  - Velocity is a Wiener process.
- State vector and motion model are summarized as:

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{v}(t) \\ \dot{\phi}(t) \\ \dot{\omega}(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} v(t)\cos(\phi(t)) \\ v(t)\sin(\phi(t)) \\ 0 \\ \omega(t) \\ 0 \end{bmatrix}}_{\tilde{\mathbf{a}}(\mathbf{x}(t))} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{\Gamma}} \underbrace{\begin{bmatrix} q_c^v(t) \\ q_c^\omega(t) \end{bmatrix}}_{\mathbf{q}_c(t)}$$



THE EULER METHOD

- According to the (modified) Euler method:

$$\mathbf{x}(t+T) \approx \mathbf{x}(t) + T\tilde{\mathbf{a}}(\mathbf{x}(t)) + \int_t^{t+T} \tilde{\mathbf{q}}(\tau)\, d\tau.$$

- From this we obtain the discrete time motion model

$$\mathbf{x}_k = \mathbf{x}_{k-1} + T\tilde{\mathbf{a}}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \Leftrightarrow \begin{bmatrix} x_k \\ y_k \\ v_k \\ \phi_k \\ \omega_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + Tv_{k-1} \cos(\phi_{k-1}) \\ y_{k-1} + Tv_{k-1} \sin(\phi_{k-1}) \\ v_{k-1} \\ \phi_{k-1} + T\omega_{k-1} \\ \omega_{k-1} \end{bmatrix} + \mathbf{q}_{k-1},$$

where $\mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and

$$\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}} = T\mathbf{\Gamma} \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix} \mathbf{\Gamma}^T = \operatorname{diag}\left(0, 0, T\sigma_v^2, 0, T\sigma_\omega^2\right)$$

EXACT SOLUTION TO DETERMINISTIC CT

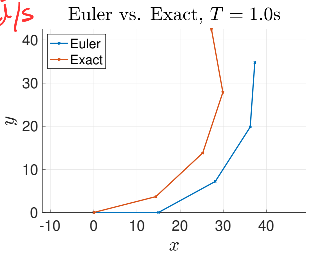
- It turns out that it is possible to solve the deterministic part analytically.
- Combined with modified Euler for the noise we get:

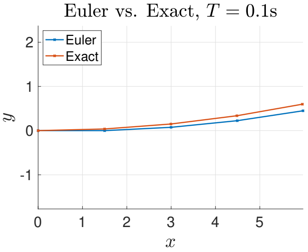
$$\begin{bmatrix} x_k \\ y_k \\ v_k \\ \phi_k \\ \omega_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \cos\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ y_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \sin\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ v_{k-1} \\ \phi_{k-1} + T\omega_{k-1} \\ \omega_{k-1} \end{bmatrix} + \mathbf{q}_{k-1},$$

where

$\mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\mathbf{Q}_{k-1} = \text{diag}(0, 0, 0, T\sigma_v^2, 0, T\sigma_\omega^2)$.

$V_0 = 15\,m/s$
$\omega_0 = .5\ rad/s$





DIRECT DISCRETIZATION OF THE NOISE

- In order to simplify the derivation of the noise covariance, it is common to assume a simpler distribution for $\tilde{\mathbf{q}}(t)$.

- Idea: assume that the noise $\tilde{\mathbf{q}}(t)$ is piecewise constant between samples, i.e., that it is constant in every interval, $[0, T]$, $[T, 2T]$, etc.
  $\rightsquigarrow$ This greatly simplifies the derivation of $\mathbf{Q}_{k-1}$ in many examples.

- **Covariance of $\tilde{\mathbf{q}}(t)$?**
  Assume that $\tilde{\mathbf{q}}(t) \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbf{Q}}/T)$.

  Why divide by T? Note that $\int_0^T \tilde{\mathbf{q}}(t)\, dt = T\tilde{\mathbf{q}}(t)$ then has covariance $\mathbb{E}\{T\tilde{\mathbf{q}}(t)\tilde{\mathbf{q}}(t)^T T\} = T\tilde{\mathbf{Q}}$.

DISCRETIZED LINEARIZATION

- Solving a nonlinear differential equation is not easy!

Linearized motion model

- We can linearize $\tilde{\mathbf{a}}$ about some estimate $\hat{\mathbf{x}}(t)$:

$$\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\hat{\mathbf{x}}(t)) + \tilde{\mathbf{a}}'(\hat{\mathbf{x}}(t))\,(\mathbf{x}(\tau) - \hat{\mathbf{x}}(t)) + \tilde{\mathbf{q}}(\tau), \quad \tau \in [t, t+T].$$

- Suppose that $\tilde{\mathbf{q}}(\tau)$ is also constant for $\tau \in (t, t+T)$.
- An accurate approximation of $\mathbf{x}(t+T)$ is obtained using:

An analytical solution for linear systems

- If $\dot{\mathbf{x}}(\tau) = \tilde{\mathbf{A}}\mathbf{x}(\tau) + \mathbf{b}$ for $\tau \in [t, t+T]$, then

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \left(\mathbf{I}T + \frac{\tilde{\mathbf{A}}T^2}{2} + \frac{\tilde{\mathbf{A}}^2T^3}{3!} + \dots\right)\mathbf{b}$$

SELF ASSESSMENT

The coordinated turn (CT) model is nonlinear and therefore harder to handle than for instance a CV or a CA model. We should thus only use it if yields better predictions. Here is a question to check if you understand some of the differences between CV and CT:

1. Without noise, the CT and CV models always describe motions along straight lines.
2. Without noise, the CT model describes the motion along a circle whereas the CV model describes the motion along a straight line.
3. Without noise, the CT model cannot describe motions along a straight line since it has a parameter ω that describes the yaw rate (that is, how fast the object is turning).

Check all that apply.