Graphs

[Ref: Discrete Mathematics, Rosen, Chapter 10.1 -10.5]

1.1 Graphs and Graph Models.

Graphs are discrete structures consisting of vertices and edges that connect the vertices. Problems of different kinds can be modeled using graphs. So study of graph is very important in the context of discrete mathematics.

What is a graph?

Definition. A graph $G = (V, E)$ consists of V, a nonempty set of vertices and E, a set of edges. Each edge connects two vertices of the graph.

Example. Consider a computer network where each point represents a data center and an edge a communication link between the two data centers.

Simple graph. In a simple graph, two conditions are met: (i) each edge connects two different vertices, i.e., there is no loop edge and (ii) no two edges connect same pair of vertices, i.e., multiple edges between same vertices pair are not allowed.

Example. Consider a graph where vertices represent persons and edges between persons represent a friendship between them. This graph can be termed as “Friendship Graph”.

Multigraphs. In multigraphs, there can be multiple edges between two vertices, however loops are not allowed.

Example. You may consider the computer network graph as a multigraph since there can be multiple communication links between two data centers.

Pseudographs. In pseudographs, multiple edges are allowed and loops are allowed.

Example. You may consider the computer network graph as a pseudograph if you allow loops as well. Loops can be modeled as feedback communication links between the same data center.

So far, we have dealt with undirected graphs where graph edges have no directions. In various problems, edges can have directions that represent special relationship. For example, in consider a family graph where vertices are persons and edges represent parent child relationship. In this graph, the edges can be directed where the direction may be used to explicitly show the parent and child.

Directed graph. A directed graph, also called digraph, consists of a nonempty set of vertices V and a set of directed edges E. Each directed edge is associated with an *ordered* pair of vertices. A directed edge associated with the ordered pair (u, v) is said to start at u and ends at v.

Simple directed graph. A directed graph with no loops and has no multiple directed edges between the same *ordered* pair of vertices.

Directed multigraph. A directed multigraph has multiple directed edges between the same *ordered* pair of vertices.

Some examples of graphs.

Hollywood graph. This graph represents actors by vertices and connects two vertices when the actors represented by these vertices have worked together on a movie. This graph is a simple (undirected) graph because its edges are undirected, it contains no loops, and it contains no multiple edges. In January 2006, this graph had 637,099 vertices and more than 20 million edges.

Web graph. The world wide web can be modeled as a directed graph where each web page is represented as a vertex and there is an edge that starts at a and ends at b if there is a link in web page a to the web page b. Currently, the web graph has more than three billion vertices and 20 billion edges.

Roadmaps. In such graphs, vertices represent intersections and edges represent roads. Undirected edges represent two-way roads and directed edges represent one-way roads. Loops and multiple edges are allowed.

Exercises Rosen 10.1: 3 – 12,

1.1 Graphs Terminology and Special Types of Graphs

Definitions for undirected graph

Adjacent. Two vertices u and v in an undirected graph are called adjacent if u and v are endpoints of an edge.

Incident. If e is an edge associated with $\{u, v\}$ in an undirected graph then e is called incident with the vertices u and v.

Degree. The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of the vertex.

Example. What are the degrees of the vertices in following graph?

Isolated. A vertex of degree zero is called in isolated vertex. It is not adjacent to any vertex.

Pendant. A vertex of degree one is called a pendant. It is adjacent to exactly one vertex.

Theorem 1. The Handshaking Theorem.

Let, $G = (V, E)$ be an undirected graph with e edges. Then $2e = \sum_{u \in V} \deg(u)$.

Theorem 2. An undirected graph has an even number of vertices of odd degree.

Proof. Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$. Then

$2e = \sum_{u \in V} \deg(u) = \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u)$.

Because $\deg(u)$ is even for $u \in V_1$, the first term is even. Furthermore, the sum of two terms in the right hand side is even, because the sum is $2e$. Hence, the second term is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree.

Definitions for directed graph

Adjacent to/from. When (u, v) is an edge, u is said to be adjacent to v and v is said to be adjacent from u. u is called the initial vertex and v is called the terminal vertex.

In-degree/Out-degree. The in-degree of a vertex v, denoted by $\deg^-(v)$ is the number of edges with v as the terminal vertex. The out-degree of a vertex v, denoted by $\deg^+(v)$ is the number of edges with v as the initial vertex.

Example. Find the in-degree and out-degree of the vertices in the following graph.

Theorem 3. Let $G = (V, E)$ be a graph with directed edges. Then $\sum_{u \in V} \deg^-(v) = \sum_{u \in V} \deg^+(v) = |E|$.

Some special simple graphs

Complete graphs. The complete graph K_n of n vertices is the simple graph that contains exactly one edge between each pair of distinct vertices.

Example. The following examples show K_3, K_4

Cycles. The cycle C_n of n vertices, where $n \geq 3$, consists of n edges $\{1,2\}, \{2,3\}, \dots, \{n-1, n\}$.

Example. The following examples show C_3, C_5.

Wheels. We obtain the wheel W_n when we add an additional vertex to Cycle C_n and connect this new vertex to every other vertices.

Example.

Cube. The n-dimensional cube Q_n is the graph that has vertices representing the corners of a cube.

Bipartite graphs. The vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset.

Example. Consider the marriage graph. Its vertex set can be divided into two subsets: a subset of men vertices and a subset of women vertices. C_6 is a bipartite graph. K_3 is not bipartite.

Theorem 4. A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof.

First, suppose that $G = (V, E)$ is a bipartite graph. Then $V = V_1 \cup V_2$. If we assign one color to all vertices of V_1 and then a second color to each vertex of V_2 then no two adjacent vertices are assigned the same color.

Now suppose that it is possible to assign colors to the vertices using just two colors so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then V_1 and V_2 are disjoint. Furthermore every edge connects a vertex in $V_1$and a vertex in V_2 because no two adjacent vertices are either in V_1 or in V_2.

Complete bipartite graph. The complete bipartite graph $K_{m,n}$ is the graph whose vertex set can be divided into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in first subset and the other vertex is in the second subset.

Example. The following graph is $K_{2,3}$.

New Graphs from Old

Subgraph. A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$. A subgraph H is a proper subgraph of G if $H \neq G$.

Example.

Union. The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Exercise 18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

Exercise 26. For which values of n, are these graphs bipartite? (a) K_n (b) C_n (c) W_n (d) Q_n.

Exercise 29. How many vertices and edges do these graphs have? (a) K_n (b) C_n (c) W_n (d) Q_n.

Exercise 31. Find the degree sequence of each of the graphs. (a) K_n (b) C_n (c) W_n (d) Q_n (e) $K_{m,n}$.

Exercise 43. How many subgraphs with at least one vertex does K_3 have?

Exercise 44. How many subgraphs with at least one vertex does W_3 have?

Exercise 46. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G and let m be the minimum degree of the vertices of G. Show that (a) $\frac{2e}{v} \geq m$ (b) $\frac{2e}{v} \leq M$.

Solve exercises 10.2: 5, 13, 17-26, 29, 31, 37, 42-45, 46, 47, 53, 57, 58, 59

1.3 Representing Graphs

Graphs can be represented in several ways. Among them two most common methods are adjacency lists and adjacency matrix.

Adjacency lists. In this method, a graph is represented using lists where there is a list for each vertex and each list specifies the list of vertices that are adjacent to the corresponding vertex.

Adjacency lists can be used to represent both simple graphs and simple directed graphs.

Example.

Adjacency matrices. Graphs can be represented using an $n \times n$ matrix which is called the adjacency matrix. Each cell (i, j) stores a 1 if $\{v_i, v_j\}$ is an edge in the Graph; otherwise stores 0. So adjacency matrix is a zero-one matrix for a simple graph.

Note that adjacency matrix is not unique for a graph. The matrix will be different if we number the vertices differently. Since there are $n!$ different numberings are possible, there are $n!$ different adjacency matrices of a graph.

The adjacency matrix of a simple graph is symmetric, that $a_{ij} = a_{ji}$, because $\{v_i, v_j\}$ is an edge implies that both (i, j) and (j, i)th entry will be one. Note that the diagonal cells of the matrix contain all 0s, since there is no loop in the simple graph.

Example.

Adjacency matrix for pseudograph. In a pseudograph, there are loops and multiple edges between the same pair of vertices. In such cases, adjacency matrix is not anymore a zero-one matrix. Instead each cell (i, j) contains a value that equals to the number of edges between the vertex pair $\{v_i, v_j\}$.

Adjacency matrix for directed graphs. For directed graphs, adjacency matrices does not have to be symmetric because (v_i, v_j) does not also imply (v_j, v_i).

Example.

Adjacency lists vs. adjacency matrices.

Space requirement. When the graph is sparse, i.e., contains fewer number of edges, then representing the graph using adjacency lists might be preferable. An adjacency matrix takes n^2 space irrespective of the number of edges present in the graph while an adjacency lists takes space proportional to the number of total edges in the graph. So from the space usage perspective, adjacency lists is better than adjacency matrix.

Finding whether $\{v_i, v_j\}$ is an edge in the graph. From the matrix, we can answer this by inspecting the matrix cell (i, j) whether it contains a 1 taking only one comparison. In case of adjacency lists, we have to scan the entire list of v_i to determine whether v_j is present taking at most n comparisons.

Finding the degree of a vertex v_i. From the matrix, we can compute this answer by counting the number of 1s in the matrix's ith row. This takes n comparisons. In case of adjacency lists, we just count the number of vertices in the list which takes only deg(v_i) operations.

Isomorphism of Graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one mapping from f from V_1 to V_2 such that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2. Such a function f is called isomorphism.

Example. Show the graphs G and H are isomorphic.

Show that two graphs are not isomorphic. Although it is very difficult to determine whether two graphs are isomorphic, it may not be hard to find whether two graphs are not isomorphic. There are certain properties of graphs that must be held in both graphs. For example, some such properties are as follows: (i) Number of vertices, (ii) Number of edges, (iii) degree sequence, (iv) For each vertex, degree of adjacent vertices must be same in both graphs.

Example. Show that the following two graphs are not isomorphic.

Show that two graphs are isomorphic. We can show this if we can define a mapping function first, and then can show that adjacency matrices of both graphs are same.

Example. Show that the following two graphs are isomorphic.

Exercises Rosen 10.3: 1-25,

1.4 Connectivity in Graphs

A path is a sequence of edges that begins at a vertex of the graph and travels from vertex to vertex along edges of the graph.

Path. Let n be a non-negative integer ad G an undirected graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \dots, e_n$ such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, with e_n associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$ because listing these vertices uniquely determines the path. A path can be length zero.

Simple path. A path is simple if it does not contain the same vertex more than once.

Circuit. A path is a circuit if it begins and ends at the same vertex and has length greater than zero.

Simple circuit. A circuit is simple if it does not contain the same vertex more than once except the first and last.

Paths and circuits in directed graphs. The same definition applies here except that edges are represented as ordered pairs, for example (x_0, x_1) instead of $\{x_0, x_1\}$.

Example. What does a path in friendship graph represent? What does a path in collaboration graph represent? What does a path in Hollywood graph represent?

In collaboration graph of mathematics, an Erdos number of a vertex is the length of a shortest path between the vertex and the prolific mathematician Paul Erdos.

In the Hollywood graph, the Bacon number of an actor c is the length of the shortest path from c to the well-known actor Kevin Bacon.

Connectedness in undirected graphs

Connected. An undirected graph is called connected if there is a path between every pair of vertices.

Example.

Theorem 1. There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof. Let u and v be two distinct vertices of the connected undirected graph G. Because G is connected, there is a path from u to v. Let $x_0, x_1, \dots, x_n$ be the vertex sequence of a path of least length where $x_0 = u$ and $x_n = v$. This path of least length is simple. To see this, suppose this is not simple. Then for some i and j two vertices x_i and x_j are same where $0 \le i < j$. Then we can find a smaller path be deleting all edges corresponding to the vertex sequence $x_i, x_{i+1}, \dots, x_j$. This gives us a new smaller path of vertex sequence $x_0, x_1, \dots, x_i, x_{j+1}, \dots, x_n$.

Connected component. A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected component of G. That is a connected component of a graph G is a maximal connected subgraph of G.

A graph G that is connected has only one connected component, i.e., the graph itself.

A graph Gthat is not connected has two or more connected components that are disjoint and have G as their union.

Example. What do the connected components of a friendship graph represent?

Connectedness in directed graphs

Strongly connected. A directed graph is strongly connected if for every pair of vertices a and b there is a path from a to b and there is a path from b to a.

Example.

Strongly connected components. The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs are called the strongly connected components of G.

Example.

Counting paths between vertices

Theorem 2: Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \dots, v_n$ of the vertices of the graph. The number of different paths of length r from v_i to v_j equals to the (i, j)th entry of A^r.

Proof. We prove this theorem using mathematical induction on r.

Base condition. For $r = 1$, the (i, j)th entry denote the number of paths from v_i to v_j because this is the number of distinct edges from v_i to v_j.

Induction. Assume that the (i, j)th entry of A^r represent the number of paths of length r from v_i to v_j. Now, $A^{r+1} = A^r * A$. Each entry of the matrix A^{r+1} is computed as follows:

$b_{i1} * a_{1j} + b_{i2} * a_{2j} + \dots + b_{in} * a_{nj}$ where b_{ik} is the (i, k)th entry of A^r. A path of length $r + 1$ from v_i to v_j is made up of a path of length r from v_i to some intermediary vertex v_k followed by an edge from v_k to v_j. By the product rule of counting, the number of such paths through v_k is the product of the number of paths from v_i to v_k (b_{ik}) and the number of edges from v_k to v_j (a_{kj}). Finally the count of all such paths is summed over all possible $v_k s$ and the result is the number of paths of length $r + 1$ from v_i to v_j.

Exercise 30: Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

Exercise 45: Show that a simple graph G with n vertices is connected if it has more than (n − 1)(n − 2)/2 edges.

Exercise Rosen 10.4: 11, 14, 15, 18, 19, 27, 28, 30, 43 - 45

1.5 Euler and Hamilton Path

Euler paths and circuits

The town of Konigsberg, Prussia, was divided into four sections by the branches of Pregel river. Seven bridges connected these regions. The town people took long walks and wondered whether it was possible to start at some location in the town, travel across all the bridges without crossing any bridge twice, and return to the starting point. The problem can be modeled using a multigraph (why not simple?) as follows.

Problem: Is there a circuit in the multigraph that contains every edge exactly once?

Euler circuit: An Euler circuit in a graph G is a circuit that contains every edge of G exactly once. An Euler path in G is a path containing every edge of G exactly once.

Example.

Necessary and sufficient condition for Euler circuit: Every vertex must have even degree.

Proof.

(*Necessary condition*) If a multigraph has en Euler circuit, then every edge of the graph must have even degree. To prove by contradiction, assume that there is at least one vertex which has odd degree. Say the vertex is v and the edges incident on v are e_1, e_2, e_3. Say, the Eucler circuit enter the vertex through e_1., then exits the vertex through e_2, and again enter the vertex through e_3, but has no way to exit it! It must exit the vertex, because it started the tour from some other vertex (why, because it enters the vertex through e_1). Otherwise, if it did start from v, then say it exits through e_1, re-enters through e_2, and again exits throught e_3,but has no way to re-enter! It must re-enter the vertex as it started the tour from v. [Contradiction proof complete]

(*Sufficient condition*) If ever edge of the multigraph has even degree, then the graph has an Eucler circuit. We will show how to construct an Euler circuit as follows. Start the circuit from any arbitrary vertex a, the continue visiting edges one by one until you are stuck. You can be stuck by two ways: you reach a vertex which has no other unvisited edge left. As we started from a, we must end at a. Why? Simply because for any other vertex, we enter the vertex using some edge, and then exit the vertex using some other edge. As every vertex has an even degree, we must have an edge to exit from an intermediate vertex. So only a is the vertex where we can get stuck.

Now two cases may occur: we have already visited all edges or there are some edges left to be visited. In the first case, we are done. In the second case, we delete the edges of the previous circuit and any vertex that is isolated after deleting edges. Then there must be at least one vertex remaining in the left over graph which was present in the previous circuit. We choose such a vertex b and continue to form another circuit similar to the earlier circuit (we can do this because remaining vertices also have even degree). Then we form a combined circuit as follows: visit the first half of the previous circuit up to vertex b, then visit the second circuit fully to return to b, then visit the last half of the first circuit and return to a. In this way, we form new circuits, and combine each new circuit with the old circuit to form a combined Eucler circuit. This process must terminate since the graph is finite and at least one edge is being deleted after every stage.

Exercise: Show that a connected multigraph has an Euler path (and not an Euler circuit) ifand only if it has exactly two vertices of odd degree. [**See book page 697 for the proof**]

Theorem 1: A connected multigraph with at least two vertices has an Eucler circuit if and only if each of its vertices has even degree.

Theorem 2: A connected multigraph has an Euler path if and only if it has exactly two vertices of odd degree.

Hamilton paths and circuits

Consider a problem similar to Euler circuit. Can we form a *simple* circuit and path that contain every vertex of the graph exactly once?

Definition 2. A simple path that passes through every vertex exactly once is called a Hamilton path, and a simple circuit that passes through every vertex exactly once is called a Hamilton circuit.

Example. The terminology comes from a game, called the Icosian Puzzle (A voyage round the world), invented in 1857 by the Irish mathematician Sir William Rowan Hamilton. This consisted of a dodecahedron with 12 faces and 20 vertices. Each vertex was labeled with the name of a city. The objective was to start at a city and travel along the edges of the dodecahedron that visits the 19 other cities exactly once and end back at the first city.

There is no necessary and sufficient condition for having Hamilton circuits. But there are two important sufficient conditions.

Dirac's theorem. If G is a simple graph with n vertices where $n \geq 3$ such that the degree of every vertex is at least $n/2$, then G has a Hamilton circuit.

Ore's theorem. If G is a simple graph with n vertices where $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices in G, then G has a Hamilton circuit.