Question 1. (5+5+10 points) Simplify each of the following expressions.

1. $C(n-2, m) + 2C(n-2, m-1) + C(n-2, m-2)$
2. $C(n-3, m) + 3C(n-3, m-1) + 3C(n-3, m-2) + C(n-3, m-3)$
3. $\sum_{k=n}^{n+r} C(k, k-n)$

Question 2. (10+10+10 points) Suppose you start at the origin of coordinates (point $(0,0)$) and flip a fair coin: You move right by 1 and simultaneously either go up by 1 (in case the coin toss was "heads") or down by 1 (if the coin toss was "tails"). Then you do the same starting from your new position, and again for a total of n such coin flips. For example, if you are at (a, b) then the next coin flip takes you to either $(a+1, b+1)$ or $(a+1, b-1)$. If all n coin flips are "heads" you end up at position (n, n) and if all are "tails" you end up at position $(n, -n)$. We say that a path is *legal* if it follows the above rule of motion, i.e., if the next step from (a, b) takes you either to $(a+1, b+1)$ or to $(a+1, b-1)$.

1. What is the probability $P_{n,m}$ that, starting at point $(0,0)$, you end up at point (n, m) after the n coin flips? Assume m is positive, and make sure you consider all the possible cases for m and n.

2. Assume that point (n, m) is reachable from point $(0,0)$ (i.e., that there is at least one legal path between them), where $m > 0$. Let r be an integer between 1 and $(n-m)/2$. Prove that the number of legal paths from $(0,0)$ to (n, m) that touch the horizontal line that is r units below the x-axis, is equal to the number of legal paths from $(0, -2r)$ to (n, m).

   Hint. It suffices to define a bijection (one-to-one and onto function) between the set of legal paths that start at $(0, -2r)$ and end at (n, m), and the set of legal paths that start at $(0,0)$ and end at (n, m) after touching (one or more times) the horizontal line that is r units below the x-axis.

3. What is the probability that, starting at point $(0,0)$, after n coin flips you end up at point (n, m) along a path that never touches the horizontal line that is r units below the x-axis? You can state your answer using the $P_{n,m}$ notation of the first question, so that your answer becomes independent of whether you correctly answered the first question about $P_{n,m}$.

Date due: April 5, 2011 at the beginning of class.