Solutions

Your Name:

CS 182
MIDTERM
Fall 2011

Left Neighbor: ____________________ Right Neighbor: ____________________

This exam contains 9 numbered pages. Check your copy and exchange it immediately if it is defective. Print your name and your student id number on the top of this page. Print the name of your left and right neighbors below your name. Good luck!

Problem	Maximum	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Bonus	5	
Total	70	

[10] **PROBLEM 1**: *Logic*

Prove that

$$p \to q \equiv p \wedge \neg q \to r \wedge \neg r$$

Truth Table

[10] **PROBLEM 2**: *Sets*

Prove (**not** using Venn's diagrams) that if A and B are sets, then

$$A - B = A \cap \bar{B}.$$

Remark. We accept a proof using Venn's diagram for **5 points**.

$A - B \subseteq A \cap \overline{B}$

$x \in A - B$

$x \in A$ & $x \notin B$

$x \in A \cap \overline{B}$

$A \cap \overline{B} \subseteq A - B$

$x \in A \cap \overline{B}$

$x \in A$ & $x \notin B$

$x \in A - B$

[10] **PROBLEM 3**: *Relations*

[8] Let xRy if x and y are *negative integers* and x divides (evenly) y. Is this relation

- reflexive, yes xRx since $x = 1 \cdot x$
- symmetric, no
- antisymmetric, yes $\rightarrow$ xRy & yRx
- transitive. yes

Prove your statements.

$y = k \cdot x$ $\cancel{x} = l \cdot y$ l, k pos

$yx = k \cdot l \cdot \cancel{x \cdot y}$

$kl = 1$

$k = 1, l = 1.$

[2] Determine whether R is an equivalence relation or a partial order relation. If it is an equivalence relation, construct the equivalence classes.

[10] **PROBLEM 4**: *A Sum*

Calculate

$$\prod_{i=1}^{1000} 2^{2i+3}.$$

$$\prod_{i=1}^{1000} (2^2)^i \cdot 2^3 = 2^{3000} \prod_{i=1}^{1000} 4^i$$

$$= 2^{3000}\, 4^{\sum_{i=1}^{1000} i}$$

$$= 2^{3000}\, 4^{\frac{1000(1001)}{2}}$$

$\underline{1004000}$

[10] **PROBLEM 5**: *Recurrence*

Solve the following recurrence

$$a_n = \sqrt{a_{n-1}}$$

assuming $a_0 = 2$.

$$a_n = a_{n-1}^{\frac{1}{2}} = (a_{n-2})^{\frac{1}{2^2}} = (a_{n-3})^{\frac{1}{2^3}}$$

$$\cdots = a_{n-n}^{\frac{1}{2^n}}$$

$$= 2^{\frac{1}{2^n}}$$

[10] **PROBLEM 6**: *Mathematical Induction*

Prove using **induction on** k that for any natural n

$$\sum_{i=1}^{n} i^k \leq \frac{n^k(n+1)}{2}.$$

$k=1$ Ok

Assumption: $\sum_{i=1}^{n} i^k \leq \frac{n^k(n+1)}{2}$

Must prove:

$$\sum_{i=1}^{n} i^{k+1} \leq \frac{n^{k+1}(n+1)}{2}$$

Proof:

$$\sum_{i=1}^{n} i^{k+1} \leq n^k \sum_{i=1}^{n} i = n^k \frac{n(n+1)}{2}$$

$i \leq n$ $\quad \sum n^k \cdot i$

$$= n^{k+1} \frac{n+1}{2}$$

Another proof (by induction)

$$n \sum_{i=1}^{n} i^k \leq n \frac{n^k(n+1)}{2} = \frac{n^{k+1}(n+1)}{2}$$

[10] **PROBLEM 7**: *Big Oh*

Show that

$$\sum_{i=1}^{n} i^3 \log n = \Theta(n^4 \log n).$$

Upper bound

$$\sum_{i=1}^{n} i^3 \log i \leq n(n^3 \cdot \log n) = n^4 \log n$$

Lowe Bound

$$\sum_{i=1}^{n} i^3 \log i \geq \sum_{i=n/2}^{n} i^3 \log i \geq \frac{n}{2}\left(\frac{n}{2}\right)^3 \log \frac{n}{2}$$

$$= \frac{n^4}{2^4} \log \frac{n}{2}$$

$$\sum_{i=1}^{n} i^3 \log i = \cancel{1^3 \log 1 + 2^3 \log 2} + \cdots + \underbrace{\left(\frac{n}{2}\right)^3 \log\left(\frac{n}{2}\right) + \cdots + n^3 \log n}_{n/2}$$

$$\geq$$

[5] **BONUS PROBLEM**: *Another Sum*

Prove

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-n} = 1.$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-n} = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{k} \cdot 1^{n-k}$$

$$= 2^{-n} (1+1)^{n} = 2^{-n} 2^{n} = 1$$