Question 1.

1. 2^{-k}
2. The net gain is 100 times the following:

   $$2^k - (2^{k-1} + 2^{k-2} + \cdots + 2^1 + 2^0) = 2^{k-1} - (2^{k-2} + \cdots + 2^1 + 2^0) = \cdots = 2^1 - 2^0 = 1$$

   where we repeatedly used the fact that $2^\ell - 2^{\ell-1} = 2^{\ell-1}$. So the answer is \$100.

Question 2. The characteristic equation is $r = 3$, so the solution is of the form $c3^n$ for some constant c. The boundary condition requires that $c = 1$, therefore the answer is $a_n = 3^n$.

Question 3. The characteristic equation is $r^2 = 4r - 4$ which has 2 as a double root, so the solution is of the form $c2^n + dn2^n$ for some constants c and d. The first boundary condition gives $c = 1$, and the second one gives $2c + 2d = 4$ which implies $d = 1$. So the answer is $2^n + n2^n$.

Question 4. For $k = 1, \ldots, 26$, let ℓ_k be the length of a longest valid (i.e., after-erasure) word that ends with the kth letter of W and whose letters are in increasing order. If there is an ℓ_k such that $6 \leq \ell_k$ then we are done (because a word that corresponds to this ℓ_k can be obtained from W using no more than 20 erasures). So suppose this is not the case, i.e., that every ℓ_k is smaller than 6. Create 5 pigeonholes numbered 1 to 5, and create 26 pigeons where, for $k = 1, \ldots, 26$, pigeon number k is a surviving word that corresponds to ℓ_k (i.e., it ends with the kth letter of W and has length ℓ_k and its letters are in increasing order — if there are many choices for such a pigeon then choose one arbitrarily). Place pigeon k in pigeonhole number ℓ_k, for $k = 1, \ldots, 26$. Because there are 26 pigeons and only 5 pigeonholes, one of the pigeonholes must contain 6 or more pigeons: For every pair of pigeons in that hole, if one of them ends with letter β and another ends with letter γ where $\beta < \gamma$, then in W letter β occurs to the right of letter γ (otherwise these two pigeons could not have had same ℓ_k as each other). Therefore the rightmost letters of the (at least 6) pigeons in that hole occur in decreasing order in W, and form a valid surviving word of length ≥ 6.