CS182 **Homework 7 Sketch of Solution** **Spring 2011**

Question 1.

1. $C(n, m)$ for the following reason. First, recall that in class we interpreted $C(n, m)$ as the number of paths that, in n horizontal or vertical positive unit steps, go from $(0, 0)$ to $(n - m, m)$. The terms of the given summation correspond to the intermediate points where such a path can be after $n - 2$ steps, i.e., 2 steps prior to reaching the destination point $(n - m, m)$. The solution follows from observing that the number of paths to $(n - m, m)$ equals:

   (the number of paths to $(n - m - 2, m)$) + (twice the number of paths to $(n - m - 1, m - 1)$) + (the number of paths to $(n - m, m - 2)$)

   where the factor of 2 in the middle term arises because there are 2 ways to go from $(n-m-1, m-1)$ to $(n-m, m)$ (whereas there is only one way to continue to $(n-m, m)$ from $(n - m - 2, m)$ or from $(n - m, m - 2)$).

2. $C(n, m)$ for a similar reason to the above one, except that now the given summation goes through the possible points reached 3 steps before arriving at the destination point $(n - m, m)$, i.e., the number of paths to $(n - m, m)$ equals:

   (the number of paths to $(n - m - 3, m)$) + (3 times the number of paths to $(n - m - 2, m - 1)$) + (3 times the number of paths to $(n - m - 1, m - 2)$) + (the number of paths to $(n - m, m - 3)$)

   and the factors of 3 in the middle terms are because there are 3 ways to go from the corresponding intermediate points to the destination $(n - m, m)$ (but only 1 such way for the intermediate points corresponding to the first and last term).

3. $C(n + r + 1, r)$ because (i) any path to $(n + 1, r)$ reaches the x coordinate of $n + 1$ for the first time coming horizontally from either $(n, 0)$ or $(n, 1)$ or ... or (n, r); and (ii) once a path to $(n + 1, r)$ first reaches the x coordinate of $n + 1$ there is only one way for it to continue to $(n + 1, r)$ (namely, straight up). The answer follows by observing that the numbers of paths to each (n, i) mentioned in (i) is one of the terms of the given summation.

Question 2.

1. Zero if $n < m$. Zero if m and n have opposite parities (i.e., one is odd and the other is even). If $n > m$ and both m and n have same parity then the probability of ending up at (n, m) is the number of paths to (n, m) divided by the number of possible n-step paths (which is 2^n). This gives:

   $$2^{-n}C(n, (n + m)/2)$$

   because, if we denote by u the number of up movements and d the number of down movements, then $u + d = n$ and $u - d = m$ together imply $u = (n + m)/2$ and $d = (n-m)/2$. Note that the answer can be equivalently stated as $2^{-n}C(n, (n-m)/2)$.

2. Let H denote the horizontal line that is r units below the x-axis. Let P be any legal path from $(0,0)$ to (n,m) that touches H. Let P' be the portion of P from its start at $(0,0)$ until the first time it touches H, and let P'' be the rest of P (i.e., $P'' = P - P'$). Then we associate with this P the legal path Q from $(0,-2r)$ to (n,m) that consists of the mirror image of P' relative to H, followed by P''. We can conversely start with Q and obtain P by taking the mirror image relative to H of the portion (call it Q') of Q from its start at $(0,-2r)$ until the first time it touches H, followed by $Q - Q'$. This defines the desired bijection.

3. The probability is zero if r is greater than the number of down steps $(n-m)/2$, otherwise it is $P_{n,m} - P_{n,m+2r}$ because the number of paths from $(0,0)$ to (n,m) that do not touch the horizontal line H is equal to the number of paths from $(0,0)$ to (n,m) minus the number of paths from $(0,0)$ to (n,m) that touch H, and the latter is (by the second question) equal to number of paths from $(0,-2r)$ to (n,m) which is $P_{n,m+2r}$.