**Solutions of HW#2**: *Language of Mathematics*

**Q1.** Prove that for any sets $A$ and $B$, $A = (A - B) \cup (A \cap B)$.

**Answer**

Note that you can use Venn diagram to prove the relation. Another way to prove that two sets are equivalent is by proving that each set is a subset of the other as follows.

i) To prove that $(A - B) \cup (A \cap B) \subseteq A$:

$$\begin{aligned} \textit{Let } x \in (A - B) \cup (A \cap B) &\Rightarrow x \in (A - B) \textit{ or } x \in (A \cap B) \\ &\Rightarrow (x \in A \textit{ and } x \notin B) \textit{ or } (x \in A \textit{ and } x \in B) \\ &\Rightarrow x \in A \textit{ and } (x \notin B \textit{ or } x \in B) \\ &\Rightarrow x \in A \end{aligned}$$

Thus, all elements of $(A - B) \cup (A \cap B)$ are also elements of $A$. That is,

$$(A - B) \cup (A \cap B) \subseteq A \cdots\cdots\cdots (1)$$

ii) To prove that $A \subseteq (A - B) \cup (A \cap B)$: we follow exactly the same steps but in the reverse order to prove that

$$A \subseteq (A - B) \cup (A \cap B) \cdots\cdots\cdots (2)$$

From $(1), (2)$ : $A = (A - B) \cup (A \cap B)$.

**Q2.** Let $x$ and $y$ be *integers*. Determine whether the following relations are reflexive, symmetric, antisymmetric, or transitive:

i) $x \equiv y \mod 7$;

ii) $xy \geq 1$;

iii) $x = y^2$.

*Justify* your statements. Finally, determine which of the above relations are equivalence and partial order relations. For equivalence relations, construct the equivalence classes.

**Answer**

i) $R = \{(x, y) | x \equiv y \mod 7\}$:

- Reflexive: **(YES)** since $x \equiv x \mod 7$.
- Symmetric: **(YES)** since $x \equiv y \mod 7 \Rightarrow y \equiv x \mod 7$.
- Antisymmetric: **(NO)** since, for example, $(1, 8)$ and $(8, 1)$ are both in $R$.
- Transitive: **(YES)** since if $(x \equiv y \mod 7)$ and $(y \equiv z \mod 7)$, then $(x \equiv z \mod 7)$

Thus, $R$ is an equivalence relation. The equivalence classes are:

$$
\begin{aligned}
[0] &= \{\cdots, -14, -7, 0, 7, 14, \cdots\} \\
[1] &= \{\cdots, -13, -6, 1, 8, 15, \cdots\} \\
[2] &= \{\cdots, -12, -5, 2, 9, 16, \cdots\} \\
[3] &= \{\cdots, -11, -4, 3, 10, 17, \cdots\} \\
[4] &= \{\cdots, -10, -3, 4, 11, 18, \cdots\} \\
[5] &= \{\cdots, -9, -2, 5, 12, 19, \cdots\} \\
[6] &= \{\cdots, -8, -1, 6, 13, 20, \cdots\}
\end{aligned}
$$

ii) $R = \{(x, y) | xy \geq 1\}$:

- Reflexive: **(NO)** since $(0, 0) \notin R$ (this is because $0^2 = 0 < 1$).
- Symmetric: **(YES)** since $xy = yx$.
- Antisymmetric: **(NO)** since, for example, $(1, 2)$ and $(2, 1)$ are both in $R$.
- Transitive: **(YES)** since if $xy \geq 1$ and $yz \geq 1$, then $xz \geq 1$. The condition mandates that the three variables have the same sign. Since $x$ and $z$ have the same sign, then $xz \geq 1$.

Thus, $R$ is neither an equivalence relation nor a partial order relation.

iii) $R = \{(x, y) | x = y^2\}$:

- Reflexive: **(NO)** since, for example, $(2, 2) \notin R$.
- Symmetric: **(NO)** since, for example, $(4, 2) \in R$ but $(2, 4) \notin R$.
- Antisymmetric: **(YES)** since if $x = y^2$ and $y = x^2$, then $x = y$.
- Transitive: **(NO)** since if $x = y^2$ and $y = z^2$, then $x = z^4 \neq z^2$.

Thus, $R$ is neither an equivalence relation nor a partial order relation.

**Q3.** Determine whether the following function is bijection from $\mathbf{R}$ to $f(\mathbf{R})$:

i) $f(x) = x^3$

ii) $f(x) = \sin^2(x)$

iii) $f(x) = \frac{x+1}{x+2}$.

**Answer**

Note that since the ranges of the functions are $f(\mathbf{R})$, all the functions cover the range and, consequently, they are onto.

i) $f(x) = x^3$:

- One-to-one: **(YES)** since if $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$.
- Onto: **(YES)** see above.
- Bijection: **(YES)** since it is both one-to-one and onto.

ii) $f(x) = \sin^2(x)$:

- One-to-one: **(NO)** since, for example, $\sin^2(0) = \sin^2(\pi) = 0$.
- Onto: **(YES)** see above.
- Bijection: **(NO)** since it is not one-to-one and onto.

iii) $f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$:

- One-to-one: **(YES)** since if $f(x_1) = f(x_2) \Rightarrow 1 - \frac{1}{x_1+2} = 1 - \frac{1}{x_2+2} \Rightarrow x_1 = x_2$.
- Onto: **(YES)** see above.
- Bijection: **(YES)** since it is both one-to-one and onto.

**Q4.** Let $g(x) = \lfloor x \rfloor$. Find

- $g^{-1}(\{0\})$;
- $g^{-1}(\{x : \ 0 < x < 1\})$

**Answer**

- $g^{-1}(\{0\}) = \{x | 0 \leq x < 1\}$ (i.e. the values whose floor is 0).
- $g^{-1}(\{x : \ 0 < x < 1\}) = \{\} = \phi$ (since the floor has to be a whole number).

**Q5.** What are the values of the following:

$$\sum_{i=1}^{500} 5^i,$$
$$\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$$
$$\sum_{j=0}^{8}(3^j - 2^j).$$

**Answer**

i)

$$\sum_{i=1}^{500} 5^i = \sum_{i=0}^{500} 5^i - 1 = \frac{5^{501}-1}{5-1} - 1$$

ii)

$$\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) = \sum_{i=1}^{2}\sum_{j=1}^{3} i + \sum_{i=1}^{2}\sum_{j=1}^{3} j = \sum_{j=1}^{3}(1+2) + \sum_{i=1}^{2}(1+2+3) = 3\times(1+2) + 2\times(1+2+3) = 21$$

iii)

$$\sum_{j=0}^{8}(3^j - 2^j) = \sum_{j=0}^{8} 3^j - \sum_{j=0}^{8} 2^j = \frac{3^9-1}{3-1} - \frac{2^9-1}{2-1} = 9330$$