Solutions of Homework #4: *Proof Techniques*

Q1. Show that $\sqrt[5]{5}$ is irrational.

Answer

Proof by contradiction: Assume that $\sqrt[5]{5} = \frac{p}{q}$ is in its simplest form, i.e., both p and q do not have a common divisor and therefore **the fraction $\frac{p}{q}$ cannot be simplified further**. Thus,

$$
\begin{aligned}
& 5 = \frac{p^5}{q^5} && (1)\\
\rightarrow\ & p^5 = 5q^5 && (2)\\
\rightarrow\ & 5|p^5 && (3)\\
\rightarrow\ & 5|p. && (4)
\end{aligned}
$$

Where $5|p$ means that p is divisible by 5. . . . . (I)

From (I), $p = 5k$ for some integer k. Substituting in (2):

$$
\begin{aligned}
& (5k)^5 = 5q^5\\
\rightarrow\ & 5^5k^5 = 5q^5\\
\rightarrow\ & q^5 = 5^4k^5\\
\rightarrow\ & 5|q^5\\
\rightarrow\ & 5|q.
\end{aligned}
$$

Thus, q is also divisible by 5. . . . . (II)

From (I) and (II), the fraction $\frac{p}{q}$ is *not* in its simplest form for it can be simplified further by dividing both the numerator and the denominator by 5 which contradicts the original assumption.

Q.2 The *harmonic number* H_n is defined as for $n \geq 1$

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove by induction that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

whenever n is nonnegative natural number.

Answer

Proof by induction on n

BASIS CASE ($n = 0$): $H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $H_{2^n} \geq 1 + \frac{n}{2}$. We need to prove that $P(n+1)$, which is $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$, is also *true*:

$$\begin{aligned}
H_{2^{n+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&= H_{2^n} + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^{n+1}} \cdots + \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + 2^n \cdot \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + \frac{1}{2} \\
&= 1 + \frac{n+1}{2}.
\end{aligned}$$

Q.3 Let A be a set of cardinality n. Let $P(A)$ be the power set, that is, the set of *all* subsets of A. Prove by induction that the cardinality of $P(A)$ is 2^n, that is

$$|P(A)| = 2^n.$$

Answer

Proof by induction on n

BASIS CASE ($n = 1$): Since $n = 1$, $|A| = 1$. Let $A = \{a\}$, then $P(A) = \{\emptyset, \{a\}\}$. Therefore, $|P(A)| = 2$.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $|A| = n \rightarrow |P(A)| = 2^n$. We need to prove that $P(n+1)$ is also *true*:

$$\begin{aligned} & A = \{a_1, a_2, \ldots, a_n\} \\ \rightarrow \quad & P(A) = \{S_1, S_2, \ldots, S_{2^n}\} \end{aligned}$$

Let

$$\begin{aligned} & B = A \cup \{a_{n+1}\} = \{a_1, a_2, \ldots, a_n, a_{n+1}\} \\ \rightarrow \quad & P(B) = \{S_1, S_2, \ldots, S_{2^n}\} \cup \{S_1 \cup \{a_{n+1}\}, S_2 \cup \{a_{n+1}\}, \ldots, S_{2^n} \cup \{a_{n+1}\}\} \\ \rightarrow \quad & |P(B)| = 2 \times |P(A)| = 2 \times 2^n = 2^{n+1} \end{aligned}$$

That is, the power set of the extended set B contains all subsets of the initial set A as well as their extensions with the added element a_{n+1}. Therefore, $P(n+1)$ is *true*.

Consider, for example, $A = \{a_1, a_2\}$, then $P(A) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. If

$$\begin{aligned} & B = A \cup \{a_3\} = \{a_1, a_2, a_3\} \\ \rightarrow \quad & P(B) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\} \\ & = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\} \cup \{\emptyset \cup \{a_3\}, \{a_1\} \cup \{a_3\}, \{a_2\} \cup \{a_3\}, \{a_1, a_2\} \cup \{a_3\}\} \\ \rightarrow \quad & |P(B)| = 2 \times |P(A)| = 2 \times 2^2 = 2^3 = 8 \end{aligned}$$

Q.4 Prove using induction that for any natural n

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

Answer

Proof by induction on n

BASIS CASE ($n = 1$): $\sum_{i=1}^{1} \frac{1}{i^2} \leq 2 - \frac{1}{1}$ and, therefore, $P(1)$ is *true*.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. We need to prove that $P(n+1)$ is also *true*, i.e., $\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}$ as follows:

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i^2} &= \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \\
&\leq (2 - \frac{1}{n}) + \frac{1}{(n+1)^2} \\
&= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\
&= 2 - \frac{n(n+1) + 1}{n(n+1)^2} \\
&= 2 - \frac{1}{n+1} - \frac{1}{n(n+1)^2} \\
&\leq 2 - \frac{1}{n+1}.
\end{aligned}$$

Q.5 Derive an explicit formula for the following recurrence for $n \geq 1$

$$a_n = \frac{n}{2}\, a_{n-1}$$

with $a_0 = 1$.

Answer

$$\begin{aligned}
a_n &= \frac{n}{2}\, a_{n-1} \\
&= \frac{n}{2} \times \frac{n-1}{2}\, a_{n-2} \\
&= \frac{n}{2} \times \frac{n-1}{2} \times \frac{n-2}{2}\, a_{n-3} \\
&\vdots \\
&= \underbrace{\frac{n}{2} \times \frac{n-1}{2} \times \frac{n-2}{2} \times \ldots \times \frac{3}{2} \times \frac{2}{2} \times \frac{1}{2}}_{n\ terms}\, a_0 \\
&= \underbrace{\frac{n!}{2^n}} \times 1 \\
&= \frac{n!}{2^n}.
\end{aligned}$$