Solutions of Homework 5: *Big* O, Ω

Important Note: $\log n \leq \sqrt{n} \leq n$.

Q1.

1. $100n + 1 \leq 100n + n = 101n = O(n) \Rightarrow c = 101, n_0 = 1.$
2. $(10n + 1)^3 \leq (10n + n)^3 = (11n)^3 = 1331n^3 = O(n^3) \Rightarrow c = 1331, n_0 = 1.$
3. $3n^3 - 5n^2 - 100 \leq 3n^3 = O(n^3) \Rightarrow c = 3, n_0 = 1.$
4. $n^2 + n + \sqrt{n} + \log n \leq n^2 + n^2 + n^2 + n^2 = 4n^2 = O(n^2) \Rightarrow c = 4, n_0 = 1.$

Q.2

i. $6n^2 - 2n \leq 6n^2 = O(n^2)$ .... (1)
$6n^2 - 2n \geq 6n^2 - 2n^2 = 4n^2 = \Omega(n^2)$ .... (2)
Form (1) and (2): $6n^2 - 2n \log n = \Theta(n^2)$.

ii. $\frac{6n^2}{\log^3 n+1} \leq 6n^2 \leq n^3 = O(n^3)$.

iii. $3n^3 + 44n^2 \geq n^2 = \Omega(n^2)$.

Q.3 Proof by contradiction: Assume that

$$
\begin{aligned}
& (\log n)^2 = O(\log n^2) \\
\rightarrow \quad & (\log n)^2 \leq c \log n^2 = k \log n \text{ (note that } \log n^2 = 2 \log n) \\
\rightarrow \quad & \log n \leq k.
\end{aligned}
$$

However, it is impossible to find such a constant k which is always greater than $\log n$ for all possible values of n (taking into account that it is a monotonically increasing function). Therefore, $(\log n)^2 \neq O(\log n^2)$.

Q.4 Note that:

- $2^{\log_2 n} = n$.
- $2^{3 \log_2 n} = 2^{\log_2 n^3} = n^3$.
- $\left(\frac{3}{2}\right)^n \leq 2^n$.
- By sketching the graphs for $\log^4 n$ and n, you can conclude that $\log^4 n \leq n$.
- A constant function (such as 100) does not grow with n as opposed to any other function and, therefore, it is upper bounded by any of these functions regardless the value of this constant.
- The exponential function a^n always dominates polynomial and logarithmic functions.

Hence, the required ascending order is:

$$100 \prec \log^4 n \prec 2^{\log_2 n} \prec 2^{3 \log_2 n} \prec n^3 \log^2 n \prec \left(\frac{3}{2}\right)^n \prec 2^n.$$