PARTIALLY ORDERED VARIETIES AND QUASIVARIETIES

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ABSTRACT. The recent development of abstract algebraic logic has led to a reconsideration of the universal algebraic theory of ordered algebras from a new perspective. The familiar equational logic of Birkhoff can be naturally viewed as one of the deductive systems that constitute the main object of study in abstract algebraic logic; technically it is a *deductive system of dimension two.* Large parts of universal algebra turn out to fit smoothly within the matrix semantics of this deductive system. Ordered algebras in turn can be viewed as special kinds of matrix models of other, closely related, 2-dimensional deductive systems. We consider here a more general notion of ordered algebra in which some operations may be anti-monotone in some arguments; this leads to many different ordered equational logics over the same language. These lectures will be a introduction to universal ordered algebra from this new perspective.

1. Introduction

Algebraic logic, in particular *abstract algebraic logic* (AAL) as presented for example in the monograph [1], can be viewed as the study of logical equivalence, or more precisely as the study of properties of logical propositions upon abstraction from logical equivalence. Traditional algebraic logic concentrates on the algebraic structures that result from this process, while AAL is more concerned with the process of abstraction itself. These lectures are the outgrowth of an attempt to apply the methods of AAL to logical implication in place of logical equivalence, with special emphasis on those circumstances in which logical implication cannot be expressed in terms of logical equivalence. This persepective results in a theory of ordered algebraic systems markedly different from the traditional one as presented for example in [14], but of course the overlap is considerable.

Underlying the development is the conception of order algebras as the reduced matrix semantics of *inequational logic*, an alternative to Birkhoff's equational logic. However the metalogical roots are] mostly kept in the background in order to simplify the exposition and avoid distracting the reader from the central part of the theory. We do however include parenthetical remarks explicitly marking connections with AAL when appropriate.

Some results can be found in the literature in a somewhat weaker form; for example, the order H-S-P theorem (Theorem 3.14) was obtained by Bloom [4] in 1976 under the condition that the fundamental operations of the algebra are monotone at all argument positions. On the other hand, most of the basic algebraic theory of partially ordered algebras presented in Section 2 can be found in a more general form in the work of A. I. Mal'cev, and that of his scientific progeny, on quasivarieties; a good appreciation of this work can be obtained

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from the monographs [15, 23]. In the Russian literature the term "quasivariety" can refer to an arbitrary strict universal Horn class, and consequently a quasivariety can mean a class of structures with relations as well as operation. The S-L-P theorem (Theorem 3.17) can be obtained from Mal'cev's well-known algebraic characterization of strict universal Horn classes [22]; the particular form of Theorem 3.17 can be found in [17]. But the Russian work always includes equality as a fundamental predicate, while inequational logic is a special kind of universal Horn logic without equality (UHLWE)—one of the so-called *equivalential* logics. Equivalential UHLWE's, and a more general class, the *protoalgebraic* UHLWE's, constitute a family of logical systems whose reduced model theory is most like algebra. These systems have been extensively investigated in the literature of AAL [2, 7, 8, 12, 13]; the papers [2, 12] contain general forms of the H-S-P theorem from which Theorem 3.14 can be easily extracted. We also call attention to the earlier papers on equivalential sentential logics [5, 24], and to the monograph [6], which is a comprehensive resource for the theory of protoalgebraic and equivalential sentential logics.

In spite of the above remarks, we feel there is value in developing the algebraic theory of order algebras from first principles and in parallel with that of ordinary, unordered, algebras rather than simply as an adjunction to the latter theory. We also think it is also important that the theory be purely algebraic rather than grafted on to the the theory of equivalential UHLWE's.

Fundamental for our work is the notion of a signature Σ augmented by a *polarity*, an arbitrary assignment of positive or negative polarity to each argument position of each operation symbol of Σ specifying whether the operation is to be monotone or antimonotone at that position. For example, in a Heyting algebra $\langle A, +, \cdot, \rightarrow, 0, 1\rangle$ with the natural order, under the natural polarity $+$ and $\cdot$ are each positive in both arguments and the implication $\rightarrow$ is positive in the second and negative in the first. Admitting signatures with arbitrary polarity greatly expands the scope of the ordered algebras to which the theory applies and is one of the major innovations. A similarly broad approach can also be found in the work of Dunn [11] and is a natural development of his metalogical investigations of relevance and other so-called substructural logics. His theory of gaggles [9, 10] he presents ordered algebra models for a very wide class of substructural logics, including linear logic, in which the polarity of the various operations play a key role.

1.1. **Outline of Lectures.** In the first section we introduce the notion of a *polarity* ρ on a signature Σ, and of a *ρ-partially ordered algebra* $\boldsymbol{\mathcal{A}}$, consisting of a Σ-algebra $\boldsymbol{A}$ and a partial ordering $\leq^{\boldsymbol{\mathcal{A}}}$ such that the fundamental operations of $\boldsymbol{A}$ are monotone or antimonotone in each argument in accordance with ρ. *Order homomorphisms* of ρ-poalgebras are homomorphisms of the underlying algebras that preserve the designated partial orders. *ρ-quasi-orders* of $\boldsymbol{\mathcal{A}}$ are quasi-orderings of the universe of the underlying algebra that include the designated partial ordering $\leq^{\boldsymbol{\mathcal{A}}}$ and respect the polarity in the same sense. They play the role in the theory of ρ-poalgebras that congruences play in universal algebra, and are similarly important to the theory. Quotients of ρ-poalgebras are taken with respect to ρ-quasi-orders, and order analogues of the homomorphism, isomorphism, and correspondence theorems of universal algebra are formulated and proved. Order subalgebras, direct products, reduced and ultraproduces, and direct limits of ρ-poalgebras are defined. Subdirect products are defined and an analogue of Birkhoff's representation as a subdirect product of subdirectly irreducible ρ-poalgebras is proved.

Inequational logic is developed in Section 3. A *quasi-inidentity* is a strict universal Horn formula whose atomic formulas is an inequality between terms; the special case of a universally quantified inequality is an *inidentity.* The class of all ρ-poalgebras that satisfy some set of quasi-inidentities is called a *quasi-povariety*; if all the quasi-inidentities are inidentities, it is called a *povariety.* The notion of a *freely generated ρ-algebra* over a class of ρ-poalgebras is discussed, and it is used in proving the order H-S-P theorem, i.e., that a class of ρ-poalgebra is an ρ-povariety iff it is closed under the formation of order homomorphic images, subalgebras, and direct products. We prove a analogous operator-theoretic characterization of quasi-povarieties as the class of ρ-poalgebras closed under order subalgebras, direct products, and direct limits.

Several examples of ρ-povarieties are given: the povariety of lattices as poalgebras and the ρ-povariety of partially ordered left residuated monoids (POLRM). Partially ordered groups form a sub-povariety of POLRM. Lattices are more commonly viewed as algebras rather than poalgebras, and in this conception are defined by identities rather than indentities. POLRMs cannot be represented as algebras in this way. This special property of the povariety of lattices, *algebraizability*, is investigated in Section 4. It is a special case of the well-known notion from AAL of the same name. In the last section we investigate the generation of ρ-quasi-orders. We prove a Mal'cev-like combinatorial lemma characterizing the ρ-quasi-ordering generated by an arbitrary set of pairs of elements on a ρ-poalgebra in terms of polynomials. This is used to obtain two characterization of those ρ-povarieties with permuting ρ-quasi-orders—one in terms of the existence of certain quasi-indentities and another, related, one in terms of inidentities with additional conditions on the polarity of the terms comprising the inidentities. The latter characterization is used to show that the ρ-quasi-orders of POLRM's permute.

POLRM's together with the various other ρ-povarieties related to substructual logics can be regarded as the paradigms ρ-povarieties from our perspective. The algebraizable ones, for example the *partially ordered commutative residuated integral monoids* (POLCRIM's) have been extensively investigated using the methods of standard universal algebra; the papers [3, 19, 20, 21, 25, 26] are representative of the publications in the area. We have some hope that the methods of ordered universal algebra will prove to be useful in the investigation of POLRM's and other non-algebraizable ρ-povarieties.

2. Basic Algebraic Theory

A reflexive and transitive relation on a set is called a *quasi-ordering* (a *qordering*). Thus $\leq \subseteq A^2$ is a qordering if

- $(\forall\, a \in A)(a \leq a)$,
- $(\forall\, a, b, c \in A)(a \leq b \text{ and } b \leq c \implies a \leq c)$.

It is a *partial ordering* (a *pordering*) if it is also antisymmetric:

- $(\forall\, a, b \in A)(a \leq b \text{ and } b \leq a \implies a = b)$.

$\langle A, \leq\rangle$ is called a *quasi-ordered set* (a *qoset*)—a *partially ordered set* (a *poset*) if $\leq$ is a *pordering*. The reverse of $\leq$ is denoted by $\geq$ or $\leq^{-1}$. Thus $a \geq b$ iff $b \leq a$.

A *signature* or *language type* is a set Σ together with an *order* or *arity function* $\mathbf{o}\colon \Sigma \to \omega$ (ω is the set of natural numbers). $\mathbf{o}(\sigma)$ is the *order* or *arity* of $\sigma \in \Sigma$ and specifies the number of arguments that σ takes. $\Sigma_n = \{\, \sigma \in \Sigma : \mathbf{o}(\sigma) = n \,\}$. A Σ-*algebra* is a system $\boldsymbol{A} = \langle A, \sigma^{\boldsymbol{A}} \rangle_{\sigma \in \Sigma}$ such that A is a nonempty set and $\sigma^{\boldsymbol{A}}$ is an operation on A of order $\mathbf{o}(\sigma)$, i.e., $\sigma^{\boldsymbol{A}}\colon A^{\mathbf{o}(\sigma)} \to A$.

A *polarity* for Σ is a fixed but arbitrary assignment of a polarity, either positive or negative, to each argument position of each operation symbol in Σ. If, as commonly done, we identify a natural number with the set of natural numbers less than it, we can define a polarity to be a bfunction

$$\rho\colon \big(\bigcup_{\sigma \in \Sigma} \sigma \times \mathbf{o}(\sigma)\big) \to \{+,-\}.$$

σ is said to be of *positive* or *negative polarity at the i-th argument* (*with respect to* ρ) if $\rho(\sigma, i)$ is $+$ or $-$, respectively. With slight abuse of notation we take $\rho^+(\sigma)$ and $\rho^-(\sigma)$ to be the sets of arguments of σ of positive and negative polarity, respectively. So $\rho^+(\sigma) = \{\, i < \mathbf{o}(\sigma) : \rho(\sigma, i) = + \,\}$ and $\rho^-(\sigma) = \{\, i < \mathbf{o}(\sigma) : \rho(\sigma, i) = - \,\}$. We also say that σ is *monotone* or *antimonotone* in the i-th argument if $\rho(\sigma, i) = +$ or $\rho(\sigma, i) = -$, respectively.

Definition 2.1. Let Σ be a signature and ρ a polarity for Σ. Let $\boldsymbol{A}$ be a Σ-algebra. A quasi-ordering $\leq$ on the universe A of $\boldsymbol{A}$ is said to be a *ρ-quasi-ordering*, a *ρ-qordering*, of $\boldsymbol{A}$ if, for every $\sigma \in \Sigma_n$ and all $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in A$,

$$\begin{aligned}&(\forall\, i \in \rho^+(\sigma))(a_i \leq b_i)) \text{ and } \forall\, j \in \rho^-(\sigma))(a_j \geq b_j))\\ &\qquad\qquad\qquad \Longrightarrow \quad \sigma^{\boldsymbol{A}}(a_0, \dots, a_{n-1}) \leq \sigma^{\boldsymbol{A}}(b_0, \dots, b_{n-1}).\end{aligned}$$

This condition is called *ρ-tonicity*. A pordering of A that satisfies it is a *ρ-partial ordering*, a *ρ-pordering*, of $\boldsymbol{A}$, and the pair $\langle \boldsymbol{A}, \leq \rangle$ is called a *ρ-partially ordered Σ-algebra*, a *ρ-poalgebra* for short.

We note for future reference that, in the presence of transitivity, to verify the ρ-tonicity condition it suffices to show that, for each $\sigma \in \Sigma_n$, each $i < n$, and all $a, b, c_0, \dots, c_{i-1}, c_{i+1}, \dots, c_{n-1} \in A$, if $i \in \rho^+(\sigma)$, then

$$a \leq b \implies \sigma^{\boldsymbol{A}}(c_0, \dots, c_{i-1}, a, c_{i+1}, \dots, c_{n-1}) \leq \sigma^{\boldsymbol{A}}(c_0, \dots, c_{i-1}, b, c_{i+1}, \dots, c_{n-1}),$$

and if $i \in \rho^-(\sigma)$, then

$$a \geq b \implies \sigma^{\boldsymbol{A}}(c_0, \dots, c_{i-1}, a, c_{i+1}, \dots, c_{n-1}) \leq \sigma^{\boldsymbol{A}}(c_0, \dots, c_{i-1}, b, c_{i+1}, \dots, c_{n-1})$$

The polarity ρ such that $\rho(\sigma, i) = +$ for every $\sigma \in \Sigma$ and every $i < \mathbf{o}(\sigma)$ is said to be *completely positive* or *completely monotone*. In this case ρ-poalgebras are called simply *poalgebras*.

The group of integers with the natural order $\big\langle \langle Z, + \rangle, \leq \big\rangle$ and the semiring of natural numbers $\big\langle \langle \omega, +, \cdot \rangle, \leq \big\rangle$ are poalgebras over the signatures $\{+\}$ and $\{+, \cdot\}$ respectively. The group of integers in the form $\big\langle \langle Z, +, - \rangle, \leq \big\rangle$ is not a poalgebra over the signature $\{+, -\}$ because $n \leq m$ implies $-n \geq -m$. It is however a ρ-poalgebra where $\rho(+, 0) = \rho(-, 1) = +$ and $\rho(-, 0) = -$. Any Boolean algebra and more generally any Heyting algebra $\big\langle \langle A, +, \cdot, -, \to, 0, 1 \rangle, \leq \big\rangle$ with the natural order given by $a \leq b$ iff $a \to b = 1$ is a ρ-poalgebra where $\rho(+, 0) = \rho(+, 1) = \rho(\cdot, 0) = \rho(\cdot, 1) = \rho(\to, 1) = +$ and $\rho(-, 0) = \rho(\to, 0) = -$.

These are special cases of a more general class of ordered algebras that we want to consider. A *partially ordered* l*eft-residuated monoid*, a *POLRM* is a structure $\big\langle\langle A, \cdot, \to, 1\rangle, \le\big\rangle$, where

- $\langle A, \cdot, 1\rangle$ is a monoid (i.e., $\cdot$ is an associative binary operation on A and $1 \cdot a = a \cdot 1 = a$ for all $a \in A$).
- $\cdot$ is monotone in both arguments, i.e., $a \le b$ implies $a \cdot c \le b \cdot c$ and $c \cdot a \le c \cdot b$, for all $a, b, c \in A$. In terms of the terminology introduced above this just says that $\big\langle\langle A, \cdot\rangle, \le\big\rangle$ is a poalgebra.
- For all $a, b \in A$, $a \to b$ is the largest element z of A such that $z \cdot a \le b$, i.e.,

$$(\forall\, z \in A)(z \cdot a \le b \Longleftrightarrow z \le a \to b).$$

This last condition is called the *left residuation condition*.

Proposition 2.2. *Every left-residuated ordered monoid is a* ρ*-poalgebra where* $\rho(\cdot, 0) = \rho(\cdot, 1) = \rho(\to, 1) = +$ *and* $\rho(\to, 0) = -$.

Proof. . $\cdot$ is monotone in both arguments by definition. To verify $\to$ is monotone in the second and anti-monotone in the first note that, from $x \to y \le x \to y$ we get

$$\big(\forall x, y\big)\big(x \cdot (x \to y) \le y\big)$$

by residuation. This condition is called *detachment* because of its strong resemblance to the detachment rule of logic.

If $a \le b$, then $c \cdot (c \to a) \le a \le b$ by detachment, and hence $c \to a \le c \to b$ by residuation. If $a \ge b$, then $b \cdot (a \to c) \le a \cdot (a \to c) \le c$ by detachment and the monotonicity of $\cdot$ in first argument. Thus $a \to c \le b \to c$ by residuation. $\square$

The identity relation on a set A is denoted by $\varDelta_A$. $\varDelta_A$ is a ρ-pordering of $\boldsymbol{A}$ for every $\varSigma$-algebra $\boldsymbol{A}$ and every polarity ρ. $\varDelta_{Z_n}$ is the only pordering of the additive group of integers modulo n for any finite nonzero n. If $\le$ is a ρ-pordering of $\boldsymbol{A}$, then so is its reverse $\ge$. Thus if there is a ρ-ordering of $\boldsymbol{A}$ different from the identity, then there are at least three distinct ρ-algebras with the the same underlying algebra $\boldsymbol{A}$ (under the assumption of course that $\boldsymbol{A}$ is non-trivial).

In the sequel explicit reference to the signature of an algebra is usually omitted when the signature is generic and all algebras are of the same signature. Also in the sequel ρ-poalgebras will be represented by boldface calligraphic letters with the corresponding boldface Roman letters for the underlying algebras. Thus we have $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \le^{\boldsymbol{\mathcal{A}}}\rangle$, $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \le^{\boldsymbol{\mathcal{A}}}\rangle$, etc. The superscript is necessary to distinguish between different ρ-porderings of the same algebra; for example $\langle \mathbb{Z}, \varDelta_Z\rangle$, $\langle \mathbb{Z}, \le\rangle$, and $\langle \mathbb{Z}, \ge\rangle$ (with $\le$ the natural ordering) are different poalgebras with the same underlying algebra. However, when no confusion is likely we normally omit the superscript.

Let $\le$ be a ρ-qordering of an algebra $\boldsymbol{A}$. A congruence relation α on $\boldsymbol{A}$ is said to be *compatible* with $\le$ if $a \le b$ implies $a' \le b'$ for all a', b' such that $a \,\alpha\, a'$ and $b \,\alpha\, b'$. It is easy to check that α is compatible with $\le$ iff $\alpha \subseteq \le$. If α is compatible with $\le$ we define the quotient $\le/\alpha$ on the quotient set A/α by the condition

$$[a]_\alpha \quad \le/\alpha \quad [b]_\alpha \qquad \text{if} \qquad a \le b,$$

where $[a]_\alpha$ is the α-equivalence class of a; note that compatibility insures that $\leq/\alpha$ is well-defined in the sense that it does not depend on the choice of the representatives a of $[a]_\alpha$ and b of $[b]_\alpha$.

Proposition 2.3. *Let $\leq$ be a ρ-qordering of an algebra $\boldsymbol{A}$ and let α be a congruence on $\boldsymbol{A}$ that is compatible with $\leq$. Then $\leq/\alpha$ is a ρ-qordering of $\boldsymbol{A}/\alpha$.* □

Let $\leq\geq = \leq \cap \geq$; then $a \leq\geq b$ iff $a \leq b$ and $b \leq a$. For every qordering of a set A, $\leq\geq$ is an equivalence relation on A, and if $\leq$ is a ρ-qordering of $\boldsymbol{A}$, then $\leq\geq$ is a congruence relation on $\boldsymbol{A}$, and in fact it is the largest congruence relation compatible with $\leq$.[1] Thus the only congruence relation compatible with a pordering is the identity congruence. $\leq\geq$ is called the *symmetrization* of $\leq$.

Definition 2.4. Let $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \leq^{\boldsymbol{\mathcal{A}}} \rangle$ and $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \leq^{\boldsymbol{\mathcal{A}}} \rangle$ be ρ-poalgebras. A homomorphism $h: \boldsymbol{A} \to \boldsymbol{B}$ is an *order homomorphism* if $h(\leq^{\boldsymbol{\mathcal{A}}}) \subseteq \leq^{\boldsymbol{\mathcal{B}}}$, i.e., $a \leq^{\boldsymbol{\mathcal{A}}} a'$, implies $h(a) \leq^{\boldsymbol{\mathcal{B}}} h(a')$ for all $a, a' \in A$; in symbols $h: \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$. The set of all order homomorphisms from $\boldsymbol{\mathcal{A}}$ to $\boldsymbol{\mathcal{B}}$ is denoted by $\mathrm{Hom}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}})$.

h is an *order monomorphism* if it is a monomorphism (i.e., an injective homomorphism) of the underlying algebras and $h(\leq^{\boldsymbol{\mathcal{A}}}) = \leq^{\boldsymbol{\mathcal{B}}} \cap B^2$. h is an *order epimorphism* if it is an epimorphism (i.e., a surjective homomorphism) of the underlying algebras. Finally, h is an *order isomorphism* if it is both an order monomorphism and an order epimorphism, in symbols $h: \boldsymbol{\mathcal{A}} \cong \boldsymbol{\mathcal{B}}$. It is easy to see that an order homomorphism h is an order-isomorphism iff it is an isomorphism of the underlying algebras and h^{-1} is an order-homomorphism between $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{A}}$.

Note that an algebra homomorphism $h: \boldsymbol{A} \to \boldsymbol{B}$ is an order homomorphism iff $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \supseteq \leq^{\boldsymbol{\mathcal{A}}}$. Thus an algebra isomorphism $h: \boldsymbol{A} \cong \boldsymbol{B}$ is an order isomorphism iff $h^{-1}(\leq^{\boldsymbol{\mathcal{A}}}) = \leq^{\boldsymbol{\mathcal{B}}}$. Let $h: \boldsymbol{A} \to \boldsymbol{B}$ be an algebra homomorphism. The congruence relation $h^{-1}(\mathit{\Delta}_B) = \{\, \langle a, a' \rangle : h(a) = h(a') \,\}$ is called the *kernel* of h and is denoted by $\ker(h)$.

The class of ρ-poalgebras together with the ordered homomorphisms constitute a category in the natural sense.

Lemma 2.5. *Let $h: \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be an order homomorphism. Then $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$ is a ρ-qordering of $\boldsymbol{A}$ such that $\leq^{\boldsymbol{\mathcal{A}}} \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$. Furthermore* $\ker h = h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{B}}})$.

Proof. Let $\leq = h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$. The verification that $\leq$ is a quasi-ordering is straightforward. Assume $a_i \leq a'_i$ for all $i \in \rho^+(\sigma)$ and $a_j \geq a'_j$ for all $j \in \rho^-(\sigma)$. Then $h(a_i) \leq^{\boldsymbol{\mathcal{B}}} h(a')_i$ for all $i \in \rho^+(\sigma)$ and $h(a_j) \geq^{\boldsymbol{\mathcal{B}}} h(a'_j)$ for all $j \in \rho^-(\sigma)$. Thus

$$\begin{aligned} h(\sigma^{\boldsymbol{A}}(a_0, \dots, a_{n-1})) = \sigma^{\boldsymbol{B}}(h(a_0), \dots, h((a_{n-1})) \\ \leq^{\boldsymbol{B}} \ \sigma^{\boldsymbol{B}}(h(a'_0), \dots, h((a'_{n-1})) = h(\sigma^{\boldsymbol{A}}(a_0, \dots, a_{n-1})). \end{aligned}$$

So $\sigma^{\boldsymbol{A}}(a_0, \dots, a_{n-1})\ h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})\ \sigma^{\boldsymbol{A}}(a_0, \dots, a_{n-1}))$. Hence $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$ is a ρ-qordering of $\boldsymbol{A}$. Finally, we have $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{B}}}) = h^{-1}(\leq^{\boldsymbol{\mathcal{B}}} \cap \geq^{\boldsymbol{\mathcal{B}}}) = h^{-1}(\mathit{\Delta}_B) = \ker(h)$. □

Definition 2.6. Let $h: \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be an order homomorphism of ρ-poalgebras. $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$ is called the *order-kernel* of h, in symbols $\mathrm{ordker}(h)$.

[1]This is called the *Leibniz congruence* of $\leq$ in the terminology of AAL. See Remarks 2.18 and 3.5 below.

Proposition 2.7. *An order homomorphism $h\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ is an order monomorphism iff its order kernel is $\leq^{\boldsymbol{\mathcal{A}}}$.*

Proof. $\Longleftarrow$ Suppose $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) = \leq^{\boldsymbol{\mathcal{A}}}$. Then $ker(h) = h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{B}}}) = \leq^{\boldsymbol{\mathcal{A}}} \cap \leq^{\boldsymbol{\mathcal{A}}} = \Delta_A$. So $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an algebra monomorphism. $\leq^{\boldsymbol{\mathcal{B}}} \cap h(A)^2 = hh^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) = h(\leq^{\boldsymbol{\mathcal{A}}})$.

$\Longrightarrow$ Suppose h is an order monomorphism, i.e., $\ker(h) = \Delta_A$ and $h(\leq^{\boldsymbol{\mathcal{A}}}) = \leq^{\boldsymbol{\mathcal{B}}} \cap h(A)^2$. Then h is algebra monomorphism and $hh^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) = \leq^{\boldsymbol{\mathcal{B}}} \cap h(A)^2 = h(\leq^{\boldsymbol{\mathcal{A}}})$. Hence $h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) = \leq^{\boldsymbol{\mathcal{A}}}$ since h is injective. $\square$

Definition 2.8. Let $\boldsymbol{\mathcal{A}}$ be a ρ-poalgebra. By a *ρ-quasi-order*, a *ρ-qorder*, of $\boldsymbol{\mathcal{A}}$ we mean a ρ-qordering $\leq$ of $\boldsymbol{A}$, the underlying algebra of $\boldsymbol{\mathcal{A}}$, such that $\leq^{\boldsymbol{\mathcal{A}}} \subseteq \leq$. The set of all ρ-qorders of $\boldsymbol{\mathcal{A}}$ is denoted by $\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$.

We will use lower case Greek letters $\alpha, \beta, \gamma, \dots$ to represent ρ-qorders of $\boldsymbol{\mathcal{A}}$ as well as congruence relations on $\boldsymbol{A}$. If α represents the ρ-qorder $\leq$, then the reverse ordering $\geq$ will be represented by α^{-1}.

Definition 2.9. Let $\boldsymbol{\mathcal{A}}$ be a ρ-poalgebra and $\alpha \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$. The ρ-poalgebra $\langle \boldsymbol{A}/\alpha \cap \alpha^{-1}, \alpha/\alpha \cap \alpha^{-1} \rangle$ is denoted by $\boldsymbol{\mathcal{A}}/\alpha$ and is called the *quotient of $\boldsymbol{\mathcal{A}}$ by α.*

Theorem 2.10 (Order Homomorphism Theorem)**.** *Let $\boldsymbol{\mathcal{A}}$ be a ρ-poalgebra and $\alpha \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$.*

(i) *The natural map $\mathbf{n}\colon \boldsymbol{A} \to \boldsymbol{A}/\alpha \cap \alpha^{-1}$ is an order epimorphism from $\boldsymbol{\mathcal{A}}$ onto $\boldsymbol{\mathcal{A}}/\alpha$ with order kernel α.*

(ii) *Let $h\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be an order homomorphism such that $\alpha \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$. Then there exists a unique order homomorphism $g\colon \boldsymbol{\mathcal{A}}/\alpha \to \boldsymbol{\mathcal{B}}$ such that $h = g \circ \mathbf{n}$. Moreover, for every $a \in A$, $g([a]_{\alpha\cap\alpha^{-1}}) = h(a)$.*

Proof. (i). $a \leq^{\boldsymbol{\mathcal{A}}} a' \implies a\,\alpha\,a' \iff [a]_{\alpha\cap\alpha^{-1}}\,\alpha/\alpha \cap \alpha^{-1}\,[a']_{\alpha\cap\alpha^{-1}}$. But $[a]_{\alpha\cap\alpha^{-1}} = \mathbf{n}(a)$ and $[a']_{\alpha\cap\alpha^{-1}} = \mathbf{n}(a')$. $\mathbf{n}^{-1}(\alpha/\alpha \cap \alpha^{-1}) = \alpha$.

(ii). By assumption, $\alpha \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$. So $\alpha \cap \alpha^{-1} \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{B}}}) = \ker(h)$. So by the unordered homomorphism theorem, there is a unique $g\colon \boldsymbol{A}/\alpha \cap \alpha^{-1} \to \boldsymbol{B}$ such that $h = g \circ \mathbf{n}$, and $g([a]_{\alpha\cap\alpha^{-1}}) = h(a)$ for each $a \in A$. It remains only to show that g is an order homomorphism. $[a]_{\alpha\cap\alpha^{-1}}\,\alpha/\alpha \cap \alpha^{-1}\,[a']_{\alpha\cap\alpha^{-1}} \iff a\,\alpha\,a'$ (since $\alpha \cap \alpha^{-1}$ is compatible with α) $\implies h(a) \leq^{\boldsymbol{\mathcal{B}}} h(a') \iff g([a]_{\alpha\cap\alpha^{-1}}) \leq^{\boldsymbol{\mathcal{B}}} g([a']_{\alpha\cap\alpha^{-1}})$. $\square$

Corollary 2.11 (Order Isomorphism Theorem)**.** *Let $h\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be an order epimorphism of ρ-poalgebras, and let $\alpha = h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})$ be the order kernel of h. Then $\boldsymbol{\mathcal{A}}/\alpha \cong \boldsymbol{\mathcal{B}}$; in particular, $g\colon \boldsymbol{\mathcal{A}}/\alpha \cong \boldsymbol{\mathcal{B}}$, where $g([a]_{\alpha\cap\alpha-1}) = h(a)$ for all $[a]_{\alpha\cap\alpha^{-1}} \in A/\alpha \cap \alpha^{-1}$.*

Proof. h is surjective, and by the theorem the mapping $g\colon [a]_{\alpha\cap\alpha^{-1}} \mapsto h(a)$ is an order homomorphism from $\boldsymbol{\mathcal{A}}/\alpha$ to $\boldsymbol{\mathcal{B}}$ such that $h = \mathbf{n} \circ g$. Moreover,

$$\begin{aligned}
[a]_{\alpha\cap\alpha^{-1}} \quad g^{-1}(\leq^{\boldsymbol{\mathcal{B}}}) \quad [a']_{\alpha\cap\alpha^{-1}} &\iff g([a]_{\alpha\cap\alpha^{-1}}) \leq^{\boldsymbol{\mathcal{B}}} g([a']_{\alpha\cap\alpha^{-1}}) \\
&\iff h(a) \leq^{\boldsymbol{\mathcal{B}}} h(a') \\
&\iff a\,h^{-1}(\leq^{\boldsymbol{\mathcal{B}}})\,a' \\
&\iff a\,\alpha\,a' \\
&\iff [a]_{\alpha\cap\alpha^{-1}} \quad \alpha/\alpha \cap \alpha^{-1} \quad [a']_{\alpha\cap\alpha^{-1}}.
\end{aligned}$$

So the order kernel of g is $\alpha/\alpha\cap\alpha^{-1}$. Hence g is an order isomorphism by Propostion 2.7. $\square$

Definition 2.12. A ρ-poalgebra $\boldsymbol{\mathcal{A}}$ is a *order subalgebra* of ρ-poalgebra $\boldsymbol{\mathcal{B}}$, in symbols $\boldsymbol{\mathcal{A}} \subseteq \boldsymbol{\mathcal{B}}$, if $\boldsymbol{A} \subseteq \boldsymbol{B}$, i.e., the underlying algebra of $\boldsymbol{\mathcal{A}}$ is a subalgebra of the underlying algebra of $\boldsymbol{\mathcal{B}}$, and $\leq^{\boldsymbol{\mathcal{A}}} = \leq^{\boldsymbol{B}} \cap B^2$.

We say that $\boldsymbol{\mathcal{A}}$ is *generated* by a set X of its elements if it is the smallest order subalgebra of $\boldsymbol{\mathcal{B}}$ that includes X; it is easy to see that such a smallest subalgebra exists.

If $h: \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ is an order homomorphism, the order subalgebra $\langle h(\boldsymbol{A}), \leq^{\boldsymbol{\mathcal{B}}} \cap h(A)^2 \rangle$ of $\boldsymbol{\mathcal{B}}$ is called the *homomorphic image* of h and is denoted by $h(\boldsymbol{\mathcal{A}})$. If h is an order monomorphism, then $\boldsymbol{\mathcal{A}}$ is order isomorphic to $h(\boldsymbol{\mathcal{A}})$. In general, if $\boldsymbol{\mathcal{A}}$ is order isomorphic to an order subalgebra of $\boldsymbol{\mathcal{B}}$ we write $\boldsymbol{\mathcal{A}} \cong ; \subseteq \boldsymbol{\mathcal{B}}$.

Definition 2.13. Let $\langle\, \boldsymbol{\mathcal{A}}_i : i \in I \,\rangle$ be a system of ρ-poalgebras. By the *order direct product* of the system we mean the ρ-poalgebra

$$\prod_{i \in I} \boldsymbol{\mathcal{A}}_i = \big\langle \prod_{i \in I} \boldsymbol{A}_i, \leq^{\prod \boldsymbol{\mathcal{A}}_i} \big\rangle,$$

where $\langle\, a_i : i \in I \,\rangle \leq^{\prod \boldsymbol{\mathcal{A}}_i} \langle\, b_i : i \in I \,\rangle$ if $a_i \leq_i^{\boldsymbol{\mathcal{A}}} b_i$ for all $i \in I$.

It is easy to check that the order direct product is a ρ-poalgebra, and that the projection function π_i: $\prod_{j \in I} \boldsymbol{A}_j \to \boldsymbol{A}_i$ is an order epimorphism for each $i \in I$.

$\prod_{i \in I} \boldsymbol{\mathcal{A}}_i$ together with the system $\langle\, \pi_i : i \in I \,\rangle$ is a product of $\boldsymbol{\mathcal{A}}_i$ in the category of ρ-poalgebras. Thus given any system $h_i: \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}_i$, $i \in I$, of order homomorphisms, there is a unique $g: \boldsymbol{\mathcal{B}} \to \prod_{i \in I} \boldsymbol{\mathcal{A}}_i$ such that, for all $i \in I$, $h_i = \pi_i \circ g$. g is denoted by $\prod_{i \in I} h_i$.

By a *filter* on a set I we mean a family $\mathcal{F}$ of subsets of I such that (1) $I \in \mathcal{F}$, (2) $J, K \in \mathcal{F}$ implies $J \cap K \in \mathcal{F}$, and (3) $J \in \mathcal{F}$ implies $K \in \mathcal{F}$ for every $K \supseteq J$. With each filter $\mathcal{F}$ on I is associated a ρ-qorder $\trianglelefteq_{\mathcal{F}}$ of the product $\prod_{i \in I} \boldsymbol{\mathcal{A}}_i$ by the condition that $\langle\, a_i : i \in I \,\rangle \trianglelefteq_{\mathcal{F}} \langle\, b_i : i \in I \,\rangle$ if $\{\, i \in I : a_i \leq_i^{\boldsymbol{\mathcal{A}}} b_i \,\} \in \mathcal{F}$. It is routine to check that $\trianglelefteq_{\mathcal{F}}$ is indeed a ρ-qorder of $\prod_{i \in I} \boldsymbol{\mathcal{A}}_i$. The quotient $\prod_{i \in I} \boldsymbol{\mathcal{A}}_i / \trianglelefteq_{\mathcal{F}}$ is called an *order reduced product* of the system $\langle\, \boldsymbol{\mathcal{A}}_i : i \in I \,\rangle$. A filter $\mathcal{F}$ over I is *consistent* if it does not contain the empty set; it is an *ultrafilter* if is maximal and consistent. A reduced product by an ultrafilter is called an *order ultraproduct*.

There is one more construction that will be used in the sequel—the *direct limit*; it is an order subalgebra of a special kind of reduced product. Let $\langle\, \boldsymbol{\mathcal{A}}_i : i \in I \,\rangle$ be a system of ρ-poalgebras indexed by a nonempty set I with an upward directed partial order $\leq$. In addition, let $\hat{h} = \langle\, h_{i,j} : i \leq j \,\rangle$ be a system of order epimorphisms indexed by the set of pairs $\leq$ with the following properties: if $i \leq j$, $h_{i,j}: \boldsymbol{\mathcal{A}}_i \to \boldsymbol{\mathcal{A}}_j$ is an order epimorphism from $\boldsymbol{\mathcal{A}}_i$ to $\boldsymbol{\mathcal{A}}_j$, $h_{i,i}$ is the identity map, and, if $i \leq j \leq k$, then $h_{j,k} \circ h_{i,j} = h_{i,k}$. For each $i \in I$, let $[i) = \{\, j \in I : i \leq j \,\}$. Note that, since I is upward directed, for each pair i, j there exists a k such that $[k) \subseteq [i) \cap [j)$. Let $\mathcal{F}$ be the set of all subsets of I that include a $[i)$ for some i. Then $\mathcal{F}$ is a filter on I, and $\langle\, a_i : i \in I \,\rangle \trianglelefteq_{\mathcal{F}} \langle\, b_i : i \in I \,\rangle$ iff, for some j, $a_i \leq^{\boldsymbol{\mathcal{A}}_i} b_i$ for all $i \geq j$. Note that, again since I is upward directed, $\langle\, a_i : i \in I \,\rangle \trianglelefteq_{\mathcal{F}} \cap \trianglelefteq_{\mathcal{F}}^{-1} \langle\, b_i : i \in I \,\rangle$ iff, for some j, $a_i = b_i$ for all $i \geq j$. Let $\boldsymbol{\mathcal{B}}$ be the order subalgebra of $\prod_{i \in I} \boldsymbol{\mathcal{A}}_i$ consisting of all elements $\langle\, a_i : i \in I \,\rangle$ with the property that, for some i, $h_{i,j}(a_i) = a_j$ for all $j \geq i$. B is clearly compatible with $\trianglelefteq_{\mathcal{F}} \cap \trianglelefteq_{\mathcal{F}}^{-1}$ in the sense that if $\langle\, a_i : i \in I \,\rangle \in B$ and $\langle\, a_i : i \in I \,\rangle \trianglelefteq_{\mathcal{F}} \cap \trianglelefteq_{\mathcal{F}}^{-1} \langle\, b_i : i \in I \,\rangle$, then $\langle\, b_i : i \in I \,\rangle \in B$. So the quotient $\boldsymbol{\mathcal{B}} / \trianglelefteq_{\mathcal{F}}$ of $\boldsymbol{\mathcal{B}}$ by $\trianglelefteq_{\mathcal{F}}$ (more precisely its restriction to B^2) is an order subalgebra of the order reduced product $\prod_{i\in I}\boldsymbol{\mathcal{A}}_i/{\trianglelefteq_{\mathcal{F}}}$. It is called the *order direct limit of* $\langle\,\boldsymbol{\mathcal{A}}_i : i\in I\,\rangle$ *by* $\hat{h}$ and is denoted by $\varinjlim^{\hat{h}}_{i\in I}\boldsymbol{\mathcal{A}}_i$.

Let K be any class of ρ-poalgebras. The class of all order subalgebras of members of K is denoted $\mathbf{S}(\mathsf{K})$, the class of all order homomorphic images of members of K is denoted by $\mathbf{H}(\mathsf{K})$, and the class of all ρ-poalgebras order isomorphic to an order direct product of systems of members of K is denoted by $\mathbf{P}(\mathsf{K})$. The classes of all ρ-poalgebras order isomorphic respectively to an order reduced product, order ultraproduct, or order direct limit of members of K are denoted by $\mathbf{P}_{\mathrm{R}}$, $\mathbf{P}_{\mathrm{U}}$, and $\underrightarrow{\mathbf{L}}(\mathsf{K})$. Finally, the class all order isomorphic images of members of K is denoted by $\mathbf{I}(\mathsf{K})$.

Theorem 2.14. *Let* K *be a class of ρ-poalgebras.*

(i) $\mathbf{SH}(\mathsf{K})\subseteq\mathbf{HS}(\mathsf{K})$, $\mathbf{PS}(\mathsf{K})\subseteq\mathbf{SP}(\mathsf{K})$, $\mathbf{PH}(\mathsf{K})\subseteq\mathbf{HP}(\mathsf{K})$. $\mathbf{P}(\mathsf{K})\subseteq\mathbf{P}_{\mathrm{R}}(\mathsf{K})$, $\mathbf{PP}_{\mathrm{U}}(\mathsf{K})\subseteq\mathbf{P}_{\mathrm{U}}\mathbf{P}(\mathsf{K})$.
(ii) $\mathsf{K}\subseteq\underrightarrow{\mathbf{L}}(\mathsf{K})\subseteq\mathbf{P}_{\mathrm{R}}(\mathsf{K})$.
(iii) $\mathbf{SP}_{\mathrm{U}}\mathbf{P}(\mathsf{K})=\mathbf{SP}_{\mathrm{R}}(\mathsf{K})$.
(iv) $\mathbf{HSP}$ *and* $\mathbf{SP}_{\mathrm{R}}(\mathsf{K})$ *are closure operators on the class of all ρ-poalgebras.*

Proof. The proofs of the inclusions of (i) are straightforward adaptations of the proofs of the corresponding results for unordered algebras. Consider for example the inclusion $\mathbf{SH}(\mathsf{K})\subseteq\mathbf{HS}(\mathsf{K})$. Suppose $h\colon\boldsymbol{\mathcal{C}}\to\boldsymbol{\mathcal{B}}$ is an order homomorphism and $\boldsymbol{\mathcal{A}}\subseteq\boldsymbol{\mathcal{B}}$. Let $\boldsymbol{\mathcal{D}}=\langle h^{-1}(\boldsymbol{A}),\leq^{\boldsymbol{\mathcal{C}}}\cap\, h^{-1}(A)^2\rangle$. Then $\boldsymbol{\mathcal{D}}\subseteq\boldsymbol{\mathcal{C}}$ and $h{\restriction}D$ is an order epimorphism from $\boldsymbol{\mathcal{D}}$ onto $\boldsymbol{\mathcal{A}}$. The two inclusions of (ii) are immediate consequences of the definition of direct limit. (iii) and (iv) follow easily from (i). □

2.1. **Lattice of quasiorders and subdirect representation.** Quasiorders play the role in the theory of ρ-poalgebras that congruences play in the theory of algebras; like congruences, they form an algebraic lattice under set-theoretical inclusion. This view of quasiorders was anticipated by [16, 17] in a more general context; see [15, page 32].

Theorem 2.15. *Let* $\boldsymbol{\mathcal{A}}=\langle\boldsymbol{A},\leq^{\boldsymbol{\mathcal{A}}}\rangle$ *be a ρ-poalgebra.* $\langle A\times A,\ \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})\rangle$ *is an algebraic closed set system.*

Proof. Let $\mathcal{K}\subseteq\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$. Then $\bigcap\mathcal{K}$ is reflexive, transitive, and satisfies the ρ-tonicity condition. Moreover, $\leq^{\boldsymbol{\mathcal{A}}}\subseteq\bigcap\mathcal{K}$. If $\mathcal{K}$ is upward directed by inclusion, then $\bigcup\mathcal{K}$ also has these properties. □

It follows from this theorem that $\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ forms an algebraic lattice in which meet of an arbitrary (possibly infinite) set of ρ-qorders is its set-theoretical intersection. The *lattice of ρ-qorders of* $\boldsymbol{\mathcal{A}}$, $\langle\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}}),\bigcap,\bigvee\rangle$, is denoted by $\mathbf{Qord}_\rho$.

Theorem 2.16. *Let* $\boldsymbol{\mathcal{A}}$ *be a ρ-poalgebra. If* $\alpha,\beta\in\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$, *then* $\alpha\vee\beta=\bigcup_{n<\omega}(\alpha\ ;\ \beta)^n$. *More generally, for every* $K\subseteq\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$,

$$\bigvee K=\bigcup\{\alpha_0\ ;\ \alpha_1\ ;\cdots;\ \alpha_{n-1} : n<\omega, \langle\alpha_0,\dots,\alpha_{n-1}\rangle\in K^n\,\},$$

where $\alpha_0\ ;\ \alpha_1\ ;\cdots;\ \alpha_{n-1}$ *is the relative product of the binary relations* $\alpha_0,\dots,\alpha_{n-1}$, *i.e., the set of all pairs* $\langle a,b\rangle$ *such that there exist* $c_0,\dots,c_n$ *such that*

$$a=c_0\,\alpha_0\,c_1\,\alpha_1\,c_2\dots c_{n-1}\,\alpha_{n-1}\,c_n=b.$$

Proof. Let $\beta = \bigcup\{\alpha_0 \,;\, \alpha_1 \,;\, \dots ;\, \alpha_{n-1} : n < \omega, \langle \alpha_0, \dots, \alpha_{n-1}\rangle \in K^n\}$. Clearly, β is reflexive and transitive. We note that $(\alpha_0 \,;\, \alpha_1 \,;\, \cdots ;\, \alpha_0)^{-1} = \alpha_{n-1}^{-1} \,;\, \cdots ;\, \alpha_1^{-1} \,;\, \alpha_0^{-1}$. It is easy to see that $\beta^{-1} = \bigcup\{\alpha_0^{-1} \,;\, \alpha_1^{-1} \,;\, \dots ;\, \alpha_{n-1}^{-1} : n < \omega, \langle \alpha_0, \dots, \alpha_{n-1}\rangle \in K^n\}$.

To see that β satisfies the ρ-tonicity condition, let σ be an operation, which for simplicity we take of order 2,n whose first and second arguments are of negative and positive polarity respectively. Assume $a\,\beta^{-1}\,b$ and let $\alpha_0, \dots, \alpha_{n-1} \in K$ and $c_0, \dots, c_n$ such that $a = c_0\,\alpha_0^{-1}\,c_1\,\alpha_1^{-1}\,c_2 \cdots c_{n-1}\,\alpha_{n-1}^{-1}\,c_n = b$ Then for any d we have

$$\sigma^{\boldsymbol{A}}(a,d) = \sigma^{\boldsymbol{A}}(c_0,d)\,\alpha_0\,\sigma^{\boldsymbol{A}}(c_1,d)\,\alpha_1\,\sigma^{\boldsymbol{A}}(c_2,d)\cdots\sigma^{\boldsymbol{A}}(c_{n-1},d)\,\alpha_{n-1}\,\sigma^{\boldsymbol{A}}(c_n,d) = \sigma^{\boldsymbol{A}}(b,d).$$

Thus $\sigma^{\boldsymbol{A}}(a,d)\,\beta\,\sigma^{\boldsymbol{A}}(b,d)$. This shows that $\sigma^{\boldsymbol{A}}$ is antimonotonic in the first argument with respect to β. In a similar way it can be shown that it is monotonic is the second argument. Thus β satisfies the ρ-tonicity condition. So β is a ρ-qorder of $\boldsymbol{\mathcal{A}}$. Since it is clearly included in every ρ-qorder that includes each $\alpha \in K$, we get that β is the least upper bound of K in the lattice $\mathbf{Qord}_\rho(\boldsymbol{\mathcal{A}})$. □

Theorem 2.17 (Order Correspondence Theorem). *Let $\boldsymbol{\mathcal{A}}$ be an ρ-poalgebra and $\alpha \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$. Then for every ρ-qorder β of $\boldsymbol{\mathcal{A}}/\alpha$ there is a unique $\hat{\beta} \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ such that*

$$\beta = \hat{\beta}/\alpha = \big\{\, \big\langle\, [a]_{\alpha\cap\alpha^{-1}},\, [b]_{\alpha\cap\alpha^{-1}} \,\big\rangle : \langle a, b\rangle \in \hat{\beta} \,\big\}.$$

The mapping $\beta \mapsto \hat{\beta}$ is a lattice isomorphism between $\mathbf{Qord}_\rho(\boldsymbol{\mathcal{A}}/\alpha)$ and the principal filter of $\mathbf{Qord}_\rho\,\boldsymbol{\mathcal{A}}$ generated by α.

Proof. Let β be a ρ-qorder of $\boldsymbol{\mathcal{A}}/\alpha = \langle \boldsymbol{A}/\alpha\cap\alpha^{-1}, \alpha/\alpha\cap\alpha^{-1}\rangle$, i.e., a ρ-qordering of $\boldsymbol{A}/\alpha\cap\alpha^{-1}$ such that $\beta \supseteq \alpha/\alpha\cap\alpha^{-1}$, and define $\hat{\beta} = \{\, \langle a, b\rangle : [a]_{\alpha\cap\alpha^{-1}}\ \beta\ [b]_{\alpha\cap\alpha^{-1}} \,\}$. It is straightforward to verify that $\hat{\beta}$ is a ρ-qordering of $\boldsymbol{A}$. For example, suppose σ is a binary operation with negative ρ-polarity in the first argument, and let $a\,\hat{\beta}^{-1}\,b$, and suppose $[a]_{\alpha\cap\alpha^{-1}}\ \beta^{-1}\ [b]_{\alpha\cap\alpha^{-1}}$. Then, for every c,

$$\begin{aligned}[\sigma^{\boldsymbol{A}}(a,c)]_{\alpha\cap\alpha^{-1}} &= \sigma^{\boldsymbol{A}/\alpha\cap\alpha^{-1}}\big([a]_{\alpha\cap\alpha^{-1}},\, [c]_{\alpha\cap\alpha^{-1}}\big)\\ &\qquad\beta\quad \sigma^{\boldsymbol{A}/\alpha\cap\alpha^{-1}}\big([b]_{\alpha\cap\alpha^{-1}},\, [c]_{\alpha\cap\alpha^{-1}}\big) = [\sigma^{\boldsymbol{A}}(b,c)]_{\alpha\cap\alpha^{-1}}.\end{aligned}$$

So $\alpha^{\boldsymbol{A}}(a,c)\,\hat{\beta}\,\sigma^{\boldsymbol{A}}(b,c)$. It remains only to show that $\hat{\beta} \supseteq \alpha$. But this follows immediately from the inclusion $\beta \supseteq \alpha/\alpha\cap\alpha^{-1}$.

Consider any $\gamma \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ that includes α. Then $\alpha\cap\alpha^{-1}$ is compatible with γ, and hence $[a]_{\alpha\cap\alpha^{-1}}\ \beta\ [b]_{\alpha\cap\alpha^{-1}}$ iff $a\,\hat{\beta}\,b$. It follows that $\gamma/\alpha\cap\alpha^{-1} = \big\{\, \big\langle\, [a]_{\alpha\cap\alpha^{-1}}, [b]_{\alpha\cap\alpha^{-1}} \,\big\rangle : \langle a, b\rangle \in \gamma \,\big\}$ is in $\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}}/\alpha)$ and that, if $\gamma, \gamma' \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ and $\gamma/\alpha\cap\alpha^{-1} = \gamma'/\alpha\cap\alpha^{-1}$, then $\gamma = \gamma'$. So the mapping $\beta \mapsto \hat{\beta}$ is a bijection between $\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}}/\alpha)$ and the set of all $\hat{\beta} \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ such that $\hat{\beta} \supseteq \alpha$. It clearly preserves the lattice ordering. □

Remark 2.18. The key property of ρ-qorderings used in the above proof is that the symmetrization of a ρ-qordering α is automatically compatible with every ρ-qordering that includes α. The deductive systems $\mathcal{L}$ considered in AAL with this property, that is the property that the Leibniz congruence of a $\mathcal{L}$-filter F is compatible with every $\mathcal{L}$-filter that includes F, are called *protoalgebraic*. The protoalgebraic deductive systems turn out to be exactly those for which a correspondence theorem analogous to Theorem 2.17 holds, and they seem to be the widest class of deductive systems for which a reasonable algebraic theory can be developed; see [6].

Definition 2.19. A ρ-poalgebra $\boldsymbol{\mathcal{B}}$ is an *order subdirect product* of a system $\langle \boldsymbol{\mathcal{A}}_i : i \in I \rangle$ of ρ-poalgebras, in symbols $\boldsymbol{\mathcal{B}} \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{A}}_i$, if

(i) $\boldsymbol{\mathcal{B}} \subseteq \prod_{i\in I} \boldsymbol{\mathcal{A}}_i$, and
(ii) $\pi_i \colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}_i$ is an order epimorphism for each $i \in I$.

Let K be a class of ρ-poalgebras. The class of all ρ-poalgebras isomorphic to a subdirect product of some system of members of K is denoted by $\mathbf{P}_{\mathrm{SD}}(\mathsf{K})$.

Proposition 2.20. $\boldsymbol{\mathcal{B}} \cong ; \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{A}}_i$ *iff there exists a* $\langle \alpha_i : i \in I \rangle \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{B}})$ *such that*

(i) $\bigcap_{i\in I} \alpha_i = \leq^{\boldsymbol{\mathcal{B}}}$, *and*
(ii) $\boldsymbol{\mathcal{B}}/\alpha_i \cong \boldsymbol{\mathcal{A}}_i$ *for every* $i \in I$.

Proof. $\Longrightarrow$ Let $h\colon \boldsymbol{\mathcal{B}} \cong \boldsymbol{\mathcal{C}} \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{B}}_i$. For every $i \in I$, $\pi_i \circ h \colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}_i$ is an order epimorphism; let α_i be its order kernel, i.e., $\alpha_i = (\pi_i \circ h)^{-1}(\leq^{\boldsymbol{\mathcal{A}}_i})$. Then $\boldsymbol{\mathcal{B}}/\alpha_i \cong \boldsymbol{\mathcal{A}}_i$ by the order isomorphism theorem.

$$
\begin{aligned}
b \textstyle\bigcap_{i\in I}\alpha_i\, b' &\iff \big(\forall\, i \in I\big)\big(\pi_i(h(b)) \leq^{\boldsymbol{\mathcal{A}}_i} \pi_i(h(b'))\big)\\
&\iff h(b) \leq^{\prod \boldsymbol{\mathcal{A}}_i} h(b')\\
&\iff h(b) \leq^{\boldsymbol{\mathcal{C}}} h(b')\\
&\iff b \leq^{\boldsymbol{\mathcal{B}}} b'.
\end{aligned}
$$

So $\bigcap_{i\in I} \alpha_i = \leq^{\boldsymbol{\mathcal{B}}}$.

$\Longleftarrow$ By the categorical product property $h = \prod_{i\in I} \mathbf{n}_i$, where $\mathbf{n}_i \colon \boldsymbol{\mathcal{A}}_i \to \boldsymbol{\mathcal{A}}_i/\alpha_i$ is the natural order epimorphism, is an order homomorphism $h\colon \boldsymbol{\mathcal{B}} \to \prod_{i\in I} \boldsymbol{\mathcal{B}}/\alpha_i$; note that $h(b) = \langle\, [b]_{\alpha_i\cap\alpha_i^{-1}} : i \in I \,\rangle$.

$$
\begin{aligned}
b \quad h^{-1}(\leq^{\prod \boldsymbol{\mathcal{A}}_i/\alpha_i}) \quad b' &\iff \langle\, [b]_{\alpha_i\cap\alpha_i^{-1}} : i \in I \,\rangle \quad \leq^{\prod \boldsymbol{\mathcal{B}}/\alpha_i} \langle\, [b']_{\alpha_i\cap\alpha_i^{-1}} : i \in I \,\rangle\\
&\iff \big(\forall\, i \in I\big)\big([b]_{\alpha_i\cap\alpha_i^{-1}} \;\; \alpha_i/\alpha_i \cap \alpha_i^{-1} \;\; [b']_{\alpha_i\cap\alpha_i^{-1}}\big)\\
&\iff (\forall\, i \in I)(b\, \alpha_i\, b')\\
&\iff b \bigcap_{i\in I} \alpha_i\, b'\\
&\iff b \leq^{\boldsymbol{\mathcal{B}}} b'.
\end{aligned}
$$

So the order kernel of h is $\leq^{\boldsymbol{\mathcal{B}}}$. Hence $\boldsymbol{B} \cong h(\boldsymbol{B}) \subseteq \prod_{i\in I} \boldsymbol{\mathcal{B}}/\alpha_i$. Since $\pi_i \circ h \colon \boldsymbol{\mathcal{B}} \to \boldsymbol{B}/\alpha_i$ is an order epimorphism for each $i \in I$ we get that $h(\boldsymbol{B}) \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{B}}/\alpha_i$. □

Definition 2.21. A ρ-poalgebra $\boldsymbol{\mathcal{B}}$ is *order subdirectly irreducible* if $\boldsymbol{\mathcal{B}} \cong ; \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{A}}_i$ implies that $\boldsymbol{\mathcal{B}} \cong \boldsymbol{\mathcal{A}}_i$ for some $i \in I$.

Theorem 2.22 (Order Subdirect Representation Theorem)**.** *Every ρ-poalgebra is isomorphic to an order subdirect product of order subdirectly irreducible ρ-poalgebras.*

Proof. Let $\boldsymbol{\mathcal{A}}$ be a ρ-poalgebra. For each pair a, b of elements of $\boldsymbol{\mathcal{A}}$ such that $a \nleq^{\boldsymbol{\mathcal{A}}} b$, let $\alpha_{a,b}$ be a maximal ρ-qorder of $\boldsymbol{\mathcal{A}}$ such that $\langle a, b\rangle \notin \alpha_{a,b}$; such a maximal ρ-qorder exists by Zorn's lemma. $\alpha_{a,b}$ is completely meet irreducible in the lattice $\mathbf{Qord}_\rho(\boldsymbol{\mathcal{A}})$, for suppose $\langle\, \beta_i : i \in I\,\rangle$ is any system of ρ-qorders such that $\alpha_{a,b} \subset \beta_i$ for all $i \in I$. Then $a\, \beta_i\, b$. Thus $\alpha_{a,b} \neq \bigcap_{i\in I} \beta_i$. Clearly $\leq^{\boldsymbol{\mathcal{A}}} = \bigcap\{\, \alpha_{a,b} : a \nleq^{\boldsymbol{\mathcal{A}}} b\,\}$. So by the Prop 2.20 $\boldsymbol{\mathcal{A}} \cong; \subseteq_{\mathrm{SD}} \prod_{a \nleq^{\boldsymbol{\mathcal{A}}} b} \boldsymbol{\mathcal{A}}/\alpha_{a,b}$. Suppose $\boldsymbol{\mathcal{A}}/\alpha_{a,b} \cong; \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{\mathcal{B}}_i$. By Prop. 2.20 and the order correspondence theorem there is a system $\langle\, \beta_i : i \in I\,\rangle$ of ρ-qorders of $\boldsymbol{\mathcal{A}}$ such that $\alpha_{a,b} = \bigcap_{i\in I} \beta_I$ and $\boldsymbol{\mathcal{A}}_i/\beta_i \cong \boldsymbol{\mathcal{B}}_i$. Since $\alpha_{a,b}$ is completely join-irreducible, $\alpha_{a,b} = \beta_i$ for some i. So $\boldsymbol{\mathcal{A}}/\alpha_{a,b} \cong \boldsymbol{\mathcal{B}}_i$, and hence each $\boldsymbol{\mathcal{A}}/\alpha_{a,b}$ is order subdirectly irreducible. □

3. Ordered Equational Logic

The set of terms over the signature Σ in the variables X is denoted by $\mathrm{Te}_\Sigma(X)$. Either of "Σ" or "X" may be omitted if they are clear from context or irrelevant. By an *inequation* we mean an ordered pair of terms $\langle t, s\rangle$, which we normally write in the form $t \preccurlyeq s$. A *quasi-inequation* is a non-empty sequence of inequations, written $t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1} \to u \preccurlyeq v$. The inequations $t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1}$ are called the *premisses* and $u \preccurlyeq v$ the *conclusion* of the quasi-inequation. Inequations are identified with quasi-inequations with an empty set of premisses.

We write a term t in the form $t(x_0, \dots, x_{n-1})$ to indicate that the variables occurring in t all appear in the list $x_0, \dots, x_{n-1}$; we often use the abbreviation $\bar{x}$ for $x_0, \dots, x_{n-1}$. Let $t(\bar{x}) \preccurlyeq s(\bar{x})$ be an inequation and $\boldsymbol{\mathcal{A}}$ a ρ-poalgebra. An assignment $\bar{a} \in A^n$ of elements of $\boldsymbol{\mathcal{A}}$ to the variables $\bar{x}$ is said to *satisfy* $t(\bar{x}) \preccurlyeq s(\bar{x})$ *in* $\boldsymbol{\mathcal{A}}$ if $t^{\boldsymbol{\mathcal{A}}}(\bar{a}) \leq^{\boldsymbol{\mathcal{A}}} s^{\boldsymbol{\mathcal{A}}}(\bar{a})$. It *satisfies* a quasi-inequation if it either fails to satisfy at least one of the premisses, or it satisfies the conclusion.

A quasi-inequation is said to be a *quasi-inidentity* of a ρ-poalgebra $\boldsymbol{\mathcal{A}}$ if it is satisfied by every assignment of elements of $\boldsymbol{\mathcal{A}}$ to the variables; in this case we also say that $\boldsymbol{\mathcal{A}}$ is a *model* of the quasi-inequation. As a special case we have the definition of the *inidentities* of $\boldsymbol{\mathcal{A}}$. For any set Q of quasi-inequations or inequations, the class of models of all members of Q is denoted by $\mathsf{Mod}(Q)$.

Definition 3.1. A class Q of ρ-poalgebras is called a *ρ-ordered quasivariety*, a *ρ-quasi-povariety* for short, if $\mathsf{V} = \mathsf{Mod}(Q)$ for some set of quasi-inequations Q. The class of models of a set of inequations is called a *ρ-ordered variety*, a *ρ-povariety.* If ρ is the completely positive polarity, Q is called simply a *quasi-povariety* or *povariety.*

Example 3.2 (Lattices As an Povariety)**.** The signature is $\Sigma = \{\wedge, \vee\}$ and, as the term "povariety" implies, the polarity is completely positive. The povariety of lattices is defined by the following six extralogical inidentities.

(1) $\qquad x \wedge y \preccurlyeq x, \qquad x \wedge y \preccurlyeq y,$

(2) $\qquad x \preccurlyeq x \vee y, \qquad y \preccurlyeq x \vee y,$

(3) $\qquad x \preccurlyeq x \wedge x, \qquad x \vee x \preccurlyeq x.$

We recall the logical inidentities and quasi-inidentities that are automatically included in the definition of any povariety.

refl	$x \preccurlyeq x$,
tran	$x \preccurlyeq y,\ y \preccurlyeq z \implies x \preccurlyeq z$,
toni$_\wedge$	$x_1 \preccurlyeq x_2,\ y_1 \preccurlyeq y_2 \implies x_1 \wedge y_1 \preccurlyeq x_2 \wedge y_2$;
toni$_\vee$	$x_1 \preccurlyeq x_2,\ y_1 \preccurlyeq y_2 \implies x_1 \vee y_1 \preccurlyeq x_2 \vee y_2$.

It is obvious that if $\langle A, \wedge, \vee\rangle$ is a lattice in the usual sense, then $\big\langle\langle A, \wedge, \vee\rangle, \leq\big\rangle$ is a model of the inidentities (1)–(3). Conversely, let $\big\langle\langle A, \wedge, \vee\rangle, \leq\big\rangle$ be a poalgebra satisfying the inidentities (1)–(3), and let $a, b \in A$. $a \wedge b \leq a, b$ by (1). Suppose $c \leq a, b$. Then by (2) and ($mono_\wedge$) we get $c \leq c \wedge c \leq a \wedge b$. So $a \wedge b$ is the greatest lower bound of a, b. Similarly, $a \vee b$ is the least upper bound of a, b. So $\langle A, \wedge, \vee\rangle$ is a lattice in the ordinary sense.

Example 3.3 (Partially Ordered Left-Residuated Monoids (POLRMS))**.** Recall that these are ρ-poalgebras $\big\langle\langle A, \cdot, \to, 1\rangle, \leq\big\rangle$ where $\rho(\cdot, 0) = \rho(\cdot, 1) = \rho(\to, 1) = +$ and $\rho(\to, 0) = -$, and such that $\langle A, \cdot, 1\rangle$ is a monoid and the residuation condition holds: for all $a, b, z \in A$, $z \cdot a \leq b$ iff $z \leq a \to b$.

Proposition 3.4. *POLRM is a ρ-povariety defined by the following set of* (*extra-logical*) *inidentities.*

$$(x \cdot y) \cdot z \preccurlyeq\succcurlyeq x \cdot (y \cdot z) \tag{4}$$

$$1 \cdot x \preccurlyeq\succcurlyeq x \tag{5}$$

$$x \cdot 1 \preccurlyeq\succcurlyeq x \tag{6}$$

$$x \cdot (x \to y) \preccurlyeq y \tag{7}$$

$$y \preccurlyeq x \to x \cdot y \tag{8}$$

Proof. That each POLRM is a ρ-algebra satisfying the inidentities (4)–(6) is clear. (7) and (8) follow respectively from the inidentities $x \to y \preccurlyeq x \to y$ and $x \cdot y \preccurlyeq x \cdot y$ by residuation.

Suppose $\big\langle\langle A, \cdot, \to, 1\rangle, \leq\big\rangle$ is a ρ-poalgebra satisfying (4)–(8). $\langle A, \cdot, 1\rangle$ is a monoid by (4)–(6). We verify the residuation condition: Suppose $a \leq b \to c$. Then $b \cdot a \leq b \cdot (b \to c) \leq c$. Suppose $b \cdot a \leq c$. Then $a \leq b \to b \cdot a \leq b \to c$. □

Partially ordered groups can be viewed as a special kind of POLRM. A partially ordered group is a group $\langle G, \cdot, {}^{-1}, e\rangle$ with a partial ordering $\leq$ of its universe with respect to which $\cdot$ is monotone in both arguments. Define $x \to y$ as $x^{-1} \cdot y$. Then $\big\langle\langle G, \cdot, \to, e\rangle, \leq\big\rangle$ is a POLRM, and the class of all partially ordered groups in this sense forms a sub-ρ-povariety of POLRM.

Remark 3.5. It should be noted that there is no equality symbol in the language of lattices as ordered algebras; this is an important feature of the metamathematics of ρ-poalgebras in our treatment. Inequational logics from this point of view can be treated as universal Horn theories with a single, binary predicate and without equality. These are in effect the *2-dimensional deductive systems* of AAL. When viewed as universal Horn theories in this way, the most general models of the inequational logic of lattices are of the form $\big\langle\langle A, \wedge, \vee\rangle, \leq\big\rangle$, where $\leq$ is a not necessarily antisymmetric quasi-ordering of $\langle A, \wedge, \vee\rangle$ satisfying the inidentities (1)–(2), i.e., its symmetrization $\langle A, \wedge, \vee\rangle/ \leq \cap \geq$ is a lattice in the ordinary sense.

The most general models of an arbitrary inequational logic are algebras with a distinguished ρ-qordering, but we restrict our attention to those special models for which the quasi-ordering is antisymmetric. These are the so-called *reduced models* of the logic when considered a 2-dimensional deductive system. Recall that the symmetrization $\leq \cap \geq$ of a given ρ-qordering $\leq$ is the largest congruence compatible with $\leq$. More generally, for each model of an arbitrary 2 dimensional deductive system, more generally still, for a *k-deductive system*, there is a largest congruence that is compatible with the so-called *designated filter* of the model; this is a binary or k-ary relation on the universe of the algebra that corresponds to the ρ-qorder of the nonreduced models of inequation logic. This congruence is called the *Leibniz congruence* of the model, and the model is *reduced* if its Leibniz congruence is the identity.

Notice that the logical axioms of inequational logic except the first are quasi-inequations. Thus strictly speaking, considered as a universal Horn theory without equality, the reduced models of a set of inequations form a quasivariety. But following the example of equational logic we refer to the reduced models of an inequational logic in which there are no proper quasi-inequations among the extralogical axioms as a variety.

Example 3.6 (The Variety of (Unordered) Algebras as a Quasi-povariety)**.** It is instructive to observe that ordinary equational logic can be viewed in a natural way as the inequational logic with a single extralogical axiom, namely the symmetry law $x \preccurlyeq y \to y \preccurlyeq x$. Its nonreduced models are of the form $\langle \boldsymbol{A}, \alpha \rangle$ where α is a congruence relation on $\boldsymbol{A}$, and its reduced models are $\langle \boldsymbol{A}, \varDelta_A \rangle$, which can be indentified with its underlying algebra $\boldsymbol{A}$.

By simply making the extralogical axiom of antisymmetry a logical axiom we get Birkhoff's well known equational logic.

Remark 3.7. The above two examples and the intervening remark illustrate two significant features of our approach to the metamathematics of ordered algebras. One is the isolation of equality from inequality in the formal development. Equality enters only peripherally by means of the Leibniz congruence and the restriction to reduced models. In the case of inequational logic the Leibniz equality is definable by the two inequations $x \preccurlyeq y, y \preccurlyeq x$, but in an arbitrary k-deductive system it may not be so readily definable. Those systems for which it is are called *equivalential logics* and constitute an extensively investgated proper subclass of the class of protoalgebraic logics.

The other point the last example illustrates is that, in our approach, the theory of ordered algebras is developed as a logical system apart from the equational logic of Birkhoff rather than by building on it—in short the two theories are developed in parallel rather than in series. A lot of work on theory of ordered algebras can be found in the literature based on the alternative approach, and most of the results we obtain here are not significantly different from what can be found in the literature, and no claim is made for their novelty. However we believe our approach is smoother and more natural exactly because of the fact it parallels traditional equational logic rather than builds on it. Furthermore, it raises new questions that lead to new and we thing interesting results of a different character. Some of will be seen in the next two sections.

Definition 3.8. Let K be an arbitrary class of ρ-poalgebras. For each ρ-poalgebra $\boldsymbol{\mathcal{A}}$ (not necessarily in K) we define

$$\mathrm{Qord}_\rho^{\mathsf{K}}(\boldsymbol{\mathcal{A}}) = \{\, \alpha \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}}) : \boldsymbol{\mathcal{A}}/\alpha \in \mathsf{K} \,\}.$$

The elements of $\mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$ are called the K-ρ-*qorders* of $\boldsymbol{\mathcal{A}}$.

Proposition 3.9. *Let* K *be a class of* ρ*-poalgebras, and let* $\boldsymbol{\mathcal{A}}$ *be a* ρ*-algebra, not necessarily in* K*.*

(i) *Assume* $\mathbf{P}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K}$*. Then* $\mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$ *is closed under arbitrary intersection. It follows that* $\mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$ *contains a smallest* ρ*-qorder.*

(ii) *Assume* $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$*. If* $\alpha \in \mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$*, then* $\beta \in \mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$ *for every* $\beta \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ *such that* $\beta \supseteq \alpha$*.*

(iii) *Assume both* $\mathbf{P}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K}$ *and* $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$*. Then* $\mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$ *is a principal filter of* $\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$*.*

Proof. (i) Let $\{\, \alpha_i : i \in I \,\} \subseteq \mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$. By Proposition 2.20 and the order correspondence theorem, $\boldsymbol{\mathcal{A}}/\bigcap_{i\in I}\alpha_i \cong; \subseteq_{\mathrm{SD}} \prod\{\, \boldsymbol{\mathcal{A}}/\alpha_i : i \in I \,\} \in \mathbf{P}_{\mathrm{SD}}\{\, \boldsymbol{\mathcal{A}}/\alpha_i : i \in I \,\} \subseteq \mathbf{P}_{\mathrm{SD}}\mathsf{K} \subseteq \mathsf{K}$.

(ii) By the order homomorphism theorem, $\boldsymbol{\mathcal{A}}/\beta$ is a homomorphic image of $\boldsymbol{\mathcal{A}}/\alpha$. Thus, since $\boldsymbol{\mathcal{A}}/\alpha \in \mathsf{K}$ by the assumption, and K is closed under homomorphic images by hypothesis, we have $\boldsymbol{\mathcal{A}}/\beta \in \mathsf{K}$ and hence $\beta \in \mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}})$.

(iii) is an immediate consequence of (i) and (ii). □

Definition 3.10. Let K be a ρ-quasivariety, and let $\boldsymbol{\mathcal{A}}$ be an arbitrary ρ-poalgebra, not necessarily in K. Let $R \subset A^2$. By the K-ρ-qorder of $\boldsymbol{\mathcal{A}}$ *generated* by R, in symbols $\Phi^{\mathsf{K}}_\rho(R)$, we mean the smallest K-ρ-qorder of $\boldsymbol{\mathcal{A}}$ that includes R, i.e., $\Phi^{\mathsf{K}}_\rho(R) = \bigcap\{\, \alpha \in \mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{A}}) : R \subseteq \alpha \,\}$.

$\Phi^{\mathsf{K}}_\rho(R)$ always exists if $\mathbf{P}_{\mathrm{SD}}(K) \subseteq \mathsf{K}$.

Definition 3.11. Let K be a class of ρ-poalgebras. A ρ-poalgebra is said to be *freely generated over* K by a set X if X generates $\boldsymbol{\mathcal{A}}$ and, for every $\boldsymbol{\mathcal{B}} \in \mathsf{K}$ and every mapping $h\colon X \to B$, there is an order homomorphism from $\boldsymbol{\mathcal{A}}$ into $\boldsymbol{\mathcal{B}}$ that extends h.

The extension is unique because of the assumption that X generates $\boldsymbol{\mathcal{A}}$; it is normally represented by the same symbol as the original map.

For any set of variables X the *algebra of terms over* X, or *term algebra over* X, is the algebra $\mathbf{Te}(X)$ whose universe is $\mathrm{Te}(X)$ and such that, for each $\sigma \in \Sigma_n$, the operation $\sigma^{\mathbf{Te}(X)}$ takes the term $\sigma(t_0, \dots, t_{n-1})$ as value for each choice of arguments $t_0, \dots, t_{n-1}$, i.e., $\sigma^{\mathbf{Te}(X)}(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, t_{n-1})$. $\mathbf{Te}(X)$ is freely generated by X over the class of all Σ-algebras. We note that the value $t^{\boldsymbol{A}}(a_0, \dots, a_{n-1})$ a term $t(x_0, \dots, x_{n-1})$ takes in an algebra $\boldsymbol{A}$ under the assignment of $a_0, \dots, a_{n-1}$ to the variables coincides with the image $h(t(x_0, \dots, x_{n-1}))$ of $t(x_0, \dots, x_{n-1})$ under any homomorphism $h\colon \mathbf{Te}(X) \to \boldsymbol{A}$ such that $h(x_0) = a_0, \dots, h(x_{n-1}) = a_{n-1}$, i.e., $h(t(x_0, \dots, x_{n-1}) = t^{\boldsymbol{A}}(h(x_0), h(x_1), \dots, h(x_{n-1}))$.

The ρ-poalgebra $\langle \mathbf{Te}(X), \Delta_{\mathrm{Te}(X)} \rangle$ over the term algebra is denoted by $\boldsymbol{\mathcal{T}e}(X)$. From the above remarks it follows that, for any set X of cardinality κ, $\boldsymbol{\mathcal{T}e}(X)$ is freely generated over the class of all ρ-poalgebras by the set X.

Theorem 3.12. *Let* K *be a class of* ρ*-poalgebras such that* $\mathbf{SP}(\mathsf{K}) \subseteq \mathsf{K}$*. Then for each cardinal* κ *there exists a* ρ*-poalgebra in* K *that is freely generated over* K *by a set of cardinality* κ*. More specifically, let* X *be any set of cardinality* κ*, and let* α *be the smallest member of* $\mathrm{Qord}^{\mathsf{K}}_\rho(\boldsymbol{\mathcal{T}e}(X))$*. Then* $\boldsymbol{\mathcal{T}e}(X)/\alpha$ *is a member of* K *and is freely generated over* K *by* $X/\alpha \cap \alpha^{-1} = \{\, [x]_{\alpha\cap\alpha^{-1}} : x \in X \,\}$*.*

Any two ρ-polagebras in K that are freely generated over K by sets of the same cardinality are isomorphic.

Proof. $\boldsymbol{\mathcal{T}e}(X)/\alpha \in \mathsf{K}$ by definition of α. Let $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and $h: X/\alpha \cap \alpha^{-1} \to A$. Let h' be the unique order homomorphism from $\boldsymbol{\mathcal{T}e}(X)$ into $\boldsymbol{A}$ such that $h'(x) = h([x]_{\alpha\cap\alpha^{-1}})$ for each $x \in X$. $\mathrm{ordker}(h') \in \mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{F}})$, so $\alpha \subseteq \mathrm{ordker}(h')$. Thus by the order homomorphism theorem there is a unique order homomorphism $g: \boldsymbol{\mathcal{T}e}(X)/\alpha \to \boldsymbol{A}$ such that, for each $x \in X$, $g([x]_{\alpha\cap\alpha^{-1}}) = h(x)$.

Suppose $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are freely generated over K by sets of the same cardinality. Then any bijection between the sets of free generators extends to an order isomorphism between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$. $\square$

Lemma 3.13. *Let K be a class of ρ-poalgebras such that $\mathbf{SP}(\mathsf{K}) \subseteq \mathsf{K}$. Let α be the smallest member of $\mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{T}e}(X))$. Then, for any pair of terms t, s over the variables X, $t \preccurlyeq s$ is an inidentity of K iff $t\,\alpha s$.*

Proof. $\Longrightarrow$ Assume $t(x_0, \dots, x_{n-1}) \preccurlyeq s(x_0, \dots, x_{n-1})$ is an inidentity of K. Since $\boldsymbol{\mathcal{T}e}(X)/\alpha \in \mathsf{K}$, we have

$$\begin{aligned}[t(x_0, \dots, x_{n-1})]_{\alpha\cap\alpha^{-1}} &= t^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}\big([x_0]_{\alpha\cap\alpha^{-1}}, \dots, [x_{n-1}]_{\alpha\cap\alpha^{-1}}\big) \\ &\leq^{\boldsymbol{\mathcal{T}e}(X)/\alpha} s^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}\big([x_0]_{\alpha\cap\alpha^{-1}}, \dots, [x_{n-1}]_{\alpha\cap\alpha^{-1}}\big) \\ &= [s(x_0, \dots, x_{n-1})]_{\alpha\cap\alpha^{-1}}.\end{aligned}$$

But $\leq^{\boldsymbol{\mathcal{T}e}(X)/\alpha} = \alpha/\alpha \cap \alpha^{-1}$. So $t(x_0, \dots, x_{n-1})\ \alpha\ (x_0, \dots, x_{n-1})$.

$\Longleftarrow$ Assume $t\,\alpha\,s$. Let $\boldsymbol{\mathcal{A}}$ be an arbitrary member of K and $h: \boldsymbol{\mathcal{T}e}(X) \to \boldsymbol{\mathcal{A}}$ any order homomorphism. $\alpha \subseteq \mathrm{ordker}(h)$ since $\mathrm{ordker}(h) \in \mathrm{Qord}^{\mathbf{Te}(X)}_{\rho}$. So $t\ h^{-1}(\leq^{\boldsymbol{\mathcal{A}}})\ s$, and hence $h(t) \leq^{\boldsymbol{\mathcal{A}}} h(s)$. Since this holds for any $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and $h: \boldsymbol{\mathcal{T}e}(X) \to \boldsymbol{\mathcal{A}}$, $t \preccurlyeq s$ is an inidentity of K. $\square$

Theorem 3.14 (Order H-S-P Theorem)**.** *A class K of ρ-poalgebras is an order variety iff it is closed under the formation of order homomorphic images, subalgebras, and direct products, i.e., iff $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$.*

Proof. $\Longrightarrow$ Assume I is a set of inidentities such that $\mathsf{K} = \mathsf{Mod}(I)$. Consider any $\boldsymbol{\mathcal{A}} \in \mathbf{H}(\mathsf{K})$ and order homomorphism $h: \boldsymbol{\mathcal{T}e}(X) \to \boldsymbol{\mathcal{A}}$. There is a $\boldsymbol{\mathcal{B}} \in \mathsf{K}$ and an order epimorphism $g: \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}$. Since g is surjective, there exists an order homomorphism $f: \boldsymbol{\mathcal{T}e}(X) \to \boldsymbol{\mathcal{B}}$ such that $h = g \circ f$. Let $t \preccurlyeq s$ be an inidentity of I. Since $\boldsymbol{\mathcal{B}} \in \mathsf{K}$ we have $f(t) \leq^{\boldsymbol{\mathcal{B}}} f(s)$, and since g is an order homomorphism, $g(f(t) \leq^{\boldsymbol{\mathcal{A}}} g(f(s))$, i.e., $h(t) \leq^{\boldsymbol{\mathcal{A}}} h(s)$. This shows that $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$. The proofs $\mathbf{S}(\mathsf{K}) \subseteq \mathsf{K}$ and $\mathbf{P}(\mathsf{K}) \subseteq \mathsf{K}$ are obtained in a similar manner; we leave the details to the interested reader. We only mention that the proof in the direct product case uses the fact the the direct product is a product in the category of ρ-poalgebras.

$\Longleftarrow$ Assume $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$, so in particular $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$ and $\mathbf{SP}(\mathsf{K}) = \mathsf{K}$. Let I be the set of all inidentitites of K. We will show that $\mathsf{K} = \mathsf{Mod}(I)$, and since the inclusion from left to right is obvious, it suffices to show that $\mathsf{Mod}(I) \subseteq \mathsf{K}$. Suppose $\boldsymbol{\mathcal{A}} \in \mathsf{Mod}(I)$. Choose X large enough so that an order epimorphism $h: \boldsymbol{\mathcal{T}e}(X) \to \boldsymbol{\mathcal{A}}$ exists. Let α be the smallest member of $\mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{T}e}(X))$. Then for all terms t, s we get, using Lemma 3.13, that $t\,\alpha\, s \implies t \preccurlyeq s \in I \implies h(t) \leq^{\boldsymbol{A}} h(s)$. Hence $\alpha \subseteq \operatorname{ordker}(h)$, and so by the order homomorphism theorem there is a $g\colon \mathbf{Te}(X)/\alpha \to \boldsymbol{A}$ such that $h = g \circ \mathbf{n}$. g is an order epimorphism since h is, and so $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{Te}(X)/\alpha$. But the latter ρ-poalgebra is in K by definition of α. Thus $\boldsymbol{A} \in \mathbf{H}(\mathsf{K}) = \mathsf{K}$. □

In the last part of the section we obtain an analogous operator-theoretic characterization for order quasivarieties that parallels Mal'cev's well known characterization of quasivarieties of unordered quasivarieties as refined in [17, 18].

Lemma 3.15. *Let K be a class of ρ-poalgebras such that $\mathbf{SP}(\mathsf{K}) \subseteq \mathsf{K}$. Let $t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1}$ be a finite set of inequations over the variables X. $\Phi^{\mathsf{K}}_{\rho}(\{\, t_i \preccurlyeq s_i : i < n \,\})$ is the smallest member α of $\operatorname{Qord}^{\mathsf{K}}_{\rho}\big(\boldsymbol{Te}(X)\big)$ such that $t_i\,\alpha\, s_i$ for all $i < n$. For any inequality $u \preccurlyeq v$, the quasi-inequation*

$$t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1} \to u \preccurlyeq v \tag{9}$$

is a quasi-inidentity of K iff $u\ \Phi^{\mathsf{K}}_{\rho}(\{\, t_i \preccurlyeq s_i : i < n \,\})\ v$.

Proof. Let $\alpha = \Phi^{\mathsf{K}}_{\rho}(\{\, t_i \preccurlyeq s_i : i < n \,\})$, and let $\bar{x} = x_0, \dots, x_{m-1}$ be the variables occurring in (9). Let $[\bar{x}]_{\alpha\cap\alpha^{-1}}$ denote the sequence $[x_0]_{\alpha\cap\alpha^{-1}}, \dots, [x_{n-1}]_{\alpha\cap\alpha^{-1}}$ of elements of $\boldsymbol{Te}(X)/\alpha$. We note first that, by definition of α, $[t_i(\bar{x})]_{\alpha\cap\alpha^{-1}} \leq^{\boldsymbol{Te}(X)/\alpha} [s_i(\bar{x})]_{\alpha\cap\alpha^{-1}}$, and hence

$$t_i^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}\big([\bar{x}]_{\alpha\cap\alpha^{-1}}\big) \leq^{\boldsymbol{Te}(X)/\alpha} s_i^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}\big([\bar{x}]_{\alpha\cap\alpha^{-1}}\big), \quad \text{for each } i < n. \tag{10}$$

$\Longrightarrow$ Assume (9) is a quasi-inidentity of K. Then by (10) and the fact that $\boldsymbol{Te}(X)/\alpha \in \mathsf{K}$ we have that $u^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}([\bar{x}]_{\alpha\cap\alpha^{-1}}) \leq^{\boldsymbol{Te}(X)/\alpha} v^{\mathbf{Te}(X)/\alpha\cap\alpha^{-1}}([\bar{x}]_{\alpha\cap\alpha^{-1}})$, and hence $u(\bar{x})\,\alpha\, v(\bar{x})$.

$\Longleftarrow$ Assume $u(\bar{x})\,\alpha\, v(\bar{x})$. Let $\boldsymbol{A} \in \mathsf{K}$ and let $h\colon \boldsymbol{Te}(X) \to \boldsymbol{A}$ be any order homomorphism such that $t_i^{\boldsymbol{A}}\big(h(\bar{x})\big) \leq^{\boldsymbol{A}} s_i^{\boldsymbol{A}}\big(h(\bar{x})\big)$ for each $i < n$. Then $\alpha \subseteq h^{-1}(\leq^{\boldsymbol{A}})$ by definition of α since $h^{-1}(\leq^{\boldsymbol{A}}) \in \operatorname{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{Te}(X))$. So $u(\bar{x})\, h^{-1}(\leq^{\boldsymbol{A}})\, v(\bar{x})$ and hence $u^{\boldsymbol{A}}(h(\bar{x})) \leq^{\boldsymbol{A}} v^{\boldsymbol{A}}(h(\bar{x}))$. Thus (9) is a quasi-inidentity of K. □

Lemma 3.16. *Let $\boldsymbol{A}$ be a ρ-poalgebra and $\mathcal{K}$ a set of ρ-qorders of $\boldsymbol{A}$ that is upward directed by inclusion so that $\bigcup\mathcal{K}$ is also a ρ-qorder of $\boldsymbol{A}$. Then $\boldsymbol{A}/\bigcup\mathcal{K}$ is isomorphic to the direct limit of the system of ρ-poalgebras $\langle\, \boldsymbol{A}/\alpha : \alpha \in \mathcal{K} \,\rangle$ by the system of order epimorphisms*

$$\hat{h} = \langle\, h_{\alpha,\beta} : \boldsymbol{A}/\alpha \to \boldsymbol{A}/\beta : \alpha, \beta \in \mathcal{K}, \alpha \subseteq \beta \,\rangle,$$

where $h_{\alpha,\beta}([a]_{\alpha\cap\alpha^{-1}} = [a]_{\beta\cap\beta^{-1}}$ for all $[a]_{\alpha\cap\alpha^{-1}} \in A/\alpha \cap \alpha^{-1}$.

Proof. Let $\mathcal{F} = \{\, \mathcal{G} \subseteq \mathcal{K} : (\exists\, \alpha)\big((\alpha] \subseteq \mathcal{G}\big) \,\}$, and let $g\colon A \to \varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}} A/\alpha \cap \alpha^{-1}$ be the map in such that, for each $a \in A$, $g(a) = \big[\big\langle\, [a]_{\alpha\cap\alpha^{-1}} : \alpha \in \mathcal{K} \,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}}$. It is clear that g maps into $\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}} A/\alpha \cap \alpha^{-1}$. To see it is an order homomorphism, consider any $\sigma \in \Sigma$, of rank

n say, and any $a_0, \dots, a_{n-1} \in A$.

$$\begin{aligned} g\big(\sigma^{\boldsymbol{A}}(a_0,\dots,a_{n-1})\big) &= \big[\big\langle\, [\sigma^{\boldsymbol{A}}(a_0,\dots,a_{n-1})]_{\alpha\cap\alpha^{-1}} : \alpha \in \mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}} \\ &= \big[\big\langle\, \sigma^{\boldsymbol{A}/\alpha\cap\alpha^{-1}}([a_0]_{\alpha\cap\alpha^{-1}},\cdots,[a_{n-1}]_{\alpha\cap\alpha^{-1}}) : \alpha \in \mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}} \\ &= \sigma^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha}\big(\big[\big\langle\,[a_0]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}},\dots \\ &\qquad\qquad \dots,\big[\big\langle\,[a_{n-1}]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}}\big) \\ &= \sigma^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha}\big(g(a_0),\dots,g(a_{n-1})\big). \end{aligned}$$

And, for all $\alpha, \beta \in \mathcal{K}$,

$$\begin{aligned} a \le^{\boldsymbol{A}} b \quad &\Longrightarrow \quad \big(\forall\,\alpha\in\mathcal{K}\big)\big([a]_{\alpha\cap\alpha^{-1}} \le^{\boldsymbol{A}/\alpha} [b]_{\alpha\cap\alpha^{-1}}\big) \\ &\Longleftrightarrow \quad \big\langle\,[a]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle \le^{\prod \boldsymbol{A}/\alpha} \big\langle\,[b]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle \\ &\Longrightarrow \quad \big[\big\langle\,[a]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}} \le^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha} \big[\big\langle\,[b]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}} \\ &\Longleftrightarrow \quad g(a) \le^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha} g(b). \end{aligned}$$

So g is an order homomorphism. To see it is surjective, recall first of all that every element of the direct limit is of the form $\big[\big\langle\,[a_\alpha]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}}$, wheren, for some $\gamma\in\mathcal{K}$, $h_{\gamma,\alpha}([a_\gamma]_{\gamma\cap\gamma^{-1}}) = [a_\alpha]_{\alpha\cap\alpha^{-1}}$ for every $\alpha\in\mathcal{K}$ such that $\gamma\subseteq\alpha$. Let $b = a_\gamma$. For, for every $\alpha\supseteq\gamma$, we have $[a_\alpha]_{\alpha\cap\alpha^{-1}} = h_{\gamma,\alpha}[b] = [b]_{\alpha\cap\alpha^{-1}}$. So $\big\langle\,[b]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle \trianglelefteq_{\mathcal{F}} \cap \trianglelefteq_{\mathcal{F}}^{-1} \big\langle\,[a_\alpha]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle$, and hence, $g(b) = \big[\big\langle\,[a_\alpha]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq_{\mathcal{F}}\cap\trianglelefteq_{\mathcal{F}}^{-1}}$. So g is an order epimorphism.

Finally, we show that its order kernel is $\bigcap\mathcal{K}$. Let $a, b\in A$.

$$\begin{aligned} g(a) \le^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha} g(b) \quad &\Longleftrightarrow \quad \big[\big\langle\,[a]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq\cap\trianglelefteq^{-1}} \le^{\varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha} \big[\big\langle\,[b]_{\alpha\cap\alpha^{-1}} : \alpha\in\mathcal{K}\,\big\rangle\big]_{\trianglelefteq\cap\trianglelefteq^{-1}} \\ &\Longleftrightarrow \quad \big(\exists\,\gamma\big)\big(\forall\,\alpha\supseteq\gamma\big)\big([a]_{\alpha\cap\alpha^{-1}} \le^{\boldsymbol{A}/\alpha} [b]_{\alpha\cap\alpha^{-1}}\big) \\ &\Longleftrightarrow \quad \big(\exists\,\gamma\big)\big(\forall\,\alpha\supseteq\gamma\big)\big(a\,\alpha\,b\big) \\ &\Longleftrightarrow \quad a \bigcup\mathcal{K}\, b. \end{aligned}$$

So, the order kernel of g is $\bigcup\mathcal{K}$, and hence $\boldsymbol{A}/\bigcup\mathcal{K} \cong \varinjlim^{\hat{h}}_{\alpha\in\mathcal{K}}\boldsymbol{A}_\alpha$ by the order isomorphism theorem. □

Theorem 3.17 (Order S-L-P Theorem)**.** *A class* K *of* ρ*-poalgebras is an* ρ*-quasi-povariety iff it is closed under the formation of subalgebras, direct limits, and products i.e., iff* $\mathbf{S}\,\underrightarrow{\mathbf{L}}\,\mathbf{P}(\mathsf{K}) = \mathsf{K}$.

Proof. $\Longrightarrow$ The proof that a ρ-quasi-povariety is a closed under order subalgebras and reduced products is straightforward. Its closed under order direct limits and product because they are special kinds of reduced products.

$\Longleftarrow$ Assume $\mathbf{S}\,\underrightarrow{\mathbf{L}}\,\mathbf{P}(\mathsf{K}) = \mathsf{K}$. Let Q be the set of all quasi-inidentities K. We will show that $\mathsf{K} = \mathsf{Mod}(Q)$. The inclusion from left to right is obvious. Suppose $\boldsymbol{A}\in\mathsf{Mod}(Q)$. Let X be a large enough set of variables such that there is a epimorphism $h\colon \boldsymbol{Te}(X)\to\boldsymbol{A}$. Let α be the order kernel of h.

Recall that, for each $R \subseteq_\omega \alpha$ (i.e, for each finite subset of α), $\Phi^{\mathsf{K}}_\rho(R)$, the ρ-K-qorder generated by R, is the smallest K-ρ-qorder of $\boldsymbol{\mathcal{T}e}(X)$ that includes R. (It exists by Proposition 3.9(i) since $\mathbf{SP}_{\mathrm{SD}}(\mathsf{K}) \subseteq \mathsf{K}$ by assumption.) Let $R = \{\, t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1} \}$, and let $u\ \Phi^{\mathsf{K}}_\rho(R)\ v$. Then, by Lemma 3.15, $t_0 \preccurlyeq s_0, \dots, t_{n-1} \preccurlyeq s_{n-1} \to u \preccurlyeq v$ is a quasi-inidentity of K. By hypothesis it is also a quasi-inidentity of $\boldsymbol{\mathcal{A}}$. $h(t_i) = h(s_i)$ for all $i < n$ since R is included in the order kernel α of h. Hence $h(u) = h(v)$, i.e., $u\,\alpha\,v$. This shows that $\Phi^{\mathsf{K}}_\rho(R) \subseteq \alpha$ for every $R \subseteq_\omega \alpha$. Thus $\alpha = \bigcup_{R \subseteq_\omega \alpha} \Phi^{\mathsf{K}}_\rho(R)$. Clearly the set of K-ρ-qorders $\Phi^{\mathsf{K}}_\rho(R)$ are upward directed by inclusion. Thus by Lemma 3.16 $\boldsymbol{\mathcal{A}}$ is the direct limit of the system $\left\langle\, \boldsymbol{\mathcal{T}e}(X)/\Phi^{\mathsf{K}}_\rho(R) : R \subseteq_\omega \alpha \,\right\rangle$. Since each $\boldsymbol{\mathcal{T}e}(X)/\Phi^{\mathsf{K}}_\rho(R) \in \mathsf{K}$, we get $\boldsymbol{\mathcal{A}} \in \underrightarrow{\mathbf{L}}(\mathsf{K}) = \mathsf{K}$. □

It follows immediately that K is a ρ-quasi-povariety iff $\mathbf{SP}_{\mathrm{R}}(\mathsf{K}) = \mathsf{K}$ iff $\mathbf{SP}_{\mathrm{U}}\mathbf{P}(\mathsf{K}) = \mathsf{K}$.

4. Algebraizable ρ-Povarieties

We define for an arbitrary ρ-poalgebra $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \leq^{\boldsymbol{\mathcal{A}}} \rangle$ the *algebra reduct* of $\boldsymbol{\mathcal{A}}$, in symbols $\mathbf{Alg}(\boldsymbol{\mathcal{A}})$, to be the underlying algebra with the ρ-pordering replaced with the identity relation, i.e., $\mathbf{Alg}(\boldsymbol{\mathcal{A}}) = \langle \boldsymbol{A}, \Delta_A \rangle$, which for convenience we identify with $\boldsymbol{A}$. (In contrast to conventional model theory the equality predicate is not assumed here to be a basic part of the definition of an algebra.) For any class of ρ-poalgebras, $\mathsf{Alg}(\mathsf{K}) = \{\, \mathbf{Alg}(\boldsymbol{\mathcal{A}}) : \boldsymbol{\mathcal{A}} \in \mathsf{K} \,\}$.

It turns out that $\mathsf{Alg}(\mathsf{K})$ is a quasivariety for every ρ-quasi-povariety K. This can be obtained directly for the well-known result of Mal'cev that the class of all substructures of a reduct of a universal Horn class is again a universal Horn class; it is easy to see that $\mathsf{Alg}(\mathsf{K})$ is closed under the formation of subalgebras. Alternatively, using another, related result of Mal'cev, the result can be verified by observing that $\mathsf{Alg}(\mathsf{K})$ is closed under subalgebras and reduced products.

Let Q be a quasivariety of algebras, and let $\boldsymbol{A}$ be an algebra not necessarily a member of Q. Those congruences α of $\boldsymbol{A}$ such that $\boldsymbol{A}/\alpha \in \mathsf{Q}$ are called Q-*congruences*. We take $\mathrm{Co}^{\mathsf{Q}}(\boldsymbol{A}) = \{\, \alpha \in \mathrm{Co}(\boldsymbol{A}) : \boldsymbol{A}/\alpha \in \mathsf{Q} \,\}$.

Proposition 4.1. *Assume* K *is a ρ-quasi-povariety. Let* $\boldsymbol{A}$ *be a Σ-algebra* (*not necessarily in* K)*. Then mapping* $\alpha \mapsto \alpha \cap \alpha^{-1}$ *is a surjective map from* $\mathrm{Qord}^{\mathsf{K}}_\rho(\langle \boldsymbol{A}, \Delta_A \rangle)$ *onto* $\mathrm{Co}^{\mathbf{Alg}(\mathsf{K})}(\boldsymbol{A})$.

Proof. If $\alpha \in \mathrm{Qord}^{\mathsf{K}}_\rho(\langle \boldsymbol{A}, \Delta_A \rangle)$, then $\boldsymbol{\mathcal{A}}/\alpha = \langle \boldsymbol{A}/\alpha \cap \alpha^{-1}, \alpha/\alpha \cap \alpha^{-1} \rangle \in \mathsf{K}$ and hence $\alpha \cap \alpha^{-1} \in \mathrm{Co}^{\mathbf{Alg}(\mathsf{K})}$. It remains to show the mapping is surjective.

Suppose $\vartheta \in \mathrm{Co}^{\mathsf{Alg}(\mathsf{K})}(\boldsymbol{A})$. $\boldsymbol{A}/\vartheta \in \mathsf{Alg}(\mathsf{K})$ by hypothesis, so there exists a ρ-ordering $\leq$ of $\boldsymbol{A}/\vartheta$. Let $\alpha = \mathbf{n}^{-1}(\leq)$, where $\mathbf{n}\colon \boldsymbol{A} \to \boldsymbol{A}/\vartheta$ is the natural map. For all $a, b \in A$ we have $a\ \alpha \cap \alpha^{-1}\ b$ iff $a\ \ \mathbf{n}^{-1}(\leq) \cap \mathbf{n}^{-1}(\geq)\ \ b$ iff $a\ \mathbf{n}^{-1}(\leq\geq)\ b$ iff $a\ \mathbf{n}^{-1}(\Delta_{A/\vartheta})\ b$ iff $a\,\vartheta\,b$. So $\alpha \cap \alpha^{-1} = \vartheta$, and hence $\alpha/\alpha \cap \alpha^{-1} = \leq$. So $\langle \boldsymbol{A}, \Delta_A \rangle/\alpha = \langle \boldsymbol{A}/\alpha \cap \alpha^{-1}, \alpha/\alpha \cap \alpha^{-1} \rangle = \langle \boldsymbol{A}/\vartheta, \leq \rangle \in \mathsf{K}$. We conclude that $\alpha \in \mathrm{Qord}^{\mathsf{K}}_\rho(\langle \boldsymbol{A}, \Delta_A \rangle)$, and because $\alpha \cap \alpha^{-1} = \vartheta$, we see that the map $\alpha \mapsto \alpha \cap \alpha^{-1}$ does indeed map $\mathrm{Qord}_\rho(\langle \boldsymbol{A}, \Delta_A)$ surjectively onto $\mathrm{Co}^{\mathsf{Alg}(\mathsf{K})}(\boldsymbol{A})$. □

If K is algebraizable, then an axiomatization of $\mathsf{Alg}(\mathsf{K})$ can be obtained from the set of defining equations, as shown in Proposition 4.6 below.

Definition 4.2. Let $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \leq^{\boldsymbol{\mathcal{A}}} \rangle$ be a ρ-poalgebra and let $(T(x,y) \approx S(x,y)) = \{\, t_i(x,y) \approx s_i(x,y) : i < n \,\}$ be a finite set of equations in two variables. $\leq^{\boldsymbol{\mathcal{A}}}$ is *definable by* $T \approx S$ if, for all $a, b \in A$,

$$a \leq^{\boldsymbol{\mathcal{A}}} b \quad \Longleftrightarrow \quad T^{\boldsymbol{A}}(a,b) = S^{\boldsymbol{A}}(a,b),$$

where there expression "$T^{\boldsymbol{A}}(a,b) = S^{\boldsymbol{A}}(a,b)$" is shorthand for "$\big(\forall\, i < n\big)\big(t_i^{\boldsymbol{A}}(a,b) = s_i^{\boldsymbol{A}}(a,b)\big)$".

A ρ-quasi-povariety K is *algebraizable* if there exists a finite set of formulas $T \approx S$ in two variables such that, for every $\boldsymbol{\mathcal{A}}$ in K, $T \approx S$ defines $\leq^{\boldsymbol{\mathcal{A}}}$. $T \approx S$ is called a *defining set of equations for* K.

Proposition 4.3. *Assume* K *is an algebraizable* ρ*-quasi-povariety with defining equations* $T \approx S$. *Then, for every* $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ *and every* $\alpha \in \mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$*, we have for all* $a, b \in A$*:*

$$a \,\alpha\, b \quad \Longleftrightarrow \quad T^{\boldsymbol{A}}(a,b)\, \alpha \cap \alpha^{-1}\, S^{\boldsymbol{A}}(a,b),$$

where the expression "$T^{\boldsymbol{A}}(a,b)\, \alpha \cap \alpha^{-1}\, S^{\boldsymbol{A}}(a,b)$*" is shorthand for the expression "*$\big(\forall\, t \approx s \in (T \approx S)\big)\big(t^{\boldsymbol{A}}(a,b)\, \alpha \cap \alpha^{-1}\, s^{\boldsymbol{A}}(a,b)\big)$*".*

Proof.

$$\begin{aligned} a \,\alpha\, b \quad &\Longleftrightarrow [a]_{\alpha\cap\alpha^{-1}} \ \leq^{\boldsymbol{A}/\alpha\cap\alpha^{-1}} \ [b]_{\alpha\cap\alpha^{-1}} \\ &\Longleftrightarrow \quad T^{\boldsymbol{A}/\alpha\cap\alpha^{-1}}\big([a]_{\alpha\cap\alpha^{-1}},\, [b]_{\alpha\cap\alpha^{-1}}\big) \ = \ S^{\boldsymbol{A}/\alpha\cap\alpha^{-1}}\big([a]_{\alpha\cap\alpha^{-1}},\, [b]_{\alpha\cap\alpha^{-1}}\big) \\ &\Longleftrightarrow \quad T^{\boldsymbol{A}}(a,b)\, \alpha \cap \alpha^{-1}\, S^{\boldsymbol{A}}(a,b). \end{aligned}$$

□

In every ρ-poalgebra equality is definable in terms of the order by symmetrization:

$$a = b \quad \Longleftrightarrow \quad a \leq^{\boldsymbol{\mathcal{A}}} b \quad \text{and} \quad b \leq^{\boldsymbol{\mathcal{A}}} a.$$

(In the terminology of AAL this means that the inequational logic of each ρ-quasi-povariety is protoalgebraic, a fact previously noted in Remark 3.7.) If $\leq^{\boldsymbol{\mathcal{A}}}$ is definable by $T \approx S$, then

$$a = b \quad \Longleftrightarrow \quad T^{\boldsymbol{A}}(a,b) = S^{\boldsymbol{A}}(a,b) \quad \text{and} \quad T^{\boldsymbol{A}}(b,a) = S^{\boldsymbol{A}}(b,a). \tag{11}$$

From (11) it follows that, if K is algebraizable with defining equations $T \approx S$, then the equations in $T(x,x) \approx S(x,x)$ are identities of $\mathsf{Alg}(\mathsf{K})$ and the quasi-equation

$$(T(x,y) \approx S(x,y)),\, (T(y,x) \approx S(y,x)) \ \to \ x \approx y$$

is a quasi-identity of $\mathsf{Alg}(\mathsf{K})$.

Corollary 4.4. *Assume* K *is an algebraizable* ρ*-quasi-povariety.*

(i) *For every* $\boldsymbol{A} \in \mathsf{Alg}(\mathsf{K})$ *there exists a unique* ρ*-pordering* $\leq$ *of* $\boldsymbol{A}$ *such that* $\langle \boldsymbol{A}, \leq \rangle \in \mathsf{K}$.
(ii) *For every* $\boldsymbol{\mathcal{A}} \in \mathsf{K}$*, the mapping* $\alpha \mapsto \alpha \cap \alpha^{-1}$ *from* $\mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$ *to* $\mathrm{Co}(\boldsymbol{A})$ *is injective.*

□

Example 4.5. Each of $\{x \wedge y \approx x\}$ and $\{x \vee y \approx y\}$ is a defining set of equations for the povariety of lattices.

The ρ-povariety of POLRMs is not algebraizable. In fact, the ρ-subpovariety of partially ordered groups is not algebraizable. To show this we need only exhibit a group with two distinct partial orderings, for example $\langle\langle \mathbb{Z}, +, -, 0\rangle, \leq\rangle$ and $\langle\langle \mathbb{Z}, +, -, 0\rangle, \Delta_{\mathbb{Z}}\rangle$, where $\leq$ is the natural ordering of the integers.

Proposition 4.6. *Assume K is an algebraizable ρ-quasi-povariety. If $T \approx S$ is a set of defining equations for K, then the quasivariety $\mathsf{Alg}(\mathsf{K})$ is defined by the following identities and quasi-identities, where I and Q are respectively any any set of inidentities and quasi-inidentities that together define K.*

$$T(x, x) \approx S(x, x). \tag{12}$$

$$T(x, y) \approx S(x, y),\, T(y, z) \approx S(y, z) \to T(x, z) \approx S(x, z). \tag{13}$$

This expression represents the set of m quasi-equations all of which have the same antecedent, the conjunction of the $2m$ equations in $T(x, y) \approx S(x, y)$ and $T(y, z) \approx S(y, z)$, and one of the equations in $T(x, z) \approx S(x, z)$ as consequent (m is the number of equations in $T \approx S$).

$$T(x, y) \approx S(x, y) \to T\big(\sigma(\bar{z}_{<i}, x, \bar{z}_{>i}\big) \approx S\big(\sigma(\bar{z}_{<i}, y, \bar{z}_{>i}\big) \quad \textit{for each } \sigma \in \Sigma_n \textit{ and } i < n \textit{ such that } \rho(\sigma, i) = +, \tag{14}$$

$$T(y, x) \approx S(y, x) \to T\big(\sigma(\bar{z}_{<i}, x, \bar{z}_{>i}\big) \approx S\big(\sigma(\bar{z}_{<i}, y, \bar{z}_{>i}\big) \quad \textit{for each } \sigma \in \Sigma_n \textit{ and } i < n \textit{ such that } \rho(\sigma, i) = -, \tag{15}$$

$$T(x, y) \approx S(x, y),\, T(y, x) \approx S(x, y) \to x \approx y. \tag{16}$$

$$T(t, s) \approx S(t, s) \quad \textit{for every } t \preccurlyeq s \textit{ in } I. \tag{17}$$

$$T(t_0, s_0) \approx S(t_0, s_0), \ldots, T(t_{n-1}, s_{n-1}) \approx S(t_{n-1}, s_{n-1}) \to T(u, v) \approx S(u, v) \quad \textit{for every } t_0 \preccurlyeq s_0, \ldots, t_{n-1} \preccurlyeq s_{n-1} \to u \preccurlyeq v \textit{ in } Q \tag{18}$$

Proof. Let E be the set of equations and quasi-equations (12)–(18). Clearly $\mathsf{Alg}(\mathsf{K}) \subseteq \mathsf{Mod}(E)$ (see the remark following Proposition 4.3). Suppose $\boldsymbol{\mathcal{A}} \in \mathsf{Mod}(E)$. Define $\leq \subseteq A^2$ by the condition that $a \leq b$ iff $T^{\boldsymbol{A}}(a, b) = S^{\boldsymbol{A}}(a, b)$. $\leq$ is a ρ-qordering of $\boldsymbol{A}$ by (12)–(15), a partial ordering by (16), and $\langle \boldsymbol{A}, \leq\rangle \in \mathsf{Mod}(\mathsf{K})$ by (17) and (18). So $\boldsymbol{A} \in \mathsf{Alg}(\mathsf{K})$. $\square$

Taking K to be the povariety of lattices, we see that (16) takes the form

$$x \wedge y \approx x,\, y \wedge x \approx y \;\to\; x \approx y.$$

This is a consequence of the lattice identity $x \wedge y \approx y \wedge x$. The quasi-identities (13)–(15) can also be replaced by identities. So $\mathsf{Alg}(\mathsf{K})$ is a variety, in fact the variety of lattices.

Proposition 4.7. *Let K be an algebraizable ρ-quasi-povariety. Then, for each $\boldsymbol{\mathcal{A}} \in \mathsf{K}$, the map $\alpha \mapsto \alpha \cap \alpha^{-1}$ is an isomorphism between the lattices $\mathbf{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$ and $\mathbf{Co}^{\mathsf{Alg}(\mathsf{K})}(\boldsymbol{A})$.*

Proof. $\square$

Theorem 4.8. *Let K be a ρ-quasi-povariety. The following are equivalent.*

(i) *K is algebraizable.*
(ii) *For each $\boldsymbol{\mathcal{A}} \in \mathsf{K}$, $\alpha \mapsto \alpha \cap \alpha^{-1}$ is an injective map from $\mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$ to $\mathrm{Co}^{\mathsf{Alg}(\mathsf{K})}(\boldsymbol{\mathcal{A}})$.*
(iii) *For each $\boldsymbol{\mathcal{A}} \in \mathsf{K}$, $\alpha \mapsto \alpha \cap \alpha^{-1}$ is an isomorphism between $\mathbf{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$ and $\mathbf{Co}^{\mathsf{Alg}(\mathsf{K})}(\boldsymbol{\mathcal{A}})$.*

Proof. (i) $\Rightarrow$ (ii): by Corollary 4.4.

(ii) $\Rightarrow$ (iii) Assume the map $\alpha \mapsto \alpha \cap \alpha^{-1}$ is injective. It is clearly order preserving. Since $\leq^{\boldsymbol{\mathcal{A}}}$ is the unique ρ-pordering $\leq$ of $\boldsymbol{A}$ such that $\langle \boldsymbol{A}, \leq\rangle \in \mathsf{K}$, we have that $\mathrm{Qord}_\rho^{\mathsf{K}}(\boldsymbol{\mathcal{A}}) = \mathrm{Qord}_\rho^{\mathsf{K}}(\langle \boldsymbol{A}, \Delta_A\rangle)$. Hence by Proposition 4.1, the mapping is surjective.

(iii) $\Rightarrow$ (i) Let α be the smallest member of $\mathrm{Qord}_\rho^{\mathsf{K}}(\boldsymbol{\mathcal{T}e}(X))$ and let $\boldsymbol{F}(X) = \mathbf{Te}(X)/\alpha \cap \alpha^{-1}$ and $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{T}e}(X)/\alpha = \langle \boldsymbol{F}(X),\ \alpha/\alpha \cap \alpha^{-1}\rangle$. $\boldsymbol{\mathcal{F}}(X)$ in a member of K and is freely generated over K by $\{\, [x]_{\alpha\cap\alpha^{-1}} : x \in X \,\}$. $[x]_{\alpha\cap\alpha^{-1}}$ and $[x']_{\alpha\cap\alpha^{-1}}$ are distinct congruence classes if x and x' are distinct (provided K is nontrivial, i.e., contains a ρ-poalgebra with more than one element). For convenience we identity $\{\, [x]_{\alpha\cap\alpha^{-1}} : x \in X \,\}$ with X.

Fix distinct variables x and y and let β be the ρ-qorder of $\boldsymbol{\mathcal{F}}(X)$ generated by $\{\langle x, y\rangle\}$. Note that $\beta \cap \beta^{-1} \in \mathrm{Co}^{\mathsf{K}}(\boldsymbol{F}(X))$, and that it is a compact element of the lattice $\mathbf{Co}^{\mathsf{K}}(\boldsymbol{F}(X))$ since it is image of a compact element of $\mathbf{Qord}_\rho^{\mathsf{K}}(\boldsymbol{\mathcal{F}}(X))$. Thus $\beta \cap \beta^{-1}$ is a finitely generated $\mathsf{Alg}(\mathsf{K})$-congruence of $\boldsymbol{F}(X)$. Let

$$\left\{\, \big\langle [t_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}},\ [s_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}} \big\rangle : i < m \,\right\}$$

be a set of generators. Let $t_i(x, y) = t_i'(x, y, x, x, \dots, x)$, that is, the term obtained from t_i' by substituting x for all occurrences of the $z_0, \dots, z_{n-1}$; similarly, let $s_i(x, y) = s_i'(x, y, x, x, \dots, x)$. We will show that $\{\, t_i(x, y) \approx s_i(x, y) : i < m \,\}$ is a defining set for K. Let $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and $a, b \in A$. Let X be a set of sufficient cardinality so that there exists an order epimorphism h from $\boldsymbol{\mathcal{F}}(X)$ onto $\boldsymbol{\mathcal{A}}$ such that $h(x) = h(z_0) = \cdots = h(z_{n-1} = a$ and $h(y) = b$.

Suppose $a \leq^{\boldsymbol{\mathcal{A}}} b$. Then $x\ h^{-1}(\leq^{\boldsymbol{A}})\ y$. So $\beta \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{A}}})$. Consequently, $\beta \cap \beta^{-1} \subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{A}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{A}}})$, the order kernel of h. So, for each $i < m$, $t_i^{\boldsymbol{A}}(a, b) = t_i'^{\boldsymbol{A}}(a, b, a, a, \dots, a) = h\big([t_i'(x, y, x, x, \dots, x)]_{\alpha\cap\alpha^{-1}}\big) = h\big([s_i'(x, y, x, x, \dots, x)]_{\alpha\cap\alpha^{-1}}\big) = s_i^{\boldsymbol{A}}(a, b)$.

Now suppose $a \not\leq^{\boldsymbol{A}} b$. Then $\langle x, y\rangle \notin h^{-1}(\leq^{\boldsymbol{A}})$. So $\beta \not\subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{A}}})$, and hence, since distinct ρ-qorders have distinct symmetrizations by hypothesis, $\beta \cap \beta^{-1} \not\subseteq h^{-1}(\leq^{\boldsymbol{\mathcal{A}}}) \cap h^{-1}(\geq^{\boldsymbol{\mathcal{A}}})$. Thus there is an $i < m$ such that the pair $\big\langle [t_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}},\ [s_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}} \big\rangle$ is not in the order kernel of h. Hence

$$t_i^{\boldsymbol{A}}(a, b) = h\big([t_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}}\big) \neq h\big([s_i'(x, y, z_0, \dots, z_{n-1})]_{\alpha\cap\alpha^{-1}}\big) = s_i^{\boldsymbol{A}}(a, b).$$

$\square$

5. Properties of the lattice of ρ-qorders

By a *polynomial form* (over a signature Σ) we mean a term $t(*, x_0, \dots, x_{n-1})$ with a distinguished variable, which we denote by $*$, that occurs at exactly one place in t; the other variables, which are assumed to be included in the list $\bar{x} = x_0, \dots, x_{n-1}$, are called *parametric variables*. The set of all polynomial forms is denoted by Pl. A polarity function ρ can be extended from fundamental operations at their various argument positions to polynomial forms in a natural way. $\rho(*) = +$. If $\rho\big(t(*, \bar{x})\big) = +$, then for any $\sigma \in \Sigma_m$ and $i < m$, $\rho\big(\sigma(y_0, \dots, y_{i-1}, t(*, \bar{x}), y_{i+1}, \dots, y_{n-1})\big) = \rho(\sigma, i)$. If $\rho\big(t(*, \bar{x})\big) = -$, then $\rho\big(\sigma(y_0, \dots, y_{i-1}, t(*, \bar{x}), y_{i+1}, \dots, y_{n-1})\big) = -\rho(\sigma, i)$. The set of all positive and negative polynomial forms are denoted by Pl_ρ^+ and Pl_ρ^-, respectively.

Let $\boldsymbol{A}$ be a Σ-algebra. A unary function $p\colon A \to A$ on the universe of $\boldsymbol{\mathcal{A}}$ is called a *polynomial function* over $\boldsymbol{A}$ if there is a polynomial form $t(*, \bar{x})$ and a fixed but arbitrary sequence $\bar{c}$ of elements of $\boldsymbol{A}$, called the *parameters* of p, such that, for every $a \in A$, $p(a) = t^{\boldsymbol{A}}(a, \bar{c})$. The set of all polynomial functions over $\boldsymbol{A}$ is denoted by $\mathrm{Pl}(\boldsymbol{A})$. Suppose now that $\boldsymbol{A}$ is the underlying algebra of a ρ-poalgebra. It is clear that, if t is of positive polarity, then p is monotone, and, if t is negative, p is antimonotone. The sets of all monotone and antimonotone polynomial functions over $\boldsymbol{\mathcal{A}}$ are denoted respectively by $\mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$ and $\mathrm{Pl}^-_\rho(\boldsymbol{\mathcal{A}})$. Let $f, f' \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$ and $g, g' \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$. Then $f \circ f', g \circ g' \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$ and $f \circ g, g \circ f \in \mathrm{Pl}^-_\rho(\boldsymbol{\mathcal{A}})$.

The proof of the following proposition is immediate.

Lemma 5.1. *A qordering α of the universe A of the a Σ-algebra $\boldsymbol{A}$ is a ρ-qordering of $\boldsymbol{A}$ iff, for all $a, b \in A$, $a \,\alpha\, b$ implies $f(a) \,\alpha\, f(a)$ for every $f \in \mathrm{Pl}^+_\rho(\boldsymbol{A})$ and $b \,\alpha^{-1}\, a$ implies $g(a) \,\alpha\, g(b)$ for every $g \in \mathrm{Pl}^-_\rho(\boldsymbol{A})$.*

For any ρ-poalgebra $\boldsymbol{\mathcal{A}}$ and set $R \subseteq A$ of pairs of elements of $\boldsymbol{\mathcal{A}}$, we denote by $\Phi_\rho(R)$ the smallest ρ-qorder of $\boldsymbol{\mathcal{A}}$ that includes R. Note that, if K is the class of all ρ-poalgebras, or more generally any ρ-povariety that contains $\boldsymbol{\mathcal{A}}$, then $\Phi^{\mathsf{K}}_\rho(R) = \Phi_\rho(R)$. We note also that $\Phi_\rho(\emptyset) = \leq^{\boldsymbol{\mathcal{A}}}$. We obtain a characterization of the ρ-qorder generated by R that is the natural analogue (and in fact a generalization) of the well-known lemma of Mal'cev characterizing congruence generation.

Proposition 5.2. *Let $\boldsymbol{\mathcal{A}}$ be a ρ-poalgebra and let $R \subseteq A^2$. Let $a, b \in A$. Then $a \,\Phi_\rho(R)\, b$ iff there exists a finite sequence*

$$a = c_0, c_1, \ldots, c_{n-1}, c_n = b \tag{19}$$

such that, for each $i < n$, one of the following conditions holds.

(i) *There exists an $\langle r, r' \rangle \in R$ and an $f \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$ such that $c_i = f(r)$ and $c_{i+1} = f(r')$, or*
(ii) *There exists an $\langle r, r' \rangle \in R^{-1}$ and an $f \in \mathrm{Pl}^-_\rho(\boldsymbol{\mathcal{A}})$ such that $c_i = f(r)$ and $c_{i+1} = f(r')$, or*
(iii) *$c_i \leq^{\boldsymbol{\mathcal{A}}} c_{i+1}$.*

Proof. Let $\hat{R}$ be the set of all pairs $\langle a, b \rangle$ such there is a sequence (19) satisfying the given conditions. Since $*$ (the identity polynomial form) is contained in Pl^+_ρ, $R \subseteq \hat{R}$. Also it is clear that $\leq^{\boldsymbol{\mathcal{A}}} \subseteq \hat{R}$. To see that $\hat{R}$ is a ρ-qordering of $\boldsymbol{A}$, suppose we have a sequence (19) such that, for each $i < n$, one of the conditions (i)–(iii) holds. Let $g \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$, and consider the sequence

$$g(a) = g(c_0), g(c_1), \ldots, g(c_{n-1}), g(c_n) = g(b). \tag{20}$$

Let $i < n$, and suppose (i) holds for the pair $\langle c_i, c_{i+1} \rangle$, i.e., there is a pair $\langle r, r' \rangle \in R$ and $f \in \mathrm{Pl}^+_\rho(\boldsymbol{A})$ such that $c_i = f(r)$ and $c_{i+1} = f(r')$. Then $g(c_i) = (g \circ f)(r)$ and $g(c_{i+1}) = (g \circ f)(r')$. But $g \circ f \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$. So (i) also holds for $\langle g(c_i), g(c_{i+1}) \rangle$. If (ii) holds for $\langle c_i, c_{i+1} \rangle$, then a similar argument using the fact that $f \in \mathrm{Pl}^-_\rho(\boldsymbol{\mathcal{A}})$ shows that (ii) also holds for the pair $\langle g(c_i), g(c_{i+1}) \rangle$. Finally, if $\langle c_i, c_{i+1} \rangle \in \leq^{\boldsymbol{\mathcal{A}}}$, $\langle g(c_i), g(c_{I+1}) \rangle \in \leq^{\boldsymbol{\mathcal{A}}}$ since $\leq^{\boldsymbol{\mathcal{A}}}$ is a ρ-qordering. So (iii) also holds for $\langle g(c_i), g(c_{i+1}) \rangle$. Thus $\langle g(a), g(b) \rangle \in \hat{R}$.

Suppose now that $g \in \mathrm{Pl}^-_\rho(\boldsymbol{A})$, and consider the sequence

$$g(b) = g(c_n), g(c_{n-1}), \ldots, g(c_1), g(c_0) = g(a). \tag{21}$$

Consider any $i < n$. Arguing as above it is easy to see that if (i), (ii), or (iii) holds for $\langle c_i, c_{i+1}\rangle$, then (ii), (i), or (iii) holds respectively for $\langle g(c_{i+1}), g(c_i)\rangle$. Thus $\langle g(b), g(a)\rangle \in \hat{R}$.

This shows that $\hat{R} \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ and $R \subseteq \hat{R}$. So $\Phi_\rho(R) \subseteq \hat{R}$. The opposite inclusion is obvious. □

Theorem 2.16 is a corollary of this result.

Proposition 5.3. *Let* K *be a* ρ*-povariety and let* $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ *and* $R \subseteq A^2$*. Then*

$$\Theta^{\mathbf{Alg}(\mathsf{K})}(R) = \big(\Phi_\rho(R) \vee \Phi_\rho(R^{-1})\big) \cap \big(\Phi_\rho(R) \vee \Phi_\rho(R^{-1})\big)^{-1}.$$

Proof. Let $\alpha = \Phi_\rho(R) \vee \Phi_\rho(R^{-1})$. Clearly $R \subseteq \alpha$ and also $R \subseteq \alpha^{-1}$ since $R^{-1} \subseteq \alpha$. So $R \subseteq \alpha \cap \alpha^{-1}$, and hence $\Theta^{\mathbf{Alg}(\mathsf{K})}(R) \subseteq \alpha \cap \alpha^{-1}$. By Proposition 4.1 there is a $\beta \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$ such that $\beta \cap \beta^{-1} = \Theta^{\mathbf{Alg}(\mathsf{K})}(R)$ and obviously $R \cup R^{-1} \subseteq \beta$. So $\alpha \subseteq \beta$ and hence $\alpha \cap \alpha^{-1} \subseteq \beta \cap \beta^{-1} = \Theta^{\mathbf{Alg}(\mathsf{K})}(R)$. □

We now investigate the permutability of ρ-qorderings. In many respects the results obtained strongly reflect the theory of permutable congruences, but with some important differences.

Definition 5.4. Let $\boldsymbol{A}$ be a Σ-poalgebra and α, β ρ-qorderings of $\boldsymbol{A}$. α and β are said to *permute* if $\alpha \,;\, \beta = \beta \,;\, \alpha$.

A ρ-quasi-povariety K is said to have *premutable* ρ*-qorders* if, for each $\boldsymbol{\mathcal{A}} \in \mathsf{K}$, any pair of K-ρ-qorders of $\boldsymbol{\mathcal{A}}$ permute.

Proposition 5.5. *Let* $\boldsymbol{\mathcal{A}}$ *be a* ρ*-poalgebra and let* $\alpha, \beta \in \mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})$*. The following are equivalent.*

(i) $\beta \,;\, \alpha \subseteq \alpha \,;\, \beta$.
(ii) $\alpha \vee^{\mathbf{Qord}_\rho(\boldsymbol{\mathcal{A}})} \beta = \alpha \,;\, \beta$.

Proof. (i) $\Longrightarrow$ (ii). Assume (i) holds. Clearly $\alpha \,;\, \beta \subseteq \alpha \vee^{\mathbf{Qord}_\rho(\boldsymbol{\mathcal{A}})} \beta$. Since $\alpha \cup \beta \subseteq \alpha \,;\, \beta$, to obtain the inclusion in the opposite direction it suffices to show that $\alpha \,;\, \beta$ is a ρ-pordering of $\boldsymbol{A}$. It is reflective because α, β are. It is transitive because $(\alpha \,;\, \beta) \,;\, (\alpha \,;\, \beta) = \alpha \,;\, \beta \,;\, \alpha \,;\, \beta \subseteq \alpha \,;\, \alpha \,;\, \beta \,;\, \beta = \alpha \,;\, \beta$. Finally, suppose $a \;\alpha \,;\, \beta\; b$, and let $c \in A$ such that $a \,\alpha\, c \,\beta\, b$. If $f \in \mathrm{Pl}^+_\rho(\boldsymbol{\mathcal{A}})$, then $f(a) \,\alpha\, f(c) \,\beta\, f(b)$ and hence $f(a) \;\alpha \,;\, \beta\; f(b)$. Now suppose $a \,(\alpha \,;\, \beta)^{-1} b$ and $f \in \mathrm{Pl}^-_\rho(\boldsymbol{A})$. Then $a \,\beta^{-1}; \alpha^{-1}\, b$. Let $c \in A$ such that $a \,\beta^{-1}\, c \,\alpha^{-1}\, b$. Then $f(a) \,\beta\, f(c) \,\alpha\, f(b)$, and hence $f(a) \;\beta \,;\, \alpha\; f(b)$. So $f(a) \;\alpha \,;\, \beta\; f(b)$ by (i).

(ii) $\Longrightarrow$ (i). $\beta \,;\, \alpha \subseteq \alpha \vee^{\mathrm{Qord}_\rho(\boldsymbol{\mathcal{A}})} \beta = \alpha \,;\, \beta$. □

It seems unlikely that the inclusion $\beta \,;\, \alpha \subseteq \alpha \,;\, \beta$ implies permutability in general, although we know of no counterexample.

We now give an analogue of the well-known Mal'cev criterion of permutable congruences.

Theorem 5.6. *Let* K *be an* ρ*-quasi-povariety.* K *has permutable* ρ*-qorders iff there exists a term* $p(x, y, z)$ *in three variables such that the following quasi-inidentities hold in* K*.*

(i) $x \preccurlyeq y \to p(x, y, z) \preccurlyeq z$.
(ii) $y \preccurlyeq z \to x \preccurlyeq p(x, y, z)$

Proof. Suppose that (i) and (ii) hold. Let $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and let $\alpha, \beta \in \mathrm{Qord}^{\mathsf{K}}_{\rho}(\boldsymbol{\mathcal{A}})$. Assume $a\ \alpha\ ; \beta\ c$. Then there exists a b such that $a\, \alpha\, b\, \beta\, c$. Thus $p^{\boldsymbol{\mathcal{A}}}(a,b,c)\, \alpha\, c$ by (i) and $a\, \beta\, p^{\boldsymbol{\mathcal{A}}}(a,b,c)$ by (ii). Hence $a\ \beta\ ; \alpha\ c$. So $\alpha\ ;\ \beta \subseteq \beta\ ;\ \alpha$, and by symmetry $\beta\ ;\ \alpha \subseteq \alpha\ ;\ \beta$.

Assume now that K has permutable ρ-qorders. Let γ be the smallest ρ-qorder of $\mathrm{Qord}^{\mathsf{K}}_{\rho}\big(\boldsymbol{\mathcal{T}e}(x,y,z)\big)$. Recall that $\boldsymbol{\mathcal{T}e}(x,y,z)/\gamma$ is freely generated over K by $[x]_{\gamma\cap\gamma^{-1}}$, $[y]_{\gamma\cap\gamma^{-1}}$, and $[z]_{\gamma\cap\gamma^{-1}}$, which we identify respectively with x, y, and z. In $\boldsymbol{\mathcal{T}e}(x,y,z)/\gamma$ we have $x\, \Phi^{\mathsf{K}}_{\rho}(x,y)\ ;\ \Phi^{\mathsf{K}}_{\rho}(y,z)\, z$. So by permutability there is a term $p(x,y,z)$ such that

$$x\ \Phi^{\mathsf{K}}_{\rho}(y,z)\ p(x,y,z)\ \Phi^{\mathsf{K}}_{\rho}(x,y)\ z.$$

Let $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and let $a,b,c \in A$ such that $a \leq b$. Let $h\colon \mathbf{Te}(x,y,z)/\gamma \to \boldsymbol{\mathcal{A}}$ such that $h(x) = a, h(y) = b, h(z) = c$. The order kernel $h^{-1}(\leq^{\boldsymbol{\mathcal{A}}})$ contains $\langle x, y\rangle$ and hence includes $\Phi_{\rho}(x,y)$. It follows that $p^{\boldsymbol{\mathcal{A}}}(a,b,c) \leq c$. So (i) is an quasi-inidentity of K. By a similar argument we get that (ii) is also a quasi-inidentity. □

Any term $p(x,y,z)$ satisfying the condition of this theorem is called an *order Mal'cev* term. Suppose $p(x,y,z)$ is a Mal'cev term for $\mathsf{Alg}(\mathsf{K})$, i.e., the four inidentities $p(x,x,z) \preccurlyeq\succcurlyeq z$ and $p(x,z,z) \preccurlyeq\succcurlyeq x$ all hold in K. Then p will be an order Mal'cev term for K provided it has either negative polarity at the y position or positive polarity at both the x and z positions. Actually, a somewhat stronger property holds, which we now formulate.

Corollary 5.7. *Let K be a ρ-quasi-povariety. Assume there is a term $p(x,y,z)$ in three variables such that the following inidentities hold.*

(i) $p(x,x,z) \preccurlyeq z$,
(ii) $x \preccurlyeq p(x,z,z)$.

Assume in addition that at least one of the following two conditions hold.

(iii) $p(x,*,z) \in \mathrm{Pl}^{-}_{\rho}$,
(iv) $p(*,y,z), p(x,y,*) \in \mathrm{Pl}^{+}_{\rho}$.

Then K has permutable ρ-qorders.

Proof. Assume (i) and (ii) hold. Let $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and let $a,b,c \in A$. If (iii) holds, then $a \leq^{\boldsymbol{\mathcal{A}}} b \implies p^{\boldsymbol{\mathcal{A}}}(a,b,c) \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(a,a,c) \leq^{\boldsymbol{\mathcal{A}}} c$ and $b \leq^{\boldsymbol{\mathcal{A}}} c \implies a \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(a,c,c) \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(a,b,c)$.

If (iv) holds, then $a \leq^{\boldsymbol{\mathcal{A}}} b \implies p^{\boldsymbol{\mathcal{A}}}(a,b,c) \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(b,b,z) \leq^{\boldsymbol{\mathcal{A}}} c$, and $b \leq^{\boldsymbol{\mathcal{A}}} c \implies a \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(a,b,b) \leq^{\boldsymbol{\mathcal{A}}} p^{\boldsymbol{\mathcal{A}}}(a,b,c)$. □

let $p(x,y,z) = x \cdot (y \to z)$. In the ρ-povariety of POLRMs we have the inidentities $p(x,x,z) = x \cdot (x \to z) \preccurlyeq z$ and $x \preccurlyeq x \cdot 1 \preccurlyeq x \cdot (z \to z) = p(x,z,z)$. Thus POLRM has permutable ρ-qorders.

References

[1] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* Amer. Math. Soc., Providence, January 1989.

[2] W. J. Blok and D. Pigozzi. Algebraic semantics for universal Horn logic without equality. In A. Romanowska and J. D. H. Smith, editors, *Universal Algebra and Quasigroup Theory*, pages 1–56. Heldermann, Berlin, 1992.

[3] W. J. Blok and J. G. Raftery. Varieties of commutative residuated integral pomonoids and their residuation subreducts. To appear.

[4] S. L. Bloom. Varieties of ordered algebras. *J. Comput. System Sci.*, 13(2):200–212, October 1976.

[5] J. Czelakowski. Equivalential logics I and II. *Studia Logica*, 40:227–236 and 355–372, 1981.
[6] J. Czelakowski. *Protoalgebraic Logics*. Kluwer, Amsterdam, 2000.
[7] P. Dellunde. Equality-free logic: the method of diagrams and preservation theorems. *Log. J. IGPL*, 7(6):717–732, 1999.
[8] P. Dellunde and R. Jansana. Some characterization theorems for infinitary universal Horn logic without equality. *J. Symbolic Logic*, 61:1242–1260, 1996.
[9] J. M. Dunn. Gaggle theory: an abstraction of Galois connections and residuations with applications to various logical operations. In *Logics in AI, Proceedings European Workshop JELIA 1990*, number 478 in Lecture Notes in Computer Science. Springer-Verlag, January 1991.
[10] J. M. Dunn. Partial-gaggles applied to logics with restricted structural rules. In P. Schröder-Heiser and K. Došen, editors, *Substrictural Logics*. Oxford University Press, 1993.
[11] J. M. Dunn. When is a logic "partially algebraizable". Manuscript, June 1999.
[12] R. Elgueta. Characterizing classes defined without equality. *Studia Logica*, 58(3):357–394, 1997.
[13] R. Elgueta. Subdirect representation theory for classes without equality. *Algebra Universalis*, 40:201–246, 1998.
[14] L. Fuchs. *Partially ordered algebraic structures*. Pergamon Press, New York, 1963.
[15] V. A. Gorbunov. *Algebraic theory of quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998.
[16] V. A. Gorbunov and V. I. Tumanov. On the structure of lattices of quasivarieties. *Soviet Math. Dokl.*, 22(2):333–336, 1980.
[17] V. A. Gorbunov and V. I. Tumanov. Construction of lattices of quasivarieties. In *Math. Logic and Theory of Algorithms*, volume 2 of *Trudy Inst. Math. Sibirsk. Otdel. Akad. SSSR*, pages 12–44. Nauka, Novosibirsk, 1982.
[18] G. Grätzer and H. Lakser. A note on the implicational class generated by a class of structures. *Canad. Math. Bull.*, 16(4):603–605, 1973.
[19] T. Kowalski. EDPC and discriminator in varieties of residuated lattices. 1999. draft.
[20] T. Kowalski and H. Ono. Residuated lattices: an algebraic glimpse at logics without contraction. preliminary report.
[21] T. Kowalski and H. Ono. Splittings in the variety of residuated lattices. June 1999. extended abstract.
[22] A. I. Mal'cev. Several remarks on quasivarieties of algebraic systems. *Algebra and Logic*, 5(3):3–9, 1966.
[23] A .I. Mal'cev. *Algebraic systems*. Springer Verlag, New York-Heidelberg, 1973.
[24] T. Prucnal and A. Wroński. An algebraic characterization of the notion of structural completeness. *Bull. Section of Logic*, 3:30–33, 1974.
[25] J. G. Raftery and C. J. Van Alten. Residuation in communtative ordered monoids with minimal zero. *Rep. Math. Logic*.
[26] C. J. Van Alten and J. G. Raftery. On the lattice of varieties of residuation algebras. *Algebra Universalis*, 41:283–315, 1999.