Universal algebra for algebraic logic

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based on notes by **Don Pigozzi** and **Kate Pałasińska** (2004)
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Logic at the intersection of Algebra, Categories and Topology

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Outline for course on (partially ordered) universal algebra

- Introduction
- First-order logic, **posets and preorders**
- (Quasi)equational theories and **(quasi)inequational theories**
- Algebras and **partially ordered algebras**
- (Quasi)varieties and **po-(quasi)varieties**
- Products, subalgebras and **po-subalgebras**
- Homomorphisms, congruences and **precongruences**
- HSP and SPP_U for classes of algebras and **po-algebras**
- Congruence lattices and **precongruence lattices**
- Subdirectly irreducible algebras and **po-algebras**
- Free algebras and **free po-algebras**
- Jónsson's Lemma for varieties and **po-varieties**

Introduction

Universal algebra is the study of (first-order) algebraic systems and their (quasi)equational theories

Algebraic logic is the study of logic by algebraic methods

It dates back to **George Boole** (1815–1864), who studied classical propositional logic in his influential book "The laws of thought", 1854, in the form of what is now called Boolean algebra.

'The "Boolean Algebra" of classes, largely originating in this classic book, has had an ever-increasing influence on all branches of mathematics.' — Garrett Birkhoff, 1955

Birkhoff (1911–1996) is credited with founding universal algebra in 1935 with his paper *On the structure of abstract algebras*

First-order logic

It all starts with unisorted **first-order logic** with equality, its **syntax** and **Tarskian semantics**.

We begin with a brief review, since first order logic is in the background of universal algebra.

During the tutorial hour we will (attempt to) use an experimental **LaTeX** interface to **Prover9**, a first-order theorem prover and model finder.

So later you will get some hands-on experience with typing (some) computer-readable first-order formulas in LaTeX.

Tarskian semantics are based on **set theory**, which we also review briefly, and to make this interesting, the experimental LaTeX interface also connects to the finite sets available in Python.

Interlude: LaTeX, Python, Jupyter, Colab

LaTeX is the mathematical typesetting language that we use for writing research papers, and we will briefly introduce the parts we need.

Python is a general purpose programming language with a convenient syntax, but we do not need to know how to program.

A **Jupyter notebook** is a web interface for using Python and for writing mathematical text.

Colab is a free cloud-based Jupyter notebook hosted at `colab.research.google.com` for collaborative research.

We will (attempt to) use Colab this week during the tutorials.

Back to first-order logic

The language of first order logic is determined by a set $\mathcal{F}$ of **function symbols** and a disjoint set $\mathcal{R}$ of **relation symbols**.

Each symbol s has an associated finite **arity** $n_s \in \omega$, where $\omega = \{0, 1, 2, \ldots\}$ is the set of finite **ordinals** (natural numbers).

For partially ordered algebras (po-algebras) we need at least $=, \leq \in \mathcal{R}$, with arity 2 (binary).

There are also first-order **variables** $var = \{x, y, z, u, v, w, x_0, x_1, \ldots\}$

terms $T_0 = var,\ T_i = \{f(t_1, \ldots, t_{n_f}) \mid t_j \in T_k, k < i, f \in \mathcal{F}\},\ T = \bigcup_{i<\omega} T_i$

atomic formulas $Fm_0 = \{r(t_1, \ldots, t_{n_r}) \mid t_j \in T, r \in \mathcal{R}\}$ and

formulas $Fm_i = \{\text{not}\,\varphi, \varphi\,\text{and}\,\psi, \varphi\,\text{or}\,\psi, \varphi \Rightarrow \psi, \varphi \Leftrightarrow \psi, \forall s\varphi, \exists s\varphi \mid s \in var, \varphi, \psi \in Fm_k, k < i\}, \qquad Fm = \bigcup_{i<\omega} Fm_i$

Operation symbols, relation symbols and metavariables

Use parentheses (,) to clarify the scope of connectives and quantifiers.

$\mathcal{R} \supseteq \{=, \leq, \geq, <, >, \subseteq, \supseteq, \subset, \supset, \sqsubseteq, \sqsupseteq, \sqsubset, \sqsupset, \equiv, \cong, |, R, S\}$ and $\mathcal{F} \supseteq$

Unary operation symbols							Binary infix operation symbols							
$^{-1}$	$'$	$\neg$	$\sim$	$-$	$\Box$	$\Diamond$	$\cdot$	$\backslash$	$/$	$-$	$+$	$\wedge$	$\vee$	$\to$
$-$	$-$	$-$	$-$	$-$	$+$	$+$	$++$	$-+$	$+-$	$+-$	$++$	$++$	$++$	$-+$
0	0	1	1	1	1	1	2	2	2	3	3	4	5	6

An operation is order-preserving $_+$ or order-reversing $_-$ in each argument.

The last row indicates the binding power (lower is stronger).

0-ary symbols are constants: $0, 1, \bot, \top \in \mathcal{F}$ and **true**, **false** $\in \mathcal{R}$

X, Y, Z, U, V, W are **set variables**, A, B, C are **sets**, a, b, c are **elements**.

p, q, r, s, t are **term variables**, φ, ψ, σ are **formula variables**.

f, g, h are **functions**, π_i are projections, $\alpha, \beta, \gamma, \delta, \theta$ are (pre)congruences.

i, j, k, l are (integer) **index variables**, m, n are **integers**.

Free variables and (quasi)(in)equational theories

Let $vr\varphi$ be the set of variables that appear in φ.

The set $fr\varphi$ of **free variables** is equal to $vr\varphi$ for **atomic** formulas φ, and

$$\begin{array}{ll} fr(\forall s\phi) = fr\phi \setminus \{s\} & fr(\text{not}\,\varphi) = fr\varphi \\ fr(\exists s\phi) = fr\phi \setminus \{s\} & fr(\varphi \star \psi) = fr\varphi \cup fr\psi \end{array}$$

for $\star \in \{\text{and}, \text{or}, \Rightarrow, \Leftrightarrow\}$, $\varphi, \psi \in Fm$.

Free variables in formulas are assumed to be **universally quantified**.

A **quasi-inequation** is a formula

$$t_1 \star_1 t_1' \text{ and} \cdots \text{and } t_n \star_n t_n' \Rightarrow t_0 \star_0 t_0'$$

where $\star_i \in \{=, \leq, \geq, \sqsubseteq, \sqsupseteq\}$. It is a **quasiequation** if $\star_i \in \{=\}$ for all i.

It is an **inequation** if $n = 0$ and an **equation** if $\star_0 \in \{=\}$.

A **(quasi)(in)equational theory** is a set of (quasi)(in)equations.

Preorders are defined by quasi-inequational theories

A **preorder** $\mathbf{P} = (P, \sqsubseteq^{\mathbf{P}})$ is a set P and a binary relation $\sqsubseteq^{\mathbf{P}} \subseteq P \times P$ such that $\sqsubseteq^{\mathbf{P}}$ is **reflexive** and **transitive**.

So preorders are defined by **Pre** $= \{x \sqsubseteq x, x \sqsubseteq y \text{ and } y \sqsubseteq z \Rightarrow x \sqsubseteq z\}$

$\mathbf{P}$ is a preorder if and only if $\mathbf{P} \models$ **Pre**.

Exercise 1: Find all preorders on $P_2 = \{1, 2\}$ and $P_3 = \{1, 2, 3\}$.
Draw directed graphs of them with dots, arrows and loops.

A **homomorphism** $h : \mathbf{P} \to \mathbf{Q}$ is a preorder-preserving function $h : P \to Q$ i.e. $x \sqsubseteq^{\mathbf{P}} y \Rightarrow h(x) \sqsubseteq^{\mathbf{Q}} h(y)$

Pre defines a **concrete category** **P**re with all preorders as **objects** and homomorphisms as **morphism**.

Every quasi-inequational theory defines a category in this way.

Exercise 2: Find all subcategories of **P**re that are defined by a quasi-inequational theory using only $\sqsubseteq, =$ as symbols.

Posets are also defined by a quasi-inequational theory

A **partially ordered set** (or poset) $\mathbf{P} = (P, \leq^{\mathbf{P}})$ is an **antisymmetric** preorder (where the symbol $\sqsubseteq$ is replaced by $\leq$).

Let $\mathbf{Pos} = \mathbf{Pre}_{\leq} \cup \{x \leq y \text{ and } y \leq x \Rightarrow x = y\}$.

Pos also defines a concrete category **P**os with the same homomorphisms.

Two preorders or posets are **isomorphic** if there exists mutually inverse homomorphisms between them.

Exercise 3: Find all posets, up to isomorphism, on P_3 and on $P_4 = \{1, 2, 3, 4\}$. Draw Hasse diagrams (loop- and triangle-free graphs with no horizontal lines and $x \leq y$ iff there is a path from x to y with each intermediate element above the previous one).

Exercise 4: (Re)discover the important connection between preorders, posets, **equivalence relations**, **partitions** and **quotients of sets by equivalence relations**. State your observations as a **Theorem**.

More Exercises

Exercise 5: Show that bijective homomorphism of posets need not be an isomorphism. Find a counterexample of minimal cardinality.

Exercise 6: Draw a Hasse diagram of the set of all preorders on a 3-element set, ordered by inclusion. Hint: there are 6 atoms and 6 coatoms and, yes, the poset is self dual.

Exercise 7: Let C_n be a chain (totally ordered poset) with n elements. Discover the simple structure of the lattice of preorders on C_n that contain this partial order.

Exercise 8: If you know enough lattice theory, prove that the set of preorders on a set forms a complete lattice under inclusion. Show that the lattice of equivalence relations on the same set is a (complete?) sublattice.

Universal algebra

Unisorted universal algebra has been developed over the last century

A course in Universal Algebra by **Burris** and **Sankappannavar** is an excellent textbook (free pdf online)

Central concepts: signature, algebras, homomorphisms, congruences, subalgebras, products, HSP, varieties, quasivarieties, free algebras, ...

An **algebra** $\mathbf{A} = (A, \mathcal{F}^{\mathbf{A}})$ is a **set** A and for each symbol $f \in \mathcal{F}$ there is an operation $f^{\mathbf{A}} : A^{n_f} \to A$ in $\mathcal{F}^{\mathbf{A}}$ (and no other operations occur in $\mathcal{F}^{\mathbf{A}}$).

Note that $\mathcal{R} = \{=\}$, and $=^{\mathbf{A}}$ is the identity relation on A.

A **partially ordered algebra** $\mathbf{A} = (A, \leq^{\mathbf{A}}, \mathcal{F}^{\mathbf{A}})$ is based on a poset $(A, \leq^{\mathbf{A}})$, also denoted by A. The **dual poset** A^∂ is $(A, \geq^{\mathbf{A}})$.

Partially ordered algebras

Each **operation symbol** $f \in \mathcal{F}$ corresponds to an operation $f^{\mathbf{A}}$ of $\mathbf{A}$ that is order-preserving **or order-reversing** in each argument.

This information is part of the **signature** $\sigma : \mathcal{F} \to \bigcup_{n<\omega}\{+,-\}^n$.

We write f^{-+} to indicate that f is binary and $f^{\mathbf{A}}{:}A^\partial \times A \to A$ is order-preserving.

The **order-type** $\tau_{f^{\mathbf{A}}}$ of an n-ary operation $f^{\mathbf{A}}$ is an n-tuple with entries from $\{+,-,\pm,\emptyset\}$. Here $\pm$ is for an argument that is both order-preserving and order-reversing, while $\emptyset$ is for one where both properties may fail.

Exercise 9: Show that if a function has type $\pm$ for some argument then it maps all elements in a connected component of the poset to the same element (in that argument, when all other inputs are fixed).

Partially ordered algebras

In standard universal algebra the base category is the category of **sets**.

For **partially ordered algebras (po-algebras)** the base category is **Pos** = the category of posets with order-preserving maps as morphisms.

However, term operations on **A** are not necessarily morphisms in **Pos**.

If f^{-+} then $f^{\mathbf{A}}(x,x)$ may **not** be order-preserving or order-reversing.

Varieties of algebras with order-preserving operations have been studied by [Bloom 1976], [Bloom and Wright 1983], [Kurz and Velebil 2017], ...

However for algebraic logic, **negation** and **residuation** are important operations, and they are **not** order-preserving.

The study of (nonorder-preserving) po-algebras is due to [Pigozzi 2004].

Motivation for studying po-algebras

Algebraic logic "is" the study of po-algebras.

$\varphi \leq \psi$ (in algebra) means φ has ψ as a consequence (in logic).

Every set is a poset ordered as an antichain, hence the study of po-algebras **includes** the study of algebras.

[Pigozzi 2004]: many standard notions naturally generalize to po-algebras.

http://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf does not work anymore, so find it on the Internet Archive.

Don Pigozzi also provides much more motivation and connections with Algebraic Logic. **Reading his notes is a required part of this course.**

Exercise 10: Get the notes and study them.

Classes of algebras defined by equational theories

Signature	Varieties of algebras
	sets $\supset$ one-element sets
$\cdot$	magmas $\supset$ semigroups $\supset$ bands $\supset$ semilattices
$\cdot, 1$	unital magmas $\supset$ monoids $\supset$ commutative monoids
$\cdot, 1, ^{-1}$	inverse property loops $\supset$ groups $\supset$ Abelian groups
$\cdot, +$	semirings $\supset$ commutative semirings
$\cdot, +, 0, -$	rings $\supset$ commutative rings
$\cdot, 1, +, 0, -$	unital rings $\supset$ commutative unital rings
$\cdot, 1, ^{-1}, +, 0, -$	skew meadows $\supset$ meadows
$f_{a\,(a \in M)}$	multi-unary algebras $\supset$ M-sets $\supset$ G-sets
$+, f_{s\,(s \in S)}$	S-semimodules
$+, 0, -, f_{r\,(r \in R)}$	R-modules $\supset$ R-vector spaces

Short names for the corresponding categories: **Set, O, Mag, Sgrp, Bnd, Slat, UMag, Mon, CMon, IPLoop, Grp, AbGrp, Srng, CSrng, Ring, CRing, URing, CURing, SkMead, Mead, MUAlg, MSet, GSet, Smod, Modu, Vec**

Equational bases for some varieties of algebras

A ssociative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, **A**$_+$: $(x + y) + z = x + (y + z)$
Id empotent: $x \cdot x = x$
C ommutative: $x \cdot y = y \cdot x$, **C**$_+$: $x + y = y + x$
U nital: $x \cdot 1 = x = 1 \cdot x$
I nverse **P** roperty: $x^{-1} \cdot (x \cdot y) = y = (y \cdot x) \cdot x^{-1}$
Inv erse: $x \cdot x^{-1} = 1$, **Minus**: $x + -x = 0$
Me adow: $(x^{-1})^{-1} = x$, $x \cdot (x \cdot x^{-1}) = x$
D istributive: $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y \cdot z$
Ac tion: $f_a(f_b(x)) = f_{ab}(x), f_1(x) = x$
Ad ditive: $f_s(x + y) = f_s(x) + f_s(y), f_r(x) + f_s(x) = f_{r+s}(x)$

Set $= \{\}$	**O** $= \{x = y\}$	**Mag** $= \{x \cdot x = x \cdot x\}$
Sgrp = **AMag**	**Bnd** = **IdSgrp**	**Slat** = **CBnd**
Mon = **USgrp**	**IPLoop** = **IPUMag**	**Grp** = **InvMon**
AbGrp = **CGrp**	**Srng** = **DA**$_+$**C**$_+$**Sgrp**	**Ring** = **MinusSrng**
SkMead = **MeURing**	**Mead** = **CSkMead**	**MuAlg** $= \{f_a(x) = f_a(x)\}$
MSet = **AcMuAlg**	$_S$**Modu** = **AdMSet**	

Classes of ordered algebras defined by equational theories

Signature	Varieties of lattice-ordered algebras
$\wedge, \vee$	lattices $\supset$ modular lattices $\supset$ distributive lattices
$\wedge, \vee, '$	i-lattices $\supset$ ortholattices
$\wedge, 1, \vee, 0, \to$	Heyting algebras $\supset$ Boolean algebras
$\wedge, \vee, \cdot, 1, \backslash$	left-residuated lattices $\supset$ ℓ-groups
$\wedge, \vee, \cdot, 1, \backslash, /$	residuated lattices $\supset$ integral residuated lattices
$\wedge, \vee, \cdot, 1, \backslash, /, 0$	Full Lambek algebras $\supset$ involutive FL-algebras
$\wedge, \vee, \cdot, 1, \sim, -$	involutive residuated lattices
$\wedge, \Box_{i(i\in I)}$	meet-semilattices with dual operators
$\vee, \Diamond_{i(i\in I)}$	join-semilattices with operators
$\wedge, 1, \vee, 0, \Diamond_{i(i\in I)}$	bounded (dist.) lattices with normal operators
$\wedge, 1, \vee, 0, \to, \Diamond_{i(i\in I)}$	Heyting algs with ops $\supset$ Boolean algs with ops

Short names for the corresponding categories: **Lat, MoLat, DLat, iLat, OL, OmL, HA, BA, LrL, LGrp, RL, IRL, FL, InFL, InRL, MsO, JsO, bLO, bDLO, HAO, BAO**

Equational bases for some varieties of ordered algebras

$\mathbf{A}_\wedge$ssociative: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $\mathbf{A}_\vee$: $(x \vee y) \vee z = x \vee (y \vee z)$
$\mathbf{C}_\wedge$ommutative: $x \wedge y = y \wedge x$, $\mathbf{C}_\vee$: $x \vee y = y \vee x$
Absorbtive: $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$, $\mathbf{L} = \mathbf{A}_\wedge \mathbf{A}_\vee \mathbf{C}_\wedge \mathbf{C}_\vee \mathbf{Abs}$
Modular: $x \wedge (y \vee (x \wedge z) = x{\wedge}y \vee x{\wedge}z$
Distributive: $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$
DeMorgan: $(x \wedge y)' = x' \vee y'$, $x'' = x$
Complemented: $x \wedge x' = 0$, $x \vee x' = 1$ **b**ounded: $x \wedge 0 = 0$, $x \vee 1 = 1$
Heyting: $x \wedge y \le z \iff y \le x \to z$ (equivalent to identities)
Boolean: $(x \to 0) \to 0 = x$
Left **r**esiduated: $x \cdot y \le z \iff y \le x \backslash z$ **I**ntegral: $x \vee 1 = 1$
Residuated: $x \le z/y \iff x \cdot y \le z \iff y \le x \backslash z$
Involutive: $0/(x \backslash 0) = (0/x) \backslash 0$ or $x{\le}{-}(y{\cdot}{\sim}z) \Leftrightarrow x{\cdot}y{\le}z \Leftrightarrow y{\le}{\sim}({-}z{\cdot}x)$

Lat = **L**	**iLat** = **DmLat**	**OL** = **Co iLat**
HA = **bHLat**	**BA** = **BHA**	**LrL** = **LrLMon**
LGrp =	**RL** = **RLMon**	**FL** = **RL** with 0

Equational bases for some varieties of ordered algebras

Operator: $\Diamond(\dots, x \vee y, \dots) = \Diamond(\dots, x, \dots) \vee \Diamond(\dots, y, \dots)$

Normal: $\Diamond(\dots, 0, \dots) = 0$, $\Diamond(\dots, 1, \dots) = 1$

dual **O**perator: $\Box(\dots, x \wedge y, \dots) = \Box(\dots, x, \dots) \wedge \Box(\dots, y, \dots)$

MsO $=$ **dOMslat**	**JsO** $=$ **OpJslat**	**bLO** $=$ **NorOpLat**
bDLO $=$ **NorOpDLat**	**HAO** $=$ **NorOpHA**	**BAO** $=$ **NorOpBA**

Other properties:

Idempotent: $x \cdot x = x$

Square **i**ncreasing: $x \wedge x{\cdot}x = x$

Square **d**ecreasing: $x \vee x{\cdot}x = x$

Cyclic: $\sim x = -x$

Classes of po-algebras defined by inequational theories

Signature	Varieties of po-algebras
$\leq$	posets
$\leq, \sim, -$	Galois posets
$\leq, \to$	implication po-algebras $\supset$ BCK algebras
$\leq, \cdot, \backslash, /$	residuated po-semigroups
$\leq, \cdot, 1, \backslash$	left-residuated po-monoids
$\leq, \cdot, 1, \backslash, /$	residuated po-monoids
$\leq \cdot, 1, \sim, -$	ipo-monoids $\supset$ pregroups $\supset$ MV-algebras
$\leq, 1, \Box_{i(i \in I)}$	1-posets with dual operators
$\leq, 0, \Diamond_{i(i \in I)}$	0-posets with operators
$\leq, q_{i(i \in I)}$	posets with quasioperators
$\leq, 0, 1, q_{i(i \in I)}$	bounded posets with quasioperators

Short names for the corresponding categories: **Pos, GaPos, ImPos, BCK, RPoSgrp, RPoMon, ipoMon, PrGrp, MV, PosdO, PosO, PosQo, bPosQo**

Axiomatic bases for some varieties of po-algebras

Pos $= \{x \le x,\ \ x \le y \text{ and } y \le x \Rightarrow x = y,\ \ x \le y \text{ and } y \le z \Rightarrow x \le z\}$

Meet-**s**emi**lat**ices $=$ **Pos** $\cup \{x \le x \wedge x,\ \ x \wedge y \le x,\ \ x \wedge y \le y\}$

Join-**s**emi**lat**ices $=$ **Pos** $\cup \{x \vee x \le x,\ \ x \le x \vee y,\ \ x \le y \vee x\}$

Lattices $=$ **Mslat** $\cup$ **Jslat** Note: this is a **third** way to define lattices

Adjunction $= \{x \le \Box\Diamond x,\ \ \Diamond\Box x \le x\}$

Galois $= \{x \le {\sim}{-}x,\ \ x \le {-}{\sim}x\}$

dual **G**alois $= \{{\sim}{-}x \le x,\ \ {-}{\sim}x \le x\}$

Left **r**esiduated $= \{x \cdot y \le z \iff y \le x\backslash z\}$

Residuated $= \{x \le z/y \iff x \cdot y \le z \iff y \le x\backslash z\}$

Involutive $= \{x \le -(y \cdot {\sim}z) \iff x \cdot y \le z \iff y \le {\sim}(-z \cdot x)\}$

Operators: $\Diamond(\dots, 0, \dots) = 0,\ \ x \le y \implies \Diamond(\dots, x, \dots) \le \Diamond(\dots, y, \dots)$

dual **O**per: $\Box(\dots, 1, \dots) = 1,\ \ x \le y \implies \Box(\dots, x, \dots) \le \Box(\dots, y, \dots)$

A **quasioperator** is order-preserving or order-reversing in each argument.

Subalgebras

A po-algebra **B** is a **subalgebra** of a po-algebra **A** if $A \subseteq B$, $\leq^{\mathbf{B}} = \leq^{\mathbf{A}} \cap B^2$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ all i.

i.e., $(B, \leq^{\mathbf{B}})$ is a subposet of $(A, \leq^{\mathbf{A}})$ **with the induced partial order** and B is closed under all operations of **A**.

A **universal formula** is a formula that has no quantifiers, so all variables are free and assumed to be universally quantified. E.g., any quasi-inequation is universal.

Lemma: If a po-algebra **A** satisfies a universal formula, then any subalgebra **B** of **A** satisfies the same formula.

For a class $\mathcal{K}$ of po-algebras, $\mathsf{S}\mathcal{K}$ denotes the class of subalgebras of members of $\mathcal{K}$.

A **universal class** is a class of po-algebras defined by universal formulas.

Corollary: Every universal class $\mathcal{K}$ of po-algebra satisfies $\mathsf{S}\mathcal{K} = \mathcal{K}$.

Products

The **direct product** $\prod_{i\in I}\mathbf{A}_i$ of a family $\{\mathbf{A}_i \mid i \in I\}$ of po-algebras is defined as for ordinary algebras (pointwise)

The partial order on the product is the pointwise order:

$$a \leq b \iff a(i) \leq^{\mathbf{A}_i} b(i) \text{ for all } i \in I$$

Lemma: If a family $\{\mathbf{A}_i \mid i \in I\}$ of po-algebras satisfy a quasi-inequation then the direct product $\prod_{i\in I}\mathbf{A}_i$ satisfies the same formula.

Exercise 11: Prove this lemma, and extend it to universal Horn classes.

For a class $\mathcal{K}$ of po-algebras, $\mathrm{P}\mathcal{K}$ is the class of products of members of $\mathcal{K}$.

A **(quasi)(in)equational class** is a class of po-algebras defined by a set Σ of (quasi)(in)equations: $\mathcal{Q} = \mathrm{Mod}(\Sigma)$

Corollary: If $\mathcal{Q}$ is a quasi-inequational class then $P\mathcal{Q} = \mathcal{Q}$.

Homomorphisms and isomorphisms

A **homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is an **order-preserving** function $h : A \to B$ (i.e., $h[\leq^{\mathbf{A}}] \subseteq \leq^{\mathbf{B}}$) and for all $f \in \mathcal{F}$

$$h(f^{\mathbf{A}}(a_1, \dots, a_{n_f})) = f^{\mathbf{B}}(h(a_1), \dots, h(a_{n_f})).$$

As usual, h is **surjective** or **onto** if $h[A] = \{h(a) \mid a \in A\} = B$.

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a **homomorphic image** of $\mathbf{A}$.

A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is an **embedding** if it is one-to-one and **order-reflecting** i.e., $h^{-1}[\leq^{\mathbf{B}}] \subseteq \leq^{\mathbf{A}}$, or $h(x) \leq^{\mathbf{B}} h(y) \implies x \leq^{\mathbf{A}} y$.

A homomorphism h is an **isomorphism** if h is a surjective embedding.

In this case $\mathbf{A}$ is said to be **isomorphic** to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$.

Exercise 12: prove that h^{-1} is an isomorphism as well. Conversely, if h is a bijection such that h and h^{-1} are homomorphisms, then h is an isomorphism.

Precongruences and quotient algebras

Recall that a **preorder** is a reflexive and transitive binary relation

A **precongruence** on a po-algebra $\mathbf{A}$ is a preorder α on A with $\leq^{\mathbf{A}} \subseteq \alpha$

$$x\alpha y \implies f^{\mathbf{A}}(z_1,\ldots,x,\ldots,z_n)\alpha f^{\mathbf{A}}(z_1,\ldots,y,\ldots,z_n) \text{ if } \sigma_f(i)=+$$
$$x\alpha y \implies f^{\mathbf{A}}(z_1,\ldots,y,\ldots,z_n)\alpha f^{\mathbf{A}}(z_1,\ldots,x,\ldots,z_n) \text{ if } \sigma_f(i)=-$$

for all $i \in \{1,\ldots,n\}$ and all fundamental operations f of $\mathbf{A}$.

The set of all precongruences of $\mathbf{A}$ is denoted by $\mathrm{Pre}(\mathbf{A})$

Every precongruence α contains a largest **congruence** $\hat{\alpha} = \alpha \cap \alpha^{-1}$

However, $\hat{\alpha}$ may not contain $\leq^{\mathbf{A}}$, so in general $\hat{\alpha}$ is not in $\mathrm{Pre}(\mathbf{A})$.

The **quotient algebra** $\mathbf{A}/\alpha$ of a po-algebra $\mathbf{A}$ modulo a precongruence α is given by $(A/\hat{\alpha}, \leq^{\mathbf{A}/\alpha}, f_1^{\mathbf{A}/\alpha}, f_2^{\mathbf{A}/\alpha}, \ldots)$, where $[x]_{\hat{\alpha}} \leq^{\mathbf{A}/\alpha} [y]_{\hat{\alpha}} \iff x\alpha y$

The isomorphism theorems

The **kernel** of a homomorphism $h : \mathbf{A} \to \mathbf{B}$ between po-algebras is $\ker h = \{(x, y) \in A^2 \mid h(x) \leq^{\mathbf{B}} h(y)\}$.

Exercise 13: Prove that $\ker h$ is a precongruence on $\mathbf{A}$.

Exercise 14: Prove that h is an embedding if and only if $\ker h = \leq^{\mathbf{A}}$.

Theorem (Exercise 15: prove the First isomorphism theorem)

Suppose $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism onto $\mathbf{B}$, and let $\gamma : \mathbf{A} \to \mathbf{A}/\ker h$ be the canonical homomorphism $\gamma(a) = [a]_{\hat{\alpha}}$. Then there is an isomorphism $g : \mathbf{A}/\ker h \to \mathbf{B}$ such that $h = g \circ \gamma$.

For $\beta \subseteq \alpha \in \mathrm{Pre}(\mathbf{A})$ let $\alpha/\beta = \{([x]_{\hat{\beta}}, [y]_{\hat{\beta}}) \in (A/\beta)^2 \mid x \,\alpha\, y\}$.

Exercise 16: Prove $\alpha/\beta \in \mathrm{Pre}(\mathbf{A}/\beta)$.

Theorem (Exercise 17: prove the Second isomorphism theorem)

For $\beta \subseteq \alpha \in \mathrm{Pre}(\mathbf{A})$ the map $h : (A/\beta)/(\alpha/\beta) \to \mathbf{A}/\alpha$ defined by $h([[x]_{\hat{\beta}}]_{\widehat{\alpha/\beta}}) = [x]_{\hat{\alpha}}$ is an isomorphism.

The isomorphism theorems

Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, $\alpha \in \mathrm{Pre}(\mathbf{A})$ and $B_\alpha = \{a \in A \mid B \cap [a]_{\hat{\alpha}}\}$.

Exercise 18: Show that B_α is closed under the operations of $\mathbf{A}$.

Define $\mathbf{B}_\alpha$ as the subalgebra of $\mathbf{A}$ and $\alpha\restriction_B = \alpha \cap B^2$.

Theorem (Exercise 19: prove the Third isomorphism theorem)

If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\alpha \in \mathrm{Pre}(\mathbf{A})$ then $\mathbf{B}/\alpha\restriction_B \cong \mathbf{B}_\alpha/\alpha\restriction_{B_\alpha}$.

Varieties of po-algebras

Let $\mathcal{K}$ be a class of po-algebras of the same signature

$\mathsf{H}_P\mathcal{K} =$ the class of **po-homomorphic images** of members of $\mathcal{K}$

$\mathsf{S}\mathcal{K} =$ the class of **subalgebras** of members of $\mathcal{K}$

$\mathsf{P}\mathcal{K} =$ the class of **products** of members of $\mathcal{K}$

$\mathcal{K}$ is a **po-variety** if $\mathcal{K}$ is closed under H_P, S, P

[Pigozzi 2004] $\mathsf{H}_P\mathsf{SP}\mathcal{K} =$ the **po-variety generated** by $\mathcal{K}$

Recall that an **inequation** is any formula $s \le t$ where s, t are terms.

$\mathrm{Mod}(\mathcal{I})$ is the class of po-algebras that satisfy all inequations in $\mathcal{I}$
(we assume the signature is exactly the symbols that appear in $\mathcal{I}$).

Theorem (Pigozzi 2004)

$\mathcal{K}$ is a po-variety if and only if $\mathcal{K} = \mathrm{Mod}(\mathcal{I})$ for a set of inequations $\mathcal{I}$

Examples of po-varieties

1. The class **Pos** of posets is a po-variety (no fundamental operations)

   For any set X, $\mathsf{H}_P(\{X\})$ = all posets of cardinality $\leq |X|$

   Pos $= \mathsf{H}_P\mathsf{SP}(\mathbf{2})$, $\quad$ $\mathrm{Mod}(x \leq y)$ = only proper po-subvariety

2. Meet-semilattices: $\sigma_\wedge = {+}{+}, \quad x \wedge y \leq x, \quad x \wedge y \leq y, \quad x \leq x \wedge x$

   Hence we get **glb**: $z \leq x$ and $z \leq y \implies z \leq z \wedge z \leq x \wedge z \leq x \wedge y$

3. Lattices: add $\sigma_\vee = {+}{+}, \quad x \leq x \vee y, \quad y \leq x \vee y, \quad x \vee x \leq x$

   Hence we get **absorption**: $x \wedge (x \vee y) \leq x \leq x \wedge x \leq x \wedge (x \vee y)$

4. Left residuated magmas: $\sigma_\backslash = {-}{+}, \sigma_\cdot = {+}{+}, x \leq y\backslash(yx), x(x\backslash y) \leq y$

5. Residuated magmas: add $\sigma_/ = {+}{-}, \quad x \leq (xy)/y, \quad (x/y)y \leq x$

   $$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

6. Partially ordered groups: $\sigma_\cdot = {+}{+}, \sigma_{-1} = -$, group axioms

Recall: quasi-inequational classes of po-algebras

A **quasi-inequation** is given by $s_1 \le t_1$ and $\cdots$ and $s_n \le t_n \Rightarrow s_0 \le t_0$

$\mathcal{K}$ is a **quasi-inequational class** if $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for a set Σ of quasi-inequations.

$\mathcal{K}$ is a **po-quasivariety** if $\mathcal{K} = \mathrm{SP}_U\mathrm{P}\mathcal{K}$ where $P_U =$ ultraproducts.

Theorem (Pigozzi 2004)

$\mathcal{K}$ is a po-quasivariety if and only if $\mathcal{K}$ is a quasi-inequational class.

The class of sets is not a po-variety, but it is a po-quasivariety.

A poset is an **antichain** if it satisfies $x \le y \implies x = y$

Set $= \mathrm{Mod}(\{x \le y \implies x = y\}) = \mathrm{SP}_U\mathrm{P}(\{0, 1\}, =)$

Inequational logic

Birkhoff's rules for equational logic give an elegant and complete system for deriving all equational consequences from a given set of identities.

Similarly, let $\mathcal{I}$ be a set of inequalities and define $\mathcal{D}$ to be the smallest set containing $\mathcal{I}$ such that for all terms r, s, t

$t \le t \in \mathcal{D}$

$s \le t, t \le s \in \mathcal{D} \implies s = t \in \mathcal{D}$

$r \le s, s \le t \in \mathcal{D} \implies r \le t \in \mathcal{D}$

$\mathcal{D}$ is closed under uniform substitution

$r = s \in \mathcal{D} \implies t(r) = t(s) \in \mathcal{D}$ for any term $t(x)$

$r \le s \in \mathcal{D} \implies t(r) \le t(s) \in \mathcal{D}$ if $t(x)$ is order-preserving

$r \le s \in \mathcal{D} \implies t(s) \le t(r) \in \mathcal{D}$ if $t(x)$ is order-reversing

Then $\mathcal{D}$ contains **all** $s \le t$ that are **true in all models** of $\mathcal{I}$.

Exercise: Prove this (see Theorem 14.19 in Burris and Sankappanavar).

Term-equivalence and clones

The choice of fundamental operation symbols for an algebra is not unique.

E.g. for Boolean algebras one can use $\{\wedge, \neg, 0\}$ or $\{\to, 0\}$ or ...

Algebras with distinct sets of fundamental operation symbols can't be isomorphic so to equate such variants we use **term-equivalence**.

For a term $t = f(t_1, \ldots, t_{n_f})$ where $f \in \mathcal{F}$ define a **term-operation** $t^{\mathbf{A}}$ by

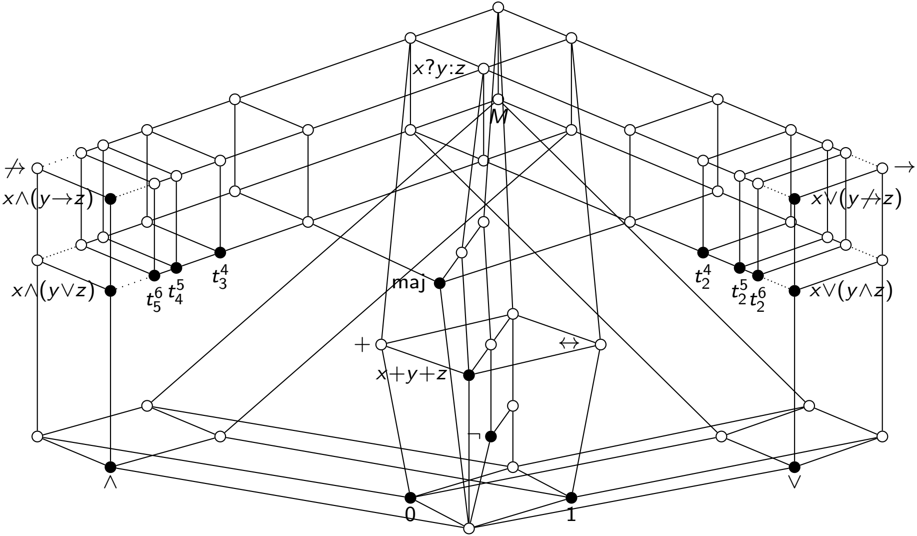
$$t^{\mathbf{A}}(x_1, \ldots, x_m) = f^{\mathbf{A}}(t_1^{\mathbf{A}}(x_1, \ldots, x_m), \ldots, t_{n_f}^{\mathbf{A}}(x_1, \ldots, x_m)).$$

The **clone of term-operations** of $\mathbf{A}$ is $\mathrm{Clo}\mathbf{A} = \{t^{\mathbf{A}} \mid t \text{ is a term}\}$.

Two (po-)algebras $\mathbf{A}, \mathbf{A}'$ are **term-equivalent** if $\mathrm{Clo}\mathbf{A} = \mathrm{Clo}\mathbf{A}'$.

Equivalently we can check that $A = A'$ and
for all $f \in \mathcal{F}$ there exist terms t' built from symbols in $\mathcal{F}'$ s.t. $f^{\mathbf{A}} = t'^{\mathbf{A}}$
for all $f' \in \mathcal{F}'$ there exist terms t built from symbols in $\mathcal{F}$ s.t. $f'^{\mathbf{A}} = t^{\mathbf{A}}$.

The Post lattice: a view of all 2-element algebras



$$t^n_k(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{maj}(x,y,z) = (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$

join-irreducibles in black

Partially ordered clones

For a poset $(P, \leq)$, a **po-clone** $(\mathcal{C}, \leq)$ is a clone $\mathcal{C}$ on P that is generated by operations that are order-preserving or order-reversing in each argument.

In the Post lattice the clones $\langle x + y + z\rangle$, $\langle +\rangle$, $\langle \leftrightarrow\rangle$ are **not** po-clones.

Exercise 20: Decide if $\langle x + y + z, \neg\rangle$, $\langle x + y + z, \mathrm{maj}\rangle$ or $\langle +, 1\rangle$ are po-clones.

The remaining clones on 2 elements are po-clones.

Joins of precongruences and the correspondence theorem

In the presentation of Exercise 8 it was shown that precongruences form a complete lattice. Here is an internal description of the complete join.

Theorem (Pigozzi 2004)

Let $\mathbf{A}$ *be a po-algebra and* $C \subseteq \mathrm{Pre}(\mathbf{A})$. *Then*

$$\bigvee C = \bigcup \{\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n \mid n < \omega, \alpha_1, \ldots, \alpha_n \in C\}.$$

Theorem (Pigozzi 2004, the Correspondence Theorem)

Let $\mathbf{A}$ *be a po-algebra and* $\alpha \in \mathrm{Pre}(\mathbf{A})$. *Then for every* $\beta \in \mathrm{Pre}(\mathbf{A}/\alpha)$ *there is a unique* $\bar{\beta} \in \mathrm{Pre}(\mathbf{A})$ *such that* $\beta = \bar{\beta}/\alpha$. *The mapping* $\beta \mapsto \bar{\beta}$ *is a lattice isomorphism from* $\mathrm{Pre}(\mathbf{A}/\alpha)$ *to the principal filter* $\uparrow\alpha$ *of* $\mathrm{Pre}(\mathbf{A})$.

Subdirect products of po-algebras

A po-algebra $\mathbf{B}$ is a **subdirect product** of a family $\{\mathbf{A}_i \mid i \in I\}$ of po-algebras if $\mathbf{B}$ is a subalgebra of $\prod_{i\in I} \mathbf{A}_i$ and $\pi_i : \mathbf{B} \to \mathbf{A}_i$ is an onto homomorphism for all $i \in I$.

Theorem

$\mathbf{B}$ is isomorphic to a subdirect product of $\{\mathbf{A}_i \mid i \in I\}$ iff there exists $\{\alpha_i \mid i \in I\} \subseteq \mathrm{Pre}(\mathbf{B})$ such that $\bigcap_{i\in I} \alpha_i = \leq^{\mathbf{B}}$ and $\mathbf{B}/\alpha_i \cong \mathbf{A}_i$ for all $i \in I$.

A po-algebra $\mathbf{B}$ is subdirectly irreducible if whenever $\mathbf{B}$ is a subdirect product of $\{\mathbf{A}_i \mid i \in I\}$ then $\mathbf{B} \cong \mathbf{A}_i$ for some $i \in I$.

Equivalently, if $\mathrm{Pre}(\mathbf{B})$ looks like a **lollipop**, i.e., has a nontrivial precongruence μ that is a lower bound for every nontrivial precongruence.

Theorem (Subdirect representation theorem for po-algebras)

Every po-algebra is isomorphic to a subdirect product of subdirectly irreducible po-algebras. (For universal algebra, due to Birkhoff [1942].)

Jónsson's Lemma for po-algebras

A po-variety $\mathcal{V}$ is precongruence distributive if every member of $\mathcal{V}$ has a distributive precongruence lattice.

$\mathcal{V}_{SI}$ denotes the class of all subdirectly irreducible algebras in $\mathcal{V}$.

Theorem (Gil-Ferez, J.)

If a po-variety $\mathcal{V}$ is precongrunce-distributive then Jónsson's Lemma holds: For any $\mathcal{K} \subseteq \mathcal{V}$, $H_P SP(\mathcal{K})_{SI} \subseteq H_P SP_U(\mathcal{K})$.

Corollary

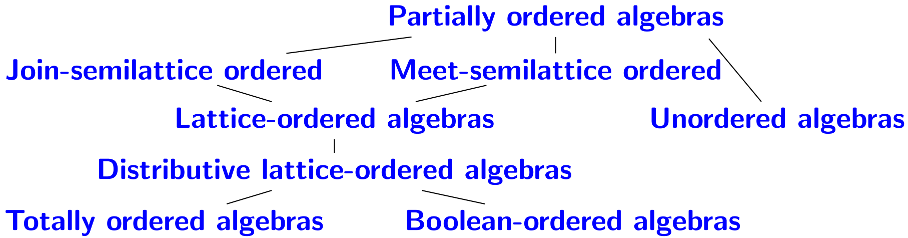
Let $\mathcal{V}$ be a precongruence distributive po-variety.

- *If two finite po-algebras in $\mathcal{V}$ are nonisomorphic then they generate distinct subvarieties.*
- *If a subvariety of $\mathcal{V}$ is generated by a finite algebra then it has only finitely many subvarieties.*

A survey of po-varieties, joint work with Bianca Newell

350 page PDF file covering $\sim$500 classes of partially ordered algebras

The survey contains an introduction and 8 chapters covering classes of a particular order type:



Each section contains definition(s) of the class of algebras, a list of properties, fine spectrum, a list of subclasses, superclasses and (some finite) algebras not in any of its subclasses.

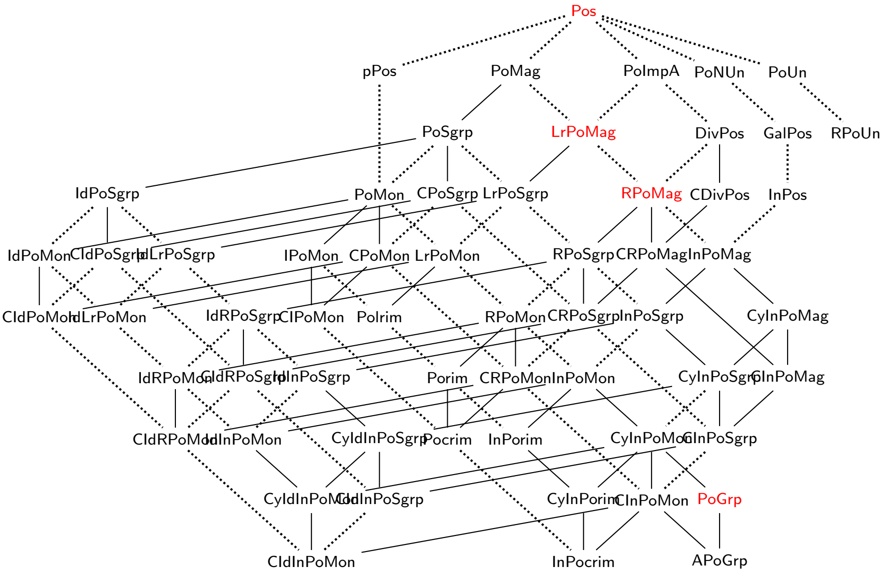


Figure: Classes of partially ordered algebras in Chapter 2

Naming Conventions

Some of the abbreviations used in the survey:

- C = commutative $\quad x \cdot y \leq y \cdot x$
- D = distributive $\quad x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$
- I = integral $\quad x \cdot y \leq x$ and $x \cdot y \leq y \quad (\Leftrightarrow x \leq 1$ if unital)
- Id = idempotent $\quad x \cdot x = x$
- J = join $\quad x \vee x \leq x,\ x \leq x \vee y,\ x \leq y \vee x$
- Lr = left-residuated $\quad xy \leq z \Leftrightarrow y \leq x\backslash z$
- M = meet $\quad x \leq x \wedge x,\ x \wedge y \leq x,\ x \wedge y \leq y$
- Po = partially ordered
- R = residuated $\quad xy \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow x \leq z/y$
- To = totally ordered $\quad x \leq y$ or $y \leq x$
- U = unital $\quad x \cdot 1 = x = 1 \cdot x$

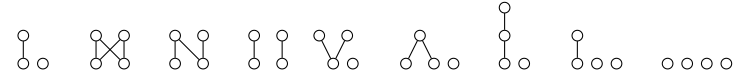
Abbreviations for the most common algebras:

- Grp = groups
- Lat = lattices
- Mon = monoids
- Pos = posets

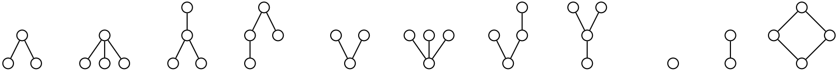
Properties recorded for the po-variety of lattices

Classtype	Variety
Equational theory	Decidable in PTIME
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes [FN1942]
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes [Jón1956]
Epimorphisms are surjective	Yes

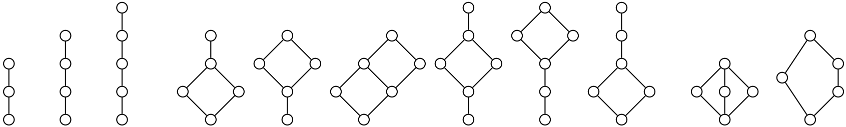
Small posets, semilattices, lattices, chains, Bool. algebras



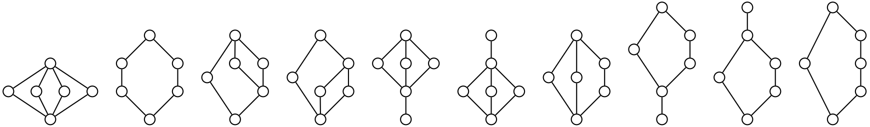
$P_{3,1}$ $P_{4,1}$ $P_{4,2}$ $P_{4,3}$ $P_{4,4}$ $P_{4,5}$ $P_{4,7}$ $P_{4,6}$ $P_{4,6}$



$J_{3,1}$ $J_{4,1}$ $J_{4,2}$ $J_{4,3}$ $M_{3,1}$ $M_{4,1}$ $M_{4,2}$ $M_{4,3}$ B_0 B_1 B_2



C_3 C_4 C_5 $D_{5,1}$ $D_{5,2}$ $D_{6,1}$ $D_{6,2}$ $D_{6,3}$ $D_{6,4}$ $L_{5,1}$ $L_{5,2}$



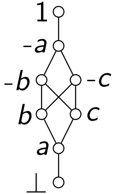
$L_{6,1}$ $L_{6,2}$ $L_{6,3}$ $L_{6,4}$ $L_{6,5}$ $L_{6,6}$ $L_{6,7}$ $L_{6,8}$ $L_{6,9}$ $L_{6,10}$

Small members in InPocrim

InPocrim
= involutive partially ordered commutative residuated integral monoids

Subclasses: CIInFL = commutative integral involutive FL-algebras

List of smallest InPocrims that are not in CIInFL



$\mathbf{Pos}_{8,1}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	a	a	a	a	a	$\bot$
$-c$	a	a	a	$\bot$	a	$\bot$
$-b$	a	a	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	a	a	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,1}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	b	b	a	a	a	$\bot$
$-c$	b	b	a	$\bot$	a	$\bot$
$-b$	a	a	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	a	a	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,2}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	$-c$	$-c$	b	a	b	$\bot$
$-c$	$-c$	$-\mathbf{c}$	b	$\bot$	b	$\bot$
$-b$	b	b	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	b	b	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,3}$

Fine spectra of classes of algebras

The **fine spectrum** of a class of models is the number of models (up to isomorphism) of each cardinality $n = 1, 2, 3, 4, \ldots$

It is an invariant for each class, preserved by term equivalence. E.g.,

Abelian groups: $f_n = 1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, \ldots$ = number of factorizations of n into prime powers.

MV-algebras: $f_n = 1, 1, 1, 2, 1, 2, 1, 3, 2, 1, 1, 4, \ldots$ = number of ways of factoring n into a product with nontrivial factors.

Monoids: $f_n = 1, 2, 7, 35, 228, 2237, 31559, 1668997, \ldots$

Lexicographic list of fine spectra (in Appendix)

Name	Fine spectrum	OEIS
PoMag	1, 16, 4051	No
PoImpA	1, 16, 3981	No
PoSgrp	1, 11, 173, 4753, 198838,...	No
Mag	1, 10, 3330, 178981952,...	A001329
Srng	1, 10, 132, 2341	No
CPoSgrp	1, 7, 83, 1468, 37248,...	No
MedMag	1, 7, 75, 3969	No
IdPoSgrp	1, 7, 69, 1035	No
MMag	1, 6, 280	
JImpA	1, 6, 245	
MImpA	1, 6, 220	
JMag	1, 6, 220	
ToMag	1, 6, 175	
ToImpA	1, 6, 175	
MultLat	1, 6, 175	
DLMag	1, 6, 175	
DLImpA	1, 6, 175	
LMag	1, 6, 175	
LImpA	1, 6, 175	
DivPos	1, 6, 123	
LrPoMag	1, 6, 110	
MSgrp	1, 6, 70, 1437	No
JSgrp	1, 6, 61, 866	No
CDivPos	1, 6, 55, 1434	No
DLSgrp	1, 6, 44, 479	No
LSgrp	1, 6, 44, 479	No
ToSgrp	1, 6, 44, 386	A084965
PoUn	1, 6, 43, 452	No
PoNUn	1, 6, 39, 386, 5203	No
BMag	1, 6, 0, 1176, 0, 0, 0	No
BImpA	1, 6, 0, 1176, 0, 0, 0	No
BSgrp	1, 6, 0, 93, 0, 0, 0	No
LrPoSgrp	1, 5, 28, 273, 3788	No
Sgrp	1, 5, 24, 188, 1915, 28634,...	A027851
DivJslat	1, 4, 281	No
DivMslat	1, 4, 216	
DivLat	1, 4, 216	
ToDivLat	1, 4, 216	
DDivLat	1, 4, 216	
CnjMag	1, 4, 215	
CMag	1, 4, 129, 43968, 254429900,...	A001425
CDivJslat	1, 4, 79, 7545	No
CDivMslat	1, 4, 64, 6208	No
CDivLat	1, 4, 64, 6208	No
PoMon	1, 4, 37, 549	No
CMSgrp	1, 4, 32, 432	??
CJSgrp	1, 4, 29, 289	No
IdMSgrp	1, 4, 28, 308, 4694	No
CPoMon	1, 4, 27, 301, 4887	No
IdJSgrp	1, 4, 23, 166, 1379	No
Srng_0	1, 4, 22, 283	No
Srng_1	1, 4, 22, 169, 1819	No
CDLSgrp	1, 4, 20, 149, 1106	No
CLSgrp	1, 4, 20, 149, 1427	No
CToSgrp	1, 4, 20, 114, 710, 4726,...	A346414
IdLSgrp	1, 4, 17, 100, 674	No
DIdLSgrp	1, 4, 17, 100, 576	No
IdToSgrp	1, 4, 17, 82, 422	??
RPoUn	1, 4, 16, 87, 562	No
GalPos	1, 4, 15, 83, 539	No
InPoMag	1, 4, 12, 77, 498	No
CyInPoMag	1, 4, 12, 76, 481	No
CInPoMag	1, 4, 12, 69, 354, 3632	No
InPoSgrp	1, 4, 10, 50, 210, 1721	No
CyInPoSgrp	1, 4, 10, 50, 196, 1397	No

Original structures database vs. current survey

"*An online database of classes of algebraic structures*", June 2003, Annual Meeting of the Assoc. for Symbolic Logic, Univ. of Illinois at Chicago

This list of mathematical structures is still at
http://math.chapman.edu/~jipsen/structures

An alphabetical list of links that point to (sometimes incomplete) axiomatic descriptions of about 300 categories of universal algebras

Current version is from a 2021 summer project with Bianca Newell to recreate this list of (partially-ordered) structures as a LaTeX document

Can be checked for consistency and updated more reliably

The po-algebra background is from Don Pigozzi, *Partially ordered varieties and quasivarieties*, 2004, unpublished lecture notes.

The current DRAFT survey

Would not exist without the tireless efforts of Bianca Newell

The survey is not finished – it's a continuously updated document

Single PDF file with many navigation links to browse the pages

Introduction (Chapter 1) is a very incomplete DRAFT

Download the latest DRAFT version of the PDF file at

http://math.chapman.edu/~jipsen/Survey-of-po-algebras-DRAFT.pdf

Software for reading theories and models into Python/Prover9/Mace4 is at

https://github.com/jipsen/Survey-of-po-algebras

Results about po-algebras

If $\leq$ is equationally definable then $\mathrm{Pre}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A})$

This coincides with the notion of **algebraizable** in algebraic logic

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

E.g., if all po-variety members $\mathbf{A}$ have lattice reducts $\mathbf{L_A}$ and $\leq^{\mathbf{A}}$ is a subrelation of $\leq^{\mathbf{L_A}}$ then Jónsson's Lemma applies

Theorem

For any po-algebra $\mathbf{A}$, the connected components of $\leq^{\mathbf{A}}$ are the kernel of a homomorphism to an unordered algebra

Residuation

Definition (Residuated po-magma)

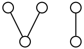
$\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a **residuated po-magma** or in **RPoMag** if

- $(A, \leq)$ is a poset and
- $\backslash, /$ are residuals: for all $x, y, z \in A$

$$x \leq z/y \iff x \cdot y \leq z \iff y \leq x \backslash z$$

Note: The residuation formula can be expressed by inequalities and implies ${}_+\cdot_+$, ${}_-\backslash_+$, ${}_+/_-$.

E.g., $yz \leq yz \Rightarrow y \leq yz/z$, hence $x \leq y \Rightarrow x \leq yz/z \Rightarrow xz \leq yz$

E.g. could  be the poset of a rpo-magma?

Lemma

For rpo-magmas, if $a, b \leq c$ *then* $(a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a, b$.

Proof.

$$
\begin{array}{lcr}
\text{Assume } a \leq c. & \text{Then} & a/(a\backslash c) \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq (a/(a\backslash c))\backslash b \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq b \\
\text{Assume } b \leq c. & \text{Then} & a\backslash c \leq a\backslash c \\
\Longleftrightarrow & & a(a\backslash c) \leq c \\
\Longleftrightarrow & & a \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq a\backslash b \leq a\backslash c \\
\Longrightarrow & & a/(a\backslash c) \leq a/((c/(a\backslash c))\backslash b) \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a
\end{array}
$$

$\square$

Finite rpo-magmas have bounded components

Lemma

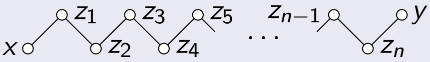
In any rpo-magma, if $d \le a, b$ *then*
$a, b \le d/(((a\backslash d)/(a\backslash(d\backslash d)))((d\backslash d)/(a\backslash(d\backslash d)))\backslash(b\backslash d)))$

Theorem

In an rpo-magma every connected component of $\le$ *is up-directed and down-directed, hence for* **finite** *rpo-magmas every connected component is* **bounded**.

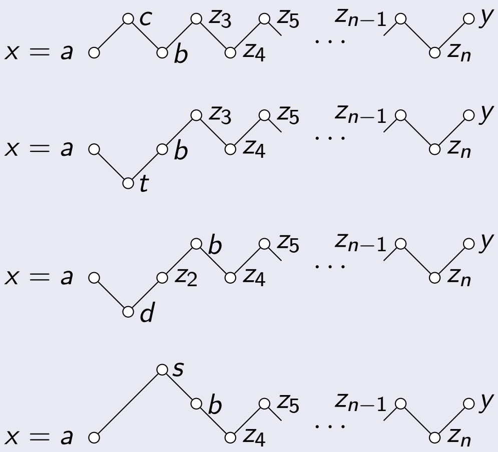
Proof.

Two elements x, y in a poset are connected iff there exists a zigzag



We need to find an upper and a lower bound of x, y.

Proof (continued).



Continue by induction to get an upper and lower bound in n steps.

What posets are possible for ipo-magmas?

Theorem

The equivalence relation on a poset that has each connected component as an equivalence class is a congruence on a rpo-magma, and the quotient algebra is a **quasigroup** *with the discrete order (i.e. $\leq$ is the equality relation).*

Conversely, from any group or quasigroup Q and a pair-wise disjoint family of **bounded** *posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q \in Q} A_q$.*

E.g. for a group Q and $x_p \in A_p$, $y_q \in A_q$ define

$$x_p \cdot y_q = \bot_{pq}, \qquad x_p \backslash y_q = \top_{p^{-1}q}, \qquad x_p / y_q = \top_{pq^{-1}}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where $\cdot$ is associative.

Note: If the order is an antichain then a rpo-semigroup is a **group**.

A **unital** rpo-magma has a constant 1 such that $x1 = x = 1x$, and a **rpo-monoid** is a unital rpo-semigroup $(A, \leq, \cdot, \sim, -, 1)$.

A **residuated lattice-ordered magma** $(A, \wedge, \vee, \cdot, \backslash, /)$ (or **rℓ-magma** for short) is a rpo-magma for which the partial order is a lattice order.

A $r\ell$-monoid is more commonly called a **residuated lattice** or **unital quantale** (if it is a complete lattice).

Involutive po-magmas

An **involutive po-magma** or **ipo-magma** $(A, \le, \cdot, \sim, -)$ is a poset $(A, \le)$ with a binary operation $\cdot$, two unary **order-reversing operations** $\sim, -$ that are an **involutive pair**: $\sim\!-x = x = -\!\sim\! x$, and for all $x, y, z \in A$

$$\text{(ires)} \quad xy \le z \iff x \le -(y \cdot \sim z) \iff y \le \sim(-z \cdot x).$$

It follows that ipo-magmas are **rpo-magmas**.

Hence $\cdot$ is order-preserving and ipo-magmas are a **po-variety**.

Models of cardinality $n =$	1	2	3	4	5	6
rpo-magmas	1	3	28	1200		
ipo-magmas	1	4	12	67	314	3029

Involutive po-magmas

A convenient **equivalent** formulation of (ires):

$$\text{(rotate)} \qquad xy \le z \iff y \cdot {\sim}z \le {\sim}x \iff -z \cdot x \le -y.$$

The po-variety of ipo-monoids includes **all partially ordered groups** where ${\sim}x = -x = x^{-1}$.

Lemma

Let $\mathbf{A} = (A, \le, \cdot, \sim, -)$ *be a poset with a binary operation and two unary operations.*

1. *If* $\cdot$ *is idempotent (i.e.* $xx = x$*) and* $\mathbf{A}$ *satisfies* (rotate) *then* $\mathbf{A}$ *is an ipo-magma.*
2. *If an ipo-magma is idempotent or unital, and* $\cdot$ *is commutative then* ${\sim}x = -x$.

Structural description for ipo-semilattices

In recent joint research with **José Gilferez and Sid Lodhia** we give a full structural description of ipo-monoids based on **Płonka sums of Boolean algebras** over a semilattice of generalized BA-homomorphisms.

Another interesting result:

[Pigozzi 2004, Cor. 5.7] shows that for left-residuated unital po-magmas $p(x, y, z) = x(y\backslash z)$ is a po-Mal'cev term: $p(x, x, y) \le y \le p(y, z, z)$ and p_{+-+}, hence precongruences are **permutable**.

Questions and open problems

Is there a characterization of po-clones other than finding a generating set of operations that are order-preserving or -reversing in each argument?
In joint research with **Reinhard Pöschel and Erkko Lehtonen**: Yes, consider certain ***S*-preclones**, where $S = \{+, -\}$ is the 2-element group.

Which po-clones are generated by operations that are **residuated**, **dually residuated**, **Galois connections** or **dual Galois connections**?

Describe the **free 1-generated residuated po-magma**.

Find more examples of po-varieties that are **precongruence-distributive**.

[Pigozzi 2004] has a Mal'cev condition for **permutability** of precongruences, but this does not imply modularity.
Can it be strengthened to give **modularity or distributivity**?

Describe the structure of **finite totally ordered bands**.

... (e.g., see ?? in table of fine spectra)

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Thanks!