ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 1

ISSUED 4 FEBRUARY 2020

1. Two linear equations in two unknowns: Cramer's rule

We want to solve a system of linear equations

$$a_1X + b_1Y = c_1$$

$$a_2X + b_2Y = c_2$$

where a_1, a_2, a_3, a_4 are either integer or rational or real numbers and X, Y are unknowns (variables).

Consider a *matrix* (square table of numbers) corresponding to this system,

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Definition 1.1. The number $\det A = a_1b_2 - a_2b_1$ is called the *determinant* of the matrix A.

Consider also the following matrices

$$A_X = \begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}, \quad A_Y = \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}.$$

Theorem 1.2 (Cramer's rule)**.** *Assume that the determinant* $\det A \neq 0$*. Then a solution* (x, y) *of the system of equations is given by the following formulas:*

$$x = \frac{\det A_X}{\det A}, \quad y = \frac{\det A_Y}{\det A}.$$

Proof. In the system of equations, replace X by

$$\frac{\det A_X}{\det A} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$$

and Y by

$$\frac{\det A_Y}{\det A} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Then a direct computation shows that the left-hand sides of both equations evaluate to the right-hand sides. □

Corollary 1.3. *If coefficients of the system of equations are rational numbers and* $\det A \neq 0$*, then there is a solution* (x, y) *of the system such that* x *and* y *are rational numbers.*

The case when the determinant $\det A$ equals to 0 is easy to analyze and conclude that then the system of equations either does not have any solutions or has infinitely many solutions which coincide with the set of all solutions of any of the two equations.

Cramer's rule generalizes to any system of linear equations in which the number of equations equals the number of unknowns. For this purpose we need to generalize the concept of a determinant. We will do this in a later section.

The example we just considered shows the usefulness of using matrices in the questions about solving linear systems. This prompts a development of a theory of matrices.

2. Matrices

An $(m \times n)$-matrix A is a rectangular table of rational or numbers having m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The numbers a_{ij}, where $i = 1, \dots m$, $j = 1, \dots n$, are called *elements* of A. We will often write a matrix indicating just its "general" element, $A = \|a_{ij}\|_{1 \le i \le m,\ 1 \le j \le n}$, or simply $A = \|a_{ij}\|$.

3. Determinants

Let $A = \|a_{ij}\|_{1 \le i,j \le n}$ be a square $(n \times n)$-matrix. Denote by A_{ij} the matrix obtained from A by striking out of A the row i and the column j.

Definition 3.1. The *determinant* $\det A$ of A is a number defined by induction as follows.

If $n = 1$ then the matrix A contains just one element a_{11} and we set $\det A = a_{11}$. Assume that $n > 1$ and that we defined determinants of $(n-1) \times (n-1)$-matrices. Fix any number i such that $1 \le i \le n$. Then define

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + \cdots + (-1)^{i+j} a_{ij} \det A_{ij} + \cdots + (-1)^{i+n} a_{in} \det A_{in}.$$

It can be proved that $\det A$ does not depend on the choice of the number i, i.e., for any $1 \le i \le n$ the number $\det A$ will be the same.

Example 3.2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We already know from the first section that $\det A = a_{11}a_{22} - a_{21}a_{12}$. Let us confirm that using Definition 3.1, in which we take $i = 1$:

$$\det A = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 3.3. Let us consider a numerical example of a (3×3)-matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Choose again in Definition 3.1 the row $i = 1$.

$$\det A = (-1)^{1+1} \det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} + (-1)^{1+2} 2 \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} + (-1)^{1+3} 3 \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} =$$
$$= -5 + 14 + 3 = 12.$$

Now choose the row $i = 2$.

$$\det A = (-1)^{2+1} 2 \det \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} + (-1)^{2+2} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} + (-1)^{2+3} 3 \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} =$$
$$= 8 - 8 + 12 = 12.$$

Exercise. Choose $i = 3$ and make sure that in this case the formula from Definition 3.1 again gives $\det A = 12$.

Let us now give another, non-inductive, definition of the determinant.

Definition 3.4. Let $i_1, i_2, \dots, i_n$ be a permutation of numbers $1, 2, \dots, n$. An *inversion* in $i_1, i_2, \dots, i_n$ is a pair (i_k, i_l) such that $k < l$ and $i_k > i_l$. For example, in the permutation $2, 3, 1$ of $1, 2, 3$ there are two inversions, $(2, 1)$ and $(3, 1)$.

In an $(n \times n)$-matrix A choose n different elements positioned in different rows and different columns, i.e., for any two elements $a_{ij}, a_{k\ell}$ among these n, we have $i \neq k$ and $j \neq \ell$. Notice that n is the maximal number of elements with such property, call the set of these elements a *configuration*. Thus, a configuration is a set of the kind $\{a_{1i_1}, a_{2i_2}, \dots, a_{ni_n}\}$ (here we have written elements of a configuration in the order of growth of the first subscript). One can easily check that there is exactly $n! = 1 \cdot 2 \cdots n$ different configurations.

Definition 3.5. The determinant $\det A$ of A is the sum over all configurations $\{a_{1i_1}, a_{2i_2}, \dots, a_{ni_n}\}$

$$\det A = \sum_{\{a_{1i_1}, a_{2i_2}, \dots, a_{ni_n}\}} (-1)^{\sigma(i_1, \dots, i_n)} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where $\sigma(i_1, \dots, i_n)$ is the number of inversions (Definition 3.4) in the permutation $i_1, i_2, \dots, i_n$.

Example 3.6. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then there are exactly two different configurations, $\{a_{11}, a_{22}\}$ and $\{a_{12}, a_{21}\}$. Hence,

$$\det A = (-1)^{\sigma(1,2)} a_{11} a_{22} + (-1)^{\sigma(2,1)} a_{12} a_{21},$$

where the numbers of inversions $\sigma(1, 2) = 0$ and $\sigma(2, 1) = 1$. We recover our original definition of the determinant of a (2×2)-matrix.

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then there are six different configurations, and the determinant $\det A$ is equal to

$$a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32}.$$

4. Cramer's rule

Now we are able to formulate the Cramer's rule for general $(n \times n)$-matrices.

Consider the system of linear equations

$$\begin{gathered} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1 \\ \cdots \quad \cdots \quad \cdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n = b_n \end{gathered} \tag{$*$}$$

Consider the matrix $A = \|a_{ij}\|_{1\le i\le n,\ 1\le j\le n}$ of this system. Assume that $\det A \neq 0$.

For each $j = 1, 2, \ldots, n$ denote by A_{X_j} the matrix obtained from A by replacing the column j, i.e., $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{nj} \end{pmatrix}$, by $\begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$.

Theorem 4.1 (Cramer's rule)**.** *A solution of the system* $(*)$ *is given by the following formulas*

$$x_j = \frac{\det A_{X_j}}{\det A} \quad \textit{for all} \quad j = 1, 2, \ldots, n.$$

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LECTURE NOTES 2

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1. Some properties of determinants

Definition 1.1. Let $A = \|a_{ij}\|$ be an $(m \times n)$-matrix. The $(n \times m)$-matrix in which the element a_{ij} of A becomes the element in row j, column i is called the *transpose* (or *transposition*) of A and is denoted by A^T. For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}.$$

In other words, rows of A become columns of A^T.

We now list some useful **properties of determinants**. Their proofs follow immediately from the definition of a determinant. Try to do these proofs (they are easy, but will not be examined). At least prove the properties for determinants of (2×2)-matrices.

(1) $\det A = \det(A^T)$ for each square matrix A.
(2) Let A' be a matrix obtained from a square matrix A by swapping the positions of some two different rows. Then $\det A' = -\det A$.
(3) Let A' be a matrix obtained from a square matrix A by multiplying each element of one of the rows of A by a number c. Then $\det A' = c \det A$.
(4) Let C be an $(n \times n)$-matrix whose row number i is of the form
$$a_{i1} + b_{i1}, \ldots, a_{in} + b_{in}.$$
Then $\det C = \det A + \det B$, where A (respectively, B) is obtained from C by replacing the row number i by $a_{i1}, \ldots, a_{in}$ (respectively, by $b_{i1}, \ldots, b_{in}$).
(5) If in an $(n \times n)$-matrix A there are two proportionate lines, i.e., lines of the kind $a_{i1}, \ldots, a_{in}$ and $ca_{i1}, \ldots, ca_{in}$ for some number c, then $\det A = 0$.
(6) The determinant $\det A$ of a square matrix A does not change if to elements of one row we add respective elements of another row, multiplied by the same number.

Because of the property (1) of determinants, all the rest of the properties are true if we replace, in their formulations, "row" by "column".

2. Operations on matrices

For two matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ of the same *format* $m \times n$ their *sum* $C = A + B$ is understood element-wise, i.e., $C = \|c_{ij}\|$ where $c_{ij} = a_{ij} + b_{ij}$. For a matrix A and a rational or real number k, the product kA of k by A is the matrix $\|ka_{ij}\|$. The difference $A - B$ of two matrices of the same format is, by the definition, the matrix $A + (-1)B$.

It's obvious that for any matrices A, B, C of the same format $m \times n$ the following properties hold true.

- $A + B = B + A$ (commutativity);
- $(A + B) + C = A + (B + C)$ (associativity);
- $A + \mathbf{0} = A$, where $\mathbf{0}$ is the $(m \times n)$-matrix whose elements are all zeroes;
- $A - A = \mathbf{0}$.

Definition 2.1. Consider two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}; \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}.$$

There is the $(m \times k)$-matrix $C = AB$, called *product* of A and B, whose (i, j)-element is

$$c_{ij} = \sum_{1 \le l \le n} a_{il} b_{lj}.$$

Note that for product to exist, the matrices A and B should "match", that is, the number of columns in A should be equal to number of rows in B.

Square matrix has equal number of rows and columns. Obviously, two square matrices of the same sizes always match.

The multiplication operation is associative, that is, $(AB)C = A(BC)$, but generally non-commutative, i.e., $AB \neq BA$, even if matrices match both ways. (Exercise: multiply two arbitrarily chosen (2×2)-matrices in different order.)

Definition 2.2. The $(n \times n)$-matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called *identity* matrix.

It is easy to check that the product of the identity matrix with any $(m \times n)$-matrix A is commutative in the sense that

$$I_m A = A I_n = A.$$

Definition 2.3. A square $(n \times n)$-matrix A is called *non-singular* (or *invertible*) if there is an $(n \times n)$-matrix, denoted by A^{-1}, such that $AA^{-1} = I_n$.

It is true but not obvious that $A^{-1}A = I_n$ (see Theorem 3.2 below).

Having a system of linear equations

$$\begin{gathered} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1 \\ \cdots \quad \cdots \quad \cdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \end{gathered} \qquad (*)$$

let $A = \|a_{ij}\|$, and set $X = (X_1, \dots X_n)^T$, $b = (b_1, \dots b_m)^T$.

Then the system $(*)$ can be represented in a form $AX = b$.

If A is a square $n \times n$ nonsingular matrix then

$$A^{-1}AX = A^{-1}b,$$

hence,

$$X = A^{-1}b.$$

It follows that to solve a system of equations $(*)$ with a square non-singular matrix it is sufficient to find the inverse matrix A^{-1}.

3. Computing inverse matrices

For an $(n \times n)$-matrix A let A_{ji} be the $((n-1) \times (n-1))$-matrix obtained from A by deleting row j and column i.

Definition 3.1. Adjugate (or adjoint) matrix $A^* = \|c_{ij}\|_{1 \le i,j \le n}$ of A is defined by the following formulas

$$c_{ij} = (-1)^{i+j} \det A_{ji}$$

(Note that the order of subscripts is inverted in A_{ji} as compared to c_{ij}).

Theorem 3.2.

- $AA^* = (\det A)I_n$ *(hence* $A^{-1} = \frac{1}{\det A} A^*$*).*
- A *is non-singular (i.e., there exists* A^{-1}*) if and only if* $\det A \neq 0$.
- *If* $AA^{-1} = I_n$ *then* $A^{-1}A = I_n$.

The proof can be found in, e.g., G. Smith's book (see the list of books), but will not be examined.

Example 3.3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Then, according to the formula for A^*,

$$A^* = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Since $\det A = -2$, dividing all elements of A by -2, we get

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$

ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 3

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1. Gaussian elimination

Let A be an $(m \times n)$-matrix. Consider the following *row operations* on A.

(1) Swap the positions any two rows.
(2) Multiply a row by any number.
(3) Add two rows.

Using these operations we can transform any matrix into *row echelon form*.

Definition 1.1. An $(m \times n)$-matrix A is said to be in a *row echelon form* if

(1) all rows consisting entirely of zeroes (if any) are at the bottom of A;
(2) the row number i, $1 \leq i < m$, containing a non-zero element, has less "first zeroes" (i.e., zeroes from the left preceding the first non-zero element) than the row $i + 1$.

If a *square* matrix is in row echelon form, and all diagonal elements $a_{ii} \neq 0$, then this matrix is called *upper triangular*.

Example 1.2. The following matrix is in row echelon form.

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_{12} \neq 0$ and $a_{24} \neq 0$.

Another example of a matrix in row echelon form is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

where all diagonal elements $a_{ii} \neq 0$. This matrix is upper triangular.

The *Gaussian elimination* is the algorithm that converts any $(m \times n)$-matrix $A = \|a_{ij}\|$ into a row echelon form, using row operations described above. It works like this. Using a sequence of row operations of type (1), the algorithm moves all zero rows the the bottom of the matrix. Keep the notation A for the resulting matrix. Choose any row of A having the smallest number of *first zeroes* (there may be many such rows, so choose any). Let a_{ij} be the first (from the left) non-zero element of that row (so-called *pivot element*). Swap this row i with the first row of A. Keep the notation A for the resulting matrix, in particular the chosen row takes the top position in A and the element a_{ij} is renamed to a_{1j}. Multiplying the first row of A by suitable numbers and adding the resulting rows to respective

other rows of A (operations of type (2) and (3)), obtain zeroes at all positions of the column j except the top position, occupied by a_{1j}.

Apply recursively the same procedure to a smaller sub-matrix of A consisting of rows $2, \dots, m$ and columns $j+1, \dots, n$ of A. The algorithm terminates when either the current sub-matrix is the zero matrix or it is a (1×1)-matrix.

Example 1.3. Let

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$$

As a pivot element a_{ij} we can take $a_{11} = 3$. Multiplying the row 1 by $-2/3$ and adding the result to the row 3 obtain

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 0 & -1/3 & 4 \end{pmatrix}.$$

Multiplying the row 1 by $2/3$ and adding the result to the row 2 obtain

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1/3 & 1 \\ 0 & -1/3 & 4 \end{pmatrix}.$$

Now apply recursively the same procedure to the sub-matrix consisting of rows 2, 3 and columns 2, 3 of A. Namely, multiply the row 2 of A by 1 and add the result to row 3 of A. Obtain

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1/3 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

At this point the Gaussian elimination terminates because the next recursive sub-matrix is (1×1)-matrix consisting of the row 3 and the column 3 of the last matrix A. Hence A is now in a row echelon (even an upper triangular) form.

2. Computing determinants with Gaussian elimination

Let A be a square matrix. Observe that Gaussian elimination algorithm, being applied to A, uses only operations of swapping of two rows and adding a row, multiplied by a number, to another row. By property (2) of determinants (Lecture Notes 2), swapping of rows changes the sign of a determinant, while by property (6), adding a row multiplied by a number to another row leaves the determinant unchanged. Therefore a row echelon form of A has the same determinant as A up to a sign. But the determinant of a square matrix in row echelon form is equal, by the definition of the determinant to the product of the diagonal elements. It follows that if A is non-singular (invertible), then the row echelon form of A is an upper triangular matrix.

Example 2.1. Question: Compute the determinant of the matrix A in the Example 1.3.

Answer: Since we know the upper triangular form of A, its determinant $\det A$ is the product of the diagonal elements of the upper triangular form, i.e., $3 \times 1/3 \times 5 = 5$.

Note that in Example 1.3 we did not need to swap rows. If during the Gaussian elimination for a non-singular matrix A we swap rows k times, then $\det A$ will be equal to the determinant of its upper triangular form multiplied by $(-1)^k$.

3. Vectors and linear independence

A matrix with one row is called a *row vector* and a matrix with one column – a *column vector*. Operations of addition, multiplication by a number and multiplication of two vectors are particular cases of the corresponding matrix operations. In the case of vectors the matrix multiplication is called the *dot product* or *scalar product*.

In this section let us deal with row vectors.

Definition 3.1. Vectors

$$a_1 = (a_{11}, \dots, a_{1n}), \dots a_m = (a_{m1}, \dots a_{mn})$$

are called *linearly dependent* if there exists a *non-zero* vector $(\lambda_1, \dots, \lambda_m)$, such that $\lambda_1 a_1 + \dots + \lambda_m a_m = (0, \dots, 0)$. If $a_1, \dots, a_m$ are not linearly dependent then they are called *linearly independent*.

Observe that if at least one of vectors $a_1, \dots, a_m$ is a zero vector, then $a_1, \dots, a_m$ are linearly dependent. For example, if $a_1 = (0, \dots, 0)$, then choose $(\lambda_1, \dots, \lambda_m) = (1, 0, \dots, 0)$.

It is clear that if $a_1, \dots, a_m$ are linearly dependent then for $\lambda_i \neq 0$ the vector a_i can be expressed via the rest:

$$a_i = -1/\lambda_i(\lambda_1 a_1 + \dots \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \dots + \lambda_m a_m).$$

Example 3.2. Vectors $(1, 2), (2, 1)$ are linearly independent. Indeed, suppose not. Then $\lambda_1(1, 2) + \lambda_2(2, 1) = 0$ for some $(\lambda_1, \lambda_2) \neq 0$. Suppose, for definiteness, $\lambda_1 \neq 0$. Then

$$(1, 2) + \frac{\lambda_2}{\lambda_1}(2, 1) = 0, \text{ so } (1, 2) = -\left(\frac{\lambda_2}{\lambda_1} 2, \frac{\lambda_2}{\lambda_1}\right).$$

It follows that

$$\frac{\lambda_2}{\lambda_1} = -\frac{1}{2} \text{ and } \frac{\lambda_2}{\lambda_1} = -2$$

which is a contradiction.

Another example. It is easy to prove that all rows in an upper triangular matrix are linearly independent. Since linear independence is preserved under row operations (1)–(3), any square matrix is non-singular if and only if its all rows are linearly independent.

4. The rank of a matrix

Definition 4.1. The largest number of linearly independent rows of an $(m \times n)$-matrix is called its *rank*.

For example, the rank of an upper triangular $(n \times n)$-matrix is n.

The rank of an arbitrary $(m \times n)$-matrix A can be computed using the Gaussian elimination: reduce A to a row echelon form, then rank A equals to the number of non zero rows in this form.

Example 4.2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 4 & 5 \end{pmatrix}.$$

Multiplying the row 1 by -3 and adding to the row 3 we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & -4 & -5 & -7 \end{pmatrix}.$$

Multiplying the row 1 by -2 and adding to the row 2 we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -7 \\ 0 & -4 & -5 & -7 \end{pmatrix}.$$

Finally, multiplying the row 2 by -1 and adding to the row 3 we get a row echelon form of the matrix A:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It has two non-zero rows, thus rank $A = 2$.

5. Columns instead of rows

We get an analogous theory if in all the above we swap, in formulations of all statements, "rows" and "columns".

In particular, a *column echelon* form of a matrix can be defined, and the corresponding Gaussian elimination algorithm follows. The *rank* of a matrix can be defined as the largest linearly independent number of columns. One can prove that the two definitions of the rank, one based on rows, another on columns, result in the same number.

6. Gaussian elimination for solving systems of linear equations with quadratic non-singular matrix

Recall the three techniques for solving systems of linear equations $A\mathbf{x} = \mathbf{b}$ with non-singular $(n \times n)$-matrices $A = \|a_{ij}\|$, column n-vector of unknowns $\mathbf{x}$, and column n-vector $\mathbf{b}$, discussed earlier.

Cramer's rule gives explicit formulae for elements of the solutions.

Inverting the matrix. Find the inverse A^{-1} of A (explicit formula), then compute $\mathbf{x} = A^{-1}\mathbf{b}$.

Gaussian elimination. Consider a new matrix obtained from A by adding to it a new column $\mathbf{b}$ on the right. Denote this new matrix by $(A, \mathbf{b})$ and call it the *extended* matrix of the system $A\mathbf{x} = \mathbf{b}$.

Reduce the extended matrix $(A, \mathbf{b})$ to the row echelon form. We get an *upper triangular* matrix extended by a column from the right. Note that the solution of the original system of equations coincides with the solution of the system obtained by Gaussian elimination, because operations (1)–(3) lead to equivalent equations.

Keep the same notation a_{ij}, b_j for elements of the obtained matrix in the row echelon form. Observe that all elements a_{ii} are different from zero.

The last equation of the system is now of the form $a_{nn}x_n = b_n$, hence $x_n = b_n/a_{nn}$. The one before last equation is of the form

$$a_{n-1n-1}x_{n-1} + a_{n-1n}x_n = b_{n-1},$$

hence

$$a_{n-1n-1}x_{n-1} + \frac{a_{n-1n}b_n}{a_{nn}} = b_{n-1},$$

therefore

$$x_{n-1} = \frac{a_{nn}b_{n-1} - a_{n-1n}b_n}{a_{nn}a_{n-1n-1}}.$$

Moving in this way up through the triangular system of equations, we will find, one by one, all x_i in the solution.

At the first glance, Gaussian elimination looks more complex than other methods but in fact it's *much* more efficient computationally.

Example 6.1.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The extended matrix is

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Its row echelon form is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \end{pmatrix}.$$

Thus, we have the following system of linear equations

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The second equation in this system is $-3x_2 = -1$, thus $x_2 = 1/3$. The first equation is $x_1 + 2x_2 = 1$, therefore $x_1 + 2/3 = 1$. It follows that $x_1 = 1/3$. The system is solved.

ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 4

ISSUED 27 FEBRUARY 2020

1. Vector spaces and their bases

The vector space $\mathbb{R}^n$ is the set of all vectors $\mathbf{x} = (x_1, \dots, x_n)$, where all x_i are real numbers. The vector space $\mathbb{R}^n$ is not just a set of vectors, but a set equipped with two operations: addition of vectors and multiplication of vectors by numbers. These operations are defined as if vectors are considered as single-row (or single-column) matrices.

In addition one can consider the operation of *dot multiplication* (also known as *scalar multiplication*) of vectors. The dot product $\mathbf{x} \cdot \mathbf{y}$ (also denoted by $\mathbf{xy}$) of two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ is defined as $x_1y_1 + \dots + x_ny_n$. This is, of course, the same as the usual product of a row matrix $\mathbf{x}$ and a column matrix $\mathbf{y}^T$, where T stands for the transposition. A vector space $\mathbb{R}^n$ equipped with the dot product is called *Euclidean space*.

Depending on the geometric context, vectors in $\mathbb{R}^n$ are sometimes called *points of* $\mathbb{R}^n$.

Theorem 1.1. *In $\mathbb{R}^n$ there are n linearly independent vectors, and this is the maximal number.*

Proof. The following n vectors

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0)$$

$$\mathbf{e}_2 = (0, 1, 0, \dots, 0)$$

$$\dots \quad \dots \quad \dots$$

$$\mathbf{e}_n = (0, 0, 0, \dots, 1)$$

are obviously linearly independent.

Let us now prove that there are no $n + 1$ linearly independent vectors in $\mathbb{R}^n$. Suppose this is not true, i.e., there are $n+1$ linearly independent vectors, in particular, non-zero, vectors in $\mathbb{R}^n$, $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_{n+1}$, where $\mathbf{a}_j = (a_{1j}, \dots, a_{nj})^T$. Write the first n vectors as columns of the $(n \times n)$-matrix A. Consider the system of linear equations $A\mathbf{x} = \mathbf{a}_{n+1}^T$, where $\mathbf{x} = (x_1, \dots, x_n)^T$ is the vector of unknowns. Since the columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent, the matrix A is non-singular. Hence there is a (unique) solution of this system of equations which is not the zero vector (because $\mathbf{a}_{n+1}$ is non-zero). We proved that the vector $\mathbf{a}_{n+1}$ is a linear combination of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ which contradicts the supposition that $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_{n+1}$ are linearly independent. □

It is easy to see that *any* vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ is a linear combination of vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ from the proof of Theorem 1.1:

$$\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_m.$$

The set of vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ is called *the standard basis* of $\mathbb{R}^n$. Any other set of linearly independent vectors in $\mathbb{R}^n$ is called a *basis*. The proof of Theorem 1.1 explains that for any basis, any vector in $\mathbb{R}^n$ is a linear combination of basis vectors.

Example 1.2. Vectors $(1, 2)$, $(3, 1)$ form a basis of $\mathbb{R}^2$.

Definition 1.3. Given a set of vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_k\} \subset \mathbb{R}^n$, its *span* (or *linear hull*) is the set of all linear combinations of vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, i.e., the set of vectors of the kind $\lambda_1\mathbf{a}_1 + \cdots + \lambda_k\mathbf{a}_k$ for all numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}$.

Note that in this definition it is possible that $k \neq n$. In the case when the set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ contains a basis (in particular, $k \geq n$), the span coincides with the whole $\mathbb{R}^n$. Otherwise, it is a proper subset of $\mathbb{R}^n$ called a *linear subspace* of $\mathbb{R}^n$.

Example 1.4.
(1) The span of $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbb{R}^3$ is the plane consisting of all points in $\mathbb{R}^3$ of the kind $(x_1, x_2, 0)$.
(2) The span of the single point $(2, 1) \in \mathbb{R}^2$ is the straight line passing through the origin $(0, 0)$ and the point $(2, 1)$.

2. Gaussian elimination for solving arbitrary systems of linear equations

Consider a general system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is $(m \times n)$-matrix, $\mathbf{x}$ is an n-vector of unknown, and b is an m-vector of numbers.

Matrix A is now not necessarily a square non- singular matrix, but A has a *rank*.

Let rank $A = r$. Reduce the matrix $(A, \mathbf{b})$ (i.e., A with added column $\mathbf{b}$) to row echelon form, and keep the same notations for elements a_{ij}, b_j in the reduced matrix. In the row $r + 1$ all elements $a_{r+1j} = 0$. If $b_{r+1} \neq 0$ then the system does not have solutions since the left hand side in the equation $r + 1$ equals to 0.

Suppose that a system of linear equations does have solutions. Let us try to answer the question: what does it mean to *solve* the system? If there is a unique solution then the answer is clear, it means to produce that unique solution. Otherwise, we need to agree on how to represent an infinite set of solutions.

Example 2.1. Let the system be already reduced to an upper echelon form:

$$x_1 + 3x_2 - x_3 + 2x_4 = 1$$
$$x_3 + 3x_4 = 2.$$

This system has infinitely many solutions. Let us first find one. Setting $x_4 = 0$, get $x_3 = 2$ from the second equation. First equation turns into $x_1 + 3x_2 - 2 = 1$. Setting $x_2 = 0$, get $x_1 = 3$. Bringing all things together, one solution of the system is: $x_1 = 3, x_2 = 0, x_3 = 2, x_4 = 0$. In the process of finding one solution, we arbitrarily assigned the value 0 to x_4 and x_2. Assigning another values to these unknowns would lead to a different concrete solution. Let us now perform this process in a general form.

The last equation implies that $x_3 = 2 - 3x_4$. Hence, from the first equation we get $x_1 + 3x_2 - 2 + 5x_4 = 1$, thus $x_1 = 3 - 3x_2 - 5x_4$. Collecting the expressions together, we obtain:

$$x_3 = 2 - 3x_4$$

$$x_1 = 3 - 3x_2 - 5x_4.$$

This system of equations is a *general solution* of the original system. Assigning arbitrary combination of values to $x_2,\ x_4$ on the right hand side, we obtain the corresponding values of $x_3,\ x_1$ on the left hand side.

We can re-write the latter system of equations in a matrix form:

$$\begin{pmatrix} 0 & -3 \\ -3 & -5 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}. \tag{2.1}$$

This represents a map from the plane $\mathbb{R}^2$ (equipped with coordinates $x_2,\ x_4$) to the plane $\mathbb{R}^2$ (equipped with coordinates $x_3,\ x_1$). Such maps are called *affine transformations* or *affine maps*. This map can serve as the *general solution* for the system of equations from the example.

Observe that this affine map is *bijective* [1]. This follows from the fact that

$$A = \begin{pmatrix} 0 & -3 \\ -3 & -5 \end{pmatrix}$$

is non-singular. Then for each concrete vector $(x_3,\ x_1)^T \in \mathbb{R}^2$ the solution of the system (2.1) exists (hence the map is surjective) and is unique (hence the map is injective), by Cramer's rule.

In general, an affine map (affine transformation) is defined by an expression $A\mathbf{x} + \mathbf{b}$ where A is an $(m \times n)$-matrix, $\mathbf{x}$ is an n-vector and $\mathbf{b}$ is an m-vector. Such an affine transformation maps the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. If $\mathbf{b} = \mathbf{0}$ (i.e., $\mathbf{b}$ is the zero vector) then the map is called *linear map* (or *linear transformation*).

Theorem 2.2. *If $n \neq m$ then the affine map is not bijective. More precisely, if $n > m$ then the map is not injective, if $n < m$ it's not surjective. When $n = m$, the map is bijective if and only if the matrix A is non-singular.*

Proof of this theorem is an easy exercise (do it!)

[1] Recall that a map $f : X \to Y$, where $X,\ Y$ are arbitrary sets, is called *surjective* if for each $y \in Y$ there is $x \in X$ such that $f(x) = y$. A map f is *injective* if for any two different $x_1,\ x_2 \in X$ we have $f(x_1) \neq f(x_2)$. A map is called *bijective* if it's simultaneously surjective and injective.

ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 5

ISSUED 7 MARCH 2020

1. Equations for straight lines in the plane

One can suggest various "analytic" (i.e., by means of formulae) definitions of a straight line in $\mathbb{R}^2$. One way to make sure each of them reflects our intuitive, geometric, understanding of a straight line, is to prove that these definitions are equivalent.

We start with the following two definitions.

A. A straight line through $\mathbf{0}$ in $\mathbb{R}^2$ is a set of all points $(x, y) \in \mathbb{R}^2$ satisfying a linear equation $ax + by = 0$ for some $a,\ b \in \mathbb{R}$ which are not simultaneously zero.
B. A straight line through $\mathbf{0}$ in $\mathbb{R}^2$ is a *span* of a single non-zero vector $(r_1,\ r_2) \in \mathbb{R}^2$.

These two definitions define the same set of points. Before proving this, let us introduce a definition.

Definition 1.1. Let $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ be two vectors in $\mathbb{R}^n$. These vectors are called *orthogonal* if their dot product, $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \dots + u_nv_n$ equals to zero.

It is easy to prove, using Pythagoras' theorem, that two non-zero vectors are orthogonal if and only if the angle between them (i.e., between the arrows from the origin in $\mathbb{R}^2$ and points $\mathbf{u}$ and $\mathbf{v}$) is $90°$.

Theorem 1.2. *Definitions A and B of a straight line in $\mathbb{R}^2$ are equivalent.*

Proof. Recall that the span of $r = (r_1,\ r_2)$ is the set of all points $(x,\ y) \in \mathbb{R}$ such that $(x, y) = (\lambda r_1,\ \lambda r_2)$ for all $\lambda \in \mathbb{R}$. Consider any non-zero vector $(a,\ b)$ orthogonal to $(r_1,\ r_2)$, i.e., $ar_1 + br_2 = 0$. Then at every point $(x,\ y)$ in the span of $r = (r_1,\ r_2)$ we also have $ax + by = 0$. Hence, the straight line in the sense of definition B lies in a straight line in the sense of definition A.

Conversely, the straight line in the sense of definition A lies in a straight line in the sense of definition B. Indeed, consider the set of points satisfying an equation $ax + by = 0$, where either a or b is non-zero. Take a particular non-zero point $(r_1,\ r_2)$ in this set, then $ar_1 + br_2 = 0$. Choose any other point $(s_1,\ s_2)$ in this set, and look at the system of linear equations

$$ar_1 + br_2 = 0$$
$$as_1 + as_2 = 0$$

in which a and b are considered as "unknowns". Observe that

$$\det \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} = 0$$

because otherwise the system of equations has, by Cramer's rule, the unique solution, namely $a = 0,\ b = 0$, which contradicts our supposition about a and b. It follows that the determinant is 0, i.e., that vectors $(r_1,\ r_2)$ and $(s_1,\ s_2)$ are linearly dependent. Thus there exists a number $\lambda \in \mathbb{R}$ such that $(s_1,\ s_2) = \lambda(r_1,\ r_2)$, i.e., $(s_1,\ s_2)$ belongs to the span of $(r_1,\ r_2)$. □

Now we can generalize the definition of a straight line in $\mathbb{R}^2$ to include lines not necessarily passing through the origin.

Definition 1.3. A straight line in $\mathbb{R}^2$ is a set of all points $(x,\ y) \in \mathbb{R}^2$ satisfying an equation $ax + by + c = 0$ where real numbers $a,\ b$ are not simultaneously equal to zero.

Now we consider another representation of a straight line in the plane. From elementary geometry we know that there exists a unique straight line passing through given two different points in the plane. Let us answer the following questions:

1. Is this property true for our "analytic" definition of a straight line?
2. If so, what is the equation for this line in terms of the two points?

Suppose that the two different points are $(x_1,\ y_1)$ and $(x_2,\ y_2)$. If there is a line through them, then

$$ax_1 + by_1 + c = 0$$

$$ax_2 + by_2 + c = 0$$

for some $a, b, c \in \mathbb{R}$, such that $a,\ b$ are not simultaneously equal to zero. The case when $c = 0$ was already considered in the proof of Theorem 1.2, so assume that $c \neq 0$. Dividing each of the equations by c, we obtain

$$\frac{a}{c}x_1 + \frac{b}{c}y_1 = -1$$

$$\frac{a}{c}x_2 + \frac{b}{c}y_2 = -1.$$

Consider the latter as a system of linear equations with "unknowns" a/c and b/c.

If

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \neq 0,$$

then, by Cramer's rule, the system has a a unique solution in a/c and b/c, i.e., there is a unique straight line through $(x_1,\ y_1)$ and $(x_2,\ y_2)$.

It remains to prove that the determinant can't be equal to zero. Indeed, if the determinant equals to 0, then its rows are linearly dependent, thus there is a number $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$ and $(x_1,\ y_1) = \lambda(x_2,\ y_2)$. The number $\lambda \neq 1$ since otherwise the two points coincide whereas we assumed them to be different. On the other hand, if $\lambda \neq 1$ then the left hand sides in the equations differ by the factor λ while the right hand sides coincide. We get a contradiction when we suppose that the determinant is zero.

Finally, let us answer the question 2. Given two different points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ in $\mathbb{R}^2$, consider the equation

$$(x - x_1)(y_2 - y_1) = (y - y_1)(x_2 - x_1),$$

which can be re-written as

$$\det \begin{pmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{pmatrix} = 0.$$

Clearly this equation can be written in a form $ax + by + c = 0$ for appropriate real numbers $a,\ b$ and c. It is straightforward to check that the points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ satisfy this equation. In case when two points are not only different but $x_1 \neq x_2$ and $y_1 \neq y_2$, the equation can be written in the form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

2. 3D and beyond

Definition 2.1. A *plane* in $\mathbb{R}^3$ passing through the origin $\mathbf{0}$ is the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying an equation $ax + by + cz = 0$, where at least one of the numbers $a,\ b,\ c$ is different from 0.

This is equivalent to the set of points which is a *span* of two linearly independent vectors in $\mathbb{R}^3$.

Example 2.2. The set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation $x - y = 0$ (note that this equation does not actually depend on z) is the same plane in $\mathbb{R}^3$ as $\text{span}\{(1, 1, 0),\ (0, 0, 1)\}$. (Draw this plane!)

Now we can define a *plane* in $\mathbb{R}^3$ (not necessarily passing through $\mathbf{0}$) as a non-empty set of points $(x, y, z) \in \mathbb{R}^3$ satisfying an equation $ax + by + cz + d = 0$, where at least one of the numbers $a,\ b,\ c$ is different from 0. Observe that the vector (a, b, c) is orthogonal to any vector on the plane $\{ax + by + cz = 0\}$, hence to any vector in the plane $\{ax + by + cz + d = 0\}$.

Recall from elementary geometry that for three points in $\mathbb{R}^3$ *not on the same straight line* there is a unique plane passing through them. This is true *for our definition of a plane in* $\mathbb{R}^3$. A proof is similar to the one in 2D case. Given three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), the linear equation of the plane passing through them is given by

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0.$$

It is easy to check that each of the three points indeed satisfies this equation.

We can also consider straight lines in $\mathbb{R}^3$. Again, we have two equivalent definitions.

Definition 2.3. A *straight line* in $\mathbb{R}^3$ is the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying a system of linear equations

$$a_1 x + b_1 y + c_1 z + d_1 = 0$$

$$a_2 x + b_2 y + c_2 z + d_2 = 0,$$

where vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly independent.

Remark 2.4. Note that there should be at least two equations defining a line since a single equation always defines a plane.

Remark 2.5. Let us understand the condition in Definition 2.3 that (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly independent.

(1) Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be linearly dependent, i.e., $(a_1, b_1, c_1) = \lambda(a_2, b_2, c_2)$ for some $\lambda \in \mathbb{R}$, while $d_1 \neq \lambda d_2$. Then the two planes, $\{a_1x + b_1y + c_1z + d_1 = 0\}$ and $\{a_2x + b_2y + c_2z + d_2 = 0\}$, are parallel (since their orthogonal vectors, (a_1, b_1, c_1) and (a_2, b_2, c_2) are collinear) with the empty intersection.
(2) Let (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) be linearly dependent. Then the planes coincide.

For a straight line in $\mathbb{R}^3$, passing through two distinct points (x_1, y_1, z_1) and (x_2, y_2, z_2), there is a formula similar to the formula for a straight line in $\mathbb{R}^2$:

$$(x - x_1)(y_2 - y_1) = (y - y_1)(x_2 - x_1), \quad (y - y_1)(z_2 - z_1) = (z - z_1)(y_2 - y_1)$$

(Note that this formula contains two separate equations.) In case when two points are not only different but $x_1 \neq x_2$, $y_1 \neq y_2$, and $z_1 \neq z_2$, the equation can be written in an elegant form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

The theory of straight lines and planes in $\mathbb{R}^3$ can be extended to the vector space $\mathbb{R}^n$ for arbitrary integer $n > 3$. The set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying a linear equation $a_1x_1 + \cdots + a_nx_n + b = 0$ is called a *hyperplane in* $\mathbb{R}^n$. The prefix "hyper" means that the dimension of the set of points is $n - 1$ (one dimension smaller than the dimension of the whole space). Each hyperplane defined by an equation $a_1x_1 + \cdots + a_nx_n = 0$ (i.e., when $b = 0$) is exactly a span of some $n - 1$ linearly independent vectors in $\mathbb{R}^n$.

Intersecting a hyperplane with another hyperplane, so that the orthogonal vectors are linearly independent, we get a plane in $\mathbb{R}^n$ of dimension $n - 2$ and so on.

ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 6

ISSUED 17 MARCH 2020

1. Linear transformations, eigenvectors and eigenvalues

We return to considering vector spaces $\mathbb{R}^n$ over real numbers. As discussed before, elements of $\mathbb{R}^n$ can be geometrically interpreted either as "points" or as "vectors" (arrows starting at the origin). We will use both terms depending on the context.

Definition 1.1. Let A be $(n \times n)$-matrix. The map $\mathbb{R}^n \to \mathbb{R}^n$ sending a point $\mathbf{x} \in \mathbb{R}^n$ to the point $A\mathbf{x} \in \mathbb{R}^n$ is called *linear transformation* of $\mathbb{R}^n$. We will sometimes denote the matrix and the corresponding linear transformation by the same symbol, e.g., A.

It is easy to prove that if the matrix A is non-singular then the corresponding linear transformation is *bijective*. (This is particularly easy to see in the 1-dimensional case, when a point $x \in \mathbb{R}$ is sent to $ax \in \mathbb{R}$ with a real number $a \neq 0$ playing the role of a non-singular (1×1)-matrix.)

Definition 1.2. Let A be an $(n \times n)$-matrix. Suppose that the equality $A\mathbf{x} = \lambda\mathbf{x}$ takes place for a *non-zero* vector $\mathbf{x}$ and a number $\lambda \in \mathbb{R}$. Then $\mathbf{x}$ is called an *eigenvector* of A, while λ is called an *eigenvalue* of A. ("Eigen-" is a German prefix meaning "own" or "unique to").

To get a geometric interpretation of an eigenvector $\mathbf{x}$ of A, consider in $\mathbb{R}^n$ a straight line ℓ through the origin, defined by $\mathbf{x}$, i.e., $\ell = \{a\mathbf{x} |\ a \in \mathbb{R}\}$. Then ℓ is "fixed" under the transformation A, i.e., the transformation maps ℓ onto itself. Indeed,

$$A(a\mathbf{x}) = aA\mathbf{x} = a\lambda\mathbf{x} = \lambda(a\mathbf{x}).$$

(The first equality in this chain is an easy exercise (do it!), while the second equality comes from the definition of the eigenvector.) This also proves the following property of eigenvectors:

> *if* $\mathbf{x}$ *is an eigenvector of* A*, then for any real* $a \neq 0$ *the vector* $a\mathbf{x}$ *is also an eigenvector of* A*.*

2. Computing eigenvalues and eigenvectors

Re-write the defining equation $A\mathbf{x} = \lambda\mathbf{x}$ as $(A - \lambda I)\mathbf{x} = 0$, where I is the *identity* $(n \times n)$-matrix.

This system of linear equations has an obvious solution, $(0, \ldots, 0)$. But, by the definition, an eigenvector is a non-zero vector. In case the matrix $A - \lambda I$ is non-singular (i.e., the determinant $\det(A - \lambda I) \neq 0$), the zero solution is *the only* solution of the system, by Cramer's rule. Therefore, for an eigenvalue to exist it's

necessary that $\det(A-\lambda I)=0$. Observe that with respect to the unknown λ the equation $\det(A-\lambda I)=0$ is an algebraic (polynomial) equation.

Definition 2.1. For a given square matrix A the polynomial $\det(A-\lambda I)$ is called its *characteristic polynomial*.

Example 2.2. Let

$$A=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}=\lambda^2-4\lambda+3.$$

Hence, the equation for λ is $\lambda^2-4\lambda+3=0$, and it has two distinct real solutions, $\lambda=1$ and $\lambda=3$.

Now it is possible to find the corresponding eigenvectors.

Let $\lambda=1$. Then the equation $(A-\lambda I)\mathbf{x}=0$ turns into

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x}=0.$$

We can re-write this system of linear equations in the form

$$x_1+x_2=0$$

$$x_1+x_2=0$$

i.e., two equations coincide. Choose $x_1=1$. Then $x_2=-1$, hence one of the solutions is $(1,-1)$. This solution is, of course, no unique: for any real $a\neq 0$ the point $(a,-a)$ is also a solution.

Let $\lambda=3$. Then we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{x}=0,$$

which has a solution, eigenvector $(1,1)$. Of course (a,a) is also an eigenvector for A for any real $a\neq 0$.

Observe that the image $A\mathbf{x}$ under the transformation A of any vector $\mathbf{x}$ on the straight line defined by $(-1,1)$ lies on the same line (since $(-1,1)$ is an eigenvector) and its length is the same as the length of $\mathbf{x}$ (since the corresponding $\lambda=1$). Similarly, the image of any vector $\mathbf{y}$ on the straight line defined by the eigenvector $(1,1)$ lies on the same line, and its length is 3 times larger than the length of $\mathbf{y}$ (since the corresponding $\lambda=3$).

3. Computing eigenvalues in general

Consider an upper triangular matrix A with (necessarily non-zero) diagonal elements

$$a_{11},a_{22},\dots,a_{nn}.$$

Since the determinant of an upper triangular matrix equals to the product of diagonal elements, we have:

$$\det(A-\lambda I)=(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda).$$

It follows that eigenvalues of A are exactly the numbers $a_{11},a_{22},\dots,a_{nn}$.

In general, to find eigenvalues of an arbitrary $(n\times n)$-matrix A one has to solve an algebraic equation of degree n, of the kind $b_0\lambda^n+b_1\lambda^{n-1}+\cdots+b_0=0$, where

$b_0\lambda^n + b_1\lambda^{n-1} + \cdots + b_0$ is the characteristic polynomial of A. Of course, solving such equations in the case $n > 1$ is a non-linear problem.

It turns out that each polynomial in one variable, of degree n is a characteristic polynomial of some $(n \times n)$-matrix. It means that to find eigenvalues of a matrix we may have to solve an arbitrary algebraic equation. When $n > 4$ there are no explicit general formulae for solutions of general algebraic equations, so one might need to use numerical, approximate algorithms.

One natural method emerges. First compute the determinant of $A-\lambda I$, and then use a numerical algorithm on the characteristic polynomial. In practice this method does not usually work well, because coefficients of the characteristic polynomial are in most cases computed approximately, while the roots of some polynomials are very sensitive of coefficients. An accurate, efficient method specifically for computing the eigenvalues was discovered only in 1960-s (so called QR algorithm).

4. Further examples

Since some algebraic equations don't have solutions in real numbers (e.g., the equation $\lambda^2 + 1 = 0$), some matrices have no real eigenvalues, and therefore no corresponding eigenvectors. Geometrically it's clear that such situations are possible, for example there are no fixed straight lines through the origin for rotations of the plane about the origin by an angle different from $k\pi$, where $k \in \mathbb{Z}$ (i.e., different from $0°, 180°, 360°$, etc.) Let us consider this and some other examples in more detail.

Example 4.1. Let

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

The linear transformation described by this matrix is the clockwise rotation of the plane about the origin by the angle α. In particular, the vector $(1, 0)$ will be carried to the vector $(\cos\alpha, \sin\alpha)$.

The characteristic polynomial is

$$\det \begin{pmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{pmatrix} = \lambda^2 - (2\cos\alpha)\lambda + 1.$$

Thus we have a quadratic equation $\lambda^2-(2\cos\alpha)\lambda+1 = 0$ for eigenvalues λ. According to the well-known formula for solving quadratic equations, the two potential solutions are: $\cos\alpha \pm \sqrt{\cos^2\alpha - 1}$. The number under the square root symbol is non-negative (actually 0) if and only if $\alpha = k\pi$, $k \in \mathbb{Z}$. Since $\cos\alpha = 1$ for $\alpha = 2k\pi$, and $\cos\alpha = -1$ for $\alpha = (2k + 1)\pi$, the eigenvalue is 1 in the first case and -1 – in the second. It follows that

$$A(2k\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

while

$$A((2k + 1)\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and all non-zero vectors in $\mathbb{R}^2$ are eigenvectors in both cases. For all other values of α the number under the square root symbol is negative, hence there are no real eigenvalues for $A(\alpha)$.

Example 4.2. Generalising the previous example when $\alpha = k\pi$, consider the linear transformation

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where real $a \neq 0$. The characteristic polynomial is $(a - \lambda)^2$, hence we have a single eigenvalue $\lambda = a$. All non-zero vectors in $\mathbb{R}^2$ are eigenvectors corresponding to this eigenvalue. Observe that every straight line through the origin is fixed under the transformation A.

Example 4.3. Let

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{R}$. This transformation is called *horizontal shear*. Since the characteristic polynomial is $(1 - \lambda)^2$, the only eigenvalue is $\lambda = 1$. Then the system of linear equations

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0$$

gives an eigenvector $(1, 0)$. When $b \neq 0$, the transformation shifts any non-zero point in $\mathbb{R}^2$ horizontally. When $b = 0$, the transformation reduces to the Example 4.2 with $a = 1$.

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LECTURE NOTES 7

ISSUED 22 MARCH 2020

1. Continuous functions

We consider functions of the kind $f : X \to \mathbb{R}$, where the domain X of f is a subset of $\mathbb{R}$. As X we will choose an interval $(a, b) := \{x \in \mathbb{R} |\ a < x < b\}$, where a or b are allowed to be ∞ or $-\infty$ (in particular, $(-\infty, \infty)$ coincides with $\mathbb{R}$). We are going to identify among all functions the "nice" ones, called *continuous*, and among continuous functions – even "nicer" ones, *differentiable* (or *smooth*).

Definition 1.1 (of continuous function)**.** A function $f : (a, b) \to \mathbb{R}$ is called *continuous* at a point $c \in (a, b)$ if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for every $x \in (a, b)$ if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. If a function is continuous at every point of the domain, it is called *continuous* in this domain (without reference to a point).

Example 1.2. (1) *Heaviside function*

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

is *not* continuous at the origin, but is continuous at every other point of $\mathbb{R}$.

(2) The function $f(x) = |x|$ is continuous (i.e., continuous at every point). After we introduce an appropriate definition we will see that $f(x)$ is not smooth at the origin.

(3) The function $f(x) = x^2$ is continuous and smooth at every point.

To see an example of how the definition works, let us prove that $f(x) = |x|$ is continuous at 0. Fix an arbitrary real $\varepsilon > 0$. Choose any positive $\delta \leq \varepsilon$. Then for any x, such that $|x| < \delta$, we have $|f(x)| = |x| < \delta \leq \varepsilon$, which means, according to Definition 1.1 that $f(x)$ is continuous.

Here is an alternative and equivalent definition of continuity. Recall the definition of a converging (Cauchy) sequence (e.g., from CM10196).

Definition 1.3 (of continuous function, alternative)**.** A function $f : (a, b) \to \mathbb{R}$ is continuous at $c \in (a, b)$ if whenever a sequence $\{x_i\}_{1 \leq i < \infty}$, where $x_i \in (a, b)$, converges to c, the sequence $\{f(x_i)\}_{1 \leq i < \infty}$ converges to $f(c)$.

Let us give yet another definition of a continuous function in the interval. First we define the concept of an *open set* in an interval (a, b).

Definition 1.4. A subset $X \subset (a, b)$ is called *open* in (a, b) if for every $c \in X$ there is an interval (a', b') such that $(a', b') \subset X$ and $c \in (a', b')$. Loosely speaking, every point in X can be surrounded by a smaller interval contained in X.

Example 1.5. (1) A union of any two (or any finite number of) intervals in $\mathbb{R}$ is an open set.
(2) The set $\{x \in \mathbb{R} | \, r < x \leq s\}$ is not an open set in $\mathbb{R}$ (choose in Definition 1.4 $(a, b) = \mathbb{R}$ and $c = s$).

Now we define continuous functions differently.

Definition 1.6. [of continuous function, another alternative] A function

$$f : (a, b) \to \mathbb{R}$$

is continuous in (a, b) if for every open subset in $\mathbb{R}$ its pre-image under f is an open set in (a, b). (Here by *pre-image of a set under* f we mean the set of all existing pre-images under f of elements of this set.)

Example 1.7. The Heaviside function is not continuous in $\mathbb{R}$ because the pre-image of the interval $(-1/2, 1/2)$ is the set $\{x \in \mathbb{R} | \, -\infty < x \leq 1\}$ which is not open in $\mathbb{R}$ by Example 1.5 (2).

All the above definitions of a continuous function are equivalent. Definition 1.6 uses, apart from the general, set-theoretic, concept of a function, only the notion of an open set. It makes this definition particularly useful for generalizations of continuous functions from functions on $\mathbb{R}$ to functions on more general sets.

Here is one of the most useful properties of continuous functions.

Theorem 1.8 (Intermediate Value Theorem)**.** *Let $f : (a, b) \to \mathbb{R}$ be a continuous function, $(a', b') \subset (a, b)$ be an interval contained in the domain (a, b), with $f(a') < f(b')$. Then for every $t \in (f(a'), f(b'))$ there is $c \in (a', b')$ such that $f(c) = t$.*

Corollary 1.9. *If under conditions of the theorem, $f(a')$ and $f(b')$ have different signs (i.e., one is positive while another is negative, or vice versa), then there exists $c \in (a', b')$ such that $f(c) = 0$.*

Example 1.10. Two cars moving along a stretch of the road from the opposite ends, and each covering the whole stretch, will meet at some point.

2. Differentiable functions

We need the following preliminary definition.

Definition 2.1 (of a limit of a function)**.** Let $f : (a, b) \to \mathbb{R}$ be a function and let $c \in (a, b)$. We say that a real ℓ is the *limit of f as x tends to c* if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for every $x \in (a, b)$, if $0 < |x - c| < \delta$ then $|f(x) - \ell| < \varepsilon$.

We denote this as $f(x) \to_{x \to c} \ell$, or as $\lim_{x \to c} f(x) = \ell$.

Example 2.2. Heaviside function does not have a limit as x tends to 0.

Remark 2.3. Think about the condition $0 < |x - c|$ in this definition. How the concept of the limit will change if we remove it?

The following theorem gives yet another equivalent definition of a continuous function.

Theorem 2.4. *A function $f : (a, b) \to \mathbb{R}$ is continuous at a point $c \in (a, b)$ if and only if $f(c) = \lim_{x \to c} f(x)$.*

Now we are able to formulate a fundamental definition of the *derivative*.

Definition 2.5 (of a derivative)**.** Consider a function $f : (a, b) \to \mathbb{R}$, and let $c \in (a, b)$. The *derivative* of f at c is the limit

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If the limit (i.e, the derivative) exists then f is said to be *differentiable* or *smooth* at c.

An alternative notation for a derivative of f at c is

$$\frac{df}{dx}(c).$$

The fraction $\frac{df}{dx}$ should be understood as a whole symbol.

Remark 2.6.
(1) One can prove that if a function is differentiable at a point in the domain, then it is also continuous at that point. The converse is not true, as the example of $f(x) = |x|$ shows.
(2) Geometrically, a function f is differentiable (smooth) at c if its graph has a tangent straight line at the point $(c, f(c))$. In this case the derivative equals to $\tan \alpha$, where α is the angle the tangent line makes with the horizontal axis.

If f is differentiable at *every* point in (a, b). i.e., $f'(c)$ exists for every point $c \in (a, b)$, then we have a new function

$$f' : (a, b) \to \mathbb{R}$$
$$c \mapsto f'(c).$$

This function is called the *derivative* of f, and f is said to be *differentiable* (without reference to a point).

Directly form Definition 2.5 one can obtain the following formulae for derivatives of standard elementary functions.

- $(\text{const})' = 0$;
- $(x^\mu)' = \mu x^{\mu-1}$, where μ is an arbitrary real number;
- $(e^x)' = e^x$, where $e = 2.71828\ldots$ is a famous transcendental constant;
- $(a^x)' = a^x \ln a$, where $\ln = \log_e$ is the logarithm to the base e;
- $(\ln x)' = 1/x$;
- $(\sin x)' = \cos x$;
- $(\cos x)' = -\sin x$;
- $(\tan x)' = 1 + \tan^2 x$.

The following rules are useful when we differentiate a complex function on (a, b).

(1) $(\alpha f)' = \alpha f'$, where α is any real number;
(2) $(f + g)' = f' + g'$;
(3) $(fg)' = f'g + g'f$;
(4)
$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$
for $x \in (a, b)$ such that $g(x) \neq 0$;
(5) "Chain rule": if $f(y)$ and $g(x)$ are two functions, then $(f(g(x))' = f'(g(x))g'(x)$.

3. Higher order derivatives

Let a function f be defined and differentiable on an interval (a, b). If we consider the derivative f' as a function on (a, b), then it is not necessarily a differentiable function or even a continuous function.

Example 3.1. Let

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x \leq 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}.$$

The function f is differentiable. However its derivative

$$f'(x) = \begin{cases} -2x & \text{if} \quad x \leq 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$$

(equivalently: $f'(x) = |2x|$) is not differentiable at 0.

Suppose that f' is nevertheless differentiable. Then we can consider $f'' := (f')'$, called *second derivative*. If f'' is differentiable, we can consider $f''' := (f'')'$, called *third derivative*, etc. Using another notations for a derivative, we get the following sequence:

$$f,\ \frac{df}{dx},\ \frac{d^2f}{dx^2},\ \frac{d^3f}{dx^3}, \dots$$

Some functions are differentiable infinitely many times, for example,

$$x^3,\ 3x^2,\ 6x,\ 6,\ 0, \dots, 0, \dots$$

or

$$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \dots$$

4. Partial derivatives

Consider a function $f(x, y)$ in two variables, defined on a rectangle

$$(a, b) \times (c, d) \subset \mathbb{R}^2.$$

Its graph is a surface in $\mathbb{R}^3$. If we fix one variable, say, set $y = \alpha$ for some $\alpha \in (c, d)$, then f turns into a function only in one variable x, namely $f(x, \alpha)$, defined on (a, b). The function $f(x, \alpha)$ may be differentiable, with the derivative $\frac{df(x,\alpha)}{dx}$. If *for every* fixed $y \in (c, d)$ the function $f(x, y)$ (as a function in one variable x) is differentiable, then we get a function

$$(a, b) \times (c, d) \to \mathbb{R}$$
$$(x, y) \mapsto \frac{df(x, y)}{dx}.$$

It is called *partial derivative of f in x*, and is denoted by

$$\frac{\partial f}{\partial x}.$$

(Note the difference in the notation d/dx for a function in one variable x and $\partial/\partial x$ for a function in two variables x, y.) In a similar way we can define *partial derivative of f in y*, denoted by

$$\frac{\partial f}{\partial y}.$$

Here is a formal definition (you may assume $n = 2$ in this definition, if it is easier to understand).

Definition 4.1. Consider a function in n variables

$$f:\ (a_1,b_1)\times\cdots\times(a_n,b_n)\to\mathbb{R}$$

defined on an n-dimensional set $(a_1,b_1)\times\cdots\times(a_n,b_n)$ in $\mathbb{R}^n$. Let $(\alpha_1,\ldots,\alpha_n)$ be a point in this set. The function f is said to have the *partial derivative in x_i at* $(\alpha_1,\ldots,\alpha_n)$ if there exists the limit

$$\frac{\partial f}{\partial x_i}(\alpha_1,\ldots,\alpha_n):=\lim_{x_i\to\alpha_i}\frac{f(\alpha_1,\ldots,\alpha_{i-1},x_i,\alpha_{i+1},\ldots,\alpha_n)-f(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)}{x_i-\alpha_i}.$$

If f has a partial derivative in x_i at every point in $(a_1,b_1)\times\cdots\times(a_n,b_n)$, we get a function on this set which is denoted by

$$\frac{\partial f}{\partial x_i}.$$

The operation of taking partial derivative may be repeated for a function in many variables. Thus, one considers partial derivatives of higher orders

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial x\partial y},\ \frac{\partial^3 f}{\partial x\partial y^2},$$

and so on.

Theorem 4.2. *If for a function $f:\ (a,b)\times(c,d)\to\mathbb{R}$ in two variables x, y, both mixed second order partial derivatives*

$$\frac{\partial^2 f}{\partial x\partial y},\quad \frac{\partial^2 f}{\partial y\partial x}$$

exist and are continuous functions, then it is the same function:

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}.$$

We conclude our discussion of partial derivatives by a generalization of the "Chain Rule". Suppose we have a function $f(y_1,\ldots,y_n)$ in n variables, and n functions $g_1(x),\ldots,g_n(x)$ in one variable. Then the composition $f(g_1(x),\ldots,g_n(x))$ is a function in one variable. The general Chain Rule:

$$\frac{df}{dx}=\frac{\partial f}{\partial g_1}\frac{dg_1}{dx}+\cdots+\frac{\partial f}{\partial g_n}\frac{dg_n}{dx}.$$

ANALYTICAL MATHEMATICS FOR APPLICATIONS 2020
LECTURE NOTES 8

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1. Series

A *series* is a formal sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Here "formal" means that we don't necessarily identify this expression with a particular numerical value, a "sum". We will see shortly that sometimes such identification makes sense, in other cases – not. A series is often abbreviated to

$$\sum_{n=1}^{\infty} a_n.$$

It is not straightforward to define the sum of a series. Naive approaches may lead to contradictions. Consider, for example, the series

$$1 + (-1) + 1 + (-1) + \cdots .$$

Grouping the terms in pairs 1–2, 3–4, 5–6, etc., we appear to get the sum zero for the series. On the other hand, grouping the terms 2–3, 4–5, 6–7, etc., we leave the first term outside groups, so the sum appears to be equal to 1.

The standard definition of a sum is the result of the following two definitions.

Definition 1.1. The nth *partial sum* of the series $\sum_{n=1}^{\infty} a_n$ is the sum of the first n terms,

$$S_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

Definition 1.2. The series *converges* if the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums converges (equivalently, is a Cauchy sequence). The limit of this sequence, $\lim_{n\to\infty} S_n$, is a real number and is called the *sum* of the series.

Example 1.3 (Geometric series)**.** Consider

$$\sum_{n=1}^{\infty} k^{n-1} = 1 + k + k^2 + k^3 + \cdots$$

where $k > 0$ and $k \neq 1$. By a well known formula (sum of geometric progression),

$$S_n = \frac{1 - k^n}{1 - k}.$$

If $k < 1$, then $\lim_{n\to\infty} k^n = 0$ hence the limit of the numerator is 1, while the denominator is constant. Thus the sum of the series exists and is equal to

$$\lim_{n\to\infty} S_n = \frac{1}{1-k}.$$

If $k > 1$, then the numerator in S_n tends to $-\infty$, while the denominator is still constant. Hence the sequence $\{S_n\}$ tends to infinity, and the series diverges.

Example 1.4 (Harmonic series)**.** This is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Intuitively it is not at all clear whether it converges or not. Here is a medieval proof of the fact that harmonic series actually diverges.

In the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

replace the part

$$\frac{1}{3} + \frac{1}{4}$$

by a smaller sum

$$\frac{1}{4} + \frac{1}{4},$$

the part

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

by a smaller sum

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

and so on. As a result, we get a series, whose partial sums, starting from the third, are smaller than corresponding partial sums of the harmonic series. The new series is the same as

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

and diverges, since its partial sums tends to infinity as $n \to \infty$. Hence the partial sums of the "larger" harmonic series tend to infinity as $n \to \infty$. Thus, the harmonic series diverges.

Example 1.5 (Series of inverse squares)**.** Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a difficult one. Euler[1] calculated that the sum of this series is $\pi^2/6$.

Example 1.6. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

converges for all integer $k \geq 2$. It is easy to prove, knowing that the series on inverse squares converges, because the partial sums of the latter a greater than the corresponding partial sums of the given series.

[1]Leonhard Euler (1707–1783) was a great Swiss mathematician. Worked mostly in Russian Imperial Academy of Sciences.

Example 1.7. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges to $\ln 2$. The proof is again non-trivial. In the further sections we will show how to prove such and similar statements in general. We take a function ($\ln x$ in this case) and try to represent it as a series, called Taylor series.

Note that this series resembles the harmonic series but the signs alternate in terms. This is called *alternating harmonic series*.

2. Operations on series

We are not proving theorems listed in this section. The proofs can be found in standard textbooks, and will *not* be examined.

There is a following relation between convergences of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$.

Theorem 2.1. *If the series*

$$\sum_{n=1}^{\infty} a_n$$

converges, then the sequence $\{a_n\}_{n=1}^{\infty}$ *converges and* $\lim_{n\to\infty} a_n = 0$.

The converse statement to this theorem is not true, for example, the harmonic series diverges while $\lim_{n\to\infty} a_n = 0$.

The converging series behave like vectors with respect to addition and multiplication by a constant.

Theorem 2.2. *If*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n$$

converge with sums a and b respectively, then for any $\alpha, \beta \in \mathbb{R}$ the series

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n))$$

converges with the sum $\alpha a + \beta b$).

The following statement is a convenient tool for proving convergence, we already used it informally in Section 1.

Theorem 2.3. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences such that* $0 \le a_n \le b_n$ *for each n, then if*

$$\sum_{n=1}^{\infty} b_n$$

converges, then

$$\sum_{n=1}^{\infty} a_n$$

converges.

3. Absolute convergence

Definition 3.1. A series

$$\sum_{n=1}^{\infty} a_n$$

is *absolutely converging* if the series of absolute values

$$\sum_{n=1}^{\infty} |a_n|$$

is converging.

One can prove that absolutely converging series converge (in the usual sense). The example of the *alternating harmonic series* shows that the converse is not true.

Example 3.2. The series[2]

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

converges absolutely for all $x \in \mathbb{R}$. This is because for any $x \in \mathbb{R}$ there exists a sufficiently large n such that

$$0 \leq \left| \frac{x^n}{n!} \right| < \frac{1}{n^2},$$

and now we can apply Theorem 2.3 to series

$$\sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right| \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^2}.$$

The sum of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we denote by e^x. Why this notation makes sense we will see in the next section.

4. Taylor series

Let a function $f : (a, b) \to \mathbb{R}$ be infinitely many times differentiable in (a, b), i.e., there exist derivatives

$$f'(x),\ f''(x), \ldots, f^{(n)}(x), \ldots$$

at every point $x \in (a, b)$.

Fix an arbitrary $x_0 \in (a, b)$ and consider the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

It is called *Taylor series of f at x_0*. In case $x_0 = 0$ the series is called *Maclaurin series*.

It might happen that for each $x \in (a, b)$ Taylor series of f at x_0 converges and the sum is $f(x)$. Functions for which this takes place are called *analytic*. There is a theorem that if a series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

[2]In the formula, $n! = 1 \cdot 2 \cdots n$, and we assume, as it is customary, that $0! = 1$.

converges to $f(x)$ at each $x \in (a,b)$, then it is a Taylor series of f at x_0, i.e., $c_n = \frac{f^{(n)}(x_0)}{n!}$ for each n. Note that there are infinitely differentiable functions on (a,b) such that their Taylor series converge at every $x \in (a,b)$ but not to the function f (examples are not straighforward).

Taylor series is a powerful instrument for representing an analytic function as a series, since we know that if such series at all exists, it has to be the Taylor series.

Example 4.1. The function e^x is analytic in $(-\infty,\infty) = \mathbb{R}$. Indeed, its Taylor series at $x_0 = 0$ can be explicitly computed:

$$f(0) = e^0 = 1,\ f''(0) = e^0 = 1, \dots$$

and substituting in the general formula we get

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

This series converges at every $x \in \mathbb{R}$ (see Example 3.2). One can also prove (we skip this) that the sum of this Taylor series is e^x

Alternatively, the sum of the series may be taken as a definition of the function e^x. In this case, taking $x = 1$ we get a definition of the number e as the sum of the series

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

Example 4.2. The function $\sin x$ is also analytic on $\mathbb{R}$. Indeed,

$$\sin 0 = 0,\ \sin' 0 = \cos 0 = 1,\ \sin'' 0 = -\sin 0 = 0,$$
$$\sin''' 0 = -\cos 0 = -1,\ \sin^{(4)} 0 = \sin 0 = 0.$$

We see that in the sequence of higher derivatives at 0 the values repeat periodically. It follows that the Taylor series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The sum of this series is indeed $\sin x$ on $\mathbb{R}$.

Example 4.3. Consider the function

$$f(x) = \frac{1}{1-x}$$

with $x \neq 1$.

We have:

$$f'(0) = \frac{1}{(1-x)^2}(0) = 1, \quad f''(0) = \frac{2}{(1-x)^3}(0) = 2,$$
$$f'''(0) = \frac{2\cdot 3}{(1-x)^4}(0) = 3!, \quad f^{(4)} = \frac{2\cdot 3\cdot 4}{(1-x)^5}(0) = 4!, \dots$$

It is easy to see that $f^{(n)}(0) = n!$

It follows that the Taylor series for $f(x)$ at $x_0 = 0$ is the *geometric series*, (see Example 1.3):

$$\sum_{n=0}^{\infty} x^n,$$

but this series does not converge for all $x \in \mathbb{R}$. It converges for all $x \in \mathbb{R}$ such that $|x| < 1$ and, as we have shown, the sum is indeed $\frac{1}{1-x}$. It follows that the function $f(x)$ is analytic and representable by the Taylor series on the interval $(-1,1) \subset \mathbb{R}$.

Example 4.4. It is straightforward to calculate that the function $\ln(1+x)$ has a Taylor series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

at $x_0 = 0$. The function is analytic and the series converges for all $x \in \mathbb{R}$ such that $-1 < x \leq 1$. In particular, for $x = 1$ we recover the series from Example 1.7.