Problem sheet 1

Issued 10 October 2022, due 31 October 2022.

Coursework forms 25% of the assessment for this unit and will be comprised of 2 problem sheets of which this is the first. In this sheet each of five problems carries the same weight.

1. For each pair of Boolean formulae P and Q state whether or not the formula $P \to Q$ is a tautology, or identically false, or neither. Prove your answer.
(i) $P = X \vee Y$, $Q = \neg(X \wedge Y)$,
(ii) $P = X \vee Y$, $Q = \neg X \wedge \neg Y$,
(iii) $P = X \to Y$, $Q = (\neg X \vee Y) \wedge (\neg X \vee X)$,
(iv) $P = X \to \neg Y$, $Q = Y \to \neg X$,
(v) $P = X \wedge (Y \vee Z)$, $Q = (X \vee Y) \wedge (X \vee Z)$,
(vi) $P = X \to Y$, $Q = \neg X \to \neg Y$,
(vii) $P = X \to Y$, $Q = \neg(Y \to X)$,
(viii) $P = (X \to Y) \wedge (Y \to Z)$, $Q = X \to Z$.

2.
(a) How many different binary logical connectives do there exist?
(b) Express the "exclusive or" (XOR) via (i) $\neg$, $\wedge$, (ii) $\neg$, $\vee$, (iii) $\neg$, $\to$.
(c) The "not or" (NOR) logical connective is defined as:

$$X \text{ NOR } Y \equiv \neg(X \vee Y).$$

Express each of connectives $\wedge$, $\vee$, $\to$, $\neg$ via only NOR.

3.

(a) For each of the following statements answer whether it is true or false.
(i) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z}(x^2 < y + 1)$

(ii) $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z}(x^2 < y + 1)$

(iii) $\exists y \in \mathbf{Z} \forall x \in \mathbf{Z}(x^2 < y + 1)$

(iv) $\forall x \in \mathbf{Z} \exists y \in \mathbf{Z}((x < y) \to (x^2 < y^2))$

(b)

(i) Prove that the statement $\forall x \in \mathbf{Z} \forall y \in \mathbf{Z}(x^2 < y + 1)$ is false by giving an example of integers x and y disproving it.

(ii) Prove that the statement $\exists x \in \mathbf{Z} \exists y \in \mathbf{Z}(x^2 < y + 1)$ is true by giving an example of integers x and y proving it.

(iii) Prove that the statement $\forall y \in \mathbf{Z} \exists x \in \mathbf{Z}(x^2 < y+1)$ is false by giving an example of an integer y disproving it.

(iv) Prove that the statement $\exists x \in \mathbf{Z} \forall y \in \mathbf{Z}((x < y) \rightarrow (x^2 < y^2))$ is true by giving an example of an integer x proving it.

4. Let $A = \{a, b, c\}$, $B = \{a, b\}$, $C = \{a, c\}$, $D = \{b, c\}$, $E = \{a\}$, $F = \{b\}$, $G = \{c\}$, $H = \emptyset$. Simplify the following expressions; in each case answer should be one of these sets.
(i) $B \cup C \cup D$
(ii) $B \cup C$
(iii) $B \cap C \cap D$
(iv) $G \setminus B$
(v) $D \setminus B$
(vi) $B \setminus A$
(vii) $(D \setminus G) \cup (G \setminus D)$
(viii) $(B \setminus C) \setminus A$

5.
(a) Determine the cardinalities of the following sets:

$$2^{\emptyset},\ 2^{\{0\}},\ 2^{\{0\}\cup\{1\}},\ 2^{\{0\}\cap\{1\}},\ 2^{\{\emptyset,0,1\}},\ 2^{2^{2^{\{0,1\}}}}.$$

(b) Let $A = \{1, 2, \ldots, n\}$. Determine the cardinalities of the following sets:

(i) $\{(x, S) | x \in S,\ S \in 2^A\}$

(ii) $\{(S, T) | S \in 2^A,\ T \in 2^A,\ S \cap T = \emptyset\}$.
Hint: Calculate the number of triples of pair-wise disjoint sets $(S, T, A \setminus (S \cup T))$.