TP6-2020-correction

April 5, 2020

1 TP6 - numpy, sympy, scipy

1.1 Une correction

1.2 Exercice 1 -- Approximation d'intégrales

On veut comparer des calculs d'intégrales "à la main" grâce à numpy et les résultats que donne sympy.

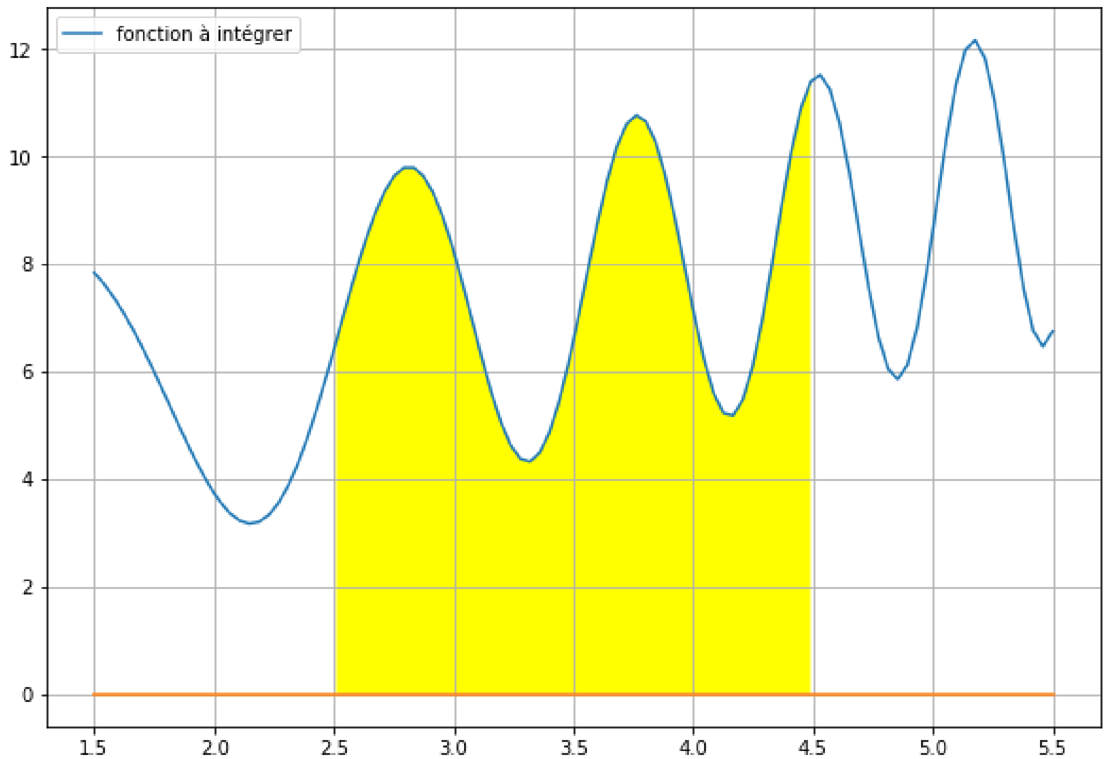
Dans un premier temps, on considère la courbe d'équation $y = f(x)$ où $f(x) = 3sin(x^2) + x + 4$

Faites le graphique ci-dessous (avec les légendes) et en coloriant l'aire situé entre la parabole et l'axe des x et pour x compris entre 2.5 et 4.5.

Cette aire est égale à L'intégrale de f entre 2.5 et 4.5.

Remarque: pour colorier entre deux courbes, utilisez la fonction fill_between (voir docs).

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

# on définit une fonction python "de base"

def fonc(x):
    return 3*np.sin(x**2)+x+4


# et un échantillon de points suffisant

x1=np.linspace(1.5,5.5,100)
z1=np.zeros_like(x1)
f,ax = plt.subplots(1,1,figsize=(10,7))
ax.plot(x1,fonc(x1),label='fonction à intégrer')
ax.plot(x1,z1)

# attention syntaxe un peu particulière!
ax.fill_between(x1, fonc(x1), z1, facecolor='yellow',where=(2.5<=x1) &(x1<=4.5))
ax.grid()
ax.legend()
print()
```



1.2.1 Avec numpy

On veut faire un calcul approché de cette aire entre a et b avec n intervalles (méthode des rectangles).

On prendra n sous-intervalles de [a,b] tous égaux tels que $(x_{i+1} - x_i) = \frac{b-a}{n}$.

On utilisera les tableaux numpy en calculant les sommes suivantes qui chacune représente une approximation de la valeur de l'aire coloriée.

$S_{a,b}(n) = \sum(x_{i+1} - x_i)f(x_i)$

$T_{(a,b)}(n) = \sum(x_{i+1} - x_i)f(x_{i+1})$

$U_{(a,b)}(n) = \sum x_{i+1} - x_i)f(m_i)$ où m_i est le milieu de x_i et x_{i+1}

Ecrire ces trois fonctions et faire une application numérique avec 10 sous-intervalles puis avec 100.

```
In [2]: def calcul1(a,b,n):
            x1=np.linspace(a,b,n+1)   # si on veut n intervalles il faut n+1 points
            return((b-a)/n)* np.sum(fonc(x1[:-1]))

        def calcul2(a,b,n):
            x1=np.linspace(a,b,n+1)
            return((b-a)/n)* np.sum(fonc(x1[1:]))
```

```
def calcul3(a,b,n):
    x1=np.linspace(a,b,n+1)
    xm=(x1[:-1]+x1[1:])/2
    return((b-a)/n)* np.sum(fonc(xm))

p=10
print("pour n =", p, "on trouve ")
print( calcul1(2.5,4.5,p), calcul2(2.5,4.5,p),calcul3(2.5,4.5,p))

p=100
print("pour n =", p, "on trouve ")
print( calcul1(2.5,4.5,p), calcul2(2.5,4.5,p),calcul3(2.5,4.5,p))
```

```
pour n = 10 on trouve
14.984155715970392 15.995378312837998 15.540863345886587
pour n = 100 on trouve
15.473061393466699 15.57418365315346 15.52414317917968
```

1.2.2 Avec sympy

Utilisez sympy pour calculer l'aire situé entre la courbe et l'axe des x pour x variant entre a et b. Puis faire l'application numérique pour a=2.5 et b=4.5

```
In [3]: # pour le alcul de l'intégrale, il faut définir une expression sympy

        import  sympy
        x,a,b = sympy.symbols('x,a,b')

        f_s = 3*sympy.sin(x**2)+x+4

        val=sympy.integrate(f_s,(x,a,b))  # intégrale entre a et b

        print("valeur de l'intégrale" ,  val.subs({a:2.5,b:4.5}).evalf())
```

```
valeur de l'intégrale 15.5239696360776
```

1.3 Exercice 2

On travaille avec deux tableaux : le premier appelé ref comporte 100 valeurs entre -1 et 2 (non réparties régulièrement).

Le second tableau se compose de 20 séries de 100 valeurs qui correspondent 20 séries de mesures correspondants aux valeurs de ref.

```
In [4]: import numpy as np
        import matplotlib.pyplot as plt
```

```
ref =np.array([-0.99      , -0.96769697, -0.93139394, -0.90109091, -0.87278788,
       -0.82848485, -0.79818182, -0.77987879, -0.73757576, -0.72127273,
       -0.6949697 , -0.65666667, -0.63636364, -0.59206061, -0.56575758,
       -0.53745455, -0.49715152, -0.46884848, -0.44654545, -0.41424242,
       -0.38193939, -0.36163636, -0.31333333, -0.2870303 , -0.26472727,
       -0.22442424, -0.20212121, -0.17781818, -0.13151515, -0.10921212,
       -0.08290909, -0.04260606, -0.01830303,  0.        ,  0.03630303,
        0.06260606,  0.10490909,  0.13121212,  0.16551515,  0.19581818,
        0.22012121,  0.25242424,  0.29072727,  0.3090303 ,  0.33533333,
        0.37163636,  0.40593939,  0.44424242,  0.46054545,  0.50084848,
        0.53315152,  0.56145455,  0.58575758,  0.60606061,  0.65636364,
        0.68066667,  0.7149697 ,  0.73727273,  0.76557576,  0.78987879,
        0.83218182,  0.86848485,  0.88678788,  0.91709091,  0.94339394,
        0.97969697,  1.016     ,  1.04630303,  1.06460606,  1.09890909,
        1.12921212,  1.15551515,  1.19781818,  1.22412121,  1.25442424,
        1.28872727,  1.3050303 ,  1.34533333,  1.37163636,  1.40793939,
        1.43824242,  1.45654545,  1.48484848,  1.52715152,  1.55345455,
        1.58375758,  1.61606061,  1.63636364,  1.67666667,  1.7109697 ,
        1.73527273,  1.76757576,  1.80787879,  1.83018182,  1.86648485,
        1.88278788,  1.90909091,  1.94139394,  1.98969697,  2.        ])
```

```
tab20=np.array([np.array([-7.03952400e-01, -7.59255455e-01, -6.33995770e-01, -5.97061662
        -5.87879628e-01, -4.40772662e-01, -5.26606616e-01, -3.61125107e-01,
        -4.13878881e-01, -3.01707792e-01, -3.11805248e-01, -2.79605866e-01,
        -2.81943651e-01, -8.63966766e-02, -5.88614656e-02, -1.70516343e-01,
        -1.38804843e-01,  7.28446756e-04, -8.05637102e-02,  1.86002161e-02,
         3.20420028e-02, -5.86581347e-03,  4.42962963e-02, -2.27773405e-02,
         7.15504409e-02,  1.84355960e-01,  1.23020341e-01,  8.69589809e-02,
         1.05652726e-01,  2.01144210e-01,  1.73756574e-01,  1.74419516e-01,
         1.46750238e-02,  1.60605070e-01,  1.60101651e-01,  2.24254792e-02,
         4.87875903e-02,  9.08028828e-02,  1.69424591e-01,  1.41358275e-02,
         2.09553869e-02,  8.73264229e-02, -2.66059393e-02,  1.04881081e-01,
         5.74944267e-02, -2.68178385e-02, -1.03740630e-01, -8.52445784e-02,
        -3.01830303e-02, -1.34898026e-01, -3.57791134e-02, -1.18061615e-02,
        -7.02789028e-02, -3.33137950e-02, -6.21813534e-02, -1.64476260e-01,
        -9.36393813e-02, -1.16558978e-02, -1.03010133e-01, -1.36626900e-01,
        -1.95846985e-01, -2.38181829e-02, -1.60763953e-01, -9.48404477e-02,
        -2.87605533e-02, -1.58902620e-01, -9.00000000e-03, -2.98078553e-03,
        -9.83806436e-02, -1.05974021e-01, -8.71191099e-02, -1.08359991e-02,
        -3.61656863e-02,  7.84456423e-02,  9.44444578e-02,  6.35729577e-02,
         9.59194457e-02,  3.71841689e-02,  9.30505812e-02,  1.08592954e-01,
         2.54506612e-01,  3.12721476e-01,  3.40584167e-01,  3.72684987e-01,
         4.34877223e-01,  4.55786908e-01,  3.64112391e-01,  5.20586026e-01,
```

```
         5.78245102e-01,  7.34118720e-01,  7.58707850e-01,  8.36309393e-01,
         8.65236988e-01,  1.07963940e+00,  1.12125821e+00,  1.07394596e+00,
         1.27055124e+00,  1.43512360e+00,  1.31800552e+00,  1.72544228e+00]),
np.array([-0.8239524 , -0.87925546, -0.68399577, -0.70706166, -0.53787963,
        -0.45077266, -0.55660662, -0.35112511, -0.33387888, -0.29170779,
        -0.23180525, -0.24960587, -0.20194365, -0.22639668, -0.16886147,
        -0.11051634, -0.11880484, -0.06927155, -0.09056371, -0.09139978,
        -0.077958  ,  0.01413419,  0.0042963 ,  0.07722266,  0.17155044,
         0.08435596,  0.02302034,  0.05695898,  0.08565273,  0.11114421,
         0.16375657,  0.01441952,  0.15467502,  0.04060507,  0.18010165,
         0.17242548,  0.18878759,  0.09080288,  0.00942459,  0.10413583,
         0.12095539,  0.13732642,  0.09339406, -0.00511892,  0.05749443,
        -0.02681784,  0.02625937, -0.07524458, -0.09018303, -0.04489803,
        -0.06577911, -0.04180616, -0.1002789 , -0.0433138 , -0.03218135,
        -0.09447626, -0.07363938, -0.1716559 , -0.17301013, -0.1366269 ,
        -0.02584698, -0.07381818, -0.12076395, -0.05484045, -0.16876055,
        -0.09890262, -0.129     , -0.01298079, -0.07838064, -0.15597402,
         0.02288089, -0.110836  , -0.11616569, -0.08155436,  0.04444446,
         0.04357296,  0.02591945,  0.15718417,  0.05305058,  0.22859295,
         0.29450661,  0.23272148,  0.28058417,  0.39268499,  0.37487722,
         0.43578691,  0.46411239,  0.60058603,  0.6682451 ,  0.73411872,
         0.73870785,  0.82630939,  0.87523699,  0.9696394 ,  1.10125821,
         1.09394596,  1.22055124,  1.3351236 ,  1.36800552,  1.75544228]),
np.array([-7.13952400e-01, -8.49255455e-01, -5.73995770e-01, -7.67061662e-01,
        -4.77879628e-01, -5.60772662e-01, -5.16606616e-01, -4.01125107e-01,
        -2.83878881e-01, -3.81707792e-01, -3.31805248e-01, -2.19605866e-01,
        -3.01943651e-01, -1.26396677e-01, -3.88614656e-02, -5.16343381e-04,
        -1.88048426e-02, -8.92715532e-02, -1.05637102e-02,  5.86002161e-02,
        -6.79579972e-02, -5.58658135e-02,  1.42962963e-02, -2.77734048e-03,
         8.15504409e-02,  2.43559596e-02,  1.93020341e-01,  5.69589809e-02,
         1.25652726e-01,  4.11442096e-02,  1.37565740e-02,  1.84419516e-01,
         4.67502381e-03, -9.39492976e-03,  5.01016511e-02,  1.32425479e-01,
        -1.21240968e-03,  6.08028828e-02, -2.05754092e-02,  1.44135828e-01,
         1.50955387e-01,  4.73264229e-02,  3.33940607e-02, -5.11891885e-03,
        -2.25055733e-02,  8.31821615e-02, -3.74063030e-03, -5.24457835e-03,
        -3.01830303e-02, -8.48980259e-02, -8.57791134e-02,  2.81938385e-02,
        -6.02789028e-02, -7.33137950e-02, -7.21813534e-02,  5.52374000e-03,
        -1.53639381e-01, -3.16558978e-02, -1.30101332e-02, -1.26626900e-01,
        -2.05846985e-01, -2.03818183e-01, -3.07639532e-02, -2.14840448e-01,
        -1.88760553e-01, -1.78902620e-01,  1.00000000e-03, -3.29807855e-02,
        -8.38064364e-03, -9.59740211e-02,  2.28808901e-02, -4.08359991e-02,
        -2.61656863e-02,  7.84456423e-02, -5.55555422e-02,  1.35729577e-02,
         9.59194457e-02,  8.71841689e-02,  1.13050581e-01,  1.08592954e-01,
         2.14506612e-01,  1.52721476e-01,  2.80584167e-01,  3.62684987e-01,
         3.64877223e-01,  4.55786908e-01,  4.64112391e-01,  4.60586026e-01,
         4.98245102e-01,  6.34118720e-01,  7.98707850e-01,  9.56309393e-01,
         9.55236988e-01,  1.09963940e+00,  1.10125821e+00,  1.22394596e+00,
         1.19055124e+00,  1.35512360e+00,  1.32800552e+00,  1.63544228e+00]),
```

```
np.array([-7.93952400e-01, -8.19255455e-01, -6.73995770e-01, -6.47061662e-01,
       -5.57879628e-01, -4.90772662e-01, -3.66606616e-01, -5.01125107e-01,
       -3.73878881e-01, -2.41707792e-01, -1.51805248e-01, -2.49605866e-01,
       -2.81943651e-01, -1.46396677e-01, -9.88614656e-02, -1.20516343e-01,
       -8.80484262e-03,  6.07284468e-02,  9.43628976e-03, -5.13997839e-02,
        1.20420028e-02,  1.24134187e-01,  1.42962963e-02, -3.27773405e-02,
        1.41550441e-01,  8.43559596e-02,  9.30203412e-02,  1.26958981e-01,
        1.45652726e-01,  1.61144210e-01,  2.37565740e-02,  3.44195161e-02,
        1.04675024e-01,  1.10605070e-01,  4.01016511e-02,  1.62425479e-01,
        1.88787590e-01,  6.08028828e-02, -5.75409234e-04,  1.44135828e-01,
        1.09553869e-02, -2.26735771e-02,  6.33940607e-02,  6.48810812e-02,
       -4.25055733e-02, -6.81783853e-03,  6.62593697e-02, -4.52445784e-02,
       -8.01830303e-02,  2.51019741e-02, -6.57791134e-02,  1.81938385e-02,
       -5.02789028e-02,  1.66862050e-02, -8.21813534e-02, -2.44762600e-02,
       -9.36393813e-02, -1.51655898e-01, -1.53010133e-01, -5.66268996e-02,
       -1.15846985e-01, -1.73818183e-01, -4.07639532e-02, -1.74840448e-01,
       -1.18760553e-01, -9.89026202e-02, -4.90000000e-02, -1.62980786e-01,
       -9.83806436e-02, -1.65974021e-01, -5.71191099e-02, -1.20835999e-01,
       -6.16568629e-03, -8.15543577e-02, -3.55555422e-02,  3.35729577e-02,
        1.05919446e-01,  1.07184169e-01,  1.43050581e-01,  4.85929543e-02,
        2.14506612e-01,  2.52721476e-01,  2.50584167e-01,  2.82684987e-01,
        4.64877223e-01,  5.25786908e-01,  4.34112391e-01,  5.10586026e-01,
        5.08245102e-01,  6.34118720e-01,  8.88707850e-01,  9.06309393e-01,
        7.95236988e-01,  1.08963940e+00,  1.02125821e+00,  1.13394596e+00,
        1.29055124e+00,  1.39512360e+00,  1.48800552e+00,  1.62544228e+00]),
np.array([-0.8139524 , -0.68925546, -0.66399577, -0.75706166, -0.55787963,
       -0.45077266, -0.39660662, -0.39112511, -0.29387888, -0.31170779,
       -0.21180525, -0.26960587, -0.26194365, -0.18639668, -0.09886147,
       -0.03051634, -0.05880484, -0.00927155, -0.04056371,  0.08860022,
        0.082042  ,  0.03413419,  0.1242963 ,  0.04722266,  0.12155044,
        0.02435596, -0.00697966,  0.13695898,  0.01565273,  0.10114421,
        0.06375657,  0.02441952,  0.11467502,  0.16060507,  0.03010165,
        0.00242548,  0.13878759,  0.08080288,  0.12942459,  0.04413583,
        0.07095539, -0.05267358,  0.10339406, -0.07511892, -0.01250557,
        0.03318216,  0.06625937,  0.07475542, -0.08018303,  0.05510197,
       -0.12577911, -0.06180616, -0.1602789 , -0.0933138 , -0.09218135,
       -0.15447626, -0.08363938, -0.0716559 , -0.13301013, -0.1766269 ,
       -0.20584698, -0.02381818, -0.14076395, -0.20484045, -0.18876055,
       -0.07890262, -0.099     , -0.05298079, -0.08838064, -0.11597402,
       -0.10711911, -0.040836  , -0.02616569,  0.03844564, -0.01555554,
       -0.04642704,  0.00591945,  0.12718417,  0.10305058,  0.08859295,
        0.29450661,  0.24272148,  0.39058417,  0.39268499,  0.42487722,
        0.49578691,  0.38411239,  0.46058603,  0.5582451 ,  0.68411872,
        0.70870785,  0.79630939,  0.95523699,  1.1496394 ,  1.22125821,
        1.26394596,  1.27055124,  1.3751236 ,  1.37800552,  1.73544228]),
np.array([-8.53952400e-01, -8.59255455e-01, -7.13995770e-01, -6.97061662e-01,
       -6.07879628e-01, -4.20772662e-01, -4.26606616e-01, -3.91125107e-01,
       -3.63878881e-01, -4.21707792e-01, -1.91805248e-01, -3.49605866e-01,
```

```
       -1.01943651e-01, -7.63966766e-02, -1.88861466e-01, -5.16343381e-04,
       -8.80484262e-03, -1.09271553e-01,  5.94362898e-02,  8.60021612e-03,
       -7.79579972e-02, -5.86581347e-03, -1.57037037e-02,  2.72226595e-02,
        1.01550441e-01, -1.56440404e-02,  1.93020341e-01,  2.69589809e-02,
        1.75652726e-01,  1.11144210e-01,  6.37565740e-02,  1.84419516e-01,
        1.74675024e-01,  1.30605070e-01,  1.70101651e-01,  1.42425479e-01,
        4.87875903e-02,  1.70802883e-01,  1.69424591e-01,  1.04135828e-01,
       -9.04461312e-03, -6.26735771e-02,  5.33940607e-02,  3.48810812e-02,
        6.74944267e-02, -8.68178385e-02,  6.62593697e-02,  8.47554216e-02,
       -3.01830303e-02,  5.51019741e-02,  2.42208866e-02, -5.18061615e-02,
       -6.02789028e-02, -1.33137950e-02, -7.21813534e-02, -4.44762600e-02,
       -1.43639381e-01, -1.81655898e-01, -1.93010133e-01, -1.76626900e-01,
       -1.05846985e-01, -1.83818183e-01, -1.30763953e-01, -2.48404477e-02,
       -1.18760553e-01, -3.89026202e-02,  1.00000000e-03, -6.29807855e-02,
       -9.83806436e-02, -6.59740211e-02, -5.71191099e-02,  2.91640009e-02,
       -6.61656863e-02, -2.15543577e-02, -4.55555422e-02, -3.64270423e-02,
        1.45919446e-01,  1.71841689e-02,  1.53050581e-01,  3.85929543e-02,
        2.94506612e-01,  2.72721476e-01,  3.50584167e-01,  4.12684987e-01,
        2.74877223e-01,  3.75786908e-01,  4.04112391e-01,  5.60586026e-01,
        6.08245102e-01,  6.44118720e-01,  8.08707850e-01,  8.26309393e-01,
        8.55236988e-01,  9.79639398e-01,  1.08125821e+00,  1.15394596e+00,
        1.25055124e+00,  1.33512360e+00,  1.49800552e+00,  1.56544228e+00]),
np.array([-0.7939524 , -0.83925546, -0.66399577, -0.70706166, -0.60787963,
       -0.60077266, -0.54660662, -0.42112511, -0.31387888, -0.33170779,
       -0.17180525, -0.26960587, -0.12194365, -0.10639668, -0.18886147,
       -0.17051634, -0.09880484,  0.03072845, -0.08056371, -0.05139978,
       -0.077958  ,  0.04413419, -0.0557037 ,  0.01722266,  0.09155044,
        0.12435596,  0.11302034,  0.09695898,  0.01565273,  0.09114421,
        0.09375657,  0.02441952,  0.18467502,  0.02060507,  0.16010165,
        0.07242548,  0.04878759,  0.17080288,  0.14942459,  0.00413583,
        0.12095539,  0.03732642,  0.03339406, -0.07511892, -0.06250557,
       -0.05681784,  0.08625937,  0.02475542, -0.04018303,  0.03510197,
       -0.14577911, -0.02180616,  0.0097211 , -0.0333138 , -0.01218135,
       -0.03447626, -0.02363938, -0.0416559 , -0.06301013, -0.0166269 ,
       -0.03584698, -0.05381818, -0.05076395, -0.07484045, -0.04876055,
       -0.19890262, -0.079     , -0.08298079, -0.01838064, -0.06597402,
       -0.13711911, -0.040836  , -0.04616569, -0.01155436, -0.00555554,
        0.09357296,  0.11591945,  0.17718417,  0.10305058,  0.11859295,
        0.14450661,  0.29272148,  0.26058417,  0.39268499,  0.39487722,
        0.46578691,  0.33411239,  0.54058603,  0.5682451 ,  0.70411872,
        0.77870785,  0.96630939,  0.79523699,  1.0896394 ,  1.10125821,
        1.10394596,  1.34055124,  1.3551236 ,  1.46800552,  1.74544228]),
np.array([-7.23952400e-01, -7.79255455e-01, -6.53995770e-01, -5.97061662e-01,
       -6.47879628e-01, -5.50772662e-01, -5.36606616e-01, -4.31125107e-01,
       -3.53878881e-01, -3.01707792e-01, -2.91805248e-01, -1.49605866e-01,
       -1.31943651e-01, -1.06396677e-01, -1.98861466e-01, -1.00516343e-01,
       -1.88048426e-02, -6.92715532e-02, -1.10563710e-01,  3.86002161e-02,
        2.20420028e-02,  1.24134187e-01,  7.42962963e-02,  1.37222660e-01,
```

```
         1.71550441e-01,  1.34355960e-01,  6.30203412e-02,  6.95898089e-03,
         9.56527256e-02,  1.01144210e-01,  5.37565740e-02,  4.41951608e-03,
         1.04675024e-01,  6.05070240e-04,  1.50101651e-01,  9.24254792e-02,
        -1.21240968e-03,  1.10802883e-01,  9.42459077e-03, -3.58641725e-02,
         4.09553869e-02,  1.73264229e-02,  4.33940607e-02,  6.48810812e-02,
         1.74944267e-02,  6.31821615e-02, -3.74063030e-03, -8.52445784e-02,
        -1.10183030e-01, -4.48980259e-02,  4.22088655e-03, -1.01806162e-01,
         9.72109724e-03, -1.23313795e-01,  7.81864659e-03, -8.44762600e-02,
        -1.13639381e-01, -8.16558978e-02, -1.03010133e-01, -2.06626900e-01,
        -2.15846985e-01, -1.63818183e-01, -1.07639532e-02, -5.48404477e-02,
        -1.88760553e-01, -7.89026202e-02, -1.69000000e-01, -9.29807855e-02,
        -8.83806436e-02, -1.35974021e-01, -4.71191099e-02, -4.08359991e-02,
        -7.61656863e-02,  7.84456423e-02,  1.04444458e-01,  1.35729577e-02,
         5.59194457e-02,  1.17184169e-01,  8.30505812e-02,  1.88592954e-01,
         3.14506612e-01,  2.72721476e-01,  2.00584167e-01,  3.32684987e-01,
         3.54877223e-01,  3.65786908e-01,  4.24112391e-01,  5.00586026e-01,
         6.48245102e-01,  7.04118720e-01,  7.78707850e-01,  9.46309393e-01,
         9.15236988e-01,  1.13963940e+00,  1.09125821e+00,  1.18394596e+00,
         1.18055124e+00,  1.41512360e+00,  1.49800552e+00,  1.62544228e+00]),
np.array([-6.83952400e-01, -8.09255455e-01, -7.13995770e-01, -6.97061662e-01,
        -5.77879628e-01, -5.00772662e-01, -3.86606616e-01, -5.11125107e-01,
        -2.83878881e-01, -3.21707792e-01, -2.51805248e-01, -3.29605866e-01,
        -2.51943651e-01, -2.66396677e-01, -1.28861466e-01, -1.70516343e-01,
         1.19515738e-03, -9.27155324e-03, -6.05637102e-02,  2.86002161e-02,
         1.22042003e-01,  1.34134187e-01,  7.42962963e-02,  4.72226595e-02,
         4.15504409e-02,  2.43559596e-02,  1.30203412e-02,  1.46958981e-01,
         1.35652726e-01,  2.01144210e-01,  1.93756574e-01,  4.44195161e-02,
         1.34675024e-01,  1.60605070e-01,  1.70101651e-01,  8.24254792e-02,
         9.87875903e-02, -9.19711718e-03,  1.59424591e-01, -1.58641725e-02,
         9.55386881e-04, -3.26735771e-02,  3.39406070e-03,  7.48810812e-02,
        -7.25055733e-02,  4.31821615e-02, -5.37406303e-02,  2.47554216e-02,
        -5.01830303e-02, -6.48980259e-02, -1.35779113e-01, -1.80616154e-03,
        -8.02789028e-02, -3.33137950e-02, -6.21813534e-02, -2.44762600e-02,
        -1.33639381e-01, -1.11655898e-01, -3.01013315e-03, -1.66268996e-02,
        -1.35846985e-01, -1.23818183e-01, -1.00763953e-01, -1.24840448e-01,
        -1.88760553e-01, -1.08902620e-01, -2.90000000e-02, -7.29807855e-02,
        -9.83806436e-02, -6.59740211e-02, -4.71191099e-02, -1.00835999e-01,
        -2.61656863e-02, -3.15543577e-02,  5.44444578e-02, -2.64270423e-02,
         1.05919446e-01,  1.47184169e-01,  9.30505812e-02,  8.85929543e-02,
         1.34506612e-01,  1.82721476e-01,  3.50584167e-01,  3.32684987e-01,
         3.94877223e-01,  3.85786908e-01,  5.14112391e-01,  4.20586026e-01,
         4.98245102e-01,  7.04118720e-01,  7.58707850e-01,  8.56309393e-01,
         8.75236988e-01,  1.09963940e+00,  1.10125821e+00,  1.24394596e+00,
         1.35055124e+00,  1.40512360e+00,  1.37800552e+00,  1.56544228e+00]),
np.array([-0.8539524 , -0.77925546, -0.57399577, -0.74706166, -0.55787963,
        -0.50077266, -0.41660662, -0.40112511, -0.36387888, -0.35170779,
        -0.21180525, -0.34960587, -0.13194365, -0.16639668, -0.02886147,
        -0.10051634, -0.07880484, -0.03927155, -0.03056371, -0.06139978,
```

```
        0.122042  ,  0.11413419,  0.1342963 ,  0.14722266,  0.17155044,
        0.12435596,  0.16302034,  0.04695898,  0.17565273,  0.02114421,
        0.14375657,  0.00441952,  0.20467502,  0.08060507,  0.12010165,
        0.13242548,  0.08878759,  0.16080288,  0.12942459,  0.09413583,
        0.05095539,  0.00732642,  0.10339406,  0.11488108, -0.02250557,
        0.00318216,  0.05625937, -0.01524458,  0.04981697, -0.03489803,
       -0.04577911, -0.07180616, -0.0102789 , -0.0733138 , -0.01218135,
       -0.14447626, -0.10363938, -0.0516559 , -0.09301013, -0.0966269 ,
       -0.20584698, -0.10381818, -0.06076395, -0.09484045, -0.16876055,
       -0.11890262, -0.139     , -0.13298079, -0.11838064, -0.08597402,
       -0.03711911, -0.130836  , -0.08616569, -0.10155436,  0.08444446,
        0.08357296,  0.11591945,  0.07718417,  0.03305058,  0.04859295,
        0.16450661,  0.14272148,  0.40058417,  0.39268499,  0.35487722,
        0.51578691,  0.46411239,  0.48058603,  0.5782451 ,  0.66411872,
        0.82870785,  0.89630939,  0.88523699,  1.1496394 ,  1.09125821,
        1.16394596,  1.21055124,  1.4251236 ,  1.48800552,  1.62544228]),
np.array([-8.53952400e-01, -8.49255455e-01, -6.13995770e-01, -6.57061662e-01,
       -5.87879628e-01, -6.00772662e-01, -5.06606616e-01, -3.81125107e-01,
       -2.83878881e-01, -3.21707792e-01, -1.51805248e-01, -1.59605866e-01,
       -2.01943651e-01, -7.63966766e-02, -1.58861466e-01, -8.05163434e-02,
        1.11951574e-02,  2.07284468e-02,  1.94362898e-02, -4.13997839e-02,
        5.20420028e-02,  1.34134187e-01,  1.04296296e-01,  1.27222660e-01,
        1.61550441e-01,  4.35595955e-03,  3.02034115e-03,  4.69589809e-02,
        1.55652726e-01,  1.14420963e-03,  4.37565740e-02,  1.54419516e-01,
        2.46750238e-02,  1.90605070e-01,  1.60101651e-01,  2.42547922e-03,
        1.87875903e-02,  2.08028828e-02,  4.94245908e-02, -2.58641725e-02,
        1.50955387e-01,  3.73264229e-02, -3.66059393e-02, -2.51189188e-02,
        6.74944267e-02, -3.68178385e-02,  5.62593697e-02,  4.47554216e-02,
        2.98169697e-02,  3.51019741e-02, -4.57791134e-02,  2.81938385e-02,
       -1.40278903e-01, -6.33137950e-02, -1.62181353e-01, -1.94476260e-01,
       -1.03639381e-01, -1.11655898e-01, -2.30101332e-02, -5.66268996e-02,
       -1.15846985e-01, -9.38181829e-02, -1.40763953e-01, -1.84840448e-01,
       -3.87605533e-02, -8.90262022e-03, -6.90000000e-02, -1.82980786e-01,
       -5.83806436e-02, -8.59740211e-02, -2.71191099e-02,  4.91640009e-02,
        6.38343137e-02,  3.84456423e-02,  3.44444578e-02, -6.64270423e-02,
        6.59194457e-02,  5.71841689e-02,  9.30505812e-02,  6.85929543e-02,
        2.44506612e-01,  2.02721476e-01,  2.80584167e-01,  3.62684987e-01,
        4.64877223e-01,  5.25786908e-01,  5.14112391e-01,  5.10586026e-01,
        4.88245102e-01,  7.84118720e-01,  8.18707850e-01,  8.96309393e-01,
        9.75236988e-01,  1.01963940e+00,  1.10125821e+00,  1.27394596e+00,
        1.25055124e+00,  1.28512360e+00,  1.49800552e+00,  1.67544228e+00]),
np.array([-7.33952400e-01, -8.39255455e-01, -7.73995770e-01, -6.87061662e-01,
       -5.57879628e-01, -5.50772662e-01, -4.36606616e-01, -4.31125107e-01,
       -3.53878881e-01, -3.51707792e-01, -2.01805248e-01, -1.59605866e-01,
       -1.41943651e-01, -1.16396677e-01, -4.88614656e-02, -1.70516343e-01,
       -7.88048426e-02, -7.92715532e-02,  1.94362898e-02, -9.13997839e-02,
       -1.79579972e-02,  1.34134187e-01, -5.70370370e-03,  1.57222660e-01,
        5.15504409e-02,  1.84355960e-01,  9.30203412e-02,  1.46958981e-01,
```

```
         1.95652726e-01,  1.81144210e-01,  1.73756574e-01,  1.44419516e-01,
         1.64675024e-01,  1.80605070e-01,  1.01651145e-04,  1.12425479e-01,
         3.87875903e-02,  1.08028828e-02,  3.94245908e-02,  1.04135828e-01,
        -9.04461312e-03,  1.37326423e-01,  1.33940607e-02, -5.51189188e-02,
        -7.25055733e-02, -8.68178385e-02, -3.74063030e-03, -7.52445784e-02,
         9.81696971e-03,  2.51019741e-02, -1.05779113e-01, -1.61806162e-01,
        -1.50278903e-01, -9.33137950e-02, -3.21813534e-02, -4.47626000e-03,
        -1.36393813e-02, -1.65589782e-03, -1.13010133e-01, -9.66268996e-02,
        -1.15846985e-01, -3.38181829e-02, -1.70763953e-01, -2.04840448e-01,
        -3.87605533e-02, -6.89026202e-02, -3.90000000e-02, -1.52980786e-01,
        -1.78380644e-01, -1.15974021e-01, -5.71191099e-02,  5.91640009e-02,
        -8.61656863e-02,  3.84456423e-02, -5.55555422e-02, -5.64270423e-02,
         5.59194457e-02,  7.71841689e-02,  4.30505812e-02,  1.48592954e-01,
         2.64506612e-01,  2.42721476e-01,  3.10584167e-01,  4.02684987e-01,
         4.34877223e-01,  5.55786908e-01,  4.14112391e-01,  4.20586026e-01,
         6.08245102e-01,  7.14118720e-01,  8.58707850e-01,  8.36309393e-01,
         7.85236988e-01,  1.14963940e+00,  1.17125821e+00,  1.07394596e+00,
         1.32055124e+00,  1.37512360e+00,  1.36800552e+00,  1.56544228e+00]),
np.array([-6.73952400e-01, -7.79255455e-01, -7.23995770e-01, -7.47061662e-01,
        -5.17879628e-01, -5.00772662e-01, -4.66606616e-01, -4.31125107e-01,
        -3.23878881e-01, -3.91707792e-01, -2.81805248e-01, -3.49605866e-01,
        -1.71943651e-01, -1.66396677e-01, -7.88614656e-02, -1.00516343e-01,
        -6.88048426e-02, -3.92715532e-02, -9.05637102e-02,  1.08600216e-01,
        -3.79579972e-02,  8.41341865e-02,  9.42962963e-02,  1.47222660e-01,
         1.15504409e-02,  2.43559596e-02,  1.83020341e-01,  3.69589809e-02,
         5.56527256e-02,  1.41144210e-01,  1.37565740e-02,  1.74419516e-01,
         1.74675024e-01,  2.06050702e-02,  6.01016511e-02,  1.02425479e-01,
         8.87875903e-02,  1.60802883e-01,  1.29424591e-01, -4.58641725e-02,
        -3.90446131e-02,  1.27326423e-01, -1.66059393e-02,  1.48810812e-02,
         1.07494427e-01,  7.31821615e-02, -4.37406303e-02, -6.52445784e-02,
         4.98169697e-02,  6.51019741e-02, -1.35779113e-01, -1.31806162e-01,
        -1.60278903e-01, -9.33137950e-02, -1.21813534e-02, -8.44762600e-02,
        -1.23639381e-01, -8.16558978e-02, -9.30101332e-02, -7.66268996e-02,
        -1.95846985e-01, -4.38181829e-02, -1.30763953e-01, -1.64840448e-01,
         1.23944674e-03, -1.78902620e-01, -3.90000000e-02, -4.29807855e-02,
        -8.83806436e-02, -4.59740211e-02, -1.37119110e-01,  4.91640009e-02,
         2.38343137e-02, -1.01554358e-01,  1.14444458e-01,  1.33572958e-01,
        -2.40805543e-02,  1.07184169e-01,  1.13050581e-01,  1.68592954e-01,
         1.74506612e-01,  2.72721476e-01,  3.30584167e-01,  2.42684987e-01,
         2.74877223e-01,  4.75786908e-01,  3.74112391e-01,  4.30586026e-01,
         6.68245102e-01,  7.04118720e-01,  7.28707850e-01,  9.26309393e-01,
         8.05236988e-01,  1.16963940e+00,  1.03125821e+00,  1.17394596e+00,
         1.17055124e+00,  1.26512360e+00,  1.30800552e+00,  1.58544228e+00]),
np.array([-7.73952400e-01, -8.79255455e-01, -5.73995770e-01, -7.67061662e-01,
        -6.37879628e-01, -4.10772662e-01, -5.06606616e-01, -4.21125107e-01,
        -3.33878881e-01, -2.31707792e-01, -1.71805248e-01, -3.29605866e-01,
        -2.41943651e-01, -1.16396677e-01, -1.58861466e-01,  9.48365662e-03,
         3.11951574e-02,  7.28446756e-04,  7.94362898e-02,  7.86002161e-02,
```

```
        -7.95799717e-03,  8.41341865e-02,  4.29629630e-03,  5.72226595e-02,
         3.15504409e-02,  3.43559596e-02,  1.53020341e-01,  3.69589809e-02,
         9.56527256e-02,  1.51144210e-01,  4.37565740e-02,  2.04419516e-01,
         1.34675024e-01,  4.06050702e-02,  1.50101651e-01,  2.42547922e-03,
         1.78787590e-01,  1.10802883e-01,  5.94245908e-02,  2.41358275e-02,
         9.55386881e-04,  1.17326423e-01,  1.03394061e-01,  6.48810812e-02,
        -5.25055733e-02,  8.31821615e-02,  7.62593697e-02,  7.47554216e-02,
        -6.01830303e-02, -4.48980259e-02, -6.57791134e-02, -6.18061615e-02,
        -1.50278903e-01, -1.13313795e-01, -8.21813534e-02, -9.44762600e-02,
        -1.33639381e-01, -6.16558978e-02, -4.30101332e-02, -4.66268996e-02,
        -1.58469848e-02, -1.43818183e-01, -5.07639532e-02, -1.44840448e-01,
        -1.68760553e-01, -8.89026202e-02,  1.00000000e-03, -5.29807855e-02,
        -1.18380644e-01, -7.59740211e-02, -4.71191099e-02, -6.08359991e-02,
        -3.61656863e-02,  1.84456423e-02,  9.44444578e-02, -6.42704232e-03,
        -1.40805543e-02,  7.18416889e-03,  1.33050581e-01,  1.38592954e-01,
         2.54506612e-01,  2.82721476e-01,  2.10584167e-01,  2.72684987e-01,
         2.74877223e-01,  4.55786908e-01,  3.84112391e-01,  6.20586026e-01,
         5.38245102e-01,  6.64118720e-01,  7.38707850e-01,  9.56309393e-01,
         8.45236988e-01,  1.03963940e+00,  1.12125821e+00,  1.10394596e+00,
         1.19055124e+00,  1.31512360e+00,  1.36800552e+00,  1.61544228e+00]),
np.array([-8.43952400e-01, -6.79255455e-01, -7.53995770e-01, -7.67061662e-01,
        -6.37879628e-01, -5.30772662e-01, -4.46606616e-01, -4.01125107e-01,
        -3.43878881e-01, -3.21707792e-01, -2.61805248e-01, -2.29605866e-01,
        -1.51943651e-01, -2.16396677e-01, -8.88614656e-02, -9.05163434e-02,
        -1.88048426e-02, -1.09271553e-01, -1.20563710e-01,  3.86002161e-02,
         8.20420028e-02,  1.24134187e-01,  4.42962963e-02,  1.37222660e-01,
        -8.44955905e-03,  8.43559596e-02,  1.33020341e-01,  1.56958981e-01,
         1.55652726e-01,  3.11442096e-02,  1.33756574e-01,  4.44195161e-02,
         9.46750238e-02,  1.60605070e-01,  1.01651145e-04,  2.42547922e-03,
         1.87875903e-02, -9.19711718e-03,  9.94245908e-02,  7.41358275e-02,
        -9.04461312e-03,  6.73264229e-02,  1.03394061e-01,  8.48810812e-02,
         6.74944267e-02, -6.68178385e-02,  6.25936970e-03,  3.47554216e-02,
        -5.01830303e-02, -1.24898026e-01, -2.57791134e-02, -1.31806162e-01,
        -1.60278903e-01,  6.68620500e-03, -1.22181353e-01,  5.52374000e-03,
        -3.63938128e-03, -6.16558978e-02, -3.01013315e-03, -1.26626900e-01,
        -7.58469848e-02, -1.73818183e-01, -7.07639532e-02, -1.24840448e-01,
        -1.88760553e-01, -4.89026202e-02, -1.90000000e-02, -4.29807855e-02,
         1.61935636e-03, -5.97402105e-03, -1.47119110e-01, -1.00835999e-01,
        -5.61656863e-02,  5.84456423e-02,  2.44444578e-02, -6.64270423e-02,
         5.91944567e-03,  1.07184169e-01,  1.53050581e-01,  5.85929543e-02,
         2.74506612e-01,  1.42721476e-01,  3.30584167e-01,  3.22684987e-01,
         4.14877223e-01,  3.75786908e-01,  4.74112391e-01,  5.80586026e-01,
         5.68245102e-01,  6.94118720e-01,  8.18707850e-01,  8.06309393e-01,
         7.95236988e-01,  1.08963940e+00,  1.19125821e+00,  1.22394596e+00,
         1.17055124e+00,  1.29512360e+00,  1.31800552e+00,  1.73544228e+00]),
np.array([-8.63952400e-01, -7.79255455e-01, -6.63995770e-01, -6.67061662e-01,
        -4.67879628e-01, -4.20772662e-01, -5.06606616e-01, -4.51125107e-01,
        -2.83878881e-01, -4.11707792e-01, -3.51805248e-01, -2.29605866e-01,
```

```
        -1.01943651e-01, -1.66396677e-01, -6.88614656e-02, -1.50516343e-01,
        -6.88048426e-02, -9.92715532e-02, -5.63710244e-04, -5.13997839e-02,
         2.20420028e-02, -5.58658135e-02,  1.24296296e-01,  1.57222660e-01,
         9.15504409e-02,  6.43559596e-02,  3.30203412e-02,  5.69589809e-02,
         9.56527256e-02,  2.11442096e-02,  1.37565740e-02,  1.94419516e-01,
         1.64675024e-01,  1.60605070e-01,  9.01016511e-02,  6.24254792e-02,
        -1.12124097e-02,  9.08028828e-02,  9.42459077e-03, -5.86417247e-03,
         5.09553869e-02, -2.67357706e-03, -5.66059393e-02,  8.48810812e-02,
        -6.25055733e-02, -3.68178385e-02,  1.62593697e-02,  6.47554216e-02,
        -8.01830303e-02, -4.48980259e-02,  4.42208866e-02, -7.18061615e-02,
        -1.50278903e-01, -3.31379500e-03, -1.42181353e-01, -1.04476260e-01,
        -1.43639381e-01, -1.65589782e-03, -1.53010133e-01, -2.66268996e-02,
        -1.45846985e-01, -5.38181829e-02, -1.70763953e-01, -1.34840448e-01,
        -1.98760553e-01, -2.89026202e-02, -6.90000000e-02, -6.29807855e-02,
        -8.38064364e-03, -3.59740211e-02, -8.71191099e-02, -9.08359991e-02,
         6.38343137e-02,  8.84456423e-02, -4.55555422e-02,  3.35729577e-02,
        -3.40805543e-02,  6.71841689e-02,  6.30505812e-02,  1.48592954e-01,
         2.14506612e-01,  3.42721476e-01,  2.20584167e-01,  2.42684987e-01,
         3.64877223e-01,  3.95786908e-01,  5.34112391e-01,  4.60586026e-01,
         6.48245102e-01,  7.04118720e-01,  8.58707850e-01,  9.26309393e-01,
         8.45236988e-01,  1.16963940e+00,  1.09125821e+00,  1.17394596e+00,
         1.22055124e+00,  1.32512360e+00,  1.46800552e+00,  1.58544228e+00]),
np.array([-7.53952400e-01, -7.39255455e-01, -7.23995770e-01, -7.97061662e-01,
        -4.57879628e-01, -4.20772662e-01, -4.76606616e-01, -3.31125107e-01,
        -3.83878881e-01, -2.81707792e-01, -1.81805248e-01, -2.29605866e-01,
        -1.21943651e-01, -2.06396677e-01, -1.58861466e-01, -1.05163434e-02,
        -1.88048426e-02, -8.92715532e-02,  5.94362898e-02, -1.39978388e-03,
        -7.79579972e-02, -5.58658135e-02,  8.42962963e-02,  9.72226595e-02,
         9.15504409e-02,  3.43559596e-02,  8.30203412e-02,  1.56958981e-01,
        -4.34727440e-03,  1.81144210e-01,  1.33756574e-01,  3.44195161e-02,
         1.04675024e-01,  1.60605070e-01,  1.70101651e-01,  1.42425479e-01,
         1.28787590e-01,  1.20802883e-01,  1.94245908e-02, -3.58641725e-02,
         9.09553869e-02, -6.26735771e-02, -4.66059393e-02,  1.48810812e-02,
         1.07494427e-01,  1.03182161e-01, -3.74063030e-03,  2.47554216e-02,
        -6.01830303e-02, -5.48980259e-02, -1.45779113e-01, -2.18061615e-02,
        -1.02789028e-02, -1.73313795e-01, -2.21813534e-02, -1.74476260e-01,
        -3.63938128e-03, -7.16558978e-02, -2.03010133e-01, -6.66268996e-02,
        -2.05846985e-01, -2.38181829e-02, -2.07639532e-02, -3.48404477e-02,
        -4.87605533e-02, -1.98902620e-01, -1.09000000e-01, -5.29807855e-02,
        -1.58380644e-01, -5.59740211e-02, -1.17119110e-01,  4.91640009e-02,
        -8.61656863e-02,  7.84456423e-02,  2.44444578e-02, -4.64270423e-02,
         1.45919446e-01,  1.57184169e-01,  1.53050581e-01,  1.18592954e-01,
         2.94506612e-01,  1.82721476e-01,  3.40584167e-01,  3.92684987e-01,
         4.64877223e-01,  3.75786908e-01,  4.34112391e-01,  5.50586026e-01,
         6.78245102e-01,  8.04118720e-01,  7.08707850e-01,  9.06309393e-01,
         9.35236988e-01,  1.12963940e+00,  1.11125821e+00,  1.16394596e+00,
         1.28055124e+00,  1.29512360e+00,  1.40800552e+00,  1.74544228e+00]),
np.array([-6.73952400e-01, -7.79255455e-01, -5.83995770e-01, -6.57061662e-01,
```

```
        -5.97879628e-01, -5.70772662e-01, -5.46606616e-01, -4.31125107e-01,
        -3.03878881e-01, -2.91707792e-01, -3.21805248e-01, -3.39605866e-01,
        -2.91943651e-01, -1.96396677e-01, -1.68861466e-01, -1.40516343e-01,
        -1.68804843e-01,  7.28446756e-04, -6.05637102e-02, -4.13997839e-02,
         5.20420028e-02,  1.04134187e-01, -2.57037037e-02,  7.72226595e-02,
         5.15504409e-02,  6.43559596e-02,  1.33020341e-01,  1.26958981e-01,
        -4.34727440e-03,  1.51144210e-01,  1.23756574e-01,  2.44195161e-02,
         1.74675024e-01,  1.10605070e-01,  1.00101651e-01,  1.62425479e-01,
         1.88787590e-01,  1.30802883e-01,  1.39424591e-01,  1.34135828e-01,
         9.09553869e-02, -5.26735771e-02,  3.33940607e-02, -7.51189188e-02,
        -2.50557327e-03,  5.31821615e-02,  1.62593697e-02, -5.24457835e-03,
        -1.20183030e-01,  1.51019741e-02, -1.57791134e-02,  2.81938385e-02,
        -9.02789028e-02, -7.33137950e-02, -9.21813534e-02, -1.34476260e-01,
        -2.03639381e-01, -4.16558978e-02, -1.43010133e-01, -1.16626900e-01,
        -8.58469848e-02, -2.03818183e-01, -8.07639532e-02, -2.48404477e-02,
        -1.58760553e-01, -4.89026202e-02, -1.89000000e-01, -5.29807855e-02,
        -8.38064364e-03, -1.35974021e-01, -1.27119110e-01, -4.08359991e-02,
        -4.61656863e-02, -1.01554358e-01,  5.44444578e-02,  3.35729577e-02,
         1.25919446e-01,  7.71841689e-02,  1.23050581e-01,  1.08592954e-01,
         3.04506612e-01,  1.92721476e-01,  3.70584167e-01,  4.12684987e-01,
         3.04877223e-01,  4.35786908e-01,  3.54112391e-01,  4.70586026e-01,
         5.68245102e-01,  6.84118720e-01,  7.48707850e-01,  8.06309393e-01,
         9.45236988e-01,  1.12963940e+00,  1.04125821e+00,  1.09394596e+00,
         1.25055124e+00,  1.32512360e+00,  1.35800552e+00,  1.73544228e+00]),
np.array([-8.23952400e-01, -8.39255455e-01, -7.73995770e-01, -6.27061662e-01,
        -6.17879628e-01, -4.90772662e-01, -5.46606616e-01, -3.41125107e-01,
        -3.63878881e-01, -3.51707792e-01, -2.61805248e-01, -3.29605866e-01,
        -2.01943651e-01, -1.16396677e-01, -1.98861466e-01, -1.40516343e-01,
         3.11951574e-02, -7.92715532e-02, -1.05637102e-02, -1.39978388e-03,
         2.20420028e-02,  3.41341865e-02,  1.44296296e-01,  7.22265952e-03,
         4.15504409e-02,  1.84355960e-01,  1.53020341e-01,  1.36958981e-01,
         1.55652726e-01,  3.11442096e-02,  1.33756574e-01,  1.54419516e-01,
         4.46750238e-02,  6.05070240e-04,  1.00101651e-01,  1.24254792e-02,
        -1.21240968e-03, -2.91971172e-02,  3.94245908e-02,  7.41358275e-02,
         4.09553869e-02,  9.73264229e-02,  9.33940607e-02, -5.11891885e-03,
        -2.50557327e-03,  9.31821615e-02, -7.37406303e-02,  6.47554216e-02,
        -1.00183030e-01, -7.48980259e-02,  4.22088655e-03, -1.61806162e-01,
        -7.02789028e-02, -1.03313795e-01, -1.62181353e-01, -7.44762600e-02,
        -1.83639381e-01, -1.41655898e-01, -1.23010133e-01, -1.56626900e-01,
        -8.58469848e-02, -1.53818183e-01, -5.07639532e-02, -1.74840448e-01,
        -5.87605533e-02, -1.38902620e-01, -7.90000000e-02, -1.62980786e-01,
        -1.68380644e-01, -1.35974021e-01, -1.57119110e-01, -1.30835999e-01,
        -2.61656863e-02,  2.84456423e-02,  5.44444578e-02,  1.13572958e-01,
         1.25919446e-01,  4.71841689e-02,  1.83050581e-01,  6.85929543e-02,
         1.54506612e-01,  3.12721476e-01,  3.80584167e-01,  3.22684987e-01,
         3.74877223e-01,  5.55786908e-01,  3.74112391e-01,  5.50586026e-01,
         5.78245102e-01,  6.64118720e-01,  8.08707850e-01,  9.56309393e-01,
         8.55236988e-01,  1.03963940e+00,  1.06125821e+00,  1.10394596e+00,
```

```
         1.28055124e+00,  1.24512360e+00,  1.34800552e+00,  1.58544228e+00]),
np.array([-8.13952400e-01, -6.79255455e-01, -6.23995770e-01, -6.47061662e-01,
        -5.67879628e-01, -5.30772662e-01, -3.76606616e-01, -5.21125107e-01,
        -4.23878881e-01, -3.71707792e-01, -2.91805248e-01, -2.09605866e-01,
        -1.11943651e-01, -9.63966766e-02, -2.28861466e-01, -1.40516343e-01,
        -1.68804843e-01,  4.07284468e-02, -1.05637102e-02,  8.60021612e-03,
        -4.79579972e-02,  1.14134187e-01, -5.70370370e-03,  4.72226595e-02,
         1.51550441e-01,  1.04355960e-01,  1.53020341e-01,  1.69589809e-02,
        -4.34727440e-03,  1.31144210e-01,  1.83756574e-01,  1.54419516e-01,
         9.46750238e-02,  2.06050702e-02,  4.01016511e-02,  1.22425479e-01,
         8.87875903e-02,  8.02882820e-04, -2.05754092e-02, -4.58641725e-02,
         1.10955387e-01,  5.73264229e-02,  1.23394061e-01,  1.14881081e-01,
        -4.25055733e-02, -3.68178385e-02,  4.62593697e-02,  3.47554216e-02,
        -2.01830303e-02, -8.48980259e-02, -1.35779113e-01, -3.18061615e-02,
        -1.20278903e-01, -1.33313795e-01, -5.21813534e-02, -7.44762600e-02,
        -2.03639381e-01, -4.16558978e-02, -2.03010133e-01, -1.96626900e-01,
        -5.58469848e-02, -1.53818183e-01, -1.20763953e-01, -5.48404477e-02,
        -4.87605533e-02, -2.89026202e-02, -9.90000000e-02, -9.29807855e-02,
        -1.68380644e-01, -1.59740211e-02, -2.71191099e-02,  5.91640009e-02,
         6.38343137e-02, -6.15543577e-02, -6.55555422e-02,  1.13572958e-01,
         1.35919446e-01,  8.71841689e-02,  4.30505812e-02,  2.28592954e-01,
         2.74506612e-01,  2.42721476e-01,  3.70584167e-01,  4.12684987e-01,
         3.54877223e-01,  5.05786908e-01,  4.34112391e-01,  4.20586026e-01,
         4.98245102e-01,  6.44118720e-01,  7.98707850e-01,  9.16309393e-01,
         9.65236988e-01,  1.09963940e+00,  1.12125821e+00,  1.26394596e+00,
         1.22055124e+00,  1.32512360e+00,  1.42800552e+00,  1.60544228e+00])])
```

Calculer le tableau moyennes qui contient les 100 valeurs moyennes des 20 séries, minimaux et maximaux qui contiennent respectivement les valeurs minimales et maximales des 20 séries (sans boucles).

On obtient donc 3 tableaux de taille 100 comme ref.

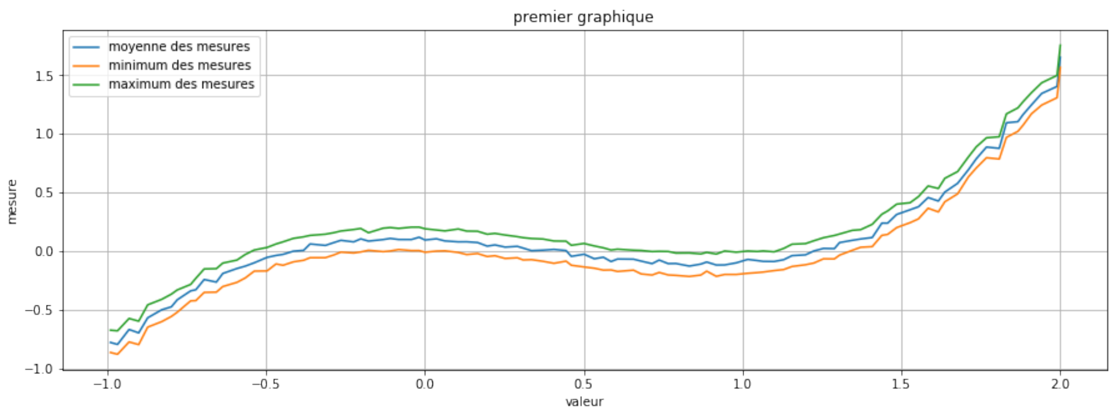
```
In [5]: tm=np.mean(tab20, axis=0)
        tmax=np.max(tab20, axis=0)
        tmin=np.min(tab20, axis=0)
```

1.3.1 1/ Ajustement de courbe

Tracer le graphique avec les 3 courbes correspondant à tm ,tmin et tmax comme sur la figure ci-dessous

```
In [6]: f,ax = plt.subplots(1,1,figsize=(15,5))


        ax.plot(ref,tm,label='moyenne des mesures')
```

```
ax.plot(ref,tmin,label='minimum des mesures')
ax.plot(ref,tmax,label='maximum des mesures')
ax.set_xlabel("valeur")
ax.set_ylabel("mesure")
ax.set_title("premier graphique ")
ax.grid()
ax.legend()
print()
```



Cette "courbe" ressemble à une courbe polynomiale. On souhaite donc trouver les paramètres $a,b,c,d $ de la fonction $f(t) = ax^3 + bx^2 + cx + d$ qui passe au mieux par les points de la courbe.

Utilisez la fonction `scipy.optimize.curve_fit` pour déterminer a,b,c,d.

```
In [7]: import scipy.optimize

        # il faut définir une fonction python avec x comme premier paramètre

        def fonc(x,a,b,c,d):
            return a*x**3+b*x**2+c*x+d

        (a,b,c,d),r = scipy.optimize.curve_fit(fonc,ref,tm)

        # on appelle curve_fitavec la fonction, les  abscisses des points, les ordonnées des poi

        print('a=',a,'b=',b,'c=',c,'d=',d)

a= 0.5035879540234752 b= -0.5883204918901609 c= -0.11581141843646665 d= 0.09863516682229863
```

On affichera le 'résidu moyen' (écart entre modèle et données) au moyen de $r = \frac{\sqrt{\Sigma(f(t)-t_i)^2}}{N}$.

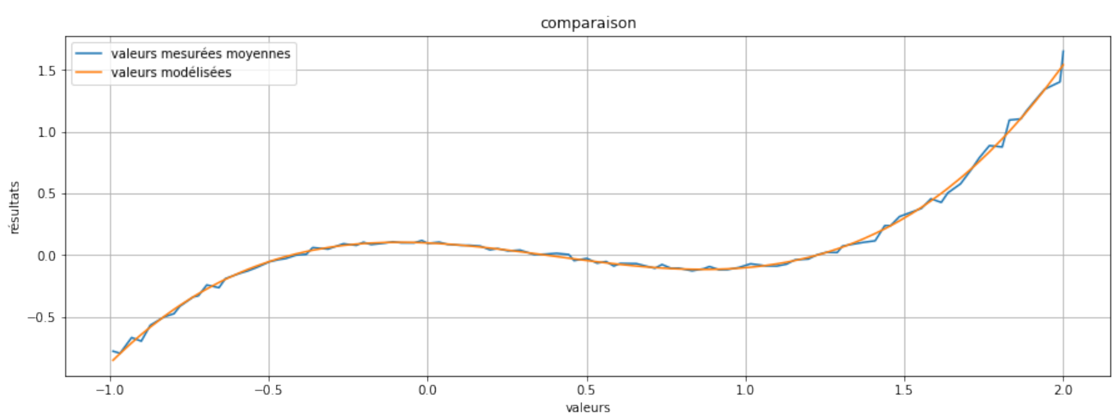
Il s'agit de la moyenne des écarts au carré entre le modèle et les données, où N est le nombre de points.

```
In [8]: # le calcul se fait aec les différences entre f(ref) et tm

        print('résidu moyen avec les moyennes:',np.sqrt(np.mean((fonc(ref,a,b,c,d)-tm)**2)))
```

```
résidu moyen avec les moyennes: 0.028297616066933175
```

Tracer sur le même shéma la moyenne des données relevées et la courbe obtenue avec la fonction d'estimation.

```
In [9]: f,ax = plt.subplots(1,1, figsize=(15,5))
        xx=np.linspace(ref[0], ref[-1], 100)
        # attention il faut suffisamment de points pour dessiner la courbe (ref ne sufit pas)

        ax.plot(ref,tm,label='valeurs mesurées moyennes')
        ax.plot(ref,fonc(ref,a,b,c,d),label='valeurs modélisées')

        ax.set_xlabel("valeurs")
        ax.set_ylabel("résultats")
        ax.set_title("comparaison")
        ax.grid()
        ax.legend()
        print()
```

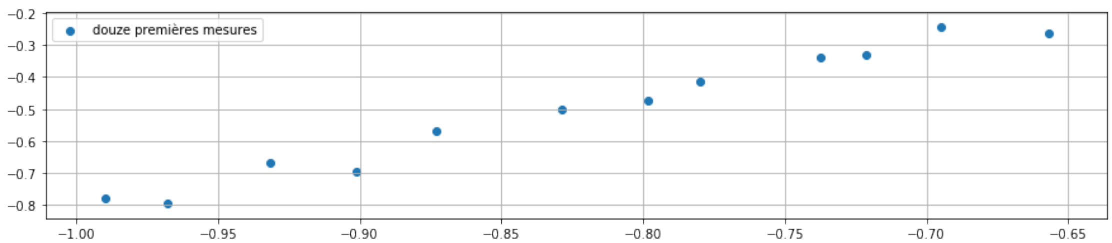


1.3.2 2/ Interpolation

Dans cette partie, on va s'interesser aux 12 premières valeurs de ref (et aux résultats associées dans tab20). Représenter les douze points correspondants sur un graphique.

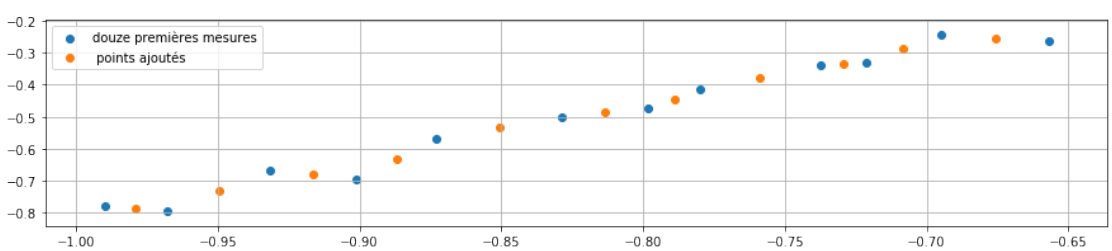
```
In [10]: # extraction de 12 valeurs
```

```
val=ref[:12]
etud = tm[:12]


f,ax = plt.subplots(1,1,figsize=(15,3))

ax.scatter(val,etud,label='douze premières mesures')

ax.grid()
ax.legend()
print()
```



Pour mieux visualiser la courbe ajouter sur le dessin les points "milieux" (en utilisant numpy).

```
In [11]: #calcul des milieux
         v= (val[1:]+val[:-1])/2
         z= (etud[1:]+etud[:-1])/2


         f,ax = plt.subplots(1,1,figsize=(15,3))

         ax.scatter(val,etud,label='douze premières mesures')
         ax.scatter(v,z,label=" points ajoutés")

         ax.grid()
         ax.legend()
         print()
```



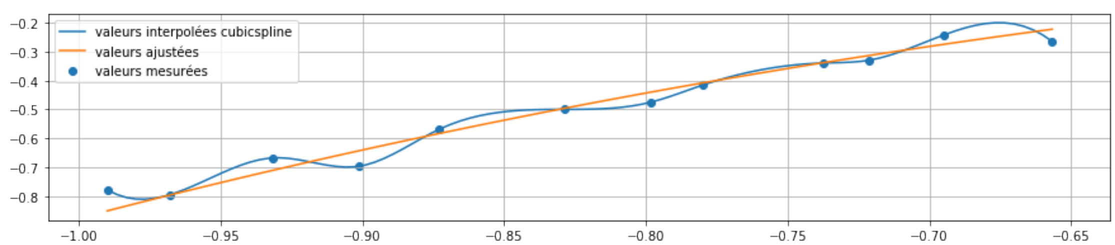
On cherche à interpoler entre les points pour estimer les valeurs entre ces points.

Interpoler au moyen de `scipy.interpolate.CubicSpline`

Tracer la courbes interpolée et les points sur le même graphique. Tracer également la courbe obtenue avec la méthode d'ajustement de la première partie.

```
In [12]: import scipy.interpolate

         cs = scipy.interpolate.CubicSpline(val,etud)

         # attention ici cs est une fonction
```

```
In [13]: f,ax = plt.subplots(1,1,figsize=(15,3))
         xx=np.linspace(val[0], val[-1], 100)  # échantillon de points suffisant
         ax.scatter(val,etud,label='valeurs mesurées')

         ax.plot(xx,cs(xx),label='valeurs interpolées cubicspline')

         ax.plot(xx,fonc(xx,a,b,c,d),label='valeurs ajustées')

         ax.grid()
         ax.legend()

         print()
```



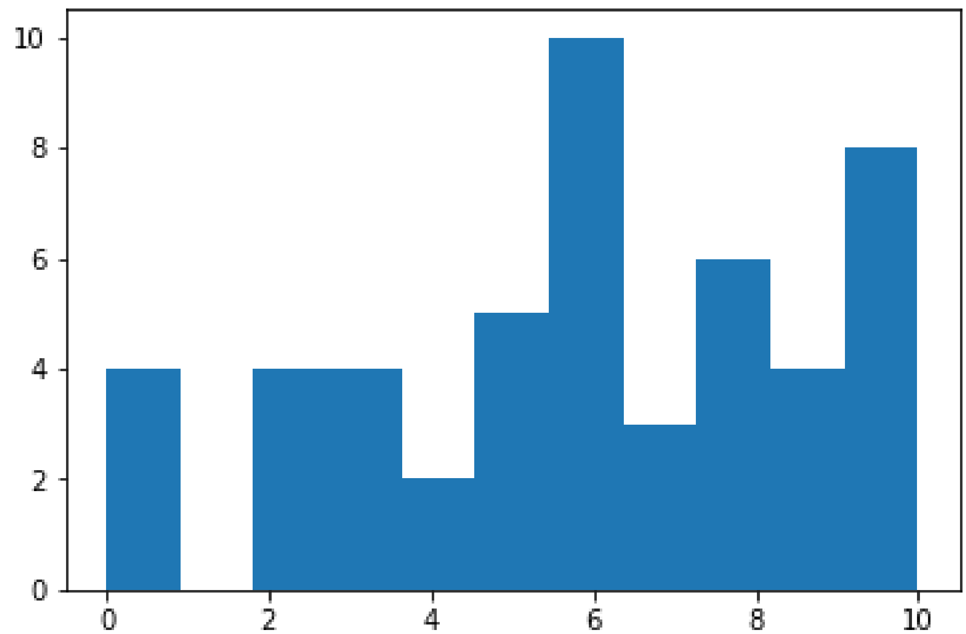
1.4 Exercice 3 : plusieurs façons d'obtenir le même histogramme

On dispose d'une liste de 50 notes toutes comprises entre 0 et 10.

```
In [14]: notes=[ 9, 10,  6,  2,  0,  9,  6,  0,  4, 10, 10, 6,  6,  5,  5,  7,
                 8,  3,  7, 10,  8,  3,  0,  2,  5,  6,  6,  7,  5,  9,  5,  2,
                10,  4,  8,  8,  9, 10,  8,  6,  6, 6,  3,  6,  2,  10,  0,  8,
                 3, 10]
```

Utilisez la fonction plt.hist avec le bon paramètre bins pour obtenir un premier graphique

```
In [15]: val=plt.hist(notes, bins=11,density=False)
```



Utilisez la fonction np.unique pour récupérer deux tableaux: une avec les notes obtenues (sans doublons) et une avec les fréquences correspondantes

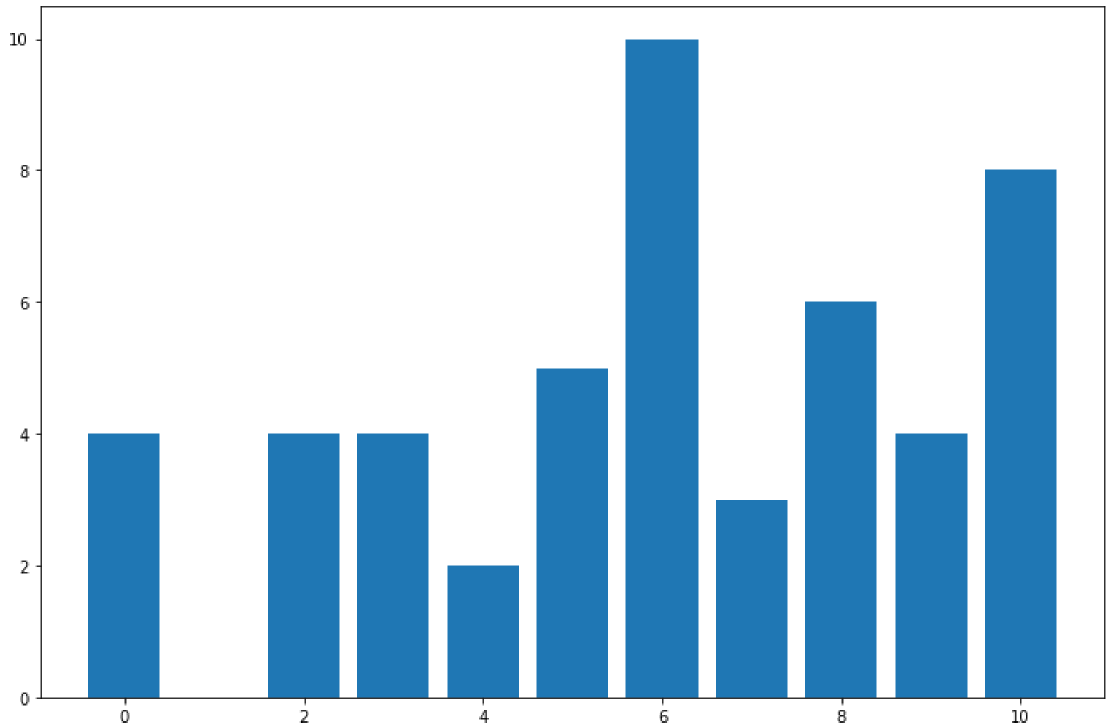
```
In [16]: unique, counts = np.unique(notes, return_counts=True)

         #np.unique renvoie deux tableaux
         print (unique, counts)

[ 0  2  3  4  5  6  7  8  9 10] [ 4  4  4  2  5 10  3  6  4  8]
```

Utilisez ce résultat pour afficher la note la plus souvent obtenue ainsi que son nombre d'occurence.

```
In [17]: i = np.argmax(counts)
         print ('val la plus fréquente :', unique[i],'nombre d\'occurences :',counts[i])

val la plus fréquente : 6 nombre d'occurences : 10
```

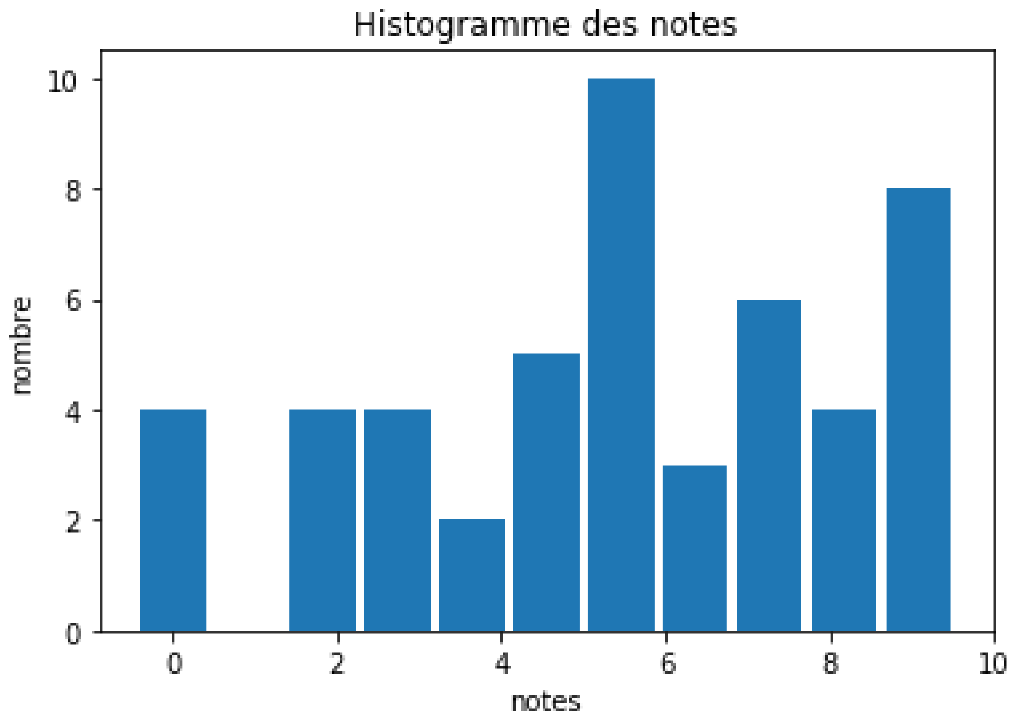
A partir de ces deux tableaux et de subplots et ax.bar, refaites un histogramme.

```
In [18]: f,ax = plt.subplots(1,1,figsize=(12,8))

         ax.bar(unique, counts)

         print()
```



Utilisez la fonction de numpy histogramme pour refaire encore ce graphique.

```
In [19]: h=np.histogram(notes,bins=11,density=False)
         fig, ax = plt.subplots()
         ax.set(title='Histogramme des notes',xlabel='notes',ylabel='nombre')
         plt.bar(h[1][0:-1],h[0])
         plt.show()
```


Histogramme des notes
nombre
notes
0
2
4
6
8
10
0
2
4
6
8
10