Resource Letter GPP-1: Geometric Phases in Physics

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ABSTRACT

This Resource Letter provides a guide to the literature on the geometric angles and phases in classical and quantum physics. Journal articles and books are cited for the following topics: anticipations of the geometric phase, foundational derivations and formulations, books and review articles on the subject, and theoretical and experimental elaborations and applications.

Suppose a system undergoes an evolution so that after some time it returns to its original state. We shall call such an evolution a *cyclic* evolution. If the system is classical, then it is impossible to say from its initial and final states that it has undergone any evolution. However, the wave function of a quantal system retains a memory of its motion in the form of a geometric phase factor. This phase factor can be measured by interfering the wave function with another coherent wave function enabling one to discern whether or not the system has undergone an evolution. Therefore geometric phase factors are 'signatures' of quantum motion. The adjective 'geometric' emphasizes that such phase factors depend only on the loop in the quantum-mechanical state space — the set of rays of the Hilbert space, sometimes called the projective Hilbert space. In particular, geometric phases are independent of parameterization of the path in the projective Hilbert space, and therefore of the speed at which it has been traversed.

As early as 1956 [1], in a classic paper on phase shifts in non-quantal polarized light [2], S. Pancharatnam anticipated the quantal geometric phases. He was only twenty-two years of age at the time. He studied the problem of determining the phase change undergone by polarized light after it has passed through a sequence of polarizers such that its final polarization is the same as its initial polarization. To describe how the phase of polarized light changes under passage through a polarizer, Pancharatnam needed to define the *phase difference* between two different polarization states. He reasoned that the most natural way to accomplish this task is to ask what would happen if two such states were brought to interfere with each other, and accordingly he proposed the following definition: the polarization states of any two monochromatic beams of light with the same momenta are *in phase* if the superposition of the two has the maximum possible intensity. Let $|A\rangle$ and $|B\rangle$ represent the polarization state-vectors of photons in the two beams. Since the intensity of their superposition is proportional to

$$(\langle A| + \langle B|)(|A\rangle + |B\rangle) = 2 + 2|\langle A|B\rangle|\cos\{\operatorname{ph}\langle A|B\rangle\},\tag{1}$$

according to his convention $|A\rangle$ and $|B\rangle$ are in phase when their scalar product $\langle A\,|\,B\rangle$ is real and positive, or equivalently, when $\mathrm{ph}\langle A\,|\,B\rangle=0$. Incidentally, since orthogonal states do not interfere, this convention breaks down for such states, and the phase difference between them remains undefined. In the general case of non-orthogonal states, it is natural to identify the phase difference between $|A\rangle$ and $|B\rangle$ with the phase $\mathrm{ph}\langle A\,|\,B\rangle$ of their scalar product.

Pancharatnam used this definition of the phase difference to analyze an experiment involving a sequence of changes in polarization of a beam of classical light by sending it through suitable polarizers. His experiment consisted of three sequential changes in polarization, from $|A\rangle$ to $|B\rangle$ to $|C\rangle$ and back to a state $|A'\rangle$ of the initial polarization. It is easy to show that in such a scheme each successive state remains in phase with the previous one. Now, the label A used here to describe a state of a polarized wave of light represents a set of values (the eigenvalues of a complete set of commuting observables) required to specify this state uniquely. In Pancharatnam's experiment all but one of these values — including the one that specifies the polarization — were returned to their original values, with the phase of polarization being the only exception. Thus, Pancharatnam's evolution was not cyclic in the sense described above. Indeed, in what follows, the classical phase difference he observed will be shown to come from the quantum mechanical phase difference between the initial and final one-photon states,

$$\langle \psi \,|\, \psi' \rangle = e^{-\frac{i}{2}\alpha_{ABC}},\tag{2}$$

where α_{ABC} is the solid angle subtended by the geodesic triangle ABC on the Poincaré sphere (whose points, as is well-known, represent all conceivable forms of polarization states). For simplicity, we ignore the dynamical phase difference due to the fixed frequency of the photon. Remarkably enough, Pancharatnam not only anticipated the quantal geometric phases, but also was able to corroborate his theory experimentally.

Another geometric phase given by a solid-angle formula analogous to (2) was put forward in 1984 by M.V. Berry (who was unaware of Pancharatnam's work) in a seminal paper on the quantum-mechanical adiabatic theorem [7]. He investigated the nonrelativistic Schrödinger evolution

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(\mathbf{R}(t))|\psi(t)\rangle$$
 (3)

of a quantal system in a slowly changing environment described by a set of N time-dependent parameters $\mathbf{R}(t) = (R_1(t), R_2(t), \dots, R_N(t))$, with the initial state

$$|\psi(0)\rangle = |n; \mathbf{R}(0)\rangle \tag{4}$$

being the stationary state given by the time-independent Schrödinger equation

$$H(\mathbf{R}(0))|n;\mathbf{R}(0)\rangle = E_n(\mathbf{R}(0))|n;\mathbf{R}(0)\rangle.$$
(5)

If $H(\mathbf{R}(t))$ is non-degenerate and slowly varying, then it is known that the time-evolving Schrödinger state $|\psi(t)\rangle$ remains an eigenstate of the instantaneous Hamiltonian $H(\mathbf{R}(t))$. More precisely,

$$|\psi(t)\rangle = e^{-i\int_0^t ds \, E_n(\mathbf{R}(s))} e^{i \, b \, [n; \mathbf{R}(t)]} |E_n(\mathbf{R}(t))\rangle, \tag{6}$$

where

$$b[n; \mathbf{R}(t)] = \int_{0}^{t} ds \langle n; \mathbf{R}(s) | i \frac{d}{ds} | n; \mathbf{R}(s) \rangle, \qquad (7)$$

or, equivalently,

$$b[n; \mathbf{R}(t)] = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R}' \cdot \langle n; \mathbf{R}' | i \nabla_{\mathbf{R}'} | n; \mathbf{R}' \rangle, \qquad (7')$$

were $\nabla_{\mathbf{R}}$ is the gradient operator in the parameter space \mathbf{R} . This is, of course, just the time-honored adiabatic theorem.

Berry's investigations, however, went beyond the usual formulation of the adiabatic theorem captured in (6) and (7). He considered the case of an adiabatic transport around a *closed* path

$$\rho_t = \{ \mathbf{R}(t) \mid \mathbf{R}(T) = \mathbf{R}(0) \, ; \, 0 < t < T \, \}$$
(8)

in the parameter space, and made the crucial observation that in such an adiabatic setup the phase factor $e^{ib[n;\mathbf{R}(t)]}$ is not integrable; i.e., in general it cannot be written as a function of \mathbf{R} , and in particular is not single-valued under continuation around the loop: $e^{ib[n;\mathbf{R}(T)]} \neq e^{ib[n;\mathbf{R}(0)]}$. Moreover, it is easy to see that (7') can be reexpressed in the form

$$b[n; \rho] = \oint_{\rho} d\mathbf{R} \cdot \langle n; \mathbf{R} | i\nabla_{\mathbf{R}} | n; \mathbf{R} \rangle, \qquad (9)$$

from which it is evident that $b[n; \mathbf{R}(T)]$, or the Berry phase as it is now called, is independent of parameterization: $b[n; \mathbf{R}(T)] = b[n; \rho]$, where ρ denotes the unparameterized loop corresponding to ρ_t . In particular, unlike the usual dynamical phase $\{-\int \mathrm{d}s \, E_n\}$, the Berry phase $b[n; \rho]$ is independent of the rate at which the state of the system traverses around ρ .

To illustrate his findings Berry analyzed the example of a spin-s particle interacting with a magnetic field ${\bf B}$ through the Hamiltonian

$$H(\mathbf{B}) = \kappa \,\mathbf{B} \cdot \mathbf{S} \,, \tag{10}$$

where κ is a constant involving the gyromagnetic ratio, and **S** is the vector spin operator whose components have 2s + 1 eigenvalues n lying between -s and +s with integer spacing. The eigenvalues of $H(\mathbf{B})$ are, of course,

$$E_n(\mathbf{B}) = \kappa B \, n \,, \tag{11}$$

with $B = |\mathbf{B}|$. Now, if one identifies the components of the external magnetic field \mathbf{B} with the parameter space \mathbf{R} , then Berry's formula is easily applicable to this case. In particular, (9) gives the geometric phase change of an eigenstate $|n; \mathbf{B}(t)\rangle$ of $H(\mathbf{B}(t))$ as $\mathbf{B}(t)$ is slowly transported — and hence the spin is slowly precessed — around a loop γ in the \mathbf{B} -space. Berry was able to show that, in that case,

$$e^{ib[n;\mathbf{B}(T)]} = e^{-in\alpha_{\gamma}},\tag{12}$$

where α_{γ} is the solid angle subtended by the loop γ at $\mathbf{B}=0$. In particular, when $s=\frac{1}{2}$ and the initial state is $|\frac{1}{2};\mathbf{B}(0)\rangle$ ('spin up' along \mathbf{B}), then the right-hand side of (12) takes the form of the right-hand side of the observation (2) of Pancharatnam — namely, $\mathrm{e}^{-\frac{1}{2}\alpha_{\gamma}}$. To establish an analogy between (2) and (12) it suffices now to identify $|\psi(0)\rangle = |\psi\rangle$ and $|\psi(T)\rangle = |\psi'\rangle$, and note that the left-hand side of (12) can be rewritten, after the dynamical phase is removed, in the form $\langle \psi(0) | \psi(T) \rangle$.

A simple explanation of the beautiful result (12) was given in 1987 by J. Anandan and L. Stodolsky [36]. They considered a sphere whose points represented the possible directions of the magnetic field **B**. In the above example of Berry, it is sufficient for the direction of **B** to trace a closed curve $\hat{\gamma}$ on this sphere in order for each eigenstate to acquire a geometric phase. In other words, it is not necessary for $\mathbf{B}(t)$ to form a closed curve; it is sufficient if merely the directions of $\mathbf{B}(0)$ and $\mathbf{B}(T)$ coincide. Anandan and Stodolsky then considered a Cartesian triad with its origin on $\hat{\gamma}(t)$ and its z-axis in the radial direction of the sphere (the direction of $\mathbf{B}(t)$ and the spin axis). If the triad is moved along $\hat{\gamma}(t)$ so that the x,y-axes are parallel-transported along the surface of the sphere, then when the triad returns to the original point $\hat{\gamma}(0) = \hat{\gamma}(T)$, it will have rotated about its z-axis by the solid angle α subtended by $\hat{\gamma}$ at the center of the sphere. Now, relative to the triad, each eigenstate individually should acquire only the usual dynamical phase factor because the triad has no angular velocity about the spin axis. Consequently, the additional phase factor acquired by the eigenstate must be interpreted as the *geometric* phase factor due to the rotation of the triad given by $e^{i\alpha J_z}|n\rangle = e^{i\alpha n}|n\rangle$, where J_z generates rotation about the z-axis of the triad.

For an arbitrary cyclic evolution in any Hilbert space, the above angle α generalizes to a set of angles $\alpha_1, \ldots, \alpha_N$. These are the geometric quantum angles introduced by Anandan [14], which perhaps provides

the deepest approach so far to the geometric phase. The geometric phases acquired by a complete set of orthogonal states $\{|n\rangle\}$ are now obtained by the action on each $|n\rangle$ by $e^{i\sum_{k=1}^{N}\alpha_k J_k}$, where the elements of the set $\{J_k\}$ commute among themselves. In the classical limit the geometric angles $\{\alpha_k\}$ reduce to the classical angles of J.H. Hannay [10], while the observables $\{J_k\}$ become the corresponding action variables that are in involution with each other.

We now illustrate the usefulness of geometric angles by providing a quantum-mechanical explanation of the above-mentioned experiment of Pancharatnam. For this purpose, we need to generalize his classical electromagnetic polarized wave, with fixed momentum **p**, passing through an arbitrary number of polarizers, such that the final polarization is the same as the initial polarization. A classical electromagnetic wave is an approximation of a coherent state in quantum electrodynamics. In the Coulomb gauge the quantized vector-potential for the electromagnetic field may be written as

$$\mathbf{A} = \sum_{\mathbf{k}} \sum_{\lambda=1}^{2} [a_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + a_{\mathbf{k},\lambda}^{\dagger} e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)}] \mathbf{e}_{\mathbf{k},\lambda} , \qquad (13)$$

where \mathbf{k} is the momentum vector, $\omega = |\mathbf{k}|$ is the frequency, $\mathbf{e}_{\mathbf{k},\lambda}$ are real orthogonal polarization vectors perpendicular to \mathbf{k} , and $a_{\mathbf{k},\lambda}$, $a_{\mathbf{k},\lambda}^{\dagger}$ are the annihilation and creation operators for the mode (\mathbf{k},λ) . The electric and magnetic fields corresponding to \mathbf{A} are $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, respectively. The coherent state corresponding to the electromagnetic wave considered by Pancharatnam is then

$$|z_1, z_2, \mathbf{p}\rangle = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \exp(z_1 a_{\mathbf{p},1}^{\dagger} + z_2 a_{\mathbf{p},2}^{\dagger}) |0\rangle,$$
 (14)

which is an eigenstate of $a_{\mathbf{p},\lambda}$ with eigenvalues z_{λ} . Therefore,

$$\langle z_1, z_2, \mathbf{p} | \mathbf{A} | z_1, z_2, \mathbf{p} \rangle = 2 \{ |z_1| \cos(\mathbf{p} \cdot \mathbf{x} - \omega t + \theta_1) \mathbf{e}_{\mathbf{p}, 1} + |z_2| \cos(\mathbf{p} \cdot \mathbf{x} - \omega t + \theta_2) \mathbf{e}_{\mathbf{p}, 2} \}, \quad (15)$$

where θ_1 and θ_2 are the phases of z_1 and z_2 respectively. It follows that $|z_1|\omega$ and $|z_2|\omega$ are the amplitudes of the electric field \mathbf{E} in the directions of $\mathbf{e}_{\mathbf{p},1}$ and $\mathbf{e}_{\mathbf{p},2}$, respectively.

We may represent the polarization state of a one-photon state $(z_1a_{\mathbf{p},1}^{\dagger}+z_2a_{\mathbf{p},2}^{\dagger})|0\rangle$ as a two-component spinor $\begin{pmatrix} z_1\\z_2 \end{pmatrix}$ in a two dimensional vector space with the usual inner product, which makes it a Hilbert space. The corresponding projective Hilbert space is the Poincaré sphere. As each photon corresponding to the mode (\mathbf{p},μ) passes through the polarizer it undergoes a transition to a state (\mathbf{p},μ') . The new state is obtained by simply projecting the old state onto the state that passes through the polarizer. It can be shown that this corresponds to parallel-transporting the old state-vector along the shorter geodesic joining the two points on the Poincaré sphere representing the two polarization states [43]. Therefore, using arguments similar to those used by Anandan and Stodolsky [36], the final state obtained after a sequence of such polarization changes that return the photons to their initial polarization state is given by the action of the operator $e^{i\alpha J}$ on the initial photon state. Here α is the solid angle subtended by the geodesic polygon defined by the sequence of the polarization states on the Poincaré sphere, and $J=\frac{N}{2}$, N being the number operator for the initial and final mode (\mathbf{p},μ) . As a result, the final state of the electromagnetic field is

$$e^{i\alpha J} |z_1, z_2, \mathbf{p}\rangle = |z_1 e^{i\frac{\alpha}{2}}, z_2 e^{i\frac{\alpha}{2}}, \mathbf{p}\rangle.$$
 (16)

Finally, in this resultant state we have

$$\langle z_1, z_2, \mathbf{p} | \mathbf{A} | z_1, z_2, \mathbf{p} \rangle = 2 \left\{ |z_1| \cos \left(\mathbf{p} \cdot \mathbf{x} - \omega t + \theta_1 + \frac{\alpha}{2} \right) \mathbf{e}_{\mathbf{p}, 1} + |z_2| \cos \left(\mathbf{p} \cdot \mathbf{x} - \omega t + \theta_2 + \frac{\alpha}{2} \right) \mathbf{e}_{\mathbf{p}, 2} \right\}$$
(17)

Comparison of this expression with equation (15) shows that $\frac{\alpha}{2}$ is the phase Pancharatnam observed in his classical experiment. A similar explanation can be given to the experiment of Tomita and Chiao [117], except in their case we would have J = N for the photon since it is a spin-1 particle.

Having obtained these results using the operator $\exp(i\alpha J)$, which depends on the geometric angle α , we may generalize them to an arbitrary superposition of number eigenstates with the same polarization. In general, such a state would not be a coherent state and cannot therefore be represented by a classical electromagnetic wave. Nevertheless, the geometric part of the evolution may be obtained by taking the expectation value of $\exp(i\alpha J)$ with respect to the initial state.

The above mentioned geometric treatment of Berry's phase by Anandan and Stodolsky suggest that the geometric phase is associated with the motion of a quantum system and not with the particular Hamiltonian used to achieve this motion. This is the basic idea used by Aharonov and Anandan [11] in obtaining a geometric phase, which, since it is associated with the motion of the quantum-mechanical state itself, does not require an adiabatically varying Hamiltonian (environment). However, if an adiabatically varying Hamiltonian is used to implement this motion, then this geometric phase is the same as Berry's phase. They defined the evolution of a normalized state $|\psi(t)\rangle$ to be cyclic in the interval [0,T] if and only if

$$|\psi(T)\rangle = e^{i\phi(0,T)}|\psi(0)\rangle,\tag{18}$$

where $\phi(0,T)$ is a real number. Equivalently, this can be reexpressed with the help of the unitary time evolution operator U(0,t) in the form

$$U(0,T)|\psi(0)\rangle = e^{i\phi(0,T)}|\psi(0)\rangle. \tag{19}$$

It follows from this equation that for an initial state to evolve cyclicly in the interval [0,T] under the timeevolution operator U(0,t), it is necessary and sufficient for it to be an eigenstate of the operator U(0,T). Incidentally, this assures the existence of cyclic evolutions as defined above at least in the finite-dimensional case. According to Aharonov and Anandan, the geometric contribution to $\phi(0,T)$, denoted by β , is

$$e^{i\beta} = e^{i\phi(0,T) + i \int_0^T ds \, \langle \psi(s) \, | i \frac{d}{ds} | \, \psi(s) \rangle}$$
(20)

or equivalently,

$$e^{i\beta} = \langle \psi(0) | \psi(T) \rangle e^{i\int_0^T ds \langle \psi(s) | i\frac{d}{ds} | \psi(s) \rangle}, \qquad (20')$$

which reduces to the Berry's phase factor in the adiabatic limit [11, 67]. What is more, just as Berry's phase, it is independent of the choice of parameterization or the speed at which the path $|\psi(t)\rangle$ is traversed. More significantly, Aharonov and Anandan demonstrated that β is projective-geometric in nature; i.e., it is the same for all paths $|\phi(t)\rangle$ that project to the same path in the projective Hilbert space. In other words, it is the same for any two motions $|\phi(t)\rangle$ and $|\psi(t)\rangle$ such that $p_{\phi(t)} = p_{\psi(t)}$, where p_{α} denotes a ray corresponding to a vector $|\alpha\rangle$, namely,

$$\mathbf{p}_{\alpha} = \{ |\beta\rangle \mid |\beta\rangle = z|\alpha\rangle; \ z \in \mathbb{C} \}. \tag{21}$$

The above two properties imply that $\beta = \beta [p_{\psi}]$, and suggest that β may have a geometric interpretation in terms of paths in the projective Hilbert space. Indeed, it may be geometrically understood as the *anholonomy* with respect to the natural connection on the projective Hilbert space. This interpretation generalizes an earlier differential-geometric interpretation of the Berry phase given by B. Simon in 1983 [8].

Berry's 1984 paper was concerned with nondegenerate states undergoing adiabatic evolution. In the same year F. Wilczek and A. Zee reported on how the theory can be generalized to include the adiabatic

evolution of degenerate quantum states [9]. They showed that in the case of a d-fold degeneracy Berry's phase factor of the nondegenerate case, $e^{ib[n;\rho]}$, is generalized to a $d \times d$ unitary matrix, which is now called the non-Abelian Berry phase or the Wilczek-Zee phase. More precisely, if the initial state is one of the eigenstates belonging to an orthonormal set of eigenstates of $H(\mathbf{R}(0))$ with a d-fold degenerate eigenvalue $E_n(\mathbf{R}(0))$, i.e.

$$H(\mathbf{R}(0))|l;\mathbf{R}(0)\rangle = E_n(\mathbf{R}(0))|l;\mathbf{R}(0)\rangle \tag{22}$$

with $l = 1, 2, \ldots, d$, then

$$|\psi(t)\rangle = e^{-i\int_0^t ds \ E_n(\mathbf{R}(s))} \sum_{l'=1}^d D_{l'l}(\mathbf{R}(t))|l'; \mathbf{R}(t)\rangle.$$
(23)

Here, the matrix D is a path-ordered exponential integral

$$D\left[\mathbf{R}(t)\right] = \mathcal{P}\exp\left\{i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R}' A(\mathbf{R}')\right\},\tag{24}$$

with

$$A_{l'l}(\mathbf{R}(t)) = \langle l'; \mathbf{R} | i \nabla_{\mathbf{R}} | l; \mathbf{R} \rangle, \qquad (25)$$

and \mathcal{P} represents the path-ordering. The non-Abelian phase factor $D[l; \mathbf{R}(T)]$ is a unitary matrix, and may be denoted by $D[l; \rho]$ because it too is independent of parameterization or the speed with which a particular path is traversed and is therefore geometric.

Berry's 1984 paper, and the other reports discussed above, were followed by a great number of papers on the subject. They can be divided into two broad groups. The first contains contributions that reformulate or generalize Berry's findings, while the second contains papers in which geometric phases are identified or measured in a great number of apparently disparate physical phenomena, or in which attempts are made to use these phases to explain unresolved physical questions. The bibliography that follows is selective and by no means exhaustive since there are hundreds of research papers on the subject.

We conclude by remarking that, there are at least four different reasons for the phenomenal success of the concepts related to geometric phases. First, these concepts are exceptionally clear and have a very elegant geometric interpretation in terms of anholonomies and connections (gauge fields). Second, geometric phases have a certain unifying character that enables one to relate many apparently disparate phenomena. Third, these phases can be observed and, indeed, various predictions of the geometric phases have been amply corroborated. Finally, and perhaps most importantly, these concepts reassert the importance and fruitfulness of geometric ideas in physical theories.

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I. ANTICIPATIONS OF THE GEOMETRIC PHASE

- 1. Actually, there are some Russian theoretical papers, collected in the reference [18] below, which anticipate the geometric phase as early as the 1940s.
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- 3. "Significance of electromagnetic potentials in the quantum theory," Y. Aharonov and D. Bohm, Phys. Rev. 115 485-491 (1959). (I)
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II. FOUNDATIONAL DERIVATIONS AND FORMULATIONS

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- 8. "Holonomy, the quantum adiabatic theorem, and Berry's phase," B. Simon, Phys. Rev. Lett. **51**, 2167-2170 (1983). (A)
- 9. "Appearance of gauge structure in simple dynamical systems," F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111-2114 (1984). (I)
- 10. "Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian," J.H. Hannay, J. Phys. A 18, 221-230 (1985). (I)
- 11. "Phase change during a cyclic quantum evolution," Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593-1596 (1987). (I)
- 12. "General setting for Berry's phase," J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339-2342 (1988). (A)
- 13. "Non-adiabatic non-abelian geometric phase," J. Anandan, Phys. Lett. A 133, 171-175 (1988). (I)
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 (I)
- 15. "Geometrical Phases from Global Gauge Invariance of Non Linear Classical Field Theories," J.C. Garrison and R.Y. Chiao, Phys. Rev. Lett. **60**, 165-168 (1988). (I)
- 16; "Comment on Geometric Phases for Classical Field Theories," J. Anandan, Phys. Rev. Lett. **60**, 2555 (1988). (I)

III. BOOKS AND REVIEW ARTICLES

- 17. **Geometric Phases in Physics**, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989). This book contains original reprints of many pioneering articles on the subject with some introductory comments on each sub-topic, from elementary to advanced. (E, I, A)
- 18. **Topological Phases in Quantum Theory**, edited by B. Markovski and S.I. Vinitsky (World Scientific, Singapore, 1989). This book traces anticipations of the geometric phase emphasizing Russian contributions to the subject and contains many less well-known reprints. (E, I, A)
- 19. "Berry's topological phase in quantum-mechanics and quantum-field theory," I.J.R. Aitchison, Physica Scripta T 23, 12-20 (1988). (I)
- "Adiabatic quantum transport in multiply connected systems," J.E. Avron, B. Zur, and A. Raveh, Rev. Mod. Phys. 60, 873-915 (1988). (A)
- 21. "Berry phase." J.W. Zwanziger, M. Koenig, and A. Pines, Annu. Rev. Phys. Chem. **41**, 601-646 (1990). This review contains an extensive list of articles up to 1990. (I)
- 22. "Anticipations of the geometric phase," M.V. Berry, Physics Today 43, 34-40 (1990). (E)
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- 24. "The geometric phase," J. Anandan, Nature **360**, 307-313 (1992). (E)
- 25. "The geometric phase in molecular-systems," C.A. Mead, Rev. Mod. Phys. 64, 51-85 (1992). (I)

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- 27. "Pancharatnam memorial issue," Guest Editors: S. Ramaseshan and R. Nityananda, Current Science, India, 67, 217-294 (1994). This special issue contains many original contributions in addition to some review articles. (E)

IV. ELABORATIONS AND APPLICATIONS

A. Theoretical articles

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- 32. "Classical and quantum adiabatic invariants," E. Gozzi, Phys. Lett. B 165, 351-354 (1985). (I)
- 33. "Quantum holonomy and the chiral gauge anomaly," A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. 55, 927-930 (1985). (A)
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- 35. "The interference of polarized light as an early example of Berry's phase," S. Ramaseshan and R. Nityananda, Current Science, India, 55, 1225-1226 (1986). (E)
- 36. "Some geometrical considerations of Berry phase," J. Anandan and L. Stodolsky, Phys. Rev. D 35, 2597-2600 (1987). (I)
- 37. "Classical adiabatic holonomy and its canonical structure," E. Gozzi and W.D. Thacker, Phys. Rev. D 35, 2398-2406 (1987). (I)
- 38. "Berry phases, magnetic monopoles, and Wess-Zumino terms or how the skyrmion got its spin," I.J.R. Aitchison, Acta Physica Polonica B 18, 207-235 (1987). (A)
- 39. "Non-abelian Berry phase effects and optical-pumping of atoms," J. Segert, Ann. Phys. 179, 294-312 (1987). (I)
- 40. "Non-abelian Berry phase, accidental degeneracy, and angular-momentum," J. Segert, J. Math. Phys. **28**, 2102-2114 (1987). (I)
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Note Added

After this resource letter was written the following interesting papers on the geometric phase have been brought to our attention:

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