Quantum Statistical Mechanics

Path integrals for bosons, fermions, and quantum spin systems

Alejandro Muramatsu

Institut für Theoretische Physik III
Universität Stuttgart

Les Houches Doctoral Training in Statistical Physics

Contents

Chapter 1

Path integrals for bosons

There is an intimate connection between statistical mechanics and quantum field-theory that is best seen by discussing the path-integral formulation of quantum mechanics. We are not going to dwell much on the historical reasons or in a detailed discussion of the equivalences but concentrate more on those aspects that are closer to our initial motivation, that is to statistical mechanics. Such a connection is most useful when discussing the critical properties of many-body quantum systems, e.g. close to a quantum phase transition, where the wealth of knowledge from statistical mechanics can be used to describe the universality class to which a given quantum critical point belongs. We will also see in later chapters, that specific properties of quantum systems rely on the topological properties of certain terms that may arise, whose geometric interpretation becomes clear in a path integral description.

We start these lectures by considering first the simplest case of a single degree of freedom and its path-integral, as introduced by Feynman [1]. Then, this chapter will be dedicated to bosons, i.e. degrees of freedom that have a classical limit. Starting from a Hamiltonian in second quantization we will arrive by means of a coherent-state representation at the corresponding path-integral.

1.1 Quantum mechanics

We discuss first the path-integral formulation for a single particle as was initially introduced by Feynman. As in the elementary lectures in quantum mechanics, we start by considering the Hamiltonian of the system

$$H = \frac{\hat{p}^2}{2m} + V(\hat{q}) \ , \tag{1.1}$$

where $\hat{p}$ and $\hat{q}$ are momentum and position operators, respectively, m is the mass of a particle and $V(\hat{q})$ a potential. Such a Hamiltonian was obtained in canonical quantization by looking at the Hamilton-function of the classical system and promoting the canonical variables p and q into operators. Since we expect that at some point the action appears, let us recall its form. The Hamilton function $H(p, q)$ and the Lagrangian $L(q, \dot{q})$ are connected by a Legendre transformation

$$H = p\dot{q} - L \; , \tag{1.2}$$

such that the Lagrangian is given by

$$L \;=\; \frac{1}{2}m\dot{q}^2 - V(q) \; , \tag{1.3}$$

so that the action is

$$S \;=\; \int \mathrm{d}t\, L(q,\dot{q},t) = \int \mathrm{d}t \left[\frac{1}{2}m\dot{q}^2 - V(q)\right] \; . \tag{1.4}$$

We will see now that starting from the Hamiltonian (1.1), we will arrive to an expression containing the action (1.4) but fully in the frame of quantum mechanics.

Let us calculate the amplitude for the system to reach the state $|q_f>$ at time t, if it was in state $|q_i>$ at time $t=0$. This amplitude can be expressed by the time evolution operator

$$\begin{aligned} &< q_f|e^{-\frac{i}{\hbar}Ht}|q_i> \\ &\quad =< q_f|e^{-iH(t-t_{N-1})/\hbar}\cdots e^{-iH(t_{j+1}-t_j)/\hbar}\cdots e^{-iHt_1/\hbar}|q_i> \; , \end{aligned} \tag{1.5}$$

where we have devided the time interval t in N segments of equal length that we denote from now on $\varepsilon \equiv t_{j+1} - t_j$. Between each pair of evolution operators for intervals ε, we can introduce the identity $\mathbf{1} = \int \mathrm{d}q_j |q_j><q_j|$, such that

$$\begin{aligned} &< q_f|e^{-\frac{i}{\hbar}Ht}|q_i> \\ &\quad = \int \prod_i^{N-1} \mathrm{d}q_i \;< q_f|e^{-i\varepsilon H/\hbar}|q_{N-1}> \cdots < q_1|e^{-i\varepsilon H/\hbar}|q_i> \;\; . \end{aligned} \tag{1.6}$$

However, since H contains both operators $\hat{p}$ and $\hat{q}$ that do not commute, we split the kinetic and potential terms making an error of $\mathcal{O}\left(\varepsilon^2\right)$, that in the limit $N \gg 1$ is small. The splitting above is known as Trotter-Suzuki slicing [2, 3], and is often used in quantum Monte Carlo algorithms (see e.g. [4]).

$$e^{-i\varepsilon H/\hbar} \;=\; \exp\left(-i\frac{\varepsilon}{\hbar}\frac{\hat{p}^2}{2m}\right) \exp\left[-i\frac{\varepsilon}{\hbar}V\left(\hat{q}\right)\right] + \mathcal{O}\left(\varepsilon^2\right) \; . \tag{1.7}$$

Here we used the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{\epsilon(A+B)} \;=\; \mathrm{e}^{\epsilon A}\mathrm{e}^{\epsilon B}\mathrm{e}^{\epsilon^2 C} \; , \tag{1.8}$$

where

$$C \;=\; \frac{1}{2}\left[A, B\right] + \mathcal{O}(\epsilon) \; . \tag{1.9}$$

The commutator $[A, B]$ leads in our case to

$$[A, B] \quad \to \quad -\frac{1}{2m\hbar^2}\left[\hat{p}^2, V(\hat{q})\right] \ . \tag{1.10}$$

Assuming that $V(\hat{q})$ is a smooth function of $\hat{q}$, we can perform a Taylor expansion such that

$$\left[\hat{p}^2, V(\hat{q})\right] \quad \to \quad [\hat{p}, \hat{q}^n] \ = \ -i\hbar n \hat{q}^{n-1} \ . \tag{1.11}$$

Since $\hat{q}$ is self-adjoint, $\mathrm{e}^{\epsilon^2 C}$ is a unitary operator that in the limit $\epsilon \to 0$ converges to the identiy. In this way we have on the one hand

$$\exp\left[-i\frac{\varepsilon}{\hbar}V\left(\hat{q}\right)\right] \mid q_j > \quad = \quad e^{-i\varepsilon V(q_j)/\hbar} \mid q_j > \ . \tag{1.12}$$

On the other hand, for the kinetic term we have to go over to eigenstates of $\hat{p}$

$$|q >= \int \frac{dp}{\sqrt{2\pi\hbar}} e^{ipq/\hbar} |p > \ , \tag{1.13}$$

such that for an infinitesimal time interval we have

$$\begin{aligned}
&< q_{j+1}| \exp\left(-i\frac{\varepsilon}{\hbar}\frac{\hat{p}^2}{2m}\right)|q_j > \\
&= \int \frac{\mathrm{d}p_{j+1}}{\sqrt{2\pi\hbar}} \int \frac{\mathrm{d}p_j}{\sqrt{2\pi\hbar}}\, e^{-i(p_{j+1}q_{j+1}-p_j q_j)/\hbar} \exp\left(-i\frac{\varepsilon}{\hbar}\frac{p_j^2}{2m}\right) \underbrace{< p_{j+1}|p_j >}_{\delta(p_{j+1}-p_j)} \\
&= \lim_{\delta \to 0^+} \frac{1}{2\pi\hbar} \int \mathrm{d}p_j \, e^{-ip_j(q_{j+1}-q_j)/\hbar} \exp\left[-i\frac{(\varepsilon - i\delta)}{\hbar}\frac{p_j^2}{2m}\right] \\
&= \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \exp\left[\frac{i}{\hbar}\frac{m}{2\epsilon}\left(q_{j+1} - q_j\right)^2\right] \ .
\end{aligned} \tag{1.14}$$

Putting everything together, and taking the limit $N \to \infty$, we have for the transition amplitude,

$$\begin{aligned}
&< q_f|e^{-\frac{i}{\hbar}Ht}|q_i > \\
&\quad = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\varepsilon}\right]^{N/2} \int \prod_i^{N-1} \mathrm{d}q_i \\
&\qquad \times \exp\left\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\left[\frac{m}{2\epsilon}\left(q_{j+1} - q_j\right)^2 - \varepsilon V(q_j)\right]\right\} \ ,
\end{aligned} \tag{1.15}$$

with the boundary conditions $q_0 = q_i$ and $q_N = q_f$. Since in the limit $N \to \infty$, $\varepsilon \to 0$, $\sum_j \varepsilon$ is a Riemann summation, and hence, asuming continuity of the variables, we can go over to an integral. In the same way, the kinetic term gives a derivative

$$\lim_{\varepsilon\to 0}\frac{1}{\epsilon}\left(q_{j+1} - q_j\right)^2 \quad = \quad \lim_{\varepsilon\to 0}\varepsilon\left(\frac{q_{j+1} - q_j}{\epsilon}\right)^2 = \varepsilon\left(\frac{\mathrm{d}q_j}{\mathrm{d}t}\right)^2 \ . \tag{1.16}$$

Finally we can write the transition amplitude in the form of a path integral

$$< q_f|e^{-\frac{i}{\hbar}Ht}|q_i > \; = \; \mathcal{C}\int \mathcal{D}q(t)\, e^{iS/\hbar} \,, \tag{1.17}$$

where the action is given by

$$S \; = \; \int_0^t \mathrm{d}t \left[\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - V(q)\right] \,, \tag{1.18}$$

and $\mathcal{C}$ takes into account all the multiplicative constants entering (1.15). We also introduced the notation

$$\int \mathcal{D}q(t) \; = \; \int \prod_i^{N-1} \mathrm{d}q_i \,, \tag{1.19}$$

denoting the integration over all possible trajectories statrting at q_i at $t = 0$, and ending at q_f at time t.

Several remarks are pertinent here.

i) The quantum mechanical amplitude for a transition can be obtained on the basis of the classical action, where not only the classical path satisfying $\delta S = 0$ is present but all possible paths are included. In this representation we do not deal with operators anymore.

ii) The classical limit is obtained in a natural way. If we take the limit $\hbar \to 0$, the saddle-point gives the only contribution, that corresponds to $\delta S = 0$.

iii) In the derivation above we assumed continuity of the dynamical variables. It is not *a priori* clear that this is true.

It is also interesting to discuss the possibility of taking imaginary times $\tau = it$. This is nothing else than an analytic continuation of (1.17) into imaginary time, also called a Wick rotation. In doing so, we have

$$< q_f|e^{-\frac{1}{\hbar}H\tau}|q_i > \; = \; \mathcal{C}'\int \mathcal{D}q(\tau)\, e^{-S_E/\hbar} \,, \tag{1.20}$$

with the *euclidean* action

$$S_E \; = \; \int_0^\tau \mathrm{d}\tau \left[\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q)\right] \,, \tag{1.21}$$

and a real constant $\mathcal{C}'$. Notice that the action contains an inverted potential ($V \to -V$). The interest on imaginary times has several reasons:

i) The expression (1.20) is purely real, and hence amenable to a direct computation, where each classical path has a positive definite weight $\exp\left(-S_E/\hbar\right)$. This renders the problem into a purely statistical one.

ii) In the case we would like to compute the partition function of a quantum mechanical problem, we have

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} \ , \tag{1.22}$$

such that we need to consider the evolution of the states entering the trace along the imaginary time $\hbar\beta$.

iii) If we are interested only in the ground-state $\mid \Psi_G >$ of a system, we can project it out of a trial wave function $\mid \Psi_T >$, as long as $< \Psi_T \mid \Psi_G > \neq 0$, since

$$\lim_{\theta\to\infty} \mathrm{e}^{-\theta H} \mid \Psi_T > \sim \mid \Psi_G > \ . \tag{1.23}$$

Also here we can view the projection as an evolution in imaginary time.

1.2 Quantum field-theory in d-dimensions

Without trying to be rigorous, let us construct by analogy the partition function for a system described by fields. Starting with a Hamiltonian in d dimensions of the following form:

$$H = \int \mathrm{d}^d r \left[\frac{1}{2}\Pi^2(r) + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \ , \tag{1.24}$$

where canonical conjugated fields satisfying $[\phi(\boldsymbol{x}), \Pi(\boldsymbol{x}')] = i\delta(\boldsymbol{x} - \boldsymbol{x}')$, are assumed (from now on $\hbar = 1$).

The partition function in imaginary time should have the form

$$Z = \int \mathcal{D}\phi \, \mathrm{e}^{-S_E} \ , \tag{1.25}$$

with the action

$$\begin{aligned} S_E &= \int \mathrm{d}^d r \int \mathrm{d}\tau \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \\ &= \int \mathrm{d}^{d+1} r \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \ . \end{aligned} \tag{1.26}$$

The expressions above should emphasize the analogy between statistical mechanics in $(d+1)$ dimensions and euclidean quantum field-theory in d dimensions, where imaginary time becomes an extra dimension for the corresponding field-theory in statistical mechanics. Below we tabulated a dictionary translating between objects in statistical mechanics and quantum field-theory.

Statistical Mechanics	**Quantum Field-Theory**
Hamiltonian	Euclidean action
Partition function Z	Generating functional
Correlation function	Expectation value of time ordered products
correlation length $\sim \xi \to e^{-\frac{r}{\xi}}$	mass $\to e^{-mr}$
free energy $F = -\ln Z$	Ground-state energy
$(T - T_c^{MF}) \sim m_0^2$	bare mass

1.3 Coherent states for bosons

Here we will follow the discussion by Negele and Orland [5]. We start by considering a system of bosons with pairwise interactions, where the Hamiltonian is given by

$$H = \sum_{i,j} T_{ij}\, b_i^\dagger b_j + \frac{1}{2} \sum_{i,j,k,\ell} V_{i,j,k,\ell}\, b_i^\dagger b_j^\dagger b_k b_\ell \ , \tag{1.27}$$

where $b_i^\dagger$ and b_i are creation and annihilation operators acting on state $|i\rangle$, respectively, obeying bosonic commutation relations

$$\left[b_i, b_j^\dagger\right] = \delta_{ij} \ , \quad \left[b_i, b_j\right] = \left[b_i^\dagger, b_j^\dagger\right] = 0 \ . \tag{1.28}$$

Now we recall that for the construction of the Feynman path-integral, an important step was to perform a Trotter splitting, where between the time-slices a complete set of states for the operators $\hat{p}$ or $\hat{q}$ was introduced. Since in the present case we have operators in second quantization, by analogy, we seek for eigenstates of bosonic creation and/or annihilation operators. The set of possible states belong to the Fock space, where a general state contains all possible occupation numbers:

$$|\phi\rangle = \sum_{n=0}^{\infty} \sum_{\alpha_1,\ldots,\alpha_n} \phi_{\alpha_1,\ldots,\alpha_n} |\alpha_1,\ldots,\alpha_n\rangle \ , \tag{1.29}$$

with the case $n=0$ corresponding to the vacuum. We consider first the possibility of $|\phi\rangle$ being an eigenstate of a creation operator $b_\alpha^\dagger$. Since $|\phi\rangle$ has a component with a minimum number of particles, if we create a particle, we increase the minimum number by one, such that the new state cannot be proportional to the old one, and hence, cannot be an eigenstate. The second possiblity is $|\phi\rangle$ being an eigenstate of an annihilation operator b_α. If we apply it to (1.29), we could in principle have a state proportional to $|\phi\rangle$, since all possible number of particles are present.

A convenient basis for the Fock-space is given by occupation number states, since they are already symmetrized and orthonormalized. In this representation, a general state can be expanded as

$$|\phi\rangle = \sum_{n_{\alpha_1},\ldots,n_{\alpha_j},\ldots} \phi_{n_{\alpha_1},\ldots,n_{\alpha_j},\ldots} |n_{\alpha_1},\ldots,n_{\alpha_j},\ldots\rangle \ . \tag{1.30}$$

If $|\phi\rangle$ is an eigenstate of an annihilation operator, it should fulfil

$$b_{\alpha_i}|\phi\rangle = \phi_{\alpha_i}|\phi\rangle \ . \tag{1.31}$$

Since $|n_{\alpha_1},\ldots,n_{\alpha_j},\ldots\rangle$ constitute a basis, the eigenvalue equation leads to the condition

$$\phi_{\alpha_i}\phi_{n_{\alpha_1},\ldots,n_{\alpha_i}-1,\ldots} = \sqrt{n_{\alpha_i}}\,\phi_{n_{\alpha_1},\ldots,n_{\alpha_i},\ldots} \ . \tag{1.32}$$

This relation leads to a recursion for $\phi_{n_{\alpha_1},\dots,n_{\alpha_i},\dots}$ until we reach $n_{\alpha_i} = 0$. By carrying out such a recursion for all the states α_i, we arrive at

$$\phi_{n_{\alpha_1},\dots,n_{\alpha_i},\dots} = \frac{\phi_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{\phi_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots . \tag{1.33}$$

On the other hand, the occupation number states can be expressed as

$$|n_{\alpha_1}, \dots, n_{\alpha_i}, \dots\rangle = \frac{{b^\dagger}_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{{b^\dagger}_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots |0\rangle , \tag{1.34}$$

such that together with (1.33) leads to the following form for an eigenstate of annihilation operators

$$|\phi\rangle = \prod_\alpha \sum_{n_\alpha} \frac{\left(\phi_\alpha {b^\dagger}_\alpha\right)^{n_\alpha}}{n_\alpha!} |0\rangle = \exp\left(\sum_\alpha \phi_\alpha {b^\dagger}_\alpha\right) |0\rangle . \tag{1.35}$$

Once we have explicitely constructed a coherent state, we can see that its adjoint is a left eigenstate of the creation operator:

$$\langle\phi| b_\alpha^\dagger = \langle\phi|\phi_\alpha^* . \tag{1.36}$$

Furthermore, the following relations hold:

$$\begin{aligned} b_\alpha^\dagger|\phi\rangle &= \frac{\partial}{\partial\phi_\alpha}|\phi\rangle , \\ \langle\phi|b_\alpha &= \frac{\partial}{\partial\phi_\alpha^*}\langle\phi| , \\ \langle\phi|\phi'\rangle &= \exp\left(\sum_\alpha \phi_\alpha^*\phi_\alpha'\right) . \end{aligned} \tag{1.37}$$

An important property we need for setting up a path integral is that the set of states we use is complete. For coherent states one speaks of the resolution of unity. It is accomplished by the following form:

$$\int \prod_\alpha \frac{\mathrm{d}\phi_\alpha^* \mathrm{d}\phi_\alpha}{2\pi i} \exp\left(-\sum_\alpha \phi_\alpha^*\phi_\alpha\right) |\phi\rangle\langle\phi| = \mathbf{1} , \tag{1.38}$$

where $\mathbf{1}$ is the identity operator in Fock-space. We can show that the above is true by directly performing the integration. Going over to the real and imaginary parts of ϕ_α, we have for the measure

$$\frac{\mathrm{d}\phi_\alpha^* \mathrm{d}\phi_\alpha}{2\pi i} = \frac{\mathrm{dRe}\phi_\alpha \mathrm{dIm}\phi_\alpha}{\pi} , \tag{1.39}$$

such that using (1.33) we obtain

$$
\begin{aligned}
\int \prod_\alpha \frac{\mathrm{d}\phi_\alpha^* \mathrm{d}\phi_\alpha}{2\pi i} \exp\left(-\sum_\alpha \phi_\alpha^* \phi_\alpha\right) |\phi\rangle\langle\phi| \\
= \sum_{\substack{n_{\alpha_1},\ldots,n_{\alpha_i},\ldots \\ n'_{\alpha_1},\ldots,n'_{\alpha_i},\ldots}} |n_{\alpha_1},\ldots,n_{\alpha_i},\ldots\rangle\langle n'_{\alpha_1},\ldots,n'_{\alpha_i},\ldots| \\
\times \prod_{\alpha_i} \int \frac{\mathrm{d}\rho_i \mathrm{d}\theta_i}{\pi} \rho_i \mathrm{e}^{-\rho_i^2} \frac{\left(\rho_i \mathrm{e}^{i\theta_i}\right)^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \frac{\left(\rho_i \mathrm{e}^{-i\theta_i}\right)^{n'_{\alpha_i}}}{\sqrt{n'_{\alpha_i}!}} \\
= \sum_{n_{\alpha_1},\ldots,n_{\alpha_i},\ldots} |n_{\alpha_1},\ldots,n_{\alpha_i},\ldots\rangle\langle n_{\alpha_1},\ldots,n_{\alpha_i},\ldots| = \mathbf{1} \; .
\end{aligned} \tag{1.40}
$$

Since as shown in (1.37) the coherent states are in general not orthogonal, the set of coherent states is overcomplete.

1.4 Path-integrals for bosons

Now we have all the ingredients to set up the path-integral for the Hamiltonian (1.27). To be specific, and since we are focusing on statistical mechanics, we consider the partition function in the grand-canonical ensemble.

$$
\begin{aligned}
Z &= \mathrm{Tr}\, \mathrm{e}^{-\beta \hat{K}} \\
&= C \int \prod_\alpha \mathrm{d}\phi_\alpha^* \mathrm{d}\phi_\alpha \exp\left(-\sum_\alpha \phi_\alpha^* \phi_\alpha\right) \langle\phi|\mathrm{e}^{-\beta\hat{K}}|\phi\rangle \; ,
\end{aligned} \tag{1.41}
$$

where we defined the operator

$$
\hat{K} \equiv H - \mu \hat{N} \; , \tag{1.42}
$$

as a short notation. As in the case of the Feynman path-integral we introduce a Trotter slicing

$$
\mathrm{e}^{-\beta\hat{K}} = \prod_{\ell=1}^{L} \mathrm{e}^{-\Delta\tau\hat{K}} \; , \tag{1.43}
$$

where $\Delta\tau = \beta/L$. Then, the resolution of unity is introduced between the time-slices and we take a normal ordered form

$$
\mathrm{e}^{-\Delta\tau\hat{K}} = \; : \mathrm{e}^{-\Delta\tau\hat{K}} : + \mathcal{O}\left(\Delta\tau^2\right) \; , \tag{1.44}
$$

where normal ordering means that all the creation operators are at the left of all annihilation operators. Here we have an error of $\mathcal{O}\left(\Delta\tau^2\right)$ that will vanish in the limit $L \to \infty$. The partition function (1.41) looks now as follows:

$$
\begin{aligned}
Z &\propto \lim_{L\to\infty} \int \prod_{\ell=1}^{L} \prod_{\alpha} \mathrm{d}\phi^*_{\alpha,\ell} \mathrm{d}\phi_{\alpha,\ell} \exp\left(- \sum_{\ell=1}^{L} \sum_{\alpha} \phi^*_{\alpha,\ell} \phi_{\alpha,\ell} \right) \\
&\quad \times \prod_{\ell=1}^{L} \langle \phi_\ell | 1 - \Delta\tau \hat{K} + \mathcal{O}\left(\Delta\tau^2\right) | \phi_{\ell-1} \rangle \ , \qquad (1.45)
\end{aligned}
$$

where we used the convention that $|\phi_0\rangle = |\phi_L\rangle$, and $\phi_{\alpha,L} = \phi_{\alpha,0} = \phi_\alpha$. The matrix elements of $\hat{K}\left(b^\dagger_\alpha, b_\beta\right)$ between coherent states are easily obtained, since those states are left and right eigenstates of $b^\dagger_\alpha$ and b_α, respectively.

$$
\langle \phi_\ell | \hat{K}\left(b^\dagger_\alpha, b_\beta\right) | \phi_{\ell-1} \rangle = K\left(\phi^*_\alpha, \phi_\beta\right) \langle \phi_\ell | \phi_{\ell-1} \rangle \ , \qquad (1.46)
$$

where now K is a complex valued function of ϕ^*_α, ϕ_α. Recalling that the overlap between coherent states is given in (1.37), we have finally,

$$
Z = \lim_{L\to\infty} \int \prod_{\ell=1}^{L} \prod_{\alpha} \mathrm{d}\phi^*_{\alpha,\ell} \, \mathrm{d}\phi_{\alpha,\ell} \, \mathrm{e}^{-S(\phi^*,\phi)} \ , \qquad (1.47)
$$

where

$$
\begin{aligned}
S(\phi^*,\phi) &= \sum_{\ell=2}^{L} \Delta\tau \left\{ \sum_{\alpha} \phi^*_{\alpha,\ell} \left[\frac{(\phi_{\alpha,\ell} - \phi_{\alpha,\ell-1})}{\Delta\tau} - \mu \phi_{\alpha,\ell-1} \right] + H\left(\phi^*_{\alpha,\ell}, \phi_{\alpha,\ell-1}\right) \right\} \\
&\quad + \Delta\tau \sum_{\alpha} \phi^*_{\alpha,1} \left[\frac{(\phi_{\alpha,1} - \phi_{\alpha,L})}{\Delta\tau} - \mu \phi_{\alpha,L} \right] + H\left(\phi^*_{\alpha,1}, \phi_{\alpha,M}\right) \ . \qquad (1.48)
\end{aligned}
$$

As will be seen in the exercises, such a discrete form is the one that should be used for actual calculations. However, the usual notation, that makes a connection with a field theory, is assuming continuity of the fields, resulting in

$$
Z = \int \mathcal{D}\phi^*_\alpha(\tau) \, \mathcal{D}\phi_\alpha(\tau) \, \mathrm{e}^{-S} \ , \qquad (1.49)
$$

where the action is given by

$$
S = \int_0^\beta \mathrm{d}\tau \sum_\alpha \left[\phi^*_\alpha(\tau) \left(\frac{\partial}{\partial\tau} - \mu \right) \phi_\alpha(\tau) + H\left(\phi^*_\alpha(\tau), \phi_\alpha(\tau)\right) \right] \ . \qquad (1.50)
$$

We also introduced the notation

$$
\int \mathcal{D}\phi^*_\alpha(\tau) \, \mathcal{D}\phi_\alpha(\tau) = \lim_{L\to\infty} \int \prod_{\ell=1}^{L} \prod_\alpha \mathrm{d}\phi^*_{\alpha,\ell} \, \mathrm{d}\phi_{\alpha,\ell} \ , \qquad (1.51)
$$

in a similar way as done for the Feynman path-integral. Due to the fact that the expression above originates from a trace, the fields obey periodic boundary conditions in imaginary time: $\phi(\beta) = \phi(0)$.

In the course of the calculation above we have discarded a multiplicative constant in the partition function. The reason is that a physical system will be completely described by its correlation functions in equilibrium. They can be obtained by attaching infinitesimal external fields and considering the response of the system to their variations. Moreover, we are generally interested in the cumulants, that is the correlations between fluctuations. For example, if we are interested in some observable $\hat{\mathcal{O}}(\boldsymbol{x})$, we may consider its expectation value $\langle \hat{\mathcal{O}}(\boldsymbol{x}) \rangle$ (one point function) or the response of the system to a field that couples to $\hat{\mathcal{O}}$ in the form of a susceptibility (two point function)

$$\chi(\boldsymbol{x}, \boldsymbol{x}') = \langle \hat{\mathcal{O}}(\boldsymbol{x}) \hat{\mathcal{O}}(\boldsymbol{x}') \rangle - \langle \hat{\mathcal{O}}(\boldsymbol{x}) \rangle \langle \hat{\mathcal{O}}(\boldsymbol{x}') \rangle \,, \tag{1.52}$$

that tells us about the fluctuations and their correlations in the system. In general we could wish to calculate an N-point function. This can be easily achieved by adding a source term to the action of the form

$$S \rightarrow S + \int \mathrm{d}\tau \, \mathrm{d}^d x \sum_\alpha \left[J^*_\alpha(\boldsymbol{x}, \tau) \phi_\alpha(\boldsymbol{x}, \tau) + \phi^*_\alpha(\boldsymbol{x}, \tau) J_\alpha(\boldsymbol{x}, \tau) \right] \,. \tag{1.53}$$

Then, for e.g. a two point correlator for bosons we have

$$\langle b_\alpha(\boldsymbol{x}, \tau) b^\dagger_\beta(\boldsymbol{x}', \tau') \rangle = \left. \frac{\delta^2 \ln Z}{\delta J^*_\alpha(\boldsymbol{x}, \tau) \delta J_\beta(\boldsymbol{x}', \tau')} \right|_{J=J^*=0} \,. \tag{1.54}$$

Since for cumulants we need to have the logarithm of the partition function, multiplicative constants do not contribute to the result, and hence, can be ignored at the outset. From the discussion above, we see that the partition function, with the inclusion of the appropriate source terms, can be used to obtain any desired correlation function. Therefore, it is called the generating functional, as we pointed out at the end of Sec. 1.2.

Problem: Partition function for the harmonic oscillator

Consider the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \,. \tag{1.55}$$

i) Give the form of the euclidean action using the expression (1.15), introducing the imaginary time $\tau = it$, and setting $N \Delta\tau = \hbar\beta$.

ii) Give explicitly the form of the partition function, showing that $x(\tau_j) = x(\tau_j + \hbar\beta)$.

iii) Due to the periodic boundary conditions of $x(\tau_j)$, it can Fourier-transformed:

$$x(\tau_j) = \frac{1}{\sqrt{\hbar\beta}} \sum_{\tilde{\omega}} x_{\tilde{\omega}} \, \mathrm{e}^{-i\tau_j\tilde{\omega}} \; . \tag{1.56}$$

Show that $\tilde{\omega}$ can only take discrete values ω_n with $n \in \mathbb{Z}$, and show which they are.

iv) Give the form of the action in terms of x_n.

v) Give the form of the partition function up to a normalization constant coming from the Fourier-transformation. Here it is convenient express it in terms of $n \geq 0$.

vi) Carry out the integrals. Since x_0 is real, it is convenient to treat it separately.

vii) Show that the result obtained can be expressed as a product of the result for $\omega = 0$, i.e. the case of a free particle and a rest. Disregard the part corresponding to the free case, lumping it into the normalization constant.

viii) Using that

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x} \; , \tag{1.57}$$

display the partition function in such a way that the eigenvalues of the harmonic oscillator are obtained.

Chapter 2

Path integrals for fermions

The main difficulty encountered in the case of fermions is the lack of a classical limit for them, in the sense that C-numbers cannot reflect the anticommuting character of fermions. We will motivate the introduction of anticommuting objects by following the same ideas we used in contructing coherent states for bosons. As in the case of bosons we follow mainly the discussion by Negele and Orland [5].

2.1 Grassmann algebras

We start, by essentially following the same path we had in the case of bosons, and consider a system of fermions with pairwise interactions, where the Hamiltonian is given by

$$H = \sum_{i,j} T_{ij}\, f_i^\dagger f_j + \frac{1}{2}\sum_{i,j,k,\ell} V_{i,j,k,\ell}\, f_i^\dagger f_j^\dagger f_\ell f_k \;, \tag{2.1}$$

where $f_i^\dagger$ and f_i are creation and annihilation operators acting on state $|i\rangle$, respectively, obeying fermionic anticommutation relations

$$\left\{f_i, f_j^\dagger\right\} = \delta_{ij} \;, \quad \left\{f_i, f_j\right\} = \left\{f_i^\dagger, f_j^\dagger\right\} = 0 \;. \tag{2.2}$$

In the bosonic case we looked for eigenstates of the annihilation operator, that turned out to be coherent states. Let us assume that such a state exists in our Fock space of interest. Then, it should fulfil

$$f_\alpha|\phi\rangle = \phi_a|\phi\rangle \;. \tag{2.3}$$

We can now consider another annihilation operator, and assuming that the eigenvalues are C-numbers, we have

$$f_\alpha f_\beta|\phi\rangle = \phi_\beta\phi_\alpha|\phi\rangle \;, \tag{2.4}$$

while for the other ordering we have

$$f_\beta f_\alpha|\phi\rangle = \phi_\alpha\phi_\beta|\phi\rangle \;, \tag{2.5}$$

but due to the anticommutation relations (2.2), the eigenvalues have also to anticommute, i.e. it should hold that $\phi_\beta\phi_\alpha = -\phi_\alpha\phi_\beta$. Therefore, they cannot be C-numbers.

We therefore consider now a set of elements $\{\xi_\alpha\}$, $\alpha = 1, \ldots, n$ with the property

$$\xi_\alpha\xi_\beta + \xi_\beta\xi_\alpha = 0 , \tag{2.6}$$

i.e. anticommuting objects, such that $\xi_\alpha^2 = 0$. They are the generators of a Grassmann algebra, that may be defined on $\mathbb{R}$ or on $\mathbb{C}$. We recall that an algebra is a set of elements with addition and product as possible operations. The generators give all the elements of the algebra by their products and linear combinations thereof. In the following we list a number of properties that will turn out to be necessary in finding the fermionic coherent states.

- Given n generators, we can construct a basis of the algebra by all possible products of the form

  $$\xi_1^{m_1}\xi_2^{m_2}\cdots\xi_n^{m_n} , \quad \text{with } m_i = 0, 1 , \tag{2.7}$$

  such that it contains 2^n elements: $\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \ldots, \xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n}\}$. A general element of the algebra is given by a linear combination of the elements of the basis.

- If n is even, we define the conjugation by assigning to each of $n/2$ generators ξ_α a generator denoted by ξ_α^* with the following properties:

  $$(\xi_\alpha)^* = \xi_\alpha^* , \quad (\xi_\alpha^*)^* = \xi_\alpha . \tag{2.8}$$

  For $\lambda \in \mathbb{C}$, we have $(\lambda\xi_\alpha)^* = \lambda^*\xi_\alpha^*$.

- For the conjugation of a product we require

  $$(\xi_{\alpha_1}\cdots\xi_{\alpha_n})^* = \xi_{\alpha_n}^*\xi_{\alpha_{n-1}}^*\cdots\xi_{\alpha_1}^* . \tag{2.9}$$

- Consider the case $n = 2$. The corresponding basis can be taken as $\{1, \xi, \xi^*, \xi^*\xi\}$.

  - If f is an analytic function (recall that an analytic complex function can be expanded in a Taylor series and depends only on z but not on z^*), then

    $$f(\xi) = f_0 + f_1\xi . \tag{2.10}$$

  - A general bilinear form is given by

    $$A(\xi^*, \xi) = a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi . \tag{2.11}$$

– Derivatives. Here we require that

$$\frac{\partial \xi_\alpha}{\partial \xi_\beta} = \delta_{\alpha\beta} \, . \tag{2.12}$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial \xi}(\xi^* \xi) &= \frac{\partial}{\partial \xi}(-\xi \xi^*) = -\xi^* \, , \\
\hookrightarrow \frac{\partial}{\partial \xi} A(\xi^*, \xi) &= a_1 - a_{12}\xi^* \, , \\
\frac{\partial}{\partial \xi^*}\frac{\partial}{\partial \xi} A(\xi^*, \xi) &= -a_{12} = -\frac{\partial}{\partial \xi}\frac{\partial}{\partial \xi^*} A(\xi^*, \xi) \, , \\
\hookrightarrow \left\{ \frac{\partial}{\partial \xi^*}, \frac{\partial}{\partial \xi} \right\} &= 0 \, .
\end{aligned} \tag{2.13}$$

– Integrals. The following rules are introduced:

$$\begin{aligned}
\int \mathrm{d}\xi = 0 \, , \qquad & \int \mathrm{d}\xi^* = 0 \, , \\
\int \mathrm{d}\xi \, \xi = 1 \, , \qquad & \int \mathrm{d}\xi^* \, \xi^* = 1 \, .
\end{aligned} \tag{2.14}$$

Then, the integral of $f(\xi)$ leads to

$$\int \mathrm{d}\xi \, f(\xi) = \int \mathrm{d}\xi \, (f_0 + f_1 \xi) = f_1 \, , \tag{2.15}$$

and for $A(\xi^*, \xi)$ we have

$$\begin{aligned}
\int \mathrm{d}\xi \, A(\xi^*, \xi) &= a_1 - a_{12}\xi^* \, , \\
\int \mathrm{d}\xi^* \, A(\xi^*, \xi) &= \bar{a}_1 + a_{12}\xi \, , \\
\int \mathrm{d}\xi^* \, \mathrm{d}\xi \, A(\xi^*, \xi) &= -a_{12} = -\int \mathrm{d}\xi \, \mathrm{d}\xi^* \, A(\xi^*, \xi) \, .
\end{aligned} \tag{2.16}$$

We see that with the adopted rules, differentiation and integration act in the same way. The generalization of the rules above to $n > 2$ and even is straightforward.

- δ-function. It is defined as

$$\delta(\xi, \xi') \equiv \int \mathrm{d}\eta \, \mathrm{e}^{-\eta(\xi - \xi')} = -(\xi - \xi') \, , \tag{2.17}$$

where η is also a Grassmann variable. We can easily verify that it works as a δ-function:

$$\int \mathrm{d}\xi' \, \delta(\xi, \xi') f(\xi') = -\int \mathrm{d}\xi' \, (\xi - \xi')(f_0 + f_1 \xi') = f(\xi) \, . \tag{2.18}$$

- Scalar product. We can define here the scalar product in a way that on the one hand resembles the form of a scalar product with bosonic coherent states, and on the other hand, allows functions of Grassmann variable to have the structure of a Hilbert space.

$$\begin{aligned}(f,g) &\equiv \int \mathrm{d}\xi^* \, \mathrm{d}\xi \, \mathrm{e}^{-\xi^*\xi} f^*(\xi^*) \, g(\xi^*) \\ &= \int \mathrm{d}\xi^* \, \mathrm{d}\xi \, (1-\xi^*\xi)(f_0^* + f_1^*\xi)(g_0 + g_1\xi^*) \\ &= f_0^* g_0 + f_1^* g_1 \; . \end{aligned} \tag{2.19}$$

2.2 Coherent states for fermions

After introducing Grassmann variables, we still need to deal with the commutation relations between Grassmann variables and fermionic operators. For that purpose we associate ξ_α to the annihilation operator f_α and ξ_α^* to $f_\alpha^\dagger$, and require

$$\{\xi_\alpha, f_\beta\} = \{\xi_\alpha^*, f_\beta\} = \{\xi_\alpha, f_\beta^\dagger\} = \{\xi_\alpha^*, f_\beta^\dagger\} = 0 \; , \tag{2.20}$$

and

$$(\xi_\alpha f_\beta)^\dagger = f_\beta^\dagger \, \xi_\alpha^* \; , \quad \text{etc} \; . \tag{2.21}$$

Since we already know how bosonic coherent states are, we try with a similar form for fermionic coherent states:

$$|\xi\rangle = \mathrm{e}^{-\sum_\alpha \xi_\alpha f_\alpha^\dagger}|0\rangle = \prod_\alpha (1 - \xi_\alpha f_\alpha^\dagger)|0\rangle \; . \tag{2.22}$$

Applying f_α on this state we have

$$\begin{aligned} f_\alpha|\xi\rangle &= f_\alpha \prod_\beta \left(1 - \xi_\beta f_\beta^\dagger\right)|0\rangle \\ &= \prod_{\beta\neq\alpha} \left(1 - \xi_\beta f_\beta^\dagger\right) \underbrace{f_\alpha \left(1 - \xi_\alpha f_\alpha^\dagger\right)|0\rangle}_{=\xi_\alpha|0\rangle = \xi_\alpha\left(1-\xi_\alpha f_\alpha^\dagger\right)|0\rangle} = \xi_\alpha|\xi\rangle \; . \end{aligned} \tag{2.23}$$

Hence $|\xi\rangle$ is an eigenstate of f_α, i.e. it is a coherent state. Once we have a coherent state, we can see that its adjoint is a left eigenstate of the creation operator:

$$\langle\xi|f_\alpha^\dagger = \langle\xi|\xi_\alpha^* \; . \tag{2.24}$$

As in the bosoniuc case we can consider the application of a creation operator on a coherent state.

$$\begin{aligned} f_\alpha^\dagger|\xi\rangle &= f_\alpha^\dagger \prod_{\beta\neq\alpha} \left(1 - \xi_\beta f_\beta^\dagger\right)|0\rangle \\ &= -\frac{\partial}{\partial\xi_\alpha}\left(1 - \xi_\alpha f_\alpha^\dagger\right) \prod_{\beta\neq\alpha}\left(1 - \xi_\beta f_\beta^\dagger\right)|0\rangle = -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle \; , \end{aligned} \tag{2.25}$$

and in the same way we have

$$\langle \xi | f_\alpha = \frac{\partial}{\partial \xi_\alpha^*} \langle \xi | \, . \tag{2.26}$$

The overlap of two coherent states is

$$\begin{aligned} \langle \xi | \xi' \rangle &= \langle 0 | \prod_\alpha \left(1 + \xi_\alpha^* f_\alpha\right) \left(1 - \xi'_\alpha f_\alpha^\dagger\right) |0\rangle \\ &= \langle 0 | \prod_\alpha \left(1 + \xi_\alpha^* \xi'_\alpha\right) |0\rangle = \mathrm{e}^{\sum_\alpha \xi_\alpha^* \xi'_\alpha} \, . \end{aligned} \tag{2.27}$$

After having seen the properties of the fermionic coherent states, we consider the resolution of unity. For that we just look at an expression similar to the one for bosons.

$$\hat{\mathcal{A}} = \int \prod_\alpha \mathrm{d}\xi_\alpha^* \mathrm{d}\xi_\alpha \, \exp\left(-\sum_\alpha \xi_\alpha^* \xi_\alpha\right) |\xi\rangle\langle\xi| \, . \tag{2.28}$$

Next let us consider the overlap between two elements in the basis of Fock-space,

$$\begin{aligned} |\alpha_1, \ldots, \alpha_n\rangle &= f_{\alpha_1}^\dagger \cdots f_{\alpha_n}^\dagger |0\rangle \, , \\ |\beta_1, \ldots, \beta_m\rangle &= f_{\beta_1}^\dagger \cdots f_{\beta_m}^\dagger |0\rangle \, . \end{aligned} \tag{2.29}$$

Their ovelap is

$$\langle \alpha_1, \ldots, \alpha_n | \beta_1, \ldots, \beta_m \rangle = \delta_{nm} \, P\begin{pmatrix} \alpha_1, \ldots, \alpha_n \\ \alpha_{i_1}, \ldots, \alpha_{i_n} \end{pmatrix} \delta_{\alpha_{i_1}\beta_1} \cdots \delta_{\alpha_{i_n}\beta_n} \, , \tag{2.30}$$

where $P\binom{\alpha_1,\ldots,\alpha_n}{\alpha_{i_1},\ldots,\alpha_{i_n}} = (-1)^p$ with p the parity of the permutation relating $\alpha_1, \ldots, \alpha_n$ and $\alpha_{i_1}, \ldots, \alpha_{i_n}$. Now we can look at the matrix element of $\hat{\mathcal{A}}$ between those two states. Here we have on the one hand

$$\langle \alpha_1, \ldots, \alpha_n | \xi \rangle = \langle 0 | f_{\alpha_n} \cdots f_{\alpha_1} | \xi \rangle = \xi_{\alpha_n} \cdots \xi_{\alpha_1} \, , \tag{2.31}$$

since the number of permutations at the end is even. In the same way we have

$$\langle \xi | \beta_1, \ldots, \beta_m \rangle = \xi_{\beta_1}^* \cdots \xi_{\beta_n}^* \, , \tag{2.32}$$

such that

$$\begin{aligned} \langle \alpha_1, \ldots, \alpha_n | \hat{\mathcal{A}} | \beta_1, \ldots, \beta_m \rangle = &\int \prod_\alpha \mathrm{d}\xi_\alpha^* \mathrm{d}\xi_\alpha \prod_\alpha \left(1 - \xi_\alpha^* \xi_\alpha\right) \\ &\times \xi_{\alpha_n} \cdots \xi_{\alpha_1} \xi_{\beta_1}^* \cdots \xi_{\beta_n}^* \, . \end{aligned} \tag{2.33}$$

For each index α, the possible combinations that can appear are

$$\int \mathrm{d}\xi_\alpha^* \mathrm{d}\xi_\alpha \, \left(1 - \xi_\alpha^* \xi_\alpha\right) \begin{Bmatrix} \xi_\alpha \xi_\alpha^* \\ \xi_\alpha^* \\ \xi_\alpha \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix} . \tag{2.34}$$

Therefore, non-vanishing results are obtained only when some α_i equals some β_j, leading to $m = n$, and such that we can pair the α_i's and β_j's. However, since we are dealing with Grassmann variables, the order matters. This can be taken into account as follows:

$$\xi^*_{\beta_1} \cdots \xi^*_{\beta_n} = P \begin{pmatrix} \alpha_1, \ldots, \alpha_n \\ \alpha_{i_1}, \ldots, \alpha_{i_n} \end{pmatrix} \delta_{\alpha_{i_1}\beta_1} \cdots \delta_{\alpha_{i_n}\beta_n} \xi^*_{\alpha_1} \cdots \xi^*_{\alpha_n} . \tag{2.35}$$

Inserting all this into (2.33), and comparing with (2.30), we have finally the resolution of unity for fermionic coherent states,

$$\int \prod_\alpha \mathrm{d}\xi^*_\alpha \mathrm{d}\xi_\alpha \exp\left(- \sum_\alpha \xi^*_\alpha \xi_\alpha \right) |\xi\rangle\langle\xi| = \mathbf{1} . \tag{2.36}$$

As a last remark before going over to the path-integral, we discuss here Gaussian integrals, both for bosons and for fermions, since such operations are the ones most commonly carried out with path-integrals.

2.2.1 Gaussian integrals

For a single real variable, the Gaussian integral is

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}} . \tag{2.37}$$

This result can be generalized for n variables multiplying a real symmetric positive definite matrix A. We consider the following integral:

$$I = \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\left(-\frac{1}{2} x_i A_{ij} x_j + x_i J_i \right) , \tag{2.38}$$

where summation over repeated indices is understood. We first displace the argument of the exponential by a change of variables

$$y_i = x_i - A^{-1}_{ij} J_j , \tag{2.39}$$

such that

$$-\frac{1}{2} x_i A_{ij} x_j + x_i J_i = -\frac{1}{2} y_i A_{ij} y_j + J_i A^{-1}_{ij} J_j . \tag{2.40}$$

We are left with the integral over y_i's. Since A is real and symmetric, it can be diagonalized by an orthonormal transformation, leading to

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}y_1 \cdots \mathrm{d}y_n \exp\left(-\frac{1}{2} y_i A_{ij} y_j \right) &= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}z_i \, \mathrm{e}^{-\frac{a_i}{2} z_i^2} \\ &= \prod_{i=1}^{n} \sqrt{\frac{2\pi}{a_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} . \end{aligned} \tag{2.41}$$

Finally, we have

$$I \;=\; \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(J_i A^{-1}_{ij} J_j\right) \; . \tag{2.42}$$

The calculations above can be easily generalized to complex (bosons) and Grassmann fields (fermions).

- Gaussian integrals for bosons. In this case a hermitic positve matrix H for the integral

$$I_b \;=\; \int \prod_{i=1}^{n} \frac{\mathrm{d}\phi_i^* \, \mathrm{d}\phi_i}{2\pi i} \, \exp\left(-\phi_i^* H_{ij} \phi_j + J_i^* \phi_i + \phi_i^* J_i\right) \; . \tag{2.43}$$

  As before, a change of variables can be performed and H can be diagonalized. For each eigenvalue we have then the integral

$$\int \frac{\mathrm{d}z_i^* \, \mathrm{d}z_i}{2\pi i} \, \mathrm{e}^{-z_i^* h_i z_i} \;=\; \int \frac{\mathrm{d}u_i \, \mathrm{d}v_i}{\pi} \, \mathrm{e}^{-h_i\left(u_i^2+v_i^2\right)} = \frac{1}{h_i} \; , \tag{2.44}$$

  where we set $z_i = u_i + iv_i$. Then, we have

$$I_b \;=\; \frac{1}{\det H} \exp\left(J_i^* H^{-1}_{ij} \, J_j\right) \; . \tag{2.45}$$

- Gaussian integrals for fermions. As in the case of bosons we consider a hermitic positive matrix H. The integral is

$$I_f \;=\; \int \prod_{i=1}^{n} \mathrm{d}\eta_i^* \, \mathrm{d}\eta_i \, \exp\left(-\eta_i^* H_{ij} \eta_j + \zeta_i^* \eta_i + \eta_i^* \zeta_i\right) \; , \tag{2.46}$$

  where η_i^*, η_i, ζ_i^*, and ζ_i are Grassmann variables. As before, a shift of variables can be performed, and after a unitary transformation that diagonalizes H, we arrive at the following integral for each eigenvalue

$$\int \mathrm{d}\xi_i^* \, \mathrm{d}\xi_i \, \mathrm{e}^{-xi_i^* h_i xi_i} \;=\; \int \mathrm{d}\xi_i^* \, \mathrm{d}\xi_i \, \left(1 - \xi_i^* h_i \xi_i\right) \;=\; h_i \; . \tag{2.47}$$

  Hence, the final result is

$$I_f \;=\; \det H \, \exp\left(\zeta_i^* H^{-1}_{ij} \, \zeta_j\right) \; . \tag{2.48}$$

2.3 Path-integrals for fermions

Now we have all the ingredients to set up the path-integral for the Hamiltonian (2.1). As in the case of bosons, we consider the partition function in the grand-canonical ensemble.

$$Z \;=\; \mathrm{Tr}\, \mathrm{e}^{-\beta \hat{K}} \; , \tag{2.49}$$

where the operator $\hat{K}$ is the equivalent of (1.42) but with the Hamiltonian (2.1). To take the trace, we consider the basis of occupation numbers in Fock-space, such that

$$
\begin{aligned}
(2.49) &= \sum_{\{|n\rangle\}} \langle n|\mathrm{e}^{-\beta \hat{K}}|n\rangle \\
&= \int \prod_{\alpha} \mathrm{d}\xi_{\alpha}^{*}\,\mathrm{d}\xi_{\alpha}\, \exp\left(-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right) \sum_{\{|n\rangle\}} \langle n|\xi\rangle\langle\xi|\mathrm{e}^{-\beta\hat{K}}|n\rangle \ . \qquad (2.50)
\end{aligned}
$$

Since we have states $|n\rangle$ with a definite occupation, they are given by

$$
|n\rangle = f_{\alpha_1}^{\dagger}\cdots f_{\alpha_n}^{\dagger}|0\rangle \ , \qquad (2.51)
$$

and the matrix elements have the form

$$
\begin{aligned}
\langle n|\xi\rangle &= \xi_{\alpha_n}\cdots\xi_{\alpha_1} \ , \\
\langle\xi|\mathrm{e}^{-\beta\hat{K}}|n\rangle &= \sum_{\{\beta_i\}} c_{\beta_1,\dots,\beta_n}\xi_{\beta_1}^{*}\cdots\xi_{\beta_n}^{*} \ . \qquad (2.52)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\langle n|\xi\rangle\langle\xi|\mathrm{e}^{-\beta\hat{K}}|n\rangle &\rightarrow \xi_{\alpha_n}\cdots\xi_{\alpha_1}\xi_{\beta_1}^{*}\cdots\xi_{\beta_n}^{*} \\
&= (-1)^n\xi_{\beta_1}^{*}\cdots\xi_{\beta_n}^{*}\xi_{\alpha_n}\cdots\xi_{\alpha_1} \rightarrow \langle -\xi|\mathrm{e}^{-\beta\hat{K}}|n\rangle\langle n|\xi\rangle \ , \qquad (2.53)
\end{aligned}
$$

such that

$$
Z = \int \prod_{\alpha}\mathrm{d}\xi_{\alpha}^{*}\,\mathrm{d}\xi_{\alpha}\,\exp\left(-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right)\sum_{\{|n\rangle\}}\langle -\xi|\mathrm{e}^{-\beta\hat{K}}|n\rangle\langle n|\xi\rangle \ . \qquad (2.54)
$$

We see from the above, that in the case of fermions we have antiperiodic boundary conditions in imaginary time, as oppossed to bosons. This feature resulting from the anticommutation relations of fermionic operators can be found in other formulations (see e.g. [6]).

Now we proceed as in the bosonic case, performing a Trotter slicing, normal ordering, and inserting the resolution of unity between the time sclices. The result is

$$
\begin{aligned}
Z = \lim_{L\to\infty} &\int \prod_{\ell=1}^{L}\prod_{\alpha}\mathrm{d}\xi_{\alpha,\ell}^{*}\,\mathrm{d}\xi_{\alpha,\ell}\,\exp\left(-\sum_{\ell=1}^{L}\sum_{\alpha}\xi_{\alpha,\ell}^{*}\xi_{\alpha,\ell}\right) \\
&\times\prod_{\ell=1}^{L}\langle\xi_{\ell}|1-\Delta\tau\hat{K}+\mathcal{O}\left(\Delta\tau^2\right)|\xi_{\ell-1}\rangle \ , \qquad (2.55)
\end{aligned}
$$

where we used the convention that $\xi_{\alpha,L} = -\xi_{\alpha,0}$, and $\xi_{\alpha L}^{*} = -\xi_{\alpha,0}$. The matrix elements of $\hat{K}\left(f_{\alpha}^{\dagger}, f_{\beta}\right)$ between coherent states are easily obtained, as in the bosonic case.

$$
\langle\xi_{\ell}|\hat{K}\left(f_{\alpha}^{\dagger}, f_{\beta}\right)|\xi_{\ell-1}\rangle = K\left(\xi_{\alpha}^{*}, \xi_{\beta}\right)\langle\xi_{\ell}|\xi_{\ell-1}\rangle \ , \qquad (2.56)
$$

where now K is a Grassmann valued function of ξ^*_α, ξ_α. Recalling that the overlap between coherent states is given in (2.27), we have finally,

$$Z = \lim_{L\to\infty} \int \prod_{\ell=1}^{L} \prod_{\alpha} \mathrm{d}\xi^*_{\alpha,\ell}\, \mathrm{d}\xi_{\alpha,\ell}\, \mathrm{e}^{-S(\xi^*,\xi)} \; , \tag{2.57}$$

where

$$\begin{aligned} S(\xi^*,\xi) &= \sum_{\ell=2}^{L} \Delta\tau \left\{ \sum_{\alpha} \xi^*_{\alpha,\ell} \left[\frac{(\xi_{\alpha,\ell} - \xi_{\alpha,\ell-1})}{\Delta\tau} - \mu\xi_{\alpha,\ell-1} \right] + H\left(\xi^*_{\alpha,\ell}, \xi_{\alpha,\ell-1}\right) \right\} \\ &\quad + \Delta\tau \sum_{\alpha} \xi^*_{\alpha,1} \left[\frac{(\xi_{\alpha,1} + \xi_{\alpha,L})}{\Delta\tau} + \mu\xi_{\alpha,L} \right] + H\left(\xi^*_{\alpha,1}, -\xi_{\alpha,M}\right) \; . \end{aligned} \tag{2.58}$$

At this point it is instructive to see the differences between the fermionic and the bosonic case. As we mentioned already for bosons, the discrete form is the one that should be used for actual calculations. Again, the usual notation, that makes a connection with a field theory, is assuming continuity of the fields, resulting in

$$Z = \int \mathcal{D}\xi^*_\alpha(\tau)\, \mathcal{D}\xi_\alpha(\tau)\, \mathrm{e}^{-S} \; , \tag{2.59}$$

with $\xi_\alpha(\beta) = -\xi_\alpha(0)$, and the action given by

$$S = \int_0^\beta \mathrm{d}\tau \sum_\alpha \left[\xi^*_\alpha(\tau) \left(\frac{\partial}{\partial\tau} - \mu \right) \xi_\alpha(\tau) + H\left(\xi^*_\alpha(\tau), \xi_\alpha(\tau)\right) \right] \; . \tag{2.60}$$

In order to obtain correlation functions, we can introduce source terms, as in the bosonic case. As already seen in Sec. 2.2.1, the source terms coupling to linear forms of Grassmann variables have to be themselves Grassmanian. The action has the form

$$S \;\to\; S + \int \mathrm{d}\tau\, \mathrm{d}^d x \sum_\alpha \left[J^*_\alpha(\boldsymbol{x},\tau)\xi_\alpha(\boldsymbol{x},\tau) + \xi^*_\alpha(\boldsymbol{x},\tau) J_\alpha(\boldsymbol{x},\tau) \right] \; , \tag{2.61}$$

where now J^*_α and J_α are Grassmann-valued functions. The derivatives of the logarithm of the generating functional have then to be taken following the rules for derivation with Grassmann variables.

Problem: Bose-Einstein and Fermi-Dirac statistics

Consider a Hamiltonian for free particles

$$H = \sum_\alpha \epsilon_\alpha c^\dagger_\alpha c_\alpha \; , \tag{2.62}$$

where $c^\dagger_\alpha$ and c_α are creation and annihilation oeprators, repectively, for either bosons or fermions. Calculate the partition function using coherent states for *a*) bosons,

b) fermions,

using the discrete form of the action (1.48) for bosons and (2.58) for fermions.

Hint: write the action as a bilinear form, and write explicitly the corresponding matrix for each case. For an $L \times L$ matrix, the determinant should be

$$\begin{aligned} \det M^{(\alpha)} &= 1 + (-1)^L (s)^L \,, \quad \text{bosons} \,, \\ \det M^{(\alpha)} &= 1 - (-1)^L (s)^L \,, \quad \text{fermions} \,, \end{aligned} \tag{2.63}$$

where s are the elements different from zero and one.

From the form obtained for the partition functions calculate the respective occupation numbers.

Chapter 3

Path integrals for spins

After having constructed path-integrals for bosons and fermions, it could seem that we should be able to set up the path integral for arbitrary systems, since bosons and fermions are the elementary degrees of freedom that constitute matter (as we know it). However, there are situations, where further degrees of freedom play the central role. This is the case for magnetic systems, where only the spin degrees of freedom are of interest. In constructing coherent states, we will have a more "classical" picture at hand, although we are going to be dealing with quantum spin systems. A more expanded exposition about coherent states related to different Lie groups can be found in the book by A. Perelomov [7].

3.1 Coherent states for spins

Let us recall some facts about spin operators in order to set the notation for later. Spin operators fulfill the SU(2) algebra

$$[S_\alpha, S_\beta] = i\hbar\varepsilon_{\alpha\beta\gamma} S_\gamma \ , \quad \alpha, \beta, \gamma = 1, 2, 3 \ (x, y, z) \ . \tag{3.1}$$

Furthermore, $\boldsymbol{S}^2 = \sum_\alpha S_\alpha^2$ commutes with $S_\alpha \ \forall \ \alpha$, such that we consider the set of states that are eigenstates of $\boldsymbol{S}^2$ and S_3:

$$\begin{aligned} \boldsymbol{S}^2|S,m\rangle &= \hbar^2 S(S+1)|S,m\rangle \ , \\ S_3|S,m\rangle &= \hbar m|S,m\rangle \ , \end{aligned} \tag{3.2}$$

where $-S \leq m \leq S$ and $S = 1/2, 1, 3/2, \ldots$. Further we can introduce operators $S_\pm = S_1 \pm iS_2$, such that

$$[S_3, S_\pm] = \pm\hbar S_\pm \ . \tag{3.3}$$

Each of the values of S corresponds to an irreducible representation, where the spin operators are represented by matrices of dimension $(2S+1)\times(2S+1)$. The complete set of states for each S can be obtained by taking the so-called lowest (highest) state

$|S,-S\rangle$ ($|S,S\rangle$) and operating on them with S_+ (S_-), where

$$\begin{aligned} S_+|S,m\rangle &= \sqrt{(S-m)(S+m+1)}\,|S,m+1\rangle\ , \\ S_-|S,m\rangle &= \sqrt{(S+m)(S-m+1)}\,|S,m-1\rangle\ . \end{aligned} \tag{3.4}$$

Having the complete set of states, we can construct the corresponding matrices by taking matrix elements of S_α in those states.

After setting up the notation, let us discuss some properties of the group SU(2) that gives us a geometrical perspective, that will be useful later, when discussing coherent states and also systems of many spins. We start by recalling that SU(2) is the group of 2×2 unitary matrices with determinant 1:

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}\ , \quad \text{with } |\alpha|^2+|\beta|^2 = 1\ . \tag{3.5}$$

With $\alpha=\alpha_1+i\alpha_2$ and $\beta=\beta_1+i\beta_2$, the condition above means $\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2=1$, such that SU(2) is homeomorphic to S^3, the surface of a sphere in 4 dimensions.

<u>Homeomorphism:</u> given T_1 and T_2 continuous manifolds, $\alpha : T_1 \to T_2$ is continuous and its inverse is also continuous.

Putting $\beta=0$, we see that we have a set that is also a group, that we call $H=\{h\}$, where

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}\ , \quad \text{with } |\alpha|^2 = 1 \ \Rightarrow \alpha = \mathrm{e}^{i\psi}\ . \tag{3.6}$$

Hence, $H\subset \mathrm{SU}(2)$, is the subgroup $H=\mathrm{U}(1)$.

We can now go back to our original interest, namely to generate coherent states for spins. We have seen that in the case of bosons and fermions, we generated the corresponding coherent states from the vacuum. Thinking in terms of the operators $S_\pm$, we can take the lowest (highest) weight state as the vacuum. For definitness we take $|S,-S\rangle$ as the state from which we expect to generate the whole Hilbert space. For the moment, let us restrict ourselves to the case $S=1/2$, and see what happens if some h operates on the lowest weight state:

$$h\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \mathrm{e}^{-i\psi}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\ , \tag{3.7}$$

i.e. such an operation brings a phase, and we are not generating a new state. In order to avoid such a trivial phase, we consider the left coset of H.

<u>Left coset of H:</u> For an element $g\in G$ and a subgroup $H\subset G$, $gH=\{gh, h\in H\}$.

We can eliminate the redundancy that the elements of H bring in, by considering all $gh \in gH$ as equivalent. Such an equivalence relation defines a *quotient space* ($\equiv$ the set of left cosets), that in our case is $X =$SU(2)/U(1). In order to see its structure, we consider explicitely its elements:

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\psi} & 0 \\ 0 & \mathrm{e}^{-i\psi} \end{pmatrix} = \begin{pmatrix} \rho_\alpha \mathrm{e}^{i(\phi_\alpha + \psi)} & \rho_\beta \mathrm{e}^{i(\phi_\beta - \psi)} \\ -\rho_\beta \mathrm{e}^{-i(\phi_\beta - \psi)} & \rho_\alpha \mathrm{e}^{-i(\phi_\alpha + \psi)} \end{pmatrix} , \tag{3.8}$$

where we can take always $\psi = -\phi_\alpha$. We can generate another set of elements disjoint from the first one by taking $\psi = -\phi_\alpha + \pi$. Hence, given the equivalence relation, the quotient space is given by the set

$$X = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \right\} , \quad \alpha \in \mathbb{R} , \quad \alpha^2 + \beta_1^2 + \beta_2^2 = 1 , \tag{3.9}$$

that is, $X \simeq S^2$. A possible parametrization of X is

$$\alpha = \cos\frac{\theta}{2} , \quad \beta = -\sin\frac{\theta}{2} \mathrm{e}^{-i\varphi} , \quad 0 \le \theta \le \pi , \quad 0 \le \varphi < 2\pi . \tag{3.10}$$

On the other hand, each point on S^2 corresponds to the unit vector

$$\boldsymbol{n} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) . \tag{3.11}$$

Therefore, we can view each point in X as obtained from the north pole by rotating $\boldsymbol{n}$ by an angle θ around an axis perpendicular to the plane formed by the line from the center of the sphere to the north pole and the vector $\boldsymbol{n}$. Given the vector $\boldsymbol{n}$, the axis on the equatorial plane with the properties described above is given by $m_1 = \sin\varphi$ and $m_2 = -\cos\varphi$. The corresponding SU(2) rotation is given by

$$g_{\boldsymbol{n}} = \exp\left[i\theta \left(m_1 \frac{\sigma_1}{2} + m_2 \frac{\sigma_2}{2} \right) \right] , \tag{3.12}$$

where σ_i, $i = 1, 2, 3$, are the Pauli Matrices. Indeed,

$$\begin{aligned} \exp\left[i\theta \left(m_1 \frac{\sigma_1}{2} + m_2 \frac{\sigma_2}{2} \right) \right] &= \sum_n \frac{1}{n!} \left(\frac{i\theta}{2} \right)^n (m_1\sigma_1 + m_2\sigma_2)^n \\ &= \sum_{n\,\mathrm{even}} \frac{1}{n!} \left(\frac{i\theta}{2} \right)^n + (m_1\sigma_1 + m_2\sigma_2) \sum_{n\,\mathrm{odd}} \frac{1}{n!} \left(\frac{i\theta}{2} \right)^n \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\mathrm{e}^{-i\varphi} \\ \sin\frac{\theta}{2}\mathrm{e}^{i\varphi} & \cos\frac{\theta}{2} \end{pmatrix} . \end{aligned} \tag{3.13}$$

Next we generalize the discussion above, that was performed using $S = 1/2$, to arbitrary values of S. Since the geometric picture we used did not depend on S, we preserve its essence in going to general S. Then, we examine states created from the lowest weight state in an analogous form as before:

$$|\boldsymbol{n}\rangle = \exp\left(i\,\theta\,\boldsymbol{m}\cdot\boldsymbol{S}\right)|S,-S\rangle\ , \tag{3.14}$$

where we recall that $\boldsymbol{m} = (\sin\varphi, -\cos\varphi, 0)$, such that

$$\boldsymbol{m}\cdot\boldsymbol{S} = m_1 S_1 + m_2 S_2 = \frac{i}{2}\left(\mathrm{e}^{-i\varphi}S_+ - \mathrm{e}^{i\varphi}S_-\right)\ . \tag{3.15}$$

Defining $\xi = -\frac{\theta}{2}\mathrm{e}^{-i\varphi}$, we can write

$$\exp\left(i\,\theta\,\boldsymbol{m}\cdot\boldsymbol{S}\right) = \exp\left(\xi S_+ - \xi^* S_-\right)\ . \tag{3.16}$$

Since $S_\pm$ act as creation and annihilation operators, it is useful to go over to a form reminiscent of normal ordering. Here we have

$$\exp\left(\xi S_+ - \xi^* S_-\right) = \mathrm{e}^{\zeta S_+}\mathrm{e}^{\eta S_3}\mathrm{e}^{\zeta' S_-}\ , \tag{3.17}$$

where we defined

$$\begin{aligned}\zeta &= -\tan\frac{\theta}{2}\,\mathrm{e}^{-i\varphi}\ ,\\ \zeta' &= -\zeta^*\ ,\\ \eta &= -2\ln\cos\frac{\theta}{2}\ .\end{aligned} \tag{3.18}$$

We can easily verify (3.17) in the case $S = 1/2$.

$$\begin{aligned}\mathrm{e}^{\zeta S_+}\mathrm{e}^{\eta S_3}\mathrm{e}^{\zeta' S_-} &= \sum_{n=0}^{\infty}\frac{\zeta^n}{n!}S_+^n\sum_{m=0}^{\infty}\frac{\eta^m}{m!}S_3^m\sum_{p=0}^{\infty}\frac{\zeta'^p}{p!}S_-^p\\ &= \left(1+\zeta S_+\right)\left(\cosh\frac{\eta}{2}+\sigma_3\sinh\frac{\eta}{2}\right)\left(1+\zeta' S_-\right)\\ &= \begin{pmatrix}\mathrm{e}^{\eta/2}+\mathrm{e}^{-\eta/2}\zeta'\zeta & \zeta\mathrm{e}^{-\eta/2}\\ \zeta'\mathrm{e}^{-\eta/2} & \mathrm{e}^{-\eta/2}\end{pmatrix} = (3.13)\ .\end{aligned} \tag{3.19}$$

Now we can consider the states generated by (3.17) out of the lowest weight state, in the general case.

$$\begin{aligned}&\exp\left(i\,\theta\,\boldsymbol{m}\cdot\boldsymbol{S}\right)|S,-S\rangle\\ &\quad= \mathrm{e}^{\zeta S_+}\mathrm{e}^{\eta S_3}\mathrm{e}^{\zeta' S_-}|S,-S\rangle\\ &\quad= \mathrm{e}^{\zeta S_+}\sum_{n=0}^{\infty}\frac{\eta^n}{n!}\underbrace{S_3^n|S,-S\rangle}_{=(-S)^n|S,-S\rangle}\\ &\quad= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}S_+^n|S,-S\rangle\\ &\quad= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}\zeta^n\left[\frac{(2S)!}{(2S-n)!n!}\right]^{1/2}|S,-S+n\rangle = |\theta,\varphi\rangle\ .\end{aligned} \tag{3.20}$$

It remains to see if the states form an overcomplete set, i.e. the resolution of unity. Since the states created are parametrized by the coordinates of the unit sphere, we have to integrate over the measure of spherical coordinates. This leads to

$$\begin{aligned}
&\int \sin\theta \mathrm{d}\theta\, \mathrm{d}\varphi\, |\theta,\varphi\rangle\langle\theta,\varphi| \\
&= \sum_{m,n=0}^{2S} \left[\frac{(2S)!}{(2S-m)!m!}\right]^{1/2} \left[\frac{(2S)!}{(2S-n)!n!}\right]^{1/2} \\
&\quad \times \int_0^{\pi} \sin\theta \mathrm{d}\theta \left(\cos\frac{\theta}{2}\right)^{4S} \left(-\tan\frac{\theta}{2}\right)^{m} \left(-\tan\frac{\theta}{2}\right)^{n} \\
&\quad \times \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{-im\varphi}\mathrm{e}^{in\varphi} |S,-S+m\rangle\langle S,-S+n| \\
&= 2\pi \sum_{n=0}^{2S} \frac{(2S)!}{(2S-n)!n!} \int_0^{\pi} \sin\theta \mathrm{d}\theta \left(\cos^2\frac{\theta}{2}\right)^{2S} \left(\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)^{n} \\
&\qquad \times |S,-S+n\rangle\langle S,-S+n| \\
&= 2\pi(2S)! \sum_{n=0}^{2S} \frac{1}{(2S-n)!n!} \int_0^{\pi} \sin\theta \mathrm{d}\theta \left(\frac{1+\cos\theta}{2}\right)^{2S} \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{n} \\
&\qquad \times |S,-S+n\rangle\langle S,-S+n| \\
&= \frac{2\pi(2S)!}{2^{2S}} \sum_{n=0}^{2S} \frac{1}{(2S-n)!n!} \underbrace{\int_{-1}^{1} \mathrm{d}u\, (1+u)^{2S-n} (1-u)^{n}}_{\equiv I} \\
&\qquad \times |S,-S+n\rangle\langle S,-S+n| \, ,
\end{aligned} \tag{3.21}$$

where

$$I = \frac{2^{2S+1}}{2S+1} \frac{(2S-n)!n!}{(2S)!} \, , \tag{3.22}$$

such that

$$(3.21) = \frac{4\pi}{2S+1} \sum_{n=0}^{2S} |S,-S+n\rangle\langle S,-S+n| \, . \tag{3.23}$$

Then, the resolution of unity is given by

$$\mathbf{1} = \frac{2S+1}{4\pi} \int \sin\theta \mathrm{d}\theta\, \mathrm{d}\varphi\, |\theta,\varphi\rangle\langle\theta,\varphi| \, . \tag{3.24}$$

3.2 Path-integrals for quantum spin systems

Once we arrived at an overcomplete set of spin-states, we can go over to a partition function. In the following we use as a short notation $|\boldsymbol{n}\rangle = |\theta,\varphi\rangle$. The partition function is given by

$$Z \;=\; \mathrm{Tr}\mathrm{e}^{-\beta H} \;=\; \int \mathrm{d}\boldsymbol{n}\,\langle \boldsymbol{n}|\mathrm{e}^{-\beta H}|\boldsymbol{n}\rangle \;, \tag{3.25}$$

where

$$\mathrm{d}\boldsymbol{n} \;=\; \frac{2S+1}{4\pi}\,\sin\theta \mathrm{d}\theta \mathrm{d}\varphi \;. \tag{3.26}$$

Now we can just repeat the same steps we had in the cases of bosons and fermions. We start by a Trotter-slicing:

$$\mathrm{e}^{-\beta H} \;=\; \prod_{\ell=1}^{L} \mathrm{e}^{-\Delta\tau H} \;, \quad \Delta\tau = \frac{\beta}{L} \;, \tag{3.27}$$

such that the partition function goes over into

$$Z \;=\; \lim_{L\leftarrow\infty} \int \prod_{\ell=1}^{L} \mathrm{d}\boldsymbol{n}_\ell \prod_{\ell=1}^{L} \langle \boldsymbol{n}_\ell|\mathrm{e}^{-\Delta\tau H}|\boldsymbol{n}_{\ell-1}\rangle \;, \tag{3.28}$$

where $\boldsymbol{n}_0 = \boldsymbol{n}$ and $\boldsymbol{n}_L = \boldsymbol{n}$, i.e. we have periodic boundary conditions in imaginary time. For the matrix elements we have up to $\mathcal{O}\left(\Delta\tau^2\right)$,

$$\begin{aligned}
\langle \boldsymbol{n}_\ell|\mathrm{e}^{-\Delta\tau H}|\boldsymbol{n}_{\ell-1}\rangle &\simeq \langle \boldsymbol{n}_\ell|1-\Delta\tau H|\boldsymbol{n}_{\ell-1}\rangle \\
&= \langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle \left[1-\Delta\tau\,\frac{\langle \boldsymbol{n}_\ell|H|\boldsymbol{n}_{\ell-1}\rangle}{\langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle}\right] \\
&\simeq \langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle \,\exp\left[-\Delta\tau\,\frac{\langle \boldsymbol{n}_\ell|H|\boldsymbol{n}_{\ell-1}\rangle}{\langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle}\right] \\
&\simeq \langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle \,\exp\left[-\Delta\tau\,\frac{\langle \boldsymbol{n}_\ell|H|\boldsymbol{n}_{\ell}\rangle}{\langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell}\rangle}\right] \;.
\end{aligned} \tag{3.29}$$

The factor in the denominator of the exponent could be a problem, but it can be easily seen that the states $|\boldsymbol{n}\rangle$ are properly normalized:

$$\begin{aligned}
\langle \boldsymbol{n}_\ell|\boldsymbol{n}_\ell\rangle &= \left(\cos\frac{\theta}{2}\right)^{4S} \sum_{n=0}^{2S} \left(\zeta^*\zeta\right)^n \frac{(2S)!}{(2S-n)!n!} \\
&= \left(\cos\frac{\theta}{2}\right)^{4S} \left(1+\zeta^*\zeta\right)^{2S} \;=\; 1 \;.
\end{aligned} \tag{3.30}$$

We can use the fact above to rewrite the overlap between coehrent states on different time slices in a more convenient form:

$$\begin{aligned}
\prod_{\ell=1}^{L} \langle \boldsymbol{n}_\ell|\boldsymbol{n}_{\ell-1}\rangle &= \prod_{\ell=1}^{L} \left[1-\langle \boldsymbol{n}_\ell|\left(|\boldsymbol{n}_\ell\rangle - |\boldsymbol{n}_{\ell-1}\rangle\right)\right] \\
&\simeq \exp\left[-\sum_{\ell=1}^{L} \Delta\tau \langle \boldsymbol{n}_\ell|\frac{\partial}{\partial\tau}|\boldsymbol{n}_\ell\rangle\right] \;.
\end{aligned} \tag{3.31}$$

Finally, the partition function can be written as

$$Z \;=\; \int \mathcal{D}\boldsymbol{n}\, \mathrm{e}^{-S} \;, \tag{3.32}$$

where

$$S \;=\; \int_0^\beta \mathrm{d}\tau \left[\langle \boldsymbol{n}| \frac{\partial}{\partial \tau} |\boldsymbol{n}\rangle + \langle \boldsymbol{n}|H|\boldsymbol{n}\rangle \right] \;. \tag{3.33}$$

Once we arrived at the partition function, we have still to examine the matrix elements appearing in the action. For a Hamiltonian containing only spin-operators, we need only to look at their expectation values in terms of coherent states.

i) $\langle \boldsymbol{n}|S_3|\boldsymbol{n}\rangle$.

$$\begin{aligned}
\langle \boldsymbol{n}|S_3|\boldsymbol{n}\rangle &= \left(\cos^2\frac{\theta}{2}\right)^{2S} \sum_{n=0}^{2S} (\zeta^*\zeta)^n \frac{(2S)!}{(2S-n)!n!} \langle S, -S+n|\, \underbrace{S_3|S,-S+n\rangle}_{=(-S+n)|S,-S+n\rangle} \\
&= -S + \left(\cos^2\frac{\theta}{2}\right)^{2S} \sum_{n=0}^{2S} (\zeta^*\zeta)^n \frac{(2S)!\, n}{(2S-n)!n!} \\
&= -S + \left(\cos^2\frac{\theta}{2}\right)^{2S} 2S \underbrace{\zeta^*\zeta}_{\sin^2\frac{\theta}{2}/\cos^2\frac{\theta}{2}} \underbrace{\sum_{n=0}^{2S-1} (\zeta^*\zeta)^n \frac{(2S-1)!\, n}{(2S-1-n)!n!}}_{(1+\zeta^*\zeta)^{2S+-1}} \\
&= -S + 2S\sin^2\frac{\theta}{2} \;=\; -S\cos\theta \;.
\end{aligned} \tag{3.34}$$

ii) $\langle \boldsymbol{n}|S_+|\boldsymbol{n}\rangle = -S\sin\theta\, \mathrm{e}^{i\varphi}$.

iii) $\langle \boldsymbol{n}|S_-|\boldsymbol{n}\rangle = -S\sin\theta\, \mathrm{e}^{-i\varphi}$.

Hence we have in short

$$\langle \boldsymbol{n}|\boldsymbol{S}|\boldsymbol{n}\rangle \;=\; -S\boldsymbol{n} \;, \tag{3.35}$$

an expression that gives a classical picture for the spin. For the Heisenberg Hamiltonian, for instance, we have

$$\begin{aligned}
\langle \boldsymbol{n}|H|\boldsymbol{n}\rangle &= J_H \sum_{\langle i,j\rangle} \langle \boldsymbol{n}|\boldsymbol{S}_i \cdot \boldsymbol{S}_j|\boldsymbol{n}\rangle \\
&= J_H S^2 \sum_{\langle i,j\rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j \;,
\end{aligned} \tag{3.36}$$

and at least the Hamiltonian part is indistinguishable from a classical Hamiltonian. In fact, without the first term in the integrand of the action (3.33), the partition function (3.32) is the one corresponding to a $d+1$ dimensional classical system, where the extra dimension comes from the imaginary time. This tells us, that the first term in the integrand of the action (3.33) contains the information about the quantum mechanical character of the system.

3.2.1 Berry phase

Due to the special role of the first term in the integrand of the action (3.33), we examine it in a section of its own. We will see that it is intimately related to a concept introduced by Michael Berry some years ago [8], when considering the phase factor that arises when a Hamiltonian $H(\boldsymbol{R})$ is transported around a circiut C in the space of the parameters $\boldsymbol{R}$.

A direct calculation (not carried out here explicitely for brevity) shows that

$$\langle \boldsymbol{n}|\frac{\partial}{\partial\tau}|\boldsymbol{n}\rangle = -iS(1-\cos\theta)\dot{\varphi}\ , \tag{3.37}$$

such that the partition function (3.32) acquires a phase

$$\exp\left(-\int_0^\beta \mathrm{d}\tau\langle \boldsymbol{n}|\frac{\partial}{\partial\tau}|\boldsymbol{n}\rangle\right) = \exp\left(iS\int_0^\beta \mathrm{d}\tau\,(1-\cos\theta)\dot{\varphi}\right)\ . \tag{3.38}$$

Moreover,

$$\int_0^\beta \mathrm{d}\tau\,\frac{\partial\varphi}{\partial\tau}(1-\cos\theta) = \int_\Gamma (1-\cos\theta)\,\mathrm{d}\varphi\ , \tag{3.39}$$

where Γ is the closed orbit (due to periodic boundary conditions in imaginary time) described by $\boldsymbol{n}$. The last expression makes evident that the phase does not depend on the speed to traverse the path Γ. Instead, the Berry phase is of purely geometrical nature.

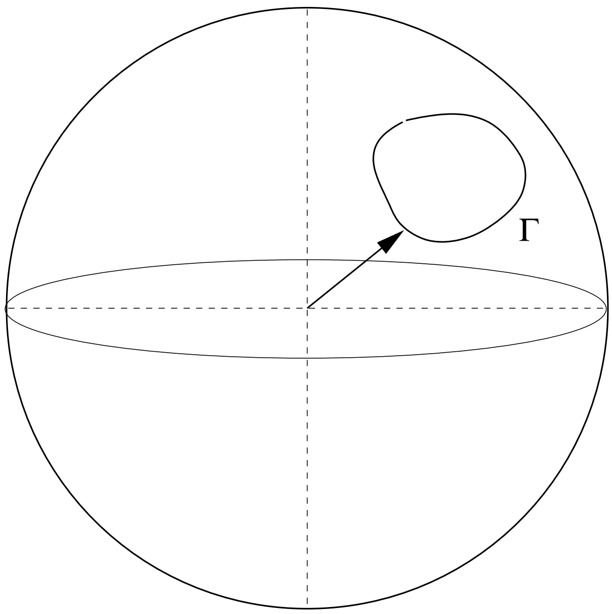

Figure 3.1: Sketch of the path of $\boldsymbol{n}$ along Γ.

Since we are considering an integral over a closed path, it is tempting to write this expression as the line integral of a vector field, that is

$$\int_\Gamma (1-\cos\theta)\,\mathrm{d}\varphi \;=\; \int_\Gamma \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{\ell}\ . \tag{3.40}$$

In fact, taking spherical coordinates on a sphere of radius 1, we can write

$$\frac{x\mathrm{d}y - y\mathrm{d}x}{z+1} \;=\; \frac{\sin^2\theta}{\cos\theta+1}\,\mathrm{d}\varphi \;=\; (1-\cos\theta)\mathrm{d}\varphi\ . \tag{3.41}$$

Hence, the vector field is

$$\boldsymbol{A} \;=\; \frac{1}{r(z+r)}\,(-y,x,0)\ , \tag{3.42}$$

where we generalized the expression to an arbitrary radius r. This generalization will allow us to understand the geometric meaning of the integral we are dealing with.

Having the circulation of a vector field on a curve in three dimensions, we can apply Stokes' theorem. For that purpose, we need the curl of the vector field, that in our case is

$$\boldsymbol{\nabla}\times\boldsymbol{A} \;=\; \frac{\boldsymbol{x}}{r^3}\ . \tag{3.43}$$

If we think of $\boldsymbol{A}$ as a gauge field, then, the corresponding magnetic field is a radial one, and hence, the field of a magnetic monopole. Summarizing the discussion above, the Berry phase in the case of a spin is given by

$$\int_\Gamma \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{\ell} \;=\; \int_F \frac{\boldsymbol{x}}{r^3}\cdot\mathrm{d}\boldsymbol{s}\ , \tag{3.44}$$

where F is the surface with boundary Γ. The expression above corresponds to the solid angle subtended by the surface $\mathrm{d}\boldsymbol{s}$, as depicted below. It is the projection on the sphere of unit radius of the surface F:

$$\int_F \frac{\boldsymbol{x}}{r^3}\cdot\mathrm{d}\boldsymbol{s} \;=\; \Omega\ , \tag{3.45}$$

such that

$$\int_0^\beta \mathrm{d}\tau\langle\boldsymbol{n}|\frac{\partial}{\partial\tau}|\boldsymbol{n}\rangle \;=\; -iS\Omega\ . \tag{3.46}$$

This last expression will put into evidence the quantum nature of the system, since the solid angle for a closed curve on S^2 has two possible values, depending on the convention we use for the direction of Γ. The two possibilities are

$$\exp\left(iS\int_0^\beta \mathrm{d}\tau\,(1-\cos\theta)\dot{\varphi}\right) \;=\; \mathrm{e}^{iS\Omega} \;=\; \mathrm{e}^{-iS\Omega'}\ , \tag{3.47}$$

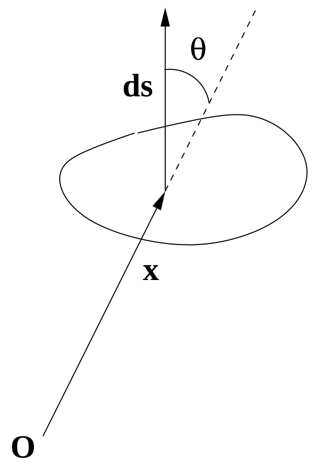

Figure 3.2: Solid angle seen from O subtended by the element of surface d$\boldsymbol{s}$.

where Ω is the solid angle for one circulation direction, and Ω' for the other one, with the due change of sign. But this means that

$$\mathrm{e}^{iS(\Omega+\Omega')} \;=\; 1\,. \tag{3.48}$$

But being on a sphere, $\Omega+\Omega'=4\pi$, and the condition above can only be fulfilled if $S=0,1/2,1,3/2,\ldots$. Hence, the Berry phase is responsible for the quantization of S, making the difference with respect to a system with classical spins. The quantization resulting from a magnetic monopole was addressed first by Dirac in order to quantize the electric charge. A short and interesting discussion is given by R. Jackiw on occassion of the Dirac Memorial in 2002 [9].

Since the Hamiltonian part could be written entirely in terms of the $\boldsymbol{n}$-fields, it would be also desirable to do so in the case of the Berry phase. In this case we have

$$\int_0^\beta \mathrm{d}\tau \langle \boldsymbol{n}|\frac{\partial}{\partial\tau}|\boldsymbol{n}\rangle \;=\; -iS\int_0^\beta \boldsymbol{A}\cdot\frac{\partial \boldsymbol{n}}{\partial\tau}\,\mathrm{d}\tau\;, \tag{3.49}$$

where the gauge field has to obey the condition

$$\varepsilon^{abc}\frac{\partial}{\partial n^b}A^c \;=\; n^a\;, \tag{3.50}$$

with $|\boldsymbol{n}|^2=1$.

Problem: Quantum Heisenberg antiferromagnet in one dimension

Consider the Hamiltonian for a quantum Heisenberg antiferromagnet in one dimension,

$$H_H \;=\; J\sum_i \boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}\;, \tag{3.51}$$

where $J > 0$, and the spin operators may correspond to $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

i) Write the corresponding partition function using coherent states.

ii) For $\beta \to \infty$ we expect that the spins are close to a Néel state, with $\boldsymbol{n}_i = (-1)^i \boldsymbol{n}$, $\boldsymbol{n}$ being a unit vector in some direction. Introduce a local order parameter $\boldsymbol{\Omega}_i \sim \boldsymbol{n}_{i+1} - \boldsymbol{n}_i$, that shows small variations on neighboring sites, and fluctuations perpendicular to it $\boldsymbol{\ell}_i \sim \boldsymbol{n}_{i+1} + \boldsymbol{n}_i$. In term of these fields we express the original spins as

$$\boldsymbol{n}_i \;=\; (-1)^i \boldsymbol{\Omega}_i \sqrt{1 - a^2 \boldsymbol{\ell}_i^2} + a \boldsymbol{\ell}_i \;, \tag{3.52}$$

where a is the lattice constant and the factor multiplying $\boldsymbol{\Omega}_i$ enforces that $\boldsymbol{\Omega}_i^2 = 1$.

a) Introduce the proposed form for the spin-fields in the interaction part of the action and assume continuity such that

$$\begin{aligned} \boldsymbol{\Omega}_{i+1} &= \boldsymbol{\Omega}_i + a\partial_x \boldsymbol{\Omega}_i + \frac{1}{2} a^2 \partial_x^2 \boldsymbol{\Omega}_i + \mathcal{O}\left(a^3\right) \;, \\ \boldsymbol{\ell}_{i+1} &= \boldsymbol{\ell}_i + a\partial_x \boldsymbol{\ell}_i + \mathcal{O}\left(a^2\right) \;, \end{aligned} \tag{3.53}$$

and the summations over lattice sites go over to integrals. Since $\boldsymbol{\Omega}_i^2 = 1$, $\boldsymbol{\Omega}_i \cdot \partial_x \boldsymbol{\Omega}_i = 0$, and $\boldsymbol{\Omega}_i \cdot \partial_x^2 \boldsymbol{\Omega}_i = -\partial_x \boldsymbol{\Omega}_i \cdot \partial_x \boldsymbol{\Omega}_i$.

b) Introduce the proposed form for the spin-fields in the Berry phase and assume continuity as well. Notice that due to (3.50) $A[\boldsymbol{\Omega}] = A[-\boldsymbol{\Omega}]$. it is useful to notice that (3.50) implies

$$\frac{\partial A^b}{\partial \Omega^a} - \frac{\partial A^a}{\partial \Omega^b} \;=\; \varepsilon^{abc}\, \Omega^c \;. \tag{3.54}$$

At the end the action should read

$$\begin{aligned} S_H \;=\;& i\frac{S}{4} \int \mathrm{d}\tau\, \mathrm{d}x\, \varepsilon^{\mu\nu} \varepsilon^{abc}\, \Omega^a \partial_\mu \Omega^b \partial_\nu \Omega^c + iS \int \mathrm{d}\tau\, \mathrm{d}x\, \boldsymbol{\ell} \cdot (\boldsymbol{\Omega} \times \partial_\tau \boldsymbol{\Omega}) \\ &+ \frac{JaS^2}{2} \int \mathrm{d}\tau\, \mathrm{d}x\, \left[\partial_x \boldsymbol{\Omega} \cdot \partial_x \boldsymbol{\Omega} + 4\boldsymbol{\ell}^2\right] \;. \end{aligned} \tag{3.55}$$

iii) Since $\boldsymbol{\ell}$ appears in a bilinear form, it can be integrated out, or what is equivalent, one can take the saddle-point with respect to $\boldsymbol{\ell}$. Replacing the solution of the saddle-point equation in the action, a field-theory for $\boldsymbol{\Omega}$ is attained. Bring it into the form

$$\begin{aligned} S \;=\;& \frac{1}{2g} \int \mathrm{d}\tau\, \mathrm{d}x \left[\frac{1}{c} \left(\partial_\tau \boldsymbol{\Omega}\right)^2 + c \left(\partial_x \boldsymbol{\Omega}\right)^2\right] \\ &+ i\frac{S}{4} \int \mathrm{d}\tau\, \mathrm{d}x\, \varepsilon^{\mu\nu} \varepsilon^{abc}\, \Omega^a \partial_\mu \Omega^b \partial_\nu \Omega^c \;. \end{aligned} \tag{3.56}$$

Which is the form of g and c in terms of the original parameters S, J, and a?

Chapter 4

Topological excitations in quantum spin systems

We have seen that the Hamiltonian part of a spin system could be written in terms of classical fields. In general, since quantum spin systems motivated from a condensed matter point of view live on a lattice, we have to deal with fields that are not continuous. This precludes in general an analytic treatment of such systems, calling for numerical methods to deal with them if accurate and controlled results are required. However, in many cases, the interest is centered on possible phase transitions, where one may suspect or knows that they are continuous ones. In this case, close to the transition, the spin-spin correlation length diverges. This sign of *criticality* may help to characterize the system further on an analytical basis, since we expect that the spins in close vecinity are strongly correlated, such that their configuration changes only weakly with distance, i.e. the fields are expected to be well described by continuous ones. A consequence of continuity can also be that topology starts to play a role, by protecting configurations that cannot be deformed continuously into trivial ones. In this chapter we will discuss some of those configurations.

4.1 The non-linear σ-model

Let us start with the Heisenberg model we shortly mentioned in the previous chapter,

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \ , \tag{4.1}$$

that for simplicity we take as a ferromagnetic one ($J > 0$). For the moment we restrict ourselves to a classical model, and we write similarly to the results obtained with coherent states in the previous chapter, $\boldsymbol{S}_i = S\boldsymbol{n}_i$, with $|\boldsymbol{n}_i|^2 = 1$. The partition function can be written as

$$Z = \sum_{\{\boldsymbol{S}_i\}} \mathrm{e}^{-\beta H} = \sum_{\{\boldsymbol{n}_i\}} \exp\left(\tilde{g} \sum_{\langle i,j \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j\right) \ , \tag{4.2}$$

where we defined $\tilde{g} = \beta J S^2$. At high temperatures ($\tilde{g} \to 0$), all configurations are equally important, so that the system is disordered. At low temperatures ($\tilde{g} \to \infty$), the favoured configuration is the one with $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = 1$, i.e. the ferromagnetic one, that breaks the rotational symmetry of the Hamiltonian.

Up to now, we did not specify the number of components of $\boldsymbol{S}_i$. If the classical spin we are considering stems from a quantum mechanical one, we would have $n = 3$ components and the Hamiltonian (4.1) would have a global O(3) symmetry. However we could generalize the model to n any natural number. We restrict us here to $n > 1$, since for $n = 1$ we have a global discrete symmetry ($\mathbb{Z}_2$), instead of a continuous one. We have also to specify the dimension d of the lattice in (4.1). Since we are considering systems with continuous global symmetries (O(n)), we recall the Mermin-Wagner theorem [10] that states that a continuous symmetry cannot be broken spontaneously at finite temperature at $d = 2$. Hence, we consider systems with $d \geq 2$. For simplicity, we assume a hypercubic lattice.

Then, we expect that there is a coupling $\tilde{g}_c$ (finite in $d \geq 3$ or zero in $d = 2$), where, on approaching it, the correlation length ξ diverges, or that $a/\xi \to 0$, where a is the lattice constant. Within the correlation length the spins are almost ordered, so that neighboring spins differ only by a small angle. This is equivalent to consider the limit $a \to 0$, and assume continuity of the spin-fields, such that a gradient expansion is possible:

$$\boldsymbol{n}_j \simeq \boldsymbol{n}_i + \partial_\mu \boldsymbol{n}_i a^\mu + \frac{1}{2} \partial_\mu^2 \boldsymbol{n}_i a^{\mu 2} + \dots \,, \tag{4.3}$$

where $\mu = x, y, z, \dots$ the component corresponding to the link between i and j. Introducing this expression into the interaction we have

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_j \simeq 1 + \underbrace{\boldsymbol{n}_i \cdot \partial_\mu \boldsymbol{n}_i}_{=0} a^\mu + \frac{1}{2} \boldsymbol{n}_i \cdot \partial_\mu^2 \boldsymbol{n}_i a^{\mu 2} + \dots \,, \tag{4.4}$$

such that

$$\begin{aligned} \sum_{\langle i,j \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j &\simeq \frac{1}{2} \sum_i a^2 \boldsymbol{n}_i \cdot \partial_\mu^2 \boldsymbol{n}_i \\ &\simeq \frac{1}{2a^{d-2}} \int \mathrm{d}^d x \, \boldsymbol{n} \cdot \partial_\mu^2 \boldsymbol{n} = -\frac{1}{2a^{d-2}} \int \mathrm{d}^d x \, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} \,, \end{aligned} \tag{4.5}$$

leading to a partition function

$$Z = \int \mathcal{D} \boldsymbol{n}(\boldsymbol{x}) \, \mathrm{e}^{-S} \,, \tag{4.6}$$

with an action

$$S = \frac{1}{2g} \int \mathrm{d}^d x \, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} \,, \tag{4.7}$$

where we defined the coupling constant

$$g = \frac{a^{d-2}}{\tilde{g}} \, . \tag{4.8}$$

This definition shows explicitely that for $d = 2$ g is dimensionless. In fact, at $d = 2$ the model is scale invariant. The action above is the O(n) non-linear σ-model.

Although the action looks like the one of a free field, it is non-trivial due to the constraint $|\boldsymbol{n}|^2 = 1$. Close to g_c we can imagine that the spins are aligned along some direction, whose component we call σ. The other components, describing transverse fluctuations around σ are called $\boldsymbol{\pi}$, such that

$$\boldsymbol{n} = (\boldsymbol{\pi}, \sigma) \quad \Rightarrow \quad \sigma = \sqrt{1 - \boldsymbol{\pi}^2} \, . \tag{4.9}$$

performing an expansion of the square root

$$\sigma \;=\; 1 - \frac{1}{2}\boldsymbol{\pi}^2 - \frac{1}{8}\left(\boldsymbol{\pi}^2\right)^2 - \frac{1}{16}\left(\boldsymbol{\pi}^2\right)^4 - \cdots \, , \tag{4.10}$$

and rescaling the fields $\boldsymbol{\pi}/\sqrt{g} \to \boldsymbol{\pi}$, the integrand of the action becomes

$$\frac{1}{2g}\partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} \;\to\; \frac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi} + \frac{g}{8}\partial_\mu \boldsymbol{\pi}^2 \cdot \partial_\mu \boldsymbol{\pi}^2 + \frac{g^2}{16}\partial_\mu \boldsymbol{\pi}^2 \cdot \partial_\mu \left(\boldsymbol{\pi}^2\right)^2 + \cdots \, , \tag{4.11}$$

such that the original rotational symmetry is realized here in a non-linear way. Hence the name of the model. Moreover, in the weak coupling limit, $g \to 0$, the action reduces to that of free fields giving quadratic fluctuations around the mean-field solution, $\sigma = 1$.

4.2 Topological excitations

We will focus here on specific cases with respect to the number of components n and the dimensionality d, that will lead to topological excitations like vortices and skyrmions.

4.2.1 The XY-model and vortices

We discuss first the case $n = 2$, such that the Hamiltonian (4.1) reduces to the XY-model. Furthermore, we restrict ourselves to $d = 2$. In this case, due to the Mermin-Wagner theorem we already mentioned, we should not expect spontaneous symmetry breaking for $g > 0$. However, as shown by Kosterlitz and Thouless [11], a transition takes place at finite temperature, where in the low temperature phase, the spin-spin correlation function decays with a power-law, and hence, the correlation length is infinite, but the order parameter is zero. The high temperature phase has a finite correlation length. Such a transition is due to the presence of vortices and antivortices that bind in the low temperature phase and unbind at the transition. We are not going to consider the transition directly (see e.g. [12]).

Let us assume that the temperature is low enough, such that the spins are almost ordered or more precisely, such that $\xi \gg a$. In that case we can use the action (4.7). For $n = 2$ we can easily solve the constraint of the field $\boldsymbol{n}$, namely $\boldsymbol{n} = (\cos\theta, \sin\theta)$. But then, the action becomes

$$S = \frac{1}{2g} \int \mathrm{d}^d x \; (\partial_\mu \theta)^2 \; . \tag{4.12}$$

In the partition function we need to have a finite action, such that the configuration is not exponentially suppressed. This means that for g small enough, θ should be a constant at infinity. First we can consider the 'classical' equation of motion, i.e. the one that corresponds to $\delta S = 0$ (since we started with a classical system, the term classical now refers to a mean-field solution, i.e. disregarding fluctuations). The result is

$$\triangle\theta = 0 \; , \tag{4.13}$$

that is the Laplace's equation. The solutions are harmonic functions. Since we are in $d = 2$, we can identify space with the complex plane, i.e. $\theta(x, y) \to \theta(z)$. Then, θ can be seen as the real part of an analytic function, since in this case the real and imaginary parts satisfy separately Laplace's equation. However, if such a function is an entire function (analytic everywhere), due to the Cauchy-Liouville theorem, it should be a constant, that is the real part corresponds to a plain ferromagnet. But there is the possibility of having isolated singularities. Since θ is a phase, in fact the phase of $n_x + i n_y$, a possibility is

$$\mathrm{e}^{i\theta} = \frac{z - z_0}{|z - z_0|} \quad \to \quad \theta = \arctan\left[\frac{\mathrm{Im}\,(z - z_0)}{\mathrm{Re}\,(z - z_0)}\right] \; . \tag{4.14}$$

One can directly corroborate that $\triangle\theta = 0$ holds in this case. For simplicity we take

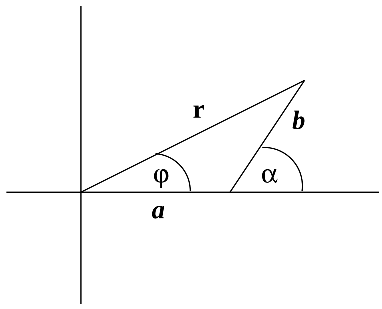

Figure 4.1: Coordinates (x, y) around the singularity at $(a, 0)$.

$z_0 = (a, 0)$, such that

$$\mathrm{Im}\,(z - z_0) = y = b \sin\alpha \; , \qquad \mathrm{Re}\,(z - z_0) = x - a = b\cos\alpha \; , \tag{4.15}$$

where b and α are displayed in Fig. 4.1. Then, we have $\theta = \alpha + 2n\pi$. This result shows on the one hand, that θ is not single-valued. This is not a problem, since the physical variable is the spin, that is single-valued. On the other hand, the result does not obey the boundary conditions we have chosen, i.e. $\theta = \theta_\infty$ at infinity. We will try to remedy this problem later. Nevertheless, we can learn from the obtained result, the physical meaning of the singularity, by looking at the spin field at some distance b from it. The spin configurations are given by $\boldsymbol{n} = (\cos\alpha, \sin\alpha)$, that is they point outwards from the circle with radius b. Since a global rotation, say by $\pi/2$, is also a possible solution, we can see now that the introduction of a singularity leads to a vortex in the spin field. Once we understood the meaning of the singularity, we introduce a configuration that respects the boundary conditions at infinity. Now we propose for θ the following form:

$$\mathrm{e}^{i\theta} \;=\; \mathrm{e}^{i\theta_\infty} \frac{(z - z^+)}{|z - z^+|} \frac{|z - z^-|}{(z - z^-)} \,, \tag{4.16}$$

where now we have two singularities located at $z^\pm$. In the limit $|z| \to \infty$ the phases cancel each other, and we have $\theta \to \theta_\infty$. In order to obtained the corresponding spin-configuration let us choose $z^\pm = (\pm a, 0)$. Then, we have

$$\begin{aligned} \theta &= \theta_\infty + \mathrm{Arg}\left[\frac{(z - z^+)}{|z - z^+|} \frac{|z - z^-|}{(z - z^-)}\right] \\ &= \theta_\infty + \arctan\left(\frac{2a\mathrm{Im}z}{|z|^2 - a^2}\right) \,. \end{aligned} \tag{4.17}$$

In order to see now the spin configuration, we consider a circle of radius b around each singularity, and a point on the circles corresponds to angles $\alpha^\pm$. Then on each circle we have

$$\begin{aligned} \mathrm{Im}z &= b \sin\alpha^\pm \,, \\ |z|^2 - a^2 &= b\left(b \pm 2a\cos\alpha^\pm\right) \,, \end{aligned} \tag{4.18}$$

such that on each circle

$$\theta^\pm = \theta_\infty + \arctan\left(\frac{2a\sin\alpha^\pm}{b \pm 2a\cos\alpha^\pm}\right) \,. \tag{4.19}$$

Assuming $b \ll 2a$, when $\alpha^\pm$ go from 0 to 2π, θ^+ increases by 2π, while θ^- decreases by 2π. Going to the spin configurations as before we can visualize the spin-fields as outgoing arrows on one circle and some ingoing ones on the other. By performing again a global rotation by $\pi/2$, we arrive at the more familiar picture for a vortex on one circle and an antivortex on the other one.

As pointed out by Kosterlitz and Thouless [11], the necessity of a vortex antivortex pair can also be explained energetically, since a single vortex would lead to a logarithmically divergent energy, a divergence that is cut off when an antivortex is included.

After seeing how vortices appear, let us put them into a general frame concerning the topological properties. For this, we introduce first in a loose way some terminology. Topology is about describing properties of spaces that can be seen as equivalent if they can be brought from one to another in a continuous way. This can be formalized with the notion of a *homotopy*.

Homotopy: given f and g continuous functions from a (topological) space X to a (topological) space Y, the continuous function $H : X \times [0,1] \to Y$, such that for $x \in X$ $H(x,0) = f(x)$ and $H(x,1) = g(x)$ is a homotopy.

With that notion we can define homotopy sectors, where each sector is given by a homotopy, i.e. all the mappings that can be continuously deformed into one another are identified in an equivalence class. On the basis of such equivalence classes, *homotopy groups* are defined, that take all the mappings from a sphere S^n onto a space X, and denoted by $\pi_n(X)$. In the case of the vortices we encountered above, we have $\pi_1(S^1) = \mathbb{Z}$, since the mapping from one circle to the other can be performed by covering it an integer number of times. For a given configuration we can obtain the corresponding homotopic sector by calculating the vorticity as follows:

$$\oint \boldsymbol{\nabla}\theta \cdot \mathrm{d}\boldsymbol{\ell} \;=\; 2\pi q \,, \tag{4.20}$$

where the vorticity $q \in \mathbb{Z}$ counts the number of times θ winds around when we go once around the vortex.

4.2.2 The O(3) non-linear σ-model and skyrmions

We consider now the case $n = 3$. Before we start to look for topological non-trivial field configurations, let us summarized how we found them in the case $n = 2$. There we had a trivial solution, where $\boldsymbol{n}$ is a constant in space, i.e. the ferromagnetic case. However, we found non-trivial finite energy solutions of the classical equations of motion with a topological character. If we consider fluctuations around those configurations, and assuming that they are small, they will amount to continuous deformations of the classical solutions, that will not be able to change the homotopy sector. Hence, it suffices to consider the classical equations of motion. We stay at $d = 2$ and consider the action (4.7). However, now with $n = 3$ it is not easy to solve the constraint. Therefore, we introduce now a Lagrange multiplier to take the constraint $|\boldsymbol{n}|^2 = 1$ into account. The action is now

$$S \;\to\; \int \mathrm{d}^2x \left[\frac{1}{2}\partial_\mu \boldsymbol{n}\cdot\partial_\mu \boldsymbol{n} + \lambda\left(\boldsymbol{n}\cdot\boldsymbol{n} - 1\right)\right] \,, \tag{4.21}$$

where we disposed of the coupling constant, since it will play no role in the discussion. The Euler-Lagrange equations are now

$$\triangle \boldsymbol{n} + \lambda \boldsymbol{n} \;=\; 0 \;, \tag{4.22}$$

that we can use to eliminate λ using the constraint on $\boldsymbol{n}$:

$$\lambda \;=\; \lambda \boldsymbol{n} \cdot \boldsymbol{n} \;=\; -\boldsymbol{n} \cdot \triangle \boldsymbol{n} \;, \tag{4.23}$$

such that the equation of motion has the form

$$\triangle \boldsymbol{n} - (\boldsymbol{n} \cdot \triangle \boldsymbol{n})\, \boldsymbol{n} \;=\; 0 \;. \tag{4.24}$$

For a given solution, the energy (or action, since the action comes originally from a classical statistical problem) is

$$E \;=\; \frac{1}{2} \int \mathrm{d}^2 x\, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} \;. \tag{4.25}$$

The trivial solution is the one with $E = 0$. As in the previous section, we look now for solutions of (4.24) with finite E. To find them, we recall a trick introduced by Belavin and Polyakov [13]. Starting from the identity

$$\int \mathrm{d}^2 x \left[(\partial_\mu \boldsymbol{n} \pm \varepsilon^{\mu\nu} \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \cdot (\partial_\mu \boldsymbol{n} \pm \varepsilon^{\mu\nu} \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \right] \geq 0 \;. \tag{4.26}$$

Carrying out the multiplication we obtain

$$\int \mathrm{d}^2 x\, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} \;\geq\; \pm \int \mathrm{d}^2 x\, \varepsilon^{\mu\nu} \boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \;, \tag{4.27}$$

such that defining

$$Q \;=\; \frac{1}{8\pi} \int \mathrm{d}^2 x\, \varepsilon^{\mu\nu} \boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \;, \tag{4.28}$$

we have a lower bound for E depending on Q:

$$E \;\geq\; 4\pi |Q| \;. \tag{4.29}$$

Our next step is to discuss the meaning of Q.

First we notice that the mixed product in (4.28) corresponds to a surface element of the sphere, since $\boldsymbol{n}$ is a unit vector. In fact, for a directed surface element we have

$$\mathrm{d}s_a \;=\; \varepsilon_{abc} \mathrm{d}n_b\, \mathrm{d}n_c \;. \tag{4.30}$$

On the other hand, in order to have configurations with a finite action, we require as in the case $n = 2$, that the fields at infinity point all in the same direction. This means that $\mathbb{R}^2$ can be compactified to S^2, the mapping between the two spaces being a stereographic projection. Then, we arrive at a situation similar to the previous one for $n = 2$, but instead of $\pi_1(S^1)$, we should be considering $\pi_2(S^2)$, i.e. the possible coverings of a sphere by another one. To show that this is so, we consider the change of variable in going from cartesian to spherical coordinates for $\boldsymbol{n}$. In doing so, the surface element above can be written as

$$\mathrm{d}s_a \;=\; \frac{1}{2}\varepsilon_{\ell m}\varepsilon_{abc}\frac{\partial n_b}{\partial \xi_\ell}\frac{\partial n_c}{\partial \xi_m}\mathrm{d}^2\xi \;, \tag{4.31}$$

where we denote the two angular variables by (ξ_1, ξ_2). Then, Q that is given by an integral over space, can be brought over to an integral over internal space (i.e. the angular variables):

$$\begin{aligned} Q &= \frac{1}{8\pi}\int \mathrm{d}^2x\, \varepsilon^{\mu\nu}\varepsilon_{abc}\, n^a\partial_\mu n^b\partial_\nu n^c \\ &= \frac{1}{8\pi}\int \mathrm{d}^2x\, \varepsilon^{\mu\nu}\varepsilon_{abc}\, n^a\frac{\partial n^b}{\partial \xi_\ell}\frac{\partial \xi_\ell}{\partial x_\mu}\frac{\partial n^c}{\partial \xi_m}\frac{\partial \xi_m}{\partial x_\nu} \\ &= \frac{1}{8\pi}\int \mathrm{d}^2\xi\, \varepsilon_{\ell m}\varepsilon_{abc}\, n^a\frac{\partial n^b}{\partial \xi_\ell}\frac{\partial n^c}{\partial \xi_m} \;, \end{aligned} \tag{4.32}$$

where we used that

$$\mathrm{d}^2x\, \varepsilon^{\mu\nu}\frac{\partial \xi_\ell}{\partial x_\mu}\frac{\partial \xi_m}{\partial x_\nu} \;=\; \varepsilon_{\ell m}\mathrm{d}^2\xi \;, \tag{4.33}$$

is the Jacobian for the transformation from (x_1, x_2) to (ξ_1, ξ_2). Then, with (4.31) we have finally

$$Q \;=\; \frac{1}{4\pi}\int n^a \mathrm{d}s_a \;=\; \frac{1}{4\pi}\int \mathrm{d}\Omega \;=\; n \in \mathbb{Z}\,. \tag{4.34}$$

The result above shows that Q counts the covering of the sphere that resulted from compactifying space by the sphere swept by $\boldsymbol{n}$. The lower bound for E that we found in (4.29) corresponds to the energy that is minimized by in each topological sector.

We still need to see how such field configurations look like. The configurations that correspond to the lower bound for E are those where the equality in eq. (4.26) is fulfilled. This leads to a differential equation for $\boldsymbol{n}$:

$$\partial_\mu \boldsymbol{n} \;=\; \mp\varepsilon^{\mu\nu}\boldsymbol{n}\times\partial_\nu\boldsymbol{n}\;. \tag{4.35}$$

Since the solutions of this differential equation minimize the energy (or action), they are also solutions of (4.24). They do not represent the most general solution, but characterize the topological sectors. Since at some point we mentioned the stereographic projection as a mean to connect the points on a sphere to points in the plane, we examine it closer.

Figure 4.2 illustrates the stereographic projection of $\boldsymbol{n}$ from the unit sphere to the plane, where the points are denoted by (ω_1, ω_2). With the conventions adopted we have

$$\tan\gamma = \frac{1-n_3}{n_1} , \tag{4.36}$$

such that

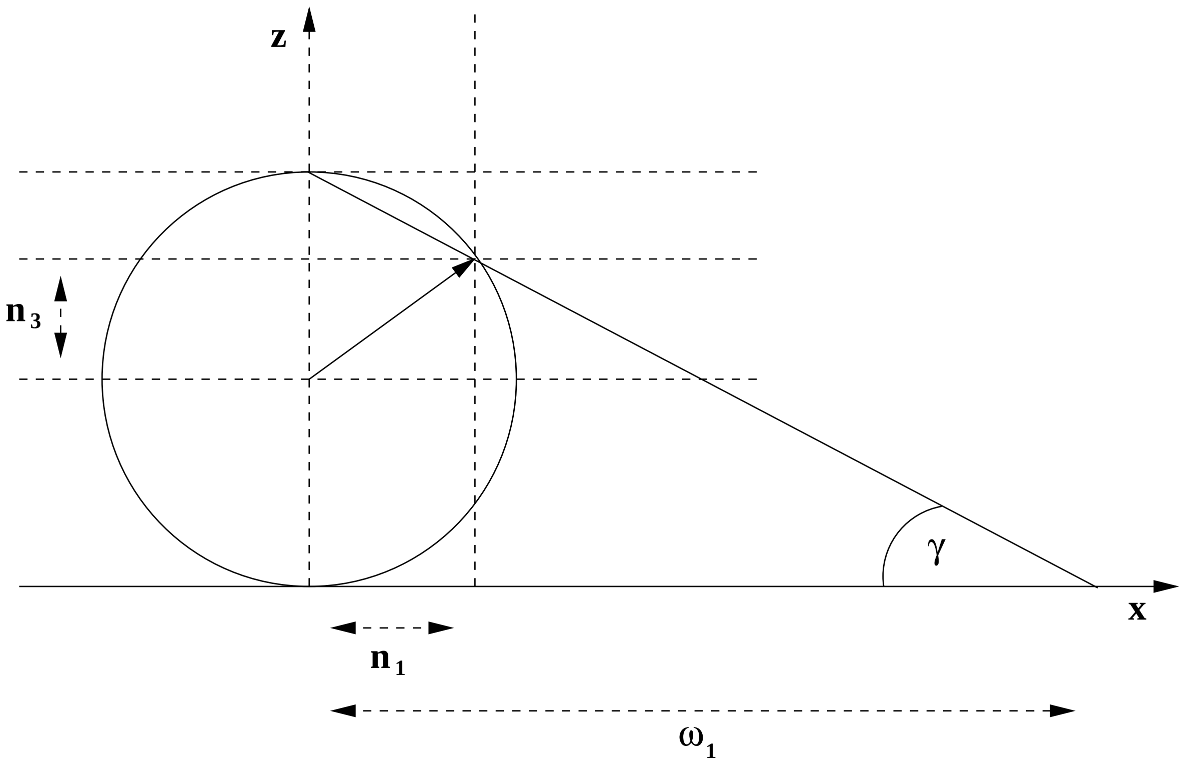

Figure 4.2: Stereographic projection.

$$\begin{aligned} \omega_1 &= \frac{2n_1}{1-n_3} , \\ \omega_2 &= \frac{2n_2}{1-n_3} . \end{aligned} \tag{4.37}$$

It is convenient to pass to the complex plane with the complex coordinate $\omega = \omega_1 + i\omega_2$. Also the space on which $\boldsymbol{n}$ lives can be viewed as the complex plane with coordinates $z = x_1 + ix_2$. Let us now go back to the differential equation (4.35) that we want to solve. For the two components entering ω in the numerator, we have

$$\begin{aligned} \partial_1 n_1 &= \mp(n_2\partial_2 n_3 - n_3\partial_2 n_2) , \\ \partial_1 n_2 &= \mp(n_3\partial_2 n_1 - n_1\partial_2 n_3) , \\ \partial_2 n_1 &= \pm(n_2\partial_1 n_3 - n_3\partial_1 n_2) , \\ \partial_2 n_2 &= \pm(n_3\partial_1 n_1 - n_1\partial_1 n_3) , \end{aligned} \tag{4.38}$$

and since

$$\partial_\mu \omega_i \;=\; \frac{2}{(1-n_3)^2}\left(\partial_\mu n_i + n_i \partial_\mu n_3 - n_3 \partial_\mu n_i\right) \;, \quad i = 1,2 \;, \quad \mu = 1,2 \;, \tag{4.39}$$

we have finally

$$\begin{aligned} \partial_1 \omega_1 &= \frac{2}{(1-n_3)^2}\left(\partial_1 n_1 \mp \partial_2 n_2\right) \;=\; \mp \partial_2 \omega_2 \;, \\ \partial_2 \omega_1 &= \frac{2}{(1-n_3)^2}\left(\partial_2 n_1 \pm \partial_1 n_2\right) \;=\; \pm \partial_1 \omega_2 \;. \end{aligned} \tag{4.40}$$

That is, the differential equation (4.35) has as solutions analytic functions $\omega(z)$, since the equations above are the Cauchy-Riemann conditions. The upper sign is for $\omega(z^*)$ and the lower sign for $\omega(z)$. However, if we are dealing with an entire function that is bounded everywhere, then due to the Cauchy-Liouville theorem, the function is a constant. Actually, we have chosen as a boundary condition $n_3 = 1$ at infinity, so that at infinity $\omega \to \infty$, therefore ω is not bounded everywhere. Since an analytic function can be expanded in a Taylor series, we can consider monomials as possible solutions:

$$\omega(z) \;=\; \left[\frac{(z - z_0)}{\lambda}\right]^n \;. \tag{4.41}$$

Actually, we can also admit isolated singularities, so that n can take both positive or negative values. In order to understand the meaning of such a solution, let us take $n = 1$, an try to see to which field $\boldsymbol{n}$ it corresponds. Since $\boldsymbol{n}$ is real, it will be convenient to consider

$$|\omega(z)|^2 \;=\; \frac{1}{\lambda^2}\left[(x - x_0)^2 + (y - y_0)^2\right] \;=\; \left(\frac{r}{\lambda}\right)^2 \;. \tag{4.42}$$

On the other hand, we have

$$|\omega|^2 \;=\; \frac{4}{(1-n_3)^2}\left(n_1^2 + n_2^2\right) \;=\; 4\,\frac{(1+n_3)}{(1-n_3)} \;, \tag{4.43}$$

such that

$$\begin{aligned} n_3 &= \frac{(|\omega|/2)^2 - 1}{(|\omega|/2)^2 + 1} \;=\; \frac{(r/2\lambda)^2 - 1}{(r/2\lambda)^2 + 1} \;, \\ n_1 &= \frac{(1-n_3)}{2}\,\omega_1 \;=\; \frac{(x - x_0)/\lambda}{(r/2\lambda)^2 + 1} \;, \\ n_2 &= \frac{(1-n_3)}{2}\,\omega_2 \;=\; \frac{(y - y_0)/\lambda}{(r/2\lambda)^2 + 1} \;. \end{aligned} \tag{4.44}$$

Using polar coordinates with center in (x_0, y_0), we have finally

$$\boldsymbol{n} \;=\; \frac{1}{1 + \left(\frac{r}{2\lambda}\right)^2}\left[\frac{r}{\lambda}\cos\varphi, \frac{r}{\lambda}\sin\varphi, \left(\frac{r}{2\lambda}\right)^2 - 1\right] \;, \tag{4.45}$$

such that at $r = 0$ the spin is pointing down, at $r \to \infty$ the spins are pointing up, and λ gives the size of the soliton solution, that goes under the name of skyrmion, after T. Skyrme, who saw such solitary solutions as a possible way to obtain nucleons.

It remains to see what happens for general n. In particular, we would like to see, how such field configurations are connected to the different topological sectors described by Q. Since we are interested in the case where the fields satisfy the lower bound, that is, when the equality in (4.27) holds, it is easier to calculate the energy density instead of calculating directly Q. First we see how the energy density can be related to ω. Let us consider

$$\begin{aligned} \left|\frac{\mathrm{d}\omega}{\mathrm{d}z}\right|^2 &= (\partial_1\omega_1)^2 + (\partial_1\omega_2)^2 \\ &= \frac{4}{(1-n_3)^4}\left[\partial_\mu n_1 \partial_\mu n_1 + \partial_\mu n_2 \partial_\mu n_2 + 2\left(\partial_1 n_1 \partial_2 n_2 - \partial_2 n_1 \partial_1 n_2\right)\right] , \end{aligned} \tag{4.46}$$

where we used the expressions (4.40) with their lower signs, since we are dealing with ω as a function of z. The last term in the square brackets can be rewritten using (4.35), as follows:

$$\partial_1 n_1 \partial_2 n_2 - \partial_2 n_1 \partial_1 n_2 = \varepsilon^{3ab} \partial_1 n_a \partial_2 n_b = n_3\, \boldsymbol{n} \cdot (\partial_1 \boldsymbol{n} \times \partial_2 \boldsymbol{n}) . \tag{4.47}$$

Using (4.27) with the lower sign, we have

$$\partial_1 n_1 \partial_2 n_2 - \partial_2 n_1 \partial_1 n_2 = -n_3\, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} . \tag{4.48}$$

On the other hand, using the relations (4.38) and the constraint of the field $\boldsymbol{n}$, one can show that

$$\partial_\mu n_3 \partial_\mu n_3 = \frac{(1-n_3^2)}{2}\, \partial_\mu \boldsymbol{n} \cdot \partial_\mu \boldsymbol{n} . \tag{4.49}$$

Putting the results above together, one arrives at

$$E = \int \mathrm{d}^2 x \, \frac{\left|\frac{\mathrm{d}\omega}{\mathrm{d}z}\right|^2}{(1+|\omega|^2/4)^2} = n^2 \lambda^{2n} \int \mathrm{d}^2 x \, \frac{|z-z_0|^{2(n-1)}}{\left(\lambda^{2n} + |z-z_0|^{2n}/4\right)^2} = 4\pi n , \tag{4.50}$$

where for the last step is more convenient to go over to polar coordinates centered at z_0. Then, we see that the powers give directly the different topological sectors of the skyrmions. The result above also shows that E does not depend on λ, the extension of the skyrmion. In fact, the field configuration obtained for $n = 1$, eq. (4.45), stays invariant upon a change of λ and a corresponding rescaling of space. Furthermore, the energy (or the action) are scale invariant, as we already pointed out in Sec. 4.1.

4.2.3 Anyons in 2+1 dimensions

Once we have seen that in two dimensions the non-linear σ-model has topological non-trivial configurations, we can ask, what happens if such configurations can travel in time. For that let us add a third dimension as euclidean time. Here we follow the discussion by Wilczek and Zee [14].

The skyrmions we have discussed previously can be seen as field configurations at a given time. Their density is the temporal part of a current

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\boldsymbol{n}\cdot(\partial_\nu\boldsymbol{n}\times\partial_\lambda\boldsymbol{n}) \ . \tag{4.51}$$

But one sees immediately, that this is a conserved current since

$$\partial_\mu j^\mu = 0 \ . \tag{4.52}$$

Being in 2+1 euclidean space, the conservation above can be written in the more familiar form $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, such that there is a gauge field $\boldsymbol{A}$ fulfilling $\boldsymbol{j} = \boldsymbol{\nabla}\times\boldsymbol{A}$, or

$$j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \ . \tag{4.53}$$

Having a gauge field, we can as usual have a term in the action connecting the current to the gauge field:

$$H = \int \mathrm{d}^3x\, j^\mu A_\mu \ , \tag{4.54}$$

and ask ourselves, if such a term makes sense.

A first interesting property is that H is invariant under global O(3) rotations as the original non-linear σ-model. To see this let us first consider the change in j^μ by a rotation $g\in$ O(3), that is, a change $\boldsymbol{n'} = g\boldsymbol{n}$. For the current we have

$$j^{\mu\prime} = \varepsilon^{\mu\nu\lambda}\varepsilon^{abc}g_{aa'}n^{a'}\partial_\nu(g_{bb'}n^{b'})\partial_\lambda(g_{cc'}n^{c'}) \ . \tag{4.55}$$

But

$$\varepsilon^{abc}g_{aa'}g_{bb'}g_{cc'} = \det(g)\,\varepsilon^{a'b'c'} \ , \tag{4.56}$$

such that

$$j^{\mu\prime} = \det(g)\, j^\mu \ . \tag{4.57}$$

Since j^μ and A_μ are related by (4.53), we have also ${A_\mu}' = \det(g)A_\mu$, such that H is O(3) invariant, and the action could in principle possess such a term. A second property of H is its scale invariance. As shown in eq. (4.8), for $d = 3$, the coupling has dimension $g \sim a$, such that the fields $\boldsymbol{n}$ remain dimensionless. Then, we have that $j^\mu \sim a^{-2}$, and consequently $A^\mu \sim a^{-1}$, such that upon rescaling distances, H remains invariant.

We consider now a general variation of the fields and see how H varies.

$$\begin{aligned}\delta H &= \int \mathrm{d}^3x\ (\delta j^\mu A_\mu + j^\mu\delta A_\mu) \\ &= \int \mathrm{d}^3x\ \left(A_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu\delta A_\lambda + j^\mu\delta A_\mu\right) \\ &= 2\int \mathrm{d}^3x\, j^\mu\delta A_\mu \ ,\end{aligned} \tag{4.58}$$

where on going from the second to the third line, we integrated by parts setting the variation to zero at the boundaries. To obtain the variation of the gauge-field, we look at

$$\delta j^{\mu} \;=\; \varepsilon^{\mu\nu\lambda}\partial_{\nu}\delta A_{\lambda} \;, \tag{4.59}$$

on the one hand, and on the other perform it directly,

$$\begin{aligned} \delta j^{\mu} &= \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon^{abc}\delta\left(n^{a}\partial_{\nu}n^{b}\partial_{\lambda}n^{c}\right) \\ &= \frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}\partial_{\nu}\left(\varepsilon^{abc}n^{a}\delta n^{b}\partial_{\lambda}n^{c}\right) \;, \end{aligned} \tag{4.60}$$

such that

$$\delta A_{\mu} \;=\; \frac{1}{4\pi}\varepsilon^{abc}n^{a}\delta n^{b}\partial_{\mu}n^{c} \;, \tag{4.61}$$

leading to

$$j^{\mu}\delta A_{\mu} \;=\; \frac{1}{32\pi^{2}}\varepsilon^{abc}\varepsilon^{def}n^{a}\delta n^{b}n^{d}\varepsilon^{\mu\nu\lambda}\partial_{\mu}n^{c}\partial_{\nu}n^{e}\partial_{\lambda}n^{f} \;=\; 0 \;. \tag{4.62}$$

To see this, notice that the product of derivatives does not vanish only when c, e, and f are all different from each other. But this implies that $c = d$. Now consider the expression for a given value for b. There are 4 possible combinations for the parity of the permutations of (a, b, c) and (c, e, f). The contributions from (even,even) and (odd,odd) cancel each other, as well as those for (even,odd) and (odd,even). The last results shows that H is an homotopic invariant, such that a topological meaning of this quantity can be expected.

In order to understand H further, we consider the curl of j^{μ}, that leads to

$$\partial_{\nu}\partial_{\nu}A^{\mu} \;=\; -\varepsilon^{\mu\lambda\rho}\partial_{\lambda}j^{\rho} \;, \tag{4.63}$$

where we chose the "Lorentz" gauge $\partial_{\mu}A^{\mu} = 0$. Hence, solving Poisson's equation above, we obtain the gauge-field

$$\begin{aligned} A^{\mu}(\boldsymbol{x}) &= \frac{1}{4\pi}\int \mathrm{d}^{3}x' \frac{\varepsilon^{\mu\lambda\rho}\partial_{\lambda'}j^{\rho}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ &= -\frac{1}{4\pi}\int \mathrm{d}^{3}x' \frac{\varepsilon^{\mu\lambda\rho}\left(x-x'\right)^{\lambda} j^{\rho}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^{3}} \;, \end{aligned} \tag{4.64}$$

where in the last line we integrated by parts. Finally, for H we have

$$H \;=\; -\frac{1}{4\pi}\int \mathrm{d}^{3}x\,\mathrm{d}^{3}x' \frac{\varepsilon^{\mu\lambda\rho}j^{\mu}(\boldsymbol{x})\left(x-x'\right)^{\lambda} j^{\rho}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^{3}} \;, \tag{4.65}$$

Let us assume now that the currents above correspond to two skyrmions well separated from each other, and let us rescale space-time such that they are reduced to their world-lines:

$$j^{\mu}\mathrm{d}^{3}x \rightarrow I\mathrm{d}\ell^{\mu} \;. \tag{4.66}$$

Then, we can write

$$H = -\frac{II'}{4\pi} \oint_{\Gamma} \oint_{\Gamma'} \frac{\mathrm{d}\boldsymbol{\ell} \cdot [(\boldsymbol{x} - \boldsymbol{x}') \times \mathrm{d}\boldsymbol{\ell}']}{|\boldsymbol{x} - \boldsymbol{x}'|^3} , \tag{4.67}$$

where we have assumed that the world-lines are closed curves, as is the case for periodic boundary conditions in time. The expression above resembles Ampères law in electrodynamics for the force between two cable loops with circulating currents. The big difference is that in the electrodynamic case we have a vector but here we have a pseudoscalar.

In order to see the meaning of the integral above, let us assume that Γ and Γ' do not intersect each other, i.e. they do not have any point in common, and rewrite (4.64) for the world-line,

$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{I'}{4\pi} \oint_{\Gamma'} \frac{(\boldsymbol{x} - \boldsymbol{x}') \times \mathrm{d}\boldsymbol{\ell}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} . \tag{4.68}$$

Here we can apply a modified form of Stokes' theorem. In its original form one has

$$\oint_{\Gamma} \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\ell} = \int_F (\boldsymbol{\nabla} \times \boldsymbol{M}) \cdot \mathrm{d}\boldsymbol{f} . \tag{4.69}$$

We can take $\boldsymbol{M} = \boldsymbol{C} \times \boldsymbol{D}$, where $\boldsymbol{D}$ is a constant vector. Then, with $\mathrm{d}\boldsymbol{\ell} = \hat{\boldsymbol{t}}\, \mathrm{d}\ell$, and $\mathrm{d}\boldsymbol{f} = \boldsymbol{n}\, \mathrm{d}f$, one has

$$D_i \oint_{\Gamma} \left(\hat{\boldsymbol{t}} \times \boldsymbol{C}\right)_i \mathrm{d}\ell = D_i \int_F \left[\left(\frac{\partial C_j}{\partial x_i}\right) n_j - \left(\frac{\partial C_j}{\partial x_j}\right) n_i\right] \mathrm{d}f , \tag{4.70}$$

that holds for any $\boldsymbol{D}$, such that finally,

$$\oint_{\Gamma} \boldsymbol{C} \times \mathrm{d}\boldsymbol{\ell} = -\int_F (\boldsymbol{n} \times \boldsymbol{\nabla}) \times \boldsymbol{C}\, \mathrm{d}f . \tag{4.71}$$

In our case we have

$$\boldsymbol{C} = \frac{(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} , \tag{4.72}$$

such that going back to (4.68), we have

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= \frac{I'}{4\pi} \int_F (\boldsymbol{n} \times \boldsymbol{\nabla}) \times \frac{(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \mathrm{d}f \\ &= \frac{I'}{4\pi} \int_F \left\{ \left[\frac{\partial}{\partial x'_i} \frac{(\boldsymbol{x} - \boldsymbol{x}')_j}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\right] \boldsymbol{n}_j - \left[\frac{\partial}{\partial x'_j} \frac{(\boldsymbol{x} - \boldsymbol{x}')_j}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\right] \boldsymbol{n}_i \right\} \mathrm{d}f . \end{aligned} \tag{4.73}$$

But since

$$\boldsymbol{\nabla}' \cdot \frac{(\boldsymbol{x} - \boldsymbol{x}')_j}{|\boldsymbol{x} - \boldsymbol{x}'|^3} = \triangle' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi\delta\left(\boldsymbol{x} - \boldsymbol{x}'\right) , \tag{4.74}$$

and we are interested on $\boldsymbol{x} \neq \boldsymbol{x}'$, the second term above can be discarded, leading to

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{\nabla}\varphi(\boldsymbol{x}) \; , \tag{4.75}$$

with

$$\begin{aligned} \varphi(\boldsymbol{x}) &= \frac{I'}{4\pi} \int_F \boldsymbol{\nabla}' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \cdot \mathrm{d}\boldsymbol{f} \\ &= \frac{I'}{4\pi} \int_F \frac{(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \cdot \mathrm{d}\boldsymbol{f} = \frac{I'}{4\pi}\Omega \; . \end{aligned} \tag{4.76}$$

Hence we see that the potential is just the solid angle subtended by the surface spanned by the curve Γ'. In order to have H, we need to perform the other line integral.

$$H = I \oint_\Gamma \mathrm{d}\boldsymbol{\ell} \cdot \boldsymbol{\nabla}\varphi = I \; \varphi|_\Gamma \tag{4.77}$$

Here we have to distinguish two cases. If the Γ and Γ' are not linked, then each integral is performed around a simply connected region, such that on a closed loop $\varphi|_\Gamma = 0$. On the other hand, if the loops are linked, the region is not any more simply connected. Going on a close loop we generate a closed surface, such that

$$\oint_\Gamma \oint_{\Gamma'} \frac{\mathrm{d}\boldsymbol{\ell} \cdot [(\boldsymbol{x} - \boldsymbol{x}') \times \mathrm{d}\boldsymbol{\ell}']}{|\boldsymbol{x} - \boldsymbol{x}'|^3} = \Omega|_\Gamma = 4\pi n \; , \tag{4.78}$$

where n is the number of times one curve winds around the other. This integral is due to Gauß. To convince oneself that this is so, we can take Γ a line extending from $-\infty$ to ∞ such that $\boldsymbol{x} = (0, 0, t_1)$ with $-\infty \leq t_1 \leq \infty$, and Γ' a circle of radius one being pierced by Γ at the center, such that $\boldsymbol{x}' = (\cos t_2, \sin t_2, 0)$, with $0 \leq t_2 \leq 2\pi$. Then,

$$\oint_\Gamma \oint_{\Gamma'} \frac{\mathrm{d}\boldsymbol{\ell} \cdot [(\boldsymbol{x} - \boldsymbol{x}') \times \mathrm{d}\boldsymbol{\ell}']}{|\boldsymbol{x} - \boldsymbol{x}'|^3} = \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_0^{2\pi} \mathrm{d}\frac{t_2}{(1 + t_1^2)^{3/2}} = 4\pi \; . \tag{4.79}$$

We have shown that H is related to the linking number of the world-lines of the skyrmions. The form (4.54) together with (4.53) is also called a Chern-Simons term, in this case for a U(1) gauge theory. If the action happens to have a term $i\theta H$, such a term in the exponential gives rise to a phase each time the skyrmions wind around each other, determining the statistics of these objects. Since there is no constraint to the factor in front of H, any statistics is possible, giving rise to anyons.

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