Mathematical Physics II

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1. Classical Mechanics

1.1. Newtonian Mechanics

Newtonian mechanics is 'Dynamical Theory' which uses time as dynamical parameter. The main target of Newtonian machanics is the point particle which has mass, lives on 3-dimensional Euclidean space and is described by time dependent coordinates. The space is called **Configuration Space**. To see the property of Configuration space, let consider a vector $\vec{v}$ on Configuration space. Then the vector is descibed by orthonormal basis.

$$\vec{v} = \sum_{i=1}^{d=3} v_i \hat{e_i}, \quad where \ \ (\hat{e_i},\ \hat{e_j}) = \delta_{ij} \ (i,j = 1,2,3)$$

Also, it's easy to see that the component can be obtained by inner product:

$$v_i = (\vec{v},\ \hat{e_i})$$

Now, we can represent the time-dependent position vector as below:

$$\vec{X}(t) = \sum_i X_i(t) \hat{e_i}$$

Because we are usually interested in inertial frame, we choose time-independent basis. Next, let's see dynamics by infinitesimal time translation from $t = 0$ to $t = \epsilon$. ($\epsilon \ll$ unit time)

$$\vec{X}(\epsilon) = \vec{X}(0) + \epsilon \, \frac{d}{dt}\vec{X}(t)\Big|_{t=0} + \mathcal{O}\left(\epsilon^2\right) \tag{1}$$

Since we can ignore higher order terms for infinitesimal ϵ, we only need $\vec{X}(0),\ \frac{d}{dt}\vec{X}(0)$. Since the latter one is velocity, we can find that only **initial position & veclocity** are needed to obtain position after infinitesimal time translation.
Now, consider $t \to \epsilon_1 + \epsilon_2$.

$$\vec{X}(\epsilon_1 + \epsilon_2) = \vec{X}(\epsilon_1) + \epsilon_2 \, \frac{d}{dt}\vec{X}(t)\Big|_{t=\epsilon_1} + \mathcal{O}\left({\epsilon_2}^2\right) \tag{2}$$

Since we have $\vec{X}(\epsilon_1)$ from given initial position & velocity, we only need $\frac{d}{dt}\vec{X}(t)\Big|_{t=\epsilon_1}$.

$$\begin{aligned} \frac{d}{dt}\vec{X}(t)\Big|_{t=\epsilon_1} &= \dot{\vec{X}}(\epsilon_1) = \dot{\vec{X}}(0) + \epsilon_1 \, \frac{d}{dt}\dot{\vec{X}}(t)\Big|_{t=0} + \mathcal{O}\left({\epsilon_1}^2\right) \\ &= \dot{\vec{X}}(0) + \epsilon_1 \, \ddot{\vec{X}}(t)\Big|_{t=0} \end{aligned} \tag{3}$$

As we can obtain $\ddot{\vec{X}}(t)$ by Force law, we can determine position at any time by only two initial conditions. Thus, Newtonian mechanics is fully deterministic dynamical theory.

a. Physical State

: At time t, $\left(\vec{X}(t), \dot{X}(t) + ForceLaw\right)$ is given, then we can determine physical state.

1.2. Lagrangian Mechanics

As seen from Newtonian mechanics, the physical state is composed of $\vec{X}(t), \dot{\vec{X}}(t)$. In Lagrangian formalism, the general coordinates and derivatives of those $(q_k(t), \dot{q}_k(t))$ [1] are ingredients of Lagrangian, which defines Action[2]:

$$S[q(t)] \equiv \int_{t_i}^{t_f} dt\ \mathcal{L}(q_k(t), \dot{q}_k(t))$$

a. Least Action Principle

> *"The path taken by the system between times t_1 and t_2 is the one for which the action is stationary (no change) to first order.[3]"*

Let q_{cl} be the classical path, then the variation of action $\delta S = S[q_{cl} + \delta q] - S[q_{cl}] = 0$ From this, we can derive the Euler-Lagrange equation.

$$\begin{aligned}
\delta S &= \delta \int_{t_i}^{t_f} dt\ \mathcal{L}(q_k(t), \dot{q}_k(t), t) \\
&= \int_{t_i}^{t_f} dt\ \delta\mathcal{L}(q_k(t), \dot{q}_k(t), t) \\
&= \int_{t_i}^{t_f} dt\ \left(\frac{\partial \mathcal{L}}{\partial q_k} \delta q_k(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta \dot{q}_k + \frac{\partial \mathcal{L}}{\partial t} \right) \\
&= \int_{t_i}^{t_f} dt\ \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta q_k \right) - \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} \right) \delta q + \frac{\partial \mathcal{L}}{\partial t} \right) = 0
\end{aligned} \tag{4}$$

In the final line of Eq(4), the first term and last term are vanished because of fixed initial, final point and no explicit time dependency of Lagrangian. Thus, we finally get the Euler-Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \tag{5}$$

And the Lagrangian is defined as $\mathcal{L} = T - V$. (T is kinetic energy, and V is potential energy.)

b. Noether's Theorem

Theorem 1.1 (Noether's Theorem)**.** *When an action has a symmetry, we can derive a conserved quantity. In other words, for symmetric transformation, there exists invariant quantity.*

Proof:

Let denote infinitesimal symmetric transformation δ_s. For this symmetric transformation, action should be invariant. Thus,

[1] k represent number of 'Degree of Freedom' (D.O.F)
[2] Action is not function - Not depends on particular point, but depends on entire trajectory. It is defined as functional.
[3] R. Penrose (2007). The Road to Reality. Vintage books. p. 474. ISBN 0-679-77631-1.

$$
\begin{aligned}
\delta_s S &= \int_{t_i}^{t_f} dt\ \delta_s \mathcal{L} \\
&= \int_{t_i}^{t_f} dt\ \left(\frac{\partial \mathcal{L}}{\partial q_k} \delta_s q_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s \dot{q}_k \right) \\
&= \int_{t_i}^{t_f} dt\ \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s q_k \right) - \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} \right) \delta_s q_k \right) \\
&= \int_{t_i}^{t_f} dt\ \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s q_k \right) \right) \\
&= \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s q_k \right) \Bigg|_{t_i}^{t_f} = 0 \quad \textit{for any period of times } (t_i, t_f) \\
\therefore\ \frac{d}{dt} &\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s q_k \right) = 0 \qquad \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta_s q_k \text{ is called } \textbf{Noether Current}\right)
\end{aligned} \tag{6}
$$

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Example 1.1. Consider relativistic free particle. With $g_{\mu\nu} = diag(1, -1, -1, -1)$ metric convention, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - d\vec{x}^2,\ ds = cdt\sqrt{1 - v^2/c^2}$

$$
\begin{aligned}
S = -mc \int ds = -mc^2 \int \sqrt{1 - \tfrac{v^2}{c^2}} dt \ \ &\equiv \int \mathcal{L} dt \\
\therefore\ \mathcal{L} = -mc^2 \sqrt{1 - \tfrac{v^2}{c^2}} \simeq -mc^2 + \frac{1}{2} m v^2 \ \ (v \ll c)
\end{aligned} \tag{7}
$$

Example 1.2. Consider relativistic charged particle under electromagnetic field. With same metric convention as above and $A^\mu = (\phi, \vec{A})$,

$$
\begin{aligned}
S &= -mc \int ds - \frac{e}{c} \int A_\mu dx^\nu \\
&= \int \left(-mc^2 \sqrt{1 - \tfrac{v^2}{c^2}} - e\phi \right) dt + \frac{e}{c} \vec{A} \cdot \vec{x} \\
&= \int \left(-mc^2 \sqrt{1 - \tfrac{v^2}{c^2}} - e\phi + \frac{e}{c} \vec{A} \cdot \vec{v} \right) dt \ \ \equiv \int \mathcal{L} dt \\
\therefore\ \mathcal{L} &= -mc^2 \sqrt{1 - \tfrac{v^2}{c^2}} - e\phi + \frac{e}{c} \vec{A} \cdot \vec{v}
\end{aligned} \tag{8}
$$

Then Euler-Lagrange equation of this lagrangian becomes:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \vec{x}} &= \frac{e}{c} \vec{\nabla} (\vec{A} \cdot \vec{v}) - e \vec{\nabla} \phi \\
&= \frac{e}{c} [(\vec{v} \cdot \vec{\nabla}) \vec{A} + \vec{v} \times (\vec{\nabla} \times \vec{A})] - e \vec{\nabla} \phi \\
\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} &= \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} \vec{A}, \quad \frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{A} \\
\therefore\ \ \frac{d\vec{p}_k}{dt} &= e \left(-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi + \frac{1}{c} \vec{v} \times (\vec{\nabla} \times \vec{A}) \right) \qquad (\vec{p}_k \equiv \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}) \\
&= e \left(\vec{E} + \tfrac{1}{c} \vec{v} \times \vec{B} \right) \qquad (\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi,\ \vec{B} = \vec{\nabla} \times \vec{A})
\end{aligned} \tag{9}
$$

In non-relativistic limit, $\vec{p}_k = m\vec{v},\ \frac{d\vec{p}_k}{dt} = m\ddot{\vec{x}} = \vec{F} = e[\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}]$ which is well known **Lorentz Force**

1.3. Hamiltonian Mechanics

Now consider rate of chage of Lagrangian.

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial q}\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} = \frac{\partial \mathcal{L}}{\partial t} + \dot{p}\dot{q} + p\ddot{q} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}(p\dot{q})$$
$$\frac{d}{dt}(p\dot{q} - \mathcal{L}) = -\frac{\partial \mathcal{L}}{\partial t} = 0 \tag{10}$$

Thus, $p\dot{q} - \mathcal{L}$ is conserved quantity and we call it **Hamiltonian**. ($\Rightarrow \; H \equiv p\dot{q} - \mathcal{L}$)

Theorem 1.2 (Legendre transform)**.** *The Hamiltonian is Legendre transform of Lagrangian.*

Proof:

$$\begin{aligned} dH &= d(p\dot{q} - \mathcal{L}) = dp\dot{q} + pd\dot{q} - d\mathcal{L} \\ &= dp\dot{q} + pd\dot{q} - (\dot{p}dq + pd\dot{q}) \\ &= \dot{q}dp - \dot{p}dq \\ &\therefore \; H = H(p, q) \end{aligned} \tag{11}$$

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And we can derive **Hamilton equation** from Eq(11).

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \; \frac{\partial H}{\partial q_k} = -\dot{p}_k \tag{12}$$

Then what difference between Hamiltonian formalism and Lagrangian formalism?

- *Lagrangian*
  - N equations & 2nd order differential equation. $\Rightarrow 2N$ constraints.
  - Useful to find symmetry.
- *Hamiltonian*
  - $2N$ equations & 1st order differential equation. $\Rightarrow 2N$ constraints.
  - Useful to solve system with conserved quantity.

Now, obtain the Hamiltonian for $T = \sum_{i,j} K_{ij}\dot{q}_i\dot{q}_j, V = V(q)$. ($K_{ij}$ is symmetric)

$$\begin{aligned} H &= \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\dot{q}_k - \mathcal{L} \\ &= 2\sum_{i,k} K_{ik}\dot{q}_i\dot{q}_k - \left(\sum_{i,j} K_{ij}\dot{q}_i\dot{q}_j - V(q)\right) \\ &= T + V = E \end{aligned} \tag{13}$$

Theorem 1.3 (Liouville's Theorem)**.** *The shape of the region in phase space would generaically change during time evolution. But the volume remains the same.*

Proof:

Consider an infinitesimal volume evolve over infinitesimal time. Then an infinitesimal volume V of a neighbourhood of the point (q_i, p_i) in phase space is :

$$V = dq_1 ... dq_n \; dp_1 ... dp_n \tag{14}$$

After infinitesimal time, phase space elements evolve to :

$$\begin{aligned} q_i \to q_i + \dot{q}_i dt = q_i + \frac{\partial H}{\partial p_i} \equiv \tilde{q}_i \\ p_i \to p_i + \dot{p}_i dt = p_i - \frac{\partial H}{\partial q_i} \equiv \tilde{p}_i \end{aligned} \tag{15}$$

So evolved volume in phase space is :

$$\tilde{V} = d\tilde{q}_1 ... d\tilde{q}_n \; d\tilde{p}_1 ... d\tilde{p}_n = (det \; \mathcal{J}) \; V \tag{16}$$

Where det $\mathcal{J}$ is the Jacobian of $2n \times 2n$ matrix

$$\begin{pmatrix} \partial\tilde{q}_i/\partial q_j & \partial\tilde{q}_i/\partial p_j \\ \partial\tilde{p}_i/\partial q_j & \partial\tilde{p}_i/\partial p_j \end{pmatrix} \tag{17}$$

And we want to show that det $\mathcal{J}$ equals to 1.

$$\begin{aligned} det \; \mathcal{J} &= det \begin{pmatrix} \delta_{ij} + (\partial^2 H/\partial p_i \partial q_j)dt & (\partial^2 H/\partial p_i \partial p_j)dt \\ -(\partial^2 H/\partial q_i \partial q_j)dt & \delta_{ij} + (\partial^2 H/\partial q_i \partial p_j)dt \end{pmatrix} \\ det \; \mathcal{J} &= 1 + \Sigma_i \left(\frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial q_i \partial p_i} \right) dt + \mathcal{O}\left(dt^2\right) = 1 + \mathcal{O}\left(dt^2\right) \\ &\Leftarrow \left(det(1 + \epsilon M) = 1 + \epsilon Tr M + \mathcal{O}\left(\epsilon^2\right) \right) \quad \text{for any matrix M, small } \epsilon \end{aligned} \tag{18}$$

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a. Poisson Bracket

Let $A(p,q)$ be a physical quantity. Then consider the rate of change of A:

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial q}\frac{dq}{dt} + \frac{\partial A}{\partial p}\frac{dp}{dt} \\ &= \frac{\partial A}{\partial q}\frac{\partial H}{\partial P} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} \\ &\equiv \{A,H\} \quad \textbf{- Poisson Bracket}\end{aligned} \tag{19}$$

$$\left(\{\alpha,\beta\} = \frac{\partial\alpha}{\partial q}\frac{\partial\beta}{\partial p} - \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial q}\right)$$

Let's see properties of Poisson bracket.

- $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$ - Jacobi Identity
- $\{A,B\} = -\{B,A\}$ - Anti-Commute
- $\{q_i,q_j\} = 0 = \{p_i,p_j\}\,,\ \{q_i,p_j\} = \delta_{ij}$ - Fundamental Commute Relation

Example 1.3. Consider 1D free particle. For this particle $H = \frac{p^2}{2m}$. Then think about time derivative of $A(p,q)$ for this particle.

$$\frac{dA}{dt} = \{A,H\} = \frac{\partial A}{\partial q}\frac{p}{m} = \begin{cases} A = p \ : \ \dfrac{dp}{dt} = 0 \\ A = q \ : \ \dfrac{dq}{dt} = \frac{p}{m} \end{cases} \tag{20}$$

Thus, it is possible to explain classical mechanics by Poisson bracket.

Example 1.4. Consider 1D simple harmonic oscillator. $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$. As before, time derivative of $A(p,q)$ is as follows.

$$\frac{dA}{dt} = \{A,H\} = \frac{\partial A}{\partial q}\frac{p}{m} - \frac{\partial A}{\partial p}\,kq = \begin{cases} A = p \ : \ \dfrac{dp}{dt} = -kq \ \text{ (Hooke's Law)} \\ A = q \ : \ \dfrac{dq}{dt} = \frac{p}{m} \end{cases} \tag{21}$$

2. Quantum Mechanics

2.1. Fundamentals of QM

a. Physical State

The physical state in QM is defined as a ray in Hilbert space. Hilbert space is complex vector space, so it has properties of vector space.

Norm : $\| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle$

Closed : $\forall \, |\psi\rangle \, , |\phi\rangle \in \mathcal{H}, \; \rightarrow \; a\,|\psi\rangle + b\,|\phi\rangle \in \mathcal{H} \quad (a, b \in \mathbb{C})$

Linear1 : $\langle\phi|a\phi_1 + b\phi_2\rangle = \langle\psi|a\phi_1\rangle + \langle\psi|b\phi_2\rangle = a\,\langle\psi|\phi_1\rangle + b\,\langle\psi|\phi_2\rangle$

C.C [1]: $\langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$

Linear2 : $\langle a\phi_1 + b\phi_2|\psi\rangle = a^*\,\langle\phi_1|\psi\rangle + b^*\,\langle\phi_2|\psi\rangle$

b. Observables

Observable is an operator on $\mathcal{H}$ ($\mathcal{O} : \mathcal{H} \rightarrow \mathcal{H}$) which is 'Hermitian'.[2] As we know, hermitian operator has real eigenvalues. So, eigenvalues of observables are real. And observables should satisfy below commutate relation.

$$[\;,\;] = i\hbar\,\{\;,\;\} \tag{22}$$

L.H.S is commutator ($[A, B] = AB - BA$) and R.H.S is Poisson Bracket.[3] Specifically, $[X, P] = i\hbar$.

c. Transition Probability

For $|\psi_1\rangle \in \mathcal{R}_1, \; |\psi_2\rangle \in \mathcal{R}_2$, the transitional amplitude of $\mathcal{R}_1 \rightarrow \mathcal{R}_2$ is given as $|\,\langle\psi_2|\psi_1\rangle\,|^2$
If $\{|\psi_i\rangle\}$ is complete, then $\sum_i |\psi_i\rangle\,\langle\psi_i| = \mathbb{1}$. Thus, we can write an arbitrary state $|\psi\rangle$ as

$$|\psi\rangle = \mathbb{1}\,|\psi\rangle = \sum_n |n\rangle\,\langle n|\psi\rangle \equiv \sum_n \psi_n\,|n\rangle \tag{23}$$

If $\langle m|n\rangle = \delta_{mn}$, then $\{n\}$ is orthonormal basis. For continuous basis, $\mathbb{1} = \int dx\,|x\rangle\,\langle x|$.

$$|\psi\rangle = \int dx\,|x\rangle\,\langle x|\psi\rangle \equiv \int dx\;\psi(x)\,|x\rangle \tag{24}$$

$\psi(x)$ is called wave function. Also, $\langle x|y\rangle = \delta(x - y)$.

[1]Complex Conjugate
[2]An operator that hermitian conjugate of the operator is same as the operator
[3]Precisely, expectation value follows this relation, not operator. Search 'Ehrenfest Theorem'

2.2. Quantization

We have seen the commutate rule at Eq(22). Then we have

$$
\begin{aligned}
[\hat{q}_i, \hat{p}_j] &= i\hbar\delta_{ij} \\
[\hat{q}_i, \hat{q}_j] &= 0 \ = [\hat{p}_i, \hat{p}_j]
\end{aligned} \tag{25}
$$

For classical mechanics, a physical quantity $A(p,q)$ satisfy $\frac{dA}{dt} = \{A, H\}$. Then, in QM,

$$\frac{d\hat{A}_H(t)}{dt} = \frac{1}{i\hbar}\left[\hat{A}_H(t), \hat{H}\right]$$

The subindex H means Heisenberg picture and this equation is called **Heisenberg E.O.M.** Then we can find simple solution:

$$\hat{A}_H(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t} \text{ [1]} \tag{26}$$

Then we can check:

$$
\begin{aligned}
\frac{d\hat{A}_H}{dt} &= e^{i\hat{H}t}\left(i\hat{H}\hat{A}(0) - \hat{A}(0)(i\hat{H})\right)e^{-i\hat{H}t} \\
&= \frac{1}{i}e^{i\hat{H}t}\left[\hat{A}(0), \hat{H}\right]e^{-i\hat{H}t} \\
&= \frac{1}{i}\left[\hat{A}_H(t), \hat{H}\right]
\end{aligned} \tag{27}
$$

a. Expectation Value

The expectation value of $\hat{A}_H(t)$ with respect to $|\psi\rangle$ is defined as:

$$
\begin{aligned}
\langle\hat{A}_H(t)\rangle_\psi &= \langle\psi|\hat{A}_H(t)|\psi\rangle \\
&= \langle\psi|e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}|\psi\rangle = \langle\psi(t)|\hat{A}(0)|\psi(t)\rangle
\end{aligned} \tag{28}
$$

$\langle\psi(t)|, |\psi(t)\rangle$ is called time-dependent state and denoted ${}_s\langle\psi|, |\psi\rangle_s$. And $\hat{A}(0)$ is called time-independent operator and denoted $\hat{A}_s$. The subindex s represent schrödinger picture.

[Please input table of difference of Schrödinger and Heisenberg]

[1] For convenience, let $\hbar = 1$ (natural unit)

2.3. Schrödinger Equation

In schrödinger picture, we have next equation:

$$\begin{aligned} i\frac{\partial}{\partial t}\left|\psi\right\rangle_s &= i\frac{\partial}{\partial t}e^{-i\hat{H}t}\left|\psi\right\rangle \\ &= i(-i\hat{H})e^{-i\hat{H}t}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle_s \\ \therefore i\frac{\partial}{\partial t}\left|\psi\right\rangle_s &= \hat{H}\left|\psi\right\rangle_s \end{aligned} \tag{29}$$

Let's see this on configuration space:

$$\begin{aligned} L.H.S\ &:\ \langle x|i\frac{\partial}{\partial t}|\psi(t)\rangle = i\frac{\partial}{\partial t}\left\langle x|\psi(t)\right\rangle \equiv i\frac{\partial}{\partial t}\psi(x,t) \\ R.H.S\ &:\ \langle x|\hat{H}|\psi(t)\rangle = \frac{1}{2m}\langle x|\hat{P}^2|\psi(t)\rangle + V(x)\psi(x,t) \end{aligned} \tag{30}$$

To solve this wave equation, we use Fourier Transform between momentum space & configuration space. First, we have next facts:

- $\hat{X}\left|x\right\rangle = x\left|x\right\rangle$
- $\hat{P}\left|p\right\rangle = p\left|p\right\rangle$
- $\langle x|\hat{X}|\psi\rangle = x\left\langle x|\psi\right\rangle = x\psi(x)$
- $\langle p|\hat{P}|\psi\rangle = p\left\langle p|\psi\right\rangle = p\psi(p)$

Second, let's see next lemma:

Lemma 2.1. *Momentum operator is a generator of spartial translation.*

Proof:
Let $\hat{U}(a) = e^{-ia\hat{P}}$. Then we can find below commutate relation.

$$\begin{aligned} \left[\hat{X}, \hat{U}(a)\right] &= \left[\hat{X}, \sum_n \frac{(-ia\hat{P})^n}{n!}\right] = \sum_n \frac{(-ia)^n}{n!}\left[\hat{X}, \hat{P}^n\right] \\ &= \sum_n \frac{a^n(-i)^{n-1}\hat{P}^{n-1}}{(n-1)!} = ae^{-ia\hat{P}} = a\hat{U}(a) \end{aligned} \tag{31}$$

Thus, we can find next property of $\hat{U}(a)$:

$$\hat{X}\hat{U}(a)\left|x\right\rangle = \left(\left[\hat{X}, \hat{U}(a)\right] + \hat{U}(a)\hat{X}\right)\left|x\right\rangle = (a+x)\hat{U}(a)\left|x\right\rangle \tag{32}$$

If we assume normalization, then we can write next equation. Then Q.E.D

$$\therefore e^{-ia\hat{P}}\left|x\right\rangle = \left|x+a\right\rangle \tag{33}$$

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a. Momentum Operator

We know momentum operator is a generator of spatial translation. Thus, for small translation denoted ϵ, we can obtain momentum operator on configuration space by Lemma(1).

$$\begin{aligned}
\hat{U}(\epsilon)\left|x\right\rangle = \left|x+\epsilon\right\rangle &\simeq \left(1-i\epsilon\hat{P}\right)\left|x\right\rangle \\
\Rightarrow \lim_{\epsilon\to 0}\frac{\left|x+\epsilon\right\rangle-\left|x\right\rangle}{-i\epsilon} &= \hat{P}\left|x\right\rangle \\
\therefore \hat{P}\left|x\right\rangle &= i\frac{\partial}{\partial x}\left|x\right\rangle
\end{aligned} \tag{34}$$

Then we can also find $\langle x|\hat{P}|\psi\rangle = -i\frac{\partial}{\partial x}\langle x|\psi\rangle = -i\frac{\partial}{\partial x}\psi(x)$. So, $\hat{P} = -i\frac{\partial}{\partial x}$.

For relativistic theory, we can rewrite momentum operator as 4-vector notation.[1]

$$P_\mu = \left(H, -P^i\right) = \left(i\frac{d}{dt}, i\frac{\partial}{\partial x^i}\right) = i\partial_\mu \quad (\partial_\mu = \frac{\partial}{\partial x^\mu}) \tag{35}$$

Now, we can write Fourier transform between momentum space & configuration space.

Lemma 2.2. $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx} = \langle p|x\rangle^*$

Proof:

We have $\langle x|\hat{P}|p\rangle = p\langle x|p\rangle \equiv p\phi_p(x)$. Then we can find $\phi_p(x)$ by using another state $|x'\rangle$

$$\begin{aligned}
\langle x|\hat{P}|p\rangle &= \int dx'\,\langle x|\hat{P}|x'\rangle\langle x'|p\rangle \\
&= \int dx' i\left(\frac{d}{dx'}\delta(x'-x)\right)\phi_p(x') \\
&= -i\int dx'\delta(x'-x)\frac{d}{dx'}(\phi_p(x')) \;\;{}^{2} \\
&= -i\frac{d}{dx}\phi_p(x) = p\phi_p(x) \\
\therefore \phi_p(x) &= \frac{1}{\sqrt{2\pi}}e^{ipx} \;\;{}^{3}
\end{aligned} \tag{36}$$

■

And we know $\langle x|\hat{P}^2|\psi\rangle = \left(-i\frac{\partial}{\partial x}\right)^2\langle x|\psi\rangle = -\frac{\partial^2}{\partial x^2}\psi(x)$. Then we can write next equation.

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \tag{37}$$

This equation is called **Schrödinger equation**.

[1] We choose natural unit : $c = \hbar = 1$

[2] Total derivative term is vanished.

[3] We choose $\frac{1}{\sqrt{2\pi}}$ to normalize eigenstates.

Since the schrödinger equation is kind of wave equation, we can solve it by separation of variables. Let $\psi(x,t) = T(t)\phi(x)$.

$$\begin{aligned}
&L.H.S : i\frac{\partial}{\partial t}\psi(x,t) = i\phi(x)\dot{T}(t)\\
&R.H.S : -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) = -\frac{1}{2m}T\phi''(x) + V(x)T(t)\phi(x)\\
&\Rightarrow \frac{i\dot{T}}{T} = \frac{\left(-\frac{1}{2m}\phi'' + V\phi\right)}{\phi} = E
\end{aligned} \tag{38}$$

Then we can get next two ordinary differential equations.

$$\begin{aligned}
\textbf{Time} &: i\dot{T}(t) = ET(t) \Rightarrow T(t) = e^{-iEt}\\
\textbf{Spatial} &: -\frac{1}{2m}\phi''(x) + V(x)\phi(x) = E\phi(x)
\end{aligned} \tag{39}$$

The second equation of Eq(39) is called time-independent schrödinger equation.

Example 2.1. ***Please input SHO example***

2.4. Quantum Symmetries

Exercise 2.1 (A^+)**.** *Prove Heisenberg Uncertainty Principle by creative way.*

Consider transformation of rays $T : \mathcal{R}_i \to \mathcal{R}'_i \quad (i = 1, 2)$. We call it symmetric transform if $P(\mathcal{R}_1 \to \mathcal{R}_2) = P(\mathcal{R}'_1 \to \mathcal{R}'_2)$.
In other words, for $|\psi_1\rangle \in \mathcal{R}_1,\ |\psi_2\rangle \in \mathcal{R}_2,\ |\psi'_1\rangle \in \mathcal{R}'_1,\ |\psi'_2\rangle \in \mathcal{R}'_2,$ if $\|\langle\psi_1|\psi_2\rangle\|^2 = \|\langle\psi'_1|\psi'_2\rangle\|^2$. then T is symmetric transform.

Theorem 2.1 (Wigner)**.** *Any symmetry transformation can be represented on the Hilbert space of physical states by an operator that is either linear and unitary or antilinear and antiunitary.* [1]

Proof:
Please input proof Kyung teacher. I believe you

■

[1] S. Weinberg, The Symmetry Representation Theorem. *The Quantum Theory of Fields, Vol.1*(pp.91-96), Cambridge

Now, consider three symmetric transformations.

$$\mathcal{R} \xrightarrow{T_1} \mathcal{R}' \xrightarrow{T_2} \mathcal{R}'' \xrightarrow{T_3} \mathcal{R}'''$$

And their representations are $\hat{U}(T_i) \quad (i = 1, 2, 3)$. Let's see below diagram.

$$\begin{tikzcd}
|\psi\rangle \arrow[r, "T_1"] \arrow[dr, "T_2T_1"] & \hat{U}(T_1)\,|\psi\rangle \arrow[d, "T_2"'] \\
& \hat{U}(T_2)\hat{U}(T_1)\,|\psi\rangle
\end{tikzcd}$$

We can find $\hat{U}(T_2T_1) = \hat{U}(T_2)\hat{U}(T_1)$. Then for three symmetric transformations, we can find below properties:

- $T_1\,(T_2T_3) = (T_1T_2)\,T_3$
- $\mathbb{1}$ is also symmetric transform T
- ${}^{\exists}T^{-1}$ such that $T^{-1}T = TT^{-1} = \mathbb{1}$

The set of operators which have above properties is called **Group**.

3. Group

3.1. Definitions & Some examples

Definition 3.1 (A Group). A set $G = \{g_1, g_2, g_3, \cdots\}$ is said to form a *group*[1] if there is an operation $\bullet$, called *group multiplication*, which associates any given (ordered) pair of elements $g_1, g_2 \in G$ with a well-defined *product* $g_1 \bullet g_2$ which is also an element of G, such that the following conditions are satisfied: [2]

i) If $g_1, g_2 \in G$ then $g_1 \bullet g_2 \in G$ -Closure

ii) $g_1 \bullet (g_2 \bullet g_3) = (g_1 \bullet g_2) \bullet g_3$ for all $g_1, g_2, g_3 \in G$ -Associative

iii) $^\exists e \in G$ such that $g \bullet e = e \bullet g = g$ for all $g \in G$ -Identity

iv) For $g \in G$, $^\exists g^{-1} \in G$ such that $g \bullet g^{-1} = g^{-1} \bullet g = e$ -Inverse

Definition 3.2 (Abelian Group). An *abelian group* G is one for which the group multiplication is commutative, i.e. $a \bullet b = b \bullet a$ for all $a, b \in G$.

Definition 3.3 (Order). The *order* of a group is the number of elements of the group (if it is finite).

Example 3.1 (Cyclic Group). *Cyclic groups are kinds of abelian group.*
(1) $C_1 = \{e\}$
(2) $C_2 = \{e, a\}$ **Please input Group multiplication table and denote example - parity, permutation**

(3) $C_3 = \{e, a, b\}$ **Please input Group multiplication table, rotation graph**

Theorem 3.1. *If the order n of a group is a prime number, it must be isomorphic to C_n.*[3]

[1] Precisely, $(G, \bullet)$ is called a group
[2] Wu-Ki Tung, Basic Group Theory. *Group Theory In Physics* (pp.12-13), World Scientific
[3] It is a corollary of Cayley theorem

Example 3.2 (Dihedral Group)**.** *The simplest non-cyclic group.*
(1) $D_2 = \{e, a, b, c\}$ **Please input Group multiplication table, Rectangle graph**

(2) $D_3 = \{e, (12), (23), (31), (123), (321)\}$ **Please input Group multiplication table, triangle graph**

Definition 3.4 (Subgroup)**.** Let $(G, \bullet)$ be a group, and let H be a subset of G. H is called a subgroup of G, written $H \leq G$, if H is a group relative to the binary operation in G. [1]

Example 3.3. *(Some basic examples).*

(1) $\{e\} \leq G$ for any group G
(2) $C_2 \leq D_2$
(3) $\{e, (12)\},\ \{e, (23)\},\ \{e, (31)\},\ \{e, (123), (321)\} \leq S_3$ (S_3 is a permutation group.)

[1] P.B.Bhattacharya et al, Subgroups and Cosets, *Basic Abstract Algebra* (pp.72), Cambridge