Higgs Inflation & Thermal History

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Abstract

Standard Model & Cosmology have common thing - Fundamental Scalar Field. In SM, it is called Higgs and in Cosmology, it is called Inflaton. The most natural thing that we can think of is they are same - Higgs-Inflaton! But it's not scientific to leave it as a hypothesis, so we need to shape our model. To match experimental results, we should use **Strong Non-minimal coupling of Higgs-Inflaton** to gravity. So, in this paper we will cover the Higgs-Inflaton as main topic and will actually match the experimental results. Of course, I will review some basic background knowledge before that. This paper is based on **?** & **?**.

1. General Relativity

1) Some Calculations of GR

1. Covariant Derivative

$$\nabla_\mu T^{\nu_1\cdots\nu_r}{}_{\rho_1\cdots\rho_s} = \partial_\mu T^{\nu_1\cdots\nu_r}{}_{\rho_1\cdots\rho_s} + \Gamma^{\nu_1}_{\mu\alpha} T^{\alpha\cdots\nu_r}{}_{\rho_1\cdots\rho_s} + \cdots + \Gamma^{\nu_r}_{\mu\alpha} T^{\nu_1\cdots\alpha}{}_{\rho_1\cdots\rho_s} - \Gamma^{\alpha}_{\mu\rho_1} T^{\nu_1\cdots\nu_s}{}_{\alpha\cdots\rho_s} - \Gamma^{\alpha}_{\mu\rho_s} T^{\nu_1\cdots\nu_s}{}_{\rho_1\cdots\alpha} \tag{1}$$

2. Connection Coefficient (Levi-Civita)

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}(g_{\nu\alpha,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}) \tag{2}$$

3. Covariant Derivative Example

$$\begin{gathered}\text{For Scalar Field } f, \quad \nabla_\mu f = \partial_\mu f \\ \text{For Vector Field } V, \quad \nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu)\end{gathered} \tag{3}$$

4. Normal Coordinates (Local Cartesian Coordinates)

$$\exists p \in \mathcal{M} \ s.t \ \Gamma^\mu_{\nu\rho}(p) = 0 \tag{4}$$

2) Einstein - Hilbert Action

Einstein-Hilbert actions is given as :

$$S_{EH} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R \tag{5}$$

To use least action principle, we should know variation rule for $\sqrt{-g}$ and R.

1. Variation of g

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = \frac{\partial}{\partial g_{\mu\nu}} \left(\Sigma_\sigma g_{\rho\sigma} \Delta^{\rho\sigma}\right) \cdot \delta g_{\mu\nu} = \Delta^{\mu\nu} \cdot \delta g_{\mu\nu} = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{6}$$

2. Variation of $g^{\mu\nu}$

$$\delta g^{\mu\nu}(g) = \frac{\partial g^{\mu\nu}}{\partial g} \delta g = \frac{\partial g^{\mu\nu}}{\partial g} g g^{\rho\sigma} \delta g_{\rho\sigma} = -\frac{\Delta^{\mu\nu}}{g^2} g g^{\rho\sigma} \delta g_{\rho\sigma} = -g^{\mu\nu} g^{\rho\sigma} \delta g_{\rho\sigma} \tag{7}$$

3. Variation of $\sqrt{-g}$

$$\delta(\sqrt{-g}) = -\frac{1}{2} \frac{1}{\sqrt{-g}} \delta g = -\frac{1}{2} \frac{1}{\sqrt{-g}} \left(-g g_{\mu\nu} \delta g^{\mu\nu}\right) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{8}$$

4. Palatini Identity

   Consider Normal Coordinate at p. Then we can find next things.

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} \\ \Rightarrow \delta R_{\mu\nu} &= \delta\Gamma^\lambda_{\mu\nu,\lambda} - \delta\Gamma^\lambda_{\mu\lambda,\nu} \\ \Rightarrow \delta R_{\mu\nu} &= \delta\Gamma^\lambda_{\mu\nu;\lambda} - \delta\Gamma^\lambda_{\mu\lambda;\nu} \end{aligned} \tag{9}$$

5. Variation of R

$$\begin{gathered} \delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = (\delta g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu}(\delta R_{\mu\nu}) = R_{\mu\nu} \delta g^{\mu\nu} \\ (g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\nu;\lambda} - \delta\Gamma^\lambda_{\mu\lambda;\nu}) = 0) \end{gathered} \tag{10}$$

   (Because last terms are Surface Terms)

6. Variation of Einstein-Hilbert Action

$$\delta(\sqrt{-g} R) = \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu}\right) \delta g^{\mu\nu} = \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu} \equiv \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \tag{11}$$

7. Einstein Equation

1) Matter Action

$$S_m = \int d^4x \sqrt{-g}\mathcal{L} \;\Rightarrow\; \delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \equiv \sqrt{-g}(-\frac{1}{2}T_{\mu\nu})\delta g^{\mu\nu} \tag{12}$$

2) Total Action

$$S = S_{EH} + S_m = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left(R + \frac{2}{M_p^2}\mathcal{L}\right) \tag{13}$$

3) Euler-Lagrange Equation

$$\begin{gathered}\delta S = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left(G_{\mu\nu} - \frac{1}{M_p^2}T_{\mu\nu}\right)\delta g^{\mu\nu} \\ \therefore G_{\mu\nu} = \frac{1}{M_p^2}T_{\mu\nu}\end{gathered} \tag{14}$$

3) FLRW Cosmology

1. Build Metric

1. Globally hyperbolic manifold with topology $\mathcal{M} = \mathbb{R}\times\Sigma$
2. Arnowitt-Deser-Misner Method & Decompose $\mathcal{M}$ into slice of Σ_t at constant time t.

$$ds^2 = \left[-N^2 + g^{ij}N_iN_j\right]dt^2 + 2N_i dt dx^i + g_{ij}dx^i dx^j \tag{15}$$

3. Spatial homogeneity & Isotropy (Rotational Invariance)

$$ds^2 = -N^2(t)dt^2 + g_{ij}(t)dx^i dx^j \tag{16}$$

4. Homogeneity & Isotropy -> Maximally Symmetric & N = 1

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right) \tag{17}$$

5. Conformal Time

$$\begin{gathered}ds^2 = a^2(\eta)\left[-d\eta^2 + \left\{d\chi^2 + f(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right\}\right] \\ f(\chi) \equiv \begin{cases}\sinh^2\chi \ (\kappa=-1) \\ \chi^2 \ (\kappa=0) \\ \sin^2\chi \ (\kappa=1)\end{cases}\end{gathered} \tag{18}$$

2. GR Calculation

0. Basic

$$g_{ii,0} = -2\frac{\dot{a}}{a}g_{ii} \tag{19}$$

1. Connection Coefficient

$$\begin{aligned}
\Gamma^i_{i0} &= \frac{1}{2}g^{ii}(g_{ii,0} + g_{i0,i} - g_{i0,i}) = \frac{1}{2}g^{ii}g_{ii,0} = \frac{\dot{a}}{a} \\
\Gamma^0_{ii} &= \frac{1}{2}g^{00}(g_{i0,i} + g_{0i,i} - g_{ii,0}) = -\frac{1}{2}g^{00}g_{ii,0} = \frac{\dot{a}}{a}g_{ii} \\
\Gamma^i_{jj} &= \frac{1}{2}g^{ii}(g_{ji,j} + g_{ij,j} - g_{jj,i}) = -\frac{1}{2}g^{ii}g_{jj,i}
\end{aligned} \tag{20}$$

2. Riemann Tensor

$$\begin{aligned}
R^{0i}{}_{0i} &= g^{ii}(R^0{}_{i0i}) = g^{ii}\left(\partial_0\Gamma^0_{ii} - \partial_i\Gamma^0_{i0} + \Gamma^0_{\alpha 0}\Gamma^\alpha_{ii} - \Gamma^0_{\alpha i}\Gamma^\alpha_{i0}\right) \\
&= g^{ii}\left(\partial_0\left(\frac{\dot{a}}{a}g_{ii}\right) - 0 + 0 - \frac{\dot{a}}{a}g_{ii}\frac{\dot{a}}{a}\right) \\
&= \frac{\ddot{a}}{a}
\end{aligned} \tag{21}$$

3. Ricci Tensor

$$R^0{}_0 = \sum_{i=1}^{3} R^{0i}{}_{0i} = 3\frac{\ddot{a}}{a} \tag{22}$$

4. Energy-Momentum Tensor

$$\begin{gathered}
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + P g_{\mu\nu} \\
\rho = u^\mu u^\nu T_{\mu\nu} \\
P = \frac{1}{3}(g^{\mu\nu}T_{\mu\nu} + \rho)
\end{gathered} \tag{23}$$

5. Energy Condition

- Null Energy Condition : $X^\mu X^\nu T_{\mu\nu} \geq 0$ for null-like vector X^μ
- Weak Energy Condition : $u^\mu u^\nu T_{\mu\nu} \geq 0$ for time-like vector u^μ
- Strong Energy Condition : $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)u^\mu u^\nu \geq 0$ for time-like vector.

If we define equation of state as $\omega(t) \equiv \frac{P(t)}{\rho(t)}$ then

- NEC : $\omega \geq -1$
- SEC : $\omega \geq -\frac{1}{3}$
- WEC : $\rho \geq 0$

6. Energy - Momentum Conservation

$$\begin{aligned}\nabla_a T^a{}_b = -\left(\frac{\partial\rho}{\partial t} + 3(\rho(t)+P(t))\frac{\dot{a}(t)}{a(t)}\right) = 0 \\ \Rightarrow\ \dot{\rho} + 3H(\rho+P) = 0\end{aligned} \tag{24}$$

7. Einstein Equation (Recommended to use CAS - Sagemath, Mathematica)

$$\begin{aligned}G_{00} = 3\frac{((\dot{a}(t))^2+\kappa)}{a(t)^2} \\ T_{00} = \rho(t)\end{aligned} \tag{25}$$

$$\begin{aligned}G_{00} = 8\pi G T_{00} \\ \Rightarrow\ H^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}\end{aligned} \tag{26}$$

8. Energy-Momentum + Eq of State

$$\dot{\rho} + 3H\rho(1+\omega) = 0\ \Rightarrow\ \rho(a) \propto a^{-3(1+\omega)} \tag{27}$$