Numerical implementation of Neutron star cooling

Tae Geun Kim

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Basic structure of Neutron star

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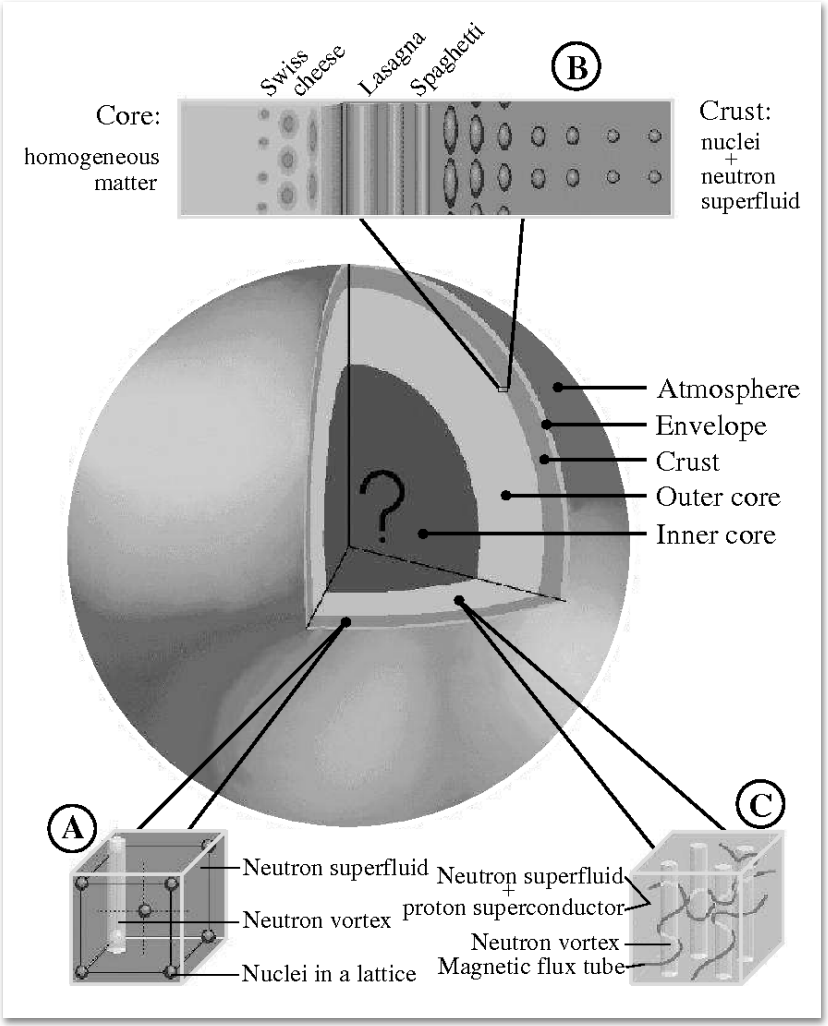
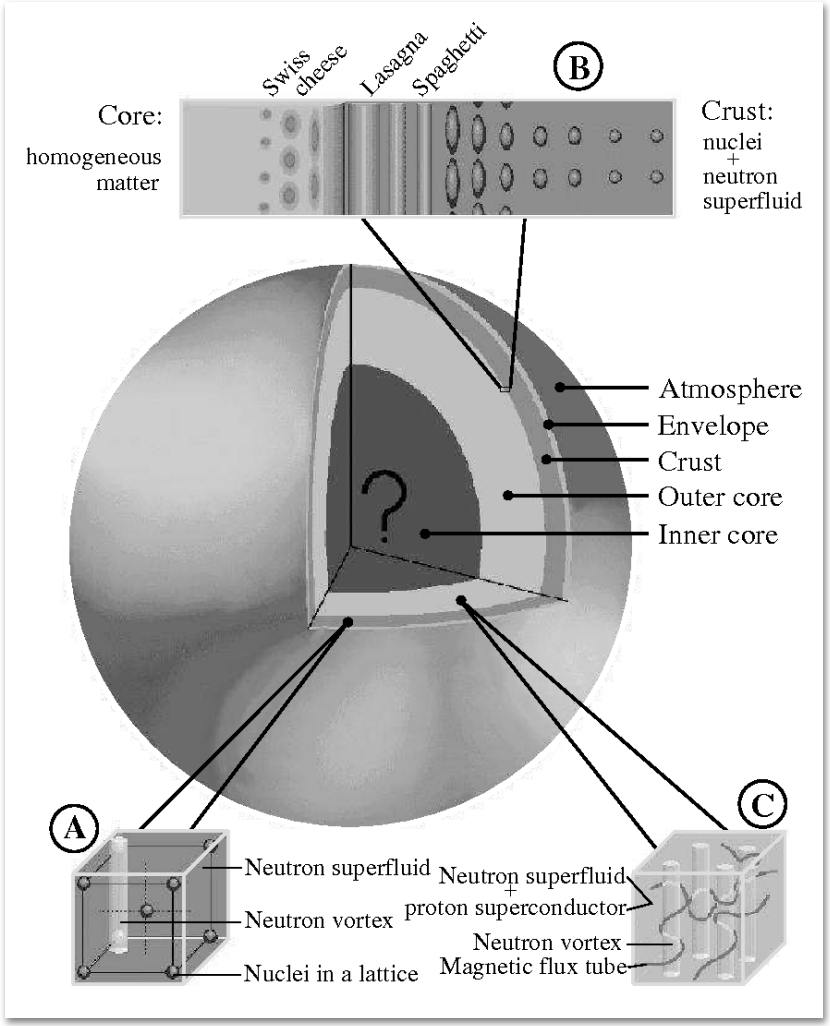


- **Atmosphere (10cm)**
  Determines the thermal radiation (the spectrum). Of upmost importance for interpretation of X-ray (and optical) observation. However it as NO effect on the thermal evolution of the star.

- **Envelope (100m)**
  Contains a huge temperature gradient - extremly important for the cooling, strongly affected by magnetic fields and the presence of *polluting* light elements

- **Crust (1km)**
  Composed of nuclei, quasi-free neutrons. The nuclei are arranged into a crystalline lattice. At bottom of crust, *pasta phases* of nuclear matter forms mantle.

Basic structure of Neutron star



- **Outer Core (10km)**

  Contains most of the stellar mass, composed of neutrons with an admixture of protons & leptons - electrons, muons ($npe\mu$ matter)

- **Inner Core (? km)**

  Exists for massive neutron star, compositions & properties are not well known. There are some proposed theoretical models:

1. **hyperonization of matter** – the appearance of various hyperons (first of all, Λ^- and Σ^- hyperons — ($npe\mu\Lambda\Sigma$)matter)
2. **pion condensation** – formation of a Bose condensate of collective interactions with the properties of π-mesons.
3. **kaon condensation** – formation of a similar condensate of K-mesons.

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  - **Tolman-Oppenheimer-Volkoff equations**
    - Describe relativistic stellar structure
  - **Thermal evolution equations**
    - Describe stellar evolution

Relativistic stellar structure

Tolman-Oppenheimer-Volkoff Equation

- For static, non-rotating and spherical symmetric star, metric is given as

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2$$

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$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

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- At interior of the star, let define new metric function for convenience

$$e^{-2\Lambda(r)} = 1 - \frac{2m(r)}{r}$$

  The quantity $m(r)$ can be interpreted as the **mass interior to radius** r.

Tolman-Oppenheimer-Volkoff Equation

- Now, consider perfect fluid matter

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \quad u = e^{-\Phi}\frac{\partial}{\partial t}$$

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- Rewrite: (index-free notation)

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- From Einstein equation, we can get two equations: Specific calculation with SageManifolds

$$(0,0) \Rightarrow \frac{dm(r)}{dr} = 4\pi r^2 \rho$$

$$(1,1) \Rightarrow \frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 P}{r^2\left(1 - \frac{2m(r)}{r}\right)}$$

Tolman-Oppenheimer-Volkoff Equation

- From energy conservation, we can get one more equation:

$$\nabla_\mu T^\mu{}_\nu = 0 \;\Rightarrow\; (\rho + P)\frac{d\Phi}{dr} + \frac{dP}{dr} = 0$$

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$$\nabla_\mu T^\mu{}_\nu = 0 \;\Rightarrow\; (\rho + P)\frac{d\Phi}{dr} + \frac{dP}{dr} = 0$$

- These three fundamental equations are called **Tolman-Oppenheimer-Volkoff Equation**

$$\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2 \rho \\
\frac{d\Phi}{dr} &= \frac{m + 4\pi r^3 P}{r^2\left(1 - \frac{2m}{r}\right)} \\
\frac{dP}{dr} &= -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2\left(1 - \frac{2m}{r}\right)}
\end{aligned}$$

Polytropic Equation of State

- Neutron stars are well modeled by polytropes

$$P = K\rho_0^{\Gamma}$$

where

K : polytropic gas constant
ρ_0 : rest mass density
$\rho_0 = m_u n_B$ (for neutron star)
m_u : nucleon mass - atomic mass unit
n_B : baryonic number density
$\rho = \rho_0(1+\epsilon)$
ϵ : internal energy density per unit mass
$\Gamma \equiv 1 + \frac{1}{n}$
n : polytropic index

Polytropic Equation of State

- By 1st law of thermodynamics with isentropic assumption, we can get

$$P = n_B^2 \frac{\partial(\rho/n_B)}{\partial n_B}$$

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- Substitute polytropic EoS:

$$\rho = \rho_0 + \frac{P}{\Gamma - 1} = \left(\frac{P}{K}\right)^{\frac{1}{\Gamma}} + \frac{\rho}{\Gamma - 1}$$

Solving TOV equation

- Total running terms

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2\left(1 - \frac{2m}{r}\right)}$$
$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r^2\left(1 - \frac{2m}{r}\right)}$$
$$\frac{dN_B}{dr} = \frac{4\pi r^2 n_B}{\sqrt{1 - \frac{2m}{r}}}$$

$$\rho_0 = \left(\frac{P}{K}\right)^{\frac{1}{\Gamma}}$$
$$\rho = \rho_0 + \frac{P}{\Gamma - 1} = \left(\frac{P}{K}\right)^{\frac{1}{\Gamma}} + \frac{P}{\Gamma - 1}$$
$$n_B = \frac{\rho_0}{m_u} = \frac{1}{m_u}\left(\frac{P}{K}\right)^{\frac{1}{\Gamma}}$$

Solving TOV equation

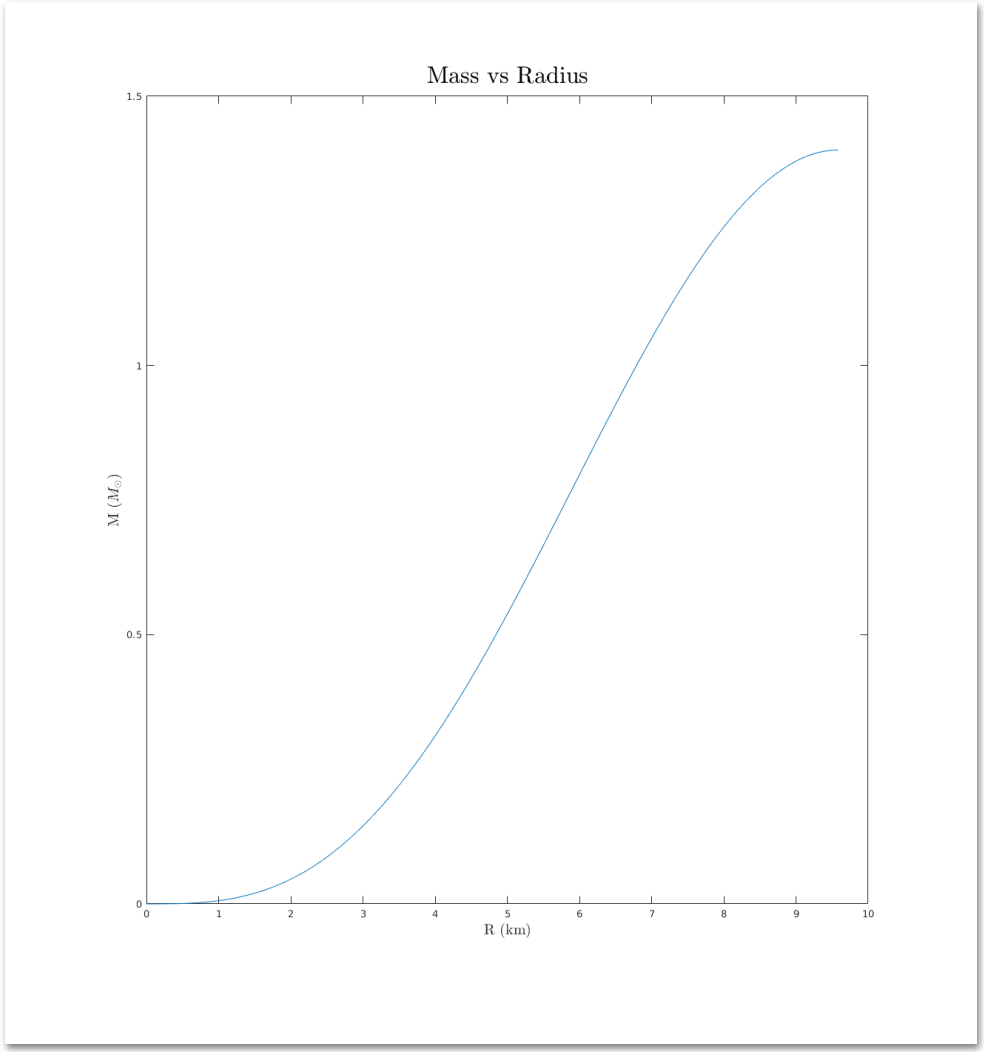
- Total boundary conditions

$$
\begin{aligned}
m(0) &= 0 \\
\rho_0(0) &= \rho_c \\
P(0) &= P_c = K\rho_c^{\Gamma} \\
n_B(0) &= n_c = \frac{\rho_c}{m_u}
\end{aligned}
$$

$$
\begin{aligned}
P(R) &= 0 \\
\Phi(R) &= \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right)
\end{aligned}
$$

Results ($K = 100, \Gamma = 2, \rho_c = 1.28 \times 10^{-3}$)

$$R = 9.5850 km$$
$$M = 1.4002 M_\odot$$



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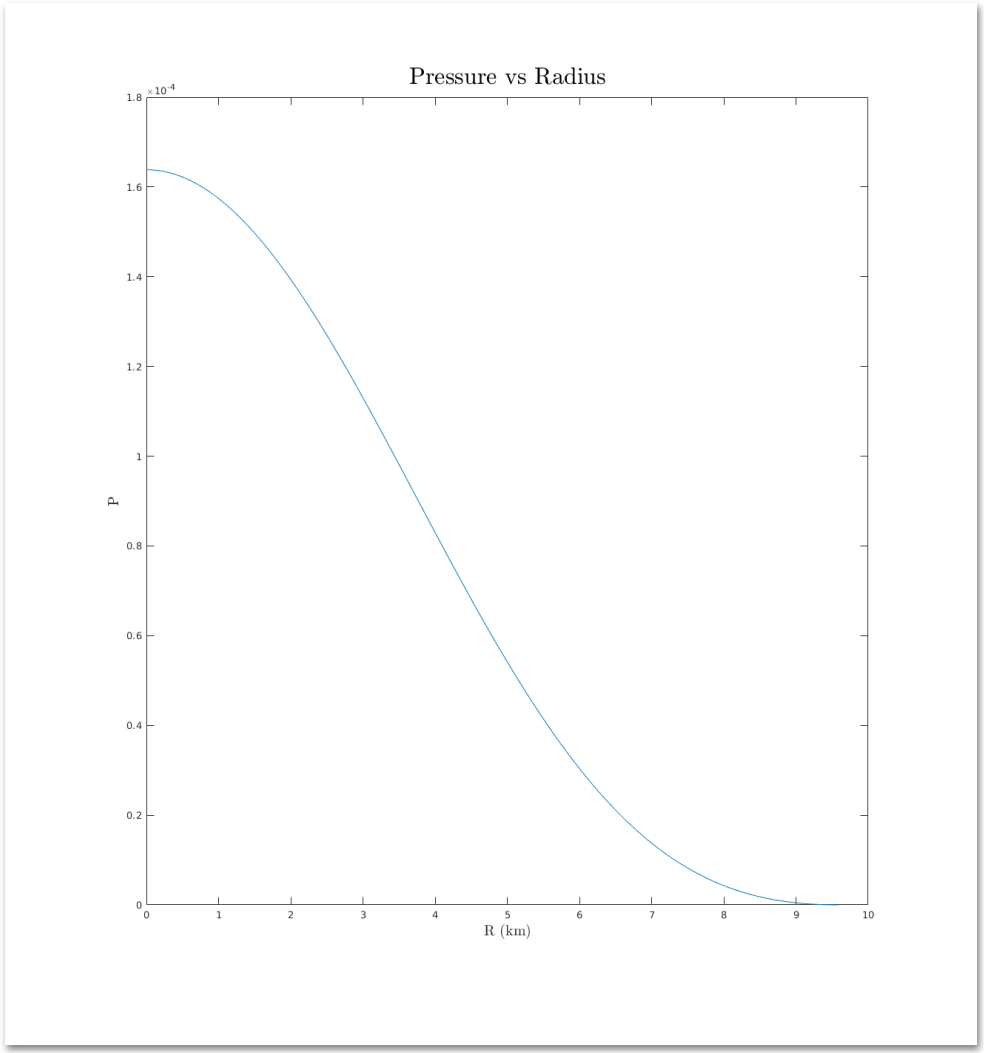
$$P(0) = P_c, \quad P(R) = 0$$



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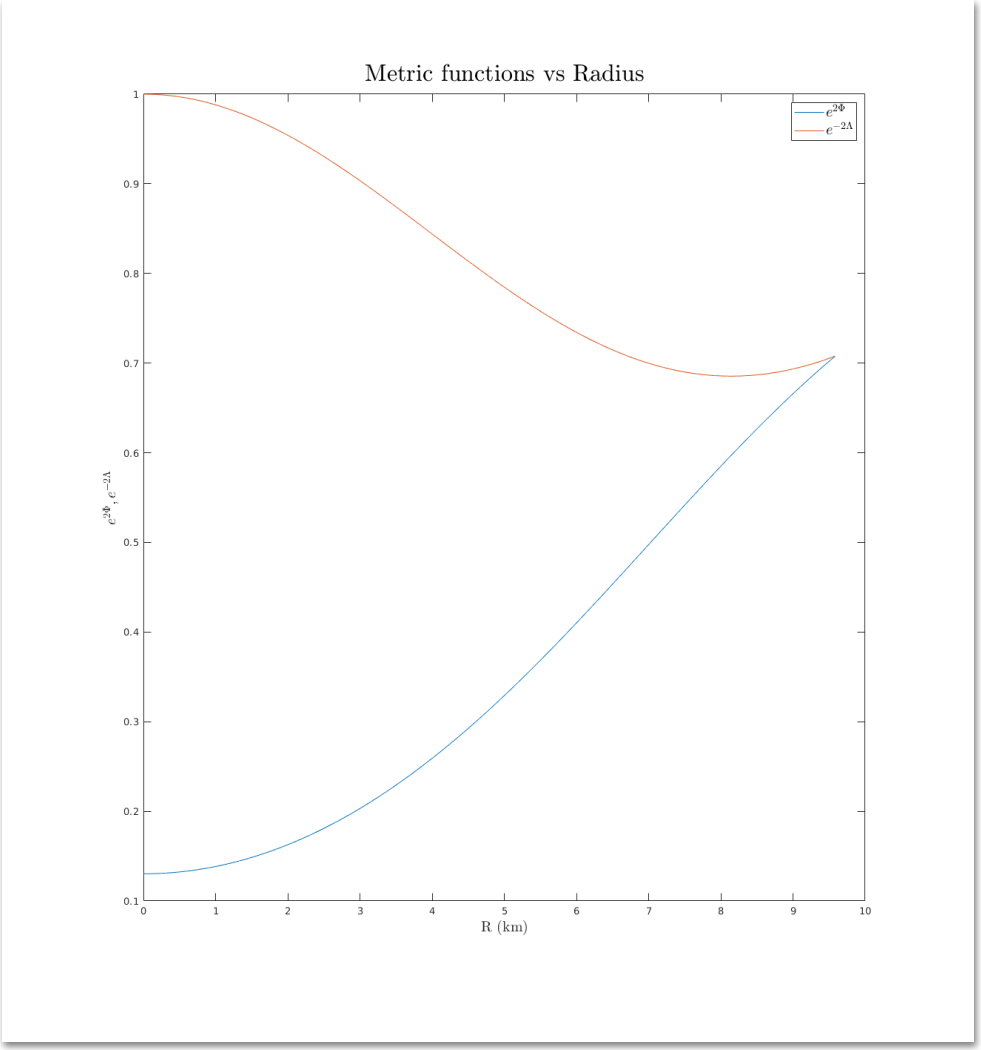
$$e^{2\Phi(R)} = e^{-2\Lambda(R)}$$



Implicit scheme to solve stiff ODE

Brief view - Thermal evolution equations

- Energy balance

$$\frac{d(Le^{2\Phi})}{dr} = -\frac{4\pi r^2 e^{\Phi}}{\sqrt{1-2m/r}}\left(C_v\frac{dT}{dt} + e^{\Phi}(Q_\nu - Q_h)\right)$$

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$$\frac{d(Te^{\Phi})}{dr} = -\frac{1}{\lambda}\frac{Le^{\Phi}}{4\pi r^2\sqrt{1-2m/r}}$$

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where

- C_v: Specific heat
- L: Diffusive luminosity at r
- λ: Thermal conductivity
- Q_ν: Neutrino emissivity
- Q_h: Heating late

Brief view - Thermal evolution equations

- Let's rewrite these equations with red-shifted functions & Lagrangian coordinate a

$$\mathcal{T} = e^{\Phi} T,\ \mathcal{L} = e^{2\Phi} L, \quad da = \frac{4\pi r^2 n_B dr}{\sqrt{1 - 2m/r^2}}$$

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- Our choice is **Backward Euler** - Most robust method (but high cost)

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- Suppose that there is non-autonomous ODE system

$$\mathbf{Y}' = f(t, \mathbf{Y})$$

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- I have achieved to reduce high cost of Backward Euler with AD & Rust (High performance programming language)

TODO

Current Progress

Symbolic

- Solve TOV from spherical symmetric metric & polytropic EM Tensor

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Numeric

- Implement Explicit RK4
- Apply ERK4 to TOV eq
- Implement Implicit Method
  - Backward Euler Method

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Numeric

- Implement more implicit methods
  - 4th order Gauss-Legendre (GL4)
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- To formulate neutrino emissivity for several Equation of States & Core model
  - Focus on Cas A (Cassiopeia A)
    - Modified Urca
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Interface

- Implement one-click (or type) installation for any OS
- Various language API (Python or even Mathematica?)

Thank you