NSCool-notes

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Chapter 1

Preface

This note is summary to implement neutron star cooling.

Chapter 2

TOV Equation

- TOV equation means **Tolman-Oppenheimer-Volkoff equation**.

2.1 Build Equation

For static, non-rotating and spherical symmetric star, metric is given as

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2$$

For isolated star, the metric must reduce to the Schwarzschild metric at outside of the star

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

At interior of the star, let define new metric function for convenience

$$e^{-2\Lambda(r)} = 1 - \frac{2m(r)}{r}$$

The quantuty $m(r)$ can be interpreted as **the mass interior to radius** r.

Now, consider perfect fluid matter:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu},\ u = e^{-\Phi}\frac{\partial}{\partial t}$$

Rewrite this to index-free notation:

$$T = \rho^{2\Phi}dt^2 + \frac{P}{1 - \frac{2m(r)}{r}}dr^2 + Pr^2 d\Omega^2$$

Solve Einstein equation with this metric & EM tensor then,

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho \tag{2.1}$$

$$\frac{d\Phi(r)}{dr} = \frac{m(r) + 4\pi r^3 P}{r^2\left(1 - \frac{2m(r)}{r}\right)} \tag{2.2}$$

From energy conservation, we can get one more equation:

$$\nabla_\mu T^\mu{}_\nu = 0 \Rightarrow (\rho + P)\frac{d\Phi}{dr} + \frac{dP}{dr} = 0$$

These three fundamental equations are called **Tolman-Oppenheimer-Volkoff Equations**

Tolman-Oppenheimer-Volkoff equation

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r^2(1 - \frac{2m}{r})}$$
$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2(1 - \frac{2m}{r})}$$

Now, use polytropic equation of state.

Polytropic Equation of State

$$P = K\rho_0^\Gamma$$

P: pressure K: polytropic gas constant ρ_0: rest mass density. for neutron star, $\rho_0 = m_u n_B$ m_u: nucleon mass with atomic mass unit n_B: baryonic number density $\rho = \rho_0(1+\epsilon)$: total mass energy density ϵ: internal energy density per unit mass $\Gamma \equiv 1 + \frac{1}{n}$ n: polytropic index

Thermodynamics process of neutron star is isentropic process. It means S, N are conserved and only V changes. First define densities as belows.

$$s = \frac{S}{V},\ n = \frac{N}{V},\ \rho = \frac{U}{V}$$

Since $dS = dN = 0$, we can get next relations.

$$\frac{ds}{s} = \frac{dn}{n} = -\frac{dV}{V}$$

Now, denote 1st law of thermodynamics.

$$dU = -PdV + TdS + \mu dN = -PdV$$

Rewrite this in terms of ρ, then we can get next relation.

$$\frac{d\rho}{\rho + P} = -\frac{dV}{V}$$

By above two relations, next equation is naturally given.

$$(\rho + P) = n\frac{\partial \rho}{\partial n}$$

Since, $\rho = \rho_0(1+\epsilon),\ \rho_0 = m_u n$,

$$\frac{\partial \rho}{\partial n} = \frac{\rho}{n} + \frac{\partial}{\partial n}\left(\frac{\rho}{n}\right)\cdot n$$

Then finally we can obtain next equation.

$$\therefore\ P = n^2 \frac{\partial}{\partial n}\left(\frac{\rho}{n}\right)$$

Let substitute P from polytropic EoS, then

$$\rho = \rho_0 + \frac{P}{\Gamma - 1}$$

Total Running Terms

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r^2(1 - \frac{2m}{r})}$$
$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2(1 - \frac{2m}{r})}$$
$$\frac{dN_B}{dr} = \frac{4\pi r^2 n_B}{\sqrt{1 - \frac{2m}{r}}}$$

where

$$\rho_0 = \left(\frac{P}{K}\right)^{\frac{1}{\Gamma}}$$
$$\rho = \rho_0 + \frac{P}{\Gamma - 1}$$
$$n_B = \frac{1}{m_u}\left(\frac{P}{K}\right)^{\frac{1}{\Gamma}}$$

Boundary Conditions

$$m(0) = 0$$
$$\Phi(0) = 0$$
$$\rho_0(0) = \rho_c$$
$$P(0) = P_c = K\rho_c^\Gamma$$
$$n_B(0) = n_c = \frac{1}{m_u}\rho_c$$
$$P(R) = 0$$
$$\Phi(R) = \frac{1}{2}\ln\left(1 - \frac{2GM}{R}\right)$$