General Relativity

By precise approach

Tae Geun Kim

2018-08-24

1 Preliminaries

Manifolds

1. Topological Manifolds

Def 1.1 Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

Why?

- **Hausdorff** : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- **Locally Euclidean** : This is the main reason that why we require manifolds.
- **Countable Basis** : We need *partition of unity* to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require *paracompactness.* And paracompactness follows from *second countability.* It is same as have countable basis.

1.1 Supplement

Thm 1.1.1 Paracompact $\simeq$ Partition of unity

Let (X, τ) be a topological space that is T_1 (all points are closed). Then the following are equivalent:

I. (X, τ) is paracompact and Hausdorff

II. Every open cover of (X, τ) admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

Def 1.1.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Def 1.1.3 Locally finite cover

Let (X, τ) be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Def 1.1.4 Refinement of open covers

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Def 1.1.5 Paracompact topological space

A topological space (X, τ) is called *paracompact* if every open cover of X has a refinement by a locally finite open cover.

Def 1.1.6 Partition of unity

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

  (where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology of the real numbers $\mathbb{R}$ regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl\left(f_i^{-1}((0, 1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

1) $\underset{i \in I}{\forall} (Supp(f_i) \subset U_i)$
2) $\{Supp(f_i) \subset X\}_{i \in I}$ is a locally finite cover
3) $\underset{x \in X}{\forall} \left(\sum_{i \in I} f_i(x) = 1\right)$

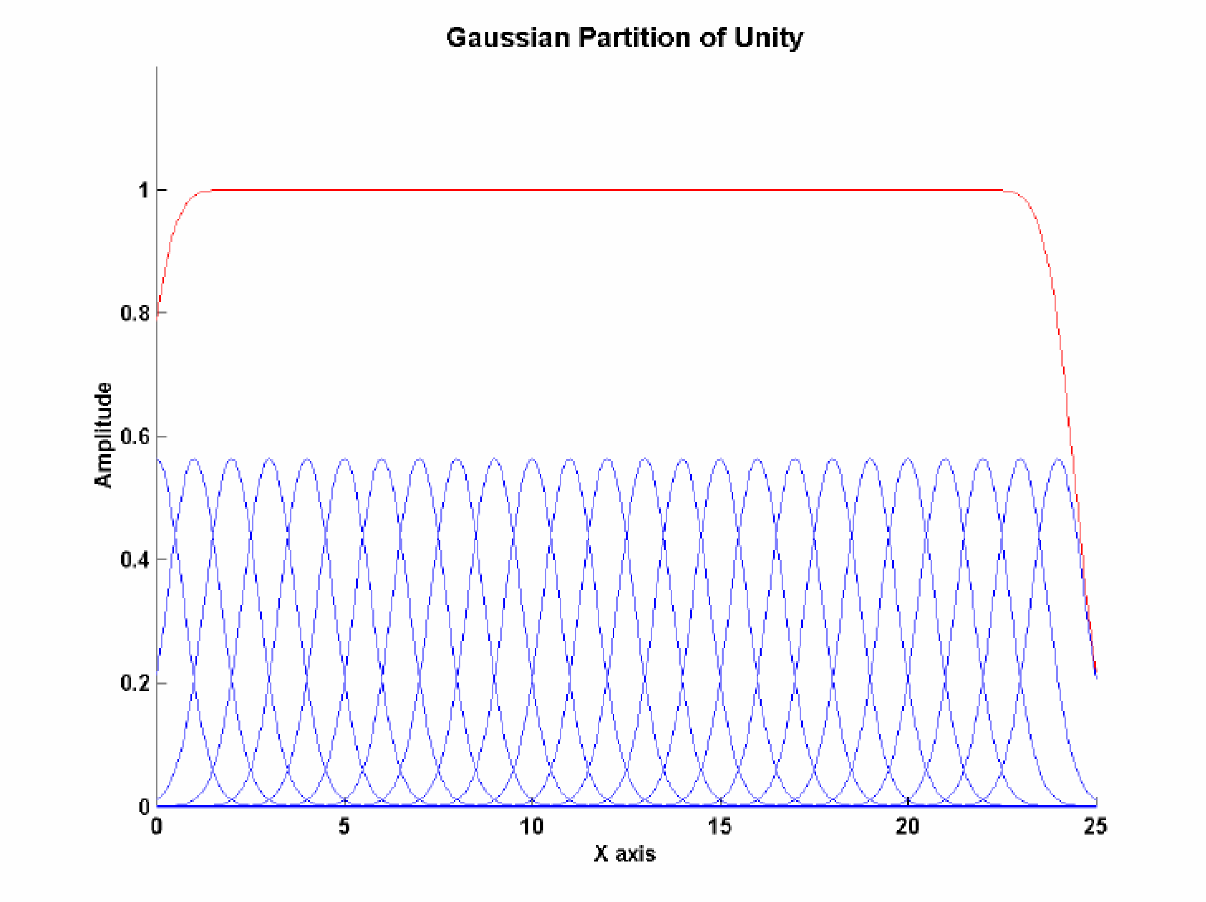

Figure 1.1: Gaussian Partition of Unity

Prop 1.1.7 Paracompact - Partition of unity

If (X, τ) is a paracompact topological space, then for every open cover $\{U_i \subset X\}_{i\in I}$ there is a subordinate partition of unity.

Proof will be given later.

Lem 1.1.8 Natural Refinement

Let (X, τ) be a topological space, $\{U_i \subset X\}_{i\in I}$ be an open cover and $\left(\phi : J \to I,\ \{V_j \subset X\}_{j\in J}\right)$ be a refinement to a locally finite cover. Then, for $\{W_i \subset X\}_{i\in I}$ with

$$W_i \equiv \left\{ \bigcup_{j\in\phi^{-1}(\{i\})} V_j \right\}$$

is still a refinement of $\{U_i \in X\}_{i\in I}$ to a locally finite cover.

Proof for 1.1.8

First we know, for V, $V_j \subset U_{\phi(j)=i}$. Conversely, $\forall j \in \phi^{-1}(\{i\})$, $V_j \subset U_i$. Thus, $W_i \in \bigcup_{j\in\phi^{-1}(\{i\})} V_j \subset U_i$.

Second, since $\{V_j \subset X\}_{j\in J}$ are locally finite, $\exists\, \mathcal{U}_x \supset \{x\}$ and a finite subset $K \subset J$ such that

$$\underset{j\in J\backslash K}{\forall} (\mathcal{U}_x \cap V_j = \emptyset)$$

(locally finite: $\mathcal{U}_x \cap V_j \neq \emptyset$ for just finite number of $j \in J$)
Then we can get by construction,

$$\underset{i\in I\backslash \phi(K)}{\forall} (\mathcal{U}_x \cap W_i = \emptyset)$$

Since $\phi(K)$ is still finite, we can find the number of i such that $\mathcal{U}_x \cap W_i = \emptyset$ is also finite. (If for $i \in K',\ \mathcal{U}_x \cap W_i = \emptyset$ then K' should be subset of $\phi(K)$.)

Therefore $\{W_i \in X\}_{i\in I}$ is locally finite.

Lem 1.1.9 Shrinking Lemma

Let X be a topological space which is normal and let $\{U_i \subset X\}_{i\in I}$ be a locally finite open cover. Assuming the axiom of choice then:

There exists another open cover $\{V_i \subset X\}_{i\in I}$ such that the topological closure $Cl(V_i)$ of its elements is contained in the original patches:

$$\underset{i\in I}{\forall} (V_i \subset Cl(V_i) \subset U_i)$$

Now, suggest some fundamental topological concepts to prove prop 1.1.7.

Def 1.1.10 Normal Spaces (T_4)

A topological space X is *normal* if for every two closed disjoint subsets $A, B \subset X$, there are neighborhoods $U \supset A,\ V \supset B$ such that $U \cap V = \emptyset$.

Prop 1.1.11 T_4 in terms of topological closure

X is normal iff for all closed subsets $C \subset X$ with open neighborhood $U \supset C$ there exists a smaller open neighborhood $V \supset C$ whose topological closure $Cl(V)$ is still contained in U:

$$C \subset V \subset Cl(V) \subset U$$

Proof for Prop 1.1.11

Suppose that (X, τ) is T_4. Consider closed subset $C \subset U$ where U is open neighborhood of C. It implies

$$C \cap X \backslash U = \emptyset$$

Since U is open, $X \backslash U$ is closed. Because of normal space, there are open neighborhoods V, W such that $C \subset V,\ X \backslash U \subset W$ and $V \cap W = \emptyset$. Because of last term, we can find $V \subset X \backslash W \subset U$. Since $X \backslash W$ is closed, we can find next relation :

$$C \subset V \subset Cl(V) \subset X \backslash W \subset U$$

In the other direction, suppose that $\forall$ open neighborhood U of closed subset C, there are smaller open neighborhood with $C \subset V \subset Cl(V) \subset U$. Now, consider disjoint closed subset $C_1, C_2 \subset X$. $C_1 \cap C_2 = \emptyset$ implies $C_1 \subset X \backslash C_2$. Since $X \backslash C_2$ is open neighborhood of C_1, there exists smaller open neighborhood V such that

$$C_1 \subset V \subset Cl(V) \subset X \backslash C_2$$

And it also implies $X \backslash Cl(V)$ is open neighborhood of C_2 where $V \cap X \backslash Cl(V) = \emptyset$. Therefore X is T_4.

Def 1.1.12 Urysohn function

Let X be a topological space, and let $A, B \subset \mathrm{X}$ be disjoint closed subsets. Then an *Urysohn function* for this situation is a continuous function

$$f : X \to [0, 1]$$

to the closed interval equipped with its Euclidean metric topology, such that

$$f(A) = \{0\} \text{ and } f(B) = \{1\}$$

Prop 1.1.13 Urysohn's Lemma

Let X be a normal topological space, and let $A, B \subset X$ be two disjoint closed subsets of X. Then there exists an *Urysohn function*.

This lemma has several **big** applications:

- **Urysohn Metrization Thm**: *If X is a normal space with a countable basis, then we can use the abundance of continuous functions from X to $[0, 1]$ to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this we see that every second countable normal space is a metric space.*
- **Tietze Extension Thm**: *Suppose A is a subset of a space X and $f : A \to [0, 1]$ is a continuous function. If X is normal and A is closed in X, then we can find a continuous function from X to $[0, 1]$ that is an extension of f.*
- **Embedding manifolds in $\mathbb{R}^n$**: Using Urysohn's lemma to develop the tool called *partitions of unity*, we can obtain the following theoerem: *Each compact n-manifold is homeomorphic to a subspace of some $\mathbb{R}^n$.*

Then let's start to prove *Urysohn's lemma.*

Proof for Urysohn's lemma

($\Leftarrow$) Suppose $f(A) = \{0\}$, $f(B) = \{1\}$ for all closed subset $A, B \subset X$. Then $A \subset f^{-1}\left([0, \frac{1}{2})\right)$ and $D \subset f^{-1}\left((\frac{1}{2}, 1]\right)$. We can find these two sets are open and disjoint.