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# General Relativity

By precise approach

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# 1 Preliminaries

## Manifolds

### 1. Topological Manifolds

#### Def 1.1 Topological Manifolds

A *manifold*  $M$  of dimension  $n$  is a topological space with the following properties.

1.  $M$  is Hausdorff
2.  $M$  is locally Euclidean of dimension  $n$
3.  $M$  has a countable basis of open sets

#### Why?

- **Hausdorff** : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- **Locally Euclidean** : This is the main reason that why we require manifolds.
- **Countable Basis** : We need *partition of unity* to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require *paracompactness*. And paracompactness follows from *second countability*. It is same as have countable basis.

### 1.1 Supplement

#### Thm 1.1.1 Paracompact $\simeq$ Partition of unity

Let  $(X, \tau)$  be a topological space that is  $T_1$  (all points are closed). Then the following are equivalent:

- $(X, \tau)$  is paracompact and Hausdorff
- Every open cover of  $(X, \tau)$  admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

**Def 1.1.2 Hausdorff**

Given points  $x$  and  $y$  of  $S$ , if  $x \neq y$ , then there exist open neighborhoods  $U$  of  $x$  and  $V$  of  $y$  in  $S$  that are disjoint: such that  $U \cap V = \emptyset$ .

**Def 1.1.3 Locally finite cover**

Let  $(X, \tau)$  be a topological space.

An open cover  $\{U_i \subset X\}_{i \in I}$  of  $X$  is called *locally finite* if  $\forall x \in X$ , there exists a neighbourhood  $U_x \supset \{x\}$  such that it intersects only finitely many elements of the cover, hence such that  $U_x \cap U_i \neq \emptyset$  for only a finite number of  $i \in I$ .

**Def 1.1.4 Refinement of open covers**

Let  $(X, \tau)$  be a topological space, and let  $\{U_i \subset X\}_{i \in I}$  be an open cover. Then a *refinement* of this open cover is a set of open subsets  $\{V_j \subset X\}_{j \in J}$  which is still an open cover in itself and such that for each  $j \in J$  there exists an  $i \in I$  with  $V_j \subset U_i$ .

**Def 1.1.5 Paracompact topological space**

A topological space  $(X, \tau)$  is called *paracompact* if every open cover of  $X$  has a refinement by a locally finite open cover.

**Def 1.1.6 Partition of unity**

Let  $(X, \tau)$  be a topological space, and let  $\{U_i \subset X\}_{i \in I}$  be an open cover. Then a *partition of unity* subordinate to the cover is

- a set  $\{f_i\}_{i \in I}$  of continuous functions

$$f_i : X \rightarrow [0, 1]$$

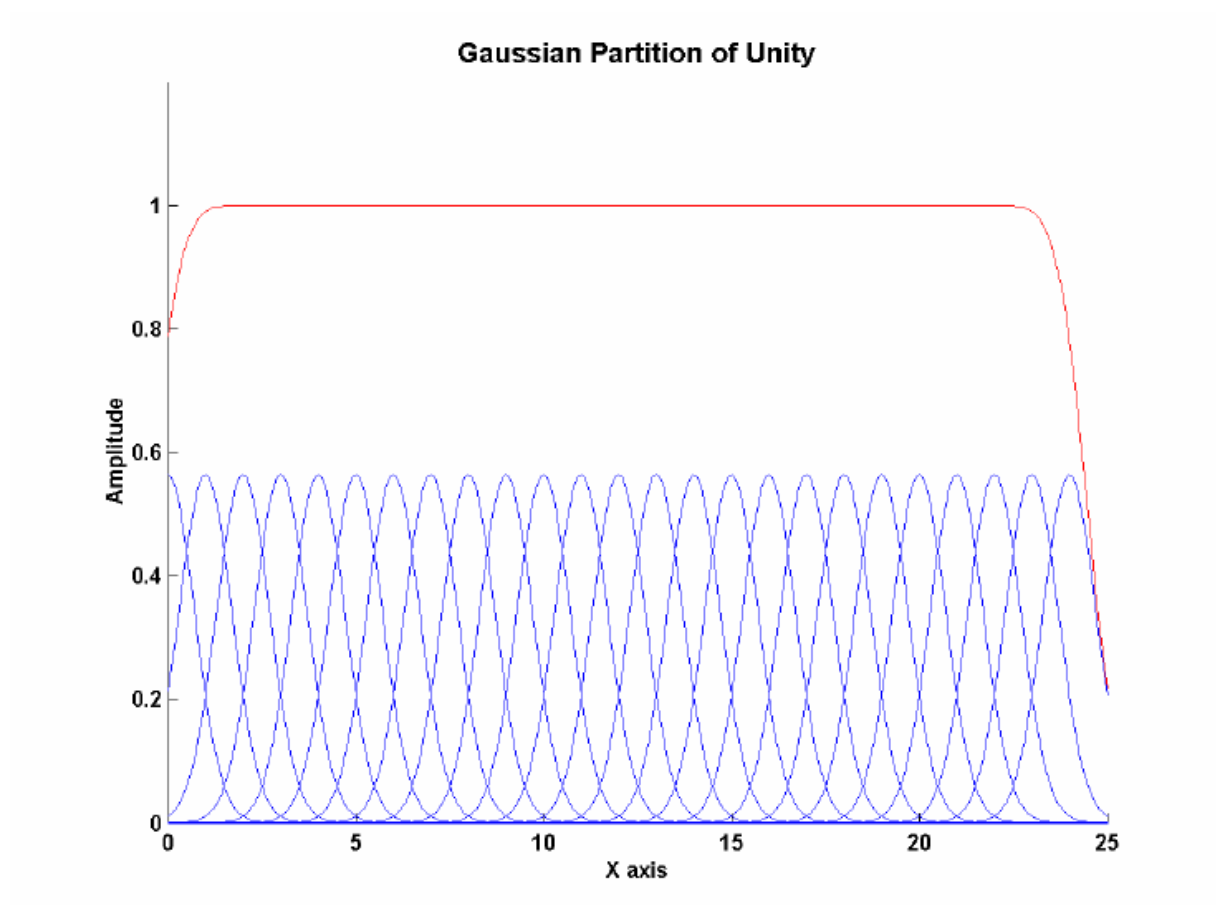
(where  $[0, 1] \subset \mathbb{R}$  is equipped with the subspace topology of the real numbers  $\mathbb{R}$  regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl(f_i^{-1}((0, 1]))$$

denoting the support of  $f_i$  (the topological closure of the subset of points on which it does not vanish) then

- 1)  $\forall_{i \in I} (Supp(f_i) \subset U_i)$
- 2)  $\{Supp(f_i) \subset X\}_{i \in I}$  is a locally finite cover
- 3)  $\forall_{x \in X} (\sum_{i \in I} f_i(x) = 1)$



**Figure 1.1:** Gaussian Partition of Unity

**Prop 1.1.7 Paracompact - Partition of unity**

If  $(X, \tau)$  is a paracompact topological space, then for every open cover  $\{U_i \subset X\}_{i \in I}$  there is a subordinate partition of unity.

Proof will be given later.

**Lem 1.1.8 Natural Refinement**

Let  $(X, \tau)$  be a topological space,  $\{U_i \subset X\}_{i \in I}$  be an open cover and  $(\phi : J \rightarrow I, \{V_j \subset X\}_{j \in J})$  be a refinement to a locally finite cover. Then, for  $\{W_i \subset X\}_{i \in I}$  with

$$W_i \equiv \left\{ \bigcup_{j \in \phi^{-1}(\{i\})} V_j \right\}$$

is still a refinement of  $\{U_i \subset X\}_{i \in I}$  to a locally finite cover.

**Proof for 1.1.8**

First we know, for  $V, V_j \subset U_{\phi(j)=i}$ . Conversely,  $\forall j \in \phi^{-1}(\{i\}), V_j \subset U_i$ . Thus,  $W_i \in \bigcup_{j \in \phi^{-1}(\{i\})} V_j \subset U_i$ .

Second, since  $\{V_j \subset X\}_{j \in J}$  are locally finite,  $\exists \mathcal{U}_x \supset \{x\}$  and a finite subset  $K \subset J$  such that

$$\forall_{j \in J \setminus K} (\mathcal{U}_x \cap V_j = \emptyset)$$

(locally finite:  $\mathcal{U}_x \cap V_j \neq \emptyset$  for just finite number of  $j \in J$ )

Then we can get by construction,

$$\forall_{i \in I \setminus \phi(K)} (\mathcal{U}_x \cap W_i = \emptyset)$$

Since  $\phi(K)$  is still finite, we can find the number of  $i$  such that  $\mathcal{U}_x \cap W_i \neq \emptyset$  is also finite. (If for  $i \in K'$ ,  $\mathcal{U}_x \cap W_i \neq \emptyset$  then  $K'$  should be subset of  $\phi(K)$ .)

Therefore  $\{W_i \subset X\}_{i \in I}$  is locally finite.

**Lem 1.1.9 Shrinking Lemma**

Let  $X$  be a topological space which is normal and let  $\{U_i \subset X\}_{i \in I}$  be a locally finite open cover. Assuming the axiom of choice then:

There exists another open cover  $\{V_i \subset X\}_{i \in I}$  such that the topological closure  $Cl(V_i)$  of its elements is contained in the original patches:

$$\forall_{i \in I} (V_i \subset Cl(V_i) \subset U_i)$$

Now, suggest some fundamental topological concepts to prove prop 1.1.7.

#### Def 1.1.10 Normal Spaces ( $T_4$ )

A topological space  $X$  is *normal* if for every two closed disjoint subsets  $A, B \subset X$ , there are neighborhoods  $U \supset A$ ,  $V \supset B$  such that  $U \cap V = \emptyset$ .

#### Prop 1.1.11 $T_4$ in terms of topological closure

$X$  is normal iff for all closed subsets  $C \subset X$  with open neighborhood  $U \supset C$  there exists a smaller open neighborhood  $V \supset C$  whose topological closure  $Cl(V)$  is still contained in  $U$ :

$$C \subset V \subset Cl(V) \subset U$$

#### Proof for Prop 1.1.11

Suppose that  $(X, \tau)$  is  $T_4$ . Consider closed subset  $C \subset U$  where  $U$  is open neighborhood of  $C$ . It implies

$$C \cap X \setminus U = \emptyset$$

Since  $U$  is open,  $X \setminus U$  is closed. Because of normal space, there are open neighborhoods  $V, W$  such that  $C \subset V$ ,  $X \setminus U \subset W$  and  $V \cap W = \emptyset$ . Because of last term, we can find  $V \subset X \setminus W \subset U$ . Since  $X \setminus W$  is closed, we can find next relation :

$$C \subset V \subset Cl(V) \subset X \setminus W \subset U$$

In the other direction, suppose that  $\forall$  open neighborhood  $U$  of closed subset  $C$ , there are smaller open neighborhood with  $C \subset V \subset Cl(V) \subset U$ . Now, consider disjoint closed subset  $C_1, C_2 \subset X$ .  $C_1 \cap C_2 = \emptyset$  implies  $C_1 \subset X \setminus C_2$ . Since  $X \setminus C_2$  is open neighborhood of  $C_1$ , there exists smaller open neighborhood  $V$  such that

$$C_1 \subset V \subset Cl(V) \subset X \setminus C_2$$

And it also implies  $X \setminus Cl(V)$  is open neighborhood of  $C_2$  where  $V \cap X \setminus Cl(V) = \emptyset$ . Therefore  $X$  is  $T_4$ .

**Def 1.1.12 Urysohn function**

Let  $X$  be a topological space, and let  $A, B \subset X$  be disjoint closed subsets. Then an *Urysohn function* for this situation is a continuous function

$$f : X \rightarrow [0, 1]$$

to the closed interval equipped with its Euclidean metric topology, such that

$$f(A) = \{0\} \text{ and } f(B) = \{1\}$$

**Prop 1.1.13 Urysohn's Lemma**

Let  $X$  be a normal topological space, and let  $A, B \subset X$  be two disjoint closed subsets of  $X$ . Then there exists an *Urysohn function*.

This lemma has several **big** applications:

- **Urysohn Metrization Thm:** *If  $X$  is a normal space with a countable basis, then we can use the abundance of continuous functions from  $X$  to  $[0, 1]$  to assign numerical coordinates to the points of  $X$  and obtain an embedding of  $X$  into  $\mathbb{R}^\omega$ . From this we see that every second countable normal space is a metric space.*
- **Tietze Extension Thm:** *Suppose  $A$  is a subset of a space  $X$  and  $f : A \rightarrow [0, 1]$  is a continuous function. If  $X$  is normal and  $A$  is closed in  $X$ , then we can find a continuous function from  $X$  to  $[0, 1]$  that is an extension of  $f$ .*
- **Embedding manifolds in  $\mathbb{R}^n$ :** Using Urysohn's lemma to develop the tool called *partitions of unity*, we can obtain the following theorem: *Each compact  $n$ -manifold is homeomorphic to a subspace of some  $\mathbb{R}^n$ .*

Then let's start to prove *Urysohn's lemma*.



**Proof for Urysohn's lemma**

( $\Leftarrow$ ) Suppose  $f(A) = \{0\}$ ,  $f(B) = \{1\}$  for all closed subset  $A, B \subset X$ . Then  $A \subset f^{-1}([0, \frac{1}{2}))$  and  $D \subset f^{-1}((\frac{1}{2}, 1])$ . We can find these two sets are open and disjoint.<sup>a</sup> Thus,  $X$  is  $T_4$ .

( $\Rightarrow$ ) Suppose that  $X$  is  $T_4$  and consider two disjoint closed sets  $A, B \subset X$ . Claim there is *Urysohn function*. To prove this, we should construct continuous function such that  $f(A) = \{0\}$ ,  $f(B) = \{1\}$ . (Maybe it's a little bit tricky.)

Since  $X$  is  $T_4$ , we can find open neighborhood for any closed subsets of  $X$  such that satisfies *prop 1.1.11*. Then we can think next idea :

Let  $\{U_p\}_{p \in [0,1] \cap \mathbb{Q}}$  be a collection of open sets such that

$$U_1 = X \setminus B, A \subset U_0 \subset Cl(U_0) \subset U_1$$

For convenience, denote  $Q = [0, 1] \cap \mathbb{Q}$ . Since  $Q$  is countable, we can enumerate it as

$$Q = \{p_n | n \in \mathbb{N}\}$$

To be more clear, we are going to define by induction a collection  $\{U_p | p \in Q\}$  of open subsets with the property:

$$p < q \Rightarrow Cl(U_p) \subset U_q$$

By definition of  $U_p$ , we know above property is satisfied when  $p = 0$ ,  $q = 1$ . Since  $Cl(U_0)$  is also subset of  $X$ , by *prop 1.1.11*, we can construct  $\{U_p\}_{p \in Q}$  completely. Also add some conditions ( $p \in (-\infty, 0) \cap \mathbb{Q} \Rightarrow U_p = \emptyset$ ,  $p \in (1, \infty) \cap \mathbb{Q} \Rightarrow U_p = X$ ), then we can extend our collection to whole  $\mathbb{Q}$ . Now, we can define new set:

$$\mathbb{Q}(x) \equiv \{p \in \mathbb{Q} | x \in U_p\}$$

Then we can find  $\mathbb{Q}(x)$  has lower bound 0.<sup>b</sup> Since  $\mathbb{Q}(x)$  has a greatest lower bound, we can define  $f : X \rightarrow [0, 1]$  by

$$f(x) = \inf \mathbb{Q}(x) = \inf \{p \in \mathbb{Q} | x \in U_p\}$$

If we show  $f$  satisfies ( ①  $0 \leq f(x) \leq 1$ , ②  $f$  is *Urysohn function* for  $A, B$ ,

③  $x \in Cl(U_p) \Rightarrow f(x) \leq p$ , ④  $x \notin U_p \Rightarrow f(x) \geq p$ , ⑤  $f$  is *continuous* ) then proof is complete.

<sup>a</sup>this will be exercise.

<sup>b</sup>this will be exercise

## Proof for Urysohn's lemma (Continued)

①  $0 \leq f(x) \leq 1$

: It's trivial because of next property :

$$f(x) = \begin{cases} 1 & \forall p > 1 \\ \text{can't define} & \forall p < 0 \end{cases}$$

②  $f$  is Urysohn function for  $A, B$ .

: Since  $A \subset U_0$ ,  $\forall x \in A$ ,  $f(x) = 0$  and  $B = X \setminus U_1$ ,  $\forall x \in B$ ,  $f(x) = \inf \{(1, \infty) \cap \mathbb{Q}\} = 1$ .

③ If  $x \in Cl(U_p)$ , then  $f(x) \leq p$

: Suppose  $x \in Cl(U_p)$ , then  $x \in Cl(U_p) \subset U_q$ ,  $\forall q \in \mathbb{Q}$ ,  $q > p$ . Thus,

$$(p, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x) \Rightarrow \inf \mathbb{Q}(x) \leq p$$

④ If  $x \notin U_p$ , then  $f(x) \geq p$

: Suppose  $x \notin U_p$ , then  $x \notin U_q$ ,  $\forall q \leq p$ . Thus,

$$(-\infty, p] \cap \mathbb{Q}(x) = \emptyset \Rightarrow p \leq \inf \mathbb{Q}(x)$$

⑤  $f$  is continuous.

: Suppose  $U = (a, b) \in \mathbb{R}$  such that  $(a, b) \cap [0, 1] \neq \emptyset$ . Claim  $f^{-1}(U)$  is open. Suppose  $x \in f^{-1}(U)$ . It means  $f(x) \in U = (a, b)$ . Since  $U$  is open, there are  $p, q \in \mathbb{Q}$  such that  $a < p < f(x) < q < b$ . By ③, ④, we know  $x \in U_q \setminus Cl(U_p)$  and  $f(U_q \setminus Cl(U_p)) \subset (a, b)$ .<sup>c</sup> Thus, we can find  $\forall x \in f^{-1}(U)$ , there are  $p, q \in \mathbb{Q}$  such that  $x \in U_q \setminus Cl(U_p) \subset f^{-1}(U)$ . Since  $U_q \setminus Cl(U_p)$  is open,  $f^{-1}(U)$  is open. Therefore,  $f$  is continuous.

Proof is complete.

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<sup>c</sup>this will be exercise.

To prove prop 1.1.7, we should know relation between *Hausdorff* and *Normal*.

**Prop 1.1.14 Dieudonné's Theorem**

Every paracompact Hausdorff space is normal.

**Proof for Dieudonné's Theorem**

Consider  $(X, \tau)$  be a paracompact Hausdorff space.

① First, claim it is regular. To show this,  $\forall x \in X$ , closed subset  $C \subset X$  such that  $x \neq C$ , there are open neighborhoods  $\mathcal{U}(x) \ni x$ ,  $\mathcal{U}(C) \supset C$  such that  $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$ . Then let's start. Since  $X$  is Hausdorff,

$$\forall c \in C, \exists \mathcal{U}_c(x) \ni x, \mathcal{U}(c) \ni c \text{ such that } \mathcal{U}_c(x) \cap \mathcal{U}(c) = \emptyset$$

We can find  $\{\mathcal{U}(c) \subset X\}_{c \in C}$  is an open cover of  $C$ , thus  $\{\mathcal{U}(c) \subset X\}_{c \in C} \cup X \setminus C$  is an open cover of  $X$ . Because of paracompactness of  $X$ , every open cover has locally finite refinement. By *lem 1.1.8 (Natural refinement)*, if there exists locally finite refinement, then there exists one with the same index set as the original cover. Thus, we can take locally finite refinement  $\mathcal{W}(c)$  such that

$$\{\mathcal{W}(c) \subset \mathcal{U}(c) \subset X\}_{c \in C}$$

Since  $\mathcal{U}(c)$  is open cover of  $C$  and  $\mathcal{W}(c)$  is refinement of  $\mathcal{U}(c)$ ,  $\bigcup_{c \in C} \mathcal{W}(c)$  is open neighborhood of  $C$ . Let it be denoted by  $\mathcal{V}(C)$ :

$$\mathcal{V}(C) = \bigcup_{c \in C} \mathcal{W}(c)$$

Now, because of locally finiteness of  $\mathcal{W}(c)$ ,  $\forall x \in X$ , there exists neighborhood  $\mathcal{W}(x)$  and finite subset  $K \subset C$  such that

$$\bigcap_{c \in C \setminus K} (\mathcal{W}(x) \cap \mathcal{W}(c)) = \emptyset$$

Let's take new neighborhood of  $x$  as follows :

$$\mathcal{U}(x) \equiv \mathcal{W}(x) \cap \left( \bigcap_{k \in K} \mathcal{U}_k(x) \right)$$

Then we can find

$$\mathcal{U}(x) \cap \mathcal{V}(C) = \emptyset^a$$

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<sup>a</sup>this will be exercise

**Proof for Dieudonné's Theorem (Continued)**

②

## 2 Appendix

### A. Topology

#### 1. Topological Spaces

##### Def A.1.1 Topological Space

A *topology* on a set  $X$  is a subset  $\mathcal{T}$  of the power set  $\mathcal{P}(X)$  with the following properties:

1.  $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$
2. Unions of elements of  $\mathcal{T}$  belong to  $\mathcal{T}$

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcup_{i \in I} U_i \in \mathcal{T}$$

3. Intersections of finitely many elements of  $\mathcal{T}$  belong to  $\mathcal{T}$ . For finite set  $I$ ,

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcap_{i \in I} U_i \in \mathcal{T}$$

A *topological space* is a set  $X$  together with a topology  $\mathcal{T}$  on  $X$ . For a topological space  $(X, \mathcal{T})$ , we call the elements of  $\mathcal{T}$  *open subsets* and their complements *closed subsets* of  $X$ .

##### Example A.1.2

- 1) Let  $X$  be a set. Then  $\mathcal{T} = \{\emptyset, X\}$ , is a topology on  $X$ , called the *trivial topology*. This is a smallest topology.
- 2) The power set  $\mathcal{P}(X)$  of a set  $X$ , is a topology on  $X$ , called the *discrete topology*. This is a largest topology.

**Exercise A.1:** Prove Example A.2.

**Def A.1.3 Basis**

Let  $\mathcal{T}$  be a topology on a set  $X$ . A subset  $\mathcal{B} \subseteq \mathcal{T}$  is called a *basis* for  $\mathcal{T}$  if every element of  $\mathcal{T}$  is a union of elements of  $\mathcal{B}$ .

**Prop A.1.4 Basis (Comfortable Definition)**

A subset  $\mathcal{B}$  of a topology  $\mathcal{T}$  on a set  $X$  is a basis of  $\mathcal{T}$  iff, for every  $U \in \mathcal{T}$  and  $x \in U$ , there is a  $V \in \mathcal{B}$  with  $x \in V \subseteq U$ .

Proof is trivial.

**Def A.1.5 Neighborhood**

Let  $X$  be a topological space,  $x \in X$ . Then  $U \subseteq X$  is called a *neighborhood* of  $x$  when there is an open set  $x \in V \subseteq U$ . We denote by  $\mathcal{U}(x)$  the set of all neighborhoods of  $x$ .

**Def A.1.6 Neighborhood Basis**

Let  $X$  be a topological space and  $x \in X$ . Then we call a subset  $\mathcal{B}(x) \subseteq \mathcal{U}(x)$  a *neighborhood basis* of  $x$  if for every neighborhood  $U$  of  $x$ , there is a  $V \in \mathcal{B}(x)$  with  $V \subseteq U$ .

**Def A.1.7 Countability**

Let  $X$  be a topological space.

- $X$  satisfies the *first countability axiom* and is called *countable* if every point in  $X$  admits a countable neighborhood basis.
- $X$  satisfies the *second countability axiom* and is called *second countable* if the topology of  $X$  admits a countable basis.

**Def A.1.8 Adherent, Interior and Boundary**

Let  $X$  be a topological space and  $Y \subseteq X$ . Then  $x \in X$  is called

1. an *adherent point* (also sometimes called a *point of closure*) of  $Y$ , if every neighborhood of  $x$  in  $X$  contains a point of  $Y$ . The set  $\bar{Y}$  of adherent points of  $Y$  is called the *closure* of  $Y$
2. an *interior point* of  $Y$  if there is a neighborhood of  $x$  in  $X$  that is contained in  $Y$ . The set  $\mathring{Y}$  of interior points of  $Y$  is called the *interior* of  $Y$
3. a *boundary point* of  $Y$  if every neighborhood of  $x$  in  $X$  contains points of  $Y$  and  $X \setminus Y$ . The set of boundary points of  $Y$  is called the *boundary* of  $Y$ , here denoted by  $\partial Y$ .

## 2. Continuous Maps

### Def A.2.1 Continuous

Let  $(X, \tau)$  and  $(Y, \tau')$  be topological spaces and  $f : X \rightarrow Y$  be a function. We call  $f$  *continuous* if  $f^{-1}(V) \in \tau$  for all  $V \in \tau'$ .

### Def A.2.2 Continuous at a point

Let  $(X, \tau)$  and  $(Y, \tau')$  be topological spaces and  $f : X \rightarrow Y$  be a function. We call  $f$  *continuous at a point*  $x \in X$  if, for every neighborhood  $V$  of  $f(x) \in Y$ , there is a neighborhood  $U$  of  $x$  with  $f(U) \subseteq V$ .

### Def A.2.3 Homeomorphism

A map  $f : X \rightarrow Y$  between topological spaces  $X$  and  $Y$  is called a *homeomorphism* if  $f$  is bijective and  $f$  and  $f^{-1}$  are continuous.

## 3. Convergence And Hausdorff Spaces

### Def A.3.1 Convergence

Let  $X$  be a topological space and  $(x_n)$  a sequence in  $X$ . Then a point  $x \in X$  is called a *limit* of the sequence  $(x_n)$  if, for every neighborhood  $\mathcal{U}(x)$  of  $x$ ,  $\exists n \in \mathbb{N}$  such that  $x_m \in \mathcal{U}(x)$ ,  $\forall m \geq n$ . We then say that the sequence *converges to*  $x$ , and we call the sequence *convergent*.

### Def A.3.2 Hausdorff

Given points  $x$  and  $y$  of  $S$ , if  $x \neq y$ , then there exist open neighborhoods  $U$  of  $x$  and  $V$  of  $y$  in  $S$  that are disjoint: such that  $U \cap V = \emptyset$ .

**Exercise A.2:** Prove that Metric spaces are Hausdorff spaces.

**Prop A.3.3 Hausdorff and Convergence**

Let  $X$  be a Hausdorff space. Then limit of sequences in  $X$  are unique if they exist.

**Exercise A.3:** Prove *prop A.3.3*.

**Def A.3.4 Regular Hausdorff ( $T_3$ )**

Let  $X$  be a topological space.  $X$  is called *regular* if given any point  $x$  and closed set  $C$ , if  $x \notin C$ , then there exist a neighborhood  $\mathcal{U}(x)$  of  $x$  and a neighborhood  $\mathcal{U}(C)$  of  $C$  such that  $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$ .



### 3 Reference

- Boothby, William Munger. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Revised Second ed., vol. 120, Academic Press, 2010.
- Ballmann, Werner. *Introduction to Geometry and Topology*. Birkhäuser, 2018.