General Relativity

By precise approach

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2018-08-24

Preliminaries

Manifolds

1. Topological Manifolds

Def 1.1 Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

Why?

- Hausdorff : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- Locally Euclidean : This is the main reason that why we require manifolds.
- Countable Basis : We need **partition of unity** to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require **paracompactness**. And paracompactness follows from **second countability**. It is same as have countable basis.

1.1 Supplement

Thm 1.1.1 Paracompact $\simeq$ Partition of unity

Let (X, τ) be a topological space that is T_1 (all points are closed). Then the following are equivalent:

I. (X, τ) is paracompact and Hausdorff
II. Every open cover of (X, τ) admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

Def 1.1.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Def 1.1.3 Locally finite cover

Let (X, τ) be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Def 1.1.4 Refinement of open covers

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Def 1.1.5 Paracompact topological space

A topological space (X, τ) is called *paracompact* if every open cover of X has a refinement by a locally finite open cover.

Def 1.1.6 Partition of unity

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

  (where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology of the real numbers $\mathbb{R}$ regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl\left(f_i^{-1}((0, 1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

1) $\underset{i \in I}{\forall}(Supp(f_i) \subset U_i)$
2) $\{Supp(f_i) \subset X\}_{i \in I}$ is a locally finite cover
3) $\underset{x \in X}{\forall}\left(\sum_{i \in I} f_i(x) = 1\right)$

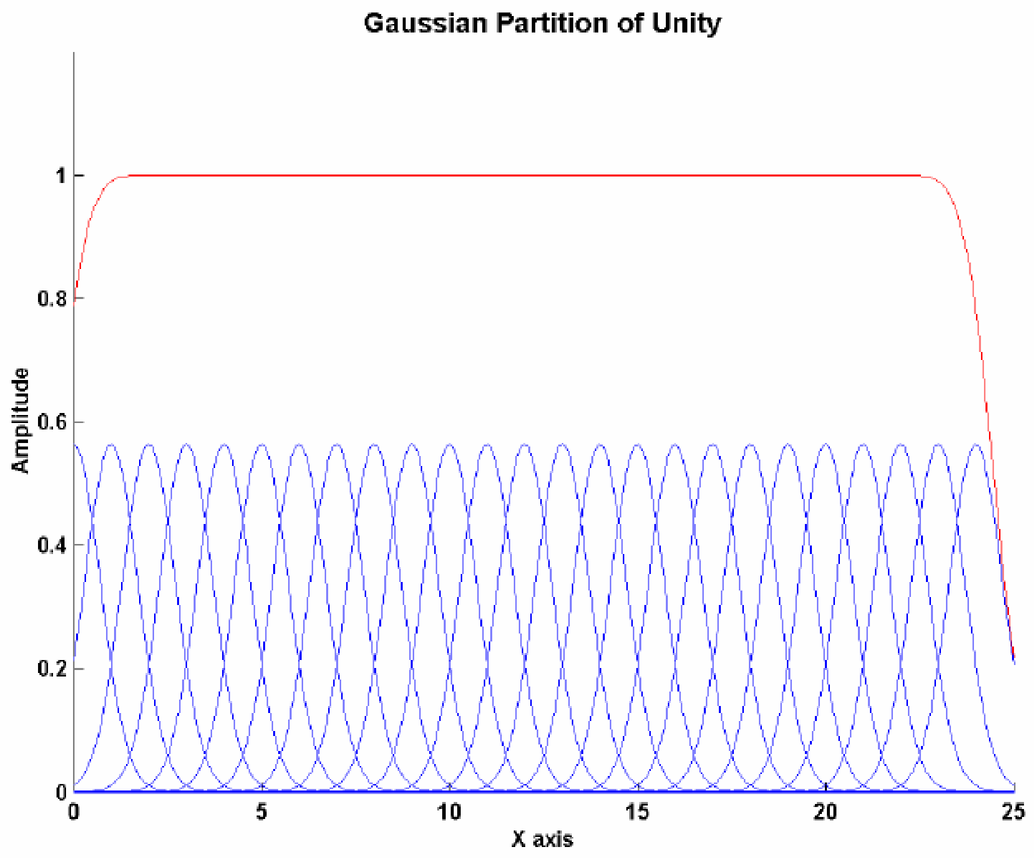

Figure 1: Gaussian Partition of Unity

Prop 1.1.7 Paracompact - Partition of unity

If (X, τ) is a paracompact topological space, then for every open cover $\{U_i \subset X\}_{i\in I}$ there is a subordinate partition of unity.

Proof will be given later.

Lem 1.1.8

Let (X, τ) be a topological space, $\{U_i \subset X\}_{i\in I}$ be an open cover and $\left(\phi : J \to I,\ \{V_j \subset X\}_{j\in J}\right)$ be a refinement to a locally finite cover. Then, for $\{W_i \subset X\}_{i\in I}$ with

$$W_i \equiv \left\{ \bigcup_{j\in\phi^{-1}(\{i\})} V_j \right\}$$

is still a refinement of $\{U_i \in X\}_{i\in I}$ to a locally finite cover.

Proof for 1.1.8

First we know, for V, $V_j \subset U_{\phi(j)=i}$. Conversely, $\forall j \in \phi^{-1}(\{i\}), \quad V_j \subset U_i$. Thus, $W_i \in \bigcup_{j\in\phi^{-1}(\{i\})} V_j \subset U_i$.

Second, since $\{V_j \subset X\}_{j\in J}$ are locally finite, $\exists\, \mathcal{U}_x \supset \{x\}$ and a finite subset $K \subset J$ such that

$$\underset{j\in J\setminus K}{\forall} (\mathcal{U}_x \cap V_j = \emptyset)$$

(locally finite: $\mathcal{U}_x \cap V_j \neq \emptyset$ for just finite number of $j \in J$)
Then we can get by construction,

$$\underset{i\in I\setminus \phi(K)}{\forall} (\mathcal{U}_x \cap W_i = \emptyset)$$

Since $\phi(K)$ is still finite, we can find the number of i such that $\mathcal{U}_x \cap W_i = \emptyset$ is also finite. (If for $i \in K',\ \mathcal{U}_x \cap W_i = \emptyset$ then K' should be subset of $\phi(K)$.)

Therefore $\{W_i \in X\}_{i\in I}$ is locally finite.