General Relativity

By precise approach

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1 Preliminaries

Manifolds

1. Topological Manifolds

Def 1.1 Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

Why?

- **Hausdorff** : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- **Locally Euclidean** : This is the main reason that why we require manifolds.
- **Countable Basis** : We need *partition of unity* to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require *paracompactness.* And paracompactness follows from *second countability.* It is same as have countable basis.

1.1 Supplement

Thm 1.1.1 Paracompact $\simeq$ Partition of unity

Let (X, τ) be a topological space that is T_1 (all points are closed). Then the following are equivalent:

- (X, τ) is paracompact and Hausdorff
- Every open cover of (X, τ) admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

Def 1.1.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Def 1.1.3 Locally finite cover

Let (X, τ) be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Def 1.1.4 Refinement of open covers

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Def 1.1.5 Paracompact topological space

A topological space (X, τ) is called *paracompact* if every open cover of X has a refinement by a locally finite open cover.

Def 1.1.6 Partition of unity

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

  (where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology of the real numbers $\mathbb{R}$ regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl\left(f_i^{-1}((0, 1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

1) $\underset{i \in I}{\forall} (Supp(f_i) \subset U_i)$
2) $\{Supp(f_i) \subset X\}_{i \in I}$ is a locally finite cover
3) $\underset{x \in X}{\forall} \left(\sum_{i \in I} f_i(x) = 1\right)$

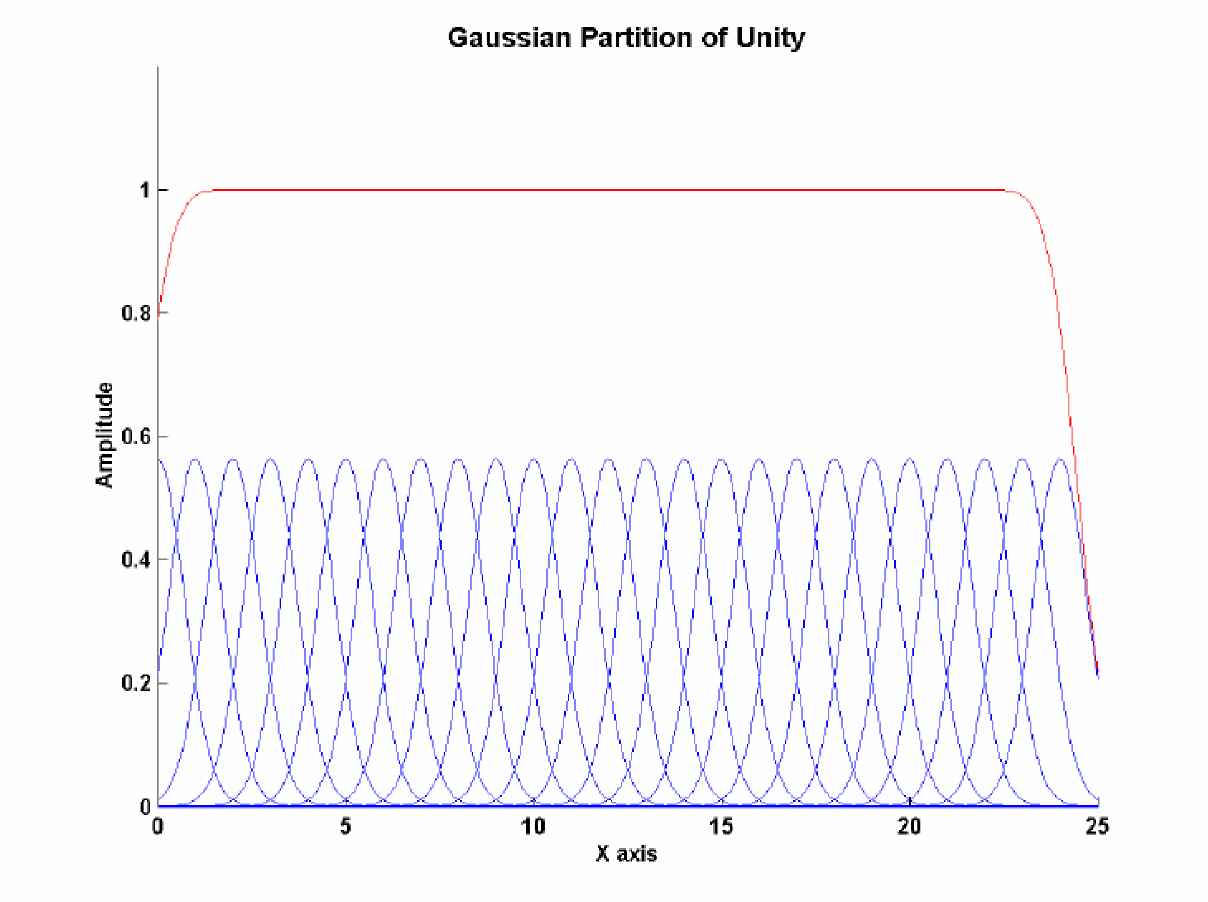

Figure 1.1: Gaussian Partition of Unity

Prop 1.1.7 Paracompact - Partition of unity

If (X, τ) is a paracompact topological space, then for every open cover $\{U_i \subset X\}_{i \in I}$ there is a subordinate partition of unity.

Proof will be given later.

Lem 1.1.8 Natural Refinement

Let (X,τ) be a topological space, $\{U_i \subset X\}_{i\in I}$ be an open cover and $\left(\phi : J \to I,\ \{V_j \subset X\}_{j\in J}\right)$ be a refinement to a locally finite cover. Then, for $\{W_i \subset X\}_{i\in I}$ with

$$W_i \equiv \left\{ \bigcup_{j\in\phi^{-1}(\{i\})} V_j \right\}$$

is still a refinement of $\{U_i \subset X\}_{i\in I}$ to a locally finite cover.

Proof for 1.1.8

First we know, for V, $V_j \subset U_{\phi(j)=i}$. Conversely, $\forall j \in \phi^{-1}(\{i\})$, $V_j \subset U_i$. Thus, $W_i \in \bigcup_{j\in\phi^{-1}(\{i\})} V_j \subset U_i$.

Second, since $\{V_j \subset X\}_{j\in J}$ are locally finite, $\exists\, \mathcal{U}_x \supset \{x\}$ and a finite subset $K \subset J$ such that

$$\underset{j\in J\setminus K}{\forall} (\mathcal{U}_x \cap V_j = \emptyset)$$

(locally finite: $\mathcal{U}_x \cap V_j \neq \emptyset$ for just finite number of $j \in J$)
Then we can get by construction,

$$\underset{i\in I\setminus\phi(K)}{\forall} (\mathcal{U}_x \cap W_i = \emptyset)$$

Since $\phi(K)$ is still finite, we can find the number of i such that $\mathcal{U}_x \cap W_i \neq \emptyset$ is also finite. (If for $i \in K',\ \mathcal{U}_x \cap W_i \neq \emptyset$ then K' should be subset of $\phi(K)$.)

Therefore $\{W_i \subset X\}_{i\in I}$ is locally finite.

Lem 1.1.9 Shrinking Lemma

Let X be a topological space which is normal and let $\{U_i \subset X\}_{i\in I}$ be a locally finite open cover. Assuming the axiom of choice then:

There exists another open cover $\{V_i \subset X\}_{i\in I}$ such that the topological closure $Cl(V_i)$ of its elements is contained in the original patches:

$$\underset{i\in I}{\forall} (V_i \subset Cl(V_i) \subset U_i)$$

Now, suggest some fundamental topological concepts to prove prop 1.1.7.

Def 1.1.10 Normal Spaces (T_4)

A topological space X is *normal* if for every two closed disjoint subsets $A, B \subset X$, there are neighborhoods $U \supset A,\ V \supset B$ such that $U \cap V = \emptyset$.

Prop 1.1.11 T_4 in terms of topological closure

X is normal iff for all closed subsets $C \subset X$ with open neighborhood $U \supset C$ there exists a smaller open neighborhood $V \supset C$ whose topological closure $Cl(V)$ is still contained in U:

$$C \subset V \subset Cl(V) \subset U$$

Proof for Prop 1.1.11

Suppose that (X, τ) is T_4. Consider closed subset $C \subset U$ where U is open neighborhood of C. It implies

$$C \cap X \backslash U = \emptyset$$

Since U is open, $X \backslash U$ is closed. Because of normal space, there are open neighborhoods V, W such that $C \subset V,\ X\backslash U \subset W$ and $V \cap W = \emptyset$. Because of last term, we can find $V \subset X\backslash W \subset U$. Since $X \backslash W$ is closed, we can find next relation :

$$C \subset V \subset Cl(V) \subset X\backslash W \subset U$$

In the other direction, suppose that $\forall$ open neighborhood U of closed subset C, there are smaller open neighborhood with $C \subset V \subset Cl(V) \subset U$. Now, consider disjoint closed subset $C_1, C_2 \subset X$. $C_1 \cap C_2 = \emptyset$ implies $C_1 \subset X\backslash C_2$. Since $X \backslash C_2$ is open neighborhood of C_1, there exists smaller open neighborhood V such that

$$C_1 \subset V \subset Cl(V) \subset X\backslash C_2$$

And it also implies $X\backslash Cl(V)$ is open neighborhood of C_2 where $V \cap X\backslash Cl(V) = \emptyset$. Therefore X is T_4.

Def 1.1.12 Urysohn function

Let X be a topological space, and let $A, B \subset$ X be disjoint closed subsets. Then an *Urysohn function* for this situation is a continuous function

$$f : X \to [0, 1]$$

to the closed interval equipped with its Euclidean metric topology, such that

$$f(A) = \{0\} \text{ and } f(B) = \{1\}$$

Prop 1.1.13 Urysohn's Lemma

Let X be a normal topological space, and let $A, B \subset X$ be two disjoint closed subsets of X. Then there exists an *Urysohn function*.

This lemma has several **big** applications:

- **Urysohn Metrization Thm**: *If X is a normal space with a countable basis, then we can use the abundance of continuous functions from X to $[0, 1]$ to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this we see that every second countable normal space is a metric space.*
- **Tietze Extension Thm**: *Suppose A is a subset of a space X and $f : A \to [0, 1]$ is a continuous function. If X is normal and A is closed in X, then we can find a continuous function from X to $[0, 1]$ that is an extension of f.*
- **Embedding manifolds in $\mathbb{R}^n$**: Using Urysohn's lemma to develop the tool called *partitions of unity*, we can obtain the following theoerem: *Each compact n-manifold is homeomorphic to a subspace of some $\mathbb{R}^n$.*

Then let's start to prove *Urysohn's lemma*.

Proof for Urysohn's lemma

($\Leftarrow$) Suppose $f(A)=\{0\}\,,\ f(B)=\{1\}$ for all closed subset $A,B\subset X$. Then $A\subset f^{-1}\left([0,\frac{1}{2})\right)$ and $D\subset f^{-1}\left((\frac{1}{2},1]\right)$. We can find these two sets are open and disjoint.[a] Thus, X is T_4.

($\Rightarrow$) Suppose that X is T_4 and consider two disjoint closed sets $A,B\subset X$. Claim there is *Urysohn function*. To prove this, we should construct continuous function such that $f(A)=\{0\}\,,\ f(B)=\{1\}$. (Maybe it's a little bit tricky.)
Since X is T_4, we can find open neighborhood for any closed subsets of X such that satisfies *prop 1.1.11*. Then we can think next idea :

Let $\{U_p\}_{p\in[0,1]\cap\mathbb{Q}}$ be a collection of open sets such that

$$U_1=X\backslash B,\ A\subset U_0\subset Cl(U_0)\subset U_1$$

For convenience, denote $Q=[0,1]\cap\mathbb{Q}$. Since Q is countable, we can enumerate it as

$$Q=\{p_n|n\in\mathbb{N}\}$$

To be more clear, we are going to define by induction a collection $\{U_p|p\in Q\}$ of open subsets with the property:

$$p<q\ \Rightarrow\ Cl(U_p)\subset U_q$$

By definition of U_p, we know above property is satisfied when $p=0,\ q=1$. Since $Cl(U_0)$ is also subset of X, by *prop 1.1.11*, we can construct $\{U_p\}_{p\in Q}$ completely. Also add some conditions ($p\in(-\infty,0)\cap\mathbb{Q}\ \Rightarrow\ U_p=\emptyset,\ p\in(1,\infty)\cap\mathbb{Q}\ \Rightarrow\ U_p=X$), then we can extend our collection to whole $\mathbb{Q}$. Now, we can define new set:

$$\mathbb{Q}(x)\equiv\{p\in\mathbb{Q}|x\in U_p\}$$

Then we can find $\mathbb{Q}(x)$ has lower bound 0.[b] Since $\mathbb{Q}(x)$ has a greatest lower bound, we can define $f:X\to[0,1]$ by

$$f(x)=\inf\mathbb{Q}(x)=\inf\{p\in\mathbb{Q}|x\in U_p\}$$

If we show f satisfies (① $0\leq f(x)\leq 1$, ② f *is Urysohn function for* A,B, ③ $x\in Cl(U_p)\Rightarrow f(x)\leq p$, ④ $x\notin U_p\Rightarrow f(x)\geq p$, ⑤ f *is continuous*) then proof is complete.

[a] this will be exercise.
[b] this will be exercise

Proof for Urysohn's lemma (Continued)

① $0 \leq f(x) \leq 1$

: It's trivial because of next property :

$$f(x) = \begin{cases} 1 & \forall p > 1 \\ \text{can't define} & \forall p < 0 \end{cases}$$

② *f is Urysohn function for A, B.*

: Since $A \subset U_0,\ \forall x \in A,\ f(x) = 0$ and $B = X \backslash U_1,\ \forall x \in B,\ f(x) = \inf\{(1, \infty) \cap \mathbb{Q}\} = 1$.

③ *If $x \in Cl(U_p)$, then $f(x) \leq p$*

: Suppose $x \in Cl(U_p)$, then $x \in Cl(U_p) \subset U_q,\ \forall q \in \mathbb{Q},\ q > p$. Thus,

$$(p, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x) \ \Rightarrow\ \inf \mathbb{Q}(x) \leq p$$

④ *If $x \notin U_p$, then $f(x) \geq p$*

: Suppose $x \notin U_p$, then $x \notin U_q,\ \forall q \leq p$. Thus,

$$(-\infty, p] \cap \mathbb{Q}(x) = \emptyset \ \Rightarrow\ p \leq \inf \mathbb{Q}(x)$$

⑤ *f is continuous.*

: Suppose $U = (a, b) \in \mathbb{R}$ such that $(a, b) \cap [0, 1] \neq \emptyset$. Claim $f^{-1}(U)$ is open. Suppose $x \in f^{-1}(U)$. It means $f(x) \in U = (a, b)$. Since U is open, there are $p, q \in \mathbb{Q}$ such that $a < p < f(x) < q < b$. By ③, ④, we know $x \in U_q \backslash Cl(U_p)$ and $f(U_q \backslash Cl(U_p)) \subset (a, b)$.[c] Thus, we can find $\forall x \in f^{-1}(U)$, there are $p, q \in \mathbb{Q}$ such that $x \in U_q \backslash Cl(U_p) \subset f^{-1}(U)$. Since $U_q \backslash Cl(U_p)$ is open, $f^{-1}(U)$ is open. Therefore, f is continuous.

Proof is complete.

[c] this will be exercise.

To prove *prop 1.1.7*, we should know relation between *Hausdorff* and *Normal*.

Prop 1.1.14 Dieudonné's Theorem

Every paracompact Hausdorff space is normal.

Proof for Dieudonné's Theorem

Consider (X, τ) be a paracompact Hausdorff space.

① First, claim it is regular. To show this, $\forall x \in X$, closed subset $C \subset X$ such that $x \notin C$, there are open neighborhoods $\mathcal{U}(x) \ni x$, $\mathcal{U}(C) \supset C$ such that $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$. Then let's start. Since X is Hausdorff,

$$\forall c \in C,\ \exists \mathcal{U}_c(x) \ni x,\ \mathcal{U}(c) \ni c \text{ such that } \mathcal{U}_c(x) \cap \mathcal{U}(c) = \emptyset$$

We can find $\{\mathcal{U}(c) \subset X\}_{c\in C}$ is an open cover of C, thus $\{\mathcal{U}(c) \subset X\}_{c\in C} \cup X\backslash C$ is an open cover of X. Because of paracompactness of X, every open cover has locally finite refinement. By *lem 1.1.8 (Natural refinement)*, if there exists locally finite refinement, then there exists one with the same index set as the original cover. Thus, we can take locally finite refinement $\mathcal{W}(c)$ such that

$$\{\mathcal{W}(c) \subset \mathcal{U}(c) \subset X\}_{c\in C}$$

Since $\mathcal{U}(c)$ is open cover of C and $\mathcal{W}(c)$ is refinement of $\mathcal{U}(c)$, $\bigcup_{c\in C} \mathcal{W}(c)$ is open neighborhood of C. Let it be denoted by $\mathcal{V}(C)$:

$$\mathcal{V}(C) = \bigcup_{c\in C} \mathcal{W}(c)$$

Now, because of locally finiteness of $\mathcal{W}(c)$, $\forall x \in X$, there exists neighborhood $\mathcal{W}(x)$ and finite subset $K \subset C$ such that

$$\underset{c\in C\backslash K}{\forall} (\mathcal{W}(x) \cap \mathcal{W}(c)) = \emptyset$$

Let's take new neighborhood of x as follows :

$$\mathcal{V}(x) \equiv \mathcal{W}(x) \cap \left(\bigcap_{k\in K} \mathcal{U}_k(x) \right)$$

Then we can find

$$\mathcal{V}(x) \cap \mathcal{V}(C) = \emptyset^a$$

[a] this will be exercise (refer to ②)

Proof for Dieudonné's Theorem (Continued)

② Claim $(X, \mathcal{T})$ is normal. Then we should prove below proposition:

$$\forall \text{disjoint closed subsets } C, D \subset X,\ \exists \text{disjoint neighborhoods } \mathcal{U}(C),\ \mathcal{U}(D) \in \mathcal{T}$$

By regularity of $(X, \mathcal{T})$, we have next proposition:

$$\forall c \in C,\ \exists \text{disjoint neighborhoods } \mathcal{U}(c) \ni c,\ \mathcal{U}_c(D) \supset D$$

Since $\{\mathcal{U}(c) \subset X\}_{c\in C} \cup X\backslash C$ is an open cover of X and paracompactness of X, we can find locally finite refinement in same index:

$$\{\mathcal{W}(c) \subset \mathcal{U}(c) \subset X\}_{c\in C}$$

Then we can find new open neighborhood of C:

$$\mathcal{V}(C) = \bigcup_{c\in C} \mathcal{W}(c)$$

By locally finiteness of $\mathcal{W}(c)$, $\forall d \in D$, $\exists$ an open neighborhood $\mathcal{W}(d)$ and finite subset $K_d \subset C$ such that

$$\underset{c\in C\backslash K_d}{\forall} (\mathcal{W}(c) \cap \mathcal{W}(d) = \emptyset)$$

So, let take new open neighborhood of $d \in D$,

$$\mathcal{V}(d) = \mathcal{W}(d) \cap \left(\bigcap_{c\in K_d} \mathcal{U}_c(D)\right) {}^{a}$$

then

$$\mathcal{V}(C) \cap \mathcal{V}(d) = \emptyset,\ \ \forall d \in D^{b}$$

Therefore take new open neighborhood of D as

$$\mathcal{V}(D) \equiv \bigcup_{d\in D} \mathcal{V}(d)$$

then

$$\mathcal{V}(C) \cap \mathcal{V}(D) = \emptyset$$

[a] Finite intersection of opensets are open

[b] For $c \in K_d$, $\mathcal{U}(c) \cap \mathcal{U}_c(D) = \emptyset$ and for $c \in X\backslash K_d$, $\mathcal{V}(C) \cap \mathcal{V}(d) = \emptyset$

2 Appendix

A. Topology

1. Topological Spaces

Def A.1.1 Topological Space

A *topology* on a set X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$ with the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. Unions of elements of $\mathcal{T}$ belong to $\mathcal{T}$

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcup_{i \in I} U_i \in \mathcal{T}$$

3. Intersections of finitely many elements of $\mathcal{T}$ belong to $\mathcal{T}$. For finite set I,

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcap_{i \in I} U_i \in \mathcal{T}$$

A *topological space* is a set X together with a topology $\mathcal{T}$ on X. For a topological space $(X, \mathcal{T})$, we call the elements of $\mathcal{T}$ *open subsets* and their complements *closed subsets* of X.

Example A.1.2

1) Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$, is a topology on X, called the *trivial topology*. This is a smallest topology.
2) The power set $\mathcal{P}(X)$ of a set X, is a topology on X, called the *discrete topology*. This is a largest topology.

Exercise A.1: Prove *Example A.2*.

Def A.1.3 Basis

Let $\mathcal{T}$ be a topology on a set X. A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Prop A.1.4 Basis (Comfortable Definition)

A subset $\mathcal{B}$ of a topology $\mathcal{T}$ on a set X is a basis of $\mathcal{T}$ iff, for every $U \in \mathcal{T}$ and $x \in U$, there is a $V \in \mathcal{B}$ with $x \in V \subseteq U$.

Proof is trivial.

Def A.1.5 Neighborhood

Let X be a topological space, $x \in X$. Then $U \subseteq X$ is called a *neighborhood* of x when there is an open set $x \in V \subseteq U$. We denote by $\mathcal{U}(x)$ the set of all neighborhoods of x.

Def A.1.6 Neighborhood Basis

Let X be a topological space and $x \in X$. Then we call a subset $\mathcal{B}(x) \subseteq \mathcal{U}(x)$ a *neighborhood basis* of x if for every neighborhood U of x, there is a $V \in \mathcal{B}(x)$ with $V \subseteq U$.

Def A.1.7 Countability

Let X be a topological space.

- X satisfies the *first countability axiom* and is called *countable* if every point in X admits a countable neighborhood basis.
- X satisfies the *second countability axiom* and is called *second countable* if the topology of X admits a countable basis.

Def A.1.8 Adherent, Interior and Boundary

Let X be a topological space and $Y \subseteq X$. Then $x \in X$ is called

1. *an adherent point* (also sometimes called a *point of closure*) of Y, if every neighborhood of x in X contains a point of Y. The set Y of adherent points of Y is called the *closure* of Y
2. *an interior point* of Y if there is a neighborhood of x in X that is contained in Y. The set $\mathring{Y}$ of interior points of Y is called the *interior* of Y
3. *a boundary point* of Y if every neighborhood of x in X contains points of Y and $X \backslash Y$. The set of boundary points of Y is called the *boundary* of Y, here denoted by ∂Y.

2. Continous Maps

Def A.2.1 Continuous

Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a function. We call f *continuous* if $f^{-1}(V) \in \tau$ for all $V \in \tau'$.

Def A.2.2 Continuous at a point

Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a function. We call f *continuous at a point* $x \in X$ if, for every neighborhood V of $f(x) \in Y$, there is a neighborhood U of x with $f(U) \subseteq V$.

Def A.2.3 Homeomorphism

A map $f : X \to Y$ between topological spaces X and Y is called a *homeomorphism* if f is bijective and f and f^{-1} are continuous.

3. Convergence And Hausdorff Spaces

Def A.3.1 Convergence

Let X be a topological space and (x_n) a sequence in X. Then a point $x \in X$ is called a *limit* of the sequence (x_n) if, for every neighborhood $\mathcal{U}(x)$ of x, $\exists n \in \mathbb{N}$ such that $x_m \in \mathcal{U}(x)$, $\forall m \geq n$. We then say that the sequence *converges to* x, and we call the sequence *convergent*.

Def A.3.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Exercise A.2: Prove that Metric spaces are Hausdorff spaces.

Prop A.3.3 Hausdorff and Convergence

Let X be a Hausdorff space. Then limit of sequences in X are unique if they exist.

Exercise A.3: Prove *prop A.3.3*.

Def A.3.4 Regular Hausdorff (T_3)

Let X be a topological space. X is called *regular* if given any point x and closed set C, if $x \notin C$, then there exist a neighborhood $\mathcal{U}(x)$ of x and a neighborhood $\mathcal{U}(C)$ of C such that $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$.

3 Reference

- Boothby, William Munger. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Revised Second ed., vol. 120, Academic Press, 2010.
- Ballmann, Werner. *Introduction to Geometry and Topology.* Birkhäuser, 2018.