General Relativity

By precise approach

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Preliminaries

Manifolds

1. Topological Manifolds

Def 1.1 Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

Why?

- Hausdorff : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- Locally Euclidean : This is the main reason that why we require manifolds.
- Countable Basis : We need **partition of unity** to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require **paracompactness**. And paracompactness follows from **second countability**. It is same as have countable basis.

1.1 Supplement

Thm 1.1.1 Paracompact $\simeq$ Partition of unity

Let (X, τ) be a topological space that is T_1 (all points are closed). Then the following are equivalent:

I. (X, τ) is paracompact and Hausdorff
II. Every open cover of (X, τ) admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

Def 1.1.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Def 1.1.3 Locally finite cover

Let (X, τ) be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Def 1.1.4 Refinement of open covers

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Def 1.1.5 Paracompact topological space

A topological space (X, τ) is called *paracompact* if every open cover of X has a refinement by a locally finite open cover.

Def 1.1.6 Partition of unity

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

  (where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology of the real numbers $\mathbb{R}$ regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl\left(f_i^{-1}((0, 1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

I. $\forall (Supp(f_i) \subset U_i)$
II. $\{Supp(f_i) \subset X\}_{i \in I}$
III. $\forall \left(\sum_{i \in I} f_i(x) = 1\right)$

2) Differentiable Manifolds

Def 2.1: C^∞ - Compatible

We say U, φ and V, ψ are C^∞-*compatible* if $U \cap V$ nonempty implies $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ to be *diffeomorphisms* of the open subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ of $\mathbb{R}^n$.

Def 2.2: Differentiable Structure

A *differentiable or* C^∞ *(or smooth) structure* on a topological manifold M is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

1. the U_α cover M,
2. $\forall \alpha, \beta$ the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible,
3. any coordinate neighborhood V, ψ compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$

Def 2.3: Differentiable Manifold

A C^∞ *manifold* is a topological manifold together with a C^∞ -differentiable structure.

Thm 2.4: Uniqueness with Hausdorff

Let M be a Hausdorff space with a countable basis of open sets. If $\{V_\beta, \psi_\beta\}$ is a covering of M by C^∞-compatible coordinate neighborhoods, then there is a unique C^∞ structure on M containing these coordinate neighborhoods.

3) Lie Group

We know $\mathbb{R}^n$ is C^∞-manifold & Abelian group with component-wise addition as group operation. And we can find next two maps are differentiable :

$$(x, y) \to x + y$$
$$x \to -x$$

Then we can generalize these facts.

Def 3.1: Lie Group

G is a Lie group provided that the mapping of $G \times G \to G$ defined by $(x, y) \mapsto x \cdot y$ where $\cdot$ is group operation of G and the mapping of $G \to G$ defined by $x \mapsto x^{-1}$ are both C^∞ mappings.

3) Vector Field and One parameter group

Def 2.1: Vector Field

A Vector field X on M is a function assigning to each point p of M a vector $X_p \in T_p(M)$

$$X : M \to T(M) = \bigcup_{p \in M} T_p(M)$$

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2. Differentiation

2.1 Tensor fields and congruences

1) Supplement for Vector

Def 1.1: Tangent Space

We define the *tangentspace* $T_p(M)$ to be the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying the two conditions

1.
2.

with the vector space operations in $T_p(M)$ defined by

1.
2.

Thm 1.2

Let $F : M \to N$ be a C^∞ map of manifolds for $p \in M$. Then there are two homomorphisms such that

$$\begin{aligned} F^* : &\quad \text{defined by } F^*(f) = \\ F_* : &\quad \text{defined by } F_*(X_p)f = \end{aligned}$$

When $F : M \to M$ is identity then F^*, F_* are isomorphism.

pf

Cor 1.4

If $F : M \to N$ is a diffeomorphism of M onto an open set $U \subset N$ and $p \in M$, then $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism onto.

Note: Coordinate reps of vector

$$\begin{aligned} X_p f &= \frac{d}{dt}\left[f \circ \gamma(t)\right] \\ &= \\ &= \end{aligned}$$

Since we know $F_*(u)f = u(f \circ F)$,

$$\frac{\partial}{\partial x^i}(f \circ \varphi^{-1}) =$$

Therefore

$$\therefore\ X_p = X_p^i E_{ip}$$