# General Relativity

By precise approach

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# Preliminaries

## Manifolds

### 1. Topological Manifolds

**Def 1.1** Topological Manifolds

A *manifold* $M$ of dimension $n$ is a topological space with the following properties.

1. $M$ is Hausdorff
2. $M$ is locally Euclidean of dimension $n$
3. $M$ has a countable basis of open sets

**Why?**

- Hausdorff : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- Locally Euclidean : This is the main reason that why we require manifolds.
- Countable Basis : We need **partition of unity** to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require **paracompactness**. And paracompactness follows from **second countability**. It is same as have countable basis.

### 1.1 Supplement

**Thm 1.1.1** Paracompact $\simeq$ Partition of unity

Let $(X, \tau)$ be a topological space that is $T_1$ (all points are closed). Then the following are equivalent:

I. $(X, \tau)$ is paracompact and Hausdorff
II. Every open cover of $(X, \tau)$ admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

**Def 1.1.2** Hausdorff

Given points $x$ and $y$ of $S$, if $x \neq y$, then there exist open neighborhoods $U$ of $x$ and $V$ of $y$ in $S$ that are disjoint: such that $U \cap V = \emptyset$.

**Def 1.1.3** Locally finite cover

Let $(X, \tau)$ be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of $X$ is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

**Def 1.1.4** Refinement of open covers

Let $(X, \tau)$ be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

**Def 1.1.5** Paracompact topological space

A topological space $(X, \tau)$ is called *paracompact* if every open cover of $X$ has a refinement by a locally finite open cover.

**Def 1.1.6** Partition of unity

Let $(X, \tau)$ be a topological space, and let $U_i \subset X_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

### 2) Differentiable Manifolds

**Def 2.1:** $C^\infty$ - Compatible

We say $U, \varphi$ and $V, \psi$ are $C^\infty$-*compatible* if $U \cap V$ nonempty implies $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ to be *diffeomorphisms* of the open subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ of $\mathbb{R}^n$.

**Def 2.2:** Differentiable Structure

A *differentiable or $C^\infty$ (or smooth) structure* on a topological manifold $M$ is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

1. the $U_\alpha$ cover $M$,
2. $\forall \alpha, \beta$ the neighborhoods $U_\alpha, \varphi_\alpha$ and $U_\beta, \varphi_\beta$ are $C^\infty$-compatible,
3. any coordinate neighborhood $V, \psi$ compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$

**Def 2.3:** Differentiable Manifold

A $C^\infty$ *manifold* is a topological manifold together with a $C^\infty$ -differentiable structure.

**Thm 2.4:** Uniqueness with Hausdorff

Let $M$ be a Hausdorff space with a countable basis of open sets. If $\{V_\beta, \psi_\beta\}$ is a covering of $M$ by $C^\infty$-compatible coordinate neighborhoods, then there is a unique $C^\infty$ structure on $M$ containing these coordinate neighborhoods.

### 3) Lie Group

We know $\mathbb{R}^n$ is $C^\infty$-manifold & Abelian group with component-wise addition as group operation. And we can find next two maps are differentiable :

$$(x, y) \to x + y$$
$$x \to -x$$

Then we can generalize these facts.

**Def 3.1:** Lie Group

$G$ is a Lie group provided that the mapping of $G \times G \to G$ defined by $(x, y) \mapsto x \cdot y$ where $\cdot$ is group operation of $G$ and the mapping of $G \to G$ defined by $x \mapsto x^{-1}$ are both $C^\infty$ mappings.

**3) Vector Field and One parameter group**

**Def 2.1:** Vector Field

A Vector field $X$ on $M$ is a function assigning to each point $p$ of $M$ a vector $X_p \in T_p(M)$

$$X : M \to T(M) = \bigcup_{p \in M} T_p(M)$$

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# 2. Differentiation

## 2.1 Tensor fields and congruences

**1) Supplement for Vector**

**Def 1.1:** Tangent Space

We define the *tangentspace* $T_p(M)$ to be the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying the two conditions

1.
2.

with the vector space operations in $T_p(M)$ defined by

1.
2.

**Thm 1.2**

Let $F : M \to N$ be a $C^\infty$ map of manifolds for $p \in M$. Then there are two homomorphisms such that

$$\begin{aligned} F^* : &\quad \text{defined by } F^*(f) = \\ F_* : &\quad \text{defined by } F_*(X_p)f = \end{aligned}$$

When $F : M \to M$ is identity then $F^*, F_*$ are isomorphism.

*pf*

**Cor 1.4**

If $F : M \to N$ is a diffeomorphism of $M$ onto an open set $U \subset N$ and $p \in M$, then $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism onto.

**Note:** Coordinate reps of vector

$$
\begin{aligned}
X_p f &= \frac{d}{dt}\left[f \circ \gamma(t)\right] \\
&= \\
&=
\end{aligned}
$$

Since we know $F_*(u)f = u(f \circ F)$,

$$\frac{\partial}{\partial x^i}(f \circ \varphi^{-1}) =$$

Therefore

$$\therefore\ X_p = X_p^i E_{ip}$$