General Relativity

By precise approach

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Preliminaries

Manifolds

1) Topological Manifolds

Def 1.1: Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

2) Differentiable Manifolds

Def 2.1: C^∞ - Compatible

We say U, φ and V, ψ are C^∞-*compatible* if $U \cap V$ nonempty implies $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ to be *diffeomorphisms* of the open subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ of $\mathbb{R}^n$.

Def 2.2: Differentiable Structure

A *differentiable or* C^∞ *(or smooth) structure* on a topological manifold M is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

1. the U_α cover M,
2. $\forall \alpha, \beta$ the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible,
3. any coordinate neighborhood V, ψ compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$

Def 2.3: Differentiable Manifold

A C^∞ *manifold* is a topological manifold together with a C^∞ -differentiable structure.

Thm 2.4: Uniqueness with Hausdorff

Let M be a Hausdorff space with a countable basis of open sets. If $\{V_\beta, \psi_\beta\}$ is a covering of M by C^∞-compatible coordinate neighborhoods, then there is a unique C^∞ structure on M containing these coordinate neighborhoods.

3) Lie Group

We know $\mathbb{R}^n$ is C^∞-manifold & Abelian group with component-wise addition as group operation. And we can find next two maps are differentiable :

$$(x, y) \to x + y$$
$$x \to -x$$

Then we can generalize these facts.

Def 3.1: Lie Group

G is a Lie group provided that the mapping of $G \times G \to G$ defined by $(x, y) \mapsto x \cdot y$ where $\cdot$ is group operation of G and the mapping of $G \to G$ defined by $x \mapsto x^{-1}$ are both C^∞ mappings.

3) Vector Field and One parameter group

Def 2.1: Vector Field

A Vector field X on M is a function assigning to each point p of M a vector $X_p \in T_p(M)$

$$X : M \to T(M) = \bigcup_{p \in M} T_p(M)$$

\begin{tcolorbox}[colback=white!5!white,colframe=white!75!black, title=**Def 2.2:** One Parameter Group]

2. Differentiation

2.1 Tensor fields and congruences

1) Supplement for Vector

Def 1.1: Tangent Space

We define the *tangentspace* $T_p(M)$ to be the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying the two conditions

1.
2.

with the vector space operations in $T_p(M)$ defined by

1.
2.

Thm 1.2

Let $F : M \to N$ be a C^∞ map of manifolds for $p \in M$. Then there are two homomorphisms such that

$$\begin{aligned} F^* &: \quad \text{defined by } F^*(f) = \\ F_* &: \quad \text{defined by } F_*(X_p)f = \end{aligned}$$

When $F : M \to M$ is identity then F^*, F_* are isomorphism.

pf

Cor 1.4

If $F : M \to N$ is a diffeomorphism of M onto an open set $U \subset N$ and $p \in M$, then $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism onto.

Note: Coordinate reps of vector

$$
\begin{aligned}
X_p f &= \frac{d}{dt}\left[f \circ \gamma(t)\right] \\
&= \\
&=
\end{aligned}
$$

Since we know $F_*(u)f = u(f \circ F)$,

$$\frac{\partial}{\partial x^i}(f \circ \varphi^{-1}) =$$

Therefore

$$\therefore\ X_p = X_p^i E_{ip}$$