General Relativity

With a rigorous approach

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Contents

1 The Background Manifold Structure

Def 1.1.0 Topological Manifolds

A *manifold* M of dimension n is a topological space with the following properties.

1. M is Hausdorff
2. M is locally Euclidean of dimension n
3. M has a countable basis of open sets

Why?

- **Hausdorff** : In Hausdorff space, convergent sequences converge to only one point. If you want to do calculus, you should need Hausdorff space.
- **Locally Euclidean** : This is the main reason that why we require manifolds.
- **Countable Basis** : We need *partition of unity* to bring many properties of Euclidean space. For Hausdorff space, existence of partition of unity require *paracompactness.* And paracompactness follows from *second countability.* It is same as have countable basis.

Now, we need some topological concepts to start *General Relativity.* If you are not familiar with topology, first read *Appendix A.*

1. Basic Concepts

Def 1.1.1 Connected

If it is not possible to write $M = A \cup B$ with, $A, B \in \mathcal{T}$ and $A \cap B = \emptyset$ then M is *connected*.

Def 1.1.2 Hausdorff

If M is connected & $\forall\, p, q \in M$, there are open neighborhoods $\mathcal{U}(p) \ni p, \mathcal{U}(q) \ni q$ such that $\mathcal{U}(p) \cap \mathcal{U}(q) = \emptyset$ then M is *Hausdorff*.

Def 1.1.3 Cover

A family $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of open sets of M is called *open cover of* M if

$$\bigcup_{\alpha \in A} \mathcal{U}_\alpha = M$$

Def 1.1.4 Compact

M is *compact* if M is Hausdorff and all open cover of M has finite refinement of M.

Def 1.1.5 Paracompact

M is *paracompact* if all open cover of M has locally finite refinement.

2. Maps

Def 1.2.1 Important Maps

Given two sets U, U' a map $\Phi : U \to U'$ is called

- *Injective* : $\forall p' \in \Phi(U),\ \exists! p \in U$ such that $\Phi(p) = p'$.
- *Surjective* : $\Phi(U) = U'$.
- *Bijective* : Φ is both injective and surjective.

Def 1.2.2 Continuous

Consider $(U, \mathcal{T}),\ (U', \mathcal{T}')$ are topological spaces. $\Phi : U \to U'$ is said to be *continuous* at a point $p \in U$ if $\Phi^{-1}(W')$ is a neighborhood of p for any neighborhood W' of $\Phi(p) \in U'$.

Def 1.2.3 Homeomorphism

If $\Phi : U \to U'$ is bijective and Φ, Φ^{-1} are continuous then Φ is called homeomorphism and U, U' are homeomorphic.

3. Coordinate Neighborhoods (Chart)

Def 1.3.1 Coordinate Neighborhood (Chart)

Given a topological space $(M, \mathcal{T})$, define *chart* of M to be a pair $(\mathcal{U}_\alpha, \varphi_\alpha)$, with $\mathcal{U}_\alpha$ an element of $\mathcal{T}$ and φ_α a homeomorphism of $\mathcal{U}_\alpha$ onto an open set of $\mathbb{R}^n$.

We usually use next notation:

- *Point on a manifold* : $p \in M$
- *Local coordinate of point* : $\varphi(p) = (x^1, \cdots, x^n) = x$

Def 1.3.2 Atlas

A familiy of charts $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ on M is said to form an *atlas* on M if $\displaystyle\bigcup_{\alpha\in A} \mathcal{U}_\alpha = M$.

Def 1.3.3 Coordinate Transform

Let $(\mathcal{U}_\alpha, \varphi_\alpha)$, $(\mathcal{U}_\beta, \varphi_\beta)$ be two charts on M with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. For a point $p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, a map (trivial homeomorphism)

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is called *Coordinate Transform*.

Def 1.3.4 C^r-atlas

An atlas on M is C^r*-atlas* if $\varphi_\beta \circ \varphi_\alpha^{-1}$ and its inverse for any pair (α, β) are $\mathbb{R}^n$ valued C^r-functions.

4. Differentiable Manifolds

Def 1.4.1 Differentiable Manifold

Differentiable Manifold of class C^r and dimension n is a Hausdorff topological space with a C^r-atlas.

We denote differentiable manifold as $(M, \mathcal{A})$ where $\mathcal{A}$ is C^r-atlas.

Def 1.4.2 Function of Manifold

A map $f : M \to \mathbb{R}$ is said to be C^k-function at $p \in M$, if for any chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ containing p, there exists open neighborhood $\mathcal{U}(p) \subset \mathcal{U}_\alpha$ of p such that the composite map

$$\tilde{f}_\alpha : \mathbb{R}^n \supset \varphi_\alpha(\mathcal{U}(p)) \to \mathbb{R}$$

defined by

$$\tilde{f}_\alpha(x) \equiv f \circ \varphi_\alpha^{-1}(x),\ x \in \mathbb{R}^n$$

is a C^k-differentiable function.

We can't define differentiability of f directly. But with chart, we can find $\mathbb{R}^n$ valued function that we already know how to determine differentiability in multi-variable real analysis. So, by using it, we can consider differentiability of f.

Def 1.4.3 Function Space

Denote by $\mathcal{F}$ the set of all differentiable functions on M with the internal operations.

1. *Multiplication* : $fg(p) = f(p)g(p)$
2. *Addition* : $(f+g)(p) = f(p) + g(p)$

It's easy to find $\mathcal{F}$ is an *Abelian Ring*.

5. Maps of Manifolds

Remark Manifold with $\mathbb{R}^n$

A manifold M is locally homeomorphic to an open set of $\mathbb{R}^n$.

Def 1.5.1 Maps between Manifolds

Let M, N be two differentiable manifolds with same dimension n and $\psi : M \to N$ a map of M into N. Suppose two points $p \in M,\ p' \in N$ such that $\psi(p) = p'$. Let $(\mathcal{U}_\alpha, \varphi_\alpha)_p,\ (\mathcal{U}'_\beta, \varphi'_\beta)_{p'}$ be two charts such that $\varphi_\alpha(p) = x \in \mathbb{R}^n,\ \varphi'_\beta(p') = x' \in \mathbb{R}^n$. By definition, $x' = \varphi'_\beta \circ \psi \circ \varphi_\alpha^{-1}(x)$. We call it by *coordinate representation* of ψ and denote by

$$\tilde{\psi}_{\alpha\beta}(x) = \varphi'_\beta \circ \psi \circ \varphi_\alpha^{-1}(x)$$

Similar to f, we also determine differentiability of ψ using by $\tilde{\psi}_{\alpha\beta}$.

Def 1.5.2 Diffeomorphism

If the map $\psi : M \to N$ is homeomorphism with both $\psi,\ \psi^{-1}$ are differentiable, then ψ is called *Diffeomorphism*.

There are some kinds of Maps.

Def 1.5.4 Immersion

If $\dim(M) > \dim(N)$, a C^r-map $\Phi : N \to M$ is said to be an *immersion* if it is locally injective and the image of $\Phi(N)$ is said to be a m-dimensional *immersed submanifold* of M
The set $\Phi(N)$ is said to be *imbedded* in M if Φ is a homeomorphism of N into its image in M, with the induced topology of M.

Def 1.5.5 Hypersurface

An imbedded submanifold of M with $m = \dim(M) - 1$ is termed a *hypersurface*.

6. The Tangent Space

To define tangent vector, we should define curve first.

Def 1.6.1 Curve

Given manifold M, a *curve* γ in M is a map with single parameter:

$$\gamma : \mathbb{R} \to M$$

Now we can define tangent vector.

Def 1.6.2 Tangent Vector

The *tangent vector* to a curve γ at a point $p = \gamma(t)$ is a map $\dot{\gamma}_p : \mathcal{F}(M) \to \mathbb{R}$ is given as

$$\dot{\gamma}_p(f) = \frac{d}{dt}\left[f \circ \gamma(t)\right]_{\gamma^{-1}(p)}, \quad f \in \mathcal{F}(M)$$

We define tangent vector as a map. It's so weird. Let's rationalize this on $\mathbb{R}^n$.

Def 1.6.3 Derivation Operator

Let M be a differentiable manifold, $p \in M$. We say that a linear function $D \in \mathcal{F}^*(M)$ defined on $\mathcal{F}(M)$ is a *derivation* of $\mathcal{F}(M)$ at p if the equality

$$D(fg) = D(f)g(p) + f(p)D(g)$$

holds for $\forall f, g \in \mathcal{F}(M)$.

We denote space of derivation operators as $\mathcal{D}_p(M)$. For $\mathbb{R}^n$, denote $\mathcal{D}_p(\mathbb{R}^n)$ as set of all derivations of $C^\infty(p)$ to $\mathbb{R}$.

Lem 1.6.4 Constant Derivation

Let $D \in \mathcal{D}_p(M)$. Then $Df = 0$ for all $f \in \mathcal{F}(M)$ such that f is constant in a neighborhood of p.

Proof for Lem 1.6.4

$$D1 = D(1 \cdot 1) = D1 \cdot 1 + 1 \cdot D1 = 2D1 \Rightarrow D1 = 0 \Rightarrow Dc = c \cdot D1 = 0$$

■

Lem 1.6.5 First Order Approximation

Let $f(x^1, \cdots x^n)$ be defined and C^∞ on some open set U. If $p \in U$, then $\exists$ spherical neighborhood $\mathcal{B}(p)$ of p such that $\mathcal{B}(p) \subset U$ and C^∞ function $g^1, \cdots, g^n$ on $\mathcal{B}(p)$ such that

1. $g^i(p) = \left(\dfrac{\partial f}{\partial x^i}\right)$
2. $f(x) = f(p) + \displaystyle\sum_{i=1}^n (x^i - p^i) g^i(x)$

Proof for Lem 1.6.5

Consider next integration.

$$\int_0^1 \frac{\partial}{\partial t} f(p + t(x-p))dt = f(x) - f(p)$$

Thus,toc-own-page: true

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) \int_0^1 \left[\frac{\partial f}{\partial x^i}\right]_{p+t(x-p)} dt$$

So, choose

$$g^i(x) = \int_0^1 \left[\frac{\partial f}{\partial x^i}\right]_{p+t(x-p)} dt$$

Then it satisfies *Lem 1.6.5*. ■

And review directional derivative.

Def 1.6.6 Directional Derivative

Let $X_p \in T_p(\mathbb{R}^n)$ such that

$$X_p = \sum_{i=1}^n \alpha^i E_{ip}$$

Then we can define a linear map $X_p^* : C^\infty(p) \to \mathbb{R}$ as

$$X_p^*(f) = \sum_{i=1}^n \alpha^i \left(\frac{\partial f}{\partial x^i}\right)_p$$

This map is called *Directional Derivative*.

Trivially, we know there is 1-1 correspondence between X_p, X_p^*. If we define space of directional derivatives, then this space has same dimension as $T_p(\mathbb{R}^n)$. Thus, they are isomorphic.

Thm 1.6.7 Tangent Vector & Derivative

$T_p(\mathbb{R}^n)$ is isomorphic to $\mathcal{D}_p(\mathbb{R}^n)$.

Proof of Thm 1.6.7

We already know relation between $X_p,\ X_p^*$. Thus, our claim is as follow:

$$\forall D \in \mathcal{D}_p(\mathbb{R}^n),\ \exists X_p \in T_p(\mathbb{R}^n) \text{ such that } X_p^* f = Df$$

By *Lem 1.6.5*, $\exists g$ such that

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g^i(x), \quad g^i(p) = \left(\frac{\partial f}{\partial x^i}\right)_p$$

Then let's use D both side,

$$\begin{aligned} Df &= D(f(p)) + \sum_{i=1}^n D(x^i - p^i) g^i(p) + \sum_{i=1}^n (p^i - p^i) D(g^i(x)) \\ &= \sum_{i=1}^n D(x^i) \left(\frac{\partial f}{\partial x^i}\right)_p \end{aligned}$$

Since $D(x^i) \in \mathbb{R}$, let $\alpha^i \equiv D(x^i)$ then proof is complete. ■

By *Thm 1.6.7*, we can identify $T_p(\mathbb{R}^n)$ & $\mathcal{D}_p(\mathbb{R}^n)$. It means we can identify canonical basis and directional derivative. Thus, from now, we use directional derivative ways rather than canonical basis.

Then let's get back to original definition.

Now, let's obtain coordinate representation of tangent vector.

$$\begin{aligned}\dot{\gamma}_p(f) &= \frac{d}{dt}\left[f \circ \gamma(t)\right]_{\gamma^{-1}(p)} \\ &= \frac{d}{dt}\left[f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right]_{\gamma^{-1}(p)} \\ &= \left(\frac{dx^i}{dt}\right)\left(\frac{\partial \tilde{f}}{\partial x^i}\right)_{\varphi(p)}\end{aligned}$$

We want to decompose tangent vector to component and basis. But we can't find directly. So, we need some awesome tool - *push forward*.

Def 1.6.8 Tangent Map (Push forward)

Let M, N be two manifolds and $\Phi : M \to N$ be a map of M into N. The induced vectors in N are given by maps:

$$\Phi_*(u) : \mathcal{F}(N) \to \mathbb{R}, \quad u \in T_p(M)$$

defined by

$$\Phi_*(u)(f) = u(f \circ \Phi), \quad f \in \mathcal{F}(N)$$

This map is called *Tangent map* and also called *Push forward*.

By *Def 1.6.8*, we can decompose tangent vector to components & bases.

$$\begin{aligned}\dot{\gamma}_p(f) &= \left(\frac{dx^i}{dt}\right)_{\gamma^{-1}(p)}\left(\frac{\partial \tilde{f}}{\partial x^i}\right)_{\varphi(p)} \\ &= \left(\frac{dx^i}{dt}\right)_{\gamma^{-1}(p)}\left(\frac{\partial}{\partial x^i}\right)_{\varphi(p)}\left(f \circ \varphi^{-1}\right) \\ &= \left(\frac{dx^i}{dt}\right)_{\gamma^{-1}(p)}\left(\varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right)_p f\end{aligned}$$

Since $\left(\varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right)_p$ is basis for $T_p(M)$, finally we can get next expression.

$$\dot{\gamma}_p = \left(\frac{dx^i}{dt}\right)_{\gamma^{-1}(p)}\left(\varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right)_p \equiv \dot{\gamma}_p^i \partial_i$$

$\dot{\gamma}_p^i$ is called *component* of tangent vector, ∂_i is called *basis* of tangent vector.

Now, let's see transformation properties of vector components.

Def 1.6.9 Change Basis

Let $\{e_i\}, \{e'_j\}$ are two bases of $T_p(M)$. From the properties of a basis, we can describe change basis as follows:

$$e'_i = A_i^{\ j} e_j$$

where $A_i^{\ j}$ form an $n \times n$ matrix of real numbers such that

$$A_i^{\ j} A^{-1}{}_j^{\ k} = \delta_i^k$$

Let's use change of basis for our tangent vector. Let choose two bases $\{e_i\}, \{e'_j\}$. Then for tangent vector $u \in T_p(M)$,

$$u = u^i e_i = u'^i e'_i$$

It's easy to find next relation.

$$(u'^i A_i^{\ j} - u^j) e_j = 0$$

By linearly indendence of bases, we can get

$$u'^i = u^j A^{-1}{}_j^{\ i}$$

Transpose both side, we finally see

$$u'^i = (A^{-1})^i_{\ j} u^j$$

Def 1.6.10 Contravariant Vector

Suppose change of basis is given as

$$e'_i = A_i^{\ j} e_j$$

If change of basis of vector u is given as

$$u'^i = (A^{-1})^i_{\ j} u^j$$

then vector u is called *contravaiant vector.*

Exercise 1.6.1: Prove that $\Phi_*(\dot{\gamma}_p) = (\Phi \dot{\circ} \gamma)_{\Phi(p)}$

7. The Cotangent Space

Def 1.7.1 Differential

Let $f \in \mathcal{F}(M)$. The *differential* of f at p is the map

$$df_p : T_p(M) \to \mathbb{R}$$

such that

$$df_p(u) = u(f) \quad \forall u \in T_p(M)$$

Exercise 1.7.1: Prove that differential is linear.

Def 1.7.2 Cotangent Space

The set of all linear maps from $T_p(M)$ into $\mathbb{R}$ is called the *cotangent space* at p. It is denoted by $T_p^*(M)$ and its general elements are *covectors*. In fact, this space is the dual of $T_p(M)$.

We denote covector as follows:

$$\omega = \omega_i e^i \quad \text{where } \omega_i \in \mathbb{R},\ e^i \in T_p^*(M)$$

Although one can choose an arbitrary basis in $T_p^*(M)$, it's convenient to link its choice uniquely to that of a basis in the tangent space. - *Dual basis*

$$e^i(e_j) = \delta^i_j$$

Prop 1.7.3 Properties of Covector

- Component: $e^i(u) = u^k e^i(e_k) = u^i \quad \forall u \in T_p(M)$
- Re-Analyze: $\omega(u) = u^k \omega(e_k) = \omega(e_k)e^k(u) \ \Rightarrow\ \omega = \omega(e_k)e^k = \omega_k e^k$
- Natural Basis: $dx^i(\partial_j) = \partial_j(x^i) = \dfrac{\partial x^i}{\partial x^j} = \delta^i_j \ \rightarrow\ dx^i$ is a natural basis for $T_p^*(M)$.
- Component of differential: $(df)_i = (df)(\partial_i) = \partial_i(f) = \dfrac{\partial \tilde{f}_\alpha}{\partial x^i} \quad \forall f \in \mathcal{F}(M)$

By above properties, we can find any covector can be written as the differential of some function.

We already know $e'_i = A_i^{\ j} e_j$. Now, let's see transformation of bases in $T_p^*(M)$.
Let $\{e'^j\}$, $\{e^k\}$ be two bases of $T_p^*(M)$.

$$e'^j(e'_k) = \delta^j_k \ \Rightarrow\ A_k^{\ i} e'^j(e_i) = \delta^j_k \ \Rightarrow\ e'^j(e_l) = \left(A^{-1}\right)_l^{\ j} = \delta^k_l \left(A^{-1}\right)_k^{\ j} e^k(e_l)$$

Therefore, we can get next two results.

$$\begin{aligned} e'^j &= \left(A^{-1}\right)_k^{\ j} e^k \\ \omega'_i &= A_i^{\ j} \omega_j \end{aligned}$$

This change of component is same as change of coordinate basis. We call this kinds of vector by *Covariant vector*.

Finally, let's see the dual tangent map.

Def 1.7.4 Dual Tangent Map (Pull Back)

Let $\Phi : M \to N$. Now we define *the dual tangent map* $\Phi^* : T^*_{\Phi(p)}(N) \to T^*_p(M)$ as

$$\left(\Phi^*(\omega)\right)(u) = \omega\left(\Phi_*(u)\right) \quad \forall u \in T_p(M)$$

Exercise 1.7.2: Prove that $\Phi^*(df) = d(f \circ \Phi)$.

8. Lie Group

Def 1.8.1 Lie Group

G is a *Lie group* provided that the mapping of $G \times G \to G$ defined by $(x, y) \to xy$ and the mapping of $G \to G$ defined by $x \to x^{-1}$ are both C^∞ mappings.

Example 1.8.2 General Linear Group

$Gl(n, \mathbb{R})$, the set of nonsingular $n \times n$ matrices, is a group with respect to matrix multiplication. Since AB is polynomial in the entries of A, B, the map $(A, B) \to AB$ is C^∞.
And for $A^{-1} = \frac{1}{detA}\tilde{a}_{ij}$, since cofactor of A is polynomial in the entries of A and $det(A) \neq 0$, entries of A^{-1} are rational functions on $Gl(n, \mathbb{R})$ with non-vanishing denominator. Thus, the map $A \to A^{-1}$ is also C^∞.
Therefore, $Gl(n, \mathbb{R})$ is a Lie group.

Example 1.8.3 Nonzero Complex Number

Let C^* be the nonzero complex numbers. Then C^* is a group with respect to multiplication of complex numbers, the inverse being $z^{-1} = \frac{1}{z}$.
Moreover C^* is a two-dimensional C^∞ manifold covered by a single coordinate neighborhood $U = C^*$ with coordinate map $z \to \varphi(z)$ given by $\varphi(x + iy) = (x, y)$ for $z = x + iy$.
Using these coordinates, the map $(z, z') \to zz'$ is given by

$$((x, y), (x', y')) \to (xx' - yy', xy' + yx')$$

and the mapping $z \to z^{-1}$ by

$$(x, y) \to \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

We can find these two maps are C^∞. Thus, C^* is Lie group.

Exercise 1.8.1: Show that if G_1, G_2 are Lie groups then the direct product $G_1 \times G_2$ of this groups with the C^∞ structure of the Cartesian product of manifolds is a Lie group.

Example 1.8.4 Toral Groups

The circle S^1 may be identified with the complex numbers of absolute value $+1$. Since $|z_1\, z_2| = |z_1|\,|z_2|$, it is a group with respect to multiplication of complex numbers – a subgroup of C^*. Thus, S_1 is also Lie group and by previous *Exercise 1.8.1*, we can see that $T^n = S^1 \times \cdots \times S^1$ is also Lie group. It is called the *toral group*.

As might be expected, the subgroups of a Lie group which are also submanifolds play a special role.

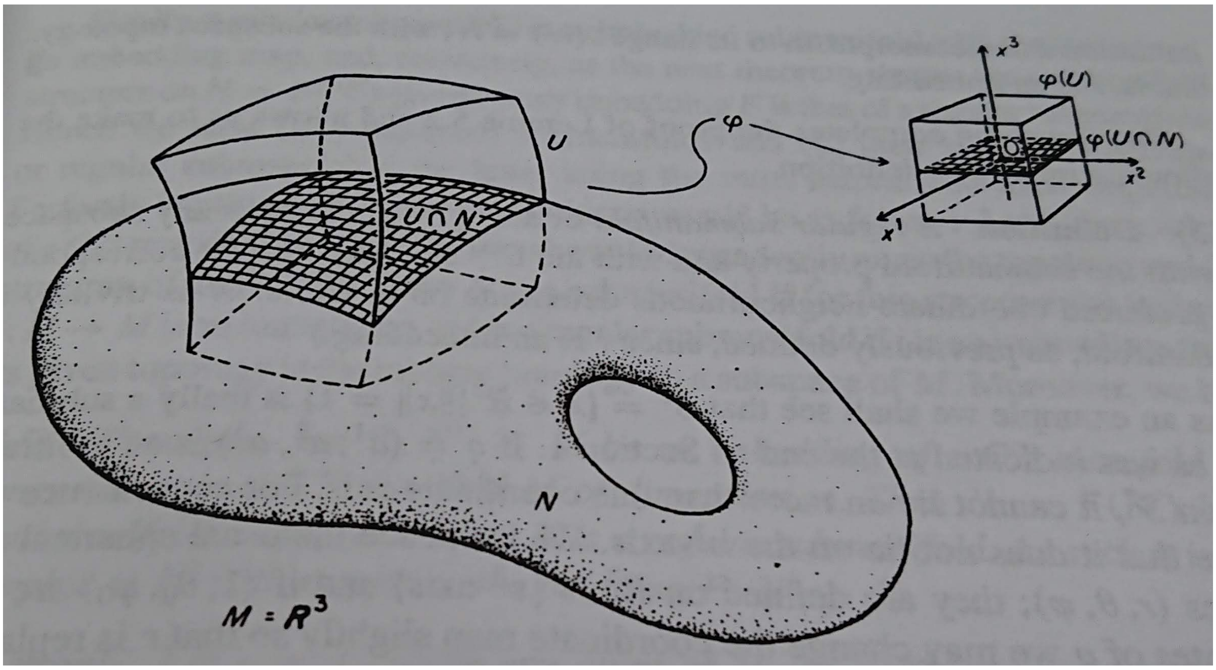
Def 1.8.5 Submanifold

A subset N of a C^∞ manifold M is said to have the *n-submanifold property* if each $p \in N$ has a coordinate neighborhood U, φ on M with local coordinates $x^1, \cdots, x^m$ such that

1. $\varphi(0) = (0, \cdots, 0)$
2. $\varphi(U) = C^m_\epsilon(0)$[a]
3. $\varphi(U \cap N) = \left\{x \in C^m_\epsilon(0)\,|\, x^{n+1} = \cdots = x^m = 0\right\}$

If N has this property, coordinate neighborhoods of this type are called *preferred* coordinates.

[a] m-dimensional cube with center zero and breadth ϵ



Our interest is not general submanifold - *Regular submanifold.*

Def 1.8.6 Regular Submanifold

A *regular submanifold* of a C^∞ manifold M is any subspace N with the C^∞ structure that the corresponding preferred coordinate neighborhoods determine on it.

In Lie group, there is an important theorem for regular submanifold.

Thm 1.8.7 Lie group & Regular submanifold

Let G be a Lie group and H a subgroup which is also a regular submanifold. Then with its differentiable structure as a submanifold H is a Lie group.

To prove above theorem, we require following lemma.

Lem 1.8.8 Regular submanifold & Differentiable Map

Let $F : A \to M$ be a C^∞ mapping of C^∞ manifolds and suppose $F(A) \subset N$, N being a regular submanifold of M. Then F is C^∞ as a mapping into N.

Proof for Lem 1.8.8

Since N is regular submanifold of M, each point of N in preferred coordinate neighborhood. Let $p \in A,\ q = F(p) \in N$ and (U, φ) be a coordinate neighborhood of $p,\ (V, \psi)$ be a coordinate of q. Then we can find next properties from definition of submanifold.

1. $\psi(q) = (0, \cdots, 0)$
2. $\psi(V) = C^m_\epsilon(0)$
3. $\psi(V \cap N) = \left\{x \in C^m_\epsilon(0) \,|\, x^{n+1} = \cdots = x^m = 0\right\}$

Let consider coordinate representation of F:

$$\begin{aligned}\tilde{F} &= \psi \circ F \circ \varphi^{-1} \\ \tilde{F}(x^1, \cdots, x^l) &= (f^1(x), f^2(x), \cdots, f^n(x), 0, \cdots, 0)\end{aligned}$$

since $F(A) \subset N,\ \psi \circ F(U) \subset V \cap N$. We can find $(V \cap N,\ \pi \circ \psi|_{V\cap N}$[a]$)$ is a coordinate neighborhood of q on N. Thus, we can consider F as a mapping into N, is given in local coordinates by

$$(x^1, \cdots, x^l) \to (f^1(x), \cdots, f^n(x))$$

Since π is also differentiable, F is C^∞ map into N. ■

[a] π is projection operator from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Proof for Thm 1.8.7

Since H is regular submanifold of G, it's easy to see $H \times H$ is a regular submanifold of $G \times G$. Thus, inclusion map[a] $F_1 : H \times H \to G \times G$ is a C^∞ imbedding. If $F_2 : G \times G \to G$ is the C^∞ mapping $(g, g') \to gg'$ and $F = F_2 \circ F_1$, then F is a C^∞ mapping from $H \times H \to G$ with image in H since H is subgroup. By *Lem 1.8.8*, F can be considered as C^∞ mapping from $H \times H$ into H. Similarly, let take a map from H to G such that $F'(h) = h^{-1}$ then its image is onto H. Thus, by *Lem 1.8.8*, it is also C^∞ mapping. Therefore the regular submanifold H of G is also Lie group. ■

[a] $\iota(x) = x$

Now, using *Thm 1.8.7*, we can find natural defined maps of a Lie group G onto itself.

1. $x \to x^{-1}$
2. Left and right translations: $L_a(x) = ax,\ R_a(x) = xa$

These maps are C^∞ by definition of Lie group and their inverses are also C^∞. So, they are, in fact, *diffeomorphisms*.

The meaning of *Thm 1.8.7* is for any regular submanifold of Lie group is also Lie group. But there is one missing link – how to see a subset is regular submanifold?

To answer this, we need fundamental concept - *rank*.

– Rank of Map

Def 1.8.9 Rank in Linear Algebra

Let A be an $m \times n$ matrix, then the *rank* is defined in four equivalent ways

1. the dimension of the subspace of V^n spanned by the rows
2. the dimension of the subspace of V^m spanned by the columns
3. the maximum order of any nonvanishing minor determinant
4. the dimension of the image

Exercise 1.8.1: Prove that rank $(AB) \leq$ rank (A).

Prop 1.8.10 Rank with invertible matrix

Let A be a $m \times n$ matrix and B be a $n \times n$ non-singular matrix. Then

$$\text{rank}\,(AB) = \text{rank}\,(A)$$

Exercise 1.8.2: Prove *prop 1.8.10.*

Def 1.8.11 Rank of C^r map

Let $F : U \to \mathbb{R}^m$ be a C^r mapping of an open set $U \in \mathbb{R}^n$, then *rank of* F *at* x is defined as the rank of $DF(x)$[a].

[a] $DF(x)$ is *Jacobian* matrix of F at x

Exercise 1.8.3: Find rank of $F(x^1, x^2) = ((x^1)^2 + (x^2)^2, 2x^1x^2)$.

And denote one of the famous theorem in Analysis - *Inverse Function Theorem.*

Thm 1.8.12 Inverse Function Theorem

Let W be an open subset of $\mathbb{R}^n$ and $F : W \to \mathbb{R}^n$ a C^r mapping, $r = 1, 2, \cdots$, or ∞. If $a \in W$ and $DF(a)$ is nonsingular, then there exists an open neighborhood U of a in W such that $V = F(U)$ is open and $F : U \to V$ is a C^r diffeomorphism. If $x \in U$ and $y = F(x)$, then we have the following formula for the derivatives of F^{-1} at y:

$$DF^{-1}(y) = (DF(x))^{-1}$$

Its proof require Analytical skills, so we skip this proof. Instead of proof, we will just use its corollary.

Cor 1.8.13 Diffeomorphism (Revisited)

A necessary and sufficient condition for the C^∞ map F to be a diffeomorphism from W to $F(W)$ is that it be one-to-one and DF be nonsingular at every point of W.

Now, denote very important theorem - *Rank Theorem*.

Thm 1.8.14 Rank Theorem

Let $A_0 \subset \mathbb{R}^n$, $B_0 \subset \mathbb{R}^m$ be open sets. $F: A_0 \to B_0$ be a C^r mapping, and suppose the rank of F on A_0 to be equal to k. If $a \in A_0$ and $b = F(a)$, then there exist open sets $A \subset A_0$ and $B \subset B_0$ with $a \in A$ and $b \in B$, and there exist C^r diffeomorphisms $G : A \to U(\text{open}) \subset \mathbb{R}^n$, $H : B \to V(\text{open}) \subset \mathbb{R}^m$ such that $H \circ F \circ G^{-1}(U) \subset V$ and such that this map has the simple form

$$H \circ F \circ G^{-1}(x^1, \cdots, x^n) = (x^1, \cdots, x^k, 0, \cdots, 0)$$

This is clearly an important theorem for it tells us that a mapping of constant rank k behaves *locally* like projection of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ to $\mathbb{R}^k$ followed by injection of $\mathbb{R}^k$ onto $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m$. This is an important tool and we shall use it frequently; we rephrase this to local coordinates:

Thm 1.8.15 Rank Theorem (Rephrased)

Let $F : N \to M$ be a differentiable mapping of C^∞ manifolds and suppose dim $N = n$, dim $M = m$ and rank $(F) = k$ at every point of N. If $p \in N$, then there exist coordinate neighborhoods (U, φ) and (V, ψ) of p and $F(p)$ such that $\varphi(p) = (0, \cdots, 0)$, $\psi(F(p)) = (0, \cdots, 0)$ and $\tilde{F} = \psi \circ F \circ \varphi^{-1}$ is given by

$$\tilde{F}(x^1, \cdots, x^n) = (x^1, \cdots, x^k, 0, \cdots, 0)$$

Then let's prove *Thm 1.8.14*.

Proof for Thm 1.8.14

Without loss of generality, let $a = 0 \in \mathbb{R}^n,\ b = 0 \in \mathbb{R}^m$. Since F has constant rank k on A_0, there exists $k \times k$ minor of nonzero determinant in $DF(a)$.

$$\frac{\partial(f^1, \cdots, f^k)}{\partial(u^1, \cdots, u^k)} = \begin{pmatrix} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^k} \\ \vdots & & \vdots \\ \frac{\partial f^k}{\partial u^1} & \cdots & \frac{\partial f^k}{\partial u^k} \end{pmatrix}_{u=a}$$

Now define C^r map $G:\ A_0 \to \mathbb{R}^n$ by

$$G(u^1, \cdots, u^n) = (f^1(u^1, \cdots, u^n), \cdots, f^k(u^1, \cdots, u^n), u^{k+1}, \cdots, u^n)$$

for $u \in A_0,\ f(u) \in B_0$ where $F(u) = (f^1(u), \cdots, f^n(u))$. Then

$$DG = \left(\begin{array}{ccc|c} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^k} & \\ \vdots & & \vdots & \bigstar^a \\ \frac{\partial f^k}{\partial u^1} & \cdots & \frac{\partial f^k}{\partial u^k} & \\ \hline & \mathbf{0} & & I_{n-k} \end{array}\right)$$

Since left upper part is non-singular at $u = a,\ DG$ is non-singular at $u = a$. Thus, there is an open subset A_1 of A_0 which contains a such that G is diffeomorphism onto an open subset $U_1 = G(A_1)$. Since $G(u) = (f^1(u), \cdots, f^k(u), u^{k+1}, \cdots, u^n)$, if we let $G^{-1}(x) = u$ then

$$x^i = \begin{cases} f^i(u) & i = 1, \cdots, k \\ u^i & i = k+1, \cdots, n \end{cases}$$

Thus,

$$\begin{aligned} F \circ G^{-1}(x) = F(u) &= (f^1(u), \cdots, f^m(u)) \\ &= (x^1, \cdots, x^k, f^{k+1}(u), \cdots, f^k(u)) \end{aligned}$$

$$\therefore\ F \circ G^{-1}(x) = (x^1, \cdots, x^k, \tilde{f}^{k+1}(x), \cdots, \tilde{f}^m(x))$$

where $\tilde{f}^{k+j}(x) = f^{k+j} \circ G^{-1}(x)$. Since G is diffeomorphism on $A_1,\ G^{-1}$ is one-to-one on U_1. Therefore it's trivial that $F \circ G^{-1}(0) = 0$.

(Continued next page)

[a] Star symbol means we need not consider this part.

Proof for Thm 1.8.14 (Continued)

$$D(F\circ G^{-1})(x) = \left(\begin{array}{c|ccc} I_k & & \mathbf{0} & \\ \hline & \dfrac{\partial \tilde{f}^{k+1}}{\partial x^{k+1}} & \cdots & \dfrac{\partial \tilde{f}^{k+1}}{\partial x^n} \\ \bigstar & \vdots & & \vdots \\ & \dfrac{\partial \tilde{f}^{m}}{\partial x^{k+1}} & \cdots & \dfrac{\partial \tilde{f}^{m}}{\partial x^n} \end{array}\right)$$

for $x \in U_1$. We know DG^{-1} is non-singular on U_1 and $G^{-1}(U_1) = A_1 \subset A_0$. Thus, by properties of rank,

$$\text{rank}\left(D(F\circ G^{-1})\right) = \text{rank}\left(DF\circ DG^{-1}\right) = \text{rank}\left(DF\right) = k$$

Since upper left part of $D(F\circ G^{-1})$ is identity - already have rank k, all components in lower right part should be zero on U_1. It means $\tilde{f}^{k+1},\cdots,\tilde{f}^m$ depend on $x^1,\cdots,x^k$ only. Now, let define function $T:\ V_1\subset\mathbb{R}^m \to B_0$ such that $0\in V_1$ as follows

$$T(y_1,\cdots,y^m) = (y^1,\cdots,y^k,y^{k+1}+\tilde{f}^{k+1}(y^1,\cdots,y^k),\cdots,y^m+\tilde{f}^m(y^1,\cdots,y^k))$$

Then the Jacobian of T is given by

$$DT(y) = \left(\begin{array}{c|c} I_k & \mathbf{0} \\ \hline \bigstar & I_{m-k}\end{array}\right)$$

Since DT is nonsingular at every point of any neighborhood V of 0 in V_1, T is a C^r diffeomorphism of V onto an open set $B\subset B_0$.
Finally, let $H=T^{-1},\ A=G^{-1}(U)$ then

$$U \xrightarrow{G^{-1}} A \xrightarrow{F} B \xrightarrow{H} V$$

are C^r maps and $G^{-1},\ H$ are C^r diffeomorphisms onto A & V. And finally we can see,

$$\begin{aligned} H\circ F\circ G^{-1}(x^1,\cdots,x^n) &= H(x^1,\cdots,x^k,\tilde{f}^{k+1}(x^1,\cdots,x^k),\cdots,\tilde{f}^m(x^1,\cdots,x^k)) \\ &= (x^1,\cdots,x^k,0,\cdots,0)\in\mathbb{R}^m \end{aligned}$$

■

Now, we ready for understanding following theorem & corollary - the most useful method of finding examples of regular submanifolds.

Thm 1.8.16 Constant Rank & Regular Submanifold

Let N be a C^∞ manifold of dimension n, M be a C^∞ manifold of dimension m, and $F: N \to M$ be a C^∞ mapping. Suppose that F has constant rank k on N and that $q \in F(N)$. Then $F^{-1}(q)$ is colosed, regular submanifold of N of dimension $n - k$.

Proof for Thm 1.8.16

Let $A = F^{-1}(q)$. Since F is C^∞ map & $\{q\}$ is closed subset of M, A is closed subset in N via continuity of F. Let $p \in A$ then since F has constant rank k on a neighborhood of p, by rank theorem, we can find coordinate neighborhoods (U, φ), (V, ψ) of p, q such that

$$\begin{gathered}\varphi(p) = 0,\ \psi(q) = 0 \\ \varphi(U) = C_\epsilon^n(0),\ \psi(V) = C_\epsilon^m(0) \\ \psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^n) = (x^1, \cdots, x^k, 0, \cdots, 0) \in \mathbb{R}^m\end{gathered}$$

Since we supposed $A = F^{-1}(q)$, $A \cap U \subset A = F^{-1}(q)$. Thus, $F(A \cap U) = q$ and then $\psi \circ (F(A \cap U)) = 0 \in \mathbb{R}^m$. Since for $x \in C_\epsilon^n(0)$, $\psi \circ F \circ \varphi^{-1}(x) = (x^1, \cdots, x^k, 0, \cdots, 0)$ and $A \cap U \subset \varphi^{-1}(C_\epsilon^n(0))$,

$$\forall x \in \varphi(A \cap U),\ x^1 = \cdots = x^k = 0$$

Thus, we can see that next properties are satisfied.

1. $A \subset N$
2. $\varphi(p) = 0$
3. $\varphi(U) = C_\epsilon^n(0)$
4. $\varphi(U \cap A) = \{x \in C_\epsilon^n(0) \,|\, x^1 = \cdots = x^k = 0\}$

Therefore $A = F^{-1}(q)$ is regular submanifold of N with dimension $n - k$.

■

Now, let's get back our focus to the Lie group.

Example 1.8.17 Special Linear Group

Special linear group is denoted by

$$Sl(n,\mathbb{R}) = \{X \in Gl(n,\mathbb{R}) \mid \det X = +1\}$$

Trivially, this is subgroup of $Gl(n,\mathbb{R})$. Claim it is submanifold of G. Let $F : Gl(n,\mathbb{R}) \to \mathbb{R}^*$ is given as $F(X) = \det X$. Since $F(XY) = \det XY = (\det X)(\det Y)$, F is algebraic homomorphism onto $\mathbb{R}^* = Gl(1,\mathbb{R})$. And F is also C^∞ since $\det X$ is given by polynomials in the entries of X. Finally, its rank is constant. Let $A \in Gl(n,\mathbb{R})$ & $a = \det A$. Then since $a \det A^{-1}X = \det X$,

$$F(X) = L_a \circ F \circ L_{A^{-1}}(X)$$

We already know L_a is diffeomorphism, DL_a is non-singular at every points. Thus,

$$\text{rank}\,(DF(X)) = \text{rank}\left(\left[aDF(A^{-1}X)DL_{A^{-1}}(X)\right]\right) = \text{rank}\left(DF(A^{-1}X)\right)$$

for all $A \in Gl(n,\mathbb{R})$. Thus, let $A = X$, then $\text{rank}\,(DF(X)) = \text{rank}\,(DF(I))$ so, constant. And we know

$$Sl(n,\mathbb{R}) = F^{-1}(+1)$$

By *Thm 1.8.16*, $Sl(n,\mathbb{R})$ is regular submanifold of $Gl(n,\mathbb{R})$. Since $Sl(n,\mathbb{R})$ is subgroup & submanifold of $Gl(n,\mathbb{R})$, by *Thm 1.8.7*, $Sl(n,\mathbb{R})$ is also Lie group.

Example 1.8.18 Orthogonal Group

Orthgonal group is denoted by

$$O(n) = \left\{X \in Gl(n,\mathbb{R}) \mid X^T X = I\right\}$$

Let $F : Gl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$ such that $F(X) = X^T X$. For $A \in Gl(n,\mathbb{R})$,

$$F(XA^{-1}) = L_{(A^{-1})^T} \circ R_{A^{-1}} \circ F(X)$$

Since L, R are diffeomorphisms, $\text{rank}\left(DF(XA^{-1})\right) = \text{rank}\,(DF(X))$. Thus, $\text{rank}\,(DF(X))$ is constant. And we can find $O(n) = F^{-1}(I)$, by *thm 1.8.7*, $O(n)$ is Lie group.

Def 1.8.19 Homomorphism

Let $F : G_1 \to G_2$ be an algebraic homomorphism of Lie group G_1, G_2. We shall call F a *homomorphism* of Lie group if F is also a C^∞ mapping.

Example 1.8.20 Det map

For $G_1 = Gl(n, \mathbb{R})$, $G_2 = Gl(1, \mathbb{R})$, the map $F : G_1 \to G_2$ given as $F(X) = \det X$ is homomorphism.

Thm 1.8.21 Fundamental Theorem of Linear Algebra

If $F : G_1 \to G_2$ is a homomorphism of Lie groups, then the rank of F is constant; the kernel is a closed regular submanifold and thus a Lie group; and

$$\dim \ker F = \dim G_1 - \text{rank } F$$

Proof for Thm 1.8.21

Let $a \in G_1$ & $b = F(a)$. Denote by $e_1,\ e_2$ the unit elements of $G_1,\ G_2$ then

$$F(x) = F(aa^{-1}(x)) = bF(a^{-1}x) = L_b \circ F \circ L_{a^{-1}}(x)$$

Then $\text{rank}\,(DF(x)) = \text{rank}\,\left(DF(a^{-1}x)\right)$, so, F has constant rank at anywhere in G_1. Since $\ker F = F^{-1}(e_2)$, by *thm 1.8.16*, $\ker F$ is closed regular submanifold of G_1 with $\dim \ker F = \dim G_1 - \text{rank } F$. By *thm 1.8.7*, $\ker F$ is Lie group.

■

9. The Action of a Lie Group on a Manifold

Def 1.9.1 Group Action

Let G be a group and X set. Then G is said to *act* on X (on the left) if there is a mapping $\theta : G \times X \to X$ satisfying two conditions:

1. If e is the identity element of G, then

$$\theta(e, x) = x \quad \text{for all } x \in X$$

2. If $g_1, g_2 \in G$, then

$$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x) \quad \text{for all } x \in X$$

When G is topological group, X is a topological space, and θ is continuous, the action is called *continuous*. When G is a Lie group, X is a C^∞ manifold, and θ is C^∞, we speak of C^∞ *action*. C^∞ action is a *fortiori continuous*.

Prop 1.9.2 Group Action & Homomorphism

If G acts on a set X, then the map $g \to \theta_g$ is a homomorphism of G into $S(X)$ where $\theta_g(x) = \theta(g, x)$. Conversly, any such homomorphism determines an action with $\theta(g, x) = \theta_g(x)$.

We note that the homomorphism is injective if and only if θ_g being the identity implies that $g = e$. If this is so, we shall call the action *effective*.

By using *prop 1.9.2*, then we can find below things:

X	G	θ	θ_g
Topological Space	Topological Group	Continuous	Homeomorphism
C^∞ manifold	Lie Group	C^∞	Diffeomorphism

Example 1.9.3 Natural Action

Very famous group action is $Gl(n, \mathbb{R})$ on $\mathbb{R}^n$. Let $G = Gl(n, \mathbb{R})$ and $X = \mathbb{R}^n$ and we define $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$ by $\theta(A, x) = Ax$, this satisfies conditions of group actions. And also trivially this action is C^∞.

Def 1.9.4 Orbit, Fixed point, Transitive

Let a group G act on a set M and suppose that $A \subset M$. Then GA denotes the set $\{ga \mid g \in G \text{ and } a \in A\}$. The *orbit* of $x \in M$ is the set Gx. If $Gx = x$, then x is a *fixed point* of G; and if $Gx = M$ for some x, then G said to be *transitive* on M. In this case $Gx = M$ for all x.

Note 1.9.5 Equivalence Relation

As a matter of notation we let G denote a Lie group, M a C^∞ manifold, and we assume a C^∞ action $\theta : G \times M \to M$. We define a relation $\sim$ on M by $p \sim q$ if for some $g \in G$ we have $q = \theta_g(p) = gp$. It is easy to see that $\sim$ is equivalence relation and that the equivalence classes coincide with the orbits of G. Obviously, $p \sim q$ implies that p and q are on the same orbit, so the equivalence class[a] $[p] \subset Gp$. Conversely, if $q \in Gp$, then $p \sim q$ so $Gp \subset [p]$. Therefore, $[p] = Gp$.

We denote by M/G the set of equivalence classes $\{[p] \mid p \in M\}$; it will often be called the *orbit space* of the action.

[a] $[p] = \{q \in M \mid p \sim q\}$

Exercise 1.9.1 : Show that $\sim$ is equivalence relation.

2 Fields

1. Vector Fields

We already defined tangent vector at a point $p \in M$, that is, an element of $T_p(M)$. Now, we extend $T_p(M)$ to TM consisting of all tangent vectors at all points of M,

$$TM = \bigcup_{p \in M} T_p(M)$$

We call this as *tangent bundle* of M. Then vector field X is a function $X : M \to TM$ given as $X(p) = X_p$.

Def 2.1.1 Vector Field

Given a differentiable manifold M, a *vector field* on M is an assignment of a tangent vector to each point in M. More precisely, a vector field X is a mapping from M into the tangent bundle TM so that $\pi \circ X = i_M$ is the identity mapping where π denotes the projection from TM to M.

Now, we can define push forward of vector field.

Def 2.1.2 F-related

If we have a vector field Y on M such that for each $q \in M$ and $p \in F^{-1}(q) \subset N$ we have $F_*(X_p) = Y_q$, then we say that the vector fields X and Y are F*-related* and we write, briefly, $Y = F_*(X)$.

Thm 2.1.3 F-related for Diffeomorphism

If $F : N \to M$ is a diffeomorphism, then each vector field X on N is F-related to a uniquely determined vector field Y on M.

3 Appendix

A. Topology

1. Topological Spaces

Def A.1.1 Topological Space

A *topology* on a set X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$ with the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. Unions of elements of $\mathcal{T}$ belong to $\mathcal{T}$

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcup_{i \in I} U_i \in \mathcal{T}$$

3. Intersections of finitely many elements of $\mathcal{T}$ belong to $\mathcal{T}$. For finite set I,

$$U_i \in \mathcal{T} \text{ for all } i \in I \implies \bigcap_{i \in I} U_i \in \mathcal{T}$$

A *topological space* is a set X together with a topology $\mathcal{T}$ on X. For a topological space $(X, \mathcal{T})$, we call the elements of $\mathcal{T}$ *open subsets* and their complements *closed subsets* of X.

Example A.1.2

1) Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$, is a topology on X, called the *trivial topology*. This is a smallest topology.
2) The power set $\mathcal{P}(X)$ of a set X, is a topology on X, called the *discrete topology*. This is a largest topology.

Exercise A.1: Prove *Example A.2.*

Def A.1.3 Basis

Let $\mathcal{T}$ be a topology on a set X. A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Prop A.1.4 Basis (Comfortable Definition)

A subset $\mathcal{B}$ of a topology $\mathcal{T}$ on a set X is a basis of $\mathcal{T}$ iff, for every $U \in \mathcal{T}$ and $x \in U$, there is a $V \in \mathcal{B}$ with $x \in V \subseteq U$.

Proof is trivial.

Def A.1.5 Neighborhood

Let X be a topological space, $x \in X$. Then $U \subseteq X$ is called a *neighborhood* of x when there is an open set $x \in V \subseteq U$. We denote by $\mathcal{U}(x)$ the set of all neighborhoods of x.

Def A.1.6 Neighborhood Basis

Let X be a topological space and $x \in X$. Then we call a subset $\mathcal{B}(x) \subseteq \mathcal{U}(x)$ a *neighborhood basis* of x if for every neighborhood U of x, there is a $V \in \mathcal{B}(x)$ with $V \subseteq U$.

Def A.1.7 Countability

Let X be a topological space.

- X satisfies the *first countability axiom* and is called *countable* if every point in X admits a countable neighborhood basis.
- X satisfies the *second countability axiom* and is called *second countable* if the topology of X admits a countable basis.

Def A.1.8 Adherent, Interior and Boundary

Let X be a topological space and $Y \subseteq X$. Then $x \in X$ is called

1. *an adherent point* (also sometimes called a *point of closure*) of Y, if every neighborhood of x in X contains a point of Y. The set $\bar{Y}$ of adherent points of Y is called the *closure* of Y
2. *an interior point* of Y if there is a neighborhood of x in X that is contained in Y. The set $\mathring{Y}$ of interior points of Y is called the *interior* of Y
3. *a boundary point* of Y if every neighborhood of x in X contains points of Y and $X \backslash Y$. The set of boundary points of Y is called the *boundary* of Y, here denoted by ∂Y.

2. Continous Maps

Def A.2.1 Continuous

Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a function. We call f *continuous* if $f^{-1}(V) \in \tau$ for all $V \in \tau'$.

Def A.2.2 Continuous at a point

Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a function. We call f *continuous at a point* $x \in X$ if, for every neighborhood V of $f(x) \in Y$, there is a neighborhood U of x with $f(U) \subseteq V$.

Def A.2.3 Homeomorphism

A map $f : X \to Y$ between topological spaces X and Y is called a *homeomorphism* if f is bijective and f and f^{-1} are continuous.

3. Convergence And Hausdorff Spaces

Def A.3.1 Convergence

Let X be a topological space and (x_n) a sequence in X. Then a point $x \in X$ is called a *limit* of the sequence (x_n) if, for every neighborhood $\mathcal{U}(x)$ of x, $\exists n \in \mathbb{N}$ such that $x_m \in \mathcal{U}(x)$, $\forall m \geq n$. We then say that the sequence *converges to* x, and we call the sequence *convergent*.

Def A.3.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Exercise A.2: Prove that Metric spaces are Hausdorff spaces.

Prop A.3.3 Hausdorff and Convergence

Let X be a Hausdorff space. Then limit of sequences in X are unique if they exist.

Exercise A.3: Prove *prop A.3.3*.

Def A.3.4 Regular Hausdorff (T_3)

Let X be a topological space. X is called *regular* if given any point x and closed set C, if $x \notin C$, then there exist a neighborhood $\mathcal{U}(x)$ of x and a neighborhood $\mathcal{U}(C)$ of C such that $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$.

4. Understand Necessarity of Second Countability in Manifold

Thm A.4.1 Paracompact $\simeq$ Partition of unity

Let (X, τ) be a topological space that is T_1 (all points are closed). Then the following are equivalent:

- (X, τ) is paracompact and Hausdorff
- Every open cover of (X, τ) admits a subordinate partition of unity

To prove this, we need a wide background knowledge.

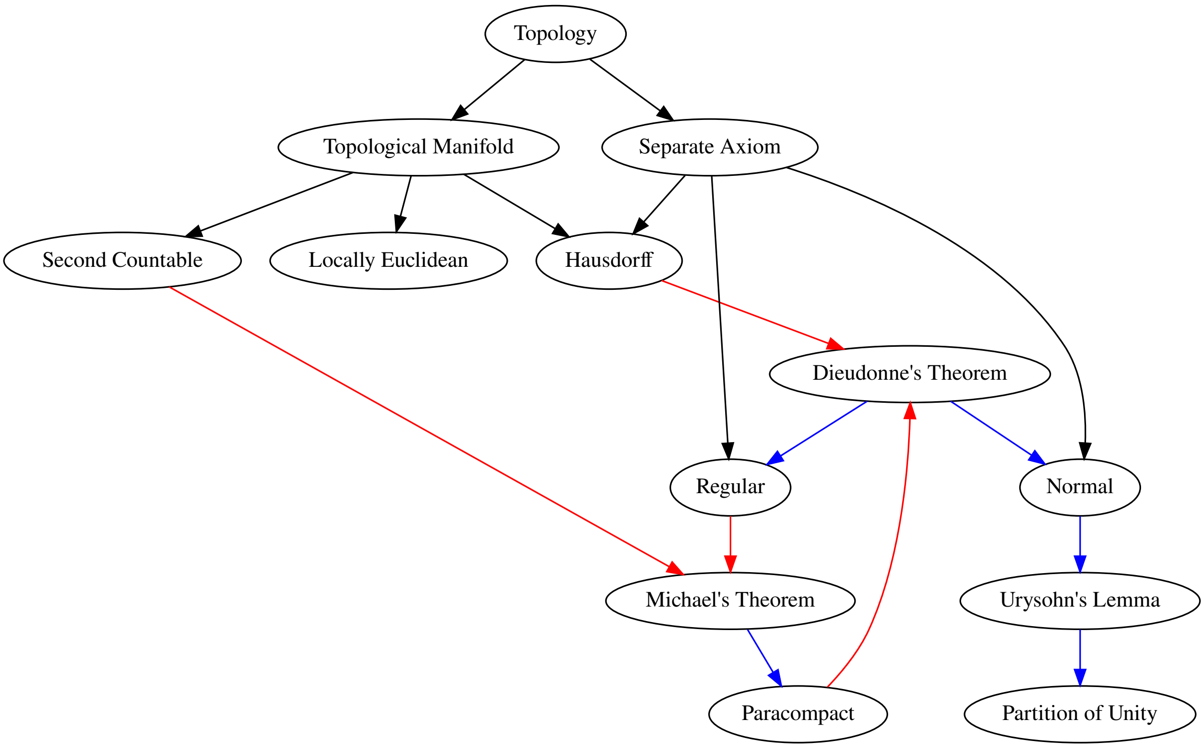


Figure 3.1: Curriculum to prove *Thm A.4.1*

Def A.4.2 Hausdorff

Given points x and y of S, if $x \neq y$, then there exist open neighborhoods U of x and V of y in S that are disjoint: such that $U \cap V = \emptyset$.

Def A.4.3 Locally finite cover

Let (X, τ) be a topological space.
An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if $\forall x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Def A.4.4 Refinement of open covers

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Def A.4.5 Paracompact topological space

A topological space (X, τ) is called *paracompact* if every open cover of X has a refinement by a locally finite open cover.

Def A.4.6 Partition of unity

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity* subordinate to the cover is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i : X \to [0, 1]$$

  (where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology of the real numbers $\mathbb{R}$ regarded as the 1D Euclidean space equipped with its metric topology)

such that with

$$Supp(f_i) := Cl\left(f_i^{-1}((0, 1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

1) $\underset{i \in I}{\forall} (Supp(f_i) \subset U_i)$
2) $\{Supp(f_i) \subset X\}_{i \in I}$ is a locally finite cover
3) $\underset{x \in X}{\forall} \left(\sum_{i \in I} f_i(x) = 1\right)$

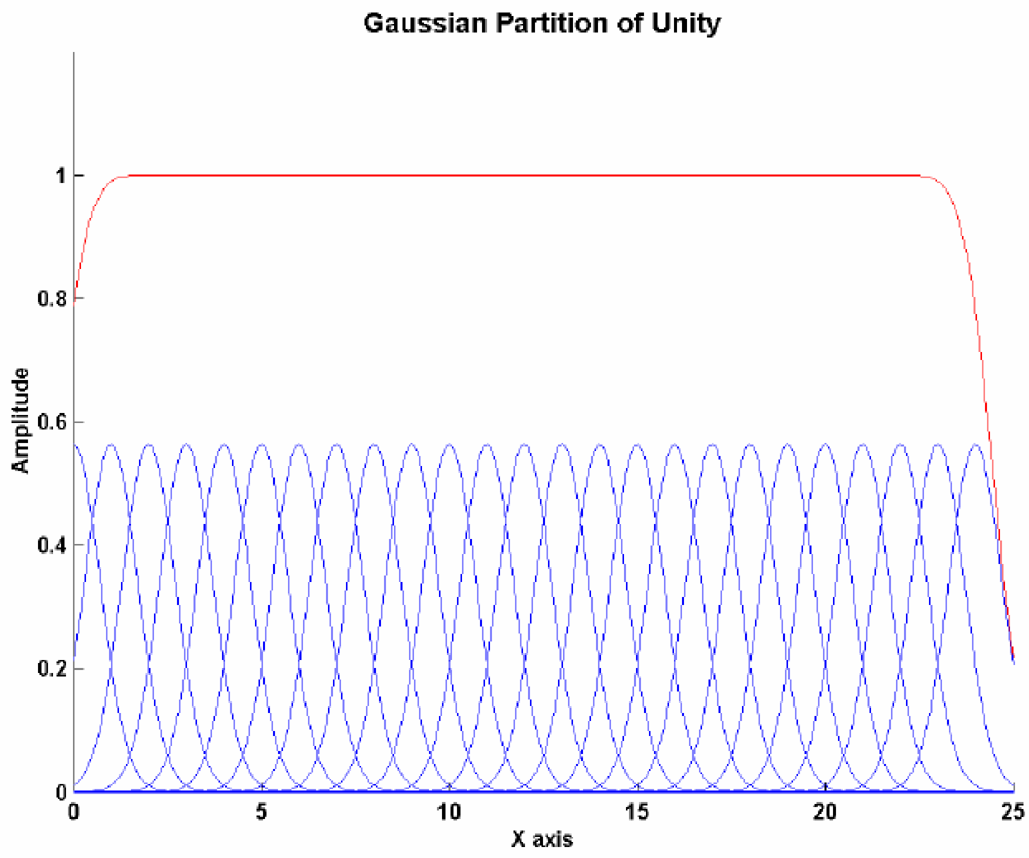


Figure 3.2: Gaussian Partition of Unity

Prop A.4.7 Paracompact - Partition of unity

If (X, τ) is a paracompact topological space, then for every open cover $\{U_i \subset X\}_{i \in I}$ there is a subordinate partition of unity.

Proof will be given later.

Lem A.4.8 Natural Refinement

Let (X, τ) be a topological space, $\{U_i \subset X\}_{i \in I}$ be an open cover and $\left(\phi : J \to I,\ \{V_j \subset X\}_{j \in J}\right)$ be a refinement to a locally finite cover. Then, for $\{W_i \subset X\}_{i \in I}$ with

$$W_i \equiv \left\{ \bigcup_{j \in \phi^{-1}(\{i\})} V_j \right\}$$

is still a refinement of $\{U_i \subset X\}_{i \in I}$ to a locally finite cover.

Proof for A.4.8

First we know, for V, $V_j \subset U_{\phi(j)=i}$. Conversely, $\forall j \in \phi^{-1}(\{i\}),\ V_j \subset U_i$. Thus, $W_i \in \bigcup_{j \in \phi^{-1}(\{i\})} V_j \subset U_i$.

Second, since $\{V_j \subset X\}_{j \in J}$ are locally finite, $\exists\, \mathcal{U}_x \supset \{x\}$ and a finite subset $K \subset J$ such that

$$\underset{j \in J \setminus K}{\forall} (\mathcal{U}_x \cap V_j = \emptyset)$$

(locally finite: $\mathcal{U}_x \cap V_j \neq \emptyset$ for just finite number of $j \in J$)
Then we can get by construction,

$$\underset{i \in I \setminus \phi(K)}{\forall} (\mathcal{U}_x \cap W_i = \emptyset)$$

Since $\phi(K)$ is still finite, we can find the number of i such that $\mathcal{U}_x \cap W_i \neq \emptyset$ is also finite. (If for $i \in K',\ \mathcal{U}_x \cap W_i \neq \emptyset$ then K' should be subset of $\phi(K)$.)

Therefore $\{W_i \subset X\}_{i \in I}$ is locally finite. ■

Lem A.4.9 Shrinking Lemma

Let X be a topological space which is normal and let $\{U_i \subset X\}_{i \in I}$ be a locally finite open cover. Assuming the axiom of choice then:

There exists another open cover $\{V_i \subset X\}_{i \in I}$ such that the topological closure $Cl(V_i)$ of its elements is contained in the original patches:

$$\underset{i \in I}{\forall} (V_i \subset Cl(V_i) \subset U_i)$$

Now, suggest some fundamental topological concepts to prove prop A.4.7.

Def A.4.10 Normal Spaces (T_4)

A topological space X is *normal* if for every two closed disjoint subsets $A, B \subset X$, there are neighborhoods $U \supset A,\ V \supset B$ such that $U \cap V = \emptyset$.

Prop A.4.11 T_4 in terms of topological closure

X is normal iff for all closed subsets $C \subset X$ with open neighborhood $U \supset C$ there exists a smaller open neighborhood $V \supset C$ whose topological closure $Cl(V)$ is still contained in U:

$$C \subset V \subset Cl(V) \subset U$$

Proof for Prop A.4.11

Suppose that (X, τ) is T_4. Consider closed subset $C \subset U$ where U is open neighborhood of C. It implies

$$C \cap X \backslash U = \emptyset$$

Since U is open, $X \backslash U$ is closed. Because of normal space, there are open neighborhoods V, W such that $C \subset V,\ X \backslash U \subset W$ and $V \cap W = \emptyset$. Because of last term, we can find $V \subset X \backslash W \subset U$. Since $X \backslash W$ is closed, we can find next relation :

$$C \subset V \subset Cl(V) \subset X \backslash W \subset U$$

In the other direction, suppose that $\forall$ open neighborhood U of closed subset C, there are smaller open neighborhood with $C \subset V \subset Cl(V) \subset U$. Now, consider disjoint closed subset $C_1, C_2 \subset X$. $C_1 \cap C_2 = \emptyset$ implies $C_1 \subset X \backslash C_2$. Since $X \backslash C_2$ is open neighborhood of C_1, there exists smaller open neighborhood V such that

$$C_1 \subset V \subset Cl(V) \subset X \backslash C_2$$

And it also implies $X \backslash Cl(V)$ is open neighborhood of C_2 where $V \cap X \backslash Cl(V) = \emptyset$. Therefore X is T_4. ■

Def A.4.12 Urysohn function

Let X be a topological space, and let $A, B \subset$ X be disjoint closed subsets. Then an *Urysohn function* for this situation is a continuous function

$$f : X \to [0, 1]$$

to the closed interval equipped with its Euclidean metric topology, such that

$$f(A) = \{0\} \text{ and } f(B) = \{1\}$$

Prop A.4.13 Urysohn's Lemma

Let X be a normal topological space, and let $A, B \subset X$ be two disjoint closed subsets of X. Then there exists an *Urysohn function*.

This lemma has several **big** applications:

- **Urysohn Metrization Thm**: *If X is a normal space with a countable basis, then we can use the abundance of continuous functions from X to $[0, 1]$ to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this we see that every second countable normal space is a metric space.*
- **Tietze Extension Thm**: *Suppose A is a subset of a space X and $f : A \to [0, 1]$ is a continuous function. If X is normal and A is closed in X, then we can find a continuous function from X to $[0, 1]$ that is an extension of f.*
- **Embedding manifolds in $\mathbb{R}^n$**: Using Urysohn's lemma to develop the tool called *partitions of unity*, we can obtain the following theoerem: *Each compact n-manifold is homeomorphic to a subspace of some $\mathbb{R}^n$.*

Then let's start to prove *Urysohn's lemma.*

Proof for Urysohn's lemma

($\Leftarrow$) Suppose $f(A) = \{0\}\,,\; f(B) = \{1\}$ for all closed subset $A, B \subset X$. Then $A \subset f^{-1}\left([0, \frac{1}{2})\right)$ and $D \subset f^{-1}\left((\frac{1}{2}, 1]\right)$. We can find these two sets are open and disjoint.[a] Thus, X is T_4.

($\Rightarrow$) Suppose that X is T_4 and consider two disjoint closed sets $A, B \subset X$. Claim there is *Urysohn function*. To prove this, we should construct continuous function such that $f(A) = \{0\}\,,\; f(B) = \{1\}$. (Maybe it's a little bit tricky.)
Since X is T_4, we can find open neighborhood for any closed subsets of X such that satisfies *prop A.4.11*. Then we can think next idea :

Let $\{U_p\}_{p\in[0,1]\cap\mathbb{Q}}$ be a collection of open sets such that

$$U_1 = X\backslash B,\; A \subset U_0 \subset Cl(U_0) \subset U_1$$

For convenience, denote $Q = [0,1] \cap \mathbb{Q}$. Since Q is countable, we can enumerate it as

$$Q = \{p_n | n \in \mathbb{N}\}$$

To be more clear, we are going to define by induction a collection $\{U_p | p \in Q\}$ of open subsets with the property:

$$p < q \;\Rightarrow\; Cl(U_p) \subset U_q$$

By definition of U_p, we know above property is satisfied when $p = 0,\; q = 1$. Since $Cl(U_0)$ is also subset of X, by *prop A.4.11*, we can construct $\{U_p\}_{p\in Q}$ completely. Also add some conditions ($p \in (-\infty, 0) \cap \mathbb{Q} \;\Rightarrow\; U_p = \emptyset,\; p \in (1, \infty) \cap \mathbb{Q} \;\Rightarrow\; U_p = X$), then we can extend our collection to whole $\mathbb{Q}$. Now, we can define new set:

$$\mathbb{Q}(x) \equiv \{p \in \mathbb{Q} | x \in U_p\}$$

Then we can find $\mathbb{Q}(x)$ has lower bound 0.[b] Since $\mathbb{Q}(x)$ has a greatest lower bound, we can define $f : X \to [0,1]$ by

$$f(x) = \inf \mathbb{Q}(x) = \inf \{p \in \mathbb{Q} | x \in U_p\}$$

If we show f satisfies (① $0 \leq f(x) \leq 1$, ② f *is Urysohn function for* A, B,
③ $x \in Cl(U_p) \Rightarrow f(x) \leq p$, ④ $x \notin U_p \Rightarrow f(x) \geq p$, ⑤ f *is continuous*) then proof is complete.

[a] this will be exercise.
[b] this will be exercise

Proof for Urysohn's lemma (Continued)

① $0 \leq f(x) \leq 1$

: It's trivial because of next property :

$$f(x) = \begin{cases} 1 & \forall p > 1 \\ \text{can't define} & \forall p < 0 \end{cases}$$

② *f is Urysohn function for A, B.*

: Since $A \subset U_0,\ \forall x \in A,\ f(x) = 0$ and $B = X \backslash U_1,\ \forall x \in B,\ f(x) = \inf\{(1, \infty) \cap \mathbb{Q}\} = 1$.

③ *If $x \in Cl(U_p)$, then $f(x) \leq p$*

: Suppose $x \in Cl(U_p)$, then $x \in Cl(U_p) \subset U_q,\ \forall q \in \mathbb{Q},\ q > p$. Thus,

$$(p, \infty) \cap \mathbb{Q} \subset \mathbb{Q}(x) \ \Rightarrow\ \inf \mathbb{Q}(x) \leq p$$

④ *If $x \notin U_p$, then $f(x) \geq p$*

: Suppose $x \notin U_p$, then $x \notin U_q,\ \forall q \leq p$. Thus,

$$(-\infty, p] \cap \mathbb{Q}(x) = \emptyset \ \Rightarrow\ p \leq \inf \mathbb{Q}(x)$$

⑤ *f is continuous.*

: Suppose $U = (a, b) \in \mathbb{R}$ such that $(a, b) \cap [0, 1] \neq \emptyset$. Claim $f^{-1}(U)$ is open. Suppose $x \in f^{-1}(U)$. It means $f(x) \in U = (a, b)$. Since U is open, there are $p, q \in \mathbb{Q}$ such that $a < p < f(x) < q < b$. By ③, ④, we know $x \in U_q \backslash Cl(U_p)$ and $f(U_q \backslash Cl(U_p)) \subset (a, b)$.[c] Thus, we can find $\forall x \in f^{-1}(U)$, there are $p, q \in \mathbb{Q}$ such that $x \in U_q \backslash Cl(U_p) \subset f^{-1}(U)$. Since $U_q \backslash Cl(U_p)$ is open, $f^{-1}(U)$ is open. Therefore, f is continuous.

Proof is complete. ■

[c] this will be exercise.

To prove *prop A.4.7*, we should know relation between *Hausdorff* and *Normal*.

Prop A.4.14 Dieudonné's Theorem

Every paracompact Hausdorff space is normal.

Proof for Dieudonné's Theorem

Consider (X, τ) be a paracompact Hausdorff space.

① First, claim it is regular. To show this, $\forall x \in X$, closed subset $C \subset X$ such that $x \notin C$, there are open neighborhoods $\mathcal{U}(x) \ni x$, $\mathcal{U}(C) \supset C$ such that $\mathcal{U}(x) \cap \mathcal{U}(C) = \emptyset$. Then let's start. Since X is Hausdorff,

$$\forall c \in C,\ \exists \mathcal{U}_c(x) \ni x,\ \mathcal{U}(c) \ni c \text{ such that } \mathcal{U}_c(x) \cap \mathcal{U}(c) = \emptyset$$

We can find $\{\mathcal{U}(c) \subset X\}_{c\in C}$ is an open cover of C, thus $\{\mathcal{U}(c) \subset X\}_{c\in C} \cup X \backslash C$ is an open cover of X. Because of paracompactness of X, every open cover has locally finite refinement. By *lem A.4.8 (Natural refinement)*, if there exists locally finite refinement, then there exists one with the same index set as the original cover. Thus, we can take locally finite refinement $\mathcal{W}(c)$ such that

$$\{\mathcal{W}(c) \subset \mathcal{U}(c) \subset X\}_{c\in C}$$

Since $\mathcal{U}(c)$ is open cover of C and $\mathcal{W}(c)$ is refinement of $\mathcal{U}(c)$, $\bigcup_{c\in C} \mathcal{W}(c)$ is open neighborhood of C. Let it be denoted by $\mathcal{V}(C)$:

$$\mathcal{V}(C) = \bigcup_{c\in C} \mathcal{W}(c)$$

Now, because of locally finiteness of $\mathcal{W}(c)$, $\forall x \in X$, there exists neighborhood $\mathcal{W}(x)$ and finite subset $K \subset C$ such that

$$\underset{c\in C\backslash K}{\forall} (\mathcal{W}(x) \cap \mathcal{W}(c)) = \emptyset$$

Let's take new neighborhood of x as follows :

$$\mathcal{V}(x) \equiv \mathcal{W}(x) \cap \left(\bigcap_{k\in K} \mathcal{U}_k(x) \right)$$

Then we can find

$$\mathcal{V}(x) \cap \mathcal{V}(C) = \emptyset^{a}$$

[a] this will be exercise (refer to ②)

Proof for Dieudonné's Theorem (Continued)

② Claim $(X, \mathcal{T})$ is normal. Then we should prove below proposition:

$$\forall \text{disjoint closed subsets } C, D \subset X,\ \exists \text{disjoint neighborhoods } \mathcal{U}(C),\ \mathcal{U}(D) \in \mathcal{T}$$

By regularity of $(X, \mathcal{T})$, we have next proposition:

$$\forall c \in C,\ \exists \text{disjoint neighborhoods } \mathcal{U}(c) \ni c,\ \mathcal{U}_c(D) \supset D$$

Since $\{\mathcal{U}(c) \subset X\}_{c \in C} \cup X \backslash C$ is an open cover of X and paracompactness of X, we can find locally finite refinement in same index:

$$\{\mathcal{W}(c) \subset \mathcal{U}(c) \subset X\}_{c \in C}$$

Then we can find new open neighborhood of C:

$$\mathcal{V}(C) = \bigcup_{c \in C} \mathcal{W}(c)$$

By locally finiteness of $\mathcal{W}(c)$, $\forall d \in D$, $\exists$ an open neighborhood $\mathcal{W}(d)$ and finite subset $K_d \subset C$ such that

$$\underset{c \in C \backslash K_d}{\forall} (\mathcal{W}(c) \cap \mathcal{W}(d) = \emptyset)$$

So, let take new open neighborhood of $d \in D$,

$$\mathcal{V}(d) = \mathcal{W}(d) \cap \left(\bigcap_{c \in K_d} \mathcal{U}_c(D) \right)^{a}$$

then

$$\mathcal{V}(C) \cap \mathcal{V}(d) = \emptyset, \ \ \forall d \in D^{b}$$

Therefore take new open neighborhood of D as

$$\mathcal{V}(D) \equiv \bigcup_{d \in D} \mathcal{V}(d)$$

then

$$\mathcal{V}(C) \cap \mathcal{V}(D) = \emptyset$$

■

[a] Finite intersection of opensets are open

[b] For $c \in K_d$, $\mathcal{U}(c) \cap \mathcal{U}_c(D) = \emptyset$ and for $c \in X \backslash K_d$, $\mathcal{V}(C) \cap \mathcal{V}(d) = \emptyset$

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