



QFT Study in 2016 Winter

Study Plan

– Main Text & References

Main text book is Peskin & Schroeder. We will deal with whole examples and final projects in Peskin. For references, maybe we will use Schwartz, Nair, Zee, Ryder and Maggiore.

– Specific Plan

- I. **4th week, November - 11/25**
: Preliminaries of QFT - Motivation of QFT, Klein-Gordon equation & SHO.
- II. **5th week, November - 11/29**
: Group Theory (Lorentz to Poincaré), Dirac equation.
- III. **2nd week, December - 12/04, 12/06, 12/09**
: Canonical Quantization of KG & Dirac Field, Weyl & Majorana Spinors, Majorana mass.
- IV. **1st week, January - 1/4, 1/8**
: Review of Spinor Fields.
- V. **2nd week, January - 1/11**
: Introduce Path integral formalism.
- VI. **3rd week, January - 1/17, 1/20**
: Cross section, LSZ reduction, Perturbative Expansion and Feynman Rule.
- VII. **4th week, January - 1/23, 1/26**
: Renormalization & Quantum Electrodynamics.
- VIII. **1st week, February - 1/31, 2/3**
: Rest of Path integral formalism and Renormalization & Implication of Unitarity.
- IX. **2nd week, February - 2/6, 2/10**
: Non-abelian gauge theory, Spontaneous symmetry breaking & Weak interaction.
- X. **3rd week, February - 2/13, 2/17**
: HEP tools (CalcHEP, Madgraph, Feynrules)
- XI. **2nd week, February - 2/20, 2/24**
: Quantum Yang-Mills Theory

The Limit of NRQM

1. Causality Violation

1) Classical Hamiltonian : $H = \frac{P^2}{2m}$

Consider the transitional amplitude - $U(t, x; 0, x_0) \equiv \langle x, t | x_0, 0 \rangle = \langle x | e^{-iHt} | x_0 \rangle$

$$\begin{aligned} U(t, x; 0, x_0) &= \langle x | e^{-i\frac{P^2}{2m}t} | x_0 \rangle = \int d^3p \langle x | e^{-i\frac{P^2}{2m}t} | p \rangle \langle p | x_0 \rangle = \frac{1}{(2\pi)^3} \int d^3p e^{-i\frac{\vec{p}^2}{2m}t} e^{i\vec{p} \cdot (\vec{x} - \vec{x}_0)} \\ &= \frac{1}{(2\pi)^3} \int d^3p e^{-\frac{it}{2m} \left(\vec{p} - \frac{m(\vec{x} - \vec{x}_0)}{t} \right)^2} e^{i\frac{m}{2t}(\vec{x} - \vec{x}_0)^2} = \left(\frac{m}{2\pi it} \right)^{\frac{3}{2}} e^{i\frac{m}{2t}(\vec{x} - \vec{x}_0)^2} \end{aligned} \quad (1)$$

\therefore Since, this expression is nonzero for all x, t , indicating that a particle can propagate between any two points in an arbitrarily short time. \Rightarrow Causality violation.

2) Relativistic Hamiltonian : $H = \sqrt{|\vec{p}|^2 + m^2}$

$$\begin{aligned} U(t, x; 0, x_0) &= \int d^3p \langle x | e^{-i\sqrt{|\vec{p}|^2 + m^2}t} | p \rangle \langle p | x_0 \rangle \\ &= \int \frac{dp}{(2\pi)^3} (2\pi) |\vec{p}|^2 e^{-it\sqrt{|\vec{p}|^2 + m^2}} \int_0^\pi d\theta \sin \theta e^{i|\vec{p}| |\vec{x} - \vec{x}_0| \cos \theta} \\ &= \frac{1}{2\pi^2} \int dp |\vec{p}| \sin(|\vec{p}| |\vec{x} - \vec{x}_0|) e^{-it\sqrt{|\vec{p}|^2 + m^2}} \end{aligned} \quad (2)$$

It can be solved by complex integration¹ or bessel function². Anyway, it is also nonzero although spacelike separated spacetime. \Rightarrow Causality violation.

3) Continuity Equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot S = 0$

In CM, the number of particles is conserved by continuity equation. But in the following cases, the number of particles of given species is not conserved.

1. $n \rightarrow \nu_e^- + e^- + p^+$ \cdots (neutron β - decay)
2. $e^+ + e^- \rightarrow 2\gamma$ \cdots (pair annihilation)

For these reasons, Quantum Field Thoery needs to come out.

¹S. Coleman, *Notes from Sidney Coleman's Physics 253a*

²M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory*

Quantum Field Theory

Let R_1, R_2 be separated. If $A(\in R_1)$ measures $obj(\in R_2)$ then they communicate faster than light, which is causality violation. It is why we should re-define observable. At first, let θ_1, θ_2 be observables at R_1, R_2 respectively. Then they should satisfy $[\theta_1, \theta_2] = 0$. The observables should be attached to each space-time points, because getting information of properties of time and space. That is, the observables are field!

Physics	Observable
Classical Mechanics	Real-valued function
Quantum Mechanics	Operator
Quantum Field Theory	Field Operator

Table 1: Physics and Observable

And then, we define $\phi(x)$ as an operator-valued function of space-time as follows:

- $[\phi(x), \phi(y)] = 0$ if $t^2 - x^2 < 0$ – Space-like separated
- $\phi(x) = \phi^\dagger$ – Hermitian
- $e^{-ipa}\phi(x)e^{ipa} = \phi(x - a)$ – Translation
- $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ – Lorentz transform

Translation could be combined with Lorentz transform, which is called as *Poincaré*.

Now, we are ready to construct QFT. Famous particles & fields are seen in the next table.

Particle	Field
Spin-zero Boson	$\phi(\vec{x}, t)$ – ϕ is a real scalar field
Spin-zero charged Boson	$\phi(\vec{x}, t)$ – ϕ is a complex scalar field
Photons (spin-1 massless boson)	$A_\mu(\vec{x}, t)$ – A_μ is a real vector field
Spin- $\frac{1}{2}$ Fermion (quarks, e^\pm)	$\psi_r(\vec{x}, t)$ – ψ_r is a spinor field

Table 2: Particle and Field

Klein-Gordon Field Theory

1. Single Particle Wave function

: A single particle wave function is marked as $u_k(\vec{x})$.

$$\begin{aligned}
 u_k(\vec{x}) &= Ae^{i\vec{k}\cdot\vec{x}} \quad \text{where} \quad k_i = \frac{2\pi}{L}n_i (i = 1, 2, 3) \\
 u_k(\vec{x}, t) &= e^{-i\omega_k t} Ae^{i\vec{k}\cdot\vec{x}} \\
 &= Ae^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \\
 &= Ae^{-ik_\mu x^\mu} = Ae^{-ik\cdot x} \quad \text{where} \quad k_\mu = (\omega_k, -\vec{k}), \quad x^\mu = (t, \vec{x})
 \end{aligned} \tag{3}$$

And for convenience, we use notation of Peskin :

$$u_p(x) = Ae^{-ip\cdot x} \quad \text{where} \quad p_\mu = (E_p, -\vec{p}), \quad E_p = \sqrt{|\vec{p}|^2 + m^2} \tag{4}$$

By using this equation, Klein-Gordon equation can be derived.

2. Klein-Gordon Equation

$$\begin{aligned}
 i\frac{\partial}{\partial t}u_p(x) &= i\frac{\partial}{\partial t}Ae^{-ip\cdot x} = i(-iE_p)u_p(x) = E_p u_p(x) \\
 -\frac{\partial^2}{\partial t^2}u_p(x) &= (|\vec{p}|^2 + m^2)u_p(x) \\
 \left(\frac{\partial^2}{\partial t^2} + |\vec{p}|^2 + m^2\right)u_p(x) &= \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)u_p(x) = 0
 \end{aligned} \tag{5}$$

$\frac{\partial^2}{\partial t^2} - \nabla^2$ can be written by using *d'Alembertian* of which symbol is \square , which is defined by $\partial_\mu \partial^\mu \equiv \square$.

$$(\square + m^2)u_p(x) = 0 \quad \dots \quad (\text{Klein-Gordon Equation}) \tag{6}$$

3. Invariant Quantity

Consider next two equations :

$$u_p^*(x)(\square + m^2)u_p(x) = 0, \quad u_p(x)(\square + m^2)u_p^*(x) = 0 \tag{7}$$

Let substitute one from another, we can obtain the next equation.

$$\partial_0(u_p^* \partial^0 u_p - \partial^0 u_p^* u_p) - \nabla(u_p^* \partial^\mu u_p - u_p \partial^\mu u_p^*) = 0 \tag{8}$$

In QM, we already knew continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot S = 0$. So, we can interpret our equation by using continuity equation.

$$\rho = i(u_p^* \partial^0 u_p - \partial^0 u_p^* u_p), \quad S = -i(u_p^* \nabla u_p - \nabla u_p^* u_p) \tag{9}$$

Since, volume integration of density is 1 (normalization or probability interpretation) we can define new inner product - *Klein Gordon Inner Product*:

$$\langle u_p | u_{p'} \rangle = i \int d^3x u_p^* \overset{\leftrightarrow}{\partial}_0 u_{p'} \quad (10)$$

4. Normalization

By Klein-Gordon Inner product, we can normalize $u_p(x)$.

$$\langle u_p | u_p \rangle = i \int d^3x u_p^* \overset{\leftrightarrow}{\partial}_0 u_p = 2i \int d^3x \left(-iE_p |A|^2 \right) = 2E_p V |A|^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2E_p V}} \quad (11)$$

$$\therefore u_p(x) = \frac{e^{-ipx}}{\sqrt{2E_p V}} \quad (12)$$

5. Generalization

Let $V \rightarrow \infty$. Then we can derive dirac delta & volume integral by kronecker delta & summation.

$$\begin{aligned} \delta_{pp'} = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{n_3 n'_3} &\Rightarrow \delta^{(3)}(p - p') = \delta^{(3)}\left(\frac{2\pi}{L}(n - n')\right) = \frac{V}{(2\pi)^3} \delta_{pp'} \Rightarrow \delta_{pp'} \rightarrow \frac{(2\pi)^3}{V} \delta^{(3)}(p - p') \\ \sum_{p'} \delta_{pp'} f_{p'} &= f_p \rightarrow \int d^3p' \delta^{(3)}(p - p') f(p') = f(p) \Rightarrow \sum_p \rightarrow \int V \frac{d^3p}{(2\pi)^3} \end{aligned} \quad (13)$$

6. Lorentz Invariant Measure

Consider a Fourier transformation - $\langle x | p \rangle \equiv \psi_p(x)$. We already defined Klein-Gordon Inner product, so,

$$\begin{aligned} \langle p | p' \rangle &= i \int d^3x \psi_p^* \overset{\leftrightarrow}{\partial}_0 \psi_{p'} = i \int d^3x \left(-iE_{p'} e^{i(p-p')x} - iE_p e^{-i(p'-p)x} \right) = 2E_p V \delta_{pp'} \\ \Rightarrow \langle p | p' \rangle &= 2E_p V \delta_{pp'} = 2E_p V \frac{(2\pi)^3}{V} \delta^{(3)}(p - p') = (2\pi)^3 2E_p \delta^{(3)}(p - p') \\ \Rightarrow \delta^{(3)}(p - p') &= \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \\ \int d^3p' \delta^{(3)}(p - p') f(p') &= \int d^3p' \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \langle p' | f \rangle \Rightarrow \langle p | f \rangle = \int d^3p' \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \langle p' | f \rangle \\ \therefore \int d^3p \frac{1}{(2\pi)^3 2E_p} |p\rangle \langle p| &= 1 \end{aligned} \quad (14)$$

But, we can't find any physical significance by just this. Thus, we should change the form.

$$\begin{aligned}
\int \frac{d^3p}{(2\pi)^3 2E_p} &= \int \frac{d^3p}{(2\pi)^3} (2\pi) \int \frac{dp^0}{2\pi} \frac{\delta(p^0 - \sqrt{|\vec{p}|^2 + m^2})}{2p^0} \\
&= \int \frac{d^3p}{(2\pi)^3} (2\pi) \int \frac{dp^0}{2\pi} \delta\left((p^0)^2 - (|\vec{p}|^2 + m^2)\right) \theta(p^0) = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \theta(p^0) \quad (15) \\
\therefore \int \frac{d^3p}{(2\pi)^3 2E_p} &= \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \theta(p^0)
\end{aligned}$$

Then we can interpret this more physically. Since $p^2 = m^2 \rightarrow (p^0)^2 - |\vec{p}|^2 = m^2$, we can think hyperboloid.

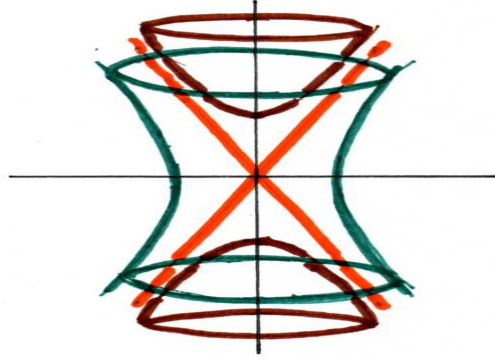


Figure 1: General Hyperboloid

Because of $\theta(p^0)$, Lorentz invariant measure is considered as integration on only positive p^0 surface of Lorentz Hyperboloid.

* Why L.I Measure's name is L.I Measure?

Consider Lorentz transform $p'_3 = \gamma(p_3 + \beta E)$, $E' = \gamma(E + \beta p_3)$.

$$\begin{aligned}
\delta^{(3)}(\vec{p} - \vec{q}) &= \frac{\delta^{(3)}(\vec{p}' - \vec{q}')}{|dp_3/dp'_3|} = \delta^{(3)}(\vec{p}' - \vec{q}') \frac{dp'_3}{dp_3} = \delta^{(3)}(\vec{p}' - \vec{q}') \gamma \left(1 + \beta \frac{dE}{dp_3}\right) \\
&= \delta^{(3)}(\vec{p}' - \vec{q}') \frac{\gamma}{E} (E + \beta p_3) = \delta^{(3)}(\vec{p}' - \vec{q}') \frac{E'}{E} \quad (16) \\
\therefore E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q}) &= E'_{\vec{p}} \delta^{(3)}(\vec{p}' - \vec{q}')
\end{aligned}$$

Thus, $E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q})$ is Lorentz invariant. And therefore, $(2\pi)^3 2E_{\vec{p}} \delta^{(3)}(p - p') = \langle p | p' \rangle$ is Lorentz invariant.

H.W. Show that $\partial_0 \langle u | v \rangle = 0$

Classical Field Theory

Because of special relativity, we use the lagrangian density instead of usual lagrangian.

$$\begin{aligned}
 S &= \int dt L = \int dt \int d^3x \mathcal{L} = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi \right) = \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right) \right] = 0 \\
 \therefore \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} &= 0
 \end{aligned} \tag{17}$$

Then we can also obtain conjugate momentum density & Hamiltonian density.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi(x)}, \quad \mathcal{H}(x) = \pi(x) \partial_0 \phi(x) - \mathcal{L} \quad (H = \int d^3x \mathcal{H}) \tag{18}$$

Lie Group

Lie group is a group whose elements depend in a continuous & differentiable way on a set of real parameters. So, we can consider Lie group as *Differentiable manifold*.

We took notation of Maggiore.

- $g(\theta)$: element of Lie group where θ^α = real parameter. (W.L.O.G $g(0) = e$)
- D_R : Linear operator from Lie group to representation.

From definition of group & these notations, we can get following properties.

- i) $D_R(e) = 1$
- ii) $D_R(g_1)D_R(g_2) = D_R(g_1g_2)$

* There is a property of a Lie group that is independent of the representation. - **Lie Algebra**

1. Generator

Consider infinitesimal θ . Then we can write unitary representation¹ of this group as $D_R(\theta) \simeq 1 + i\theta_a T_R^a$.
 $\Rightarrow T_R^a = -i \left. \frac{\partial D_R}{\partial \theta_a} \right|_{\theta=0}$ T_R^a is called the generator of the representation D_R of group $g(\theta)$.

¹ $D_R(g(\theta)) = \exp i\theta_a T_R^a$

2. Lie Algebra

Let g_1, g_2 be the elements of group g . Then if the group can be represented as unitary, we can write following reps.

$$D_R(g_1) = e^{i\alpha_a T_R^a}, D_R(g_2) = e^{i\beta_a T_R^a} \quad (19)$$

By properties of representation, we can get next equation.

$$D_R(g_1)D_R(g_2) = D_R(g_1g_2) = D_R(g_1g_2) \Rightarrow e^{i\alpha_a T_R^a} e^{i\beta_a T_R^a} = e^{i\delta_a T_R^a} \quad (20)$$

By BCH formula, for matrix A, B , $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$.

$$\begin{aligned} e^{i\alpha_a T_R^a} e^{i\beta_b T_R^b} &= e^{i(\alpha_a + \beta_a) T_R^a - \frac{\alpha_a \beta_b}{2} [T_R^a, T_R^b]} \equiv e^{i\delta_c T_R^c} \\ \Rightarrow \alpha_a \beta_b [T_R^a, T_R^b] &= i [-2\{\delta_c(\alpha, \beta) - \alpha_c - \beta_c\}] \end{aligned} \quad (21)$$

It's easy to verify $\delta(\alpha, \beta)$ is linear for each α, β . ($\delta_c(\alpha, 0) = \alpha_c$, $\delta_c(0, \beta) = \beta_c$) Thus, we can write δ_c as $\delta_c(\alpha, \beta) = \alpha_c + \beta_c + C^{ab}_c \alpha_a \beta_b$ where C^{ab}_c is constant. Thus, the last equation of (21) becomes $\alpha_a \beta_b [T_R^a, T_R^b] = i(-2C^{ab}_c \alpha_a \beta_b) T_R^c \equiv i f^{ab}_c \alpha_a \beta_b T_R^c$. Therefore we get the next relation.

$$\therefore [T^a, T^b] = i f^{ab}_c T^c \quad (22)$$

We called f^{ab}_c as structure constant.

Example 0.1

1. $f^{ij}_k = \epsilon_{ijk}$ for $SO(3)$, $SU(2)$
2. $f^{ab}_c = 0$ for abelian group. (This is the fundamental reason for why we can't quantize a charge of $U(1)$).

* Non-Compact Groups have no unitary representations of finite dimensions

: If a group is non-compact then to identify its generator we need infinite dimensional representations. (Hilbert space of 1-particle state)

Lorentz Group

1. Rotation Group in 3D – SO(3)

Let R be a representation of rotation and r be a n -dimensional vector, then simply, $r \rightarrow r' = R r$ where $R^T R = 1$. Since $(R_1 R_2)^T \cdot R_1 R_2 = R_2^T (R_1^T R_1) R_2 = 1$, it is closed of multiplication. And trivially identity & inverse are exist, R forms a group. We called this group as $O(n)$. (In addition, if $\det R = 1$, then we call it $SO(n)$.)

* Generator

$$R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow R_z(\delta\theta) = \begin{pmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 - i\delta\theta \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv 1 - i\delta\theta J_z \quad (23)$$

Thus, we can find all of generators.

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (24)$$

Then we can write unitary representation of Rotation group.

$$\therefore R(\theta) = e^{-i\vec{\theta} \cdot \vec{J}} \quad \text{where } [J_i, J_j] = i\epsilon_{ijk} J_k \quad (25)$$

2. Unitary Group – SU(2)

Let U be a representation of this group, then it should be satisfied $U^\dagger U = 1$. We called this group as $U(n)$. (In addition, if $\det U = 1$, then we call it $SU(n)$.) Then U can be written as below. (Check!)

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{where } |a|^2 + |b|^2 = 1 \quad (26)$$

We can take $a = e^{-i\alpha} \cos \gamma$, $b = e^{i\beta} \sin \gamma$, thus,

$$\begin{aligned} U(\delta) &= \begin{pmatrix} 1 - i\alpha & -\gamma - i\beta\gamma \\ \gamma - i\beta\gamma & 1 + i\alpha \end{pmatrix} \xrightarrow{\beta\gamma \rightarrow \beta} \alpha^2 + \beta^2 + \gamma^2 = 0 \\ \Rightarrow U(\delta) &= 1 - i\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 1 - i\vec{\eta} \cdot \frac{\vec{\sigma}}{2} \end{aligned} \quad (27)$$

Since σ is pauli matrices, it satisfy $\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i\epsilon^{ijk} \frac{\sigma^k}{2}$ Thus, we can write

$$\therefore U(\eta) = e^{-i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}} \quad (28)$$

3. Lorentz Group

We denote coordinates of Lorentz transformed frame by x' . Then we already knew $x^{0'} = \gamma(x^0 + \beta x^i)$, $x^{i'} = \gamma(x^i + \beta x^0)$ where $\gamma^2 - \beta^2 \gamma^2 = 1$. So, we can substitute $\gamma \equiv \cosh \eta$, $\gamma\beta \equiv \sinh \eta$

$$\text{then } x^{\mu'} = \Lambda^\mu_{\nu'} x^\nu \text{ where } (\Lambda^\mu_{\nu'})_x = \begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For infinitesimal η , we can represent $(\Lambda^\mu_{\nu'})^i = 1 - i\eta K^i$ where

$$K_x = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_z = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (29)$$

For example, commutator of K_x , K_y follows next relation.

$$[K_x, K_y] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -iJ_z \quad (4D \text{ representation}) \quad (30)$$

So, pure Lorentz boost can't form a group \Rightarrow They need rotation to form a group.

We can obtain next commutation relation for J , K .

$$[J_i, J_j] = i\epsilon_{ijk} J_k, [J_i, K_j] = i\epsilon_{ijk} K_k, [K_i, K_j] = -i\epsilon_{ijk} J_k \quad (31)$$

So, there are 3 Boost, 3 Rotation generators forming group. And we call this group as **Lorentz Group**.

Now, consider next two new generators.

$$A = \frac{1}{2}(J + iK), B = \frac{1}{2}(J - iK) \quad (32)$$

Then we can find the commutation relation.

$$[A_i, A_j] = i\epsilon_{ijk} A_k, [B_i, B_j] = i\epsilon_{ijk} B_k, [A_i, B_j] = 0 \quad (33)$$

So, A, B obey $SU(2)$ algebra. Thus the Lorentz group is isomorphic to $SU(2) \otimes SU(2)$. Therefore we can label the angular momentum of Lorentz group as $(2j_+ + 1, 2j_- + 1)$

4. Decomposition of Lorentz tensors under SO(3)

Let j be the angular momentum of tensor representation of SO(3). For j , dimension of the representation is $2j + 1$. (j is integer) Then let notice some representations which have different angular momentum.

- $j = 0$: $\dim = 0 \rightarrow \text{Scalar}$
- $j = 1$: $\dim = 3 \rightarrow \text{Spatial vector}$

For convenience, let ϕ be a scalar then we describe this as $\phi \in \mathbf{0}$. Also, let V^i be a spatial vector then we describe it as $V^i \in \mathbf{1}$.

Now, consider 4-vector. $x^\mu = (x^0, x^i)$ where $x^0 \in \mathbf{0}$, $x^i \in \mathbf{1}$. Thus, we write this as $x^\mu \in \mathbf{0} \oplus \mathbf{1}$. Then let consider the angular momentum of generic tensor $T^{\mu\nu}$.

$$T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) = (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2})^1 \quad (34)$$

This result means, the generic tensor $T^{\mu\nu}$ can be decomposed by 1 scalar, 2 spatial vectors and the tensor which has 9 degrees of freedom. (dimension of $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ is $1 + 3 + 5 = 9$). And it's easy to find what kinds of these elements are - trace, anti-symmetric tensor ($A^{\mu\nu}$) and traceless symmetric tensor ($S^{\mu\nu}$).

- $\text{tr}(T) \in \mathbf{0}$
- $A^{\mu\nu}$: six components. $A^{0i}, \frac{1}{2}\epsilon_{ijk}A^{jk}$ are spatial vectors. $\Rightarrow A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$
- $S^{\mu\nu}$: dimension of $S^{\mu\nu}$ is $(4 - 1) + 6 = 9$ (4 is diagonal, -1 is traceless, 6 is symmetric components)

Therefore we can decompose the second rank generic tensor as trace, anti-symmetric tensor and traceless symmetric tensor.

* How to represent spinor as above notation?

: we know SU(2) has same algebra with SO(3).² So, spinor also has angular momentum j but it is half integer. For $j = \frac{1}{2}$, dimension is 2 and $J^i = \frac{\sigma^i}{2}$ where σ^i is pauli matrices.

Since $\frac{1}{2} \otimes \mathbf{0} = \mathbf{0} \otimes \frac{1}{2} = \frac{1}{2}$, $\frac{1}{2} \otimes \frac{1}{2} = \mathbf{0} \oplus \mathbf{1}$, we can find $\text{SU}(2) \otimes \text{SU}(2) = \text{SO}(1,3)$.

¹For (j_1, j_2) , we can obtain total angular momentum by counting angular momentums between $|j_1 - j_2|$, $j_1 + j_2$.
For example, $\mathbf{0} \otimes \mathbf{0} = \mathbf{0}$, $\mathbf{0} \otimes \mathbf{1} = \mathbf{1}$, $\mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$

²We explain this issue at appendix A

5. Lorentz Algebra

1) Scalar Representation

Since $\vec{L} = \vec{x} \times \vec{p}$, for Quantum mechanically, we can write $L^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$. Then for spacetime, we can write $L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$. We call it as scalar representation of Lorentz group. Trivially, $L^{\mu\nu}$ is antisymmetric. So, there are 6 degrees of freedom: L^{0i} = boost, L^{ij} = rotation. And we can find next commutation relation.

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i(\eta^{\mu\rho} L^{\nu\sigma} - \eta^{\mu\sigma} L^{\nu\rho} - \eta^{\nu\rho} L^{\mu\sigma} + \eta^{\nu\sigma} L^{\mu\rho}) \quad (35)$$

We call it as commutation rule of Lorentz algebra. Then any matrices that are represent this algebra must obey same commutation rules.

2) Vector Representation

Consider $(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i(\eta^{\mu\alpha} \delta^\nu_\beta - \eta^{\nu\alpha} \delta^\mu_\beta)$.¹ Then we can check $\mathcal{J}^{\mu\nu}$ also satisfies the commutation rule of Lorentz algebra. So, $\mathcal{J}^{\mu\nu}$ are also representations of Lorentz group. We call it as the vector representation of Lorentz group. Because we wrote the Lorentz transform of 4-vector as

$$V^\alpha \rightarrow \left(\delta^\alpha_\beta - \frac{i}{2} \omega_{\mu\nu} (\mathcal{J}^{\mu\nu})^\alpha{}_\beta \right) V^\beta \equiv \Lambda^\alpha{}_\beta V^\beta \quad (36)$$

where $\omega_{\mu\nu}$ is anti-symmetric variable. And we can find \mathcal{J}^{0i} are boosts and \mathcal{J}^{ij} are rotations where $\omega_{0i} = \beta$, $\omega_{ij} = \theta$. (It's easy to find. If you have some troubles to deal this, then see Peskin & Schroeder.)

3) Spinor Representation

Consider $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$ where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Then it is also generator of Lorentz group. Thus, $1 - \frac{i}{2} \omega_{\rho\sigma} S^{\rho\sigma}$ is also representation of Lorentz transform. We call it **Spinor Representation** and denote it by $\Lambda_{\frac{1}{2}}$.

We can find next relations:

- Boost : $S^{0j} = \frac{i}{4} [\gamma^0, \gamma^j] = -\frac{i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$
- Rotation : $S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2} \epsilon^{ijk} \Sigma^k$

⁰All proofs of commutation relation of representations are noted at appendix.

¹ μ, ν represent 6 matrices and α, β represent components

A SU(2) & O(3)

Consider two component spinor $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ which is transformed by $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $|a|^2 + |b|^2 = 1$.

Since $\xi \rightarrow U\xi$, $\xi^\dagger \rightarrow \xi^\dagger U^\dagger$, we can find $\xi^\dagger \xi = |\xi_1|^2 + |\xi_2|^2$ is invariant under U . Now, consider transformation of this spinor by specific components.

$$\begin{aligned} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &\rightarrow \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} = \begin{pmatrix} a\xi_1 + b\xi_2 \\ -b^*\xi_1 + a^*\xi_2 \end{pmatrix}, \quad (\xi_1^* \quad \xi_2^*) = (\xi_1^* \quad \xi_2^*) \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = (a^*\xi_1^* + b^*\xi_2^* \quad -b\xi_1^* + a\xi_2^*) \\ \Rightarrow \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} &= \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} \Rightarrow \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = U \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} \end{aligned} \quad (37)$$

Thus, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\tilde{\xi} = \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}$ are states which are transformed by U .¹ So, we can guess there is some relationship between ξ and $\tilde{\xi}$. We denote it \sim . Then now consider ξ^\dagger .

$$\xi^\dagger \sim (\tilde{\xi})^\dagger = (-\xi_2 \quad \xi_1) \Rightarrow \xi \xi^\dagger \sim \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (-\xi_2 \quad \xi_1) = \begin{pmatrix} -\xi_1 \xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_2 \xi_1 \end{pmatrix} \equiv h \quad (38)$$

We know $\xi \xi^\dagger \rightarrow U \xi \xi^\dagger U^\dagger \Rightarrow h \rightarrow U h U^\dagger$, thus, we can find $\det h = \det h'$. Let $h = \vec{\sigma} \cdot \vec{r}$ where $\vec{r} = (x, y, z)$, $\vec{\sigma}$ are pauli matrices.² Then

$$h = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \Rightarrow z = -\xi_1 \xi_2, \quad x = \frac{1}{2} (\xi_1^2 - \xi_2^2), \quad y = \frac{-1}{2i} (\xi_1^2 + \xi_2^2) \quad (39)$$

We know $\det h$ is invariant under this transformation, thus, $\det h = -(x^2 + y^2 + z^2)$ is invariant. \Rightarrow O(3) Transform!

$$\therefore \text{An SU(2) transformation on } \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \text{O(3) transformation on } \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore U(\eta) = e^{-i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}} \longleftrightarrow R(\theta) = e^{-i\vec{\theta} \cdot \vec{J}} \quad (40)$$

Therefore we can conclude the rotation generator of spinor is $\vec{J} = \frac{\vec{\sigma}}{2}$.

¹This procedure is called as *Flip Spin* $\tilde{\xi} = -i\sigma_2 \xi^*$

²It is position vector in SU(2)

B $SL(2, \mathbb{C})$

Consider the next transform for $X = x_\mu \sigma^\mu$ ($\sigma^\mu = (1, \vec{\sigma})$):

$$X \rightarrow X' = SXS^\dagger \quad (41)$$

If $S \in SL(2, \mathbb{C})$, then this transform describes the Lorentz transformation.

Proof :

The definition of $SL(2, \mathbb{C})$ is $\forall A \in SL(2, \mathbb{C}), A_{ij} \in \mathbb{C} \ \& \ \det A = 1$. So, this transform preserves determinant (Symmetric Transform). Then let consider $\det X$.

$$X = x_\mu \sigma^\mu = x^0 \cdot 1 - x^1 \sigma^1 - x^2 \sigma^2 - x^3 \sigma^3 \quad (42)$$

$$= x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - x^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (43)$$

$$= \begin{pmatrix} x^0 - x^3 & -x^1 - ix^2 \\ -x^1 + ix^2 & x^0 + x^3 \end{pmatrix} \quad (44)$$

$$\therefore \det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^2 \quad (45)$$

Since the transform preserves determinant, we can get next relation.

$$x'^2 = x^2 \quad (46)$$

which is same as Lorentz transform.

Now, let find matrix elements of Lorentz transform on this representation.

First, let's see next lemma

Lemma B.1 $x^\mu = \frac{1}{2} \text{tr}(\bar{\sigma}^\mu x_\nu \sigma^\nu)$

