QFT Study in 2016 Winter

Study Plan

– Main Text & References

Main text book is Peskin & Schroeder. We will deal with whole examples and final projects in Peskin. For references, maybe we will use Schwartz, Nair, Zee, Ryder and Maggiore.

– Specific Plan

I. **4th week, November - 11/25**
: Preliminaries of QFT - Motivation of QFT, Klein-Gordon equation & SHO.

II. **5th week, November - 11/29**
: Group Theory (Lorentz to Poincaré), Dirac equation.

III. **2nd week, December - 12/04, 12/06, 12/09**
: Canonical Quantization of KG & Dirac Field, Weyl & Majorana Spinors, Majorana mass.

IV. **1st week, January - 1/4, 1/8**
: Review of Spinor Fields.

V. **2nd week, January - 1/11**
: Introduce Path integral formalism.

VI. **3rd week, January - 1/17, 1/20**
: Cross section, LSZ reduction, Perturbative Expansion and Feynman Rule.

VII. **4th week, January - 1/23, 1/26**
: Renormalization & Quantum Electrodynamics.

VIII. **1st week, Feburary - 1/31, 2/3**
: Rest of Path integral formalism and Renomalization & Implication of Unitarity.

IX. **2nd week, Feburary - 2/6, 2/10**
: Non-abelian gauge theory, Spontaneous symmetry breaking & Weak interaction.

X. **3rd week, Feburary - 2/13, 2/17**
: HEP tools (CalcHEP, Madgraph, Feynrules)

XI. **2nd week, Feburary - 2/20, 2/24**
: Quantum Yang-Mills Theory

The Limit of NRQM

1. Causality Violation

1) Classical Hamiltonian : $H = \dfrac{P^2}{2m}$

Consider the transitional amplitude - $U(t,x;0,x_0) \equiv \langle x,t|x_0,0\rangle = \langle x|\, e^{-iHt}\,|x_0\rangle$

$$\begin{aligned} U(t,x;0,x_0) &= \langle x|\, e^{-i\frac{P^2}{2m}t}\,|x_0\rangle = \int d^3p\ \langle x|\, e^{-i\frac{P^2}{2m}t}\,|p\rangle\,\langle p|x_0\rangle = \frac{1}{(2\pi)^3}\int d^3p\; e^{-i\frac{\vec{p}^{\,2}}{2m}t} \bullet e^{i\vec{p}\cdot(\vec{x}-\vec{x_0})} \\ &= \frac{1}{(2\pi)^3}\int d^3p\; e^{-\frac{it}{2m}\left(\vec{p}-\frac{m(\vec{x}-\vec{x_0})}{t}\right)^2} e^{i\frac{m}{2t}(\vec{x}-\vec{x_0})^2} = \left(\frac{m}{2\pi it}\right)^{\frac{3}{2}} e^{\frac{im}{2t}(\vec{x}-\vec{x_0})^2} \end{aligned} \tag{1}$$

$\therefore$ Since, this expression is nonzero for all x, t, indicating that a particle can propagate between any two points in an arbitrarily short time. $\Rightarrow$ Causality violation.

2) Relativistic Hamiltonian : $H = \sqrt{|\vec{p}\,|^2 + m^2}$

$$\begin{aligned} U(t,x;0,x_0) &= \int d^3p\ \langle x|\, e^{-i\sqrt{|\vec{p}\,|^2+m^2}}\,|p\rangle\,\langle p|x_0\rangle \\ &= \int \frac{dp}{(2\pi)^3}(2\pi)\,|\vec{p}\,|^2\, e^{-it\sqrt{|\vec{p}\,|^2+m^2}} \int_0^{\pi} d\theta \sin\theta\; e^{i|\vec{p}\,||\vec{x}-\vec{x_0}|\cos\theta} \\ &= \frac{1}{2\pi^2}\int dp\,|\vec{p}\,|\sin(|\vec{p}\,|\,|\vec{x}-\vec{x_0}|)e^{-it\sqrt{|\vec{p}\,|^2+m^2}} \end{aligned} \tag{2}$$

It can be solved by complex integration[1] or bessel function[2]. Anyway, it is also nonzero although spacelike separated spacetime. $\Rightarrow$ Causality violation.

3) Continuity Equation : $\dfrac{\partial\rho}{\partial t} + \nabla\cdot S = 0$

In CM, the number of particles is conserved by continuity equation. But in the following cases, the number of particles of given species is not conserved.

1. $n \to \nu_e^- + e^- + p^+$ $\cdots$ (neutron β - decay)
2. $e^+ + e^- \to 2\gamma$ $\cdots$ (pair annihilation)

For these reasons, Quantum Field Thoery needs to come out.

[1] S. Coleman, *Notes from Sidney Coleman's Physics 253a*
[2] M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory*

Quantum Field Theory

Let R_1, R_2 be separated. If A($\in R_1$) measures obj($\in R_2$) then they communicate faster than light, which is causality violation. It is why we should re-define observable. At first, let θ_1, θ_2 be observables at R_1, R_2 respectively. Then they should satisfy $[\theta_1, \theta_2] = 0$. The observables should be attached to each space-time points, because getting information of properties of time and space. That is, the observables are field!

Physics	**Observable**
Classical Mechanics	Real-valued function
Quantum Mechanics	Operator
Quantum Field Theory	Field Operator

Table 1: Physics and Observable

And then, we define $\phi(x)$ as an operator-valued function of space-time as follows:

- $[\phi(x), \phi(y)] = 0 \text{ if } t^2 - x^2 < 0$ – Space-like separated
- $\phi(x) = \phi^\dagger$ – Hermitian
- $e^{-ipa}\phi(x)e^{ipa} = \phi(x - a)$ – Translation
- $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ – Lorentz transform

Translation could be combined with Lorentz transform, which is called as *Poincaré*.
Now, we are ready to construct QFT. Famous particles & fields are seen in the next table.

Particle	**Field**
Spin-zero Boson	$\phi(\vec{x}, t)$ – ϕ is a real scalar field
Spin-zero charged Boson	$\phi(\vec{x}, t)$ – ϕ is a complex scalar field
Photons (spin-1 massless boson	$A_\mu(\vec{x}, t)$ – A_μ is a real vector field
Spin-$\frac{1}{2}$ Fermion (quarks, $e^\pm$)	$\psi_r(\vec{x}, t)$ – ψ_r is a spinor field

Table 2: Particle and Field

Klein-Gordon Field Theory

1. Single Particle Wave function

: A single particle wave function is marked as $u_k(\vec{x})$.

$$\begin{aligned} u_k(\vec{x}) &= Ae^{i\vec{k}\cdot\vec{x}} \quad where \quad k_i = \frac{2\pi}{L}n_i(i=1,2,3) \\ u_k(\vec{x},t) &= e^{-i\omega_k t}Ae^{i\vec{k}\cdot\vec{x}} \\ &= Ae^{-i(\omega_k t-\vec{k}\cdot\vec{x})} \\ &= Ae^{-ik_\mu x^\mu} = Ae^{-ik\cdot x} \quad where \quad k_\mu = (\omega_k, -\vec{k}), \quad x^\mu = (t,\vec{x}) \end{aligned} \tag{3}$$

And for convenience, we use notation of Peskin :

$$u_p(x) = Ae^{-ip\cdot x} \quad where \quad p_\mu = (E_p, -\vec{p}\ ), \quad E_p = \sqrt{|\vec{p}\,|^2+m^2} \tag{4}$$

By using this equation, Klein-Gordon equation can be derived.

2. Klein-Gordon Equation

$$\begin{aligned} i\frac{\partial}{\partial t}u_p(x) = i\frac{\partial}{\partial t}Ae^{-ip\cdot x} &= i(-iE_p)u_p(x) = E_p u_p(x) \\ -\frac{\partial^2}{\partial t^2}u_p(x) &= (|\vec{p}\,|^2+m^2)u_p(x) \\ \left(\frac{\partial^2}{\partial t^2}+|\vec{p}\,|^2+m^2\right)u_p(x) &= \left(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2\right)u_p(x) = 0 \end{aligned} \tag{5}$$

$\frac{\partial^2}{\partial t^2} - \nabla^2$ can be written by using *d'Alembertian* of which symbol is $\Box$, which is defined by $\partial_\mu\partial^\mu \equiv \Box$.

$$(\Box + m^2)u_p(x) = 0 \qquad \cdots \qquad \text{(Klein-Gordon Equation)} \tag{6}$$

3. Invariant Quantity

Consider next two equations :

$$u_p^*(x)(\Box+m^2)u_p(x) = 0, \;\; u_p(x)(\Box+m^2)u_p^*(x) = 0 \tag{7}$$

Let substitute one from another, we can obtain the next equation.

$$\partial_0(u_p^*\partial^0 u_p - \partial^0 u_p^* u_p) - \nabla(u_p^*\partial^\mu u_p - u_p\partial^\mu u_p^*) = 0 \tag{8}$$

In QM, we already knew continuity equation : $\frac{\partial\rho}{\partial t} + \nabla\bullet S = 0$. So, we can interpret our equation by using continuity equation.

$$\rho = i(u_p^*\partial^0 u_p - \partial^0 u_p^* u_p), \;\; S = -i(u_p^*\nabla u_p - \nabla u_p^* u_p) \tag{9}$$

Since, volume integration of density is 1 (normalization or probability interpretation) we can define new inner product - *Klein Gordon Inner Product*:

$$\langle u_p | u_{p'} \rangle = i \int d^3x \; u_p^* \overleftrightarrow{\partial_0} u_{p'} \tag{10}$$

4. Normalization

By Klein-Gordon Inner product, we can normalize $u_p(x)$.

$$\langle u_p | u_p \rangle = i \int d^3x \; u_p^* \overleftrightarrow{\partial_0} u_p = 2i \int d^3x \; \left(-iE_p \, |A|^2 \right) = 2E_p V \, |A|^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2E_p V}} \tag{11}$$

$$\therefore u_p(x) = \frac{e^{-ipx}}{\sqrt{2E_p V}} \tag{12}$$

5. Generalization

Let $V \to \infty$. Then we can derive dirac delta & volume integral by kronecker delta & summation.

$$\delta_{pp'} = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \delta_{n_3 n_3'} \;\Rightarrow\; \delta^{(3)}(p - p') = \delta^{(3)} \left(\frac{2\pi}{L}(n - n') \right) = \frac{V}{(2\pi)^3} \delta_{pp'} \;\Rightarrow\; \delta_{pp'} \to \frac{(2\pi)^3}{V} \delta^{(3)}(p - p')$$

$$\sum_{p'} \delta_{pp'} f_{p'} = f_p \;\to\; \int d^3p' \; \delta^{(3)}(p - p') f(p') = f(p) \;\Rightarrow\; \sum_p \to \int V \frac{d^3p}{(2\pi)^3} \tag{13}$$

6. Lorentz Invariant Measure

Consider a Fourier transformation - $\langle x | p \rangle \equiv \psi_p(x)$. We already defined Klein-Gordon Inner product, so,

$$\begin{aligned}
&\langle p | p' \rangle = i \int d^3x \; \psi_p^* \overleftrightarrow{\partial_0} \psi_{p'} = i \int d^3x \; \left(-iE_{p'} e^{i(p - p')x} - iE_p e^{-i(p' - p)x} \right) = 2E_p V \delta_{pp'} \\
&\Rightarrow \langle p | p' \rangle = 2E_p V \delta_{pp'} = 2E_p V \frac{(2\pi)^3}{V} \delta^{(3)}(p - p') = (2\pi)^3 2E_p \delta^{(3)}(p - p') \\
&\Rightarrow \delta^{(3)}(p - p') = \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \\
&\int d^3p' \; \delta^{(3)}(p - p') f(p') = \int d^3p' \; \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \langle p' | f \rangle \Rightarrow \langle p | f \rangle = \int d^3p' \; \frac{1}{(2\pi)^3 2E_p} \langle p | p' \rangle \langle p' | f \rangle \\
&\therefore \int d^3p \; \frac{1}{(2\pi)^3 2E_p} |p\rangle \langle p| = 1
\end{aligned} \tag{14}$$

But, we can't find any physical significance by just this. Thus, we should change the form.

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3 2E_p} &= \int \frac{d^3p}{(2\pi)^3}(2\pi)\int \frac{dp^0}{2\pi}\frac{\delta(p^0-\sqrt{|\vec{p}\,|^2+m^2})}{2p^0} \\ &= \int \frac{d^3p}{(2\pi)^3}(2\pi)\int \frac{dp^0}{2\pi}\delta\left((p^0)^2-(|\vec{p}\,|^2+m^2)\right)\theta(p^0) = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\theta(p^0) \\ \therefore \int \frac{d^3p}{(2\pi)^3 2E_p} &= \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\theta(p^0)\end{aligned} \tag{15}$$

Then we can interpret this more physically. Since $p^2 = m^2 \ \rightarrow (p^0)^2 - |\vec{p}\,|^2 = m^2$, we can think hyperboloid.

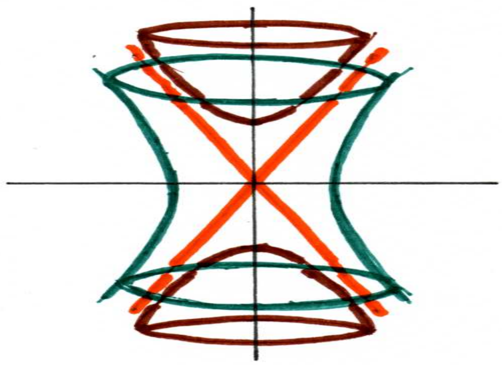

Figure 1: General Hyperboloid

Becaue of $\theta(p^0)$, Lorentz invariant measure is considered as integration on only positive p^0 surface of Lorentz Hyperboloid.

*** Why L.I Measure's name is L.I Measure?**

Consider Lorentz transform $p_3' = \gamma(p_3 + \beta E),\ E' = \gamma(E + \beta p_3)$.

$$\begin{aligned}\delta^{(3)}(\vec{p}-\vec{q}) &= \frac{\delta^{(3)}(\vec{p}\,'-\vec{q}\,')}{|dp_3/dp_3'|} = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{dp_3'}{dp_3} = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\gamma\left(1+\beta\frac{dE}{dp_3}\right) \\ &= \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{\gamma}{E}(E+\beta p_3) = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{E'}{E} \\ &\therefore E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}\,) = E_{\vec{p}}'\ \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\end{aligned} \tag{16}$$

Thus, $E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}\,)$ is Lorentz invariant. And therefore, $(2\pi)^3 2E_{\vec{p}}\,\delta^{(3)}(p-p') = \langle p|p'\rangle$ is Lorentz invariant.

H.W. Show that $\partial_0 \langle u|v\rangle = 0$

Classical Field Theory

Because of special relativity, we use the lagrangian density instead of usual lagrangian.

$$S = \int dt\, L = \int dt \int d^3x \mathcal{L} = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$
$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\partial_\mu\phi \right) = \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \right) \right] = 0 \tag{17}$$
$$\therefore \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

Then we can also obtain conjugate momentum density & Hamiltonian density.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi(x)}, \quad \mathcal{H}(x) = \pi(x)\partial_0\phi(x) - \mathcal{L} \qquad (H = \int d^3x\, \mathcal{H}) \tag{18}$$

Lie Group

Lie group is a group whose elements depend in a continuous & differentiable way on a set of real parameters. So, we can consider Lie group as *Differentiable manifold.*

We took notation of Maggiore.

- $g(\theta)$: element of Lie group where θ^α = real parameter. (W.L.O.G $g(0) = e$)
- D_R : Linear operator from Lie group to representation.

From definition of group & these notations, we can get following properties.

i) $D_R(e) = 1$

ii) $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$

$*$ There is a property of a Lie group that is independent of the representation. - **Lie Algebra**

1. Generator

Consider infinitesimal θ. Then we can write unitary representation[1] of this group as $D_R(\theta) \simeq 1 + i\theta_a T_R^a$.
$\Rightarrow \; T_R^a = -i \left.\frac{\partial D_R}{\partial \theta_a}\right|_{\theta=0}$ T_R^a is called the generator of the representation D_R of group $g(\theta)$.

[1] $D_R(g(\theta)) = \exp i\theta_a T_R^a$

2. Lie Algebra

Let g_1, g_2 be the elements of group g. Then if the group can be represented as unitary, we can write following reps.

$$D_R(g_1) = e^{i\alpha_a T_R^a},\ D_R(g_2) = e^{i\beta_a T_R^a} \tag{19}$$

By properties of representation, we can get next equation.

$$D_R(g_1)D_R(g_2) = D_R(g_1g_2) = D_R(g_1g_2) \ \Rightarrow\ e^{i\alpha_a T_R^a}e^{i\beta_a T_R^a} = e^{i\delta_a T_R^a} \tag{20}$$

By BCH formula, for matrix A, B, $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$.

$$\begin{aligned} e^{i\alpha_a T_R^a}e^{i\beta_b T_R^b} = e^{i(\alpha_a+\beta_a)T_R^a - \frac{\alpha_a\beta_b}{2}\left[T_R^a, T_R^b\right]} \equiv e^{i\delta_c T_R^c} \\ \Rightarrow \alpha_a\beta_b\left[T_R^a, T_R^b\right] = i\left[-2\{\delta_c(\alpha,\beta) - \alpha_c - \beta_c\}\right] \end{aligned} \tag{21}$$

It's easy to verify $\delta(\alpha,\beta)$ is linear for each α, β. ($\delta_c(\alpha, 0) = \alpha_c$, $\delta_c(0,\beta) = \beta_c$) Thus, we can write δ_c as $\delta_c(\alpha,\beta) = \alpha_c + \beta_c + C^{ab}{}_c\alpha_a\beta_b$ where $C^{ab}{}_c$ is constant. Thus, the last equation of (21) becomes $\alpha_a\beta_b\left[T_R^a, T_R^b\right] = i(-2C^{ab}{}_c\alpha_a\beta_b)T_R^c \equiv if^{ab}{}_c\alpha_a\beta_b T_R^c$. Therefore we get the next relation.

$$\therefore \left[T^a, T^b\right] = if^{ab}{}_c T^c \tag{22}$$

We called $f^{ab}{}_c$ as structure constant.

Example 0.1

1. $f^{ij}{}_k = \epsilon_{ijk}$ for SO(3), SU(2)
2. $f^{ab}{}_c = 0$ for abelian group. (This is the fundamental reason for why we can't quantize a charge of U(1).

∗ Non-Compact Groups have no unitary representations of finite dimensions

: If a group is non-compact then to identify its generator we need infinite dimensional representations. (Hibert space of 1-particle state)

Lorentz Group

1. Rotation Group in 3D – SO(3)

Let R be a representation of rotation and r be a n-dimensional vector, then simply, $r \to r' = R\,r$ where $R^T R = 1$. Since $(R_1 R_2)^T \bullet R_1 R_2 = R_2^T (R_1^T R_1) R_2 = 1$, it is closed of multiplication. And trivially identity & inverse are exist, R forms a group. We called this group as $\mathrm{O}(n)$. (In addition, if $\det R = 1$, then we call it $\mathrm{SO}(n)$.)

$*$ **Generator**

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow R_z(\delta\theta) = \begin{pmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 - i\delta\theta \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv 1 - i\delta\theta J_z \tag{23}$$

Thus, we can find all of generators.

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

Then we can write unitary representation of Rotation group.

$$\therefore \;\; R(\theta) = e^{-i\vec{\theta}\bullet\vec{J}} \;\; \text{where } [J_i, J_j] = i\epsilon_{ijk} J_k \tag{25}$$

2. Unitary Group – SU(2)

Let U be a representation of this group, then it shoud be satisfied $U^\dagger U = 1$. We called this group as $\mathrm{U}(n)$. (In addition, if $\det U = 1$, then we call it $\mathrm{SU}(n)$.) Then U can be written as below. (Check!)

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \;\; \text{where } |a|^2 + |b|^2 = 1 \tag{26}$$

We can take $a = e^{-i\alpha}\cos\gamma,\; b = e^{i\beta}\sin\gamma$, thus,

$$\begin{aligned} U(\delta) &= \begin{pmatrix} 1 - i\alpha & -\gamma - i\beta\gamma \\ \gamma - i\beta\gamma & 1 + i\alpha \end{pmatrix} \overset{\beta\gamma\to\beta}{\Longrightarrow} \alpha^2 + \beta^2 + \gamma^2 = 0 \\ \Rightarrow\; U(\delta) &= 1 - i\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 1 - i\eta\bullet\frac{\vec{\sigma}}{2} \end{aligned} \tag{27}$$

Since σ is pauli matrices, it satisfy $\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2}$ Thus, we can write

$$\therefore\; U(\eta) = e^{-i\eta\bullet\frac{\sigma}{2}} \tag{28}$$

3. Lorentz Group

We denote coordinates of Lorentz transformed frame by x'. Then we already knew $x^{0'} = \gamma(x^0 + \beta x^i),\ x^{i'} = \gamma(x^i + \beta x^0)$ where $\gamma^2 - \beta^2\gamma^2 = 1$. So, we can substitute $\gamma \equiv \cosh\eta,\ \gamma\beta \equiv \sinh\eta$ then $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$ where $(\Lambda^\mu{}_\nu)_x = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

For infinitesimal η, we can represent $(\Lambda^\mu{}_\nu)^i = 1 - i\eta K^i$ where

$$K_x = -i\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & & \\ & & & \end{pmatrix},\ K_y = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix},\ K_z = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix} \tag{29}$$

For example, commutator of $K_x,\ K_y$ follows next relation.

$$[K_x, K_y] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & & \\ & 0 & -1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix} = -iJ_z \quad \text{(4D representation)} \tag{30}$$

So, pure Lorentz boost can't form a group $\Rightarrow$ They need rotation to form a group.
We can obtain next commutation relation for $J,\ K$.

$$[J_i, J_j] = i\epsilon_{ijk}J_k,\ [J_i, K_j] = i\epsilon_{ijk}K_k,\ [K_i, K_j] = -i\epsilon_{ijk}J_k \tag{31}$$

So, there are 3 Boost, 3 Rotation generators forming group. And we call this group as **Lorentz Group**.

Now, consider next two new generators.

$$A = \frac{1}{2}(J + iK),\ \ B = \frac{1}{2}(J - iK) \tag{32}$$

Then we can find the commutation relation.

$$[A_i, A_j] = i\epsilon_{ijk}A_k,\ \ [B_i, B_j] = i\epsilon_{ijk}B_k,\ \ [A_i, B_j] = 0 \tag{33}$$

So, A, B obey SU(2) algebra. Thus the Lorentz group is isomorphic to SU(2)$\otimes$SU(2). Therefore we can label the angular momentum of Lorentz group as $(2j_+ + 1,\ 2j_- + 1)$

4. Decomposition of Lorentz tensors under SO(3)

Let j be the angular momentum of tensor representation of SO(3). For j, dimension of the representation is $2j+1$. (j is integer) Then let notice some representations which have different angular momentum.

- $j = 0$: dim $= 0 \rightarrow$ Scalar
- $j = 1$: dim $= 3 \rightarrow$ Spartial vector

For convenience, let ϕ be a scalar then we describe this as $\phi \in \mathbf{0}$. Also, let V^i be a spartial vector then we describe it as $V^i \in \mathbf{1}$.

Now, consider 4-vector. $x^\mu = (x^0, x^i)$ where $x^0 \in \mathbf{0}$, $x^i \in \mathbf{1}$. Thus, we write this as $x^\mu \in \mathbf{0} \oplus \mathbf{1}$. Then let consider the angular momentum of generic tensor $T^{\mu\nu}$.

$$T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) = (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2})^1 \tag{34}$$

This result means, the generic tensor $T^{\mu\nu}$ can be decomposed by 1 scalar, 2 spartial vectors and the tensor which has 9 degrees of freedom. (dimension of $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ is $1 + 3 + 5 = 9$). And it's easy to find what kinds of these elements are - trace, anti-symmetric tensor ($A^{\mu\nu}$) and traceless symmetric tensor ($S^{\mu\nu}$).

- $tr(T) \in \mathbf{0}$
- $A^{\mu\nu}$: six components. A^{0i}, $\frac{1}{2}\epsilon_{ijk}A^{jk}$ are spartial vectors. $\Rightarrow$ $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$
- $S^{\mu\nu}$: dimension of $S^{\mu\nu}$ is $(4-1)+6 = 9$ (4 is diagonal, -1 is traceless, 6 is symmetric components)

Therefore we can decompose the second rank generic tensor as trace, anti-symmetric tensor and traceless symmetric tensor.

$*$ **How to represent spinor as above notation?**

: we know SU(2) has same algebra with SO(3).[2] So, spinor also has angular momentum j but it is half integer. For $j = \frac{1}{2}$, dimension is 2 and $J^i = \frac{\sigma^i}{2}$ where σ^i is pauli matrices.

Since $\mathbf{\frac{1}{2}} \otimes \mathbf{0} = \mathbf{0} \otimes \mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}}$, $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$, we can find SU(2)$\otimes$SU(2) = SO(1,3).

[1]For (j_1, j_2), we can obtain total angular momentum by counting angular momentums between $|j_1 - j_2|$, $j_1 + j_2$. For example, $\mathbf{0} \otimes \mathbf{0} = 0$, $\mathbf{0} \otimes \mathbf{1} = 1$, $\mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$

[2]We explain this issue at appendix A

5. Lorentz Algebra

1) Scalar Representation

Since $\vec{L} = \vec{x} \times \vec{p}$, for Quantum mechanically, we can write $L^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$. Then for spacetime, we can write $L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$. We call it as scalar representation of Lorentz group. Trivially, $L^{\mu\nu}$ is antisymmetric. So, there are 6 degrees of freedom: L^{0i} = boost, L^{ij} = rotation. And we can find next commutation relation.

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i(\eta^{\mu\rho} L^{\nu\sigma} - \eta^{\mu\sigma} L^{\nu\rho} - \eta^{\nu\rho} L^{\mu\sigma} + \eta^{\nu\sigma} L^{\mu\rho}) \tag{35}$$

We call it as commutation rule of Lorentz algebra. Then any matrices that are represent this algebra must obey same commutation rules.

2) Vector Representation

Consider $(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i(\eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta})$. [1] Then we can check $\mathcal{J}^{\mu\nu}$ also satisfies the commutation rule of Lorentz algebra. So, $\mathcal{J}^{\mu\nu}$ are also representations of Lorentz group. We call it as the vector representation of Lorentz group. Because we wrote the Lorentz transform of 4-vector as

$$V^\alpha \;\rightarrow\; \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^{\alpha}{}_{\beta}\right) V^\beta \equiv \Lambda^\alpha{}_\beta V^\beta \tag{36}$$

where $\omega_{\mu\nu}$ is anti-symmetric variable. And we can find $\mathcal{J}^{0i}$ are boosts and $\mathcal{J}^{ij}$ are rotations where $\omega_{0i} = \beta$, $\omega_{ij} = \theta$. (It's easy to find. If you have some troubles to deal this, then see Peskin & Schroeder.)

3) Spinor Representation

Consider $S^{\mu\nu} = \dfrac{i}{4}[\gamma^\mu, \gamma^\nu]$ where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Then it is also generator of Lorentz group. Thus, $1 - \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}$ is also representation of Lorentz transform. We call it **Spinor Representation** and denote it by $\Lambda_{\frac{1}{2}}$.

We can find next relations:

- Boost : $S^{0j} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$
- Rotation : $S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k$

[0] All proofs of commutation relation of representations are noted at appendix.

[1] μ, ν represent 6 matrices and α, β represent components

A SU(2) & O(3)

Consider two component spinor $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ which is transformed by $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $|a|^2 + |b|^2 = 1$. Since $\xi \to U\xi$, $\xi^\dagger \to \xi^\dagger U^\dagger$, we can find $\xi^\dagger \xi = |\xi_1|^2 + |\xi_2|^2$ is invariant under U. Now, consider transformation of this spinor by specific components.

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \to \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} a\xi_1 + b\xi_2 \\ -b^*\xi_1 + a^*\xi_2 \end{pmatrix}, \ \begin{pmatrix} {\xi_1}^{*\prime} & {\xi_2}^{*\prime} \end{pmatrix} = \begin{pmatrix} \xi_1^* & \xi_2^* \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} a^*\xi_1^* + b^*\xi_2^* & -b\xi_1^* + a\xi_2^* \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} {\xi_1}^{*\prime} \\ {\xi_2}^{*\prime} \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} \ \Rightarrow \ \begin{pmatrix} -{\xi_2}^{*\prime} \\ {\xi_1}^{*\prime} \end{pmatrix} = U \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} \tag{37}$$

Thus, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\tilde{\xi} = \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}$ are states which are transformed by U.[1] So, we can guess there is some relationship between ξ and $\tilde{\xi}$. We denote it $\sim$. Then now consider $\xi^\dagger$.

$$\xi^\dagger \sim \left(\tilde{\xi}\right)^\dagger = \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} \ \Rightarrow \ \xi\xi^\dagger \sim \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} = \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_2\xi_1 \end{pmatrix} \equiv h \tag{38}$$

We know $\xi\xi^\dagger \to U\xi\xi^\dagger U^\dagger \Rightarrow h \to UhU^\dagger$, thus, we can find $\det h = \det h'$. Let $h = \vec{\sigma} \bullet \vec{r}$ where $\vec{r} = (x, y, z)$, $\vec{\sigma}$ are pauli matrices. [2] Then

$$h = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \Rightarrow z = -\xi_1\xi_2, \ x = \frac{1}{2}\left(\xi_1^2 - \xi_2^2\right), \ y = \frac{-1}{2i}\left(\xi_1^2 + \xi_2^2\right) \tag{39}$$

We know $\det h$ is invariant under this transformation, thus, $\det h = -(x^2 + y^2 + z^2)$ is invariant. $\Rightarrow$ O(3) Transform!

$$\therefore \text{An SU(2) transformation on } \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \text{O(3) transformation on } \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore U(\eta) = e^{-i\vec{\eta}\cdot\frac{\vec{\sigma}}{2}} \ \longleftrightarrow \ R(\theta) = e^{-i\vec{\theta}\bullet\vec{J}} \tag{40}$$

Therefore we can conclude the rotation generator of spinor is $\vec{J} = \dfrac{\vec{\sigma}}{2}$.

[1] This procedure is called as *Flip Spin* $\tilde{\xi} = -i\sigma_2\xi^*$

[2] It is position vector in SU(2)

B SL(2, $\mathbb{C}$)

Consider the next transform for $X = x_\mu \sigma^\mu$ ($\sigma^\mu = (1, \vec{\sigma})$):

$$X \to X' = SXS^\dagger \tag{41}$$

If $S \in SL(2, \mathbb{C})$, then this transform describes the Lorentz transformation.

Proof :

The definition of $SL(2, \mathbb{C})$ is $\forall A \in SL(2, \mathbb{C})$, $A_{ij} \in \mathbb{C}$ & $\det A = 1$. So, this transform preserves determinant (Symmetric Transform). Then let consider $\det X$.

$$\begin{aligned} X &= x_\mu \sigma^\mu = x^0 \cdot 1 - x^1 \sigma^1 - x^2 \sigma^2 - x^3 \sigma^3 && (42) \\ &= x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - x^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (43) \\ &= \begin{pmatrix} x^0 - x^3 & -x^1 - ix^2 \\ -x^1 + ix^2 & x^0 + x^3 \end{pmatrix} && (44) \\ \therefore \det X &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^2 && (45) \end{aligned}$$

Since the transform preserves determinant, we can get next relation.

$$x'^2 = x^2 \tag{46}$$

which is same as Lorentz transform.

Now, let find matrix elements of Lorentz transform on this representation.

First, let's see next lemma

Lemma B.1 $x^\mu = \frac{1}{2} tr(\bar{\sigma}^\mu x_\nu \sigma^\nu)$

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[ur] \arrow[r] & C \end{tikzcd}$$