EFFECTIVE ACTION IN QUANTUM GRAVITY

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Contents

PART 2

Preface

The investigation of quantum aspects of the gravitational interaction plays a leading role in modern theoretical high energy physics. The questions of the quantum gravity are naturally connected with early universe cosmology and Grand Unification Theories. In principle, all recent achievements of high-energy theoretical physics, namely the discovery of the supergravity models and development of the superstring theory, were inspired by the hope of constructing the quantum gravitational theory.

In spite of numerous efforts the general problems of quantum gravity still remain unsolved. In these conditions the consideration of different quantum gravity models is a necessary stage and may be the only chance to study the quantum aspects of gravitational interaction.

The natural way to construct quantum gravity models is to apply quantum field theory methods to the theories of classical gravitational fields interacting with matter. Under this approach the effective action of the quantum field must be a central object of quantum gravity. Therefore the investigation of quantum gravity models is concerned with the computation of the effective action and the study of its properties.

This book is devoted to some selected problems of quantum gravity where the effective action plays a major role. Certainly, it does not embrace the whole subject of quantum gravity, all its aspects and trends. In our opinion is is impossible to include all the various material on quantum gravity into a single-volume book. The present book reflects the authors' interests and investigations.

The book is intended for researchers in high energy theoretical physics and gravitational theory, as well as the graduate and post-graduate students, specializing in those subjects. We assume that reader has studied standard textbooks on quantum field theory and gravity. Therefore in the cases connected with some well-known quantum field theoretical or general relativistic material we will sometimes use expressions of the type: 'one can show', 'it is easy to see', 'we shall find' and so on. We expect that the reader will understand us.

The book consists of three parts. The first part is pedagogical. It contains a short introduction to the field theoretical models including gravity. Also in this part we give a sufficiently detailed description of the functional methods of quantum field theory. In particular we discuss the broad spectrum of questions concerning the problem of effective action. In our opinion, the content of this part may be of interest for graduate and postgraduate students as some addition (but not a replacement) to a textbook on quantum field theory. Here one should note that there are very extensive references given on the questions which are considered in the first part of the book. To help the reader to find their way among the corresponding books and papers we decided to give commentaries on references in the chapters of this part (and only this part) due to its pedagogical character.

The second part of the book is devoted to the quantum theory of interacting fields in curved space–time (with torsion, generally speaking). Here the consideration is based on the renormalization group, allowing us to research the asymptotics of the effective couplings (including the couplings which describe the interaction with an external gravitational field), to compute the effective action and to find its asymptotic behaviour.

In the third part of the book we consider some problems of quantum gravitational field theory, namely, the quantum theory of higher-derivative gravity and quantum Kaluza–Klein theories. A number of questions concerning the separate gravitational aspects of quantum theory of strings and membranes are also included in this part.

This book is based, in the main, on the authors research. The problems under consideration here have been discussed at different times with many of our collegues, friends and co-authors from Moscow and Tomsk. We are very grateful to A O Barvinsky, E S Fradkin, P M Lavrov, S L Lyakhovich, O K Kalashnikov, S M Kuzenko, A A Tseytlin, I V Tyutin, G A Vilkovisky, B L Voronov and Yu Yu Wolfengaut for their fruitful cooperation and interest in our work.

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PART 1

ELEMENTS OF QUANTUM FIELD THEORY

1 The Basic Models of Quantum Field Theory

1.1 General background

Let us consider the four-dimensional Minkowski space–time with the coordinates $x^\mu; \mu = 0, 1, 2, 3$ and the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. A real or complex function, which is defined on Minkowski space–time is called a field . Usually it is accepted that the fields are the geometrical objects, i.e. are the tensors of some rank or spinors relative to the Lorentz transformations. In quantum field theory it is postulated that the fields are the fundamental physical concepts in terms of which the interactions of the elementary particles are described.

The field theory is characterized by the fundamental physical quantity which is called the action S. The action is the functional of the fields and has the following general form

$$S = \int \mathrm{d}x \, \mathcal{L}(x). \tag{1.1}$$

$\mathcal{L}(x)$ is a real scalar local function of fields and their space–time derivatives and is called the Lagrangian. A model of field theory is chosen if the set of the fields $\Phi(x)$ and the Lagrangian $\mathcal{L}(x)$ are defined. Sometimes we shall speak simply of the theory instead of the model of the field theory.

Usually it is accepted that the Lagrangian contains the field derivatives no higher than the first-order. Recently theories with higher derivatives where the Lagrangians depend on the derivatives

$$\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\Phi \qquad n \geqslant 1 \qquad (\partial_\mu \equiv \partial/\partial x^\mu)$$

became of interest.

The fundamental equations of the classical field theory are the equations of motion. These equations follow from the stationary action principle and have the form

$$\frac{\delta S}{\delta \Phi(x)} = 0. \tag{1.2}$$

The Lagrangians of the simplest field theory models are polynomials of definite degree in the field Φ which also depend on the first derivatives $\partial_\mu \Phi$ not more than quadratically. As regard the field Φ, it is natural to consider tensors of the low ranks, i.e. scalars φ, vectors A_μ, second-rank tensors $B_{\mu\nu}$ and spinors.

Let us consider Φ as one of the fields $\varphi, A_\mu, B_{\mu\nu}$. It is proposed that the quadratic parts in the fields and their derivatives in the Lagrangian must have the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \Phi K^{\mu\nu} \partial_\nu \Phi - \tfrac{1}{2}\Phi M^2 \Phi. \tag{1.3}$$

Here $K^{\mu\nu}, M^2$ are constant matrices in the linear space of the fields Φ. In this case one says that the Lagrangian (1.3) describes the model of a free field. How can we find $K^{\mu\nu}$ and M^2?

Consider the equation of motion for the free field Φ

$$K^{\mu\nu}\partial_\mu \partial_\nu \Phi + M^2 \Phi = 0. \tag{1.4}$$

Let us introduce the Fourier-transformation of field $\Phi(x)$.

$$\Phi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^2} \mathrm{e}^{\mathrm{i}px} \tilde{\Phi}(p).$$

Here $px \equiv x^\mu p_\mu$. Then equation (1.4) takes the form

$$(K^{\mu\nu} p_\mu p_\nu - M^2)\tilde{\Phi}(p) = 0. \tag{1.5}$$

It is well-known that the condition of existence of a solution to a linear homogeneous set of equations can be written as

$$\det(K^{\mu\nu} p_\mu p_\nu - M^2) = 0. \tag{1.6}$$

Relation (1.6) is the basic one for finding the matrices $K^{\mu\nu}$ and M^2. Namely, relation (1.6) should lead to the following expressions

$$p^2 = m_i^2 \qquad i = 1, 2, \ldots . \tag{1.7}$$

Here $p^2 = \eta^{\mu\nu} p_\mu p_\nu$ and m_i are the masses of the particles which are described by the given field Φ. Besides using it for the construction of matrices $K^{\mu\nu}, M^2$, one can use the invariant tensors $\eta_{\mu\nu}$, $\varepsilon_{\mu\nu\alpha\beta}$ and if the fields Φ have the internal indices one can use the invariant tensors connected with these indices. Moreover, one must demand that the energy, corresponding to the Lagrangian (1.3), be positive. However, the above-mentioned conditions are not enough for some free-field theory models. In this case the additional considerations are connected with the use of a specific model.

Usually the first term in the Lagrangian (1.3) is called the kinetic term and the second is called the massive term. The inclusion of the field's interactions in the simplest case is carried by adding the polynomials over the fields and their derivatives of general degree higher than two to the Lagrangian (1.3).

Let us consider briefly, the simple field theory models taking into account first scalars, then vectors and finally second-rank tensors and spinors.

1.2 Models of the scalar field theory

The real scalar field $\varphi(x)$ has neither Lorentzian nor internal indices. The only possible expression for Lagrangian (1.3) in this case has the form

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m^2\varphi^2. \tag{1.8}$$

Equality (1.6) now becomes $p^2 = m^2$ so the Lagrangian (1.8) describes one particle with a mass m.

Including the interaction leads to the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m^2\varphi^2 - V(\varphi). \tag{1.9}$$

Here $\eta^{\mu\nu}$ is the matrix inverse to the matrix $\eta_{\mu\nu}, V(\varphi)$ is the polynomial over the field φ (higher than second degree). $V(\varphi)$ is called the interaction potential. However, renormalizability considerations lead to the fact that the degree of this polynomial is not higher than four.

The simplest generalization of Lagrangian (1.9) is the Lagrangian of the complex scalar field theory $\varphi = \varphi_1 + i\varphi_2$. Here φ_1, φ_2 are real scalar fields. In this case

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi - m^2\varphi^*\varphi - V(\varphi^*, \varphi). \tag{1.10}$$

Lagrangian (1.10) describes two interacting particles with equal masses m. The corresponding free Lagrangian is the sum of the free Lagrangians (1.8) for the scalar fields φ_1 and φ_2.

The more complicated generalization is the theory of multicomponent scalar fields $\varphi^i; i = 1, 2, \ldots, n$. The typical Lagrangian of this theory is

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\delta_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j - \tfrac{1}{2}m^2\delta_{ij}\varphi^i\varphi^j - \tfrac{1}{3!}f_{ijk}\varphi^i\varphi^j\varphi^k \\ - \tfrac{1}{4!}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l.\end{aligned} \tag{1.11}$$

Here the summation convention over the repeated indices is adopted. The values f_{ijk}, f_{ijkl} are called the scalar coupling constants. Degrees higher than fourth are forbidden by the renormalizability condition.

The so-called sigma-model also belongs to the class of scalar-field theories. This theory is described by the n-component scalar field φ^i with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}g_{ij}(\varphi)\partial_\mu\varphi^i\partial_\nu\varphi^j. \tag{1.12}$$

Here $g_{ij}(\varphi)$ is the Riemannian metric on the manifold with the coordinates φ^i. The Lagrangian (1.12) is invariant not only under the Lorentz transformations but under the scalar field φ^i reparametrizations $\varphi'^i = f^i(\varphi)$ and the corresponding transformations of the metric

$$g'_{ij}(\varphi') = \frac{\partial\varphi^k}{\partial\varphi'^i}\frac{\partial\varphi^l}{\partial\varphi'^j}g_{kl}(\varphi).$$

In general this Lagrangian is not a polynomial of finite degree over the fields φ^i.

The equation of motion for the sigma-model may be written in the geometrical terms (as the Lagrangian) and has the form

$$\Box\varphi^i + \Gamma^i_{jk}(\varphi)\eta^{\mu\nu}\partial_\mu\varphi^j\,\partial_\nu\varphi^k = 0. \tag{1.13}$$

Here $\Gamma^i_{jk}(\varphi)$ are the Christoffel symbols, constructing over the metric $g_{ij}(\varphi)$, $\Box \equiv \partial_\mu\partial^\mu$. The corresponding free equation of motion follows from equation (1.13) in the form $\Box\varphi^i = 0$, i.e. corresponds to the n-component massless scalar field theory.

In four-dimensional space–time the sigma-model is not renormalizable and therefore it is not very often used as a quantum field theory. However, the two-dimensional sigma-model is attractive now because it has extensive applications in the string theory. Let us note that one can generalize the sigma-model by introducing the higher derivatives so that the corresponding quantum theory can be formulated in geometrical terms and also be renormalizable.

1.3 Vector field theory models

Let us consider the vector field $A_\alpha(x)$. In accordance with relation (1.3) the kinetic term in the Lagrangian must be

$$\tfrac{1}{2}K^{\mu\nu,\alpha\beta}\,\partial_\mu A_\alpha\,\partial_\nu A_\beta. \tag{1.14}$$

The matrix $K^{\mu\nu,\alpha\beta}$ may be constructed from the invariant tensor $\eta^{\alpha\beta}$. Taking into account the symmetry properties of the expression (1.14) one can find the only possible form for $K^{\mu\nu,\alpha\beta}$

$$K^{\mu\nu,\alpha\beta} = C_1\eta^{\mu\nu}\eta^{\alpha\beta} + C_2(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}).$$

Here C_1, C_2 are arbitrary constants. Using this equality and discarding non-essential total divergences we may write relation (1.14) in the form

$$\tfrac{1}{2}C_1\partial_\alpha A_\beta\partial^\alpha A^\beta + C_2\partial_\alpha A^\alpha\,\partial_\beta A^\beta. \tag{1.15}$$

Let us divide the vector A_α into longitudinal, $A^{\|}_\alpha$, and transverse, $A^{\perp}_\alpha$, parts, that is $A_\alpha = A^{\perp}_\alpha + A^{\|}_\alpha$ and $\partial^\alpha A^{\perp}_\alpha \equiv 0$. After again discarding non-essential total divergences one obtains relation (1.15) in the form

$$\tfrac{1}{2}C_1\partial_\alpha A^{\perp}_\beta\partial^\alpha A^{\perp\beta} + (\tfrac{1}{2}C_1 + C_2)\,\partial_\alpha A^{\|\alpha}\partial_\beta A^{\|\beta}. \tag{1.16}$$

Relation (1.16) shows that the field model with the kinetic term (1.14) is the theory of the two fields, namely the transverse vector field and the longitudinal vector field.

Let us find the Lagrangian describing the transverse vector field $A^{\perp}_\alpha$ only. It may be done if one assumes that $C_2 = -C_1/2$. Returning to relation (1.15) and rejecting non-essential total divergences one can once again obtain

$$\tfrac{1}{2}C_1(\partial_\alpha A_\beta\,\partial^\alpha A^\beta - \partial_\alpha A^\alpha\,\partial_\beta A^\beta) = \tfrac{1}{4}C_1F_{\alpha\beta}F^{\alpha\beta}. \tag{1.17}$$

Here

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.18}$$

Let us choose $C_1 = -1$† in relation (1.17). Then equality (1.17) together with equality (1.18) show that the theory of a free vector

† We can see that by redefining the field A_α one can obtain $|C_1| = 1$. So one must consider the cases $C_1 = \pm 1$ only. However, if $C_1 = 1$ then the energy is not positive.

field based on the above conjecture is a combination of the Maxwell electrodynamics with the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta}. \tag{1.19}$$

Let us note that the model with Lagrangian (1.19) has a remarkable property. Due to equality (1.18) the Lagrangian is invariant under the transformations

$$A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \xi(x). \tag{1.20}$$

Here the scalar field $\xi(x)$ is the parameter of this transformation. The transformation (1.20) is called a gauge transformation. The appearance of such invariance is connected with the fact that the longitudinal part of vector A_α is not contained in the Lagrangian. Hence it can be changed in an arbitrary way without changing of the Lagrangian.

The equation of motion for Lagrangian (1.19) has the form

$$\partial_\nu F^{\mu\nu} = 0. \tag{1.21}$$

Of course, the equation of motion is also invariant under the gauge transformations (1.20). The corresponding relation (1.6) which defines the particle spectrum is a trivial identity in this case. The reason for this is well-known and discussed in standard textbooks on the quantum field theory.

The problem is solved in the following way. One must break the Lagrangian invariance under gauge transformation (1.20). It may be done by using the gauge fixing, for example, by adding to the Lagrangian the terms which depend on the longitudinal part of the vector A_α. Taking into account the relativistic covariance we obtain the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \tfrac{1}{2} C \partial_\alpha A^\alpha \, \partial_\beta A^\beta. \tag{1.22}$$

Here C is an arbitrary constant which is not equal to zero. The physical quantities depend on $A^\perp_\alpha$ only by definition and are not affected by the above-mentioned modification of the Lagrangian.

Relation (1.6) corresponding to Lagrangian (1.22) then has the form

$$\det \left(p^2 \delta^\mu_\nu - (1 - C) p^\mu p_\nu\right) = 0.$$

The computation of this determinant leads to the equation $(p^2)^4 C = 0$. Hence, Lagrangian (1.22) describes four massless particles. However, only two of them are physical ones and correspond to the two polarization states of the photon. The other two, corresponding to the so-called time-like and longitudinal photons, are gauge ones and do not contribute to the physical quantities.

Let us consider the question of the mass term for the vector field. The corresponding modification of the Lagrangian seems very natural.

$$\mathcal{L} = -\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \tfrac{1}{2}m^2 A^\alpha A_\alpha. \tag{1.23}$$

However, the mass term contains the longitudinal part of vector A_α, but the kinetic term depends only on the transversal part of this vector. Can we write the mass term, depending on the transverse part of A_α only? In principle the answer to this question is positive, that is, we can write

$$m^2 A^{\perp}_\alpha A^{\perp\alpha} = m^2 A_\alpha P^{\alpha\beta} A_\beta \equiv m^2 A_\alpha(\eta^{\alpha\beta} - \tfrac{1}{\Box}\partial^\alpha\partial^\beta)A_\beta. \tag{1.24}$$

Here $P^{\alpha\beta} = \eta^{\alpha\beta} - \frac{1}{\Box}\partial^\alpha\partial^\beta$ is a projection operator on the transverse vectors. $\frac{1}{\Box}$ is the operator inverse to the operator $\Box$. But relation (1.24) is not local and that is why it is not suitable for our purposes. So the mass term in the vector-field Lagrangian is incompatible with the gauge invariance†.

Using scalars and vectors one can construct more complicated models. For example, let us write the Lagrangian of scalar electrodynamics.

$$\mathcal{L} = \eta^{\mu\nu}(\partial_\mu + \mathrm{i}eA_\mu)\varphi^*(\partial_\nu - \mathrm{i}eA_\nu)\varphi - m^2\varphi^*\varphi - V(\varphi^*, \varphi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.25}$$

Here e is the electric charge. It is not difficult to check that the Lagrangian (1.25) is invariant under the following gauge transformations

$$A'_\alpha = A_\alpha + \partial_\alpha\xi \qquad \varphi' = \mathrm{e}^{\mathrm{i}e\xi}\varphi \qquad \varphi'^* = \mathrm{e}^{-\mathrm{i}e\xi}\varphi^* \tag{1.26}$$

for ξ on arbitrary scalar field.

There is a general principle, allowing us to obtain the Lagrangian (1.25) starting from the Lagrangian of the complex scalar field (1.10). Expression (1.10) shows that the Lagrangian of the complex scalar field is invariant under the global transformations $\varphi' = \mathrm{e}^{\mathrm{i}e\xi}\varphi, \varphi'^* = \mathrm{e}^{-\mathrm{i}e\xi}\varphi^*$, where ξ is constant. Let us consider the problem of constructing the Lagrangian which is invariant under these transformations with the parameter ξ depending on x. Naturally the mass

† It is interesting to note that in three-dimensional space–time one can introduce the vector-field mass term without conflict with gauge invariance. The corresponding mass term is well-known and has the form $m\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}$, here $\alpha, \beta, \gamma = 0, 1, 2$ and $\varepsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor.

term and interaction potential $V(\varphi^*, \varphi)$ are invariant as before. The sources of the non-invariance are the derivatives $\partial_\mu\varphi, \partial_\mu\varphi^*$. Indeed

$$\partial_\mu\varphi' = \mathrm{e}^{\mathrm{i}e\xi}(\partial_\mu\varphi + \mathrm{i}e\varphi\,\partial_\mu\xi).$$

Let us modify the derivative so that the term $\mathrm{i}e\varphi\partial_\mu\xi$ might be eliminated. We introduce the so-called covariant derivative

$$D_\mu\varphi = \partial_\mu\varphi - \mathrm{i}eA_\mu\varphi \tag{1.27}$$

where A_μ is some vector field. The following relation must hold:

$$(D_\mu\varphi)' = \mathrm{e}^{\mathrm{i}e\xi(x)}D_\mu\varphi. \tag{1.28}$$

Equality (1.28) shows that the field A_α must transform as $A'_\alpha = A_\alpha + \partial_\alpha\xi$. As a result we obtain the gauge transformations (1.26). The expression $\eta^{\mu\nu}D_\mu\varphi^*D_\nu\varphi$ is suitable for the kinetic term in the Lagrangian. However, we have a new field A_α in the model and it is necessary to add kinetic terms depending on this field to the Lagrangian. We know how to write the vector-field Lagrangian and as a result we obtain Lagrangian (1.25).

Let us give some general remarks. First of all it is obvious that the transformations $\varphi' = \mathrm{e}^{\mathrm{i}e\xi}\varphi$, $\varphi'^* = \mathrm{e}^{-\mathrm{i}e\xi}\varphi^*$ form the group $U(1)$ with the generator $T = 1$. Secondly, the covariant derivative may be written as $D_\mu\varphi = \partial_\mu\varphi - \mathrm{i}eA_\mu T\varphi$. And finally $TF_{\mu\nu} = (\mathrm{i}/e)[D_\mu, D_\nu]$. We have demonstrated that a theory which is invariant under the Lie group of global transformations may be modified so that it will be invariant under the same group but with the local parameters. Hence, the covariant derivative and vector field appear in the theory and this vector field is defined up to gauge transformation. The Lagrangian of such a vector field is constructed in terms of the commutator of the covariant derivatives. This is the content of the gauge invariance principle.

We now discuss the gauge invariance principle. Let us have some model of the field theory with the set of the fields Φ^i and Lagrangian $\mathcal{L}(\Phi, \partial_\mu\Phi)$. Suppose that the fields Φ^i are transforming according to the representation of some compact semi-simple Lie group (non-Abelian in general) with the constant parameters $\xi^a; a = 1, 2, \ldots, n$. In this case one says that the group is a global one. Let us assume that the Lagrangian is invariant under this transformation group so $\mathcal{L}(\Phi, \partial_\mu\Phi) = \mathcal{L}(\Phi', \partial_\mu\Phi')$ where $\Phi'^i = h^i_j(\xi)\Phi^j$. The matrixes h^i_j form the group representation and $h(\xi) = \exp(\mathrm{i}g\xi^aT^a)$, here T^a are the representation of this group generators, and g is a constant.

According to the gauge invariance principle one can formulate the theory in such a way that the Lagrangian (action in general) may be invariant under the same group but with the parameters depending on the coordinates. So we have the local transformation

$$\Phi'^i = h^i_j(\xi)\Phi^j \qquad \xi^a = \xi^a(x). \tag{1.29}$$

It is evident that the source of the non-invariance of the Lagrangian under the transformations (1.29) may only be the derivative $\partial_\mu \Phi^i$. For the Lagrangian invariance the equality $\partial_\mu \Phi' = h\partial_\mu \Phi$ must be fulfilled. Really we have

$$\partial_\mu \Phi' = h(\partial_\mu \Phi + h^{-1}\partial_\mu h \Phi).$$

Let us act in analogy with electrodynamics.

First of all we introduce the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu. \tag{1.30}$$

Here $A_\mu(x)$ is the new vector field belonging to the representation of the Lie algebra with the generators T^a. Since the generators T^a are the basis of the representation we shall have $A_\mu = A^a_\mu T^a$ where A^a_μ is the set of the n vector fields.

Let us demand that the derivative D_μ is a covariantly transforming one, that is

$$D'_\mu \Phi' = hD_\mu \Phi. \tag{1.31}$$

Relation (1.31) leads to the transformation law for the A_μ field

$$A'_\mu = hA_\mu h^{-1} + (\mathrm{i}/g)h\,\partial_\mu h^{-1}. \tag{1.32}$$

The vector field A_μ, introduced by means of the covariant derivative (1.30), is called a Yang–Mills gauge field. It is easy to show that the set of the transformations (1.29, 1.32) forms the group, which is called the group of the gauge transformations.

Now one can construct the Lagrangian which is invariant under the gauge transformations (1.32). For this we must replace the usual derivative $\partial_\mu \Phi$ by the covariant derivative $D_\mu \Phi$ in the Lagrangian $\mathcal{L}(\Phi, \partial_\mu \Phi)$. As a result we shall obtain the Lagrangian $\mathcal{L}(\Phi, D_\mu \Phi)$ which is invariant under the transformation (1.32). Let us note that the interaction of the initial fields Φ with the new vector field has appeared. So the gauge invariance principle is a method of introducing interactions into the theory.

As we have introduced the new field A_μ we must find the corresponding Lagrangian for it. We shall act in an analogous way to electrodynamics. Let us compute the commutator of the covariant derivatives,

$$[D_\mu, D_\nu] = -\mathrm{i}g\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]\Big)$$

and let us denote

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{1.33}$$

Then

$$[D_\mu, D_\nu] = -igG_{\mu\nu}.$$

The expression $G_{\mu\nu} in(1.33)$ is called the Yang–Mills field strength tensor. It is easy to show that the tensor $G_{\mu\nu}$ transforms under the gauge transformations (1.32) as follows

$$G'_{\mu\nu} = hG_{\mu\nu}h^{-1}. \tag{1.34}$$

Since $A_\mu = A^a_\mu T^a$ and the generators T^a form a Lie algebra representation we can write

$$[T^a, T^b] = if^{cab}T^c \tag{1.35}$$

where f^{abc} are structure constants of the group. Let us note that for the semi-simple groups f^{abc} are totally antisymmetric. Using the relations (1.33, 1.35) one can show that

$$\begin{aligned} G_{\mu\nu} &= G^a_{\mu\nu}T^a \\ G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \end{aligned} \tag{1.36}$$

Consider

$$\mathrm{Tr}\, G'_{\mu\nu}G'^{\mu\nu} = \mathrm{Tr}\, hG_{\mu\nu}h^{-1}hG^{\mu\nu}h^{-1} = \mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu}.$$

This relation shows that the $\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu}$ is invariant under the transformations (1.32). On the other hand one can write

$$\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} = G^a_{\mu\nu}G^{b\mu\nu}\, \mathrm{Tr}\, T^aT^b.$$

The normalization condition $\mathrm{Tr} T^aT^b \propto \delta^{ab}$ is fulfilled for the generators of the compact simple groups. Therefore, $\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} \propto G^a_{\mu\nu}G^{a\mu\nu}$. The last expression is used as the Yang–Mills field Lagrangian.

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}. \tag{1.37}$$

The total Lagrangian of the initial fields Φ^i and the gauge field A^a_μ which is invariant under the gauge transformations (1.29, 1.32) has the form

$$\mathcal{L}(\Phi, D_\mu\Phi) + \mathcal{L}_{\mathrm{YM}}.$$

Let us also write the infinitesimal form of the gauge transformations (1.32):

$$\begin{aligned} \delta A^a_\mu &= D^{ab}_\mu \xi^b \\ D^{ab}_\mu &= \delta^{ab}\partial_\mu + g f^{acb} A^c_\mu . \end{aligned} \tag{1.38}$$

This relation will be used below.

Consider the construction of the field models for second-rank tensor $B_{\mu\nu}$. It is well-known that the arbitrary tensor $B_{\mu\nu}$ may be written in the form $B_{\mu\nu} = h_{\mu\nu} + \omega_{\mu\nu}$, where $h_{\mu\nu}$ is the symmetric tensor and $\omega_{\mu\nu}$ is the antisymmetric one. The tensors $h_{\mu\nu}, \omega_{\mu\nu}$ transform independently under the Lorentz transformation and therefore it is natural to construct the independent Lagrangians for them.

1.4 The theory of the antisymmetric second-rank tensor field

The antisymmetric tensor field $\omega_{\mu\nu}$ satisfies the evident relation $\omega^\mu_\mu = 0$ and has six independent components (in every point). For the Lagrangian construction we shall use the same analogy as in electrodynamics. Let us introduce the projection operators P^μ_ν and $Q^\mu_\nu = \delta^\mu_\nu - P^\mu_\nu$ and divide the tensor $\omega_{\mu\nu}$ on the longitudinal and transverse parts over the every index. Then

$$\omega_{\mu\nu} = \omega^{\perp\perp}_{\mu\nu} + \omega^{\perp\|}_{\mu\nu} + \omega^{\|\perp}_{\mu\nu}. \tag{1.39}$$

Here

$$\partial^\mu \omega^{\perp}_{\mu\nu} = 0.$$

Let us note that the term $\omega^{\|\|}_{\mu\nu}$ is absent in relation (1.39) because this term is equal to zero.

Assume that the Lagrangian was dependent on $\omega^{\perp\perp}_{\mu\nu}$ only. This means that if this Lagrangian is written in terms of the initial field $\omega_{\mu\nu}$ it must be invariant under the addition to $\omega_{\mu\nu}$ of the arbitrary antisymmetric tensor, which is longitudinal in any of its indices. It means that the theory of the antisymmetric tensor field $\omega_{\mu\nu}$ contains the gauge transformation

$$\omega'_{\alpha\beta} = \omega_{\alpha\beta} + \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha \tag{1.40}$$

where $\xi_\alpha(x)$ is the arbitrary vector field.

Let us introduce the field strength $F_{\alpha\beta\gamma}$ by the rule

$$F_{\alpha\beta\gamma} = \partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta}. \tag{1.41}$$

It is easy to check that the tensor $F_{\alpha\beta\gamma}$ is invariant under the transformations (1.40).

Now the Lagrangian construction is obvious. It has the form

$$\mathcal{L} = -\tfrac{1}{12} F_{\alpha\beta\gamma} F^{\alpha\beta\gamma}. \tag{1.42}$$

The equation of motion corresponding to Lagrangian (1.42) may be written as

$$\partial_\gamma F^{\alpha\beta\gamma} = 0. \tag{1.43}$$

It is evident that the equation of motion is also gauge invariant.

The gauge transformations in the antisymmetric second-rank tensor field theory have an interesting property which is absent in the electrodynamics and Yang–Mill's field theory. Let us represent transformation (1.40) in the form

$$\delta\omega_{\alpha\beta}(x) = \int \mathrm{d}y R^{\mu}_{\alpha\beta}(x,y)\,\xi_\mu(y) \tag{1.44}$$

where

$$R^{\mu}_{\alpha\beta}(x,y) = (\delta^{\mu}_{\beta}\partial_\alpha - \delta^{\mu}_{\alpha}\partial_\beta)\delta(x-y). \tag{1.45}$$

Here the derivatives act on the first argument of the delta-function. The expressions $R^{\mu}_{\alpha\beta}(x,y)$ are called the generators of gauge transformations (1.40). We have from the relation (1.45)

$$\int \mathrm{d}y R^{\mu}_{\alpha\beta}(x,y)\partial_\mu \delta(y-z) \equiv 0.$$

This relation may be written in a shorter form as

$$R^{\mu}_{\alpha\beta}\partial_\mu \equiv 0. \tag{1.46}$$

Equality (1.46) shows that the generators of the gauge transformations (1.40) are linearly dependent.

The reason for the linear dependence of the generators (1.45) may be easily explained. Really the parameters ξ_α in (1.40) are non-unique, but with accuracy up to the transformation

$$\xi'_\alpha = \xi_\alpha + \partial_\alpha \xi \tag{1.47}$$

where $\xi(x)$ is the arbitrary scalar field.

Let us consider the question of the spectrum of the particles described by the field $\omega_{\mu\nu}$ with Lagrangian (1.42). Here it is impossible to use equation (1.6) due to the gauge freedom in the definition of the field $\omega_{\mu\nu}$. As in electrodynamics, it is necessary to break the gauge invariance of Lagrangian (1.42) by adding to it terms which are not invariant under the transformations (1.40) and taking into account the relation (1.47). Then it is necessary to use equation (1.6) and to decide which particles are physical.

We shall, however, choose another way. Let us show explicitly that the theory with Lagrangian (1.42) is classically equivalent to the theory of the massless free scalar field. Consider the model of the field theory, containing the scalar field φ and the vector field V_μ with the following action

$$S = \int \mathrm{d}x(\partial_\mu \varphi V^\mu - \tfrac{1}{2} V^\mu V_\mu). \tag{1.48}$$

The equations of motion corresponding to this theory are

$$\begin{aligned} V_\mu &= \partial_\mu \varphi \\ \partial_\mu V^\mu &= 0. \end{aligned} \tag{1.49}$$

By substituting the first of equations (1.49) into the second we obtain

$$\Box \varphi = 0$$

which is the free massless scalar field equation of motion. If the first of the equations in (1.49) is substituted into the action (1.48), we obtain the standard action of the free massless scalar field theory. Therefore the action (1.48) is the free massless scalar field theory action in the first-order formalism (the action is a linear one over the first-order derivatives only).

From the other side the solution of the second of the equations (1.49) has the general form

$$V^\mu = \varepsilon^{\mu\nu\alpha\beta} \partial_\nu \omega_{\alpha\beta} \tag{1.50}$$

where $\omega_{\alpha\beta}$ is some antisymmetric second-rank tensor field. By substituting (1.50) into the first of equations (1.49) we get

$$\varepsilon^{\mu\nu\alpha\beta} \partial_\nu \omega_{\alpha\beta} = \partial^\mu \varphi. \tag{1.51}$$

It is evident that equation (1.51) is consistent if and only if $\Box \varphi = 0$.

Equation (1.51) leads (together with relation (1.41)) to

$$F^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma\delta} \partial_\delta \varphi \tag{1.52}$$

where φ satisfies the equation $\Box \varphi = 0$. Then $\partial_\gamma F^{\alpha\beta\gamma} = 0$, that is the equation of motion of the field $\omega_{\mu\nu}$. So the equation for $\Box \varphi$ leads to equation (1.43).

On the other hand, if equation (1.43) is fulfilled,

$$F^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma\delta}\partial_\delta\varphi$$

where φ is some scalar field and we have

$$\partial_\mu\varphi \propto \varepsilon_{\mu\alpha\beta\gamma}F^{\alpha\beta\gamma}$$

so that $\Box\varphi = 0$. Hence equation (1.43) leads to the equation $\Box\varphi = 0$. Therefore the theory of the field $\omega_{\mu\nu}$ and the theory of the free massless scalar field are classically equivalent. If we substitute expression (1.50) into action (1.48), we can obtain the action of the field $\omega_{\mu\nu}$ with Lagrangian (1.42). The equivalence of these theories means that they have the same particle spectrum. Therefore the theory of the antisymmetric second-rank tensor field describes just one massless particle.

How can we generalize Lagrangian (1.42)? We can add the mass term $m^2\omega_{\alpha\beta}\omega^{\alpha\beta}$ but this breaks the gauge invariance and we shall not consider this case. The inclusion of the interaction may be done in the usual way in the supergravity models. Let us also note that there is an antisymmetric second-rank tensor field model containing non-Abelian gauge invariance but its consideration is beyond the scope of this book.

1.5 The theory of the symmetric second-rank tensor field

In order to find the free Lagrangian of a symmetric second-rank tensor field $h_{\mu\nu}(x)$ we shall follow the way which is analogous to constructing the vector field and antisymmetric second-rank tensor field theories.

Let us represent the tensor $h_{\mu\nu}$ as the sum of the tensors of the same rank, which are either longitudinal or transverse over every index

$$h_{\mu\nu} = h^{\perp\perp}_{\mu\nu} + h^{\perp\|}_{\mu\nu} + h^{\|\perp}_{\mu\nu} + h^{\|\|}_{\mu\nu} \qquad \partial^\mu h^{\perp}_{\mu\nu} = 0. \tag{1.53}$$

Let the Lagrangian have the structure (1.3) with $M^2 = 0$ but only be dependent on the field $h^{\perp\perp}_{\mu\nu}$. If we rewrite such a Lagrangian in terms of the initial field $h_{\mu\nu}$, it is easy to see that this Lagrangian must be invariant under the shift of $h_{\mu\nu}$ on the arbitrary symmetric second-rank tensor which is longitudinal over the every index. It means that the theory of the free massless symmetric second-rank tensor includes the gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{1.54}$$

where $\xi_\mu(x)$ is the arbitrary vector field.

A free Lagrangian should have the general structure (1.3)

$$\tfrac{1}{2} K^{\mu\nu,\alpha\beta,\gamma\delta} \partial_\mu h_{\alpha\beta} \partial_\nu h_{\gamma\delta} \tag{1.55}$$

and be invariant under transformations (1.54). One can construct the matrix $K^{\mu\nu,\alpha\beta,\gamma\delta}$ from the invariant tensors $\eta^{\mu\nu}$. Taking into account, that $K^{\mu\nu,\alpha\beta,\gamma\delta}$ has a symmetry which is dictated by the form of expression (1.55). This circumstance leads to the following Lagrangian

$$\mathcal{L} = \tfrac{1}{4}(\eta^{\mu\nu} \partial_\mu h_{\alpha\beta}\, \partial_\nu h^{\alpha\beta} - 2\partial_\mu h^\mu_\alpha\, \partial_\beta h^{\alpha\beta} + 2\partial_\mu h^\alpha_\alpha\, \partial_\nu h^{\mu\nu} - \eta^{\mu\nu} \partial_\mu h^\alpha_\alpha\, \partial_\nu h^\beta_\beta). \tag{1.56}$$

We can show that Lagrangian (1.56) corresponds to the linearized approximation in Einstein gravity. It is well-known that the action of the Einstein gravitation theory may be written in the form

$$S = -\frac{1}{2\kappa^2} \int \mathrm{d}x \sqrt{-g} R. \tag{1.57}$$

Here $g_{\mu\nu}$ is the four-dimensional Riemannian manifold metric with the signature equal to -2, $g \equiv \det g_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature. The Ricci tensor $R_{\mu\nu}$ is defined as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} \tag{1.58}$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols, corresponding to the metric $g_{\mu\nu}$. $g^{\mu\nu}$ is the matrix which is the inverse of the matrix $g_{\mu\nu}$ and κ^2 is the gravitational constant.

Let us represent $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and expand action (1.57) in a Taylor series over the $h_{\mu\nu}$ taking into account the quadratic terms only. As a result we obtain Lagrangian (1.56). Let us note that in this case the gauge transformations (1.54) represent the linearized form of gauge transformations for Einstein gravity.

Thus the symmetric second-rank tensor field theory is connected with the description of the gravitational interaction. It is attractive to formulate the symmetric second-rank tensor field theory in the framework of the unified gauge approach. There are various variants of the construction of the Einstein gravity in the gauge field theory formalism. We shall consider one such variant.

The gauge approach is based on localization of some global transformation group. Since gravity is connected with space–time properties it is natural to start with the symmetry group of the space–time transformations of Minkowski space, that is the Poincaré group. The corresponding Poincaré algebra contains 10 generators P_a, M_{ab}; $a, b = 0, 1, 2, 3$. Here P_a are the generators of the space–time translations and M_{ab} are the generators of the Lorentzian rotations. The algebra of these generators has the form

$$\begin{aligned}
[P_a, P_b] &= 0 \\
[M_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b \\
[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad} - \eta_{ad}M_{bc}.
\end{aligned} \tag{1.59}$$

Let us begin to construct the gauge theory, corresponding to the Poincaré algebra (1.59). Let us introduce the covariant derivatives

$$\begin{aligned}
D_\mu &= \partial_\mu - e^a_\mu P_a - \tfrac{1}{2}\omega^{ab}_\mu M_{ab} \\
\omega^{ab}_\mu &= -\omega^{ba}_\mu .
\end{aligned}$$

Owing to these relations, the theory under consideration must contain ten vector fields $e^a_\mu(x)$, $\omega^{ab}_\mu(x)$.

Let us calculate the commutator of the covariant derivatives taking into account Poincaré algebra (1.59). As a result we get

$$[D_\mu, D_\nu] = -R^a_{\mu\nu}P_a - \tfrac{1}{2}R^{ab}_{\mu\nu}M_{ab}. \tag{1.60}$$

Here we used the notation

$$\begin{aligned}
R^a_{\mu\nu} &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + e_{\mu b}\omega^{ab}_\nu - e_{\nu b}\omega^{ab}_\mu \\
R^{ab}_{\mu\nu} &= \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu .
\end{aligned} \tag{1.61}$$

$R^a_{\mu\nu}, R^{ab}_{\mu\nu}$ may be called the strength tensors of the fields e^a_μ, ω^{ab}_μ or the curvature tensors corresponding to the generators P_a, M_{ab}.

Let us find the infinitesimal form of the gauge transformations. The general rule for the covariant derivative transformation follows from relation (1.61) as $D'_\mu = hD_\mu h^{-1}$, where the group element in our case is $h = \exp(\xi^a P_a + \frac{1}{2}\lambda^{ab}M_{ab})$ and $\xi^a(x), \lambda^{ab}(x)$ are the local parameters of the transformation and $\lambda^{ab} = -\lambda^{ba}$. In the infinitesimal form we have $h = 1 + \xi^a P_a + \frac{1}{2}\lambda^{ab}M_{ab}$. Then using direct calculation we obtain

$$\begin{aligned}
e'^{\,a}_\mu &= e^a_\mu + \delta e^a_\mu \qquad \omega'^{\,ab}_\mu = \omega^{ab}_\mu + \delta\omega^{ab}_\mu \\
\delta e^a_\mu &= \partial_\mu \xi^a - \omega^{ab}_\mu \xi_b + \lambda^{ab} e_{\mu b} \\
\delta\omega^{ab}_\mu &= \partial_\mu \lambda^{ab} - \omega^a_{\mu c}\lambda^{cb} + \omega^b_{\mu c}\lambda^{ca}.
\end{aligned} \tag{1.62}$$

The indices a, b, c are raised and lowered using η^{ab}, η_{ab}. The relations (1.62) are the infinitesimal gauge transformations of the fields e^a_μ, ω^{ab}_μ.

Let us introduce the symmetric tensor field $g_{\mu\nu}$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = g_{\nu\mu}. \tag{1.63}$$

Using the relations (1.63, 1.62) one can show that the field $g_{\mu\nu}$ transforms under the gauge transformations in the following way

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu} + \delta g_{\mu\nu} \\ \delta g_{\mu\nu} &= e_{\nu a}\partial_\mu \xi^a + e_{\mu a}\partial_\nu \xi^a - (\omega^{ab}_\mu e_{\nu a} + \omega^{ab}_\nu e_{\mu a})\xi_b. \end{aligned} \tag{1.64}$$

Let us introduce the parameters $\xi^\mu(x) = e^\mu_a(x)\xi^a(x)$, (where e^μ_a satisfy the equations $e^\mu_a e^a_\nu = \delta^\mu_\nu$) instead of the parameters $\xi^a(x)$. In terms of the parameters $\xi^\mu(x)$ the transformation (1.64) may be written in the form

$$\begin{aligned} \delta g_{\mu\nu} = {} & g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha \\ & - (\omega^{ab}_\mu e_{\nu a} e_{\alpha b} + \omega^{ab}_\nu e_{\mu a} e_{\alpha b} - e_{\nu a}\partial_\mu e^a_\alpha - e_{\mu a}\partial_\nu e^a_\alpha)\xi^\alpha. \end{aligned} \tag{1.65}$$

Let us note that the parameters λ^{ab} are absent in the transformations (1.64, 1.65) of the field $g_{\mu\nu}$.

Our aim is to find a way leading to the standard formulation of a gravitational theory. It is well-known that the dynamical variables in this case are tetrad fields e^a_μ (or metric fields $g_{\alpha\beta}$). But so far we have the formulation where besides the tetrads e^a_μ there are additional variables ω^{ab}_μ. To decrease the number of dynamic variables, it is natural to impose some constraints on these variables e^a_μ and ω^{ab}_μ. How can one find the equations of constraints? It is obvious that the necessary constraints must be covariant objects, constructed from the fields e^a_μ and ω^{ab}_μ. The only such objects are the curvature tensors $R^a_{\mu\nu}$ and $R^{ab}_{\mu\nu}$ (1.61). It is easy to understand that it would be senseless to impose the constraints $R^a_{\mu\nu} = 0$ and $R^{ab}_{\mu\nu} = 0$ simultaneously because the whole dynamic content of the theory is then absent.

Since we want to construct a theory where the variables are expressed in terms of variables e^a_μ, it is natural to have the constraints containing the variables ω^{ab}_μ without derivatives. The only candidate to serve as such a constraint is the equation $R^a_{\mu\nu} = 0$. Notice that the above constraint assumes that torsion is absent from the theory. We shall consider this equation then as a necessary constraint. Hence, we impose the constraints on the field e^a_μ, ω^{ab}_μ in the form

$$R^a_{\mu\nu} = 0$$

or

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + e_{\mu b}\omega^{ab}_\nu - e_{\nu b}\omega^{ab}_\mu = 0. \tag{1.66}$$

Equation (1.66) may be solved for ω^{ab}_μ and thus the fields ω^{ab}_μ may be found as functions of e^a_μ. Let us denote these functions by $\omega^{ab}_\mu(e^c_\nu) = \overline{\omega}^{ab}_\mu$. Note that equation (1.66) is the standard relation between the tetrad and spinor connection in Riemannian geometry. Substituting the expression $\overline{\omega}^{ab}_\mu$ into transformation law (1.66), one can show, by direct calculations, that $R^a_{\mu\nu} = 0$

$$\overline{\omega}_{\mu\nu\alpha} + \overline{\omega}_{\nu\mu\alpha} - e_{\nu a}\partial_\mu e^a_\alpha - e_{\mu a}\partial_\nu e^a_\alpha = -\partial_\alpha g_{\mu\nu} \tag{1.67}$$

where

$$\omega_{\mu\alpha\beta} = \omega^{ab}_\mu e_{\alpha a} e_{\beta b}.$$

Taking into account equality (1.67) one can rewrite relation (1.65) in the form

$$\delta g_{\mu\nu} = g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha + \partial_\alpha g_{\mu\nu}\xi^\alpha.$$

Relation (1.68) represents the infinitesimal general-coordinate transformation of the symmetric second-rank tensor. This transformation is accepted as the gauge transformation in Einstein gravity. Therefore, one can consider that the field $g_{\mu\nu}(x)$ (1.63) plays the role of the Riemannian metric and the parameters $\xi^\alpha(x)$ define the infinitesimal transformation of the coordinates $x'^\mu = x^\mu + \xi^\mu(x)$.

Let us consider

$$\overline{R}^{ab}_{\mu\nu} = \partial_\mu \overline{\omega}^{ab}_\nu - \partial_\nu \overline{\omega}^{ab}_\mu + \overline{\omega}^a_{\mu c}\overline{\omega}^{cb}_\nu - \overline{\omega}^a_{\nu c}\overline{\omega}^{cb}_\mu$$

and define $R_{\mu\nu\alpha\beta} = \overline{R}^{ab}_{\mu\nu} e_{a\alpha} e_{b\beta}$. We can show that $R_{\mu\nu\alpha\beta}$ depends on $g_{\mu\nu}$ only and coincides with the well-known Riemannian curvature tensor. Let us also introduce $R_{\mu\nu} = R_{\mu\alpha\nu\beta}g^{\alpha\beta}$ and $R = g^{\mu\nu}R_{\mu\nu}$. It is obvious that the tensor $R_{\mu\nu}$ is given by equality (1.58).

Let us write the action of the theory under consideration. In order to do it one can use the above tensors $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R. The only action constructed from these tensors and leading to second-order equations of motion has the form

$$S = -\frac{1}{2\kappa^2}\int \mathrm{d}x\sqrt{-g}\,(R - \Lambda)$$

where Λ is a constant which is called the cosmological constant. At $\Lambda = 0$, we obtain action (1.57). Using the tensors $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R, one can construct more complicated actions, for instance,

$$S = \int \mathrm{d}x\sqrt{-g}\Big(aR_{\mu\nu}R^{\mu\nu} + bR^2 - \frac{1}{2\kappa^2}(R - \Lambda)\Big).$$

This action defines the theory which is known as higher-derivative gravity.

1.6 The theory of the Dirac spinor field

Let us consider the Dirac spinor field defined by the known transformation properties under the Lorentz transformation. For example, in infinitesimal form the spinor transformation law may be written as

$$\psi'(x') = S(\Lambda)\psi(x) \tag{1.69}$$

where

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$
$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}{}_{\nu} \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}.$$

Here $\varepsilon_{\mu\nu}$ are the parameters of the Lorentz rotations. The matrix $S(\Lambda)$ has the form

$$S(\Lambda) = 1 + \tfrac{1}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu} \qquad \sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

where 1 is the 4×4 unit matrix and γ_μ are the Dirac matrices, which satisfy the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1. \tag{1.70}$$

How is it possible to write the Dirac field Lagrangian? One can show with the help of the standard arguments that the expression $\overline{\psi}\psi$ is the scalar and $\overline{\psi}\gamma^\mu\psi$ is a vector where $\overline{\psi} = \psi^+\gamma_0$ is the Dirac conjugated spinor. The simplest real scalar which can be constructed from $\overline{\psi}$, ψ, γ^μ and ∂_μ and which is a bilinear combination of the fields is

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \tag{1.71}$$

where m is a parameter with dimensions of mass. The expression (1.71) is the free Dirac field Lagrangian.

The corresponding equation of motion has the form

$$\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi = 0. \tag{1.72}$$

We should check now that equation (1.72) leads to a relativistic relation (1.6). Let us introduce the Fourier-transformation $\tilde{\psi}(p)$ of $\psi(x)$. Then we have from (1.72)

$$(\gamma^\mu p_\mu + m)\tilde{\psi} = 0$$

and

$$(\gamma^\nu p_\nu - m)(\gamma^\mu p_\mu + m)\tilde{\psi} = 0.$$

The left side of this relation may be transformed on the basis of relation (1.69) to the form

$$(p^2 - m^2)\tilde{\psi} = 0.$$

Thus, for the spinor field the relativistic relation $p^2 = m^2$ is derived. Hence, we can consider the parameter m in Lagrangian (1.71) as the mass of the particles, which are described by the spinor field.

The next question is how is it possible to introduce the interaction in the spinor field Lagrangian? One of the ways is dictated by the gauge invariance principle. Lagrangian (1.71) is invariant under the global transformations of the $U(1)$ group,

$$\psi' = \mathrm{e}^{\mathrm{i}\xi}\psi \qquad \overline{\psi'} = \overline{\psi}\mathrm{e}^{-\mathrm{i}\xi}$$

where ξ is constant. The localization of these transformations leads to the necessity of the vector field A_μ introduced by means of the ordinary derivative changing to the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$, where e is charge. The vector field Lagrangian has the form (1.19). As a result we obtain a theory with the Lagrangian

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi - m\overline{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \qquad (1.73)$$

The field theory model with the Lagrangian (1.73) is called the spinor electrodynamics.

If the spinor fields have the internal indices and transform over the representation of some Lie group, one can introduce the interaction by also using the gauge principle. In this case the Yang–Mills fields A^a_μ appear in the theory. This type of procedure for the interaction introduction was described in section 3.

It is not difficult to construct the interaction of the spinor and scalar fields using the Lorentzian scalar $\overline{\psi}\varphi\psi$. The corresponding Lagrangian has the form

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - m_1\overline{\psi}\psi + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m_2^2\varphi^2 - h\overline{\psi}\varphi\psi. \qquad (1.74)$$

Here m_1, m_2 are the masses of the spinor and scalar fields respectively. The term $h\overline{\psi}\psi\varphi$ is called the Yukawa coupling and the dimensionless constant h is called the Yukawa coupling constant.

Using the scalar, spinor and vector fields one can construct many of the field theory models. Some of these models will be considered in subsequent chapters of this book.

1.7 Construction of the field theory models interacting with gravity

Let us consider the theory of some fields Φ in the Minkowski space–time. The action is

$$S = \int dx \mathcal{L}(\Phi, \partial_\mu \Phi). \tag{1.75}$$

The inclusion of the interaction with the gravitational field, i.e. the theory formulation in curved space–time is performed in the following way. Let us introduce the Riemannian manifold with the coordinates x^μ and metric $g_{\mu\nu}(x)$ the signature of which is -2. The field Φ, contained in action (1.75) must be generalized so that it is defined on a Riemannian manifold and forms a geometrical object under general covariant transformations. All the natural constructions on Minkowski space–time must be generalized in an analogous way. Furthermore, all the ordinary derivatives ∂_μ must be changed by the general covariant derivatives ∇_μ. It is also necessary to ensure that the Lagrangian is the scalar under the general coordinate transformations. Finally, the integration in the expression for the action must be performed over the invariant volume. The above-mentioned procedure, based on the generally-coordinate covariance, is called the minimal interaction for gravity and leads to the action

$$S = \int dx \sqrt{-g} \mathcal{L}(\Phi, \nabla_\mu \Phi). \tag{1.76}$$

However, general-coordinate covariance does not forbid the adding to the Lagrangian invariant terms which are vanishing in flat space–time. All such terms in the action describe the non-minimal interaction with the gravity. Therefore, the theory under consideration must be written in the form

$$S = \int dx \sqrt{-g} \left(\mathcal{L}(\Phi, \nabla_\mu \Phi) + \text{non-minimal interaction}\right). \tag{1.77}$$

The expressions for the terms describing the non-minimal interaction cannot be found on the basis of the theory in Minkowski space–time. It is necessary to have additional considerations.

We will fix the non-minimal interaction terms on the basis of the following natural conditions. Firstly, it is desirable to conserve in curved space–time as many symmetries of the initial theory as possible. Secondly, let us accept that the terms describing the non-minimal interaction must not introduce new dimensional parameters to the theory. Finally, let us demand that the terms describing the non-minimal interaction must admit the massless limit. These three conditions fix the non-minimal interaction structure in the action (1.77).

As examples, let us apply the above-mentioned procedure for finding the non-minimal interaction in the scalar field theory and the spinor electrodynamics actions in curved space–time.

The Lagrangian of scalar field theory in the Minkowski space–time has the form (1.9). It is evident that the only admissible term, describing the non-minimal interaction with gravity is $\frac{1}{2}\xi R\varphi^2$ where R is the scalar curvature, and ξ is a dimensionless parameter. We will call this parameter the non-minimal coupling constant. The final form of the action of the scalar field theory may be written in the form

$$S = \int \mathrm{d}x\sqrt{-g}\{\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{2}\xi R\varphi^2 - V(\varphi)\}. \quad (1.78)$$

The value $\xi = 0$ of the non-minimal coupling constant leads to a theory which is minimally coupled with gravity. However, there is another interesting value of the parameter ξ.

Let us consider the free Lagrangian corresponding to (1.78) when $V(\varphi) = 0$ and let $m = 0, \xi = 1/6$. Then one can show that the action is invariant not only under the generally-coordinate transformations but also under the conformal transformations

$$g'_{\mu\nu}(x) = \mathrm{e}^{2\sigma(x)}g_{\mu\nu}(x) \qquad \varphi'(x) = \mathrm{e}^{-\sigma(x)}\varphi(x). \quad (1.79)$$

Here $\sigma(x)$ is an arbitrary scalar field. Therefore we can call the non-minimal coupling conformal coupling if $\xi = 1/6$ and σ is the parameter of this conformal transformation. Let us note that the standard potential $V(\varphi) = (1/4!)f\varphi^4$ where f is the scalar coupling constant does not break the conformal invariance of the action.

The Lagrangian of the spinor electrodynamics in the Minkowski space–time has the form (1.73). For the action writing in curved space–time it is necessary to define the spinors and the Dirac matrices in Riemannian space.

It is well-known that in Riemannian space the spinor transforms as the scalar under the generally-coordinate transformations and transforms in accordance with (1.69) under the local Lorentzian rotations. The matrix $S(\Lambda)$ in the transformation law now has the following form:

$$\begin{aligned} &S(\Lambda) = 1 + \tfrac{1}{4}\varepsilon^{ab}\sigma_{ab} \qquad \sigma_{ab} = \tfrac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a) \\ &a, b = 0, 1, 2, 3. \end{aligned} \quad (1.80)$$

Here $\varepsilon^{ab}(x)$ are the local Lorentzian rotation parameters and γ_a are the ordinary Dirac matrices, which satisfy the relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}1.$$

Now it is necessary to introduce the analogy of the Dirac matrices for the Riemannian space. It may be found in the following way. Let us define the tetrad $e^a_\mu(x)$ which satisfies the relations $\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu}$, $g^{\mu\nu}e^a_\mu e^b_\nu = \eta^{ab}$. With the help of it one can construct the matrices $\gamma_\mu(x)$ by the rule

$$\gamma_\mu(x) = e^a_\mu(x)\gamma_a.$$

It is not difficult to show that the matrices $\gamma_\mu(x)$ satisfy the equation

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2g_{\mu\nu}(x)1.$$

The matrices $\gamma_\mu(x)$ are the appropriate generalization of the Dirac matrices in the spinor field theory formulation in the curved space–time.

The following step consists in defining the generally-covariant derivative of the spinor field. This derivative is defined as

$$\nabla_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\omega^{ab}_\mu\sigma_{ab}\psi$$

where the matrices σ_{ab} are given by the relation (1.80) and ω^{ab}_μ is the spinor connection which satisfies equation (1.66).

The Lagrangian of spinor electrodynamics contains the field A_μ which must be considered as the vector under the generally-coordinate transformations. The final action of spinor electrodynamics in curved space–time has the form

$$S = \int dx\sqrt{-g}\{\mathrm{i}\bar{\psi}\gamma^\mu(x)(\nabla_\mu - \mathrm{i}eA_\mu)\psi - m\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\}. \quad (1.81)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\bar{\psi} = \psi^+\gamma_{(0)}$, $\gamma_{(0)} = \gamma_a|_{a=0}$ as before. The action (1.81) is invariant under the gauge transformations

$$\psi' = \mathrm{e}^{\mathrm{i}\xi(x)}\psi \qquad \overline{\psi'} = \bar{\psi}\mathrm{e}^{-\mathrm{i}\xi(x)} \qquad A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\xi(x).$$

where $\xi(x)$ is the arbitrary scalar field. The above-mentioned three conditions concerning the structure of the non-minimal interaction indicate that the Lagrangian (1.81) is fixed unambiguously.

The formulation of an arbitrary field theory model in curved space–time may be carried out in the same way. We will consider many such models in further chapters without any special discussions of their Lagrangian structure.

1.8 Comments

1. A detailed exposition of the largest part of the material under consideration in this chapter is given in the all textbooks on the quantum field theory. See, for example, Bogoliubov and Shirkov [1], De Witt [2], Schweber [3], Gasiorowicz [4], Itzykson and Zuber [5].
2. The construction of the renormalizable four-dimensional sigma-model with fourth derivatives is given in [6].
3. A detailed discussion of a particle spectrum in the explicit covariant formulation of the Maxwell electrodynamics is given, for example, in the books by Schweber [3] and Gasiorowicz [4].
4. The classical theory of the Yang–Mills fields is considered in all modern textbooks on the quantum field theory. See, for example, De Witt [2], Itzykson and Zuber [5], Faddeev and Slavnov [7], Abers and Lee [8].
5. The theory of the free antisymmetric second-rank tensor field was first studied by Ogievetsky and Polubarinov [9].
6. The gauge treatment of the gravity was first developed by Utiyama [11] and Kibble [12]. In this book we follow the paper by Kaku *et al* [10].
7. The spinor analysis in curved space–time is considered, for example, in the book by Weinberg [13].
8. The formulation of the field theory models in curved space–time is discussed in the review paper [14].
9. The conformal invariance of the massless scalar field at the non-minimal coupling constant $\xi = 1/6$ was first noted by Penrose [15] (see also the paper by Chernikov and Tagirov [16]).

2 Effective Action in Quantum Field Theory

2.1 Canonical quantization of the scalar field

Let us consider the scalar field theory with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m^2\varphi^2 - V(\varphi). \tag{2.1}$$

The construction of the quantum theory is based on the canonical quantization procedure, which first assumes the Hamiltonian formulation (Hamiltonization) of the classical theory.

To discuss the Hamiltonization of the theory with Lagrangian (2.1), we write the action in the form

$$S = \int \mathrm{d}t\, L \tag{2.2}$$

where

$$L = \int \mathrm{d}^3x \mathcal{L}(\varphi, \partial_\mu\varphi). \tag{2.3}$$

Here $t \equiv x^0$ is a time coordinate of Minkowski space. We will consider the field $\varphi(x) \equiv \varphi(\boldsymbol{x}, t)$ to be the set of the coordinates $\varphi_{\boldsymbol{x}}(t)$ of the dynamical system, where the vector $\boldsymbol{x}$ plays the role of an index. Relation (2.3) shows that L is a functional of $\varphi(\boldsymbol{x}, t)$ and $\dot{\varphi}(\boldsymbol{x}, t) \equiv \partial\varphi/\partial t$. Then in accordance with the relation (2.2) one can understand L as the Lagrangian function of the dynamical system.

Let us introduce the momenta $\pi(\boldsymbol{x}, t)$, corresponding to the coordinates $\varphi(\boldsymbol{x}, t)$

$$\pi(\boldsymbol{x}, t) = \frac{\delta_t L}{\delta\dot{\varphi}(\boldsymbol{x}, t)} \tag{2.4}$$

where the functional derivative on the right side is taken at a fixed value of t. Due to the locality of the Lagrangian we have from relation (2.4)

$$\pi(\boldsymbol{x},t)=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}(\boldsymbol{x},t)}. \tag{2.5}$$

Assume now that the following relation is fulfilled

$$\det\frac{\delta_t^2\mathcal{L}}{\delta\dot{\varphi}(\boldsymbol{x},t)\delta\dot{\varphi}(\boldsymbol{y},t)}\neq 0 \tag{2.6}$$

then relation (2.4) can be solved in terms of the velocities $\dot{\varphi}=f(\pi,\varphi)$.

Let us construct the Hamiltonian H by the rule

$$H=\int \mathrm{d}^3x(\pi(\boldsymbol{x},t)\dot{\varphi}(\boldsymbol{x},t)-\mathcal{L})|_{\dot{\varphi}=f(\pi,\varphi)}. \tag{2.7}$$

As a result the Hamiltonian is the functional of the coordinates φ and momenta π. Standard considerations show that the classical equations of motion can be written as

$$\begin{aligned}\dot{\varphi}(\boldsymbol{x},t)&=\frac{\delta_t H}{\delta\pi(\boldsymbol{x},t)}\\ \dot{\pi}(\boldsymbol{x},t)&=-\frac{\delta_t H}{\delta\varphi(\boldsymbol{x},t)}.\end{aligned} \tag{2.8}$$

Let us note that equations (2.8) can be obtained from the action principle for the action S_H in the canonical formalism

$$S_H=\int \mathrm{d}t\left(\int \mathrm{d}^3x\,\pi(\boldsymbol{x},t)\dot{\varphi}(\boldsymbol{x},t)-H\right). \tag{2.9}$$

Equations of motion for (2.9) have the simple form

$$\frac{\delta S_H}{\delta\varphi}=0 \qquad \frac{\delta S_H}{\delta\pi}=0. \tag{2.10}$$

For finding the solution of these equations it is necessary to introduce the initial conditions

$$\varphi(\boldsymbol{x},t)|_{t=t_0}=\varphi(\boldsymbol{x}) \qquad \pi(\boldsymbol{x},t)|_{t=t_0}=\pi(\boldsymbol{x}).$$

The quantization of the theory is carried out with the help of the following postulates:

(*a*) The system state is described by the vector $|\psi\rangle$ in the Hilbert space. The physical observables are represented by the Hermitian operators acting in this Hilbert space.

(*b*) The expectation value of the observable A in the state $|\psi\rangle$ is given by $\langle\psi|\hat{A}|\psi\rangle$ where $\hat{A}$ is the operator corresponding to A and $\langle\psi_1|\psi_2\rangle$ is the inner product in the Hilbert space.

(*c*) The initial coordinates and momenta $\varphi(\boldsymbol{x})$, $\pi(\boldsymbol{x})$ are described by the Hermitian operators $\hat{\varphi}(\boldsymbol{x})$, $\hat{\pi}(\boldsymbol{x})$, which satisfy the canonical commutation relations

$$\begin{aligned} \left[\hat{\varphi}(\boldsymbol{x}), \hat{\varphi}(\boldsymbol{y})\right] &= 0 \\ \left[\hat{\pi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})\right] &= 0 \\ \left[\hat{\varphi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})\right] &= \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{2.11}$$

(*d*) The evolution of the state is defined by the Schrödinger equation

$$\mathrm{i}\,\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle \tag{2.12}$$

where the Hermitian Hamiltonian operator $\hat{H}$ is obtained from the classical Hamiltonian H by substituting the operator $\hat{\varphi}$, $\hat{\pi}$ instead of coordinates φ and momenta π. Some arrangement of the non-commuting operators $\hat{\varphi}$, $\hat{\pi}$ is assumed.

The postulates (a, b, c, d) form the contents of the canonical quantization procedure. Unfortunately, this procedure in the field theory is not strict from a mathematical point of view. In particular, $\hat{\varphi}(\boldsymbol{x})$ does not represent the operator but an operator valued generalized function. Due to the locality the Hamiltonian contains the products of the fields at the same point. But the multiplication of generalized functions is, in general, not defined. As a result one can expect the appearance of divergences in the field theory.

Let us consider the theory with Lagrangian (2.1). In this case we have from equalities (2.1), (2.5) that $\pi = \dot{\varphi}$. Therefore, using relation (2.7), one can obtain the Hamiltonian operator in the form

$$\begin{aligned} \hat{H} = \int \mathrm{d}^3x[&\tfrac{1}{2}\hat{\pi}^2(\boldsymbol{x}) + \tfrac{1}{2}\partial_i\hat{\varphi}(\boldsymbol{x})\partial_i\hat{\varphi}(\boldsymbol{x}) \\ &+ \tfrac{1}{2}m^2\hat{\varphi}^2(\boldsymbol{x}) + V(\hat{\varphi}(\boldsymbol{x}))]. \end{aligned} \tag{2.13}$$

The canonical quantization procedure defines the quantum theory in the so-called Schrödinger picture where the operators do not depend on time, but the dynamics of the system is caused by the dependence of the vectors of state on the time. We can proceed to the so-called Heisenberg picture where the states are always stationary ones but the system dynamics is caused by the operators' dependence on time according to the following rule:

$$\hat{A}(t) = \exp[-\mathrm{i}(t - t_0)\hat{H}]\hat{A}\exp[\mathrm{i}(t - t_0)\hat{H}]. \tag{2.14}$$

Here $\hat{A}$ is an arbitrary operator and operator $\hat{H}$ is the Hamiltonian. In the case under consideration $\hat{H}$ has the form (2.13). From equation (2.14) we have the equation of motion for the operator $\hat{A}(t)$

$$\mathrm{i}\dot{\hat{A}}(t) = \left[\hat{A}(t), \hat{H}\right]. \tag{2.15}$$

Let us write equation (2.15) for the operators $\hat{\varphi}(\boldsymbol{x}, t) \equiv \hat{\varphi}(x)$, $\hat{\pi}(\boldsymbol{x}, t) \equiv \hat{\pi}(x)$. Using the expression for Hamiltonian (2.13) and commutation relations (2.11) we obtain

$$\begin{aligned} \dot{\hat{\varphi}}(x) &= \hat{\pi}(x) \\ \dot{\hat{\pi}}(x) &= \Delta\hat{\varphi}(x) - m^2\hat{\varphi}(x) + V'(\hat{\varphi}(\boldsymbol{x})) \end{aligned} \tag{2.16}$$

where

$$V'(\hat{\varphi}) = \left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=\hat{\varphi}}.$$

It is not difficult to see that the operator equations of motion (2.16) have a form coinciding with the classical Hamiltonian equations (2.8). Therefore one can write

$$\left.\frac{\delta S_H}{\delta \pi(x)}\right|_{\varphi=\hat{\varphi},\pi=\hat{\pi}} = 0 \qquad \left.\frac{\delta S_H}{\delta \varphi(x)}\right|_{\varphi=\hat{\varphi},\pi=\hat{\pi}} = 0. \tag{2.17}$$

Let us consider equations (2.16) for the free theory when $V(\varphi) = 0$ in more detail. The solution of these equations can be written in the form

$$\begin{aligned} \hat{\varphi}(x) &= \int \frac{\mathrm{d}^3 p}{\sqrt{2(2\pi)^3\varepsilon(\boldsymbol{p})}}[\hat{a}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + \hat{a}^+(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}] \\ \hat{\pi}(x) &= -\mathrm{i}\int \frac{\mathrm{d}^3 p}{\sqrt{2(2\pi)^3\varepsilon(\boldsymbol{p})}}\,\varepsilon(\boldsymbol{p})[\hat{a}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} - \hat{a}^+(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}]. \end{aligned} \tag{2.18}$$

Here $p_0 \equiv \varepsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ is the energy of the relativistic particle with momentum $\boldsymbol{p}$ and mass m.

Let us substitute equations (2.18) into commutation relations (2.11). Then it is easy to show that the operators $\hat{a}(\boldsymbol{p})$, $\hat{a}^+(\boldsymbol{p})$ satisfy the commutation relations for the Bose creation and annihilation operators

$$\begin{aligned} [\hat{a}(\boldsymbol{p}), \hat{a}(\boldsymbol{p}')] &= 0 \\ [\hat{a}(\boldsymbol{p}), \hat{a}^+(\boldsymbol{p}')] &= 0 \\ [\hat{a}(\boldsymbol{p}), \hat{a}^+(\boldsymbol{p}')] &= \delta(\boldsymbol{p} - \boldsymbol{p}'). \end{aligned} \tag{2.19}$$

Let us express the Hamiltonian H corresponding to the free theory in the terms of creation and annihilation operators. To do this we substitute equalities (2.18) in relation (2.13) for $V(\varphi) = 0$. Then after direct calculation we obtain

$$\hat{H}_0 = \tfrac{1}{2} \int \mathrm{d}^3 p\, \varepsilon(\boldsymbol{p})(\hat{a}^+(\boldsymbol{p})\hat{a}(\boldsymbol{p}') + \hat{a}(\boldsymbol{p})\hat{a}^+(\boldsymbol{p}')). \tag{2.20}$$

Let $|0\rangle$ be a state vector, which satisfies the equation

$$\hat{a}(\boldsymbol{p})|0\rangle = 0.$$

Using the vector $|0\rangle$ we construct the infinite set of the state vectors (unnormalized)

$$\begin{aligned} &|n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle \\ &\quad = (\hat{a}^+(\boldsymbol{p}_1))^{n_1} (\hat{a}^+(\boldsymbol{p}_2))^{n_2} \ldots (\hat{a}^+(\boldsymbol{p}_k))^{n_k} |0\rangle \\ &\quad k = 1, 2, \ldots; \qquad n_1, n_2, \ldots, n_k = 0, 1, 2, \ldots . \end{aligned} \tag{2.21}$$

Let us introduce the operator

$$\hat{N} = \int \mathrm{d}^3 p\, \hat{a}^+(\boldsymbol{p})\hat{a}(\boldsymbol{p}). \tag{2.22}$$

It is easy to check with the help of direct calculation that

$$\begin{aligned} &\hat{N}|n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle \\ &\quad = (n_1 + n_2 + \ldots + n_k)|n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle. \end{aligned} \tag{2.23}$$

Since the eigenvalues of the operator $\hat{N}$ are the natural numbers this operator is called the particles number operator. The state $|n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle$ is called the n-particle state where $n = n_1 + n_2 + \ldots + n_k$. In particular, $|0\rangle$ is the state without particles.

Let us consider the action of the operator $\hat{H}_0$ (2.20) on the vector $|n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle$. Using definition (2.21) and relation (2.19), we obtain

$$\begin{aligned}\hat{H}_0 |n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle \\ = (E_0 + n_1 \varepsilon(\boldsymbol{p}_1) + n_2 \varepsilon(\boldsymbol{p}_2) + \ldots \\ + n_k \varepsilon(\boldsymbol{p}_k)) \mid n_1, \boldsymbol{p}_1; n_2, \boldsymbol{p}_2; \ldots; n_k, \boldsymbol{p}_k\rangle\end{aligned} \tag{2.24}$$

where

$$E_0 = \tfrac{1}{2} \int \mathrm{d}^3 p \, \varepsilon(\boldsymbol{p}).$$

Relation (2.24) shows that the n-particle states are the eigenvalues of the free Hamiltonian. The energy spectrum is bounded from below by the value E_0, corresponding to the state without the particles. In the case under consideration the state without the particles is the ground state of the field. The field state $|0\rangle$ with the lowest energy is called the vacuum state. Relations (2.23) and (2.24) serve as a basis for the interpretation of the field states in terms of particles.

Let us note that the ground state energy E_0 is expressed through a divergent integral and formally it is infinite. Usually the free-field Hamiltonian is defined in another way, so that the ground state energy is equal to zero. To obtain the definition of $\hat{H}_0$, the known procedure of the normal ordering of the creation and annihilation operators is used. It is equivalent to use the expression $\hat{H}_0 - E_0$ as the free Hamiltonian where $\hat{H}_0$ is given by relation (2.20).

In conclusion let us make one important remark. The canonical quantization procedure described in this section can only be applied literally to the theories of scalar fields. In the other theories, for instance, the spinor field theories, the vector field theories, and the second-rank tensor field theories, relation (2.6) is broken. Therefore, from the momenta definition (2.4) it is impossible to express all the velocities, and constraints between the coordinates and momenta appear. Hence, the standard hamiltonization procedure does not work and the problem of quantum theory construction arises. For the canonical formalism construction one can use the Dirac method where the constraints in phase space play an important role. We shall not be concerned with the Dirac method here. However, let us note that for the quantization of most popular field theory models, different indirect approaches are developed which can be derived from the Dirac method.

2.2 The generating functional of Green's functions

The fundamental objects of the quantum field theory are the Green's functions of the Heisenberg field operators

$$G_n(x_1, x_2, \ldots, x_k) = \langle 0|T\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_k)|0\rangle. \tag{2.25}$$

Here symbol T defines the so-called chronological or T-ordering which means that fields must be arranged from left to right in order of decreasing time arguments. Moreover under the T-product sign all the fields $\hat{\varphi}(x_i)$ are commutating ones. The set of the Green's functions $G_n(x_1, x_2, \ldots, x_n)$ $(n = 1, 2, \ldots)$ contains all the information which is of interest in the quantum field theory.

Instead of considering the individual Green's functions, G_n, one can introduce the object including all the Green's functions G_n, $n = 1, 2, \ldots$. Let $J(x)$ be a c-number scalar field, which is called the external source. The reason for this name will be explained later. We define the functional $Z[J]$ by the rule

$$\begin{aligned} Z[J] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int & \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n G_n(x_1, x_2, \ldots, x_n) \\ & \times J(x_1)J(x_2)\ldots J(x_n) \end{aligned} \tag{2.26}$$

and from this definition we have

$$\left.\frac{\delta^n Z[J]}{\delta \mathrm{i}J(x_1)\delta \mathrm{i}J(x_2)\ldots\delta \mathrm{i}J(x_n)}\right|_{J=0} = G_n(x_1, x_2, \ldots, x_n) \tag{2.27}$$

where $n = 1, 2, \ldots$. Relations (2.27) show that the functional $Z[J]$(2.26) is the generating functional of the Green's functions.

Let us substitute expression (2.25) for the Green's functions G_n into relation (2.26). Then we obtain

$$\begin{aligned} Z[J] &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \langle 0 \mid T \int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \hat{\varphi}(x_1)J(x_1) \\ &\quad \times \hat{\varphi}(x_2)J(x_2)\ldots\hat{\varphi}(x_n)J(x_n)|0\rangle \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \langle 0|T\Big(\mathrm{i}\int \mathrm{d}x\hat{\varphi}(x)J(x)\Big)^n|0\rangle \\ &= \langle 0|T\exp\Big(\mathrm{i}\int \mathrm{d}x\, \hat{\varphi}(x)J(x)\Big)|0\rangle. \end{aligned}$$

So the generating functional of the Green's functions may be written in the form

$$Z[J] = \langle 0|T \exp\left(\mathrm{i}\int \mathrm{d}x \hat{\varphi}(x)J(x)\right)|0\rangle. \qquad (2.28)$$

Later on we will work with the relation (2.28) for $Z[J]$.

It is covenient to introduce the more general generating functional depending not only on the source $J(x)$ but also on one more source $K(x)$,

$$Z[J,K] = \langle 0|T \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\pi}(x)K(x))\right)|0\rangle. \qquad (2.29)$$

It is evident that

$$Z[J] = Z[J,K]|_{K=0}. \qquad (2.30)$$

Let us find the equation of motion for the functional $Z[J,K]$, using the operatorial equations of motion (2.16) and (2.18). From the definition of $Z[J,K]$(2.29) it follows that

$$\begin{aligned}
\frac{\delta Z[J,K]}{\delta \mathrm{i} J(x)} &= \langle 0|T\hat{\varphi}(x) \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\pi}(x)K(x))\right)|0\rangle \\
\frac{\delta Z[J,K]}{\delta \mathrm{i} K(x)} &= \langle 0|T\hat{\pi}(x) \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\pi}(x)K(x))\right)|0\rangle.
\end{aligned} \qquad (2.31)$$

Differentiating these equations with respect to time and using the T-product properties we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} J(x)} &= \langle 0|T\dot{\hat{\varphi}}(x) \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\pi}(x)K(x))\right)|0\rangle \\
&\quad - K(x)Z[J,K] \\
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} K(x)} &= \langle 0|T\dot{\hat{\pi}}(x) \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\pi}(x)K(x))\right)|0\rangle \\
&\quad + J(x)Z[J,K].
\end{aligned} \qquad (2.32)$$

Using the operatorial equations of motion (2.16), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} J(x)} &= \langle 0|T\hat{\pi}(x) \\
&\quad \times \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x)+\hat{\pi}(x)K(x))\right)|0\rangle \\
&\quad - K(x)Z[J,K] \\
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} K(x)} &= \langle 0|T(\Delta\hat{\varphi}(x) - m^2\hat{\varphi}(x) - V'(\hat{\varphi}(x)) \\
&\quad \times \exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x)+\hat{\pi}(x)K(x))\right)|0\rangle \\
&\quad + J(x)Z[J,K].
\end{aligned} \tag{2.33}$$

Let us apply the identity

$$\begin{aligned}
&\langle 0|TF(\hat{\varphi}(x),\hat{\pi}(x))\exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x)+\hat{\pi}(x)K(x))\right)|0\rangle \\
&\quad = F\left(\frac{\delta}{\delta \mathrm{i} J},\frac{\delta}{\delta \mathrm{i} K}\right)Z[J,K]
\end{aligned} \tag{2.34}$$

to the right part of relations (2.33). This equation can be proved by expanding

$$F\left(\frac{\delta}{\delta \mathrm{i} J},\frac{\delta}{\delta \mathrm{i} K}\right)$$

in the functional Taylor series. As a result we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} J(x)} - \frac{\delta Z[J,K]}{\delta i K(x)} &= -K(x)Z[J,K] \\
\frac{\partial}{\partial t}\frac{\delta Z[J,K]}{\delta \mathrm{i} K(x)} - (\Delta - m^2)\frac{\delta Z[J,K]}{\delta \mathrm{i} J(x)} + V'\left(\frac{\delta}{\delta \mathrm{i} J(x)}\right)Z[J,K] & \\
&= J(x)Z[J,K].
\end{aligned} \tag{2.35}$$

Equations (2.35) are the equations of motion for the generating functional $Z[J,K]$.

These equations may be rewritten in a form where their universal structure is clear. Let us note that equations (2.35) are analogous to the classical Hamiltonian equations (2.28) if the Hamiltonian contains the term

$$\int \mathrm{d}^3x(\varphi(x)J(x)+\pi(x)K(x))$$

describing the interaction of the fields φ, π by the external sources J, K. This analogy explains the origin of the terms in the generating functional $Z[J,K]$(2.34). The above-mentioned remark shows that equation (2.35) may be written in the form

$$\left.\frac{\delta S_H}{\delta\varphi(x)}\right|_{\varphi=\frac{\delta}{\delta iJ},\pi=\frac{\delta}{\delta iK}} Z[J,K]=J(x)Z[J,K]$$
$$\left.\frac{\delta S_H}{\delta\pi(x)}\right|_{\varphi=\frac{\delta}{\delta iJ},\pi=\frac{\delta}{\delta iK}} Z[J,K]=K(x)Z[J,K]. \tag{2.36}$$

Here S_H is the action in the canonical formalism (2.9). The equations of motion for the functional $Z[J,K]$(2.36) contain the classical action S_H which is the fundamental object of the theory. Therefore equations (2.36) have a universal sense and they can be derived for the arbitrary canonically quantized theory in general terms.

Now let us find the solution of equations (2.36). For this we shall use the functional integral technique, which has broad application in the quantum field theory. We can perform the functional Fourier transformation of the generating functional $Z[J,K]$

$$Z[J,K]=\int \mathrm{D}\varphi D\pi \exp\left(\mathrm{i}\int \mathrm{d}x(\varphi(x)J(x)+\pi(x)K(x)\right)\tilde{Z}[\varphi,\pi]. \tag{2.37}$$

Here the functional $\tilde{Z}[\varphi,\pi]$ is the Fourier tramsform of the functional $Z[J,K]$ depending on the classical fields $\varphi(x),\pi(x)$. Substituting relation (2.37) into equations (2.26) we obtain

$$\frac{\delta S_H}{\delta\varphi(x)}\tilde{Z}[\varphi,\pi]=-\mathrm{i}\frac{\delta\tilde{Z}[\varphi,\pi]}{\delta\varphi(x)}$$
$$\frac{\delta S_H}{\delta\pi(x)}Z[\varphi,\pi]=-\mathrm{i}\frac{\delta\tilde{Z}[\varphi,\pi]}{\delta\pi(x)}. \tag{2.38}$$

It follows from these relations that

$$\tilde{Z}[\varphi,\pi]=N\mathrm{e}^{\mathrm{i}S_H[\varphi,\pi]} \tag{2.39}$$

where N is an arbitrary constant. We shall assume that N is equal to unity or it is included in the definition of $\mathrm{D}\varphi\mathrm{D}\pi$.

So the solution of the equations of motion (2.36) for the generating functional $Z[J,K]$ has the form

$$Z[J,K]=\int \mathrm{D}\varphi\mathrm{D}\pi \exp\left(\mathrm{i}(S_H+\int \mathrm{d}x(\varphi(x)J(x)+\pi(x)K(x))\right). \tag{2.40}$$

Let us suppose $K = 0$. As a result we shall have the expression for the generating functional of the Green's functions

$$Z[J] = \int \mathrm{D}\varphi \mathrm{D}\pi \exp\left[\mathrm{i}\left(S_H + \int \mathrm{d}x \varphi(x) J(x)\right)\right]. \tag{2.41}$$

The generating functional $Z[J]$(2.41) is the solution of universal equations (2.36), obtained on the basis of the canonical quantization only. Therefore, one can say that relation (2.41) is a consequence of the canonical quantization. Expression (2.41) contains the action S_H which is the fundamental object of the classical theory. This is also true for equations (2.36). Therefore this expression has a universal character. Relation (2.41) is sometimes called the Green's functions generating functional, obtained by functional integration over the classical phase space. This representation, which is analogous to (2.41), may be derived for arbitrary canonically quantized theory in general terms.

Let us consider expression (2.41) and let us try to carry out integration over the momenta. If we perform the replacement of the variables $\pi \to \Pi + \pi$, we obtain

$$\begin{aligned} Z[J] &= \int \mathrm{D}\varphi \mathrm{D}\pi \exp[\mathrm{i}(S_H[\varphi, \Pi + \pi] + \int \mathrm{d}x\, \varphi(x) J(x)] \\ &= \int \mathrm{D}\varphi \mathrm{D}\pi \exp[\mathrm{i}(S_H[\varphi, \Pi] + \int \mathrm{d}x \frac{\delta S_H}{\delta \pi(x)}\bigg|_{\pi=\Pi} \pi(x) \\ &\quad + \Delta S_H[\varphi, \Pi, \pi] + \int \mathrm{d}x\, \varphi(x) J(x)]. \end{aligned} \tag{2.42}$$

We expanded $S_H[\varphi, \Pi + \pi]$ in a functional power series in $\pi(x)$ and denote the terms containing the powers of $\pi(x)$ of at least second order $\Delta S_H[\varphi, \Pi, \pi]$. Let us assume that

$$\frac{\delta S_H}{\delta \pi(x)}\bigg|_{\pi=\Pi} = 0.$$

This is one of the classical Hamiltonian equations of motion. We assume from this equation that Π may be expressed as a function of φ, $\dot{\varphi}$. After the substitution of such a function in the expression for $S_H[\varphi, \Pi]$ we obtain the classical action $S[\varphi]$. As a matter of fact the transition from the Hamiltonian to the Lagrangian description has been performed. From relation (2.42) one has the factorization of the integration over the momenta in the form

$$\int \mathrm{D}\pi \exp\left(\mathrm{i}\Delta S_H[\varphi, \pi]\right) = \mu[\varphi]. \tag{2.43}$$

Hence

$$Z[J] = \int \mathrm{D}\,\varphi\mu(\varphi)\exp\mathrm{i}\Big(S[\varphi] + \int \mathrm{d}x\,\varphi(x)J(x)\Big). \tag{2.44}$$

The functional $\mu[\varphi]$ is called the functional measure. Expression (2.44) is called the generating functional of the Green's functions and is found by means of the functional integral over configuration space. This representation contains the initial action $S[\varphi]$ which is the fundamental object of classical theory. However, the specific feature of the functional integral over the configuration space is the appearance of the local functional measure.

Expression (2.44) may be rewritten in the form

$$Z[J] = \int \mathrm{D}\varphi\exp\mathrm{i}\Big(S[\varphi] + \Delta S[\varphi] + \int \mathrm{d}x\,\varphi(x)J(x)\Big). \tag{2.45}$$

Here $\Delta S[\varphi] = -\mathrm{i}\,\mathrm{Tr}\ln\mu[\varphi]$ is the extra term which must be added to the initial classical action in the quantum theory construction. The symbol Tr denotes the functional trace

$$\mathrm{Tr}\,A = \int \mathrm{d}x A(x,x)$$

where $A(x,x) = A(x,y)|_{y=x}$, and $A(x,y)$ is the kernel of the operator A.

Usually, the kernel $\mu(x,y)$, corresponding to the functional measure, has the form $\mu(x,y) = \delta(x-y)\mu(\varphi(x))$, where $\mu(\varphi(x))$ is a function of field $\varphi(x)$. Then

$$\mathrm{Tr}\ln\mu = \delta^{(4)}(0)\int \mathrm{d}x\ln\mu(\varphi(x)).$$

That is why the functional measure always contains the ill-defined $\delta^{(4)}(0)$ and therefore it demands some regularization. In the simplest models one can find that $\mu(\varphi) = 1$ and the question about the functional measure is absent. But in the essentially non-linear theories such as the sigma-model or gravity theories, the functional measure is not trivial. The most popular regularization in quantum field theory is the dimensional one where $\delta^{(4)}(0) = 0$ and the question about the functional measure is also absent. However, if one uses the other regularization schemes the functional measure must be taken into account in expression (2.45).

As an example let us consider the functional measure computation in the theory with Lagrangian (2.1) and in the sigma-model. The

Hamiltonian of scalar field theory has the form (2.13). Due to this circumstance the expression ΔS_H in relation (2.42) is

$$-\int \tfrac{1}{2}\pi^2 \mathrm{d}x$$

and the functional measure (2.43) is a trivial constant.

Let us consider the sigma-model with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{ab}(\varphi)\eta^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b, \qquad a, b = 1, 2, \ldots, n. \tag{2.46}$$

The corresponding Hamiltonian is

$$H = \int \mathrm{d}^3x\Big(\tfrac{1}{2}g^{ab}(\varphi)\pi_a\pi_b + \tfrac{1}{2}g_{ab}(\varphi)\partial_i\varphi^a\partial_i\varphi^a\Big) \tag{2.47}$$

where π_a is the momentum conjugate to the field φ^a. The derivation of representation (2.41) for the theory under consideration requires no modification. The generating functional of Green's functions is given by expression (2.44) where functional measure (2.43) is equal to

$$\begin{aligned}\mu[\varphi] &= \int \mathrm{D}\pi \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}x g^{ab}(\varphi(x))\pi_a(x)\pi_b(x)\right] \\ &= \prod_x \det{}^{-1/2} g_{ab}(\varphi(x)).\end{aligned}$$

That is why the functional measure in the sigma-model is not trivial one.

We shall assume the use of dimensional regularization and shall not take the functional measure into account.

The generating functional $Z[J]$ computation is produced on the basis of the perturbation theory. Let us consider expression (2.44) (with $\mu[\varphi] = 1$) and let the action S correspond to the theory with Lagrangian (2.1). Then using the identity

$$F(\varphi)\exp\Big(\mathrm{i}\int \mathrm{d}x\, \varphi(x)J(x)\Big) = F\Big(\frac{\delta}{\delta \mathrm{i}J}\Big)\exp\Big(\mathrm{i}\int \mathrm{d}x\, \varphi(x)J(x)\Big)$$

we obtain

$$Z[J] = \exp\Big[-\mathrm{i}\int \mathrm{d}x\, V\Big(\frac{\delta}{\delta \mathrm{i}J}\Big)\Big]Z_0[J] \tag{2.48}$$

where

$$Z_0[J] = \int \mathrm{D}\varphi \exp\Big(\mathrm{i}\int \mathrm{d}x[\varphi(x)(-\Box - m^2 + \mathrm{i}\varepsilon)\varphi(x) + \varphi(x)J(x)]\Big). \tag{2.49}$$

Here integration by parts was performed and the term $-\varepsilon\varphi^2$, ($\varepsilon \to +0$), providing the convergence of the Gaussian functional integral was introduced. It is obvious that expression (2.49) is the generating functional of the free theory Green's functions.

Using the standard rules for the Gaussian functional integral calculation we obtain

$$Z_0[J] = \det{}^{-1/2}(\Box + m^2 - \mathrm{i}\varepsilon) \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}x\, \mathrm{d}y\, J(x)G(x,y)J(y)\right). \tag{2.50}$$

Here $G(x,y)$ is the Feynman propagator, which satisfies the equation

$$(\Box + m^2 - \mathrm{i}\varepsilon)G(x,y) = -\delta(x-y). \tag{2.51}$$

The expression $\det^{-1/2}(\Box + m^2 - \mathrm{i}\varepsilon)$ does not depend on the sources $J(x)$ and it is a constant; we shall not take this constant into account.

As a result relations (2.48), (2.50) are

$$Z[J] = \exp\left[-\mathrm{i}\int \mathrm{d}x V\left(\frac{\delta}{\delta \mathrm{i} J}\right)\right] \exp\left(\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, J(x)G(x,y)J(y)\right). \tag{2.52}$$

Equality (2.52) is the basis of the perturbation calculation of $Z[J]$. Let us expand the expression

$$\exp\left[-\mathrm{i}\int \mathrm{d}x\, V\left(\frac{\delta}{\delta \mathrm{i} J}\right)\right]$$

in the powers of interaction potential V then we obtain a perturbation series. These series have a structure such that each term can be represented by standard Feynman diagrams.

Let us give some remarks about the generating functional of the Green's functions in the presence of the spinor fields. In this case it is necessary to consider the following Green's function

$$\begin{aligned}\langle 0|T\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_k)\hat{\psi}(y_1)\hat{\psi}(y_2)\\ \ldots\hat{\psi}(y_m)\hat{\bar{\psi}}(z_1)\hat{\bar{\psi}}(z_2)\ldots\hat{\bar{\psi}}(z_m)|0\rangle.\end{aligned} \tag{2.53}$$

Note that the spinor fields are anticommutating ones under the T-product. The generating functional of Green's functions (2.53) has the form

$$Z[J,\eta,\bar{\eta}] = \langle 0|T\exp\left(\mathrm{i}\int \mathrm{d}x(\hat{\varphi}(x)J(x) + \hat{\bar{\psi}}(x)\eta(x) + \bar{\eta}(x)\hat{\psi}(x))\right)|0\rangle. \tag{2.54}$$

Here $\eta(x)$, $\bar{\eta}(x)$ are the sources for fields $\hat{\bar{\psi}}(x)$ and $\hat{\psi}(x)$, respectively. From the mathematical point of view the sources $\eta(x)$ and $\bar{\eta}(x)$ are infinite-dimensional Grassmann's algebra elements. These are anticommutating with each other and with $\hat{\bar{\psi}}(x)$, $\hat{\psi}(x)$. One can obtain the equations of motion for the generating functional (2.54) and find their solution by means of the functional integral. Omitting the details which are analogous to the scalar field theory let us give the final result

$$Z[J,\eta,\bar{\eta}] = \int \mathrm{D}\varphi \mathrm{D}\psi \mathrm{D}\bar{\psi} \exp\Big[\mathrm{i}(S[\varphi,\psi,\bar{\psi}] + \int \mathrm{d}x(\varphi(x)J(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x))\Big]. \tag{2.55}$$

Here $\psi(x)$, $\bar{\psi}(x)$ are the classical fields belonging to an infinite-dimensional Grassmann algebra. Therefore, the fields $\bar{\psi}(x)$ and $\psi(x)$ and sources $\eta(x), \bar{\eta}(x)$ are anticommutating with each other. $S[\varphi,\psi,\bar{\psi}]$ is the classical action, containing the scalar and spinor fields. For finding the Green's functions it is necessary to remember that the derivatives with respect to $\bar{\eta}$ are always the ones on the left and the derivatives with respect to η are always the ones on the right.

The perturbative computation of the generating functional (2.55) is based on the relation which is analogous to relation (2.48). The corresponding generating functional of the free spinor Green's functions has the form

$$\begin{aligned} Z_0[\eta,\bar{\eta}] &= \int \mathrm{D}\psi \mathrm{D}\bar{\psi} \exp\Big(\mathrm{i}\int \mathrm{d}x[\bar{\psi}(x)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) \\ &\quad + \bar{\eta}(x)\psi(x)) + \bar{\psi}(x)\eta(x)]\Big) \\ &= \exp\Big(\mathrm{i}\int \mathrm{d}x\mathrm{d}x\bar{\eta}(x)S(x,y)\eta(y)\Big). \end{aligned} \tag{2.56}$$

Here $S(x,y)$ is the Feynman propagator of the spinor field which satisfies the equation

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)S(x,y) = -\delta(x-y). \tag{2.57}$$

The perturbation theory for the $Z[J,\eta,\bar{\eta}]$ as in the scalar field theory leads to standard Feynman diagrams.

2.3 Effective action

Let us consider the generating functional of the Green's functions $Z[J]$. We define the Green's functions, depending on the source J by the rule

$$G_n(x_1, x_2, \ldots, x_n|J) = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta \mathrm{i} J(x_1) \delta \mathrm{i} J(x_2) \ldots \delta \mathrm{i} J(x_n)}. \tag{2.58}$$

The usual Green's function can be obtained from the function $G_n(x_1, \ldots, x_n|J)$ at $J = 0$.

We introduce the connected Green's functions $G_n^c(x_1, x_2, \ldots, x_n|J)$ by induction

$$\begin{aligned}
G_0(J) &= G_0^c(J) = 1 \\
G_1(x|J) &= \mathrm{i}G_1^c(x|J) \\
G_2(x_1, x_2|J) &= \mathrm{i}G_1^c(x_1|J)\mathrm{i}G_1^c(x_2|J) + \mathrm{i}G_2^c(x_1, x_2|J) \\
G_3(x_1, x_2, x_3|J) &= \mathrm{i}G_1^c(x_1|J)\mathrm{i}G_1^c(x_2|J)\mathrm{i}G_1^c(x_3|J) \\
&\quad + \mathrm{i}G_2^c(x_1, x_2|J)\mathrm{i}G_1^c(x_3|J) \\
&\quad + \mathrm{i}G_2^c(x_1, x_3|J)\mathrm{i}G_1^c(x_2|J) \\
&\quad + \mathrm{i}G_2^c(x_2, x_3|J)\mathrm{i}G_1^c(x_1|J) \\
&\quad + \mathrm{i}G_3^c(x_1, x_2, x_3|J)
\end{aligned} \tag{2.59}$$

$$G_n(x_1, x_2, \ldots, x_n|J) = \sum^{(n)} \prod \mathrm{i}G_k^c(x_1, \ldots, x_k|J) + \mathrm{i}G_n^c(x_1, \ldots, x_n|J).$$

Here the symbol $\Sigma^{(n)}$ denotes the summation over the types of splitting of the n arguments $x_1, x_2, \ldots, x_n$ on the non-intersecting sets and the symbol $\prod$ denotes the product of Green's functions with the arguments from those uncrossing sets (every Green's function has the multiple i). Note that in accordance with definition (2.58) all the Green's functions are symmetrical, therefore the connected Green's functions are also symmetrical functions.

Let us introduce the generating functional of the connected Green's functions $W[J]$ so that

$$G_n^c(x_1, x_2, \ldots, x_n|J) = \frac{\delta^n W[J]}{\delta \mathrm{i} J(x_1) \delta \mathrm{i} J(x_2) \ldots \delta \mathrm{i} J(x_n)}. \tag{2.60}$$

We want to show that the following relation takes place

$$Z[J] = \mathrm{e}^{\mathrm{i}W[J]}. \tag{2.61}$$

We shall prove this relation by induction. Let $n = 1$. Differentiating relation (2.61) with respect to the source we obtain

$$\frac{\delta Z[J]}{\delta \mathrm{i} J(x)} = \mathrm{e}^{\mathrm{i}W[J]}\,\mathrm{i}\frac{\delta W[J]}{\delta \mathrm{i} J(x)}.$$

Using relation (2.61) we can write

$$G_1(x|J) = \mathrm{i}\frac{\delta W}{\delta \mathrm{i} J(x)}.$$

Comparing this relation with relation (2.59) we conclude that

$$G_1^{\mathrm{c}}(x|J) = \frac{\delta W[J]}{\delta \mathrm{i} J(x)}.$$

Let $n = 2$. Then

$$\begin{aligned} G_n(x_1, x_2|J) &= \frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta \mathrm{i} J(x_1)\delta \mathrm{i} J(x_2)} \\ &= \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta \mathrm{i} J(x_1)}\,\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta \mathrm{i} J(x_2)} + \mathrm{i}\frac{\delta^2 W[J]}{\delta \mathrm{i} J(x_1)\delta \mathrm{i} J(x_2)} \\ &= \mathrm{i}G_1^{\mathrm{c}}(x_1|J)\mathrm{i}G_2^{\mathrm{c}}(x_2|J) + \mathrm{i}\frac{\delta^2 W[J]}{\delta \mathrm{i} J(x_1)\delta \mathrm{i} J(x_2)}. \end{aligned}$$

Comparing this relation with relation (2.59) we conclude that

$$G_2^{\mathrm{c}}(x_1, x_2|J) = \frac{\delta^2 W}{\delta \mathrm{i} J(x_1)\delta \mathrm{i} J(x_2)}.$$

Let the following equality be performed for all $k \leqslant n$:

$$G_n^{\mathrm{c}}(x_1, x_2, \ldots, x_k|J) = \frac{\delta^k W[J]}{\delta \mathrm{i} J(x_1)\delta \mathrm{i} J(x_2)\ldots\delta \mathrm{i} J(x_k)}. \tag{2.62}$$

Using relations (2.58) and (2.59) we have

$$\begin{aligned} \frac{\delta^n Z[J]}{\delta \mathrm{i} J(x_1)\ldots\delta \mathrm{i} J(x_n)} = Z[J]\Big(&\sum^{(n)}\prod \mathrm{i}G_k^{\mathrm{c}}(x_1, \ldots, x_k|J) \\ &+ \mathrm{i}G_n^{\mathrm{c}}(x_1, \ldots, x_n|J)\Big). \end{aligned}$$

From this we can write

$$
\begin{aligned}
&G_{n+1}(x_1,\dots,x_n,x_{n+1}|J)\\
&\quad=\frac{1}{Z[J]}\frac{\delta^{n+1}Z[J]}{\delta \mathrm{i}J(x_1)\dots\delta\mathrm{i}J(x_{n+1})}\\
&\quad=\frac{1}{Z[J]}\frac{\delta}{\delta\mathrm{i}J(x_{n+1})}Z[J]\\
&\quad\times\Big(\sum^{(n)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}G^{\mathrm{c}}_n(x_1,\dots,x_n|J)\Big)\\
&\quad=\mathrm{i}G^{\mathrm{c}}_1(x_{n+1}|J)\Big(\sum^{(n)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}G^{\mathrm{c}}_n(x_1,\dots,x_n|J)\Big)\\
&\quad+\sum^{(n)}\frac{\delta}{\delta\mathrm{i}J(x_{n+1})}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}\frac{\delta G^{\mathrm{c}}_n(x_1,\dots,x_n|J)}{\delta\mathrm{i}J(x_{n+1})}\\
&\quad=\mathrm{i}G^{\mathrm{c}}_1(x_{n+1}|J)\Big(\sum^{(n)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)\\
&\quad+\mathrm{i}G^{\mathrm{c}}_n(x_1,\dots,x_n|J)\Big)\\
&\quad+\sum^{(n+1)'}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}\frac{\delta^{n+1}W[J]}{\delta\mathrm{i}J(x_1)\dots\delta\mathrm{i}J(x_n)\delta\mathrm{i}J(x_{n+1})}.
\end{aligned}
$$

Here equalities (2.62) were used. The expression

$$\sum^{(n+1)'}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)$$

contains the summation over all types of splitting of the $n+1$ arguments $x_1,x_2,\dots,x_{n+1}$ over the non-intersecting sets but the argument x_{n+1} enters an arbitrary set of this type at least with one of the other arguments $x_1,x_2,\dots,x_n$. Then

$$
\begin{aligned}
&\mathrm{i}G^{\mathrm{c}}_1(x_{n+1}|J)\Big(\sum^{(n)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}G^{\mathrm{c}}_n(x_1,\dots,x_n|J)\Big)\\
&\quad+\sum^{(n+1)'}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)=\sum^{(n+1)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&G^{\mathrm{c}}_{n+1}(x_1,\dots,x_{n+1}|J)\\
&\qquad=\sum^{(n+1)}\prod \mathrm{i}G^{\mathrm{c}}_k(x_1,\dots,x_k|J)+\mathrm{i}\frac{\delta^{n+1}W[J]}{\delta\mathrm{i}J(x_1)\dots\delta\mathrm{i}J(x_{n+1})}.
\end{aligned}
$$

Comparing this relation with relation (2.59) we conclude that

$$G^c_{n+1}(x_1,\ldots,x_{n+1}|J)=\frac{\delta^{n+1}W[J]}{\delta \mathrm{i}J(x_1)\ldots\delta \mathrm{i}J(x_{n+1})}.$$

Now we can make the final conclusion that the functional $W[J]$ defined by equality (2.61) really is the generating functional of the connected Green's functions.

In the language of the Feynman diagrams the connected Green's function is represented by a connected diagram. Remember that the diagram is called a connected one if it is impossible to separate the parts which are not joined by the lines.

Let us consider

$$\frac{\delta W[J]}{J(x)}=\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta \mathrm{i}J(x)}=\frac{\langle 0|T\hat{\varphi}(x)\exp\left(\mathrm{i}\int \mathrm{d}x\hat{\varphi}(x)J(x)\right)|0\rangle}{\langle 0|T\exp\left(\int \mathrm{d}x\hat{\varphi}(x)J(x)\right)|0\rangle}\equiv\varphi(x).$$

The scalar field $\varphi(x)$ which is defined by means of this relation is called the mean field and is dependent on the source. The relation

$$\varphi(x)=\frac{\delta W[J]}{\delta \mathrm{i}J(x)} \tag{2.63}$$

we consider to be an equation for finding the source $J(x)$. Assume that this equation is solved, then $J(x)=J(x|\varphi)$. That is why the mean field can be considered as the independent functional argument and the source is the functional of it.

Let us introduce the functional $\Gamma[\varphi]$ using the rule

$$\Gamma[\varphi]=W[J]-\int \mathrm{d}x\,\varphi(x)J(x) \tag{2.64}$$

where the source $J(x)$ is expressed from equation (2.63) as the functional of the $\varphi(x)$; $J(x)\equiv J(x|\varphi)$. Hence the functional $\Gamma[\varphi]$ is the functional of the mean field $\varphi(x)$. Note the passage from the functional $W[J]$ to the functional $\Gamma[\varphi]$ with the help of relations (2.64) and (2.63), representing the functional Legendre transformation.

Let us note the two important properties of the functional $\Gamma[\varphi]$.

1. Let us consider

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}=\int \mathrm{d}y\frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\varphi(x)}-J(x)-\int \mathrm{d}y\,\varphi(y)\frac{\delta J(y)}{\delta\varphi(x)}.$$

From relation (2.63) we have $\delta W[J]/\delta J(y) = \varphi(y)$. Then we obtain

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x). \tag{2.65}$$

Relation (2.65) is the basis of another analogy. Let there be a classical field theory with the action, containing the interaction of the field φ with the external source $J(x)$

$$S_0[\varphi] = S[\varphi] + \int \mathrm{d}x\, \varphi(x)J(x).$$

The classical equation of motion corresponding to this action is

$$\frac{\delta S[\varphi]}{\delta\varphi} = -J(x). \tag{2.66}$$

Let us compare relations (2.65) and (2.66). One can see that the functional $\Gamma[\varphi]$ in the quantum theory plays the same role as the classical action $S[\varphi]$ in the classical theory. By reason of this the functional $\Gamma[\varphi]$ is called the effective action. Equation (2.65) can be understood as the equation of motion for the mean field.

2. Let us consider

$$\int \mathrm{d}y \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\,\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(y)\delta\varphi(z)} = -\int \mathrm{d}y \frac{\delta\varphi(x)}{\delta J(y)}\,\frac{\delta J(y)}{\delta\varphi(z)} = -\delta(x-z).$$

Here relations (2.63) and (2.65) are used.

However, from relation (2.60) we have

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = -G_2^c(x,y|J).$$

Then

$$\int \mathrm{d}z\, G_2^c(x,y|J)\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(y)\delta\varphi(z)} = \delta(x-z).$$

Thus

$$\frac{\delta^2\Gamma[\varphi]}{\delta J(x)\,\delta J(y)} = G_2^{c^{-1}}(x,y|J). \tag{2.67}$$

Let us introduce the so-called n-point vertex functions

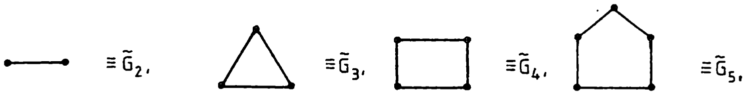
$$\Gamma_n(x_1, x_2, \ldots, x_n) \quad n \geqslant 2.$$

The definition of these functions can be given by induction.

For the simplification of the intermediate transformations we can use the following notation:

$$\mathrm{i}^n G_n^c(x_1, \ldots, x_n | J) \equiv \widetilde{G}_n^c(x_1, \ldots, x_n) = \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)}. \tag{2.68}$$

Let us represent the functions $\widetilde{G}_n$ by means of polygons. For example,



and so on.

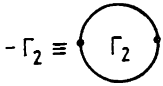
The two-point vertex function $\Gamma_2(x_1, x_2)$ is defined by the relation

$$\widetilde{G}_2(x_1, x_2) = -\int \mathrm{d}y_1 \mathrm{d}y_2 \widetilde{G}_2(x_1, y_1) \widetilde{G}_2(x_2, y_2) \Gamma_2(y_1, y_2). \tag{2.69}$$

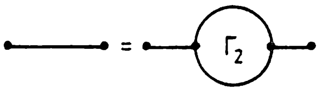
From this relation we have

$$\Gamma_2(x_1, y_2) = -\widetilde{G}_2^{-1}(x_1, x_2) = G_2^{c\,-1}(x_1, x_2). \tag{2.70}$$

If we represent



then relation (2.69) can be represented in the form



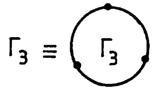
(2.71)

Relations (2.70) and (2.71) show that the two-point vertex function is obtained (with arbitrariness up to a sign) by amputating the external lines from the function $\widetilde{G}_2$.

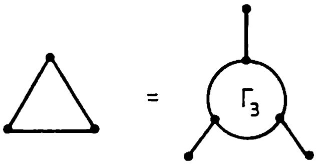
The three-point vertex function $\Gamma_3(x_1, x_2, x_3)$ is defined by the relation

$$\widetilde{G}_3(x_1, x_2, x_3) = \int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \times \widetilde{G}_2(x_1, y_1) \widetilde{G}_2(x_2, y_2) \widetilde{G}_2(x_3, y_3) \Gamma_3(y_1, y_2, y_3). \tag{2.72}$$

Let us represent



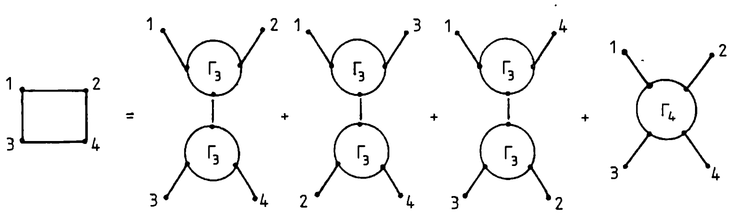
then relation (2.72) can be represented as



(2.73)

We see that the three-point vertex function is obtained by amputating the external lines from the function $\widetilde{G}_3$.

The higher vertex functions have the most convenient definition on the basis of diagrammatic language. Let us represent $\Gamma_n, n \geqslant 3$ as a circle with n points. Then, for example,



(2.74)

Relation (2.74) is the definition of the four-point vertex function. We see that the four-point vertex function can be obtained from the function $\widetilde{G}_4$ rejecting the diagram which can be made unconnected by disconnecting one internal line and amputating the external lines of the remaining diagrams.

The general definition of the n-point vertex function can be formulated in diagram language in the following way:

1. represent all n-point connected diagrams with external lines amputated;
2. reject diagrams which contain k-point vertex functions at $k < n$ and which can be made disconnected by breaking one internal line (these diagrams are called one-particle reducible);

3. the remaining n-point diagrams are connected, they do not contain the external lines and are one-particle irreducible. Such diagrams correspond to the n-point vertex functions.

Let us show that the effective action is the generating functional of the vertex functions. That is, we will show that the following relation is fulfilled

$$\Gamma_n(x_1, x_2, \ldots, x_n) = \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_2)\ldots\delta\varphi(x_n)}. \tag{2.75}$$

The proof of relation (2.75) is made by induction.

At $n = 2$ we obtain from relations (2.70), (2.67) that relation (2.75) is fulfilled. Let us proceed to the cases where $n > 2$. First of all we obtain a useful relation. Consider

$$\begin{aligned}\frac{\delta}{\delta J(x)} &= \int \mathrm{d}y\, \frac{\delta\varphi(y)}{\delta J(x)}\frac{\delta}{\delta\varphi(y)} \\ &= \int \mathrm{d}y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\frac{\delta}{\delta\varphi(y)} = \int \mathrm{d}y\, \widetilde{G}_2(x,y)\frac{\delta}{\delta\varphi(y)}.\end{aligned}$$

This equality may be reduced to

$$\frac{\delta}{\delta J} = \widetilde{G}_2\frac{\delta}{\delta\varphi}$$

or

$\frac{\delta}{\delta J} =$ $\frac{\delta}{\delta\varphi}$

This relation shows that the arbitrary differentiation of the diagram with respect to the source adds an external line to the diagram. Let us compute

$\frac{\delta}{\delta\varphi}($ $)$

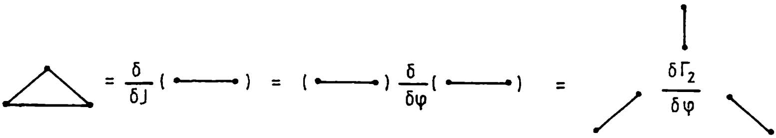
Using relation (2.70) we have

$$\begin{aligned}\frac{\delta\widetilde{G}_2(x,y)}{\delta\varphi(z)} &= -\frac{\delta\Gamma_2^{-1}(x,y)}{\delta\varphi(z)} \\ &= \int \mathrm{d}u_1\mathrm{d}u_2\, \widetilde{G}_2(x,u_1)\frac{\delta\Gamma_2(u_1,u_2)}{\delta\varphi(z)}\widetilde{G}_2(u_2,y).\end{aligned}$$

This relation can be represented as

(2.76)

Let us consider



(2.77)

We have from this relation

$$\widetilde{G}_3(x_1,x_2,x_3) = \int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \widetilde{G}_2(x_1,y_1)\widetilde{G}_2(x_2,y_2)\widetilde{G}_2(x_3,y_3) \times \frac{\delta\Gamma_2(y_1,y_2)}{\delta\varphi(y_3)}. \tag{2.78}$$

But we have obtained that

$$\Gamma_2(y_1,y_2) = \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(y_1)\delta\varphi(y_2)}.$$

Then

$$\frac{\delta\Gamma_2(y_1,y_2)}{\delta\varphi(y_1)} = \frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y_1)\delta\varphi(y_2)\delta\varphi(y_3)}.$$

Relation (2.78) can be rewritten in the form

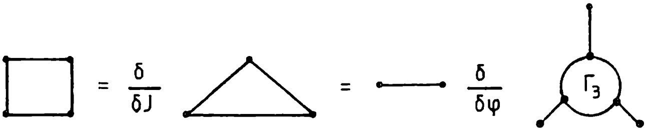
$$\widetilde{G}_3(x_1,x_2,x_3) = \int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \widetilde{G}_2(x_1,y_1)\widetilde{G}_2(x_2,y_2)\widetilde{G}_2(x_3,y_3) \times \frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y_1)\delta\varphi(y_2)\varphi(y_3)}. \tag{2.79}$$

Taking into account equalities (2.72), (2.78) and (2.73), (2.77) we get

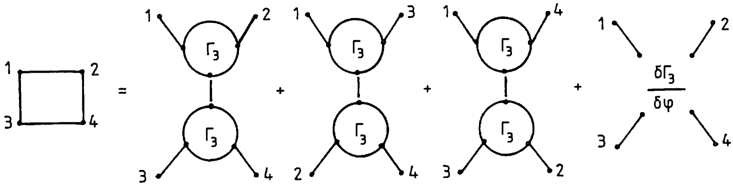
$$\Gamma_3(y_1,y_2,y_3) = \frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y_1)\delta\varphi(y_2)\delta\varphi(y_3)}.$$

Relation (2.75) is therefore proved for $n = 3$.

Let us consider



The derivative $\delta/\delta\varphi$ may act at every one of the external lines and at Γ_3. The action on the external line is given by relation (2.76) taking into account that $\delta\Gamma_2/\delta\varphi = \Gamma_3$. Then



(2.80)

Here the last term is

$$\int dy_1 dy_2 dy_3 dy_4 \widetilde{G}_2(x_1, y_1)\widetilde{G}_2(x_2, y_2)\widetilde{G}_2(x_3, y_3)\widetilde{G}_2(x_4, y_4) \times \frac{\delta\Gamma_3(y_1, y_2, y_3)}{\delta\varphi(y_4)}. \tag{2.81}$$

Then we have from relations (2.74), (2.80) and (2.81) that

$$\Gamma_4(y_1, y_2, y_3, y_4) = \frac{\delta\Gamma_3(y_1, y_2, y_3)}{\delta\varphi(y_4)}.$$

Taking into account relation (2.75) at $n = 3$ we obtain

$$\Gamma_4(y_1, y_2, y_3, y_4) = \frac{\delta^4\Gamma[\varphi]}{\delta\varphi(y_1)\delta\varphi(y_2)\delta\varphi(y_3)\delta\varphi(y_4)}.$$

Thus, relation (2.75) is proved for $n = 4$.

The proof of relation (2.75) for other values of n is carried out in the same way and we will not present it here.

Note, in conclusion, one more analogy between the classical action $S[\varphi]$ and effective action $\Gamma[\varphi]$. The classical action is the generating functional of the classical n-point vertex functions. Really the relations

$$S_n(x_1, x_2, \ldots, x_n) \equiv \frac{\delta^n S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)\ldots\delta\varphi(x_n)} \qquad n \geqslant 3$$

describe the field interaction and play the role of the vertices in the construction of Feynman diagrams. The effective action is the generating functional of the quantum vertex functions including the classical vertex functions and all their quantum corrections. However, due to the locality of the classical Lagrangian the classical vertex functions are the local ones in space–time, but generally the quantum vertex functions are not local. Moreover, in the simplest field theory models, where the Lagrangian is a polynomial in the fields and their derivatives we have a finite number of the classical vertex functions only †. As to the quantum vertex functions, their number is always infinite.

Let us note now a remark about the effective action when the spinor fields are present. Consider the generating functional $Z[J, \eta, \bar{\eta}]$ (2.54) and (2.55). We introduce the generating functional of the connected Green's functions $W[J, \eta, \bar{\eta}]$ as $Z[J, \eta, \bar{\eta}] = \exp(\mathrm{i}W[J, \eta, \bar{\eta}])$ and make the Legendre functional transformation

$$\Gamma[\varphi, \psi, \bar{\psi}] = W[J, \eta, \bar{\eta}] - \int \mathrm{d}x(\varphi(x)J(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x))$$

where the sources $J(x)$, $\eta(x)$ and $\bar{\eta}(x)$ are expressed as functions of the mean field $\varphi(x)$, $\psi(x)$ and $\bar{\psi}(x)$ from the equations

$$\frac{\delta\Gamma[\varphi, \psi, \bar{\psi}]}{\delta\varphi(x)} = -J(x)$$
$$\frac{\delta\Gamma[\varphi, \psi, \bar{\psi}]}{\delta\psi(x)} = -\bar{\eta}(x)$$
$$\frac{\delta\Gamma[\varphi, \psi, \bar{\psi}]}{\delta\bar{\psi}(x)} = -\eta(x).$$

Here the derivatives with respect to ψ are those on the right-hand side and those with respect to $\bar{\psi}$ are on the left. Further consideration is made in the analogy with the scalar field case.

† In the essentially non-linear models, such as the sigma-model or gravity model, the number of the classical vertex functions is infinite.

2.4 The loop expansion

Let us consider the generating functional of Green's functions. It can be written in the following form (taking into account (2.61) and (2.44)):

$$\exp\left(\frac{\mathrm{i}W[J]}{\hbar}\right) = \int \mathrm{D}\varphi \exp\left[\frac{\mathrm{i}}{\hbar}\Big(S[\varphi] + \int \mathrm{d}x\, \varphi(x)J(x)\Big)\right]. \qquad (2.82)$$

Note that the functional measure is not taken into account for the above-mentioned reasons. In principle we can also make all these considerations with functional measure, adding to the classical action the value $\Delta S = -\mathrm{i}\,\mathrm{Tr}\ln\mu$ (see (2.45)).

We introduce the Planck constant $\hbar$, which was previously taken to be unity, into relation (2.82). Such a dimensional constant is necessary to give the correct dimensions for all the quantities in relation (2.82). Really, the expression under the exponent is dimensionless, but $S[\varphi]$ and $\int \mathrm{d}x\varphi(x)J(x), W[J]$ have the dimension of the action. Therefore, we must take into account the fundamental constant of the action dimension. There is one such constant which is $\hbar$. Of course, if we take into account the constant $\hbar$ from the beginning of the derivation of the $Z[J]$ expression, we obtain relation (2.82) automatically.

Let us introduce the effective action $\Gamma[\varphi]$ on the basis of relations (2.63) and (2.64). Then we have

$$\exp\left[\frac{\mathrm{i}}{\hbar}\Big(\Gamma[\phi] + \int \mathrm{d}x\phi(x)J(x)\Big)\right] = \int \mathrm{D}\varphi \exp\left[\frac{\mathrm{i}}{\hbar}\Big(S[\varphi] + \int \mathrm{d}x\varphi(x)J(x)\Big)\right]. \qquad (2.83)$$

Now the source $J(x)$ is the functional of the mean field $\phi(x)$ and satisfies equation (2.65) $J(x) = -\delta\Gamma[\phi]/\delta\phi(x)$.

Let us make the change of variable $\varphi \to \varphi + \phi$, where ϕ is the mean field. Then we get

$$\exp\frac{\mathrm{i}}{\hbar}\Big(\Gamma[\phi] + \int \mathrm{d}x\, \phi(x)J(x)\Big) = \int \mathrm{D}\varphi \exp\frac{\mathrm{i}}{\hbar}\Big(S[\varphi+\phi] + \int \mathrm{d}x(\varphi(x)+\phi(x))J(x)\Big).$$

or

$$\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma[\phi]\right) = \int \mathrm{D}\varphi \exp\left[\frac{\mathrm{i}}{\hbar}\Big(S[\varphi+\phi] - \int \mathrm{d}x\, \varphi(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\Big)\right]. \qquad (2.84)$$

The functional $S[\phi+\varphi]$ may be expanded in a functional Taylor series on the fields $\phi(x)$

$$S[\phi+\varphi] = S[\phi]+\sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\, S_n(x_1,\ldots,x_n|\phi)\varphi(x_1)\ldots\varphi(x_n) \tag{2.85}$$

where

$$S_n(x_1,\ldots,x_n|\phi) = \frac{\delta^n S[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}$$

are the classical vertex functions, depending on the mean field $\phi(x)$.

To simplify the form of further relations we introduce the following notation:

$$\int \mathrm{d}x_1\ldots\mathrm{d}x_n S(x_1,\ldots,x_n|\phi)\varphi(x_1)\ldots\varphi(x_n) \equiv S_n[\phi]\varphi^n. \tag{2.86}$$

and

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} \equiv \Gamma_1[\phi] \quad \int \mathrm{d}x\varphi(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)} \equiv \varphi\Gamma_1[\phi]. \tag{2.87}$$

Taking into account relations (2.85) and (2.86), we can rewrite (2.84) as

$$\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma[\phi]\right) = \int \mathrm{D}\varphi\exp\left(\frac{\mathrm{i}}{\hbar}(S[\phi]+\tfrac{1}{2}S_2[\phi]\varphi^2 + \sum_{n=3}^{\infty}\frac{1}{n!}S_n[\phi]\varphi^n - \varphi(\Gamma_1[\phi]-S_1[\phi]))\right). \tag{2.88}$$

Let us make the substitution $\varphi = \hbar^{1/2}\varphi$. Then

$$\exp\left(\frac{\mathrm{i}}{\hbar}(\Gamma[\phi]-S[\phi])\right) = \int \mathrm{D}\varphi\exp\left[\mathrm{i}\left(\frac{1}{2}S_2[\phi]\varphi^2 + \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\phi]\varphi^n - \hbar^{-\frac{1}{2}}\varphi(\Gamma_1[\phi]-S_1[\phi])\right)\right]. \tag{2.89}$$

Note that the effective action $\Gamma[\phi]$ appears in relation (2.89) only in the combination $\Gamma[\phi]-S[\phi]\equiv\bar{\Gamma}[\phi]$. The functional $\bar{\Gamma}[\phi]$ is a power series in $\hbar$,

$$\bar{\Gamma}[\phi] = \sum_{n=1}^{\infty}\hbar^n\bar{\Gamma}^{(n)}[\phi]. \tag{2.90}$$

Hence the functional $\bar{\Gamma}[\phi]$ represents all the quantum corrections to the classical action $S[\phi]$. Therefore, we can say that the effective action is the classical action with all quantum corrections made to it.

Taking equality (2.90) into account we can write the relation (2.89) in the form

$$\exp\left(\mathrm{i}\sum_{n=1}^{\infty}\hbar^{n-1}\bar{\Gamma}^{(n)}[\phi]\right) = \int \mathrm{D}\varphi \exp\left[\mathrm{i}\left(\tfrac{1}{2}S_2[\phi]\varphi^2 + \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\phi]\varphi^n \right.\right.$$
$$\left.\left. - \sum_{n=1}^{\infty}\hbar^{-\frac{1}{2}+n}\varphi\bar{\Gamma}_1^{(n)}[\phi]\right)\right]. \qquad (2.91)$$

Relation (2.91) is the basis for the construction of a perturbation theory in powers of $\hbar$. Really the functional under the exponent on the right side of this relation can be understood as the action of some theory where the quadratic term $\frac{1}{2}S_2[\phi]\varphi^2$ defines the propagator and the other terms describe the fields interactions. Expanding the exponent in (2.91) in powers of $\hbar$ we obtain the perturbation series where the expansion parameter is $\hbar$. Every term of this series may be represented by the standard Feynman diagrams where the propagator and the vertices depend on the mean field ϕ. As a result the perturbation series for the quantum corrections $\bar{\Gamma}[\phi]$ to the effective action can be consistently calculated.

Two remarks can be made concerning the structure on the right-hand side of relation (2.91). One might think that the perturbation series would contain non-integer powers of $\hbar$. However, the non-zero contribution to the functional integral

$$\int \mathrm{D}\varphi \exp(\mathrm{i}\tfrac{1}{2}S_2[\phi]\varphi^2)\varphi(x_1)\varphi(x_2)\ldots\varphi(x_N) \qquad (2.92)$$

is caused by even N only and because of this the non-integer powers of $\hbar$ are absent in the perturbation series.

What is the role of the terms

$$-\sum_{n=1}^{\infty}\hbar^{n-1/2}\varphi\bar{\Gamma}_1^{(n)}[\phi]$$

in expression (2.91)? After expansion of the series these terms only lead to one-particle reducible diagrams. They cancel the one-particle reducible diagrams appearing from the other terms of the expansion. As a result the right part of (2.91) will contain only one-particle irreducible diagrams.

The perturbation series in powers of $\hbar$ is called the loop expansion or the expansion on the number of loops in the Feynman diagrams. This name is connected with the fact that the expansion in powers of $\hbar$ correspond to the series expansion where every term contains only the diagrams with the given number of the loops (that is with the given number of independent integrals over the four-momenta).

Let us consider the application of relation (2.91) for the first (one-loop) and second (two-loop) corrections to the effective action. (Sometimes, we write EA for effective action.)

In the one-loop approximation relation (2.91) has the form

$$\exp(\mathrm{i}\bar{\Gamma}^{(1)}[\phi]) = \int \mathrm{D}\varphi \exp\left(\frac{\mathrm{i}}{2} S_2[\phi]\varphi^2\right).$$

Then

$$\exp(\mathrm{i}\bar{\Gamma}^{(1)}[\phi]) = \det{}^{-1/2} S_2[\phi]$$

and hence

$$\bar{\Gamma}^{(1)}[\phi] = \frac{\mathrm{i}}{2} \operatorname{Tr}\ln S_2[\phi] = \frac{\mathrm{i}}{2} \ln\det S_2[\phi]. \tag{2.93}$$

Using this relation we can write the EA in the one-loop approximation as

$$\Gamma^{(1)}[\phi] = S[\phi] + \bar{\Gamma}^{(1)}[\phi] = S_2[\phi] + \frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln S_2[\phi]. \tag{2.94}$$

In the various models of quantum field theory the expressions $S_2[\phi]$ are the kernels of some differential operators, depending on the mean field. Therefore, finding the one-loop effective action reduces to the mathematical problem of the calculation of determinants of differential operators.

For example, in the scalar field theory with the Lagrangian (2.1) the kernel of the corresponding operator has the form

$$S_2[\phi] \equiv S_2(x_1, x_2|\phi) = -[\Box + m^2 + V''(\phi(x_1))]\delta(x_1 - x_2).$$

Let us now consider the two-loop approximation. From relation (2.91) we have

$$\begin{aligned}
&\exp[\mathrm{i}(\bar{\Gamma}^{(1)}[\phi] + \hbar\bar{\Gamma}^{(2)}[\phi])] \\
&\qquad = \int \mathrm{D}\varphi \exp\left[\left(\mathrm{i}(\tfrac{1}{2} S_2[\phi]\varphi^2 + \frac{\hbar^{\frac{1}{2}}}{3!} S_3[\phi]\varphi^3\right.\right. \\
&\qquad\qquad \left.\left. + \frac{\hbar}{4!} S_4[\phi]\varphi^4 - \hbar^{1/2}\varphi\bar{\Gamma}_1^{(1)}[\phi]\right)\right].
\end{aligned} \tag{2.95}$$

Here the terms which give a contribution to $\bar{\Gamma}^{(2)}$ have been retained. Expanding both parts of this relation to first order in $\hbar$ we obtain

$$\bar{\Gamma}^{(2)}[\phi] = \frac{\mathrm{i}}{2(3!)^2}\langle (S_3[\phi]\varphi^3)(S_3[\phi]\varphi^3)\rangle + \frac{1}{4!}\langle S_4[\phi]\varphi^4\rangle + \frac{\mathrm{i}}{2}\langle (\Gamma_1^{(1)}[\phi]\varphi)(\Gamma_1^{(1)}[\phi]\varphi)\rangle. \tag{2.96}$$

Here we denote

$$\langle \ldots \rangle = \frac{\int \mathrm{D}\varphi \exp(\frac{\mathrm{i}}{2}S_2[\phi]\varphi^2)(\ldots)}{\int \mathrm{D}\varphi \exp(\frac{i}{2}S_2[\phi]\varphi^2)}.$$

Note that

$$\langle \varphi(x)\varphi(y)\rangle = \mathrm{i}G(x,y|\phi)$$

where $G(x,y|\phi)$ is the Green's function, depending on the mean field $\phi(x)$ and satisfying the equation

$$\int \mathrm{d}z\, S_2(x,z|\phi)G(z,y|\phi) = \delta(x-y)$$

or in a compact form

$$S_2[\phi]G[\phi] = 1 \qquad G[\phi] = S_2[\phi]^{-1}. \tag{2.97}$$

The following notation is used in relation (2.96)

$$\langle (S_3[\phi]\varphi^3)S_3[\phi]\varphi^3)\rangle \equiv \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 S_3(x_1,x_2,x_3|\phi) \times \langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(y_1)\varphi(y_2)\varphi(y_3)\rangle S_3(y_1,y_2,y_3|\phi).$$

$$\langle S_4[\phi]\varphi^4\rangle \equiv \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 S_4(x_1,x_2,x_3,x_4|\phi)\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

$$\langle (\Gamma_1^{(1)}[\phi]\varphi)(\Gamma_1^{(1)}[\phi]\varphi)\rangle = \int \mathrm{d}x_1\mathrm{d}x_2 \frac{\delta\bar{\Gamma}^{(1)}[\phi]}{\delta\varphi(x_1)}\langle \varphi(x_1)\varphi(x_2)\rangle \frac{\delta\bar{\Gamma}^{(1)}[\phi]}{\delta\varphi(x_2)}. \tag{2.98}$$

The right-hand side of relation (2.96) may be transformed with the help of Wick's theorem (which in the functional integral language is reduced simply to the calculation of functional integrals like (2.92)).

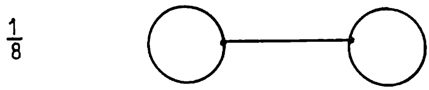
Let us consider first of all the last term in relation (2.96). Using the explicit form of $\bar{\Gamma}^{(1)}[\phi]$(2.93) we obtain

$$\bar{\Gamma}_1^{(1)}[\phi] = \frac{\mathrm{i}}{2} \operatorname{Tr} S_2[\phi]^{-1} S_3[\phi] = \frac{\mathrm{i}}{2} \operatorname{Tr} G_2[\phi] S_3[\phi].$$

Hence

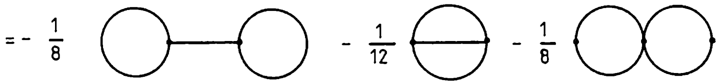
$$\frac{\mathrm{i}}{2}\langle(\Gamma_1^{(1)}[\phi]\varphi)(\Gamma_1^{(1)}[\phi]\varphi)\rangle = \frac{1}{8} S_3[\phi]G[\phi]G[\phi]G[\phi]S_3[\phi]$$
$$\equiv \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 S_3(x_1,x_2,x_3|\phi)G(x_1,x_2|\phi)G(x_3,y_3|\phi)$$
$$\times\, G(y_1,y_2|\phi)S_3(y_1,y_2,y_3|\phi). \tag{2.99}$$

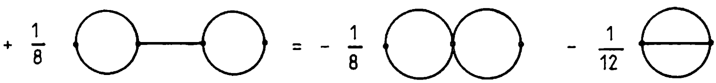
This expression is represented by the following diagram:



Now consider the other terms on the right-hand side of (2.96). Using Wick's theorem we get

$\bar{\Gamma}^{(2)}[\phi]$





$$= -\frac{1}{8}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 S_4(x_1,x_2,x_3,x_4|\phi)G(x_1,x_2|\phi)G(x_3,x_4|\phi)$$
$$-\frac{1}{12}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 S_3(x_1,x_2,x_3|\phi)$$
$$\times\, G(x_1,y_1|\phi)G(x_2,y_2|\phi)G(x_3,y_3|\phi)S_3(y_1,y_2,y_3|\phi). \tag{2.100}$$

The relation (2.100) represent the expression for the two-loop correction to the EA.

2.5 The effective potential

According to the results of section 2.4 the calculation of the effective action is made in the framework of the loop expansion. For finding the one-loop EA it is necessary to calculate the determinants of the differential operators, dependent on the mean field. For higher order corrections it is necessary to find first of all the Green's function $G(x,y|\phi)$, depending on the mean field. Neither the first problem nor the second can be solved in the general case.

Therefore the problem of development of the approximate methods for the effective action calculation has arisen. We shall consider one of these methods, allowing us to compute the effective action for the slowly changing mean fields when we can neglect their derivatives.

Let the mean field $\varphi(x)$ change slowly in space–time. Then the effective action which is the essentially non-local object may be found as a series in the mean field derivatives. For a slowly changing field we can leave in these series a finite number of terms that lead to the so-called local approximation for the effective action, i.e.

$$\Gamma[\varphi] = \int dx[-V_{\text{eff}}(\varphi) + \tfrac{1}{2}Z(\varphi)\eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \ldots]. \qquad (2.101)$$

Every term in the right-hand side of (2.101) is a function of the mean field $\varphi(x)$. The first term $V_{\text{eff}}(\varphi)$ in the local approximation for EA (2.101) is called the effective potential. It is obvious that the calculation of the effective potential is connected only with finding the effective action at the constant mean field, $\varphi(x) = \varphi = \text{const}$.

In principle the calculation of the effective potential in the arbitrary order of the loop expansion is a real problem. For such calculations it is necessary to find the determinants of the differential operators with a constant coefficient and to find the Green's functions in the constant mean field. Both of these problems in principle are solvable. Of course, the computation of the effective potential in higher orders of the loop expansion may be accompanied by severe technical difficulties.

Let us consider the mean field equation of motion (2.65) when the source is switched off. For the constant mean field this equation, taking account of (2.101), leads to

$$\frac{\partial V_{\text{eff}}(\varphi)}{\partial\varphi} = 0. \qquad (2.102)$$

Relation (2.102) is the exact equation which allows us to find the constant field leading to the minimum of the effective action. Such a constant field is of interest in the problems of symmetry breaking and vacuum stability.

Relations (2.91) and (2.101) show that the effective potential can be calculated in the framework of the loop expansion. We shall consider the one-loop effective potential calculation for the scalar field theory with Lagrangian (2.1). In the one-loop approximation the effective action is given by (2.94), where $S_2(x, y|\phi) = -(\Box + m^2 + V''(\phi))\delta(x-y)$. The problem reduces to the calculation of $\det S_2[\varphi]$.

For the calculation of $\det S_2[\varphi]$ we shall use the method of the generalized zeta-function, which is frequently encountered (see also Chapter 6).

First of all we shall give the appropriate definitions. Let A be the Hermitian differential operator with a positive spectrum. Consider the eigenvalues problem

$$A f_n = \lambda_n f_n. \tag{2.103}$$

Here f_n are the eigenfunctions and λ_n are the eigenvalues which are assumed to be discrete. Suppose that the functions form the orthonormal and complete set, that is

$$\int \mathrm{d}x\, f_n(x) f_m(x) = \delta_{nm} \qquad \sum_n f_n(x) f_n(y) = \delta(x-y). \tag{2.104}$$

Let us introduce the function $\zeta(z|A)$, depending on the complex variable z by the rule

$$\zeta(z|A) = \sum_n \frac{1}{\lambda_n^z}. \tag{2.105}$$

In the regions of the complex plane where these series are divergent the function $\zeta(z|A)$ is defined by means of the analytic continuation. The function $\zeta(z|A)$ is called the generalized zeta-function connected with the operator A.

We define

$$\det A = \prod_n \lambda_n. \tag{2.106}$$

and note that

$$\left.\frac{\mathrm{d}\zeta(z|A)}{\mathrm{d}z}\right|_{z=0} = -\ln \prod_n \lambda_n.$$

Then we have

$$\det A = \exp[-\zeta'(0|A)]. \tag{2.107}$$

This relation is the main one for the calculation of differential operator determinants.

Hence, our problem consists in finding $\zeta(z|A)$. Usually the explicit calculation of $\zeta(z|A)$ by means of formula (2.105) is very inconvenient. However, finding $\zeta(z|A)$ is connected with the solution of some differential equation.

For the given operator A let us define the differential equation

$$\mathrm{i}\frac{\partial K(x,y|s)}{\partial s} = AK(x,y|s) \tag{2.108}$$

where operator A acts on the first argument of $K(x,y|s)$. Suppose that the function $K(x,y|s)$ satisfies the following initial condition

$$K(x,y|0) = \delta(x-y). \tag{2.109}$$

Expand the function $K(x,y|s)$ in the complete system of the eigenfunctions of the operator A. Then we have

$$K(x,y|s) = \sum_n g_n(s) f_n(x) f_n(y)$$

and in accordance with (2.109) and (2.104) we can find $g_n(0) = 1$. Let us substitute this expansion in (2.108) and take into account (2.103). Then we obtain $g_n(s) = \exp[-\lambda_n s]$. Hence

$$K(x,y|s) = \sum_n f_n(x) f_n(y) \exp(-\lambda_n s). \tag{2.110}$$

It is not difficult to note that

$$\int \mathrm{d}x K(x,x|s) = \sum_n \int \mathrm{d}x f_n(x) f_n(x) \exp(-\lambda_n s) = \sum_n \exp(-\lambda_n s)$$

where the condition of orthonormality (2.104) was used. Taking into account the equality

$$\sum_n \int_0^\infty \mathrm{i}\,\mathrm{d}s(\mathrm{i}s)^{z-1} \exp[-\lambda_n s] = \sum_n \frac{1}{\lambda_n^z}\Gamma(z) \tag{2.111}$$

where $\Gamma(z)$ is the gamma-function. The relations (2.108)–(2.111) and (2.105) show that the generalized zeta-function may be represented in the form

$$\zeta(z|A) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{i}\mathrm{d}s(\mathrm{i}s)^{z-1} \int \mathrm{d}x K(x,x|s) \tag{2.112}$$

where $K(x,x|s) = K(x,y|s)|_{y=x}$ and $K(x,y|s)$ is the solution of equation (2.108) with initial condition (2.109).

Now let us consider relation (2.112) as the definition of the generalized zeta-function for an arbitrary differential operator A. Consider (2.106) as the definition of the determinant of an arbitrary differential operator A. In relation (2.112) the integral over s should be computed for those values of z for which this integral is convergent. For other values of z the function $\zeta(z|A)$ is defined by means of analytic continuation.

Let us apply the method of the generalized zeta-function for finding a one-loop effective potential of the scalar field theory with the Lagrangian (2.1). The one-loop EA may be represented in the form

$$\Gamma^{(1)}[\varphi] = S[\varphi] + \frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln\frac{A}{\mu^2} = S[\varphi] + \frac{\mathrm{i}}{2}\hbar \ln\det\frac{A}{\mu^2}$$
$$A = -(\Box + m^2 + V''(\varphi)) \qquad \varphi = \text{constant}. \tag{2.113}$$

The operator A has the dimension 2 in mass units, but the expression under the logarithm must be dimensionless. For this reason we introduced the arbitrary constant μ having the dimension of mass. In principle the parameter μ must be written in relation (2.93).

From (2.113) and (2.107) we have

$$\Gamma^{(1)}[\varphi] = S[\varphi] + \frac{\mathrm{i}}{2}\hbar\zeta'\left(0\middle|\frac{A}{\mu^2}\right).$$

Using relation (2.112) and the equalities

$$\Gamma^{(1)}[\varphi] = -\int \mathrm{d}x V^{(1)}_{\text{eff}}(\varphi) + \dots$$
$$S[\varphi] = -\int \mathrm{d}x(V(\varphi) + \tfrac{1}{2}m^2\varphi^2 + \dots)$$

we obtain

$$V^{(1)}_{\text{eff}} = \tfrac{1}{2}m^2\varphi^2 + V(\varphi) + \frac{\mathrm{i}\hbar}{2}\zeta'\left(x,0\middle|\frac{A}{\mu^2}\right)$$
$$\zeta\left(x,z\middle|\frac{A}{\mu^2}\right) = \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{i}\,\mathrm{d}s\,(\mathrm{i}s)^{z-1}K(x,x|s). \tag{2.114}$$

Thus, for the one-loop effective potential computation it is necessary to find $K(x,y|s)$.

Let us write equation (2.108) where the operator A is given by (2.113)

$$\mathrm{i}\frac{\partial K(x,y|s)}{\partial s} = -\frac{1}{\mu^2}(\Box + m^2 + V''(\varphi))K(x,y|s). \tag{2.115}$$

The solution of this equation, which satisfies initial condition (2.109) has the form

$$K(x,y|s) = -\frac{\mathrm{i}\mu^4}{(4\pi \mathrm{i}s)^2}\exp\left(\frac{\mathrm{i}\mu^2}{4s}(x-y)^2 + \frac{\mathrm{i}}{\mu^2}(m^2 + V''(\varphi))s\right). \tag{2.116}$$

Here $(x-y)^2 = \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)$. Then the function $\zeta(x,z|A)$ is

$$\begin{aligned}\zeta\left(x,z\left|\frac{A}{\mu^2}\right.\right) &= -\frac{\mathrm{i}\mu^4}{(4\pi)^2\Gamma(z)}\int_0^\infty \frac{\mathrm{i}ds}{(\mathrm{i}s)^2}(\mathrm{i}s)^{z-1}\exp\left(\frac{\mathrm{i}}{\mu^2}(m^2 + V''(\varphi))s\right)\\ &= \frac{\mathrm{i}\mu^4}{(4\pi)^2}\left(\frac{m^2 + V''(\varphi)}{\mu^2}\right)^{z-2}\frac{\Gamma(z-2)}{\Gamma(z)}.\end{aligned} \tag{2.117}$$

Hence,

$$\begin{aligned}V_{\mathrm{eff}}^{(1)} &= \tfrac{1}{2}m^2\varphi^2 + V(\varphi)\\ &\quad - \frac{\hbar}{2(4\pi)^2}\mu^4\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\frac{m^2 + V''(\varphi)}{\mu^2}\right)^{z-2}\frac{\Gamma(z-2)}{\Gamma(z)}\right].\end{aligned} \tag{2.118}$$

Using the known properties of the gamma-function we obtain $\Gamma(z-2)/\Gamma(z) = 1/(z-1)(z-2)$. Substituting this relation into (2.118) and carrying out elementary transformations we obtain

$$V_{\mathrm{eff}}^{(1)} = \tfrac{1}{2}m^2\varphi^2 + V(\varphi) + \frac{\hbar}{64\pi^2}(m^2 + V''(\varphi))\left(\ln\frac{m^2 + V''(\varphi)}{\mu^2} - \frac{3}{2}\right). \tag{2.119}$$

Relation (2.119) should be considered as the final one. However, it is still incomplete in one respect. The effective potential (2.119) contains the arbitrary constant μ^2. Note that it contains other constants too. Let, for example, $V(\varphi) = (1/4!)f\varphi^4$, where f is the scalar coupling constant. Then the effective potential depends on the parameters m^2 and f. These parameters, m^2 and f, are not the observable mass and coupling constant. For the introduction of the observable mass and coupling constant it is necessary to define these parameters taking into account the conditions of the measurement. It is known that the additional relations must be incorporated in to the effective potential and on the basis of it we can express the effective potential in terms of the observable mass and coupling constant.

Let m_R^2 and f_R be the observable values of the mass and coupling constant. Let us define them with the help of the following conditions

$$\left.\frac{d^2 V_{\text{eff}}^{(1)}(\varphi)}{d\varphi^2}\right|_{\varphi=\varphi_1} = m_R^2 \qquad \left.\frac{d^4 V_{\text{eff}}^{(1)}(\varphi)}{d\varphi^4}\right|_{\varphi=\varphi_2} = f_R. \tag{2.120}$$

Here φ_1 and φ_2 are constants of the mass dimension, reflecting the energy scales on which the mass and coupling constant are measured.

Let us consider how (2.120) works. Suppose for simplicity that $m = 0$. Then the effective potential (2.119) is

$$V_{\text{eff}}^{(1)}(\varphi) = \frac{f\varphi^4}{4!} + \frac{\hbar f^2 \varphi^4}{256\pi^2}\left(\ln\frac{f\varphi^2}{2\mu^2} - \frac{3}{2}\right). \tag{2.121}$$

Let

$$\left.\frac{d^2 V_{\text{eff}}^{(1)}(\varphi)}{d\varphi^2}\right|_{\varphi=0} = 0 \qquad \left.\frac{d^4 V_{\text{eff}}^{(1)}(\varphi)}{d\varphi^4}\right|_{\varphi=\varphi_0} = f_R. \tag{2.122}$$

Let us substitute expression (2.121) into (2.122). The first of these relations is fulfilled automatically. From the second relation we find

$$f = f_R - \frac{3 f_R \hbar}{32\pi^2}\left(\ln\frac{f_R\varphi_0^2}{2\mu^2} + \frac{8}{3}\right). \tag{2.123}$$

We now use relation (2.123) in (2.121) and obtain that

$$V_{\text{eff}}^{(1)}(\varphi) = \frac{1}{24} f_R \varphi^4 + \frac{\hbar f_R^2}{(16\pi)^2}\left(\ln\frac{\varphi^2}{\varphi_0^2} - \frac{25}{6}\right). \tag{2.124}$$

This relation is the final one. Expression (2.124) is called the Coleman–Weinberg effective potential.

The additional relations (2.120) or (2.122) are called the normalization conditions. It appears that the form of effective potential (2.124) depends on the choice of φ_0 but it is not so. Due to normalization conditions (2.122) being fixed, the coupling constant f_R must also depend on φ_0. Let us consider the transformation $\varphi_0 \to \varphi_0'$. Then it is easy to find the corresponding transformation $f_R \to f_R'$, such that the effective potential be form-invariant, $V_{\text{eff}}^{(1)}(f_R, \varphi_0) = V_{\text{eff}}^{(1)}(f_R', \varphi_0')$. The last relation leads to the well-known renormalization group equation for f_R but we do not consider this aspect here.

2.6 Quantization of gauge theories

Let us consider the theory with the set of boson fields $\phi^A(x)$ and with the action $S[\phi]$. The index A includes the internal as well as relativistic indices. In this section we will use the condensed notations by De Witt. Let us write

$$\phi^A(x) \equiv \phi^i$$

where the index i includes the indexes A and space-time variables x. Let $F[\phi]$ be the functional of ϕ^i. We denote

$$\frac{\delta F[\phi]}{\delta\phi^A(x)} \equiv F_{,i}[\phi]$$

$$\int \mathrm{d}x \frac{\delta F[\phi]}{\delta\phi^A(x)} \delta\phi^A(x) \equiv F_{,i}[\phi]\delta\phi^i.$$

In this notation the equation of motion is

$$S_{,i}[\phi] = 0. \tag{2.125}$$

Let us suppose that infinitesimal transformation of fields ϕ^i with parameters $\xi^a(x) \equiv \xi^\alpha$, $\alpha = \{a, x\}$ is given by

$$\delta\phi^i = R^i_\alpha[\phi]\xi^\alpha. \tag{2.126}$$

Here the right-hand side includes summation over a and integration over x. Let us transform the action in accordance with (2.126)

$$\delta S[\phi] = S_{,i}R^i_\alpha[\phi]\xi^\alpha. \tag{2.127}$$

We will assume that the relation

$$S_{,i}R^i_\alpha[\phi] = 0 \tag{2.128}$$

is fulfilled without using equations of motion (2.125). In this case the theory with action $S[\phi]$ is called a gauge theory, transformations (2.126) are called gauge transformations, $R^i_\alpha[\phi]$ are called generators of gauge transformations, and ϕ^i are called gauge fields.

As fields ϕ^i, we will consider the vector field $A_\mu(x)$, the Yang–Mills field $A^a_\mu(x)$ and the gravitational field $g_{\mu\nu}(x)$. Corresponding actions are given with the help of Lagrangians (1.19), (1.37) and (1.57). The generators of gauge transformations for ϕ^i have the following form.

1. Electromagnetic field:

$$R_\mu(x,y) = \partial_\mu \delta(x-y).$$

2. Yang–Mills field:

$$R^a_\mu(x,y) = D^a_\mu \delta(x-y) = \Big(\partial_\mu \delta^{ab} + \mathrm{i} g f^{acb} A^c_\mu\Big)\delta(x-y).$$

3. Gravitational field:

$$R_{\mu\nu,\alpha}(x,y) = \Big(g_{\mu\alpha}(x)\partial_\nu + g_{\nu\alpha}(x)\partial_\mu + \partial_\alpha g_{\mu\nu}(x)\Big)\delta(x-y). \quad (2.129)$$

These relations follow from (1.20), (1.32) and (1.68). The antisymmetric second-rank tensor field $\omega_{\mu\nu}(x)$ is also a gauge field. The generators of corresponding gauge transformations are given by (1.45). Besides the fields ϕ^i, we will also consider the multicomponent scalar field φ^I. In this case the generators of gauge transformations are $R^I_{Ja}(x,y) = \mathrm{i}(T^a)^I_J \varphi^J(x)\delta(x-y)$. Here $(T^a)^I_J$ are the matrices of the corresponding generators of Lie algebra in the representation where determined by fields φ^I.

Let us discuss the general properties of generators $R^i_\alpha[\phi]$. First of all, we make two gauge transformations δ_1, δ_2 using parameters ξ^α_1 and ξ^β_2, respectively. After that we find the commutator of these two transformations. Then we get

$$[\delta_1, \delta_2]\,\phi^i = (R^i_{\beta,j}[\phi]R^j_\alpha[\phi] - R^i_{\alpha,j}[\phi]R^j_\beta[\phi])\xi^\alpha_1\xi^\beta_2. \quad (2.130)$$

The expression

$$R^i_{\beta,j}[\phi]R^j_\alpha[\phi] - R^i_{\alpha,j}[\phi]R^j_\beta[\phi]$$

is called the commutator of the generators of gauge transformations.

Let us consider $[\,\delta_1, \delta_2\,]\,S[\phi]$. It is equal to zero because of (2.128). On the other hand, we have

$$\begin{aligned}[\delta_1, \delta_2]\,S[\phi] &= S_{,i}[\phi]\,[\delta_1, \delta_2]\,\phi^i \\ &= S_{,i}[\phi]\Big(R^i_{\beta,j}[\phi]R^j_\alpha[\phi] - R^i_{\alpha,j}[\phi]R^j_\beta[\phi]\Big)\xi^\alpha_1\xi^\beta_2 = 0.\end{aligned}$$

The parameters $\xi^\alpha_1, \xi^\beta_2$ are arbitrary. It means that we get the equation for the commutator of generators of gauge transformations

$$S_{,i}[\phi]\Big(R^i_{\beta,j}[\phi]R^j_\alpha[\phi] - R^i_{\alpha,j}[\phi]R^j_\beta[\phi]\Big) = 0. \quad (2.131)$$

The general solution of this equation is

$$R^i_{\beta,j}[\phi]R^j_\alpha[\phi] - R^i_{\alpha,j}[\phi]R^j_\beta[\phi] \\ = f^\gamma_{\alpha\beta}[\phi]R^i_\gamma[\phi] + F^{ij}_{\alpha\beta}[\phi]S_{,j}[\phi]. \qquad (2.132)$$

The functionals $f^\gamma_{\alpha\beta}[\phi]$, $F^{ij}_{\alpha\beta}[\phi]$ have the following properties:

$$\begin{aligned} f^\gamma_{\alpha\beta}[\phi] &= -f^\gamma_{\beta\alpha}[\phi] \\ F^{ij}_{\alpha\beta}[\phi] &= -F^{ij}_{\beta\alpha}[\phi] \\ F^{ij}_{\alpha\beta}[\phi] &= -F^{ji}_{\alpha\beta}[\phi]. \end{aligned} \qquad (2.133)$$

Using (2.132) in (2.131), and taking account of (2.133), we can check that (2.132) is a solution of (2.131).

It is said that relation (2.132) defines a gauge algebra. If the functionals $F^{ij}_{\alpha\beta}[\phi]$ are not equal to zero then the gauge algebra is called open, otherwise the gauge algebra is said to be closed. The functionals $f^\gamma_{\alpha\beta}[\phi]$ are usually called the structural coefficients of gauge algebra. Note that gauge algebra is always closed if the equations of motion $S_{,i}[\phi] = 0$ are fulfilled. The theory for which only relations (2.128) are satisfied is usually called the general gauge theory.

We will consider here gauge theories of a special kind. Firstly, we will assume that the gauge algebra is closed, i.e. $F^{ij}_{\alpha\beta}[\phi] = 0$. Secondly, we will assume that functionals $f^\gamma_{\alpha\beta}[\phi]$ do not depend on ϕ^i. This means that $f^\gamma_{\alpha\beta}[\phi]$ are really constants. In this case the generators of gauge transformations form the set of relations typical for Lie algebra. That is why the generators define a group which is called a gauge. For example, we can check that Jacobi identities are valid for the structure constants which were discussed above. Let us introduce the operators $R_\alpha[\phi]$ acting on arbitrary functional $F[\phi]$ in the following way:

$$R_\alpha[\phi]F[\phi] = R^i_\alpha[\phi]F_{,i}[\phi].$$

Let

$$[\,R_\alpha[\phi], R_\beta[\phi]\,] = R_\alpha[\phi]R_\beta[\phi] - R_\beta[\phi]R_\alpha[\phi]$$

be the commutator of these operators. Direct calculation gives

$$[R_\alpha[\phi],[R_\beta[\phi],R_\gamma[\phi]]\,] + [R_\beta[\phi],[R_\gamma[\phi],R_\alpha[\phi]]\,] \\ + [R_\gamma[\phi],[R_\alpha[\phi],R_\beta[\phi]]\,] = 0.$$

Using (2.132) here when $F^{ij}_{\alpha\beta}[\phi] = 0$ and $f^\alpha_{\beta\gamma}[\phi] =$ constant we get the Jacobi identity for structure constants

$$f^\alpha_{\beta\gamma}f^\gamma_{\rho\sigma} + f^\alpha_{\rho\gamma}f^\gamma_{\sigma\beta} + f^\alpha_{\sigma\gamma}f^\gamma_{\beta\rho} = 0.$$

In the theories of fields $A_\mu(x), A_\mu^a(x), \omega_{\mu\nu}(x), g_{\mu\nu}(x)$ these constants have the following form

1. Electromagnetic field:

$$f^\gamma_{\alpha\beta}(x,y,z) = 0$$

2. Yang–Mills field:

$$f^\gamma_{\alpha\beta}(x,y,z) = g f^{\mathrm{cab}} \delta(x-y)\delta(x-z)$$

3. Antisymmetric second-rank tensor field:

$$f^\gamma_{\alpha\beta}(x,y,z) = 0$$

4. Gravitational field:

$$f^\gamma_{\alpha\beta}(x,y,z) = \delta^\rho_\tau \delta(x-z)\partial_\sigma \delta(x-y) + \delta^\rho_\sigma(x)\partial_\tau \delta(x-z). \quad (2.134)$$

Here the derivative acts on the first argument of the δ-function.

The generators of gauge transformations are called linearly independent if the equation

$$R^i_\alpha[\phi] n^\alpha[\phi] = 0 \quad (2.135)$$

has only a single solution $n^\alpha[\phi] = 0$. Otherwise, the generators are called linearly dependent. The theories of the electromagnetic field, the Yang–Mills field and the gravitational field are theories with linearly independent generators. The theory of the antisymmetric second-rank tensor is the theory with linearly dependent generators due to relation (1.46).

The canonical quantization of gauge theories may be performed with the help of Dirac's method, which is not discussed here. However, for the Yang–Mills theory and gravity, Faddeev, Popov and De Witt have found independently the true representation for the generating functional of Green's functions in terms of a functional integral, which has recently also been found on the basis of canonical quantization. Recently Batalin, Fradkin and Vilkovisky have developed the generalized canonical quantization method which gives the possibility of constructing the representation for the generating functional of Green's functions in general gauge theory†.

† Recently a general approach to quantization has been developed, which does not use the canonical methods and applies to the action only. This approach is called Lagrangian's quantization of general gauge theories.

For the quantization of the Yang–Mills theory Faddeev and Popov have used a heuristic method which can be applied to gauge theories with a closed gauge algebra, linearly independent generators and with functionals $f^{\alpha}_{\beta\gamma}[\phi]$ independent of ϕ^i. We consider only such models of field theory in this book. That is why we will present here the derivation of a representation for the generating functional of Green's functions in terms of a functional integral, using the Faddeev and Popov method.

Consider the space M of fields ϕ^i. Let G be a group of gauge transformations and $\phi^i_0 \in M$. Then the set ${}^G\phi^i_0$, which is obtained by the action of all elements of G on ϕ^i_0, is called gauge group G orbit with the representative ϕ^i_0. One can show that the orbit is given by one of its representatives. Let us introduce factor-space M/G the elements of which are different orbits, denoted as $\{\phi\}, \{\phi\} \in M/G$.

Because of the gauge invariance of physical quantities these quantities are the functionals of orbits only and they do not depend on the choice of concrete representative of the orbit. For example, $S[\phi] = S[\{\phi\}]$. That is why it is natural to call $\{\phi\}$ physical fields and the space M/G can be called the space of physical fields.

Let us try to write the functional integral corresponding to the generating functional of Green's functions at zero sources. It is natural to expect that it contains the integration only over physical fields $\{\phi\}$ and that is why it should be

$$\int \mathrm{D}\{\phi\} \exp(\mathrm{i}S[\phi]). \tag{2.136}$$

On the other hand, we can consider the naive expression

$$\int \mathrm{D}\phi \exp(\mathrm{i}S[\phi]). \tag{2.137}$$

It is evident that it contains the integration over different orbits (over $\{\phi\}$) as well as integration along the orbit, i.e. over the gauge group). Because of the gauge invariance of $S[\phi]$ the integration over the gauge group should be factorized giving the general multiplicator which is the volume of the gauge group Vol G. Hence,

$$\int \mathrm{D}\phi \exp(\mathrm{i}S[\phi]) = \ \mathrm{Vol}\, G \int \mathrm{D}\{\phi\} \exp(\mathrm{i}S[\phi]). \tag{2.138}$$

It is natural to rewrite (2.136) in terms of the original fields ϕ^i. We will consider this in the following way. To find the subspace of the physical fields M/G of space M, we introduce the functional $\chi^\alpha[\phi]$ which is called the gauge and impose the equation

$$\chi^\alpha[\phi] - l^\alpha = 0 \tag{2.139}$$

Here l^α are arbitrary functions of x. This equation can be understood as the equation of surface in the space M. Let us demand this surface crosses any orbit only one time. In other words, for any ϕ the element $h \in G$ exists, such that if $\chi^\alpha[\phi] - l^\alpha \neq 0$ then $\chi^\alpha[{}^h\phi] - l^\alpha = 0$. Here ${}^h\phi^i$ is the result of the action of element $h (h \in G)$ on field ϕ^i. Each element of the gauge group is given by parameters ξ^α, that is why (2.139) is the equation for finding the corresponding parameters. Hence, the number of equations (2.139) is equal to the number of gauge parameters.

To have the integrand of the functional integral on all fields ϕ^i constrained to the surface defined by (2.139), it must contain $\delta(\chi^\alpha[\phi] - l^\alpha)$. Hence, it is natural to expect that this integral is

$$\int \mathrm{D}\phi \exp[\mathrm{i}S[\phi]]\delta(\chi^\alpha[\phi] - l^\alpha)\Delta[\phi] \tag{2.140}$$

for some functional $\Delta[\phi]$. The main problem is to find this functional $\Delta[\phi]$. This problem is solved with the help of the Faddeev–Popov ansatz.

Let us try to represent the functional integral on the left-hand side of (2.138) as a product of two factors. One of them is Vol G, then the second one must be the explicit expression for the integral (2.136) in terms of the original fields ϕ^i. We will see that this integral is written as (2.140) and will also find the explicit expression for the functional $\Delta[\phi]$.

Let us start the account of the Faddeev–Popov ansatz. We introduce the functional $\Delta[\phi]$ with the help of equation

$$\Delta[\phi] \int \mathrm{D}\, h\delta(\chi^\alpha[{}^h\phi] - l^\alpha) = 1. \tag{2.141}$$

Here the integral over h is formal over the gauge group G and it is known that

$$\int \mathrm{D}h\, 1 = \mathrm{Vol}\, G. \tag{2.142}$$

Let us assume that the formal integral over the gauge group has the properties which are typical for an integral over the Haar measure. In particular, it is invariant under a group translation.

Let us show that functional $\Delta[\phi]$ which is defined by (2.141) is gauge invariant. Consider

$$\left(\Delta\left[{}^{h'}\phi\right]\right)^{-1} \int \mathrm{D}h\, \delta\left(\chi^\alpha\left[{}^{hh'}\phi\right] - l^\alpha\right)^{-1} = (\Delta[\phi])^{-1}. \tag{2.143}$$

Here we used the integral's invariance under translations on the group.

Let us consider the 'naive' expression (2.137). Using expression (2.141) we multiply it by unity and obtain

$$\int \mathrm{D}\phi \exp(\mathrm{i}S[\phi]) = \int \mathrm{D}\phi \mathrm{D}h \exp(\mathrm{i}S[\phi])\delta(\chi^\alpha[{}^h\phi] - l^\alpha)\Delta[\phi].$$

In this expression we change the variables ${}^h\phi^i \to \phi^i$. In the theories (Yang–Mills theory, gravity) which are of interest to us the parametrization of fields ϕ^i exists where gauge transformations generators depend on these fields linearly (see, for example (2.129)). In this case the Jacobian of the above replacement is some non-essential constant. This change of variables is some gauge transformation for which functionals $S[\phi]$ and $\Delta[\phi]$ are invariant. Then

$$\int \mathrm{D}\phi \exp(\mathrm{i}S[\phi]) = \int \mathrm{D}h \int \mathrm{D}\phi \exp(\mathrm{i}S[\phi])\delta(\chi^\alpha[\phi] - l^\alpha)\Delta[\phi]$$
$$= \mathrm{Vol}\, G \int \mathrm{D}\phi \exp(\mathrm{i}S[\phi])\delta(\chi^\alpha[\phi] - l^\alpha)\Delta[\phi]. \quad (2.144)$$

Comparing (2.138) and (2.144) we can write

$$\int \mathrm{D}\{\phi\} \exp(\mathrm{i}S[\phi]) = \int \mathrm{D}\phi \exp[\mathrm{i}S[\phi]]\delta(\chi^\alpha[\phi] - l^\alpha)\Delta[\phi]. \quad (2.145)$$

Equality (2.145) gives us the possibility of stating that the functional integral for gauge theory is

$$\int \mathrm{D}\phi \exp(\mathrm{i}S[\phi])\delta(\chi^\alpha[\phi] - l^\alpha)\Delta[\phi]. \quad (2.146)$$

Now it is necessary to find $\Delta[\phi]$ defined by (2.141). We see from (2.146) that this functional should be calculated only for fields ϕ^i which satisfy the equations $\chi^\alpha[\phi] - l^\alpha = 0$. This means that it is enough to integrate over h in (2.141) in the neighbourhood of the identity element for such fields ϕ^i in (2.141). Then

$$({}^h\phi)^i = \phi^i + R^i_\alpha[\phi]\xi^\alpha.$$

An integral over the group in the neighbourhood of the identity element is the integral over the parameters (so that $\mathrm{D}h \to \mathrm{D}\xi$, up to a constant factor). Then we get

$$\begin{aligned}1 &= \Delta[\phi]\int \mathrm{D}\xi\,\delta(\chi^\alpha[\phi^i + R^i_\alpha\xi^\alpha] - l^\alpha)\\ &= \Delta[\phi]\int \mathrm{D}\xi\,\delta(\chi^\alpha[\phi] - l^\alpha + \chi^\alpha_{,i}[\phi]R^i_\beta\xi^\beta)\\ &= \Delta[\phi]\int \mathrm{D}\xi\,\delta(\xi^\alpha)\det{}^{-1}\chi^\alpha_{,i}[\phi]R^i_\beta[\phi].\end{aligned} \tag{2.147}$$

We used the fact that the fields ϕ^i satisfy the equations $\chi^\alpha[\phi] - l^\alpha = 0$. Equation (2.147) shows that

$$\Delta[\phi] = \det M^\alpha_\beta[\phi]. \tag{2.148}$$

Here

$$M^\alpha_\beta[\phi] = \chi^\alpha_{,i}[\phi]R^i_\beta[\phi]. \tag{2.149}$$

As a result, the functional integral (2.146) has the form

$$\int \mathrm{D}\phi\exp(\mathrm{i}S[\phi])\delta(\chi^\alpha[\phi] - l^\alpha)\det M^\alpha_\beta[\phi]. \tag{2.150}$$

We can rewrite this expression in a more convenient form.

Let us introduce an arbitrary non-degenerate functional $G_{\alpha\beta}[\phi]$ and multiply expression (2.150) by unity in the form

$$1 = \det{}^{1/2} G_{\alpha\beta}[\phi]\int \mathrm{D}l\exp\left(\frac{\mathrm{i}}{2}l^\alpha G_{\alpha\beta}[\phi]l^\beta\right).$$

Then we get

$$\int \mathrm{D}\phi\exp\left\{\mathrm{i}S[\phi] + \tfrac{1}{2}\chi^\alpha[\phi]G_{\alpha\beta}[\phi]\chi^\beta[\phi]\right\}\det M^\alpha_\beta[\phi]\det{}^{1/2} G_{\alpha\beta}[\phi]. \tag{2.151}$$

The expression for the functional integral was obtained in this form by Faddeev and Popov (when $G_{\alpha\beta}[\phi] = \delta_{\alpha\beta}$) and independently by De Witt. One can represent $\det M^\alpha_\beta$, with the help of the functional integral over anticommutating fields $\bar{C}_\alpha$ and C^β, as

$$\det M^\alpha_\beta[\phi] = \int \mathrm{D}\bar{C}\mathrm{D}C\exp\left(\frac{\mathrm{i}}{2}\bar{C}_\alpha M^\alpha_\beta C^\beta\right).$$

The same procedure can be applied to $\det^{1/2} G_{\alpha\beta}[\phi]$:

$$\det{}^{1/2} G_{\alpha\beta}[\phi] = \int \mathrm{D}b\exp\left(\frac{\mathrm{i}}{2}b^\alpha G_{\alpha\beta}b^\beta\right).$$

Then, we can rewrite (2.151) in the form

$$\int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C\mathrm{D}b \exp \mathrm{i}(S[\phi] + S_{\mathrm{GF}}[\phi] + S_{\mathrm{gh}}[\phi, \bar{C}, C, b]). \tag{2.152}$$

Here $S[\phi]$ is the initial action and

$$\begin{aligned} S_{\mathrm{GF}}[\phi] &= \tfrac{1}{2}\chi^{\alpha}[\phi]G_{\alpha\beta}[\phi]\chi^{\beta}[\phi] \\ S_{\mathrm{gh}}[\phi, \bar{C}, C, b] &= \bar{C}_{\alpha}M^{\alpha}_{\beta}[\phi]C^{\beta} + \tfrac{1}{2}b^{\alpha}G_{\alpha\beta}[\phi]b^{\beta}. \end{aligned} \tag{2.153}$$

The functional $S_{\mathrm{GF}}[\phi]$ is caused by the contribution of gauge, the functional $S_{\mathrm{gh}}[\phi, \bar{C}, C, b]$ is called the ghost action, unphysical anticommutating fields $\bar{C}_{\alpha}$ and C^{β} are called Faddeev–Popov ghosts, anticommutating field b^{α} is called the third ghost.

Relations (2.152) and (2.153) serve as the basis for a definition of Green's functions generating functional of the gauge theory in the form

$$Z[J] = \int \mathrm{D}\phi \mathrm{D}D\bar{C}\mathrm{D}C\mathrm{D}b \exp[\mathrm{i}(S[\phi] + S_{\mathrm{GF}}[\phi] + S_{\mathrm{gh}}[\phi, \bar{C}, C, b] + \phi^{i}J_{i})] \tag{2.154}$$

or in the form

$$Z[J] = \int \mathrm{D}\phi \exp[\mathrm{i}(S[\phi] + \phi^{i}J_{i})]\delta(\chi^{\alpha}[\phi] - l^{\alpha}) \det M^{\alpha}_{\beta}[\phi].$$

Let us make some remarks concerning (2.154).

1. Usually the functional $G_{\alpha\beta}[\phi]$ is constant (does not depend on ϕ^{i}) or it does not contain derivatives. Then $\det{}^{1/2}G_{\alpha\beta}[\phi]$ is some constant and this constant can be omitted. That is why the third ghost is not essential for usual considerations. However, there is a good example — higher-derivative gravity — where $\det{}^{1/2}G_{\alpha\beta}[\phi]$ should be taken into account. In the following chapters we do not take into account $\det{}^{1/2}G_{\alpha\beta}[\phi]$ (excluding special cases such as higher-derivative gravity).
2. Relation (2.154) shows that the generating functional of Green's functions is written in the form

$$Z[J] = \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp[\mathrm{i}(S_{\mathrm{total}}[\phi, \bar{C}, C] + \phi^{i}J_{i})] \tag{2.155}$$

where

$$S_{\mathrm{total}}[\phi, \bar{C}, C] = S[\phi] + S_{\mathrm{GF}}[\phi] + S_{\mathrm{gh}}[\phi, \bar{C}, C]$$

and $S_{\mathrm{GF}}[\phi]$, $S_{\mathrm{gh}}[\phi, \bar{C}, C]$ are given by (2.153) (at $b^\alpha = 0$). The generating functional (2.154) has the standard theoretical functional integral representation for the theory with action $S_{\mathrm{total}}[\phi, \bar{C}, C]$. That is why it can be studied by standard methods of perturbation theory. We note that (2.155) defines Green's functions only for fields ϕ^i. Therefore the fields $\bar{C}_\alpha$ and C^α could not appear as the external lines of diagrams. More exactly, these fields are contained only in loops so $\bar{C}_\alpha$ and C^α are called the ghosts.

3. To obtain (2.145) we introduced an arbitrary element in the theory — gauge $\chi^\alpha[\phi]$. The left-hand side of (2.145) does not depend on the choice of gauge, so the right side also does not depend on the gauge. We can prove this. Let $\chi'^\alpha[\phi]$ be another gauge and let the functionals $\Delta_\chi[\phi]$ and $\Delta_{\chi'}[\phi]$ be constructed with the help of $\chi^\alpha[\phi]$ and $\chi'^\alpha[\phi]$ respectively. Let us multiply the integrand on the right-hand side of (2.145) by the unity of the following form

$$1 = \Delta_{\chi'}[\phi] \int \mathrm{D}h\delta\left[\chi'^\alpha\left[{}^h\phi\right] - l'^\alpha\right]. \tag{2.156}$$

Then we get

$$\int \mathrm{D}\phi\,\mathrm{D}h\delta(\chi^\alpha[\phi] - l^\alpha)\Delta_\chi[\phi]\delta(\chi'^\alpha[{}^h\phi] - l'^\alpha)\Delta_{\chi'}[\phi]\exp(\mathrm{i}S[\phi]).$$

Changing the variables $({}^h\phi)^i \to \phi^i$ and replacing $h^{-1} \to h$ we obtain

$$\int \mathrm{D}\phi\mathrm{D}h\delta(\chi^\alpha[{}^h\phi] - l^\alpha)\Delta_\chi[\phi]\delta(\chi'^\alpha[\phi] - l'^\alpha)\Delta_{\chi'}[\phi]\exp(\mathrm{i}S[\phi])$$
$$= \int \mathrm{D}\phi\delta(\chi'^\alpha[\phi] - l'^\alpha)\Delta_{\chi'}[\phi]\exp(\mathrm{i}S[\phi]). \tag{2.157}$$

Thus,

$$\int \mathrm{D}\phi\delta(\chi^\alpha[\phi] - l^\alpha)\Delta_\chi[\phi]\exp(\mathrm{i}S[\phi])$$
$$= \int \mathrm{D}\phi\delta(\chi'^\alpha[\phi] - l'^\alpha)\Delta_{\chi'}[\phi]\exp(\mathrm{i}S[\phi]).$$

This shows the gauge independence of the right-hand side of (2.145) explicitly.

Let us consider the generating functional of Green's functions, where term $\phi^i J_i$ is included

$$Z_\chi[J] = \int \mathrm{D}\phi\delta(\chi^\alpha[\phi] - l^\alpha)\Delta_\chi[\phi]\exp[\mathrm{i}(S[\phi] + \phi^i J_i)]. \qquad (2.158)$$

Here we noted that functional (2.158) is constructed in the gauge $\chi^\alpha[\phi]$.

We substitute (2.156) into the integrand in (2.158) and change the variables $({}^h\phi)^i \to \phi^i$, $h^{-1} \to h$. Then we obtain

$$Z_\chi[J] = \int \mathrm{D}\phi\mathrm{D}h\, \delta(\chi^\alpha[{}^h\phi] - l^\alpha)\Delta_\chi[\phi]$$
$$\times\, \delta(\chi'^\alpha[\phi] - l'^\alpha)\Delta_{\chi'}[\phi]\exp[\mathrm{i}(S[\phi] + [{}^{h^{-1}}\phi]^i J_i)].$$

We see that the integration over the gauge group is not factorized. This means that the generating functional of Green's functions depends on the choice of gauge. Hence, all Green's functions are gauge dependent objects. Note that the reason for the gauge dependence of $Z[J]$ is caused by the term $\phi^i J_i$. However, we can show that the S-matrix which is constructed from the Green's functions with the help of known rules does not depend on the choice of gauge. This follows from the theorem of equivalence, which allows the possibility of substituting $\phi^i J_i$ instead of $[{}^{h^{-1}}\phi]^i J_i$ in the functional integral on the mass shell.

2.7 Effective action in gauge theories and Ward identities

Let us consider the generating functional of Green's function (2.155)

$$Z[J] = \int \mathrm{D}\phi\mathrm{D}\bar{C}\mathrm{D}C\exp[\mathrm{i}(S_{\text{total}}[\phi, \bar{C}, C] + \phi^i J_i)].$$
$$S_{\text{total}}[\phi, \bar{C}, C] = S[\phi] + \tfrac{1}{2}\chi^\alpha[\phi]G_{\alpha\beta}\chi^\beta[\phi] + \bar{C}_\alpha\chi^\alpha_i[\phi]R^i_\beta[\phi]C^\beta \qquad (2.159)$$

$Z[J]$ has the standard structure (we omit the constant contribution $\det^{1/2} G_{\alpha\beta}$), so the effective action $\Gamma[\phi]$ can be presented as in section 2.3.

Let

$$Z[J] = \mathrm{e}^{\mathrm{i}W[J]}$$

and introduce the mean field

$$\bar{\phi}^i = W^{,i}[J]. \tag{2.160}$$

Then we define

$$\Gamma[\bar{\phi}] = W[J] - \bar{\phi}^i J_i \tag{2.161}$$

where J_i is expressed in terms of the mean field from (2.160). Relations (2.160) and (2.161) lead to the equation of motion for the mean field

$$\Gamma_{,i}[\bar{\phi}] = -J_i.$$

The loop expansion can be constructed using the general scheme. According to the results of section 2.4 we have

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma[\bar{\phi}]\right) &= \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp\left(\frac{\mathrm{i}}{\hbar}(S_{\text{total}}[\bar{\phi}+\phi, \bar{C}, C] - \phi^i \Gamma_{,i}[\bar{\phi}])\right) \\ &= \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp\left\{\frac{\mathrm{i}}{\hbar}(S[\bar{\phi}+\phi] + \tfrac{1}{2}\chi^\alpha[\bar{\phi}+\phi]\right. \\ &\quad \times G_{\alpha\beta}\chi^\beta[\bar{\phi}+\phi] + \bar{C}_\alpha \chi^\alpha_{,i}[\bar{\phi}+\phi] R^i_\beta[\bar{\phi}+\phi] C^\beta \\ &\quad \left. - \phi^i \Gamma_{,i}[\bar{\phi}])\right\}.\end{aligned} \tag{2.162}$$

Here we restored the $\hbar$-dependence. This expression is the basic one for the calculation of $\Gamma[\bar{\phi}]$ as an expansion in a power series in $\hbar$.

As an example we will consider the calculation of the one-loop quantum correction $\bar{\Gamma}^{(1)}[\bar{\phi}]$. The same considerations as in section 2.4 lead to the following functional integral

$$\begin{aligned}\exp(\mathrm{i}\bar{\Gamma}^{(1)}[\bar{\phi}]) &= \int \mathrm{D}\phi \mathrm{D}C \mathrm{D}\bar{C} \exp\left\{\mathrm{i}(\tfrac{1}{2}S_{,ij}[\bar{\phi}]\phi^i\phi^j\right. \\ &\quad \left. + \tfrac{1}{2}S_{\mathrm{GF},ij}[\bar{\phi}]\phi^i\phi^j + \bar{C}_\alpha M^\alpha_\beta[\bar{\phi}]C^\beta)\right\} \\ &= \det{}^{-1/2}(S_{,ij}[\bar{\phi}] + S_{\mathrm{GF},ij}[\bar{\phi}]) \det M^\alpha_\beta[\bar{\phi}].\end{aligned}$$

Hence

$$\bar{\Gamma}^{(1)}[\bar{\phi}] = \frac{\mathrm{i}}{2}\operatorname{Tr}\ln(S_{,ij}[\bar{\phi}] + S_{\mathrm{GF},ij}[\bar{\phi}]) - \mathrm{i}\operatorname{Tr}\ln M^\alpha_\beta[\bar{\phi}]. \tag{2.163}$$

Now let us consider relation (2.159) again. We shall see that the action $S_{\text{total}}[\phi, \bar{C}, C]$ has a wonderful global symmetry which has been found by Becchi, Rouet and Stora, and independently by Tyutin.

Let us carry out the gauge transformation with the parameters $\xi^\alpha = C^\alpha\mu$ where μ is a constant anticommutating parameter, i.e.

$\mu^2 = 0, C^\alpha\mu = -\mu C^\alpha$, $\bar{C}_\alpha\mu = -\mu\bar{C}_\alpha$. Such transformations have the form $\delta\phi^i = R^i_\alpha C^\alpha\mu$. The action does not change under this transformation. For $\frac{1}{2}\chi^\alpha[\phi]G_{\alpha\beta}\chi^\beta[\phi]$ we obtain

$$\delta(\tfrac{1}{2}\chi^\alpha[\phi]G_{\alpha\beta}\chi^\beta[\phi]) = \chi^\alpha[\phi]G_{\alpha\beta}\chi^\beta_{,i}[\phi]R^i_\gamma C^\gamma\mu. \tag{2.164}$$

We will define the transformation of $\bar{C}_\alpha$ according to the rule $\delta\bar{C}_\alpha = G_{\alpha\beta}\chi^\beta[\phi]\mu$. Then

$$(\delta\bar{C}_\alpha)M^\alpha_\beta[\phi]C^\beta = -G_{\alpha\beta}\chi^\beta[\phi]M^i_\gamma[\phi]C^\gamma\mu. \tag{2.165}$$

Thus

$$\begin{aligned}&\delta(\tfrac{1}{2}\chi^\alpha[\phi]G_{\alpha\beta}\chi^\beta[\phi]) + (\delta\bar{C}_\alpha)M^\alpha_\beta[\phi]C^\beta\\ &\quad= \chi^\alpha[\phi]G_{\alpha\beta}M^\beta_\gamma[\phi]C^\gamma\mu - \chi^\alpha[\phi]G_{\alpha\beta}M^\beta_\gamma[\phi]C^\gamma\mu\\ &\quad= 0.\end{aligned}$$

Let us define the transformation of C^α according to the rule $\delta C^\alpha = \frac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu$. Consider then changing the remaining part of $S_{\text{total}}[\phi, \bar{C}, C]$ to obtain

$$\begin{aligned}&\bar{C}_\alpha(\delta M^\alpha_\beta[\phi])C^\beta + \bar{C}_\alpha M^\alpha_\beta[\phi]\delta C^\beta\\ &\quad= \bar{C}_\alpha(\chi^\alpha_{,ij}[\phi]R^i_\beta[\phi]R^j_\gamma[\phi]C^\gamma\mu + \chi^\alpha_{,i}[\phi]R^i_{\beta,j}[\phi]R^j_\gamma[\phi]C^\gamma\mu)C^\beta\\ &\qquad+ \bar{C}_\alpha M^\alpha_\beta\tfrac{1}{2}f^\beta_{\gamma\delta}C^\gamma C^\delta\mu\\ &\quad= -\tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,ij}[\phi](R^i_\beta[\phi]R^j_\gamma[\phi] - R^i_\gamma[\phi]R^j_\beta[\phi])C^\beta C^\gamma\mu\\ &\qquad+ \tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,i}[\phi](R^i_{\beta,j}[\phi]R^j_\gamma[\phi] - R^i_{\gamma,j}[\phi]R^j_\beta[\phi])C^\beta C^\gamma\mu\\ &\qquad+ \tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,i}[\phi]R^i_\beta[\phi]f^\beta_{\gamma\delta}C^\gamma C^\delta\mu\\ &\quad= \tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,i}f^\delta_{\gamma\beta}R^i_\delta[\phi]C^\beta C^\gamma\mu + \tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,i}[\phi]R^i_\beta[\phi]f^\beta_{\gamma\delta}C^\gamma C^\delta\mu\\ &\quad= \tfrac{1}{2}\bar{C}_\alpha\chi^\alpha_{,i}[\phi]R^i_\beta[\phi]C^\delta C^\gamma\mu(f^\beta_{\gamma\delta} + f^\beta_{\delta\gamma}) = 0.\end{aligned} \tag{2.166}$$

Here, we have used the relations

$$\begin{aligned}R^i_{\beta,j}[\phi]R^j_\gamma[\phi] - R^i_{\gamma,j}[\phi]R^j_\beta[\phi] &= f^\delta_{\gamma\beta}R^i_\delta[\phi]\\ f^\beta_{\gamma\delta} &= -f^\beta_{\delta\gamma}.\end{aligned}$$

Thus the transformations

$$\begin{aligned}\delta\phi^i &= R^i_\alpha[\phi]C^\alpha\mu\\ \delta\bar{C}^\alpha &= G_{\alpha\beta}\chi^\beta[\phi]\mu\\ \delta C^\alpha &= \tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu\end{aligned} \tag{2.167}$$

do not change the action $S_{\text{total}}[\phi, \bar{C}, C]$ (2.159). It is invariant! The transformations (2.167) are called Becchi–Rouet–Stora–Tyutin transformations or BRST-transformations. The invariance of the action induced by transformations (2.167) is called BRST-invariance.

The BRST-invariance has some interesting properties. Let us show that $R^i_\alpha[\phi]C^\alpha\mu$ and $f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu$ are BRST-invariants as well as $S_{\text{total}}[\phi, \bar{C}, C]$.

1. Let us consider

$$\begin{aligned}
\delta(R^i_\alpha[\phi]C^\alpha) &= R^i_{\alpha,j}[\phi]R^j_\beta[\phi]C^\beta\mu C^\alpha + R^i_\alpha[\phi]\tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu \\
&= -\tfrac{1}{2}(R^i_{\alpha,j}[\phi]R^j_\beta[\phi] - R^i_{\beta,j}[\phi]R^j_\alpha[\phi])C^\beta C^\alpha\mu \\
&\quad + \tfrac{1}{2}f^\alpha_{\beta\gamma}R^i_\alpha[\phi]C^\beta C^\gamma\mu \\
&= -\tfrac{1}{2}f^\gamma_{\beta\alpha}R^i_\gamma[\phi]C^\beta C^\alpha\mu + \tfrac{1}{2}R^i_\alpha[\phi]f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu \\
&= -\tfrac{1}{2}R^i_\alpha[\phi]f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu + \tfrac{1}{2}R^i_\alpha[\phi]f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu = 0.
\end{aligned} \tag{2.168}$$

2. Let us consider

$$\begin{aligned}
\delta(\tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu) &= \tfrac{1}{2}f^\alpha_{\beta\gamma}(\tfrac{1}{2}f^\beta_{\rho\sigma}C^\rho C^\sigma\mu C^\gamma + \tfrac{1}{2}f^\gamma_{\rho\sigma}C^\beta C^\rho C^\sigma\mu) \\
&= \tfrac{1}{2}f^\alpha_{\beta\gamma}f^\gamma_{\rho\sigma}C^\rho C^\sigma C^\beta\mu \\
&= \tfrac{1}{6}f^\alpha_{\beta\gamma}f^\gamma_{\rho\sigma}(C^\rho C^\sigma C^\beta + C^\beta C^\rho C^\sigma + C^\sigma C^\beta C^\rho)\mu \\
&= \tfrac{1}{6}(f^\alpha_{\beta\gamma}f^\gamma_{\rho\sigma} + f^\alpha_{\rho\gamma}f^\gamma_{\sigma\beta} + f^\alpha_{\sigma\gamma}f^\gamma_{\beta\rho})C^\rho C^\sigma C^\beta\mu \\
&= 0.
\end{aligned} \tag{2.169}$$

Here we used a Jacobi identity for structural constants.

Let us show that the Berezinian of the BRST-transformation is equal to unity. The Berezinian is the generalization of the determinant of the matrix containing elements of different Grassmanian parity. In our case it has the following structure

$$J = \operatorname{Ber}\frac{\delta(\phi'^i, \bar{C}'_\alpha, C'^\beta)}{\delta(\phi^j, \bar{C}_\rho, C^\sigma)} = \operatorname{Ber}\begin{pmatrix} \frac{\delta\phi'^i}{\delta\phi^j} & \frac{\delta\phi'^i}{\delta\bar{C}_\rho} & \frac{\delta\phi'^i}{\delta C^\sigma} \\ \frac{\delta\bar{C}'_\alpha}{\delta\phi^j} & \frac{\delta\bar{C}'_\alpha}{\delta\bar{C}_\rho} & \frac{\delta\bar{C}'_\alpha}{\delta C^\sigma} \\ \frac{\delta C'^\beta}{\delta\phi^j} & \frac{\delta C'^\beta}{\delta\bar{C}_\rho} & \frac{\delta C'^\beta}{\delta C^\sigma} \end{pmatrix}$$

All the derivatives act on the anticommutating fields as the left derivatives. Using an explicit form of BRST-transformations we get

$$J = \operatorname{Ber}\begin{pmatrix} \delta^i_j + R^i_{\alpha,j}[\phi]C^\alpha\mu & 0 & R^i_\sigma[\phi]\mu \\ G_{\alpha\gamma}\chi^\gamma_{,j}[\phi]\mu & \delta^\rho_\alpha & 0 \\ 0 & 0 & \delta^\beta_\sigma + f^\beta_{\sigma\gamma}C^\gamma\mu \end{pmatrix}.$$

We represent the matrix under consideration as $1 + B\mu$, where 1 is the identity matrix of the corresponding dimension. Then because $\mu^2 = 0$ we get

$$J = 1 + \operatorname{Tr} B\mu = 1 + \left(R^i_{\alpha,i}[\phi]C^\alpha + f^\beta_{\beta\gamma}C^\gamma\right)\mu. \tag{2.170}$$

The generators $R^i_\alpha[\phi]$ and structural constants $f^\alpha_{\beta\gamma}$ are local functionals in the quantum field theory. That is why expressions like $R^i_{\alpha,i}[\phi], f^\beta_{\beta\gamma}$ behave as $(\partial_{\mu_1}\ldots\partial_{\mu_n}\delta(x))_{x=0}$ for some $n \geqslant 0$. However, derivatives of δ-function at zero are not defined. It is necessary to use some regularization. In this book we shall assume in all cases the use of dimensional regularization where $(\partial_{\mu_1}\ldots\partial_{\mu_n}\delta(x))_{x=0} = 0$. In view of this remark $R^i_{\alpha,i}[\phi] = 0$ and $f^\beta_{\beta\gamma} = 0$. Hence, from (2.170) it follows that the Berezinian of the BRST-transformation is equal to unity.

The BRST-invariance of $S_{\text{total}}[\phi,\bar{C},C]$ is the basis for the modern derivation of Ward indentities in gauge theories. In general, any relations between Green's functions or vertex functions for the same theory are called Ward identities. The appearance of these identities is caused by gauge invariance of the original action $S[\phi]$.

Let us introduce the functional

$$Z[J,\eta,\bar{\eta},K,L] = \int \mathrm{D}\phi\mathrm{D}\bar{C}\mathrm{D}C \exp\Big[\mathrm{i}\Big(S_{\text{total}}[\phi,\bar{C},C] + \phi^i J_i + \eta_\alpha C^\alpha + \bar{\eta}^\alpha\bar{C}_\alpha + K^i R_{i\alpha}[\phi]C^\alpha + L_\alpha \tfrac{1}{2} f^\alpha_{\beta\gamma}C^\beta C^\gamma\Big)\Big]. \tag{2.171}$$

It is evident that the functional (2.171) at $\eta_\alpha = 0$, $\bar{\eta}^\alpha = 0$, $K^i = 0$ and $L_\alpha = 0$ is equal to the generating functional of Green's functions for gauge theory (2.159). However, we will show that the introduction of sources to ghosts $\bar{C}_\alpha$, C^α and to BRST-invariants $R_{i\alpha}[\phi]C^\alpha$ and $\frac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma$ is a very convenient trick. Let us assume that gauges $\chi^\alpha[\phi]$ are linear on ϕ^i, that is

$$\chi^\alpha[\phi] = t^\alpha_i\phi^i + t^\alpha. \tag{2.172}$$

Here t^α_i, t^α do not depend on ϕ^i.

We denote

$$\bar{S}[\phi.\bar{C},C,K,L] = S_{\text{total}}[\phi,\bar{C},C] + K^i R_{i\alpha}[\phi]C^\alpha + L_\alpha\tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma. \tag{2.173}$$

From (2.171) we obtain

$$Z[J,\eta,\bar{\eta},K,L] = \int \mathrm{D}\phi\mathrm{D}\bar{C}\mathrm{D}C \exp[\mathrm{i}(\bar{S}[\phi,\bar{C},C,K,L] + \phi^i J_i + \eta_\alpha C^\alpha + \bar{\eta}\bar{C}_\alpha)]. \tag{2.174}$$

Let us perform the replacement of variables in accordance with the BRST-transformations:

$$\begin{aligned}
\phi'^i &= \phi^i + R^i_\alpha[\phi]C^\alpha\mu \\
\bar{C}'_\alpha &= \bar{C}_\alpha + G_{\alpha\beta}\chi^\beta[\phi]\mu \\
C'^\alpha &= C^\alpha + \tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma\mu.
\end{aligned}$$

The functional Berezinian of this replacement is equal to unity because of (2.170). The functional $\bar{S}[\phi,\bar{C},C,K,L]$ is BRST-invariant on construction. Taking into account $\mu^2 = 0$ we obtain from (2.174)

$$\begin{aligned}
\int \mathrm{D}\phi\mathrm{D}\bar{C}\mathrm{D}C \exp[\mathrm{i}(\bar{S}[\phi,\bar{C},C,K,L] + \phi^i J_i + \eta_\alpha C^\alpha + \bar{\eta}^\alpha\bar{C}_\alpha)]& \\
\times (J_i R^i_\alpha[\phi]C^\alpha + \eta_\alpha\tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma + \bar{\eta}^\alpha G_{\alpha\beta}\chi^\beta[\phi]) = 0.&
\end{aligned} \tag{2.175}$$

Note that

$$R^i_\alpha[\phi]C^\alpha = \frac{\delta\bar{S}}{\delta K_i} \qquad \tfrac{1}{2}f^\alpha_{\beta\gamma}C^\beta C^\gamma = \frac{\delta\bar{S}}{\delta L_\alpha}$$

where derivatives on K_i and L_α are on the left-hand side. We introduce the notation

$$\langle A\rangle \equiv \int \mathrm{D}\phi\,\mathrm{D}\bar{C}\,\mathrm{D}C\,A\exp[\mathrm{i}(\bar{S} + \phi^i J_i + \eta_\alpha C^\alpha + \bar{\eta}^\alpha\bar{C}_\alpha).$$

Then we can rewrite (2.175) as

$$J_i\left\langle\frac{\delta\bar{S}}{\delta K_i}\right\rangle + \eta_\alpha\left\langle\frac{\delta\bar{S}}{\delta L_\alpha}\right\rangle + \bar{\eta}^\alpha G_{\alpha\beta}\langle t^\beta_i\phi^i + t^\beta\rangle = 0. \tag{2.176}$$

Consider

$$\begin{aligned}
\left\langle\frac{\delta\bar{S}}{\delta K_i}\right\rangle &= \int \mathrm{D}\phi\,\mathrm{D}\bar{C}\,\mathrm{D}C\frac{\delta\bar{S}}{\delta K_i}\exp[\mathrm{i}(\bar{S} + \phi^i J_i + \eta_\alpha C^\alpha + \bar{\eta}^\alpha\bar{C}_\alpha)] \\
&= -\mathrm{i}\frac{\delta Z[J,\eta,\bar{\eta},K,L]}{\delta K_i} \equiv -\mathrm{i}\frac{\delta Z}{\delta K_i}.
\end{aligned} \tag{2.177}$$

In the same way

$$\left\langle\frac{\delta\bar{S}}{\delta L_\alpha}\right\rangle = -\mathrm{i}\frac{\delta Z}{\delta L_\alpha}. \tag{2.178}$$

and

$$\langle t^\beta_i\phi^i + t^\beta\rangle = -\mathrm{i}t^\beta\frac{\delta Z}{\delta J_i} + t^\beta Z. \tag{2.179}$$

Relations (2.176)–(2.179) lead to the identity

$$J_i \frac{\delta Z}{\delta \mathrm{i} K_i} + \eta_\alpha \frac{\delta Z}{\delta \mathrm{i} L_\alpha} + \bar{\eta}^\alpha G_{\alpha\beta}(t_i^\beta \frac{\delta Z}{\delta \mathrm{i} J_i} + t^\beta Z) = 0. \tag{2.180}$$

Let us introduce the generating functional of connected Green's functions $W[J, \eta, \bar{\eta}, K, L] \equiv W$ in the usual way

$$Z = \mathrm{e}^{\mathrm{i}W}.$$

Identity (2.180) can be rewritten as

$$J_i \frac{\delta W}{\delta K_i} + \eta_\alpha \frac{\delta W}{\delta L_\alpha} + \bar{\eta}^\alpha G_{\alpha\beta}(t_i^\beta \frac{\delta W}{\delta J_i} + t^\beta W) = 0. \tag{2.181}$$

Let us find the Legendre transform of W,

$$\Gamma[\phi, C, \bar{C}, K, L] = W[J, \eta, \bar{\eta}, K, L] - \phi^i J_i - \eta_\alpha C^\alpha - \bar{\eta}^\alpha \bar{C}_\alpha. \tag{2.182}$$

Here the sources J, η and $\bar{\eta}$, are expressed in terms of mean fields ϕ, C and $\bar{C}$ from the equations

$$\frac{\delta W}{\delta J_i} = \phi^i \qquad \frac{\delta W}{\delta \eta_\alpha} = C_\alpha \qquad \frac{\delta W}{\delta \bar{\eta}^\alpha} = \bar{C}_\alpha. \tag{2.183}$$

All derivatives of the sources are on the left-hand sides. Then, as usual,

$$\frac{\delta \Gamma}{\delta \phi^i} = -J_i \qquad \frac{\delta \Gamma}{\delta C^\alpha} = -\eta_\alpha \qquad \frac{\delta \Gamma}{\delta \bar{C}_\alpha} = -\bar{\eta}^\alpha. \tag{2.184}$$

All derivatives of the fields are on the right-hand sides.

Now we will find some relations

$$\begin{aligned}
\frac{\delta W}{\delta K_i} &= \frac{\delta \Gamma}{\delta K_i} + \frac{\delta \Gamma}{\delta \phi^j}\frac{\delta \phi^j}{\delta K_i} + \frac{\delta \Gamma}{\delta \bar{C}^\alpha}\frac{\delta \bar{C}^\alpha}{\delta K_i} + \frac{\delta \Gamma}{\delta C^\alpha}\frac{\delta C^\alpha}{\delta K_i} \\
&\quad + \frac{\delta \phi^j}{\delta K_i} J_j + \eta_\alpha \frac{\delta C^\alpha}{\delta K_i} + \bar{\eta}^\alpha \frac{\delta \bar{C}_\alpha}{\delta K_i} \\
&= \frac{\delta \Gamma}{\delta K_i} - J_j \frac{\delta \phi^j}{\delta K_i} - \eta_\alpha \frac{\delta C^\alpha}{\delta K_i} - \bar{\eta}^\alpha \frac{\delta \bar{C}^\alpha}{\delta K_i} + J_j \frac{\delta \phi^j}{\delta K_i} \\
&\quad + \eta_\alpha \frac{\delta C^\alpha}{\delta K_i} + \bar{\eta}^\alpha \frac{\delta \bar{C}_\alpha}{\delta K_i} \\
&= \frac{\delta \Gamma}{\delta K_i}.
\end{aligned} \tag{2.185}$$

In the same way

$$\frac{\delta W}{\delta L_\alpha} = \frac{\delta \Gamma}{\delta L_\alpha}. \tag{2.186}$$

Using (2.184)–(2.186) we can rewrite the identity (2.181) as

$$\frac{\delta \Gamma}{\delta \phi^i}\frac{\delta \Gamma}{\delta K_i} + \frac{\delta \Gamma}{\delta C^\alpha}\frac{\delta \Gamma}{\delta L_\alpha} + \frac{\delta \Gamma}{\delta \bar{C}^\alpha} G_{\alpha\beta}\chi^\beta[\phi] = 0. \tag{2.187}$$

Now we return to (2.174). Change the variables according to the rule $\bar{C}_\alpha = C_\alpha + \lambda_\alpha$ where λ_α is an infinitesimal anticommutating constant parameter. Taking into account the explicit form of $S_{\text{total}}[\phi, \bar{C}, C]$ (2.159) we get

$$\langle \chi^\alpha_{,i}[\phi] R^i_\beta[\phi] C^\beta - \bar{\eta}^\alpha \rangle = 0.$$

Then taking account of (2.177) we obtain

$$\langle \chi^\alpha_{,i}[\phi] \frac{\delta \bar{S}}{\delta K_i} \rangle - \bar{\eta}^\alpha Z = 0$$

or

$$t^\alpha_i \frac{\delta Z}{\delta \mathrm{i} K_i} - \bar{\eta}^\alpha Z = 0. \tag{2.188}$$

One can rewrite identity (2.188) in terms of effective action (2.182) in the form

$$t^\alpha_i \frac{\delta \Gamma}{\delta K_i} + \frac{\delta \Gamma}{\delta \bar{C}^\alpha} = 0. \tag{2.189}$$

Using this identity in (2.187) we get

$$\frac{\delta \Gamma}{\delta \phi^i}\frac{\delta \Gamma}{\delta K_i} + \frac{\delta \Gamma}{\delta C^\alpha}\frac{\delta \Gamma}{\delta L_\alpha} - \chi^\alpha_{,i}[\phi]\frac{\delta \Gamma}{\delta K_i} G_{\alpha\beta}\chi^\beta[\phi] = 0. \tag{2.190}$$

Let us introduce the functional $\widetilde{\Gamma}[\phi, C, \bar{C}, K, L]$ according to the rule

$$\Gamma[\phi, C, \bar{C}, K, L] = \widetilde{\Gamma}[\phi, C, \bar{C}, K, L] + \tfrac{1}{2}\chi^\alpha[\phi] G_{\alpha\beta}\chi^\beta[\phi].$$

Using this expression in (2.190) and (2.189) we obtain

$$\begin{gathered} \frac{\delta \widetilde{\Gamma}}{\delta \phi^i}\frac{\delta \widetilde{\Gamma}}{\delta K_i} + \frac{\delta \widetilde{\Gamma}}{\delta C^\alpha}\frac{\delta \widetilde{\Gamma}}{\delta L_\alpha} = 0 \\ \chi^\alpha_{,i}[\phi]\frac{\delta \widetilde{\Gamma}}{\delta K_i} + \frac{\delta \widetilde{\Gamma}}{\delta \bar{C}^\alpha} = 0. \end{gathered} \tag{2.191}$$

The relations (2.191) are final and give the universal form of Ward identities. Differentiating these equations over sources K and L and fields ϕ, C and $\bar{C}$ and choosing fields and sources equal to zero we can find different relations between vertex functions. Let us note that equations (2.191) play a central role in the investigations of the general structure of renormalizations in gauge theories.

2.8 Effective action of antisymmetric second-rank tensor field

Let us consider the theory of antisymmetric second-rank tensor field $\omega_{\mu\nu}$ with the action

$$S = -\tfrac{1}{12}\int dx\sqrt{-g}g^{\rho\alpha}g^{\tau\beta}g^{\sigma\gamma}F_{\alpha\beta\gamma}F_{\rho\tau\sigma}. \tag{2.192}$$

Here

$$F_{\alpha\beta\gamma} = \nabla_\alpha\omega_{\beta\gamma} + \nabla_\beta\omega_{\gamma\alpha} + \nabla_\gamma\omega_{\alpha\beta}. \tag{2.193}$$

The action of this theory is written in curved space–time with metric $g_{\mu\nu}$. It is evident that the effective action of the corresponding theory in flat space–time has a trivial structure. Therefore we consider the theory in an external gravitational field and assume that the sources for fields $\omega_{\alpha\beta}$ vanish. In this case the metric $g_{\mu\nu}$ will be a single argument of the effective action.

As was noted in section 1.4, the theory with action (2.192) is a gauge one. The gauge transformations are

$$\delta\omega_{\mu\nu} = \nabla_\mu\xi_\nu - \nabla_\mu\xi_\mu \tag{2.194}$$

and form an Abelian gauge group. However, the generators of gauge transformations (2.194)

$$R^\alpha_{\mu\nu}(x,y) = (\delta^\alpha_\nu\nabla_\mu - \delta^\alpha_\nu\nabla_\mu)\delta(x-y)$$

are linearly dependent because $R^\alpha_{\mu\nu}\nabla_\alpha\xi \equiv 0$ where ξ is an arbitrary scalar field. This is why it is impossible to define the generating functional of Green functions with the help of functional integral (2.146) because $M^\alpha_\beta[\phi]$(2.149) is degenerate and the functional $\Delta[\phi]$(2.148) does not exist. Indeed, the natural covariant gauge in this theory is

$$\chi_\mu = \nabla^\nu\omega_{\nu\mu}. \tag{2.195}$$

The corresponding matrix M^ν_μ constructed according to rule (2.149) has the form

$$M^\nu_\mu = \delta^\nu_\mu\nabla^\alpha\nabla_\alpha - \nabla^\nu\nabla_\mu. \tag{2.196}$$

This matrix has the evident zero-vector $n_\nu = \nabla_\nu\xi$ and so it is degenerate.

At present there are general methods of quantization for theories with linearly dependent generators. From the point of view of these methods the quantization of the theory with action (2.192) is a trivial exercise. However, this theory is quite simple. That is why we can find the functional integral in the theory under consideration with the help of procedures such as the Faddeev–Popov ansatz. Indeed, before the appearance of general methods of quantization the theory with action (2.192) had been investigated by Schwarz in geometrical terms. He suggested a suitable procedure for picking out the gauge group volume and obtained the correct expression for effective action of this theory. We shall find the effective action in the theory under consideration by a generalization of the Faddeev–Popov ansatz (without paying attention to the geometrical structure of the theory).

The basic relations for the construction of the generating functional in section 2.6 were relations (2.141) and (2.146). Let us try to find the corresponding analogues in the theory under consideration. First of all, we shall write formally $\delta(\chi_\mu[\omega])$, where gauge χ_μ is given by (2.195). However, we can show that

$$\nabla^\mu \chi_\mu = 0. \tag{2.197}$$

Then χ_μ is a covariantly transverse vector and $\delta(\chi_\mu) \propto \delta(0)$. It means that $\delta(\chi_\mu)$ is not defined. Indeed, $\delta(\chi_\mu) \propto \delta^{(3)}(\chi_\mu^\perp)\delta(\chi_\mu^\parallel)$ and $\chi_\mu^\parallel = 0$. Thus, $\delta(\chi_\mu) \propto \delta(0)$.

Let us try to find the analogue of the δ-function for transverse vectors. One can formally write

$$\delta(\chi_\mu) = \int \mathrm{D}K \exp\left(\mathrm{i} \int \mathrm{d}x \sqrt{-g} K^\mu \chi_\mu\right). \tag{2.198}$$

The functional in the integrand is invariant under the gauge transformation

$$K^\mu \to {}^f K^\mu = K^\mu + \nabla^\mu f \tag{2.199}$$

where f is a scalar field. According to the results of section 2.6 to find the correct functional integral of $\exp(\mathrm{i} \int \mathrm{d}x \sqrt{-g} K^\mu \chi_\mu)$ we should isolate the volume of the gauge group in (2.198). For this purpose it is convenient to use the relation

$$\begin{aligned} 1 &= \int \mathrm{D}f \delta(\nabla^f_\mu K^\mu) \det \Box_0 \\ \Box_0 &= \nabla^\mu \nabla_\mu \end{aligned} \tag{2.200}$$

in the integrand of (2.198). As a result we find the functional integral

$$\hat{\delta}(\chi_\mu) = \int \mathrm{D}K \exp\left(\mathrm{i} \int \mathrm{d}x \sqrt{-g} K^\mu \chi_\mu\right) \delta(\nabla^\mu K_\mu) \det \Box_0. \tag{2.201}$$

By definition, $\hat{\delta}(\chi_\mu)$ is called the δ-function of a transverse vector. Let

$$\delta(\nabla^\mu K_\mu) = \int \mathrm{D}\varphi \exp\left(\mathrm{i}\int \mathrm{d}x\sqrt{-g}\nabla_\mu K^\mu \varphi\right)$$

and substitute into (2.201). Then we obtain

$$\hat{\delta}(\chi_\mu) = \int \mathrm{D}\varphi\delta(\chi_\mu + \nabla_\mu\varphi)\det\Box_0. \tag{2.202}$$

In addition we will use $\hat{\delta}(\chi_\mu)$ as the δ-function of the gauge condition.

We will now find the functional Δ which is needed for writing (2.141). Consider

$$\int \mathrm{D}\xi\hat{\delta}(\chi_\mu({}^\xi\omega)) \tag{2.203}$$

where $({}^\xi\omega)_{\alpha\beta} = \omega_{\alpha\beta} + \nabla_\alpha\xi_\beta - \nabla_\beta\xi_\alpha$. The expression $\hat{\delta}(\chi_\mu({}^\xi\omega)) = \hat{\delta}(\nabla^\nu\omega_{\mu\nu} + \Box\xi_\mu - \nabla^\nu\nabla_\mu\xi_\nu)$ is invariant under the gauge transformations

$$\xi_\alpha \to {}^h\xi_\alpha = \xi_\alpha + \nabla_\alpha h$$

where h is a scalar field. One should isolate the volume gauge group in (2.203) in order to define the corresponding functional integral over ξ_μ correctly. We can do this in the same way as in the derivation of (2.201). As a result we define the functional integral of $\hat{\delta}(\chi_\mu({}^\xi\omega))$ as

$$\int \mathrm{D}\xi\hat{\delta}(\chi_\mu({}^\xi\omega))\delta(\nabla^\alpha\xi_\alpha)\det\Box_0. \tag{2.204}$$

Let us introduce the functional Δ by the relation

$$\Delta\int \mathrm{D}\xi\hat{\delta}(\chi_\mu({}^\xi\omega))\delta(\nabla^\alpha\xi_\alpha)\det\Box_0 = 1. \tag{2.205}$$

It follows that

$$\Delta = \det\Box_1\det{}^{-2}\Box_0 \tag{2.206}$$

where

$$\Box_1\xi_\mu = \nabla^\alpha\nabla_\alpha\xi_\mu - \nabla^\nu\nabla_\mu\xi_\nu + \nabla_\mu\nabla^\nu\xi_\nu.$$

To prove relation (2.206), we consider

$$\int \mathrm{D}\xi\hat{\delta}(\chi_\mu({}^\xi\omega))\delta(\nabla^\alpha\xi_\alpha)\det\Box_0$$
$$= \int \mathrm{D}K\mathrm{D}\xi\delta(\nabla_\alpha K^\alpha - l_1)\delta(\nabla^\nu\xi_\nu - l_2)\det{}^2\Box_0$$

$$\times \exp\left(\int \mathrm{d}x\sqrt{-g}K^{\mu}\chi_{\mu}({}^{\xi}\omega)\right)$$

$$= \int \mathrm{D}K\mathrm{D}\xi\delta(\nabla_{\alpha}K^{\alpha} - l_1)\delta(\nabla^{\nu}\xi_{\nu} - l_2)\det{}^2\Box_0$$

$$\times \exp\left(\int \mathrm{d}x\sqrt{-g}K^{\mu}(\chi_{\mu}(\omega) + \Box\xi_{\mu} - \nabla^{\nu}\nabla_{\mu}\xi_{\nu}\right.$$

$$\left. - \nabla_{\mu}\nabla^{\nu}\xi_{\nu} + \nabla_{\mu}\nabla^{\nu}\xi_{\nu}\right)$$

$$= \int \mathrm{D}K\mathrm{D}\xi\delta(\nabla_{\alpha}K^{\alpha} - l_1)\delta(\nabla^{\nu}\xi_{\nu} - l_2)\det{}^2\Box_0$$

$$\exp\left(\int \mathrm{d}x\sqrt{-g}(K^{\mu}(\chi_{\mu}(\omega) + \Box_1\xi_{\mu}) + \nabla_{\mu}K^{\mu}\nabla^{\nu}\xi_{\nu})\right)$$

$$= \int \mathrm{D}K D\xi\delta(\nabla_{\alpha}K^{\alpha} - l_1)\delta(\nabla^{\nu}\xi_{\nu} - l_2)\det{}^2\Box_0$$

$$\times \exp\left(\int \mathrm{d}x\sqrt{-g}K^{\mu}(\chi_{\mu}(\omega) + \Box\xi_{\mu} + \Box_1\xi_{\mu}) + l_1 l_2\right). \tag{2.207}$$

Here scalar fields l_1 and l_2, were introduced. Equation (2.207) does not depend on these scalar fields by construction. Besides, we used the fact that $\nabla_{\mu}K^{\mu} = l_1$ and $\nabla^{\nu}\xi_{\nu} = l_2$ in the integrand. If we change the variables $\xi_{\mu} \to \xi_{\mu} - \frac{1}{\Box_1}\chi_{\mu}(\omega)$, multiply by $\exp(-\mathrm{i}\int \mathrm{d}x\sqrt{-g}l_1 l_2)$ and integrate over l_1, l_2, we obtain

$$\int \mathrm{D}K\mathrm{D}\xi\det{}^2\Box_0\exp\int \mathrm{d}x\sqrt{-g}K^{\mu}\Box_1\xi_{\mu} = \det{}^2\Box_0\det{}^{-1}\Box_1.$$

This relation leads to (2.206).

Let us now define the effective action. Consider a naive expression

$$\int \mathrm{D}\omega\exp(\mathrm{i}S). \tag{2.208}$$

Here S is given by (2.192). We can introduce (2.205) into the integrand of (2.208) and obtain

$$\int \mathrm{D}\omega\mathrm{D}\xi\exp(\mathrm{i}S)\Delta\hat{\delta}(\chi_{\mu}({}^{\xi}\omega))\delta(\nabla^{\mu}\xi_{\mu})\det\,\Box_0$$

$$= \int \mathrm{D}\omega\mathrm{D}\xi\mathrm{D}\varphi\exp(\mathrm{i}S)\Delta\delta(\chi_{\mu}({}^{\xi}\omega) + \nabla_{\mu}\varphi - V_{\mu})\det{}^2\Box_0.$$

Here we used (2.202) and introduced the vector field V_{μ} on which this expression does not depend. (The construction of this expression does not depend on $V_\mu^\perp$. Changing the variable φ we can show that the longitudinal component of V_μ also vanishes.) We change the variables ${}^\xi\omega \to \omega$. After that the volume of gauge group (see (2.204))

$$\mathrm{Vol}G = \int \mathrm{D}\xi\delta(\nabla^\mu\xi_\mu)\det\,\Box_0 \tag{2.209}$$

can be isolated.

As a result we define the effective action Γ_ω for the theory under consideration as

$$\exp(\mathrm{i}\Gamma_\omega) = \int \mathrm{D}\omega\mathrm{D}\varphi\exp(\mathrm{i}S)\delta(\chi_\mu(\omega)+\nabla_\mu\varphi - V_\mu)\det\Box_0\Delta. \tag{2.210}$$

Let us rewrite (2.210) by integrating over V_μ with the weight $(-\mathrm{i}\int \mathrm{d}x\sqrt{-g}V_\mu V^\mu)$ and use the explicit form of Δ (2.206). Then we obtain

$$\exp(\mathrm{i}\Gamma_\omega) = \int \mathrm{D}\omega\mathrm{D}\varphi\exp\left(\mathrm{i}\int \mathrm{d}x\sqrt{-g}(\tfrac{1}{12}F_{\alpha\beta\phi}F^{\alpha\beta\gamma} - \tfrac{1}{2}\chi_\mu(\omega)\chi^\mu(\omega)\right.$$
$$\left. + \frac{1}{2}\varphi\Box_0\varphi\right)\det{}^{-1}\Box_0\det\Box_1 \tag{2.211}$$

It is easy to show that

$$\int \mathrm{d}x\sqrt{-g}(-\tfrac{1}{12}F_{\alpha\beta\gamma}F^{\alpha\beta\gamma} - \tfrac{1}{2}\chi_\mu(\omega)\chi^\mu(\omega))$$
$$= \tfrac{1}{4}\int \mathrm{d}x\sqrt{-g}\omega^{\alpha\beta}\Box_2\omega_{\alpha\beta}$$

where

$$\Box_2\omega_{\alpha\beta} = \nabla^\mu\nabla_\mu\omega_{\alpha\beta} - [\nabla^\gamma,\nabla_\beta]\omega_{\alpha\gamma} - [\nabla^\gamma,\nabla_\alpha]\omega_{\gamma\beta}. \tag{2.212}$$

Thus,

$$\exp(\mathrm{i}\Gamma_\omega) = \int \mathrm{D}\omega\mathrm{D}\varphi\det\,\Box_1\det{}^{-1}\Box_0$$
$$\times\exp\left(\mathrm{i}\int \mathrm{d}x\sqrt{-g}(\tfrac{1}{2}\omega^{\alpha\beta}\Box_2\omega_{\alpha\beta} - \tfrac{1}{2}\varphi\Box_0\varphi)\right)$$
$$= \det{}^{-1/2}\Box_2\det\,\Box_1\det{}^{-3/2}\Box_0. \tag{2.213}$$

Hence, (2.213) yields the final expression for the effective action

$$\Gamma_\omega = \Gamma_\varphi + \frac{\mathrm{i}}{2}(\mathrm{Tr}\ln\Box_2 - 2\,\mathrm{Tr}\ln\Box_1 + 2\,\mathrm{Tr}\ln\Box_0). \tag{2.214}$$

Here $\Gamma_\varphi = (\mathrm{i}/2)\,\mathrm{Tr}\ln\Box_0$ is the effective action of a massless scalar field minimally interacting with gravity.

Note that we can show that the theory of an antisymmetric second-rank tensor field in curved space is classically equivalent to the theory of a massless scalar field minimally interacting with gravity. Moreover, we can show that these theories are equivalent on a quantum level because the second term in (2.214) is topological invariant.

2.9 Gauge dependence of the effective action

Let us consider the generating functional of Green's functions for gauge theory (see section 2.6)

$$\begin{aligned} Z_{G,\chi}[J] = \int \mathrm{D}\phi \exp\Big[\mathrm{i}(S[\phi] + S_{\mathrm{GF}}[\phi] \\ -\mathrm{i}\,\mathrm{Tr}\ln\chi^\alpha_{,i}[\phi]R^i_\beta[\phi] - \frac{\mathrm{i}}{2}\mathrm{Tr}\ln G_{\alpha\beta} + \phi^i J_i)\Big] \\ S_{\mathrm{GF}} = \tfrac{1}{2}G_{\alpha\beta}\chi^\alpha[\phi]\chi^\beta[\phi]. \end{aligned} \tag{2.215}$$

It is evident that the generating functional depends on the choice of functionals $G_{\alpha\beta}$ and $\chi^\alpha[\phi]$ which are chosen for gauge fixing. The subscripts G,χ in $Z_{G,\chi}[J]$ indicate that the generating functional is calculated with some definite values of $G_{\alpha\beta}$ and $\chi^\alpha[\phi]$.

We will choose the linear gauge $\chi^\alpha = t^\alpha_i\phi^i$ and $t^\alpha_{i,j} = 0$. Let us change $G_{\alpha\beta}$ and t^α_i by $\delta G_{\alpha\beta}, \delta t^\alpha_i$ respectively. Then for the generating functional we obtain

$$\begin{aligned} Z_{G+\delta G,t+\delta t}[J] = \int \mathrm{D}\phi \exp\Big(\mathrm{i}\{S[\phi] + S_{\mathrm{GF}}[\phi] \\ + \tfrac{1}{2}\delta G_{\alpha\beta}t^\alpha_i t^\beta_j\phi^i\phi^j + \tfrac{1}{2}G_{\alpha\beta}(\delta t^\alpha_i t^\beta_j + t^\alpha_i\delta t^\beta_j)\phi^i\phi^j \\ - \mathrm{i}\,\mathrm{Tr}\ln M^\alpha_\beta - \mathrm{i}M^{-1}{}^\beta_\alpha\delta t^\alpha_i R^i_\beta - \frac{\mathrm{i}}{2}\mathrm{Tr}\ln G_{\alpha\beta} \\ - \frac{\mathrm{i}}{2}G^{\alpha\beta}\delta G_{\alpha\beta} + \phi^i J_i\}\Big). \end{aligned} \tag{2.216}$$

Here $\delta G_{\alpha\beta}$ and δt^α_i are infinitesimal variations. In the functional integral (2.216) we change the variables according to the corresponding gauge transformations

$$\begin{aligned} \phi^i \to \phi^i + R^i_\gamma\xi^\gamma \\ \xi^\gamma = -M^{-1}{}^\gamma_\delta(\delta t^\delta_i + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau}t^\tau_i)\phi^i. \end{aligned} \tag{2.217}$$

The correponding functional Jacobian is

$$J = \exp(1 + R^i_\gamma \xi^\gamma_{,i}). \tag{2.218}$$

Here we omitted $R^i_{\gamma,i}\xi^\gamma = 0$ because of the locality of the generators R^i_j (see discussion below (2.170)). We will assume that the generators of gauge transformations are linear ones on fields ϕ^i, hence $R^i_{\gamma,jk} = 0$. Using (2.217) in (2.216) we obtain

$$\begin{aligned} Z_{G+\delta G,t+\delta t}[J] = \int & \mathrm{D}\phi \exp\big(\mathrm{i}\{S[\phi] + S_{\mathrm{GF}}[\phi] \\ & - \mathrm{i}\,\mathrm{Tr}\ln M^\alpha_\beta - \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln G_{\alpha\beta} \\ & + \tfrac{1}{2}(\delta G_{\alpha\beta} t^\alpha_i t^\beta_j \phi^i\phi^j + G_{\alpha\beta}\delta t^\alpha_i t^\beta_j + G_{\alpha\beta} t^\alpha_i \delta t^\beta_j)\phi^i\phi^j \\ & - \mathrm{i} M^{-1}{}^\beta_\alpha \delta t^\alpha_i R^i_\beta \\ & - \tfrac{\mathrm{i}}{2} G^{\alpha\beta}\delta G_{\alpha\beta} + \tfrac{1}{2} G_{\alpha\beta} t^\alpha_i t^\beta_j (R^i_\gamma t^\gamma_k \phi^k + t^i_k \phi^k R^j_\gamma)\xi^\gamma \\ & - \mathrm{i} R^i_\gamma \xi^\gamma_{,i} - \mathrm{i} M^{-1}{}^\alpha_\beta t^\beta_i R^i_{\alpha,j} R^j_\gamma \xi^\gamma + \phi^i J_i + R^i_j \xi^j J_i\}\big)\,. \end{aligned} \tag{2.219}$$

Here ξ^γ is given by (2.217).

Consider

$$\begin{aligned} & G_{\alpha\beta} t^\alpha_i t^\beta_j (R^i_\gamma \phi^j + \phi^i R^j_\gamma)\xi^\gamma \\ & \quad = -G_{\alpha\beta}(t^\beta_j \phi^j M^\alpha_\gamma + t^\alpha_i \phi^i M^\beta_\gamma) M^\gamma_\delta (\delta t^\delta_k + \tfrac{1}{2} G^{\delta\lambda}\delta G_{\lambda\tau} t^\tau_k)\phi^k \\ & \quad = -\left[\delta G_{\alpha\beta} t^\alpha_i t^\beta_j + G_{\alpha\beta}(\delta t^\alpha_i t^\beta_j + t^\alpha_i \delta t^\beta_j)\right]\phi^i\phi^j\,. \end{aligned}$$

Thus

$$\begin{aligned} & [\delta G_{\alpha\beta} t^\alpha_i t^\beta_j + G_{\alpha\beta}(\delta t^\alpha_i t^\beta_j + t^\alpha_i \delta t^\beta_j)]\phi^i\phi^j + G_{\alpha\beta} t^\alpha_i t^\beta_j (R^i_\gamma \phi^j + \phi^i R^j_\gamma)\xi^\gamma \\ & \quad = 0. \end{aligned} \tag{2.220}$$

Consider

$$\begin{aligned} R^i_\gamma \xi^\gamma_{,i} = {} & R^i_\alpha M^{-1}{}^\alpha_\beta t^\beta_j R^j_{\gamma,i} M^{-1}{}^\gamma_\delta (\delta t^\delta_k + \tfrac{1}{2} G^{\delta\lambda}\delta G_{\lambda\tau} t^\tau_k)\phi^k \\ & - R^i_\alpha M^{-1}{}^\alpha_\delta (\delta t^\delta_i + \tfrac{1}{2} G^{\delta\lambda}\delta G_{\lambda\tau} t^\tau_i) \\ = {} & -R^i_\alpha M^{-1}{}^\alpha_\delta \delta t^\delta_i - \tfrac{1}{2} G^{\delta\lambda}\delta G_{\lambda\delta} \\ & + R^i_\alpha M^{-1}{}^\alpha_\beta t^\beta_j R^j_{\gamma,i} M^{-1}{}^\gamma_\delta (\delta t^\delta_k + \tfrac{1}{2} G^{\delta\lambda}\delta G_{\lambda\tau} t^\tau_k)\phi^k. \end{aligned}$$

Therefore

$$\begin{aligned}
&M^{-1}{}^{\alpha}_{\beta}\delta t^{\beta}_{i} R^{i}_{\alpha} + \tfrac{1}{2}G^{\alpha\beta}\delta G_{\alpha\beta} + R^{i}_{\gamma}\xi^{\gamma}_{,i} + M^{-1}{}^{\alpha}_{\beta} t^{\beta}_{i} R^{i}_{\alpha,j} R^{j}_{\gamma}\xi^{\gamma} \\
&\quad = M^{-1}{}^{\alpha}_{\beta}\delta t^{\beta}_{i} R^{i}_{\alpha} + \tfrac{1}{2}G^{\alpha\beta}\delta G_{\alpha\beta} - R^{i}_{\alpha} M^{-1}{}^{\alpha}_{\beta}\delta t^{\beta}_{i} - \tfrac{1}{2}G^{\alpha\beta}\delta G_{\alpha\beta} \\
&\qquad + R^{i}_{\alpha} M^{-1}{}^{\alpha}_{\beta} t^{\beta}_{j} R^{j}_{\gamma,i} M^{-1}{}^{\gamma}_{\delta}(\delta t^{\delta}_{k} + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k} \\
&\qquad - M^{-1}{}^{\alpha}_{\beta} t^{\beta}_{i} R^{i}_{\alpha,j} R^{j}_{\gamma} M^{-1}{}^{\gamma}_{\delta}(\delta t^{\delta}_{k} + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k} \\
&\quad = M^{-1}{}^{\alpha}_{\beta} t^{\beta}_{i}(R^{j}_{\alpha} R^{i}_{\gamma,j} - R^{j}_{\gamma} R^{i}_{\alpha,j}) M^{-1}{}^{\gamma}_{\delta}(\delta t^{\delta}_{k} + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k} \\
&\quad = M^{-1}{}^{\alpha}_{\beta} f^{\sigma}_{\alpha\gamma} M^{\beta}_{\sigma} M^{-1}{}^{\gamma}_{\delta}(\delta t^{\delta}_{k} + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k} \\
&\quad = f^{\alpha}_{\alpha\gamma} M^{-1}{}^{\gamma}_{\delta}(\delta t^{\delta}_{k} + \tfrac{1}{2}G^{\delta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k} \\
&\quad = 0.
\end{aligned} \tag{2.221}$$

Expression (2.221) is equal to zero because of the locality of structural constants $f^{\alpha}_{\alpha\gamma} = 0$ (see discussion below (2.170)).

Using (2.219)–(2.221) we obtain

$$\begin{aligned}
&Z_{G+\delta G,t+\delta t}[J] \\
&\qquad = Z_{G,t}[J]\Big[1 - \mathrm{i}J_i\langle R^{i}_{\alpha} M^{-1}{}^{\alpha}_{\beta}(\delta t^{\beta}_{k} + G^{\beta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k}\rangle_J\Big].
\end{aligned} \tag{2.222}$$

Here the notation used is

$$\begin{aligned}
\langle A\rangle_J = \frac{1}{Z_{G,t}[J]}\int \mathrm{D}\phi A \exp\Big[\mathrm{i}(S[\phi] + S_{\mathrm{GF}}[\phi] \\
- \mathrm{i}\,\mathrm{Tr}\ln M^{\alpha}_{\beta} - \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln G_{\alpha\beta} + \phi^{i}J_i\}\Big].
\end{aligned}$$

One can rewrite the above expressions introducing the generating functional of connected Green's functions $W_{G,t}[J]$ and also the effective action $\Gamma[\phi]$. In terms of effective action, expression (2.222) is

$$\begin{aligned}
&\frac{\delta\Gamma}{\delta G_{\alpha\beta}}\delta G_{\alpha\beta} + \frac{\delta\Gamma}{\delta t^{\alpha}_{i}}\delta t^{\alpha}_{i} \\
&\qquad = \frac{\delta\Gamma}{\delta\phi^{i}}\langle R^{i}_{\alpha} M^{-1}{}^{\alpha}_{\beta}(\delta t^{\beta}_{k} + G^{\beta\lambda}\delta G_{\lambda\tau} t^{\tau}_{k})\phi^{k}\rangle.
\end{aligned}$$

From this relation it follows that

$$\begin{aligned}
\frac{\delta\Gamma[\phi]}{\delta G_{\alpha\beta}} &= \Gamma_{,i}[\phi]\langle R^{i}_{\gamma} M^{-1}{}^{\gamma}_{\delta} G^{\delta(\alpha} t^{\beta)}_{k}\phi^{k}\rangle \\
\frac{\delta\Gamma[\phi]}{\delta t^{\alpha}_{i}} &= \Gamma_{,j}[\phi]\langle R^{j}_{\beta}(M^{-1})^{\beta}_{\alpha}\phi^{i}\rangle.
\end{aligned} \tag{2.223}$$

Expressions (2.223) indicate explicitly how the effective action is changing when the gauge-fixing functionals $G_{\alpha\beta}$ and t_i^α are changed. One can see that the effective action depends generally on the gauge.

Let the mean field satisfy the equation $\Gamma_{,i}[\phi] = 0$. Then from (2.223) it follows that $\delta\Gamma[\phi]/\delta G_{\alpha\beta} = 0$ and $\delta\Gamma[\phi]/\delta t_i^\alpha = 0$. Thus, the effective action at its extrema does not depend on the gauge.

2.10 Effective action in the background field gauge

Let us consider the gauge theory with action $S[\phi]$, closed algebra and linearly independent generators. We limit ourselves to the theories in which the generators $R^i_\alpha[\phi]$ have the following form

$$
\begin{aligned}
R^i_\alpha[\phi] &= r^i_\alpha + R^i_{\alpha,j}\phi^j \\
r^i_{\alpha,j} &= 0 \\
R^i_{\alpha,jk} &= 0.
\end{aligned}
\tag{2.224}
$$

According to the results of section 2.6 the gauge function$\chi^\alpha[\phi]$ should be chosen for quantization. We assume the choice of linear gauge

$$
\chi^\alpha[\phi] = t_i^\alpha\phi^i \tag{2.225}
$$

where $t^\alpha_{i,j} = 0$.

The generating functional of connected Green's functions is given by

$$
\begin{aligned}
e^{iW[J]} = \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp \Big[&\mathrm{i}(S[\phi] + \tfrac{1}{2}G_{\alpha\beta}t_i^\alpha t_j^\beta\phi^i\phi^j \\
&+\bar{C}_\alpha t_i^\alpha R^i_\beta[\phi]C^\beta - \frac{\mathrm{i}}{2}\operatorname{Tr}\ln G_{\alpha\beta} + \phi^i J_i.)\Big]
\end{aligned}
\tag{2.226}
$$

We will now pass to the effective action $\Gamma[\bar{\phi}]$ corresponding to $W[J]$. The construction of the effective action depends on the gauge (on the choice of $G_{\alpha\beta}$ and t_i^α). Moreover, the effective action is not a gauge invariant functional, i.e. $\Gamma_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}] \neq 0$ in general. In addition, the S-matrix does not depend on the gauge. Therefore, we can try to find a proper choice of $G_{\alpha\beta}$ and t^i_α in order to construct the gauge invariant effective action (with the same S-matrix).

We introduce the functionals $G_{\alpha\beta}[\widetilde{\phi}], t_i^\alpha[\widetilde{\phi}]$ depending on some background field $\widetilde{\phi}^i$. Following De Witt we shall choose the transformation law for the above functionals under the gauge transformations of background field $\delta\widetilde{\phi}^i = R^i_\alpha[\widetilde{\phi}]\xi^\alpha$ in the following form

$$
\begin{aligned}
\delta t_i^\alpha &= (f^\alpha_{\beta\gamma}t_i^\gamma - t_j^\alpha R^j_{\beta,i})\xi^\beta \\
\delta G_{\alpha\beta} &= -(G_{\alpha\delta}f^\delta_{\gamma\beta} + G_{\delta\beta}f^\delta_{\gamma\alpha})\xi^\gamma
\end{aligned}
\tag{2.227}
$$

where $f^{\alpha}_{\beta\gamma}$ are structure constants of the gauge algebra. In this gauge the generating functional of connected Green's functions depends on the background field $\widetilde{\phi}^i$,

$$\exp[\mathrm{i}W[J,\widetilde{\phi}]] = \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp[\mathrm{i}(S[\phi] + \tfrac{1}{2}G_{\alpha\beta}[\widetilde{\phi}]t^{\alpha}_i[\widetilde{\phi}]t^{\beta}_j[\widetilde{\phi}]\phi^i\phi^j + \bar{C}_\alpha t^{\alpha}_i[\widetilde{\phi}]R^i_\beta[\phi]C^\beta - \tfrac{1}{2}\operatorname{Tr}\ln G_{\alpha\beta}[\widetilde{\phi}] + \phi^i J_i)] \tag{2.228}$$

We define the mean field $\bar{\phi}^i$

$$\bar{\phi}^i = \frac{\delta W[J,\widetilde{\phi}]}{\delta J^i} \tag{2.229}$$

and introduce the effective action $\Gamma[\bar{\phi},\widetilde{\phi}]$,

$$\Gamma[\bar{\phi},\widetilde{\phi}] = W[J,\widetilde{\phi}] - \bar{\phi}^i J_i$$

where the source J_i is expressed from (2.228) in terms of mean fields $\bar{\phi}^i$. Changing the variable $\phi \to \phi + \bar{\phi}$ in the functional integral (2.227) we obtain

$$\exp(\mathrm{i}\Gamma[\bar{\phi},\widetilde{\phi}]) = \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp\Big[\mathrm{i}\Big(S[\bar{\phi}+\phi] + \tfrac{1}{2}G_{\alpha\beta}[\widetilde{\phi}]t^{\alpha}_i[\widetilde{\phi}]t^{\beta}_j[\widetilde{\phi}](\phi^i+\bar{\phi}^i)(\phi^j+\bar{\phi}^j) + \bar{C}_\alpha t^{\alpha}_i[\widetilde{\phi}]R^i_\beta[\phi+\bar{\phi}]C^\beta - \tfrac{1}{2}\operatorname{Tr}\ln G_{\alpha\beta}[\widetilde{\phi}] + \phi^i \frac{\delta\Gamma[\bar{\phi},\widetilde{\phi}]}{\delta\bar{\phi}^i}\Big)\Big]. \tag{2.230}$$

We put $\widetilde{\phi} = \bar{\phi}$ in (2.230) and denote $\Gamma[\bar{\phi},\widetilde{\phi}]|_{\widetilde{\phi}=\bar{\phi}} = \Gamma[\bar{\phi}]$. The functional $\Gamma[\bar{\phi}]$ is the effective action which is calculated in a special gauge depending on the mean field $\bar{\phi}$. From (2.230) it follows that

$$\exp(\mathrm{i}\Gamma[\bar{\phi}]) = \int \mathrm{D}\phi \mathrm{D}\bar{C}\mathrm{D}C \exp\Big[\mathrm{i}\Big(S[\bar{\phi}+\phi] + \tfrac{1}{2}G_{\alpha\beta}[\bar{\phi}]t^{\alpha}_i[\bar{\phi}]t^{\beta}_j[\bar{\phi}](\bar{\phi}^i+\phi^i)(\bar{\phi}^j+\phi^j) + \bar{C}_\alpha t^{\alpha}_i[\bar{\phi}]R^i_\beta[\bar{\phi}+\phi]C^\beta - \tfrac{1}{2}\operatorname{Tr}\ln G_{\alpha\beta}[\bar{\phi}] + \phi^i \left(\frac{\delta\Gamma[\bar{\phi},\widetilde{\phi}]}{\delta\bar{\phi}^i}\right)\Big|_{\widetilde{\phi}=\bar{\phi}}\Big)\Big]. \tag{2.231}$$

Now we can discuss the properties of the effective action $\Gamma[\bar\phi]$ (2.231). Let

$$\Gamma[\bar\phi] = S[\bar\phi] + \tfrac{1}{2}G_{\alpha\beta}[\bar\phi]t_i^\alpha[\bar\phi]t_j^\beta[\bar\phi]\bar\phi^i\bar\phi^j + \bar\Gamma[\bar\phi] \tag{2.232}$$

where $\bar\Gamma[\bar\phi] = \sum_{n=1}^{\infty}\bar\Gamma^{(n)}[\bar\phi]$. The functional $\bar\Gamma^{(n)}[\bar\phi]$ corresponds to the n-loop contribution to the effective action.

First of all we shall find the one-loop contribution. Substituting (2.232) into (2.231) and repeating the discussion leading to (2.93) we obtain

$$\begin{aligned}\bar\Gamma^{(1)}[\bar\phi] = {} & \frac{\mathrm{i}}{2}\operatorname{Tr}\ln F_{ij}[\bar\phi] \\ & - \mathrm{i}\operatorname{Tr}\ln M^\alpha_\beta[\bar\phi] - \frac{\mathrm{i}}{2}\operatorname{Tr}\ln G_{\alpha\beta}[\bar\phi]\end{aligned} \tag{2.233}$$

where

$$\begin{aligned}F_{ij}[\bar\phi] &= S_{,ij}[\bar\phi] + G_{\alpha\beta}[\bar\phi]t_i^\alpha[\bar\phi]t_j^\beta[\bar\phi] \\ M^\alpha_\beta[\bar\phi] &= t_i^\alpha[\bar\phi]R^i_\beta[\bar\phi].\end{aligned} \tag{2.234}$$

Let us find the transformation properties of $\bar\Gamma^{(1)}[\bar\phi]$ under the transformation $\delta\bar\phi^i = R^i_\alpha[\bar\phi]\xi^\alpha$. First of all, we will determine the corresponding change of $F_{ij}[\bar\phi]$ and $M^\alpha_\beta[\bar\phi]$

$$\begin{aligned}\delta F_{ij}[\bar\phi] = {} & S_{,ijk}[\bar\phi]R^k_\gamma[\bar\phi]\xi^\gamma + \delta G_{\alpha\beta}[\bar\phi]t_i^\alpha[\bar\phi]t_j^\beta[\bar\phi] \\ & + G_{\alpha\beta}[\bar\phi](\delta t_i^\alpha[\bar\phi]t_j^\beta[\bar\phi] + t_i^\alpha[\bar\phi]\delta t_j^\beta[\bar\phi]).\end{aligned} \tag{2.235}$$

Differentiating the relation $S_{,k}[\bar\phi]R^k_\gamma[\bar\phi] = 0$ twice and taking into account (2.224) we obtain

$$S_{,ijk}[\bar\phi]R^k_\gamma[\bar\phi] = -S_{,ik}[\bar\phi]R^k_{\gamma,j} + S_{,jk}[\bar\phi]R^k_{\gamma,i}. \tag{2.236}$$

Substituting (2.227) and (2.236) into (2.235) we have

$$\delta F_{ij}[\bar\phi] = -(F_{ik}[\bar\phi]R^k_{\gamma,j} + F_{jk}[\bar\phi]R^k_{\gamma,i})\xi^\gamma. \tag{2.237}$$

Consider

$$\begin{aligned}\delta M^\alpha_\beta[\bar\phi] &= \delta t_i^\alpha[\bar\phi]R^i_\beta[\bar\phi] + t_i^\alpha[\bar\phi]R^i_{\beta,j}[\bar\phi]R^j_\gamma[\bar\phi]\xi^\gamma \\ &= (f^\alpha_{\gamma\lambda}[\bar\phi]t_i^\lambda[\bar\phi] - t_j^\alpha[\bar\phi]R^j_{\gamma,i})R^i_\beta[\bar\phi]\xi^\gamma \\ &\quad + t_i^\alpha[\bar\phi]R^i_{\beta,j}[\bar\phi]R^j_\gamma[\bar\phi]\xi^\gamma \\ &= f^\alpha_{\gamma\lambda}M^\lambda_\beta[\bar\phi]\xi^\gamma + t_i^\alpha[\bar\phi](R^j_\gamma[\bar\phi]R^i_{\beta,j} - R^j_\beta[\bar\phi]R^i_{\gamma,j}[\bar\phi])\xi^\gamma \\ &= (f^\alpha_{\gamma\lambda}M^\lambda_\beta[\bar\phi] - f^\lambda_{\gamma\beta}M^\alpha_\lambda[\bar\phi])\xi^\gamma.\end{aligned} \tag{2.238}$$

Let us calculate

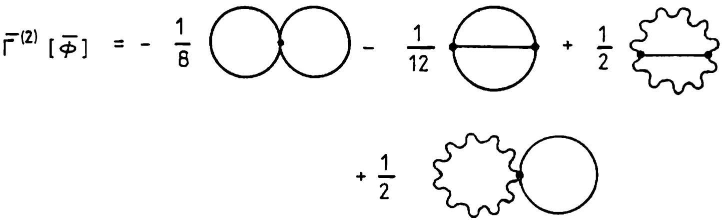
$$\bar{\Gamma}^{(1)}_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}]\xi^\alpha = \frac{\mathrm{i}}{2}F^{-1^{ij}}[\bar{\phi}]\delta F_{ji}[\bar{\phi}] - \mathrm{i}M^{-1}{}^\alpha_\beta[\bar{\phi}]\delta M^\beta_\alpha[\bar{\phi}] - \mathrm{i}G^{\alpha\beta}[\bar{\phi}]\delta G_{\alpha\beta}[\bar{\phi}].$$

Using (2.227), (2.237) and (2.238) we obtain

$$\begin{aligned}\bar{\Gamma}^{(1)}_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}]\xi^\alpha &= -\frac{\mathrm{i}}{2}(R^i_{\gamma,i} + R^j_{\gamma,j})\xi^\alpha \\ &\quad - \mathrm{i}(f^\beta_{\gamma\beta} - f^\alpha_{\gamma\alpha})\xi^\gamma \\ &\quad - \mathrm{i}(f^\beta_{\gamma\beta} - f^\alpha_{\gamma\alpha})\xi^\gamma = 0. \end{aligned} \tag{2.239}$$

Here we used the relations $R^i_{\gamma,i} = 0$ and $f^\alpha_{\gamma\alpha} = 0$ (see discussion below (2.170)). Thus, the one-loop quantum correction to the effective action $\Gamma[\bar{\phi}]$ (2.231) is gauge invariant.

Consider now the two-loop correction. In this case using the results of section 2.6 we obtain



(2.240)

Here the solid line is the propagator $F^{-1^{ij}}[\bar{\phi}]$ and the wavy line is the propagator $M^{-1}{}^\alpha_\beta[\bar{\phi}]$. Using (2.227), (2.236)–(2.238) we can show by direct but tedious calculations that

$$\bar{\Gamma}^{(2)}_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}] = 0. \tag{2.241}$$

The two-loop quantum correction to effective action (2.231) is therefore also gauge invariant.

In the same way (with the help of (2.227), (2.236)–(2.238)) we can prove that

$$\bar{\Gamma}^{(k)}_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}] = 0.$$

The arbitrary-order loop correction to the effective action $\Gamma[\bar{\phi}]$ is gauge-invariant:

$$\bar{\Gamma}_{,i}[\bar{\phi}]R^i_\alpha[\bar{\phi}] = 0. \tag{2.242}$$

Therefore, the functional

$$\widetilde{\Gamma}[\bar{\phi}] = \Gamma[\bar{\phi}] - \tfrac{1}{2}G_{\alpha\beta}[\bar{\phi}]t_i^\alpha[\bar{\phi}]t_j^\beta[\bar{\phi}]\bar{\phi}^i\bar{\phi}^j = S[\bar{\phi}] + \bar{\Gamma}[\bar{\phi}] \qquad (2.243)$$

is also gauge invariant due to (2.242). This functional is called gauge-invariant effective action. The use of the gauge-invariant effective action is very convenient for calculations. It guarantees the possibility of having explicit gauge invariance at all stages of calculations. The gauge-invariant effective action approach based on the background field-dependent gauges is sometimes called the background-field method. Of course, we must understand that the gauge invariant effective action is dependent on the quantum field gauge.

2.11 Comments

1. Canonical quantization is considered in many textbooks on quantum field theory. See, for example, books by Schweber [3] and Itzykson and Zuber [5]. A consistent discussion of canonical quantization for constrained systems is given in [159].
2. The functional method in quantum field theory has been considered in books by De Wit [2], Itzykson and Zuber [5], Vasiliev [160] and in lectures by De Witt [161] and by Abers and Lee [18].
3. Our consideration of the generating functionals of connected and one-particle irreducible Green's functions follows the unpublished lectures by Tyutin [163].
4. The generalized ξ-function regularization has been introduced by Dowker and Critchley [164] and Hawking [165]. For a discussion, see the review by De Witt [166].
5. The foundation of the general approach to gauge theories has been developed by De Witt [2]. The structure of the commutator for generators of gauge transformations (expression (2.132)) has been discussed in [167].
6. The functional integral for the generating functional of Green's functions in gauge theories with closed algebra and linearly independent generators was first found by De Witt [103] and Faddeev and Popov [169]. The Faddeev–Popov method has been considered in all modern textbooks on quantum field theory (see [5, 7, 8, 160]). We follow review [103].

The approaches to the covariant quantization of gauge theories have been developed in papers by Mandelstam [170], Fradkin and Tyutin [171], Kallosh [172], De Witt and van Holten [173], Batalin and Vilkovisky [174], Voronov and Tyutin [175].

7. The theorem of equivalence has been proved by Kallosh and Tyutin [176].

8. The fundamental ideas of canonical quantization of gauge theories were considered by Dirac [177]. The approach to canonical quantization of gauge theories in terms of functional integrals was found by Faddeev [178] and Fradkin [179]. During the last few years Batalin, Fradkin and Vilkovisky have developed a method of generalized canonical quantization, which made it possible to use the relativistic gauges and apply them to gauge theories with open gauge algebra and with dependent generators (see papers [180]–[184]).
9. BRST-transformations were found by Becchi, Rouet and Stora [70] and Tyutin [71].
10. The Ward identities in the form (2.191) were first investigated in [185].
11. The covariant quantization of the antisymmetric tensor field was first developed by Schwarz [186]. The covariant quantization of arbitrary gauge theories with linearly dependent generators was first expounded in [174], [183] and [188]. The canonical quantization of such theories was considered in [188] and [189].
12. The replacement of variables in functional integral (2.217) was introduced by De Witt [103]. Relations (2.223) which define the gauge dependence of effective action are given in [162] (see also [168]).

PART 2

QUANTUM FIELD THEORY IN CURVED SPACE–TIME AND THE EFFECTIVE ACTION

3 Renormalization and Renormalization Group Equations in Curved Space–time

3.1 The basic ideas of renormalization theory

The effective action $\Gamma[\varphi]$ and vertex functions are usually calculated in terms of perturbation theory on the basis of the loop expansion. Nevertheless, the typical situation occurs when vertex functions are given in terms of Feynman integrals which are divergent at large momenta. It means that the theory contains ultraviolet divergences.

Renormalization is the special procedure for the reconstructuion of the theory under consideration so that the divergences are absent and the vertex functions are finite. Renormalization theory is discussed in many books on quantum field theory (for example, [1, 5, 7, 21, 474]). We will present here only the basic ideas of renormalization (which will be used in the course of this book).

To obtain the finite theory it is necessary, first of all, to give the sense of the divergent Feynman integrals which are used for writing the vertex functions in the loop expansion. This can be done by means of the so-called regularization procedure. Any regularization consists of changing the original divergent Feynman integral into another finite integral, depending on some parameter of regularization Λ. When the parameter Λ tends to some definite value (regularization turned off), the regularized integral formally reduces to the original integral.

In quantum field theory there are some well-known regularizations, for example, dimensional regularization, Pauli-Villars regularization, analytical regularization, generalized ζ-function regularization and its many-loop generalization-operator regularization, regularization with the help of higher derivatives, cutoff of the Feynman momenta integrals at the upper limit and cutoff of the proper-time integrals at the lower limit.

We will use mostly dimensional regularization for which the dimensionality of space–time n is the parameter of regularization. The advantage of such a regularization scheme is the possibility of introducing the regularization parameter directly into the action, defining it in the space–time of n dimensions. Then we can investigate some general problems using only regularized action.

Let us consider now the theory of fields φ_0^i with the action $S_0[\varphi_0, p_0]$ where p_0 is the set of all parameters of the theory (i.e. masses and coupling constants). The renormalization is described in the following way.

Another theory with fields φ^i, parameters p and action $S[\varphi, p, \Lambda]$ is introduced instead of the original theory with the action $S_0[\varphi_0, p_0, \Lambda]$, where

$$S[\varphi, p] = S_0[\varphi, p] + \Delta S[\varphi, p, \Lambda]. \tag{3.1}$$

Here $S_0[\varphi, p]$ is the original action (n is introduced into the action S_0 in the dimensional regularization using the regularization parameter n), where φ_0 and p_0 are replaced by φ and p. The action ΔS has the following structure

$$\Delta S[\varphi, p, \Lambda] = \sum_{r=1}^{\infty} \hbar^r \Delta S_r[\varphi, p, \Lambda]. \tag{3.2}$$

The functionals ΔS_r are chosen in a way leading to the finite effective action in the theory with action (3.1) in each order of the loop expansion when regularization is turned off. This functional $\Delta S[\varphi, p, \Lambda]$ is called the counterterm action (or counterterms). The fundamental result of the renormalization theory consists of the fact that corresponding functionals ΔS_r exist and are local functionals of the fields φ^i in local quantum field theory.

It is very often the case that the functional $\Delta S[\varphi, p, \Lambda]$ has the same structure as the action $S_0[\varphi, p]$, in the following sense. Let

$$\begin{aligned} S_0[\varphi_0, p_0] = {}& \tfrac{1}{2} S_{2 i_1 i_2} \varphi_0^{i_1} \varphi_0^{i_2} \\ & + \sum_{m=2}^{\infty} P^{(m)}_{0 i_1 i_2 \ldots i_m} \varphi_0^{i_1} \varphi_0^{i_2} \ldots \varphi_0^{i_m}. \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} S_0[\varphi, p] = {}& \tfrac{1}{2} S_{2 i_1 i_2} \varphi^{i_1} \varphi^{i_2} \\ & + \sum_{m=2}^{\infty} P^{(m)}_{i_1 i_2 \ldots i_m} \varphi^{i_1} \varphi^{i_2} \ldots \varphi^{i_m}. \end{aligned} \tag{3.4}$$

Here $P^{(m)}_{0i_1i_2\ldots i_m}$ and $P^{(m)}_{i_1i_2\ldots i_m}$ are the parameters of actions (3.3) and (3.4), respectively. (In the simplest models of quantum field theories m is finite. However, it is not the case, for example, in quantum gravity. We use here condensed notations of Chapter 2, section 6.) We will say that the structure of $\Delta S[\varphi, p, \Lambda]$ is the same as the structure of $S_0[\varphi, p]$ if $\Delta S[\varphi, p, \Lambda]$ is

$$\Delta S[\varphi, p, \Lambda] = \tfrac{1}{2} S_{2j_1j_2} (\Delta Z_1)^{j_1j_2}_{i_1i_2} \varphi^{i_1}\varphi^{i_2} + \sum_{m=2}^{\infty} P^{(m)}_{j_1j_2\ldots j_m} (\Delta Z_m)^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} \varphi^{i_1}\varphi^{i_2}\ldots\varphi^{i_m}. \quad (3.5)$$

It is important that the coefficients $S_{2i_1i_2}$ and $P^{(m)}_{i_1i_2\ldots i_m}$ are the same in (3.4) and in (3.5). In this case

$$\begin{aligned} S[\varphi, p, \Lambda] &= S_0[\varphi, p] + \Delta S[\varphi, p, \Lambda] \\ &= \tfrac{1}{2}\left(\delta^{j_1j_2}_{i_1i_2} + (\Delta Z_1)^{j_1j_2}_{i_1i_2}\right) S_{2j_1j_2}\varphi^{i_1}\varphi^{i_2} \\ &\quad + \sum_{m=2}^{\infty}\left(\delta^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} + (\Delta Z_m)^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m}\right) \\ &\quad \times P^{(m)}_{j_1j_2\ldots j_m}\varphi^{i_1}\varphi^{i_2}\ldots\varphi^{i_m}. \end{aligned} \quad (3.6)$$

Let us denote

$$\begin{aligned} Z_1{}^{j_1j_2}_{i_1i_2} &= \delta^{j_1j_2}_{i_1i_2} + (\Delta Z_1)^{j_1j_2}_{i_1i_2} \\ (\bar{Z}_m)^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} &= \delta^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} \\ &\quad + (\Delta Z_m)^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m}. \end{aligned} \quad (3.7)$$

Here $\delta^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m}$ is the symmetrized product of delta symbols δ^j_i.

As a result we can rewrite (3.6) in the form

$$S[\varphi, p, \Lambda] = \tfrac{1}{2} Z_1{}^{j_1j_2}_{i_1i_2} S_{2j_1j_2}\varphi^{i_1}\varphi^{i_2} + \sum_{m=2}^{\infty} \bar{Z}_m{}^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} P^{(m)}_{j_1j_2\ldots j_m}\varphi^{i_1}\varphi^{i_2}\ldots\varphi^{i_m}. \quad (3.8)$$

Let $Z^{j_1j_2}_{1i_1i_2}$ have the following structure: $Z^{j_1j_2}_{1i_1i_2} = Z^{j_1}_{1(i_1} Z^{j_2}_{1i_2)}$. Then it is evident that the action $S[\varphi, p, \Lambda]$(3.8) is formed from the original action $S_0[\varphi_0, p_0]$ with the help of the transformations

$$\begin{aligned} \varphi^i_0 &= (Z_1^{1/2})^i_j \varphi^j \\ P^{(m)}_{0i_1i_2\ldots i_m} &= Z_m{}^{j_1j_2\ldots j_m}_{i_1i_2\ldots i_m} P^{(m)}_{j_1j_2\ldots j_m} \end{aligned} \quad (3.9)$$

where

$$Z_{m i_1 i_2 \ldots i_m}^{\ \ j_1 j_2 \ldots j_m} = \bar{Z}_{m i_1 i_2 \ldots i_m}^{\ \ k_1 k_2 \ldots k_m} \times (Z_1^{-1/2})^{j_1}_{k_1} (Z_1^{-1/2})^{j_2}_{k_2} \ldots (Z_1^{-1/2})^{j_m}_{k_m}. \tag{3.10}$$

Expressions (3.9) are called renormalization transformations, and Z_1 and Z_m (3.7) and (3.10) are called the renormalization constants. We use the following terminology: $S_0[\varphi_0, p_0]$ is bare action, φ_0 and p_0 are bare fields and parameters. In this case the renormalized action is obtained from the bare one with the help of transformations of renormalization. The theories which have this property are usually called the multiplicatively renormalized theories.

Let us consider the theory which is not multiplicatively renormalized. Suppose that multiplicatively renormalized action can be found by the addition of a finite number of some local functionals to the original action S_0. Then we shall say that this theory can be reduced to a multiplicatively renormalized theory.

The multiplicatively renormalized theories have a number of remarkable properties. In particular, to study these theories we can use the powerful method of the renormalization group. In this chapter we assume that the field theoretical models under consideration can be reduced to multiplicatively renormalizable theories.

In many known cases the sum over m in (3.3)–(3.5) contains a finite number of terms or leads to a finite number of local geometrical invariants. (For example, in gravity with the action $\int d^4x\sqrt{-g}(aR_{\mu\nu}R^{\mu\nu} + bR^2)$ the counterterms are of the same structure as the original action (in a special gauge). However, if we represent metric $g_{\mu\nu}$ in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then the action and counterterms are an infinite series like (3.3)–(3.5) in terms of the fields $h_{\mu\nu}$.) That is why we can say that the theory is renormalized if it contains a finite number of counterterms, and the theory is not renormalized in the opposite case. Non-renormalizability of field theory does not mean, of course, that the theory is not interesting.

Is it possible to find the explicit structure of the counterterms based only on the original action $S_0[\varphi_0, p_0]$ without the calculation of Feynman's diagrams? To answer this question it is convenient to use the superficial degree of divergence of Feynman's diagrams. Let G be an arbitrary diagram for some vertex function in four-dimensional space–time. The superficial degree of divergence $\omega(G)$ for this diagram is called

$$\omega(G) = 4 - \sum_{v} d_{g_v} - \sum_{L_{ext}} d_\varphi. \tag{3.11}$$

Here d_{g_v} is the dimensionality of the coupling constant (in terms of mass dimensionality), the sum over v is the sum of the number of vertices for graph G, d_φ is the canonical dimensionality of fields φ (in units of mass), $\sum_{L_{ext}}$ is the sum over numbers of points to which the external lines must be joined for the creation of the Green's function from the vertex function. It is well-known from the theory of renormalization that the given diagram G is superficially divergent if $\omega(G) \geqslant 0$. If $\omega(G) < 0$ then this diagram is finite. (Provided, of course, that all the subdiagrams of G are finite.)

From $\omega(G) \geqslant 0$ we can find the values of N for which vertex functions $\Gamma_N(x_1, \ldots, x_N)$ are divergent when regularization is turned off. Then the corresponding contribution to the counterterms is proportional to the local functional

$$\int dx_1 \ldots dx_N \Gamma_N^{(div)}(x_1, \ldots, x_N)\varphi(x_1)\ldots\varphi(x_N)$$

where $\Gamma_N^{(\mathrm{div})}(x_1, \ldots, x_N)$ is the divergent part of the vertex function $\Gamma_N(x_1, \ldots, x_N)$. Considering the suitable values of N we can choose all necessary counterterms with accuracy up to finite contributions. In this case it is known that multiplicative renormalizability requires dimensionless coupling constants.

Additional aspects appear in connection with the renormalization of gauge theories. Let the gauge theory under consideration be multiplicatively renormalized. During the quantization we introduce the terms which break gauge invariance. Then we can expect that the corresponding counterterms are not gauge invariant. Nevertheless, using the Ward identities we can prove that renormalization is gauge-invariant (possibly with the deformation of gauge algebra). Of course, the existence of regularization which preserves gauge invariance is assumed.

As was noted above, the regularization scheme is a very important element of the theory of renormalization. The demands for any regularization are obvious enough. Regularization must preserve as many properties of the original action as possible, such as the gauge invariance, the Poincaré invariance, the global symmetry etc. Otherwise it is necessary to prove that these properties are preserved in the renormalized theory. Unfortunately, a regularization which preserves all the properties of the theory does not exist. It leads to the appearance of anomalies in the theory.

Let us consider now some aspects of the dimensional regularization (see, e.g., [69]). (A modern account of the renormalization theory in the framework of dimensional regularization is given in [21].) A regularization supposes that the original theory is given in n-dimensional space–time. In this case the divergences of the Feynman diagrams are the expressions containing poles $1/(n-4)^k$ and $k = 1, 2, \ldots$. The limit $n \to 4$ is taken in Green's functions after renormalization is fulfilled.

Let us discuss the theory containing scalar fields φ, spinor fields ψ, vectors A_μ and the coupling constants corresponding to scalar coupling f, Yukawa coupling h and gauge coupling g. Their dimensions are $d_\varphi = 1$, $d_\psi = 3/2$, $d_A = 1$, $d_f = d_h = d_g = 0$ in four-dimensional space–time and $d_\varphi = (n-2)/2$, $d_\psi = (n-1)/2$, $d_A = (n-2)/2$, $d_f = 4-n$, $d_h = (4-n)/2$, $d_g = (4-n)/2$ in n-dimensional space–time. It is convenient to consider that the dimensions of renormalized fields and parameters correspond to four-dimensional space–time. Then renormalization transformations have the following form

$$
\begin{aligned}
\varphi_0 &= \mu^{(n-4)/2} Z_1^{1/2} \varphi & \psi_0 &= \mu^{(n-4)} \bar{Z}_1^{1/2} \psi \\
A_{0\mu} &= \mu^{(n-4)/2} \bar{\bar{Z}}_1^{1/2} A_\mu & m_0 &= Z_m m \\
h_0 &= \mu^{(4-n)/2} Z_h h & g_0 &= \mu^{(4-n)/2} Z_g g \\
f_0 &= \mu^{(4-n)} Z_f f. & &
\end{aligned}
\tag{3.12}
$$

Here μ is an arbitrary parameter with dimension of mass, m are the masses of scalars and spinors. All the constants of renormalization are dimensionless and depend only on dimensionless coupling constants.

It is known that the procedure of renormalization has some arbitrariness. This means that it is possible to add finite (quasi)local functionals to the counterterms in each order of the loop expansion. Some additional conditions have to be used in the theory for the fixing of these finite counterterms. They are connected with the conditions for the measurement of theoretical parameters and are called normalization conditions. For example, in dimensional regularization when $n \to 4$ the counterterms contain the finite terms as well as poles. We will use the minimal subtraction scheme where contributions to the counterterms are given only by pole parts. Finite counterterms are absent.

It is evident that dimensional regularization affects those properties of the four-dimensional theory which are impossible to formulate in n-dimensional space–time. Initially, these properties are the chiral and scale symmetries. These symmetries take place in massless theories in four dimensional space–time and break in n-dimensions. That is why we can expect the appearance of chiral and scale anomalies in the renormalized theory. Of course, it is not right to think that if some symmetry breaks in the regularized theory then an anomaly must appear. It indicates that the question of the existence of corresponding symmetry in the renormalized theory must be investigated carefully. As to chiral and scale symmetries the corresponding anomalies exist.

3.2 Structure of renormalization in curved space–time

Let us consider the theory containing scalars $\varphi(x)$, spinors $\psi(x)$ and vectors $A_\mu(x)$ in curved space–time with the metric $g_{\mu\nu}$. Renormalization and other connected problems for free field theories in curved space–time have been investigated by many authors and discussed for example, in books [17–20]. Here we will consider the interacting theory, containing the scalar couplings constants f, Yukawa couplings h and gauge couplings g and assume that the theory has to be multiplicatively renormalized in flat space–time.

Let the space–time under consideration be asymptotically flat. Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Then we have a theory interacting with external field $h_{\mu\nu}$ in flat space. The field $h_{\mu\nu}$ is dimensionless, its presence or absence does not influence the superficial degree of divergence. Hence, we can expect that in curved space–time the vertex functions Γ_N with the same N as in flat space-time are divergent. The question of interest is only the structure of counterterms.

Let us make one remark. In flat space the superficial degree of divergence indicates that the zero-point vertex function (effective action at zero fields) is also divergent. Usually this object is a constant and therefore can be removed. The effective action in curved space–time is $\Gamma[g_{\mu\nu}, \Phi]$ where $\Phi^A \equiv (\varphi, \psi, A_\nu)$. The corresponding zero-point vertex function $\Gamma[g_{\mu\nu}] = \Gamma[g_{\mu\nu}, \Phi]_{\ \Phi=0}$ depends on the external field $g_{\mu\nu}$; the functional $\Gamma[g_{\mu\nu}]$ is of special interest. This functional is also divergent and therefore it must be renormalized as well.

In dimensional regularization we have to formulate the theory in n-dimensional curved space–time. This is not difficult, if we consider $g_{\mu\nu}$ as an n-dimensional Riemannian metric, $\mu, \nu = 0, 1, \ldots, n-1$. It is evident that n-dimensional general coordinate invariance is valid at any stage of the calculations. Hence, the counterterms are generally coordinate invariant. (There are theories in $4k+2$ dimensions, $k = 0, 1, 2, \ldots$, where this property does not occur for special reasons. It leads to the appearance of gravitational anomalies [66, 67].) Thus, within the framework of dimensional regularization the renormalized action is generally coordinate invariant. (This is true, of course, if a general coordinate invariant gauge is chosen for Yang–Mills gauge field.)

The general coordinate invariance of renormalized action leads to the corresponding invariance of effective action. Indeed, due to the general coordinate invariance

$$\frac{\delta S}{\delta g_{\mu\nu}} R_{\mu\nu,\alpha} + \frac{\delta S}{\delta \Phi^A} R^A_\alpha = 0. \tag{3.13}$$

Here $R_{\mu\nu,\alpha}$ and R^A_α are the generators of general coordinate transformations for metric and fields Φ^A respectively, $R^A_\alpha \equiv R^A_{B\alpha}\Phi^B$ and $R^A_{B\alpha}$ does not depend on Φ.

Let us write the generating functional of the renormalized Green's function:

$$\exp(\mathrm{i}W[g_{\mu\nu}, J]) = \int \mathrm{D}\Phi \exp[\mathrm{i}(S[g_{\mu\nu}, \Phi] + \Phi^A J_A)] \tag{3.14}$$

Consider

$$\begin{aligned}\exp(\mathrm{i}W[g_{\mu\nu} + \delta g_{\mu\nu}, J]) &= \int \mathrm{D}\Phi \exp[\mathrm{i}(S[g_{\mu\nu} + \delta g_{\mu\nu}, \Phi] + \Phi^A J_A)] \\ &= \int \mathrm{D}\Phi \exp[\mathrm{i}(S[g_{\mu\nu}, \Phi] + \Phi^A J_A \\ &\quad + R^A_{B\alpha}\Phi^B J_A \xi^\alpha)].\end{aligned} \tag{3.15}$$

Here $\delta g_{\mu\nu} = R_{\mu\nu,\alpha}\xi^\alpha, \xi^\alpha$ are infinitesimal parameters of general coordinate transformations. We have changed the variables $\Phi^A \to \Phi^A + \delta\Phi^A$ and $\delta\Phi^A = R^A_{B\alpha}\Phi^B\xi^\alpha$ and have taken into account that $S[g_{\mu\nu} + \delta g_{\mu\nu}, \Phi + \delta\Phi] = S[g_{\mu\nu}, \Phi]$. We have also used the fact that the Jacobian of this transformation is $1 + R^A_{A\alpha}\xi^\alpha = 1$ because $R^A_{A\alpha}$ contains $\delta(0)$ or $\partial_\mu \delta(x)|_{x=0}$ and due to this is equal to zero in dimensional regularization. From (3.15) we obtain

$$\frac{\delta W}{\delta g_{\mu\nu}} R_{\mu\nu,\alpha} = R^A_{B\alpha}\langle\Phi^B\rangle J_A \tag{3.16}$$

where

$$\langle\Phi^B\rangle = \frac{\delta W}{\delta J_B} \equiv \bar{\Phi}^B. \tag{3.17}$$

We introduce the effective action by

$$\Gamma[g_{\mu\nu}, \bar{\Phi}] = W[g_{\mu\nu}, J] - \bar{\Phi}^A J_A. \tag{3.18}$$

Making use of identity (3.16) we obtain

$$\frac{\delta \Gamma}{\delta g_{\mu\nu}} R_{\mu\nu,\alpha} + \frac{\delta \Gamma}{\delta \bar{\Phi}^A} R^A_\alpha = 0. \tag{3.19}$$

Thus, the effective action is a general coordinate invariant functional.

Let us consider now the renormalization in curved space–time. Before beginning, we will present a short review of references in this field. The renormalization of free fields in curved space–time has been discussed by many authors, for a review, see [17–20]. An investigation of the renormalization of interacting field theory in curved space–time was started in [22, 23] in the weak gravitational field limit. The divergences of Green's functions containing insertions of the improved energy–momentum tensor [27] were studied in [24–26]. The renormalization of scalar interacting theories was the subject of [28–39, 49–53]. The renormalization of gauge theories (with fermions) has been discussed in was investigated in [40–48, 68]. The renormalization and explicit one-loop counterterms calculations in gauge theories with scalars and spinors were considered in [54–63]. The diagram technique for Green's functions in curved space–time has been formulated in [64] (see also [65]).

First of all, we will consider the real $f\varphi^4$ theory, which is very convenient for the description of the main problems of renormalization in curved space–time [49]. Then, we will discuss the theories with scalars, spinors and gauge fields.

The action of the theory under consideration is

$$S_0 = \int \mathrm{d}^n x\sqrt{-g}\left\{\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 - \tfrac{1}{2}(m_0^2 - \xi_0 R)\varphi_0^2 - \tfrac{1}{4!}f_0\varphi_0^4\right\}. \quad (3.20)$$

The term $\xi_0 R\varphi_0^2$ corresponds to a nonminimal interaction with the gravitational field, ξ_0 is the non-minimal coupling constant, subscipt zero denotes bare action, field and parameters.

To analyse the possible counterterms we will use the following considerations.

(a) Superficial degree of divergence.
(b) General coordinate invariance.
(c) Locality.
(d) Independence of logarithmic divergent diagrams from dimensional parameters in minimal substraction scheme [85].

Evaluation of the superficial degree of divergence (3.11) shows that only zero-point, two-point and four-point vertex functions diverge. Let Γ_0 be the effective action of non-renormalized (regularized) theory. We represent it in the form $\Gamma_0 = S_0 + \bar{\Gamma}_0$ where $\bar{\Gamma}_0$ includes all quantum corrections to effective action. It is not difficult to show that

$$
\begin{aligned}
\frac{\partial \bar{\Gamma}_0}{\partial m_0^2} &= -\tfrac{1}{2}\int \mathrm{d}^n x\sqrt{-g}\langle\varphi_0^2(x)\rangle_c \\
\frac{\partial \bar{\Gamma}_0}{\partial \xi_0} &= \tfrac{1}{2}\int \mathrm{d}^n x\sqrt{-g}R(x)\langle\varphi_0^2(x)\rangle_c .
\end{aligned}
\tag{3.21}
$$

Here $\langle\varphi_0^2(x)\rangle_c = -\mathrm{i}G_c(x,y)|_{y=x}$ and $G_c(x,y)$ is the connected Green's function in the presence of a source, which is expressed in terms of the mean field from relation (3.17).

Let us introduce vertex functions $\Gamma_2(y_1,y_2)$, $\Gamma_4(y_1,y_2,y_3,y_4)$ and the following notations

$$
\begin{aligned}
F_2(x\mid y_1,y_2) &= \left.\frac{\delta^2\langle\varphi^2(x)\rangle_c}{\delta\varphi(y_1)\delta\varphi(y_2)}\right|_{\varphi=0} \\
F_4(x\mid y_1,y_2,y_3,y_4) &= \left.\frac{\delta^4\langle\varphi^2(x)\rangle_c}{\delta\varphi(y_1)\delta\varphi(y_2)\delta\varphi(y_3)\delta\varphi(y_4)}\right|_{\varphi=0} .
\end{aligned}
\tag{3.22}
$$

Then from (3.21) and (3.22) we obtain relations for quantum corrections to the vertex functions

$$
\begin{aligned}
\frac{\partial \bar{\Gamma}_{02}(y_1,y_2)}{\partial m^2} &= -\tfrac{1}{2}\int \mathrm{d}^n x\sqrt{-g}F_2(x\mid y_1,y_2) \\
\frac{\partial \bar{\Gamma}_{02}(y_1,y_2)}{\partial \xi_0} &= +\tfrac{1}{2}\int d^n x\sqrt{-g}F_2(x\mid y_1,y_2)R(x)
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\frac{\partial \bar{\Gamma}_{04}(y_1,y_2,y_3,y_4)}{\partial m^2} &= -\tfrac{1}{2}\int \mathrm{d}^n x\sqrt{-g}F_4(x\mid y_1,y_2,y_3,y_4) \\
\frac{\partial \bar{\Gamma}_{04}(y_1,y_2,y_3,y_4)}{\partial \xi_0} &= +\tfrac{1}{2}\int \mathrm{d}^n x\sqrt{-g}R(x)F_4(x\mid y_1,y_2,y_3,y_4).
\end{aligned}
\tag{3.24}
$$

As an additional consideration, we shall use the induction over the number of loops.

3.2.1 One-loop approximation

Let us represent $\bar{\Gamma}_0 = \bar{\Gamma}_{0\mathrm{fin}} + \bar{\Gamma}_{0\mathrm{div}}$ where $\bar{\Gamma}_{0\mathrm{fin}}$ and $\bar{\Gamma}_{0\mathrm{div}}$ are the finite and divergent parts of $\bar{\Gamma}_0$, respectively. From (3.23) and (3.24) follow similar relations for $\bar{\Gamma}^{(1)}_{02\mathrm{div}}$ and $\bar{\Gamma}^{(1)}_{04\mathrm{div}}$.

Taking into account the superficial degree of divergence one can show that $F_{4\mathrm{div}}$ is finite. Therefore, $\bar{\Gamma}^{(1)}_{04\mathrm{div}}$ does not depend upon m_0^2 and ξ_0. Due to the locality and general covariance it has the form

$a^{(1)}(f_0)\delta(y_1,\ldots,y_4)$. Here the index (1) indicates one-loop approximation, $a_0^{(1)}(f_0)$ is a divergent function of f_0 and

$$\delta(y_1,...,y_4) = \delta(y_1,y_2)\delta(y_1,y_3)\delta(y_2,y_3)\delta(y_3,y_4)$$
$$\delta(y_1,y_2) = \frac{\delta(y_1-y_2)}{\sqrt{-g(y_1)}}.$$

In the one-loop approximation it is possible to replace bare parameters m_0, ξ_0 and f_0 by the renormalized parameters m, ξ and f which differ from the bare ones by the pole parts. Then $\bar{\Gamma}^{(1)}_{04\mathrm{div}}$ does not depend on m^2 and ξ, and $a^{(1)}$ depends on f only.

Since $F^{(1)}_{02}$ is logarithmically divergent, its divergent part does not depend on dimensional parameters $m, \xi R$. Due to the locality and general-coordinate invariance it follows that

$$F^{(1)}_{02\mathrm{div}}(x \mid y_1,y_2) = -2b_1(f)\delta(x,y_1)\delta(x,y_2)$$

where b_1 is a divergent function of f.

From (3.23) we obtain

$$\begin{aligned}\frac{\partial\bar{\Gamma}^{(1)}_{02\mathrm{div}}(y_1,y_2)}{\partial m^2} &= b^{(1)}\delta(y_1,y_2)\\ \frac{\partial\bar{\Gamma}^{(1)}_{02\mathrm{div}}(y_1,y_2)}{\partial\xi} &= -b^{(1)}R\delta(y_1,y_2).\end{aligned} \tag{3.25}$$

The solution of these equations is

$$\bar{\Gamma}^{(1)}_{02\mathrm{div}}(y_1,y_2) = [b^{(1)}(m^2-\xi R) + \tilde{b}^{(1)}R + \tilde{\tilde{b}}^{(1)}\Box]\delta(y_1,y_2). \tag{3.26}$$

Here $\Box = \nabla^\mu\nabla_\mu$ acts upon the first argument of δ-function, $\tilde{b}^{(1)}$ and $b^{(1)}$ are the divergent functions of f.

To make two- and four-point vertex functions finite at the one-loop level it is necessary to introduce one-loop counterterms which are built using $\bar{\Gamma}^{(1)}_{04\mathrm{div}}$ and $\bar{\Gamma}^{(1)}_{02\mathrm{div}}$. This is equivalent to replacing the bare field φ_0 and parameters m_0, ξ_0 and f_0 in action (3.20) by the renormalized ones in the form:

$$\begin{aligned}\varphi_0\Box\varphi_0 &= \mu^{n-4}(\varphi\Box\varphi - \tilde{\tilde{b}}^{(1)}\varphi\Box\varphi) \equiv \mu^{n-4}Z_1^{(1)}\varphi\Box\varphi\\ m_0^2\varphi_0^2 &= \mu^{n-4}(m^2\varphi^2 - b^{(1)}m^2\varphi^2) \equiv \mu^{n-4}Z_1^{(1)}Z_2^{(1)}m^2\varphi^2\\ \xi_0R\varphi_0^2 &= \mu^{n-4}(\xi\varphi^2 - b^{(1)}\xi\varphi^2 - \tilde{b}^{(1)}\varphi^2)R\\ &\equiv \mu^{n-4}Z_1^{(1)}(Z_2^{(1)}\xi + Z_3^{(1)})R\varphi^2\\ f_0\varphi_0^4 &= \mu^{n-4}(f - a^{(1)})\varphi^4 \equiv \mu^{n-4}(Z_1^{(1)})^2Z_4^{(1)}f\varphi^4.\end{aligned} \tag{3.27}$$

These relations are definitions of one-loop renormalization constants $Z_1^{(1)}$, $Z_2^{(1)}$, $Z_3^{(1)}$ and $Z_4^{(1)}$. The parameter μ is introduced in order to take into account that the bare field and parameters have dimensions corresponding to n-dimensional space–time and renormalized ones have dimensions corresponding to four-dimensional space–time. In accordance with this we must make a simple redefinition of the functions $a^{(1)}$, $b^{(1)}$, $\tilde{b}^{(1)}$ and $\tilde{\tilde{b}}^{(1)}$. All renormalization constants are dimensionless and depend on f only.

3.2.2 $(k+1)$-loop approximation

Let us suppose that the analysis which led us to (3.27) is carried out for a k-loop approximation and it is shown that four- and two-point functions become finite under the following renormalization transformations

$$
\begin{aligned}
&\varphi_0 = \mu^{(n-4)/2}(Z_1^{(k)})^{1/2}\varphi \qquad m_0^2 = Z_2^{(k)}m^2 \\
&\xi_0 = Z_2^{(k)}\xi + Z_3^{(k)} \\
&f_0 = \mu^{4-n}Z_4^{(k)}f.
\end{aligned}
\tag{3.28}
$$

The index k indicates a k-loop approximation. The action leading to the finite two- and four-point functions up to k loops is

$$
\begin{aligned}
S^{(k)} = \mu^{n-4}\int \mathrm{d}^4x\sqrt{-g}\Big\{&\tfrac{1}{2}Z_1^{(k)}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\
&-\tfrac{1}{2}Z_1^{(k)}Z_2^{(k)}(m^2-\xi R)\varphi^2 - \tfrac{1}{2}Z_1^{(k)}Z_3^{(k)}R\varphi^2 \\
&-\tfrac{1}{4!}(Z_1^{(k)})^2Z_4^{(k)}f\varphi^4\Big\}.
\end{aligned}
\tag{3.29}
$$

Let us introduce for this action the generating functional of vertex functions and write the system of equations like (3.23), (3.24) in the $(k+1)$-loop approximation

$$
\begin{aligned}
&\frac{\partial\bar{\Gamma}_{2\mathrm{div}}^{(k+1)}(y_1,y_2)}{\partial m^2} \\
&\quad = -\tfrac{1}{2}\mu^{n-4}\int \mathrm{d}^nx\sqrt{-g}\left[Z_1^{(k)}Z_2^{(k)}F_2(x\mid y_1,y_2)\right]_{\mathrm{div}}^{(k+1)} \\
&\frac{\partial\bar{\Gamma}_{2\mathrm{div}}^{(k+1)}(y_1,y_2)}{\partial\xi} \\
&\quad = \tfrac{1}{2}\mu^{n-4}\int \mathrm{d}^nx\sqrt{-g}R\left[Z_1^{(k)}Z_2^{(k)}F_2(x\mid y_1,y_2)\right]_{\mathrm{div}}^{(k+1)}
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
\frac{\partial \bar{\Gamma}^{(k+1)}_{4\mathrm{div}}(y_1,\ldots,y_4)}{\partial m^2} &= -\tfrac{1}{2}\mu^{n-4}\int \mathrm{d}^n x\sqrt{-g} \\
&\quad\times\left[Z_1^{(k)}Z_2^{(k)}F_4(x\mid y_1,\ldots,y_4)\right]^{(k+1)}_{\mathrm{div}} \\
\frac{\partial \bar{\Gamma}^{(k+1)}_{4\mathrm{div}}(y_1,\ldots,y_4)}{\partial \xi} &= \tfrac{1}{2}\mu^{n-4}\int \mathrm{d}^n x\sqrt{-g}R \\
&\quad\times\left[Z_1^{(k)}Z_2^{(k)}F_4(x\mid y_1,\ldots,y_4)\right]^{(k+1)}_{\mathrm{div}}.
\end{aligned}
\tag{3.31}
$$

In expressions $[Z_1^{(k)}Z_2^{(k)}F_{2,4}]_{\mathrm{div}}$ the term $Z_1^{(k)}Z_2^{(k)}$ is introduced in order to make finite all subdiagrams. Then $[\,Z_1^{(k)}Z_2^{(k)}F_4(x\mid y_1,\ldots,y_4)\,]^{(k+1)}$ is finite apart from the overall divergence. This means that $\bar{\Gamma}^{(k+1)}_{4\mathrm{div}}$ does not depend on m^2,ξ and due to the locality and general coordinate invariance it has the form $\mu^{n-4}a^{(k+1)}\delta(y_1,\ldots,y_4)$, where $a^{(k+1)}$ is a divergent function of f.

The expression $[Z_1^{(k)}Z_2^{(k)}F_2(x\mid y_1,y_2)]^{(k+1)}$ is logarithmically divergent and thus it does not depend on the dimensional parameters m^2 and ξR. From here and from (3.30) it follows that

$$
\bar{\Gamma}^{(k+1)}_{2\mathrm{div}}(y_1,y_2) = \mu^{n-4}[b^{(k+1)}(m^2-\xi R)+\tilde{b}^{(k+1)}R+\tilde{\tilde{b}}^{(k+1)}\Box]\delta(y_1,y_2).
\tag{3.32}
$$

We have from this that two- and four-point functions are finite in the $k+1$-loop approximation if the following renormalization transformations take place

$$
\begin{aligned}
\varphi_0\Box\varphi_0 &= \mu^{n-4}(Z_1^{(k)}-\tilde{\tilde{b}}^{(k+1)})\varphi\Box\varphi \equiv \mu^{n-4}Z_1^{(k+1)}\varphi\Box\varphi \\
m_0^2\varphi_0^2 &= \mu^{n-4}(Z_1^{(k)}Z_2^{(k)}-b^{(k+1)})m^2\varphi^2 \equiv \mu^{n-4}Z_1^{(k+1)}Z_2^{(k+1)}m^2\varphi^2 \\
\xi_0 R\varphi_0^2 &= \mu^{n-4}(Z_1^{(k)}(Z_2^{(k)}\xi+Z_3^{(k)})-b^{(1)}\xi-\tilde{b}^{(1)})R\varphi^2 \\
&\equiv \mu^{n-4}Z_1^{(k+1)}(Z_2^{(k+1)}\xi+Z_3^{(k+1)})R\varphi^2 \\
f_0\varphi_0^4 &= \mu^{n-4}((Z_1^{(k)})^2Z_4^{(k)}f-a^{(k+1)})\varphi^4 \equiv \mu^{n-4}(Z_1^{(k+1)})^2Z_4^{(k+1)}f\varphi^4.
\end{aligned}
\tag{3.33}
$$

Comparing (3.28) and (3.33) we conclude, by the induction principle, that the renormalization transformation giving the finite two- and four-point vertex functions is

$$
\begin{aligned}
\varphi_0 &= \mu^{n-4}Z_1^{1/2}\varphi & m_0^2 &= Z_2 m^2 \\
\xi_0 &= Z_2\xi+Z_3 & f_0 &= \mu^{4-n}Z_4 f.
\end{aligned}
\tag{3.34}
$$

The general form of the renormalization constants is

$$Z = 1 + \sum_{r=1}^{\infty} \frac{C_r}{(n-4)^r} \tag{3.35}$$

and C_r depends only on f. The renormalized action leading to the finite k-point ($k > 0$) vertex functions is

$$\begin{aligned} S' = \mu^{n-4} \int \mathrm{d}^4x\sqrt{-g}\Big\{ & \tfrac{1}{2} Z_1 g^{\mu\nu} \partial_\mu\varphi\partial_\nu\varphi \\ & - \tfrac{1}{2} Z_1 Z_2 (m^2 - \xi R)\varphi^2 - \tfrac{1}{2} Z_1 Z_3 R\varphi^2 - \tfrac{1}{4!} f Z_1^2 Z_4 \varphi^4 \Big\}. \end{aligned} \tag{3.36}$$

Thus, we have found the renormalization structure in the scalar field sector.

Let us proceed to the consideration of divergences of zero-point vertex function (or vacuum effective action) $\Gamma[g_{\mu\nu}] = \Gamma[g_{\mu\nu}, \varphi]|_{\varphi=0}$. We denote $\Gamma[g_{\mu\nu}] = \Gamma^{(0)}$ and $Z_1 Z_2 \langle\varphi^2(x)\rangle_c|_{\varphi=0} = F(x)$. From relations of type (3.31), taking into account the action S' (3.36) we obtain

$$\begin{aligned} \frac{\partial\Gamma^{(0)}_{\mathrm{div}}}{\partial m^2} &= -\tfrac{1}{2}\mu^{n-4} \int \mathrm{d}^n x\sqrt{-g} F_{\mathrm{div}} \\ \frac{\partial\Gamma^{(0)}_{\mathrm{div}}}{\partial \xi} &= \tfrac{1}{2}\mu^{n-4} \int \mathrm{d}^n x\sqrt{-g} R F_{\mathrm{div}} \end{aligned} \tag{3.37}$$

$$\begin{aligned} \frac{\partial F_{\mathrm{div}}}{\partial m^2} = -\tfrac{1}{2}\mu^{n-4} \int \mathrm{d}^n y\sqrt{-g}\Big\{ & (Z_1 Z_2)[2i(\langle\varphi(x)\varphi(y)\rangle_c^{(0)})^2 \\ & + \langle\varphi^2(x)\varphi^2(y)\rangle_c^{(0)}]\Big\}_{\mathrm{div}} \\ \frac{\partial F_{\mathrm{div}}}{\partial \xi} = \tfrac{1}{2}\mu^{n-4} \int \mathrm{d}^n y\sqrt{-g} R\Big\{ & (Z_1 Z_2)^2 \Big[2\mathrm{i}(\langle\varphi(x)\varphi(y)\rangle_c^{(0)})^2 \\ & + \langle\varphi^2(x)\varphi^2(y)\rangle_c^{(0)}\Big]\Big\}_{\mathrm{div}}. \end{aligned} \tag{3.38}$$

The index 0 shows that the mean field is equal to zero in the corresponding term. The expression

$$(Z_1 Z_2)^2 \left[2\mathrm{i}(\langle\varphi(x)\varphi(y)\rangle_c^{(0)})^2 + \langle\varphi^2(x)\varphi^2(y)\rangle_c^{(0)}\right]$$

is logarithmically divergent and does not depend on dimensional parameters m^2 and ξR. ($(Z_1 Z_2)^2$ makes the subdiagrams finite ones.) From dimensional considerations, locality and general coordinate invariance it follows that

$$\left\{(Z_1 Z_2)^2[2\mathrm{i}(\langle\varphi(x)\varphi(y)\rangle_c^{(0)})^2 + \langle\varphi^2(x)\varphi^2(y)\rangle_c^{(0)}]\right\}_{\mathrm{div}} = C_1 \delta(x, y)$$

where C_1 is a divergent function of f. Substituting this in (3.33) we find that

$$F_{\text{div}} = \mu^{n-4}[(m^2 - \xi R)C_1 + RC_2] \tag{3.39}$$

where C_2 is a dimensionless divergent function of f.

Making use of (3.39) in (3.37) and solving this equation we obtain

$$\Gamma^{(0)}_{\text{div}} = -\mu^{n-4} \int d^4x\sqrt{-g}\Big\{(m^2 - \xi R)^2 Z_\Lambda + (m^2 - \xi R)RZ_\kappa + Z_{a_1}R^2 + Z_{a_2}R_{\mu\nu}R^{\mu\nu} + Z_{a_3}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + Z_{a_4}\Box R\Big\}. \tag{3.40}$$

Here Z_Λ, Z_κ, $Z_{a_i}, i = 1,\ldots,4$ are dimensionless divergent functions of f. We denote $C_1 \equiv Z_\Lambda/2$, $C_2 \equiv Z_\kappa$. (Note that Z_{a_4} might also depend on ξ.)

To make the vacuum energy finite it is necessary to change S'(3.36) to $S' - \Gamma^{(0)}_{\text{div}}$. Then it is easy to see that in order to preserve multiplicative renormalizability of the theory in the vacuum energy sector (in other words, to avoid divergences by re-defining theoretical parameters) we should add the action of the external fields to the bare action (3.20)

$$S_{\text{ext}} = \int d^4x\sqrt{-g}\Big\{\Lambda_0 - \frac{1}{\kappa_0^2}R + a_{01}R^2 + a_{02}R_{\mu\nu}R^{\mu\nu} + a_{03}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + a_{04}\Box R\Big\}. \tag{3.41}$$

Here $a_{01},\ldots,a_{04},\Lambda_0$ and κ_0 are bare parameters. From (3.40) and (3.41) it follows that all the vacuum divergences can be eliminated by the redefinition of the parameters $\Lambda_0, \kappa_0, a_{01},\ldots,a_{04}$.

As a result the renormalized action is

$$\begin{aligned} S = \mu^{n-4}\int d^4x\sqrt{-g}\Big\{&\Lambda - \frac{1}{\kappa^2}R + a_1R^2 + a_2R_{\mu\nu}R^{\mu\nu} \\ &+ a_3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + a_4\Box R + (m^2 - \xi R)^2 Z_\Lambda + (m^2 - \xi R)RZ_\kappa \\ &+ Z_{a_1}R^2 + Z_{a_2}R_{\mu\nu}R^{\mu\nu} + Z_{a_3}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ &+ Z_{a_4}\Box R + \tfrac{1}{2}Z_1 g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}Z_1Z_2(m^2 - \xi R)\varphi^2 \\ &- \tfrac{1}{2}Z_1Z_3R\varphi^2 - \tfrac{1}{4!}fZ_1^2Z_4\varphi^4\Big\}. \end{aligned} \tag{3.42}$$

Renormalized action (3.42) is obtained from the bare action $S_0 + S_{\text{ext}}$ (3.20) and (3.41) by following renormalization transformations:

$$\Lambda_0 = \mu^{n-4}(\Lambda + m^4 Z_\Lambda) \qquad \kappa_0^{-2} = \mu^{n-4}(\kappa^{-2} - m^2Z_\kappa + 2m^2Z_\Lambda)$$

$$
\begin{aligned}
&a_{01} = \mu^{n-4}(a_1 + Z_{a_1} + \xi^2 Z_\Lambda - \xi Z_\kappa) \qquad a_{02} = \mu^{n-4}(a_2 + Z_{a_2}) \\
&a_{03} = \mu^{n-4}(a_3 + Z_{a_3}) \qquad a_{04} = \mu^{n-4}(a_4 + Z_{a_4}) \\
&\varphi_0 = \mu^{(n-4)/2} Z_1 \varphi \qquad m_0^2 = Z_2 m^2 \qquad \xi_0 = Z_2 \xi + Z_3 \\
&f_0 = \mu^{4-n} Z_4 f.
\end{aligned} \tag{3.43}
$$

Relations (3.43) form a complete set of renormalization transformations for the theory under consideration.

All renormalization constants depending on f and Z_{a_4} can also depend on ξ. We have also found the explicit form for the counterterms' dependence on m^2 and ξ.

Since renormalization constants do not depend on the external gravitational field metric their form does not change if we put $g_{\mu\nu} = \eta_{\mu\nu}$. Then the renormalization constants Z_1, Z_2 and Z_4 have the same form as in flat space. The renormalization constants Z_Λ, Z_κ, Z_{a_1}, Z_{a_2}, Z_{a_3}, Z_{a_4} and Z_3 are characteristic only of the theory in the curved space–time.

Using only dimensional and general coordinate invariance considerations we could expect the renormalization of ξ_0 in the form $\xi_0 = Z_2'\,\xi + Z_3$ where Z_2' and Z_3 are relevant only for curved space. The above consideration shows that $Z_2' = Z_2$ where Z_2 is the mass renormalization constant.

Let us suppose that non-minimal interaction of the theory with gravity is missed from the very beginning. The general analysis shows that counterterms $R\varphi^2$ may appear. Then the principle of renormalizability demands the addition of $\xi R\varphi^2$ to the original action term. The same is valid for vacuum action (3.41). Its addition to (3.20) is caused by the principle of renormalizability. We note also that the above analysis is based on reference [49] (see also, review paper [14]).

Let us investigate now a general situation. Consider the theory, containing the fields $\Phi^A \equiv (\varphi, \psi, A_\mu)$ in an external gravitational field. Assume that this theory is multiplicatively renormalized in flat space–time.

The structure of renormalization of the theory in curved space–time can be found in the same way as was done for the scalar theory. In order to avoid the repetition of the above details we will formulate the final results and discuss the features of the renormalization of the theory in curved space–time and its differences from the flat case. We assume that scalar and spinor masses are included in the Lagrangian in the form $\frac{1}{2}m_{ij}^2\varphi^i\varphi^j + M_{rs}\bar{\psi}^r\psi^s$ and non-minimal interaction with gravity has the form $\frac{1}{2}\xi_{ij}R\varphi^i\varphi^j$.

(*a*) The investigation of the possible structure of counterterms in the sector of matter fields Φ^A leads to the following conclusions. The renormalization of fields Φ^A and parameters having flat analogues (in other words, all parameters except ξ_{ij}) is carried out using the same renormalization constants as in flat space. In particular, for the scalar mass matrix we have $m^2_{0ij} = Z^{kl}_{2ij} m^2_{kl} + \ldots$ where we do not write explicitly the fermion mass contribution. As to non-minimal coupling constant ξ_{ij} dimensional and general coordinate invariance considerations lead to the relation $\xi_{0ij} = \tilde{Z}^{kl}_{2ij}\xi_{kl} + Z_{3ij}$ where $\tilde{Z}_2$ and Z_3 are typical only for curved space–time. However, writing relations like (3.30) and (3.31) as above it is easy to show that

$$\xi_{0ij} = Z^{kl}_{2ij}\xi_{kl} + Z_{3ij}. \tag{3.44}$$

We see that in the matter fields sector, the constants Z_{3ij} are required only for the theory in curved space–time.
(*b*) From dimensional, general coordinate invariance and locality considerations it is easy to find the general expression for vacuum divergences

$$\begin{aligned}\Gamma^{(0)}_{\rm div} = -\mu^{n-4} \int d^4x\sqrt{-g}\Big\{&C_1 m^4 + C_2 M^4 + C_3 m^2 M^2 \\ &+ C_4 m^2 R + C_5 M^2 R + C_6 R^2 + C_7 R_{\mu\nu}R^{\mu\nu} \\ &+ C_8 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + C_9 \Box R\Big\}.\end{aligned} \tag{3.45}$$

Here $C_1, \ldots, C_9$ are divergent functions of Yukawa h, gauge g and scalar f coupling constants (and possibly of ξ). We will find some relations between $C_1, \ldots, C_9$ and its dependence on ξ. (The possible matrix indices of $C_1, \ldots, C_9$ are omitted.)

Let us write the relations analogous to (3.37) and (3.38) taking into account additional parameters — fermion masses. Then we obtain

$$\begin{aligned}\frac{\partial\Gamma^{(0)}_{\rm div}}{\partial m^2} &= -\mu^{n-4}\int d^nx\sqrt{-g}(d_1(m^2 - \xi R) + d_2 R + d_3 M^2) \\ \frac{\partial\Gamma^{(0)}_{\rm div}}{\partial \xi} &= \tfrac{1}{2}\mu^{n-4}\int d^nx\sqrt{-g}(d_1(m^2 - \xi R) + d_2 R + d_3 M^2)R\end{aligned} \tag{3.46}$$

where d_1, d_2 and d_3 are divergent functions of f, h, g. Substituting (3.45) into (3.46) we obtain

$$\begin{aligned}&C_1 = d_1/2 \qquad C_4 = -d_1\xi + d_2 \qquad C_3 = d_3 \\ &C_6 = -d_3\xi + Z^{(2)}_\kappa \qquad C_5 = \tfrac{1}{2}d_1\xi^2 - d_2\xi + Z_{a_1}.\end{aligned}$$

Moreover we find that C_3, C_7 and C_8 do not depend on ξ. Here Z_{a_1} and $Z_\kappa^{(0)}$ are divergent functions of f, g, h which are independent of ξ. Now we will denote for uniformity

$$\tfrac{1}{2}d_1 = Z_\Lambda^{(1)} \qquad d_2 = Z_\kappa^{(1)} \qquad d_3 = Z_\Lambda^{(2)}$$
$$C_2 = Z_\Lambda^{(3)} \qquad C_7 = Z_{a_2} \qquad C_8 = Z_{a_3} \qquad C_9 = Z_{a_4}.$$

Then

$$\begin{aligned}\Gamma_{\text{div}}^{(0)} = -\mu^{n-4}\int d^n x\sqrt{-g}\Big\{&[Z_\Lambda^{(1)}(m^2-\xi R)+Z_\Lambda^{(2)}M^2] \\ &\times(m^2-\xi R)+Z_\Lambda^{(3)}M^4+[Z^{(1)}(m^2-\xi R)+Z_\Lambda^{(2)}M^2]R \\ &+Z_{a_1}R^2+Z_{a_2}R_{\mu\nu}R^{\mu\nu}+Z_{a_3}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+Z_{a_4}\Box R\Big\}. \qquad (3.47)\end{aligned}$$

This completely defines the dependence of vacuum counterterms on ξ (excepting Z_{a_4}). In obtaining (3.47) we did not write indices for m^2, M and ξ in order to avoid over complication of formulae. We can restore these indices in expression (3.47) if we need to.

The structure of (3.47) shows that for multiplicative renormalizability of the vacuum energy it is necessary to add the external fields action S_{ext} (analogous to (3.41)) to S_0. Then the renormalized action is formed from the bare action $S_{\text{ext}} + S_0$ with the help of renormalization transformations to the theory in flat space and the following additional transformations:

$$\begin{aligned}\xi_{0ij} &= Z_{2ij}^{kl}\xi_{kl} + Z_{3kl} \\ \Lambda_0 &= \mu^{n-4}(\Lambda + m^4 Z_\Lambda^{(1)} + m^2M^2Z_\Lambda^{(2)} + M^4 Z_\Lambda^{(3)}) \\ \kappa_0^{-2} &= \mu^{n-4}(\kappa^{-2} - m^2Z_\kappa^{(1)} - M^2Z_\kappa^{(2)} + 2\xi m^2 Z_\Lambda^{(1)} - \xi M^2 Z_\Lambda^{(2)}) \\ a_{01} &= \mu^{n-4}(a_1 + \xi^2 Z_\Lambda^{(1)} - \xi Z_\kappa^{(1)}) \\ a_{02} &= \mu^{n-4}(a_2 + Z_{a_2}) \\ a_{03} &= \mu^{n-4}(a_3 + Z_{a_3}) \\ a_{04} &= \mu^{n-4}(a_4 + Z_{a_4}).\end{aligned} \qquad (3.48)$$

Here Λ, κ^2, a_1, a_2, a_3 and a_4 are the renormalized parameters of the external field's action. Thus, renormalization in curved space–time is defined by the additional (with respect to the flat space) dimensionless renormalization constants Z_3, $Z_\Lambda^{(1)}$, $Z_\Lambda^{(2)}$, $Z_\Lambda^{(3)}$, $Z_\kappa^{(1)}$, $Z_\kappa^{(2)}$, Z_{a_1}, Z_{a_2} and Z_{a_3} depending only on h, f, g and Z_{a_4} depending on h, f, g and possibly on ξ.

3.3 One-loop renormalization

From the results of section 3.2 it follows that the theory, multiplicatively renormalizable in flat space, can be reformulated to be multiplicatively renormalizable in curved space. However, this means that new renormalization constants appear in the theory and there are only new renormalization constants Z_3 in the matter sector.

In the one-loop approximation we can find a connection between the constants Z_2 and Z_3. This means that new (in comparison with flat space) calculations are not necessary in the matter sector in the one-loop approximation. This connection between Z_2 and Z_3 for the theory containing scalars, spinors and vectors was first reported in [54]. In [56] this connection has been proved for a theory of a general type. Now we will present the results of [56] (see also, review paper [14]). We note that renormalization of ξ_{ij} does not depend on masses which is why it is enough to consider massless theory.

Let us consider the massless theory action S_0 depending on $\Phi^A \equiv (\varphi_0^i, \psi_0^k$ and $A_{0\mu}^a)$. In four dimensions the theory has one very important property.

Let $\xi_{0ij} = \frac{1}{6}\delta_{ij}$. Then the massless theory is conformally invariant. The corresponding transformations of the fields are

$$g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad \varphi'_0 = \mathrm{e}^{-\sigma}\varphi_0$$
$$\psi'_0 = \mathrm{e}^{-3\sigma/2}\psi_0 \qquad A'_{0\mu} = A_\mu .$$

The infinitesimal conformal transformations can be formulated in n-dimensional space–time

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu} \qquad \delta\varphi_0 = -\frac{n-2}{2}\sigma\varphi_0$$
$$\delta\psi_0 = -\frac{n-1}{2}\sigma\psi_0 \qquad \delta A_{0\mu} = 0 \tag{3.49}$$

or, in a compact notation

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu} \qquad \delta\Phi_0^A = \left(k^{(A)} + (n-4)r^{(A)}\right)\sigma\Phi_0^A . \tag{3.50}$$

Here $k^{(A)} = -1, -3/2, 0, r^{(A)} = -1/2, -1/2, 0$ for scalars, spinors and vectors, respectively. There is no summation over A in (3.50).

The action is generally not conformally invariant in n-dimensions; transformations (3.50) lead to

$$2g_{\mu\nu}(x)\frac{\delta S_0}{\delta g_{\mu\nu}(x)} + \Phi_0^A(x)\left(k^{(A)} + (n-4)r^{(A)}\right)\frac{\delta S_0}{\delta\Phi_0^A(x)}$$
$$\equiv K_n S = \left(\xi_{0ij} - \tfrac{1}{6}\delta_{ij}\right)E^{ij}(\Phi_0) + (n-4)F(\Phi_0). \tag{3.51}$$

there is summation over A where, E^{ij} and F are functionals for which the explicit form does not matter. For $\xi_{0ij} = \frac{1}{6}\delta_{ij}$ and $n = 4$ the action is conformally invariant. K_n is defined by (3.51) and is called the generator of conformal transformations.

Now let us write the action corresponding to the quantum field theory

$$S_{0\,\text{total}} = S_0 + \int \mathrm{d}^n x\sqrt{-g}\Big\{\frac{1}{2\omega}(\nabla_\mu A_0^{a\mu})^2 + \bar{C}_a\nabla_\mu(\delta^{ab}\nabla^\mu + gf^{acb}A_0^{c\mu})C^b\Big\}.$$

Here the general covariant gauge $\chi^a = \nabla_\mu A_0^{a\mu}$ is used. We apply K_n to $S_{0\,\text{total}}$ and introduce the effective action and obtain

$$\begin{aligned} K_n\Gamma_0 = &-2\nabla_\mu\Big(\frac{1}{\omega}\langle A_0^{a\mu}\nabla_\nu A^{a\nu}\rangle + \langle\bar{C}_a \mathrm{D}^{\mu ab}C^b\rangle\Big) \\ &+ (\xi_{ij} - \tfrac{1}{6}\delta_{ij})\langle E^{ij}(\Phi_0)\rangle \\ &+ (n-4)[\langle F(\Phi_0)\rangle + \frac{1}{2\omega}\langle(\nabla_\mu A_0^{a\mu})(\nabla_\nu A_0^{a\nu})\rangle \\ &- \nabla_\mu(\frac{1}{\omega}\langle A_0^{a\mu}\nabla_\nu A_0^{a\nu}\rangle + \langle\bar{C}_a\mathrm{D}^{\mu ab}C^b\rangle)]. \end{aligned} \tag{3.52}$$

Here

$$\begin{aligned} \langle\ldots\rangle &= \mathrm{e}^{-\mathrm{i}W}\int \mathrm{D}\Phi_0\,\mathrm{D}\bar{C}\,\mathrm{D}C\exp\left[\mathrm{i}(S_{0\,\text{total}} + J_A\Phi_0^A)\right](\ldots) \\ \mathrm{D}^{\mu ab} &= \delta^{ab}\nabla_\mu - gf^{acb}A_{0\mu}^c \end{aligned}$$

and W is the generating functional of connected Green's functions. The source J_A is expressed by using (3.17) in terms of mean field. In obtaining (3.52) we have used the fact that $\langle\bar{C}_a\nabla_\mu \mathrm{D}^{\mu ab}C^b\rangle = 0$ which could be easily shown in terms of dimensional regularization [69]. Indeed,

$$\begin{aligned} &\int \mathrm{D}\Phi_0\,\mathrm{D}\bar{C}\,\mathrm{D}C\,\exp\Big[\mathrm{i}\Big(S_0 + \int \mathrm{d}^4x\sqrt{-g}(\frac{1}{2\omega}(\nabla_\mu A_0^{a\mu})^2 \\ &\qquad + \bar{C}_a\nabla_\mu\mathrm{D}^{\mu ab}C^b) + \Phi_0^A J_A\Big)\Big](\bar{C}_a\nabla_\mu\mathrm{D}^{\mu ab}C^b) \\ &= -\mathrm{i}\int \mathrm{D}\Phi\mathrm{D}\bar{C}\mathrm{D}C\bar{C}_a\frac{\delta}{\delta\bar{C}_a}\exp\left[\mathrm{i}(S_{0\,\text{total}} + \Phi_0^A J_A)\right] \\ &= -\mathrm{i}\int \mathrm{D}\Phi\mathrm{D}\bar{C}\mathrm{D}C\frac{\delta\bar{C}_a}{\delta\bar{C}_a}\exp\left[\mathrm{i}(S_{0\,\text{total}} + \Phi_0^A J_A)\right]. \end{aligned}$$

However, $\delta\bar{C}_a/\delta\bar{C}_a \propto \delta(0)$ which is zero in dimensional regularization.

We now introduce a generating functional $Z[J,\eta,\bar{\eta}]$ in the following form:

$$Z[J,\eta,\bar{\eta}] = \int \mathrm{D}\Phi_0\,\mathrm{D}\bar{C}\,\mathrm{D}C\ \exp\Big[\mathrm{i}\{S_{0\,\mathrm{total}} + \int \mathrm{d}^n x(\Phi_0^A J_A + \bar{\eta}^a\bar{C}_a + \eta_a C^a)\}\Big]. \tag{3.53}$$

Let us perform the replacement of the variables in this integral which correspond to BRST-transformations [70, 71]

$$\begin{aligned} \Phi'^A &= \Phi_0^A + R_b^A C^b\mu \\ \bar{C}'_a &= \bar{C}_a + \frac{1}{\omega}\nabla_\nu A_0^{a\nu}\mu \\ C'^a &= C^a + \tfrac{1}{2}f^{abc}C^bC^c\mu. \end{aligned} \tag{3.54}$$

We have shown in section 2.7 that the Berezenian of these transformations is equal to unity. Using BRST-invariance of action S_{total} we obtain

$$J^{a\mu}\langle \mathrm{D}_\mu^{ab}C^b\rangle + I_i\mathrm{i}(T^a)^i_j\langle\varphi_0^j C^a\rangle + \tfrac{1}{2}\eta^a f^{abc}\langle C^bC^c\rangle + \frac{1}{\omega}\bar{\eta}^a\langle\nabla_\mu A_0^{a\mu}\rangle = 0. \tag{3.55}$$

Here $J^{a\mu}$, I_i are sources for $A^a_{0\mu}$ and φ^i_0, respectivelly, and $(T^a)^i_j$ is a representation of the generators of gauge transformations of the scalar fields. Differentiating (3.55) with respect to $J^{a\mu}$ and $\bar{\eta}^a$ and putting $\eta = \bar{\eta} = 0$ we obtain

$$\langle\bar{C}_a\mathrm{D}^{\mu ab}C^b\rangle + \frac{1}{\omega}\langle A_0^{a\mu}\nabla_\nu A_0^{a\nu}\rangle = -\frac{\delta\Gamma_0}{\delta\varphi_0^i}(T^b)^i_j\langle A^a_{0\mu}\bar{C}_a\varphi^j C^b\rangle. \tag{3.56}$$

Here

$$\frac{\delta\Gamma_0}{\delta\varphi_0^i}\langle A^a_{0\mu}\bar{C}_a\varphi^j C^b\rangle \equiv \int \mathrm{d}^n y\frac{\delta\Gamma_0}{\delta\varphi_0^i(y)}\langle A^a_{0\mu}(x)\bar{C}_a(x)\varphi^j(y)C^b(y)\rangle$$

and the sources J and I are expressed in terms of the mean fields $A^a_{0\mu}$ and φ^i_0, so that $I_i = -\delta\Gamma/\delta\varphi^i_0$. Substituting (3.56) into (3.52) we obtain

$$\begin{aligned} K_n\Gamma_0 = {}& 2\nabla_\mu\left(\frac{\delta\Gamma_0}{\delta\varphi_0^i}(T^b)^i_j\langle A_0^{a\mu}\bar{C}_a\varphi^j C^b\rangle\right) + (\xi_{ij} - \tfrac{1}{6}\delta_{ij})\langle E^{ij}(\Phi_0)\rangle \\ & + (n-4)[\langle F(\Phi_0)\rangle + \frac{1}{2\omega}\langle(\nabla_\mu A_0^{a\mu})(\nabla_\nu A_0^{a\nu})\rangle \\ & + \nabla_\mu\frac{\delta\Gamma_0}{\delta\varphi_0^i}(T^b)^i_j\langle A_0^{a\mu}\bar{C}_a\varphi^j C^b\rangle]. \end{aligned} \tag{3.57}$$

This is a consequence of conformal invariance of action at $\xi_{0ij} = \frac{1}{6}\delta_{ij}$ and $n = 4$ and it may be called the conformal Ward identity.

We will apply (3.57) for investigation of one-loop divergences of effective action. Let $\xi_{0ij} = \frac{1}{6}\delta_{ij}$ and $n = 4$ in (3.57). Then

$$K_4\bar{\Gamma}^{(1)}_{0\text{div}} = 2\nabla_\mu\left(\frac{\delta\Gamma_0}{\delta\varphi^i_0}(T^b)^i_j\langle A^{a\mu}_0\bar{C}_a\varphi^j C^b\rangle\right)^{(1)}_{\text{div}}.$$

But the expression $\langle A^a_{0\mu}\bar{C}_a\varphi^j C^b\rangle$ is already one-loop. This means that we can replace $\delta\Gamma_0/\delta\varphi^i_0$ with $\delta S_0/\delta\varphi^i_0$. We obtain

$$K_4\bar{\Gamma}^{(1)}_{0\text{div}} = 2\nabla_\mu\left(\frac{\delta S_0}{\delta\varphi^i_0}(T^b)^i_j\langle A^{a\mu}_0\bar{C}_a\varphi^j C^b\rangle\right)^{(1)}_{\text{div}}. \qquad (3.58)$$

Only $\langle A^a_{0\mu}\bar{C}_a\varphi^j C^b\rangle$ contains divergences. It is easy to show that only two diagrams give any contribution to $\langle A^a_{0\mu}\bar{C}_a\varphi^j C^b\rangle$ at the one-loop level. One of these diagrams is finite and the other is proportional to $f^{abb} = 0$. Hence $K_4\bar{\Gamma}^{(1)}_{0\text{div}} = 0$. One-loop divergences in the scalar field sector are conformally invariant.

Let us take into account the locality, the general coordinate invariance and the above result. Then we shall have for $\Gamma^{(1)}_{0\text{div}}$ the following relation:

$$\bar{\Gamma}^{(1)}_{0\text{div}} = \Delta Z\int d^n x\sqrt{-g}\Big(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 + \tfrac{1}{12}R\varphi_0^2 + \ldots\Big) \qquad (3.59)$$

where ... stands for terms which are not essential for our purposes and ΔZ is a divergent function, having the form $C/(n-4)$ where C depends on the gauge, Yukawa and scalar coupling constants. Thus, in the scalar field sector the one-loop renormalized action has the form

$$S^{(1)} = \mu^{n-4}\int d^n x\sqrt{-g}Z^{(1)}_1\Big(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi^i\,\partial_\nu\varphi^i + \tfrac{1}{12}R\varphi^i\varphi^i + \ldots\Big). \quad (3.60)$$

This expression shows that the value $\xi_{ij} = \frac{1}{6}\delta_{ij}$ is a fixed point of renormalization transformations in the one-loop approximation.

On the other hand, according to results of section 3.2

$$\xi_{0ij} = Z^{(1)kl}_{2\;ij}\xi_{kl} + Z^{(1)}_{3ij}. \qquad (3.61)$$

Since $\xi_{ij} = \frac{1}{6}\delta_{ij}$ is a fixed point we obtain

$$\tfrac{1}{6}\delta_{ij} = Z^{(1)kl}_{2\;ij}\delta_{kl} + Z^{(1)}_{3ij}.$$

Then

$$Z^{(1)}_{3ij} = -\tfrac{1}{6}(Z^{(1)kl}_{2\,ij}\delta_{kl} - \delta_{ij}) \qquad (3.62)$$

or $Z^{(1)}_3 = -\frac{1}{6}(Z^{(1)}_2 - 1)$. The constant Z_2 connected with the renormalization of scalar field mass and is known from flat space–time calculations. Thus, in the one-loop approximation the renormalization of all matter parameters including the non-minimal coupling constant, is completely defined by the renormalization constants of the corresponding theory in flat space.

3.4 Renormalization group equations

Renormalization group equations (RGEs) are the direct consequences of the multiplicative renormalizability of quantum field theory. There are numerous applications of RGEs in high-energy physics mainly because it is possible to use them to study the asymptotic behaviour of scattering amplitudes [1, 5, 21, 72].

RGEs have been applied in quantum field theory in curved space–time by many authors for the investigation of the renormalization structure of composite operators and the calculation of conformal anomalies [33, 34, 48, 73–75]. The correct form of RGEs for vertex functions in the $f\varphi^4$-theory has been formulated in the first paper of [52]. It has been shown in [53] that a new RGE, which has no analogue in flat space, appears in curved space–time (vacuum energy RGE). The RGEs and asymptotic form of vertex functions in asymptotically free theories containing scalars, spinors and vectors in curved space were first investigated in [54] (see also [56, 58]). Later, the general discussion of RGEs and calculations for concrete models in curved space–time were reported in [55–63, 76–78] (for a review, see [14, 56]).

Let us derive the renormalization group equations in curved space–time. We will follow references [14, 56]. Let $S_0[\phi_0, p_0]$, $S[\phi, p]$ be the bare and renormalized actions respectively, ϕ_0 and p_0 are bare fields and parameters, while ϕ and p are renormalized ones and $\phi = (\varphi, \psi, A_\mu)$. ϕ_0 and p_0 are connected with ϕ and p by the renormalization transformations. The generating functionals of bare and renormalized Green's functions are written in the form

$$e^{iW_0[J_0]} = \int D\phi_0 \exp\{i(S_0[\phi_0, p_0] + \phi_0 J_0)\} \qquad (3.63)$$

$$e^{iW[J]} = \int D\phi \exp\{i(S[\phi, p] + \phi J)\}. \qquad (3.64)$$

Multiplicative renormalizability means $S[\phi_0, p_0] = S[\phi, p]$. We will change the variables $\phi_0 \to Z_1^{1/2}\phi$ in (3.63). Then from (3.63) and (3.64) it follows that $W_0[J_0] = W[J]$. Here $J = \mu^{(n-4)/2} Z_1^{1/2} J_0$. Now

$$\phi_0 = \frac{\delta W_0}{\delta J_0} = \frac{\delta W}{\delta J} Z_1^{1/2} \mu^{n-4} = \mu^{(n-4)/2} Z_1^{1/2} \phi$$

where ϕ_0 and ϕ are corresponding mean fields. Hence

$$\Gamma_0[\phi_0] = W_0[J_0] - \phi_0 J_0 = W[J] - \phi \mu^{(n-4)/2} Z_1^{1/2} \mu^{(n-4)/2} Z_1^{-1/2} J = \Gamma[\phi].$$

Thus,

$$\Gamma_0[g_{\alpha\beta}, \phi_0, p_0, n] = \Gamma[g_{\alpha\beta}, \phi, p, \mu, n]. \tag{3.65}$$

where the arguments of bare and renormalized effective actions are written explicitly. Remember that bare ϕ_0 and p_0 and renormalized ϕ and p are connected by renormalization transformations (3.12).

It is evident from (3.65) that

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \Gamma[g_{\alpha\beta}, \phi, p, \mu, n] = 0.$$

From here it follows that

$$\left[\mu \frac{\partial}{\partial \mu} + \mu \frac{\mathrm{d}p}{\mathrm{d}\mu} \frac{\partial}{\partial p} + \int \mathrm{d}^n x \mu \frac{\mathrm{d}\phi(x)}{\mathrm{d}\mu} \frac{\delta}{\delta \phi(x)}\right]$$
$$\times \Gamma[g_{\alpha\beta}, \phi, p, \mu, n] = 0. \tag{3.66}$$

Here $\mu \mathrm{d}p/\mathrm{d}\mu$, $\mu d\phi/d\mu$ are calculated at fixed bare ϕ_0 and p_0. We denote

$$\mu \frac{dp}{d\mu} = \beta_p(n) \qquad \mu \frac{d\phi(x)}{d\mu} = \gamma(n)\phi(x)$$
$$\beta_p = \beta_p(4) \qquad \gamma = \gamma(4).$$

Thus,

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \beta_p(n) \frac{\partial}{\partial p} + \gamma(n) \int \mathrm{d}^n x \phi(x) \frac{\delta}{\delta \phi(x)}\right]$$
$$\times \Gamma[g_{\alpha\beta}, \phi, p, \mu, n] = 0. \tag{3.67}$$

The renormalized action for the theory under consideration is (see section 3.2)

$$S = \mu^{n-4} \int \mathrm{d}^n x \sqrt{-g} \Big\{ \Lambda + m^4 Z_\Lambda^{(1)} + m^2 M^2 Z_\Lambda^{(2)} + M^4 Z_\Lambda^{(3)}$$

$$
\begin{aligned}
&- R\Big(\frac{1}{\kappa^2} - m^2 Z_\kappa^{(1)} - M^2 Z_\kappa^{(2)} + 2\xi m^2 Z_\Lambda^{(1)} - \xi M^2 Z_\Lambda^{(2)}\Big) \\
&+ \Big(a_1 + \xi^2 Z_\Lambda^{(1)} - \xi Z_\kappa^{(1)}\Big) R^2 + (a_2 + Z_{a_2}) R_{\mu\nu} R^{\mu\nu} \\
&+ (a_3 + Z_{a_3}) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \\
&+ (a_4 + Z_{a_4}) \Box R + \tfrac{1}{2} Z_1 g^{\mu\nu} D_\mu\varphi D_\nu\varphi \\
&- \tfrac{1}{2} Z_1 Z_2 (m^2 - \xi R)\varphi^2 + \tfrac{1}{2} Z_1 Z_3 R\varphi^2 \\
&- \frac{Z_1^2 Z_4}{4!} f\varphi^4 + Z_1' \bar{\psi} \mathrm{i}\gamma^\mu \mathrm{D}_\mu \psi - Z_1' Z_2' M \bar{\psi}\psi \\
&- Z_h Z_1' Z_1^{1/2} h\bar{\psi}\varphi\psi - Z_1'' \tfrac{1}{4} G_{\mu\nu}^a G^{\mu\nu}.
\end{aligned} \tag{3.68}
$$

The meanings of Z_1' , Z_1'' , Z_h and Z_2' are evident, $D_\mu\varphi$ and $D_\mu\psi$ are general coordinate covariant and gauge covariant derivatives, internal indexes are dropped.

From (3.68) it is easy to show that

$$
S[g_{\alpha\beta}, \phi, p, \mu, n] = S[k^2 g_{\alpha\beta}, k^{-d_\phi}\phi, k^{-d_p} p, k^{-1}\mu, n] \tag{3.69}
$$

where k is constant, d_ϕ and d_p are canonical dimensions for ϕ and p at $n = 4$. It is easy to show that the same relation is valid for Γ:

$$
\Gamma[g_{\alpha\beta}, \phi, p, \mu, n] = \Gamma[k^2 g_{\alpha\beta}, k^{-d_\phi}\phi, k^{-d_p} p, k^{-1}\mu, n]. \tag{3.70}
$$

From relation (3.70) we obtain

$$
\begin{aligned}
\Big[\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial \mu} + d_p p \frac{\partial}{\partial p} + d_\phi \int \mathrm{d}^n x \phi(x) \frac{\delta}{\delta\phi(x)}\Big] \\
\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, p, \mu, n] = 0.
\end{aligned} \tag{3.71}
$$

for $t = \ln k$.

Solving for $\mu\partial\Gamma/\partial\mu$ from (3.71) and substituting the result into equation (3.67) which is written for $\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, p, \mu]$ we obtain in the limit $n \to 4$

$$
\begin{aligned}
\Big[\frac{\partial}{\partial t} - (\beta_p - p d_p)\frac{\partial}{\partial p} - (\gamma - d_\phi) \int \mathrm{d}^n x \phi(x) \frac{\delta}{\delta\phi(x)}\Big] \\
\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, p, \mu] = 0.
\end{aligned} \tag{3.72}
$$

This is the final RGE for the effective action in curved space–time.

The solution of (3.72) has the form

$$
\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, p, \mu] = \Gamma[g_{\alpha\beta}, \phi(t), p(t), \mu]. \tag{3.73}
$$

Here $\phi(t)$ and $p(t)$ satisfy the equations

$$\begin{aligned} \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} &= (\gamma(t) - d_\phi)\phi(t) \qquad \phi(0) = \phi. \\ \frac{\mathrm{d}p(t)}{\mathrm{d}t} &= \beta_p(t) - d_p p(t) \qquad p(0) = p. \end{aligned} \tag{3.74}$$

Note that $\phi(t)$ and $p(t)$ are called the effective field and the effective charges or effective coupling constants, $\phi(t)$ and $p(t)$ depend on the coupling constants p. Here $\gamma(t) \equiv \gamma(p(t))$, $\beta_p(t) \equiv \beta_p(p(t))$.

Equation (3.73) connects the effective action of two theories with the metrics $\mathrm{e}^{-2t}g_{\alpha\beta}$ and $g_{\alpha\beta}$. Now we will consider the scale transformation of metric $g_{\alpha\beta} \to \mathrm{e}^{-2t}g_{\alpha\beta}$ and t = constant. Then the invariants of curvature are transformed in the following way:

$$\begin{aligned} R &\to \mathrm{e}^{2t}R \qquad R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to \mathrm{e}^{4t}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ R_{\mu\nu}R^{\mu\nu} &\to \mathrm{e}^{4t}R_{\mu\nu}R^{\mu\nu} \qquad R^2 \to \mathrm{e}^{4t}R^2. \end{aligned}$$

Hence, the limit $t \to \infty$ corresponds to the strong scalar curvature limit. On the other hand, the transformation $g_{\alpha\beta} \to \mathrm{e}^{-2t}g_{\alpha\beta}$ shows that the limit $t \to \infty$ corresponds to a limit of small distances or large momenta. Thus, the relations (3.73) and (3.74) give the possibility of investigating the asymptotic form of the effective action in the limit of small distances or strong scalar curvature [52].

Let us consider the structure of (3.73) in detail. Let $p \equiv (\lambda, \bar{M}, \xi, \Lambda, \kappa, a_i)$ where λ are dimensionless coupling constants of quantum fields, $\bar{M}$ is the set of scalar and spinor masses, Λ, κ, a_i ($i = 1, \ldots 4$) are the coupling constants of the external fields action S_{ext}. (In general, there are gauge parameters among the coupling constants p. We shall assume that gauge conditions are chosen such that they do not demand the renormalization of gauge parameters.) Equations (3.74) are rewritten as

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = \beta_\lambda(\lambda(t)) \qquad \lambda(0) = \lambda \tag{3.75}$$

$$\begin{cases} \dfrac{\mathrm{d}\bar{M}(t)}{\mathrm{d}t} = (\gamma_{\bar{M}}(\lambda(t)) - 1)\bar{M}(t) \qquad \bar{M}(0) = \bar{M} \\ \dfrac{\mathrm{d}\xi(t)}{\mathrm{d}t} = \beta_\xi(\lambda(t), \xi(t)) \qquad \xi(0) = \xi \\ \dfrac{\mathrm{d}\phi(t)}{\mathrm{d}t} = (\gamma(\lambda(t)) - \mathrm{d}_\phi)\phi(t) \qquad \phi(0) = \phi \end{cases} \tag{3.76}$$

$$\begin{cases} \dfrac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = \beta_\Lambda(t) - 4\Lambda(t) \qquad \Lambda(0) = \Lambda \\ \dfrac{\mathrm{d}\kappa^{-2}(t)}{\mathrm{d}t} = \beta_{\kappa^{-2}}(t) - 2\kappa^{-2}(t) \qquad \kappa^2(0) = \kappa^2 \\ \dfrac{\mathrm{d}a_i(t)}{\mathrm{d}t} = \beta_{a_i}(t) \qquad a_i(0) = a_i. \end{cases} \tag{3.77}$$

In these equations the fact that β_λ depends only on λ and $\beta_{\bar{M}} = \gamma_{\bar{M}}\bar{M}$, $\gamma_{\bar{M}}$ depends only on λ was used. As it was shown in section 3.2, $\xi_{0ij} = Z^{kl}_{2ij}\xi_{kl} + Z_{3ij}$. Hence $\beta_{\xi_{ij}} = \gamma^{kl}_{2ij}\xi_{kl} + \gamma_{3kl}$ and γ^{kl}_{2ij} corresponds to the mass matrix of scalars and γ_2 and γ_3 depend only on λ. From (3.48) it follows that $\beta_\Lambda = \bar{M}^4\gamma_\Lambda$, $\beta_{\kappa^{-2}} = \bar{M}^2\gamma_{\kappa^{-2}}$ and γ_Λ, $\gamma_{\kappa^{-2}}$, β_{a_i} depend only on λ and ξ. From the above considerations we see that equations (3.75)–(3.77) can naturally be divided into three groups.

1. Equations (3.75) for $\lambda(t)$. These equations have the same form as in flat space–time.
2. Equations (3.76) for effective masses $\bar{M}(t)$, effective coupling constants $\xi(t)$ and effective fields $\phi(t)$. These equations should be solved after solving those of the first group (3.75). The equations for $\bar{M}(t)$, $\phi(t)$ have the same form as in flat space–time. A new equation appears only for $\xi(t)$. Note that in the the one-loop approximation the expression for $\beta_{\xi_{ij}}$ is simply $\beta_{\xi_{ij}} = \gamma^{kl}_{2ij}(\xi_{kl} - \frac{1}{6}\delta_{kl})$.
3. Equations (3.77) for $\Lambda(t)$, $\kappa(t)$, $a_i(t)$. These equations should be solved after solving equations (3.75) and (3.76).

RGEs for vertex functions are obtained from (3.73) by differentiating over ϕ, taking account of (3.76) for the effective field $\phi(t)$

$$\begin{aligned} \Gamma_k&[x_1, \ldots, x_k; \mathrm{e}^{-2t}g_{\alpha\beta}, \lambda, \bar{M}, \xi, \mu] \\ &= \Gamma_k[x_1, \ldots, x_k; g_{\alpha\beta}, \lambda(t), \bar{M}(t), \xi(t), \mu] \\ &\quad \times \exp\Big[-kd_\phi t + k\int_0^t \mathrm{d}t'\gamma(\lambda(t'))\Big]. \end{aligned} \tag{3.78}$$

Here $k = 1, 2, \ldots$, and the constants Λ, κ, a_i are contained only in the external fields action and these do not contribute to vertex functions.

Let $\phi = 0$ in (3.73) then we have the RGE for vacuum effective action $\Gamma^{(0)}[g_{\alpha\beta}, p, \mu]$ (see [53, 56, 14]). Let $\Gamma^{(0)} = S_{\text{ext}} + \bar{\Gamma}^{(0)}$ where $\bar{\Gamma}^{(0)}$ contains quantum corrections. From (3.73) we have

$$\Gamma^{(0)}[\mathrm{e}^{-2t}g_{\alpha\beta}, \lambda, \bar{M}, \xi, \Lambda, \kappa, a_i, \mu]$$

$$= \int d^n x \sqrt{-g} \Big\{ \Lambda(t) - \frac{1}{\kappa^2(t)} R + a_1(t) R^2 + a_2(t) R_{\mu\nu} R^{\mu\nu}$$
$$+ a_3(t) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + a_4(t) \Box R \Big\}$$
$$+ \bar{\Gamma}^{(0)}[g_{\alpha\beta}, \lambda(t), \bar{M}(t), \xi(t), \mu]. \tag{3.79}$$

From (3.79) we obtain

$$\Lambda(t) = \Lambda e^{-4t} + \int_0^t dt' e^{-4(t-t')} \bar{M}^4(t') \gamma_\Lambda(\lambda(t'), \xi(t'))$$
$$\frac{1}{\kappa^2(t)} = e^{-2t} \frac{1}{\kappa^2} + \int_0^t dt' e^{-2(t-t')} \gamma_{\kappa^{-2}}(\lambda(t'), \xi(t')) \bar{M}^2(t') \tag{3.80}$$
$$a_i(t) = a_i + \int_0^t dt' \beta_{a_i}(\lambda(t'), \xi(t')).$$

Expressions (3.79) and (3.80) define the behaviour of vacuum effective action in the limit of strong scalar curvature. These expressions have no analogues in flat space–time. RGE (3.79) and relations (3.80) have been obtained in [53] and also investigated in [54, 56, 57, 76].

RGE (3.79) allows us to obtain a general expression for conformal anomaly. Let $\tilde{\Gamma}^{(0)}[g_{\alpha\beta}, p, \mu] = \Gamma^{(0)}[g_{\alpha\beta}, p, \mu]_{(0)}$, where (0) means that $\bar{M} = 0$, $\xi = 1/6$, $\Lambda = 0$, $\kappa^{-2} = 0$ and a_1, a_2, a_3 are chosen to have

$$S_{\text{ext}} = \int d^n x \sqrt{-g} (c_1 C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + c_2 G + a_4 \Box R).$$

Here $C_{\mu\nu\alpha\beta}$ is a Weyl tensor, G is the Gauss–Bonnet invariant. Note that

$$\frac{\partial}{\partial t} \tilde{\Gamma}^{(0)} \left(e^{-2t} g_{\alpha\beta}, \lambda, \mu \right) \Big|_{t=0}$$
$$= - \int d^4 x \, g_{\alpha\beta}(x) \frac{\delta \tilde{\Gamma}^{(0)}[g_{\alpha\beta}, \lambda, \mu]}{\delta g_{\alpha\beta}(x)} = -\tfrac{1}{2} \int d^4 x \, T. \tag{3.81}$$

Here T is the mean value of the energy–momentum tensor trace. The original renormalized action is conformally invariant at $\bar{M} = 0$, $\xi = 1/6$, $\Lambda = 0$, $\kappa^{-2} = 0$ and suitable values of a_1, a_2, a_3. Hence T corresponds to an anomalous trace of the energy–momentum tensor. Expression (3.72) yields

$$\tfrac{1}{2} \int d^n x \sqrt{-g} T = - \int d^n x \sqrt{-g} \Big(\beta_{c_1} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \beta_{c_2} G + \beta_{a_4} \Box R \Big)$$
$$- \left[\beta_\lambda \frac{\partial}{\partial \lambda} + \beta_\xi \frac{\partial}{\partial \xi} \right] \tilde{\Gamma}^{(0)}[g_{\alpha\beta}, \lambda, \xi, \mu]. \tag{3.82}$$

Here $\bar{\Gamma}^{(0)}$ are quantum corrections to the vacuum effective action. $\tilde{\Gamma}[g_{\alpha\beta}, \lambda, \xi, \mu]$, $\beta_\xi = \gamma_2\xi + \gamma_3$, $\beta_\lambda, \gamma_2, \gamma_3$ depend on λ ; $\beta_{c_1} = \beta_{a_2} + 4\beta_{a_3}$, $\beta_{c_2} = -(\beta_{a_2} + 3\beta_{a_3})$. Expression (3.82) gives a general expression for the integral of the anomalous trace of the energy–momentum tensor.

3.5 Asymptotic conformal invariance

According to (3.73), the behaviour of the effective action, $\Gamma[\mathrm{e}^{-2t}g_{\alpha\beta}, \phi, p, \mu]$, for $t \to \infty$ is defined by the behaviour of corresponding effective coupling constants and $\phi(t)$. So we will now study equations (3.74)–(3.77).

First of all, let us consider the equations for $\lambda(t)$, $\bar{M}(t)$. These equations are identical to those in flat space–time. It is well-known that the limit $t \to \infty$ for all the effective coupling constants in field models including scalars, spinors and non-Abelian gauge fields exists only in definite asymptotically free theories [79–84]. Then it is enough to use the one-loop approximation for analysis of the behaviour of the effective couplings.

We will consider here the theories in which all effective coupling constants $\lambda(t) \propto g^2(t)$ [85–89]. The equation for the gauge effective coupling constant $g(t)$ in the one-loop approximation has the following well-known solution [72, 79–84]

$$g^2(t) = g^2/(1 + b^2g^2t) \tag{3.83}$$

where b^2 = constant. It is evident that $g^2(t) \to 0$ and all $\lambda(t) \to 0$ when $t \to \infty$. The theory is asymptotically free in all coupling constants.

The explicit construction of asymptotically free field models in which $\lambda(t) \propto g^2(t)$ is an art. It is necessary to provide rigid limitations in the relations between coupling constants and in the multiplet structure in these theories [81–89]. Because all effective couplings are proportional to one effective coupling $g^2(t)$, we will say that asymptotic freedom in the models under consideration is realized in a special solution of the RGEs.

Consider (3.76) for effective masses. In the one-loop approximation $\lambda \propto g^2$ therefore $\gamma_{\bar{M}} = C_M g^2$, C_M = constant. Then the solution of equation (3.76) shows that $\bar{M}(t) \to 0$ when $t \to \infty$.

In the matter field sector only the coupling constant ξ has no analogy in flat space–time. That is why the investigation of the equation for $\xi(t)$ is of special interest.

In asymptotically free theories where all of $\lambda(t) \to 0$ when $t \to \infty$ we can limit ourselves to the one-loop approximation. We obtain

$$\frac{\mathrm{d}\xi_{ij}(t)}{\mathrm{d}t} = \gamma^{kl}_{2ij}(\lambda(t))(\xi_{kl}(t) - \tfrac{1}{6}\delta_{kl}). \tag{3.84}$$

In the one-loop approximation $\gamma^{kl}_{2ij} = g^2 C^{kl}_{ij}$ where C^{kl}_{ij} is a constant matrix. The solution of (3.84) has the form

$$\xi_{ij}(t) = \tfrac{1}{6}\delta_{ij} + \sum_s P^{kl}_{sij}(\xi_{kl} - \tfrac{1}{6}\delta_{kl})\left(\frac{g^2(t)}{g^2}\right)^{-\lambda_s/b^2} \tag{3.85}$$

Here λ_s are the eigenvalues of the matrix $C^B_A (A \equiv (i,j),\ B \equiv (i',j'))$ and P^B_{sA} are the corresponding projectors. C^B_A is a finite-dimensional matrix, so the range of s is finite.

Let us consider some consequences of (3.85).

1. Let all $\lambda_s < 0$. From (3.85) it follows that $\xi_{ij}(t) \to \frac{1}{6}\delta_{ij}$ when $t \to \infty$ independently of initial values of ξ_{ij}. The value $\xi_{ij} = \delta_{ij}/6$ corresponds to the conformally invariant theory. Due to the asymptotic freedom $\lambda(t) \to 0$, $\bar{M}(t) \to 0$. That is why by using (3.73) we can write

$$\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, \lambda, \bar{M}, \xi, \ldots, \mu] = \Gamma[g_{\alpha\beta}, \phi(t), 0, 0, \tfrac{1}{6}, \ldots, \mu]. \tag{3.86}$$

Here the coupling constants contained in S_{ext} are denoted by dots. These constants are non-essential in the matter sector.

Thus, the asymptotic form of the effective action in the matter fields sector is defined by the free massless classical action where the value of the non-minimal coupling constant is dictated by radiative corrections. Independently of the choice of initial values for ξ_{ij} when $t \to \infty$ the asymptotic form of vertex functions is described by the free massless conformally-invariant classical theory. So as $\Gamma = S + \bar{\Gamma}$ and $\bar{\Gamma} \propto g^2$ due to asymptotic freedom, the leading term in the effective action is S with the interaction turned off. From here it follows that in the limit of strong scalar curvature (or high energies) all particles in the theory under consideration can be described by free massless conformally-invariant equations of motion. The phenomena where the asymptotic form of the effective action is defined by the classical free conformally-invariant theory is called asymptotic conformal invariance. This has been reported in [54] (see, also [14, 56, 58]).

2. Let all $\lambda_s > 0$. Then from (3.85) it follows that when $t \to \infty$, $|\xi_{ij}(t)| \to \infty$ independently of the initial values of ξ_{ij}. Here $\lambda(t) \to 0$, $\bar{M}(t) \to 0$ as above. Then we obtain from (3.73) that

$$\Gamma[\mathrm{e}^{-2t} g_{\alpha\beta}, \phi, \lambda, \bar{M}, \xi, \ldots, \mu] \propto \Gamma[g_{\alpha\beta}, \phi(t), 0, 0, \xi, \ldots, \mu]|_{|\xi| \to \infty}. \tag{3.87}$$

Thus the asymptotic form of the vertex functions is described by a free massless classical action where non-minimal coupling constants increase (in absolute value) when $t \to \infty$. Hence, all the particles of the theory under consideration can be described by free massless field equations. The equations for scalars must contain large (in modulus) parameters of non-minimal coupling. Note that such behaviour of $\xi(t)$ has been found independently in [54, 57] (see, also [14, 56, 58–63]).

3. Let all $\lambda_s = 0$. Then $\xi_{ij}(t) = \frac{1}{6}\delta_{ij}$. It follows that in this case the coupling constant ξ is not renormalized in the one-loop approximation.

4. Let $b^2 = 0$. It corresponds to so-called finite theories (see, for example, [90–99] and references therein). These theories have been intensively investigated during recent years. Thus, in finite theories $g^2(t) = g^2$. (We note that some finite theories are finite in all orders of perturbation theory.)

From (3.85) we obtain

$$\xi_{ij}(t) = \tfrac{1}{6}\delta_{ij} + \sum_s P_{s\,ij}^{\;kl}\left(\xi_{kl} - \tfrac{1}{6}\delta_{kl}\right) e^{\lambda_s g^2 t}. \tag{3.88}$$

If $g^2 \ll 1$ then we can limit ourselves by the one-loop approximation, and $\bar{M}(t)_{t\to\infty} \to 0$ as above. Let $\lambda_s < 0$. Then from (3.88) it follows that $t \to \infty$, $\xi_{ij}(t) \to \frac{1}{6}\delta_{ij}$. This new kind of asymptotic conformal invariance is reported in [100–102]. For $\lambda_s < 0$ or $\lambda_s = 0$ the situation is the same as above in asymptotically free theories. We will discuss the behaviour of effective coupling constants for finite theories in more detail later.

We turn now to the vacuum effective action $\Gamma^{(0)} = S_{\text{ext}} + \bar{\Gamma}^{(0)}$. Expression (3.73) yields

$$\begin{aligned}\Gamma^{(0)}&[e^{-2t}g_{\alpha\beta}, \lambda, \bar{M}, \xi, \Lambda, \kappa, a_i, \mu] \\ &= \Gamma^{(0)}[g_{\alpha\beta}, \lambda(t), \bar{M}(t), \xi(t), \Lambda(t), \kappa(t), a_i(t)].\end{aligned} \tag{3.89}$$

The case of interest to us are asymptotically conformally invariant theories. Due to the asymptotic freedom $\lambda(t) \to 0$, $\bar{M}(t) \to 0$ and because of asymptotic conformal invariance $\xi_{ij}(t) \to \frac{1}{6}\delta_{ij}$ in the high-energy limit. Hence, the contribution to $\bar{\Gamma}^{(0)}$ is defined by the massless free theory with $\xi = 1/6$

$$\bar{\Gamma}^{(0)}[g_{\alpha\beta}, \lambda(t), \bar{M}(t), \xi(t), \mu] \propto \bar{\Gamma}^{(0)}[g_{\alpha\beta}, 0, 0, \tfrac{1}{6}, \mu].$$

This does not depend on t.

It is interesting now to investigate the behaviour of effective coupling constants $\Lambda(t), \kappa(t), a_i(t)$. We know that $\beta_\Lambda = \bar{M}^4\gamma_\Lambda$, $\beta_{\kappa^{-2}} = \bar{M}^2\gamma_{\kappa^{-2}}$ and in the one-loop approximation $\gamma_\Lambda = \gamma_\Lambda^{(0)} + g^2\gamma_\Lambda^{(1)}$, $\gamma_{\kappa^{-2}} = \gamma_{\kappa^{-2}}^{(0)} + g^2\gamma_{\kappa^{-2}}^{(1)}$ where $\gamma_\Lambda^{(0)}$, $\gamma_\Lambda^{(1)}$, $\gamma_{\kappa^{-2}}^{(0)}$, $\gamma_{\kappa^{-2}}^{(1)}$ do not depend on g^2. Then from (3.77) taking acount of $g^2(t) \to 0$ we find $\Lambda(t) \propto \mathrm{e}^{-4t}$, $\kappa^{-2}(t) \propto \mathrm{e}^{-2t}$. Hence, $\Lambda(t) \to 0$, $\kappa^{-2}(t) \to 0$.

In the one-loop approximation $\beta_{a_i} = \beta_{a_i}^{(0)} + g^2\beta_{a_i}^{(1)}$, where $\beta_{a_i}^{(0)}$ and $\beta_{a_i}^{(1)}$ do not depend on g^2. We can find $\beta_{a_i}^{(0)}$ from the calculation of vacuum counterterms for free theory [2, 17–19]. Equations (3.77) show that $a_i(t) \propto t\beta_{a_i}^{(0)}$ which is independent of the initial values of a_i. Thus,

$$\begin{aligned} &S_{\text{ext}}[g_{\alpha\beta}, \xi(t), \Lambda(t), \kappa(t), a_i(t)] \\ &\quad \propto S_{\text{ext}}[g_{\alpha\beta}, \tfrac{1}{6}, 0, 0, t\beta_{a_i}^{(0)}] \\ &\quad = t\int \mathrm{d}^n x\sqrt{-g}\Big(\beta_{a_1}^{(0)}R^2 + \beta_{a_2}^{(0)}R_{\mu\nu}R^{\mu\nu} \\ &\qquad + \beta_{a_3}^{(0)}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \beta_{a_4}^{(0)}\Box R\Big)\Big|_{\xi=1/6}. \end{aligned} \tag{3.90}$$

Here only $\beta_{a_1}^{(0)}$ and possibly $\beta_{a_4}^{(0)}$ depend on ξ. Therefore, $\bar{\Gamma}^{(0)}[g_{\alpha\beta}, \lambda(t), \bar{M}(t), \xi(t), \mu]$ behaves as a constant when $t \to \infty$ and $S_{\text{ext}}[g_{\alpha\beta}, \xi(t), \Lambda(t), \kappa(t), a_i(t)]$ increases according to (3.90). Hence S_{ext} gives the main contribution to the asymptotic behaviour of vacuum effective action (3.89).

To find the asymptotic (3.90) explicitly it is necessary to find $\beta_{a_i}^{(0)}$ for $\xi = 1/6$. For this we have to calculate the one-loop vacuum divergences of free conformally-invariant theory. As a result we obtain

$$\begin{aligned} &\Gamma^{(0)}[\mathrm{e}^{-2t}g_{\alpha\beta}, \lambda, \bar{M}, \xi, \Lambda, \kappa, a_i, \mu] \\ &\qquad \propto t\int \mathrm{d}^n x\sqrt{-g}(uC_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + vG + w\Box R). \end{aligned} \tag{3.91}$$

Here $C_{\mu\nu\alpha\beta}$ is a Weyl tensor, Gauss–Bonnet invariant $G = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2$, and from [17–19]

$$\begin{aligned} u &= -\frac{1}{10(4\pi)^2}[\tfrac{1}{12}(N_1 + 2N_2) + \tfrac{1}{2}N_F + N_A] \\ v &= \frac{1}{180(4\pi)^2}[\tfrac{1}{2}(N_1 + 2N_2) + \tfrac{11}{2}N_F + 31N_A] \\ w &= -\frac{1}{15(4\pi)^2}[\tfrac{1}{12}(N_1 + 2N_2) + \tfrac{1}{2}N_F + N_A]. \end{aligned} \tag{3.92}$$

Here N_1, N_2, N_F and N_A are the numbers of real and complex scalars, Dirac spinors and vectors contained in the model under consideration. Expression (3.91) shows that the asymptotic of vacuum energy up to the total divergence is given by the square of the Weyl tensor.

3.6 Techniques for calculating counterterms

There are different methods for the calculation of divergences of effective action in an external gravitational field. We will only discuss here methods which are general, namely which are valid for theories in a general curved space–time.

3.6.1 Perturbation theory over external fields

Let us represent $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In this case we can consider the theory in curved space–time as the theory containing an infinite number of vertices of interaction with external tensor field $h_{\mu\nu}$ in flat space–time (see, for example, [22–26, 103]). The calculation of divergences is done in the same way as in flat space with standard Feynman diagrams. However, explicit general coordinate invariance is not manifest in this case. It is necessary to do some work to write counterterms in general covariant form.

3.6.2 The method of local momentum representation of propagators in curved space–time

This method has been developed in references [28, 31] and has been applied for direct calculations, for example, in [29, 30, 32, 37, 40, 41, 57]. According to this method the propagators are written as an expansion on Riemann normal coordinates. To find the divergences of effective action in renormalized theories it is enough to consider some of the first terms of this expansion. As a result, we have to work only with standard Feynman momentum integrals.

3.6.3 Schwinger–De Witt technique

(See, for example, reviews [2, 104, 105].) The Schwinger proper time representation is used for propagators in curved space–time. The integrand in a proper time integral is written in standard form following De Witt [2]. One can show that to find the divergences of effective action it is necessary to know only the first few De Witt coefficients at coincidence points. The elegance of this technique is the possibility of securing general covariance at all stages of the counterterm computations.

As has already been noted, the one-loop effective action is given by the determinant of some differential operator. Very often this operator has canonical form

$$\widehat{H}_{AB} = \hat{1\!\!1}_{AB} g^{\mu\nu} \nabla_\mu \nabla_\nu + 2\hat{h}^\mu_{AB} \nabla_\mu + \widehat{\Pi}_{AB}. \tag{3.93}$$

Here $\hat{1\!\!1}_{AB}$ is unity in the space of fields Λ_A, ∇_μ is the covariant derivative of Λ^A, $\hat{h}^\mu_{AB}$, $\widehat{\Pi}_{AB}$ are some matrices acting in the space of fields Λ^A. The corresponding one-loop contribution to effective action is

$$\Gamma^{(1)} = -\frac{i}{2} \operatorname{Tr} \ln \widehat{H}_{AB} \tag{3.94}$$

where Tr denotes the functional trace taking into account the statistics of the fields.

A general algorithm for the calculation of divergences of expression (3.94) has been given in [104, 105]. According to this,

$$\begin{aligned} \Gamma^{(1)}_{\text{div}} = \frac{1}{(4\pi)^2(n-4)} \int \mathrm{d}^n x \sqrt{-g}\, \mathrm{tr} \Big[\tfrac{1}{2} \hat{P} \cdot \hat{P} + \tfrac{1}{12} \hat{S}_{\alpha\beta} \hat{S}^{\alpha\beta} \\ + \tfrac{1}{6} \Box P + \frac{\hat{1\!\!1}}{180} \Big(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu} + \Box R \Big) \Big]. \end{aligned} \tag{3.95}$$

Here

$$\begin{aligned} \hat{P}_{AB} &= \widehat{\Pi}_{AB} + \hat{1\!\!1}_{AB} \frac{R}{6} - \nabla_\mu \hat{h}^\mu_{AB} - \hat{h}_{\mu AC} \hat{h}^\mu_{CB} \\ \hat{S}_{AB\alpha\beta} &= (\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) \hat{1\!\!1}_{AB} + \nabla_\beta \hat{h}_{\alpha AB} \\ &\quad - \nabla_\alpha \hat{h}_{\alpha AB} + \hat{h}_{\beta AC} \hat{h}_{\alpha CB} - \hat{h}_{\alpha AC} h_{\beta CB}. \end{aligned} \tag{3.96}$$

The trace over indices A and B takes into account the statistics of field Λ^A. Expressions (3.95) and (3.96) solve the problem of finding one-loop counterterms in a general form. According to (3.95) and (3.96) the explicit computation of one-loop counterterms is reduced to the commutation of covariant derivatives, the differentiation and multiplication of matrices.

In our opinion the use of expressions (3.95) and (3.96) is the simplest way of calculating one-loop counterterms for arbitrary theory in curved space–time. As an example of an application of this algorithm we will calculate the one-loop divergences for the theory with the action

$$S = \int \mathrm{d}^n x \sqrt{-g}\,\Big(-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a - \tfrac{1}{2}(m^2 - \xi R)\varphi^a\varphi^a - \tfrac{1}{4!}f(\varphi^a\varphi^a)^2 + \sum_{k=1}^{s}\bar{\psi}^a_k(\mathrm{i}\gamma^\mu(x)D^{ab}_\mu - M\delta^{ab} - \mathrm{i}h\varepsilon^{acb}\varphi^c)\psi^b_k\Big). \tag{3.97}$$

Action (3.97) describes a gauge theory with gauge group SU(2). This theory contains scalars φ^a and spinors ψ^a belonging to the adjoint representation of the gauge group and gauge fields A^a_μ; $a = 1,2,3$, $k = 1,\ldots,s$, s is the number of spinor multiplets.

Following the background field method, we divide all fields in (3.97) into two parts: background field $\Phi^A \equiv (\varphi^a, A^a_\mu, \psi^a_k)$ and quantum fields $\Lambda^A \equiv (\sigma^a, B^a_\mu, \chi^a_k)$ according to the following rule

$$\varphi^a \to \varphi^a + \sigma^a \qquad A^a_\mu \to A^a_\mu + B^a_\mu \qquad \psi^a_k \to \psi^a_k + \chi^a_k. \tag{3.98}$$

One-loop divergences of effective action are defined by the bilinear (with respect to quantum fields) part of the action S. We are interested in the counterterms, which depend essentially on the external gravitational field. Since the renormalization of the wave function of the gauge field and gauge coupling constant is the same as in flat space–time we can take $A^a_\mu = 0$. Then we can restore the dependence of counterterms on A^a_μ by using gauge invariance. For the quantum field B^a_μ we choose the general-coordinate invariant gauge $\nabla_\mu B^{a\mu}$. As a result the bilinear part of action (3.97) (with the gauge fixing terms) is

$$\begin{aligned} S^{(2)} + S_{\mathrm{GF}} = \int \mathrm{d}^n x\sqrt{-g}\Big[&-\tfrac{1}{2}\sigma^a\Box\sigma^a + \tfrac{1}{2}B^{a\mu}\Big[\delta^{ab}(\delta^\nu_\mu\Box - R^\nu_\mu) \\ &+ g^2\varepsilon^{adc}\varepsilon^{bdl}\varphi^c\varphi^l\delta^\nu_\mu\Big]B^b_\nu + g\varepsilon^{acb}B^{c\mu}(\partial_\mu\sigma^a\varphi^b + \partial_\mu\varphi^a\sigma^b) \\ &- \tfrac{1}{12}f\sigma^a(2\varphi^a\varphi^b + \delta^{ab}\varphi^c\varphi^c)\sigma^b - \tfrac{1}{2}\sigma^a(m^2 - \xi R)\sigma^a \\ &+ \mathrm{i}\sum_{k=1}^{s}\Big(\bar{\chi}^a_k\gamma^\mu(x)\nabla_\mu\chi^a_k - h\bar{\chi}^a_k\varepsilon^{acb}\varphi^c\chi^b_k - h\bar{\chi}^a_k\varepsilon^{acb}\sigma^c\psi^b_k \\ &- h\bar{\psi}^a_k\varepsilon^{acb}\sigma^c\chi^b_k + \mathrm{i}M\bar{\chi}^a_k\chi^a_k\Big)\Big]. \end{aligned} \tag{3.99}$$

Let us change the variables in the functional integral

$$\sigma^a = \mathrm{i}\tilde{\sigma}^a \qquad \chi^b_k = -\tfrac{1}{2}(\gamma^\nu(x)\nabla_\nu - \mathrm{i}M)\eta^b_k. \tag{3.100}$$

The Jacobian of this transformation only gives a contribution in the external field sector and it is not important for our purposes. As a result

$$S^{(2)} + S_{\rm GF} = \tfrac{1}{2}\int d^n x\sqrt{-g}\,(\,\tilde{\sigma}^a \quad B^{a\mu} \quad \bar{\chi}^a_k\,)\,(\hat{H})\begin{pmatrix}\tilde{\sigma}^b\\ B^b_\nu\\ \eta^b_k\end{pmatrix}. \qquad (3.101)$$

The differential operator $\hat{H}$ has form (3.93) where

$$\hat{h}^\alpha = \begin{pmatrix} 0 & \frac{1}{2}g\varepsilon^{acb}g^{\nu\alpha}\varphi^c & -\frac{1}{2}h\varepsilon^{cab}\bar{\psi}^c_k\gamma^\alpha \\ \frac{i}{2}g\varepsilon^{acb}\delta^\alpha_\mu\varphi^c & 0 & \frac{1}{2}g\varepsilon^{acb}\bar{\psi}^c_k\gamma_\mu\gamma^\alpha \\ 0 & 0 & -\frac{1}{2}h\varepsilon^{acb}\varepsilon^c\gamma^\alpha \end{pmatrix} \qquad (3.102)$$

$$\hat{\Pi} = \begin{pmatrix} \delta^{ab}(m^2-\xi R)+\frac{1}{6}f(2\varphi^a\varphi^b+\delta^{ab}\varphi^c\varphi^c) & 2ig\varepsilon^{acb}g^{\nu\alpha}\partial_\alpha\varphi^c & -h\varepsilon^{cab}\bar{\psi}^c_k M \\ -ig\varepsilon^{acb}\delta^\alpha{}_\mu\partial_\alpha\varphi^c & -\delta^{ab}R^\nu_\mu + g^2\delta^\nu_\mu(\delta^{ab}\varphi^c\varphi^c-\varphi^a\varphi^b) & ig\varepsilon^{cab}\bar{\psi}^c_k\gamma_\mu M \\ 2h\varepsilon^{abc}\psi^c_k & 2ig\varepsilon^{abc}\gamma^\nu\psi^c_k & \delta^{ab}(M^2-\frac{1}{4}R)+ih\varepsilon^{acb}\varphi^c M \end{pmatrix}.$$

Here we used the relation

$$\gamma^\nu(x)\nabla_\nu\gamma^\mu(x)\nabla_\mu = g^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{1}{4}R.$$

The one-loop effective action is

$$\Gamma^{(1)} = -\frac{i}{2}\,{\rm Tr}\ln\hat{H} + i\,{\rm Tr}\ln\hat{H}_{\rm gh} + \frac{i}{2}\,{\rm Tr}\ln(i\gamma^\mu\nabla_\mu + M)^2. \qquad (3.103)$$

Here $\hat{H}_{\rm gh}$ is a ghost operator which is not essential because $A^a_\mu = 0$. (It gives a contribution to the external field sector.) The last term in (3.103) is caused by the replacement of variables (3.100). The operators $\hat{H}_{\rm gh} = \delta^{ab}\Box$ and

$$(i\gamma^\mu\nabla_\mu + M)^2 = -g^{\mu\nu}\nabla_\mu\nabla_\nu + 2iM\gamma^\mu\nabla_\nu + M^2 + \tfrac{1}{4}R$$

have the same structure as $\hat{H}$.

We will now apply (3.95) and (3.96) to find the divergences of $-\frac{1}{2}\,{\rm Tr}\ln H$. Firstly we consider

$$\hat{h}_\beta\hat{h}_\alpha = \begin{pmatrix} 0 & \frac{i}{2}g\varepsilon^{acd}\delta^\lambda{}_\beta\varphi^c & \frac{i}{2}h\varepsilon^{cab}\bar{\psi}^c_k\gamma_\beta \\ \frac{i}{2}g\varepsilon^{acb}\varphi^c g_{\mu\beta} & 0 & -\frac{1}{2}g\varepsilon^{acb}\bar{\psi}^c_k\gamma_\mu\gamma_\beta \\ 0 & 0 & \frac{1}{2}h\varepsilon^{acb}\varphi^c\gamma_\beta \end{pmatrix}$$

$$
\times \begin{pmatrix} 0 & \frac{\mathrm{i}}{2} g\varepsilon^{deb}\varphi^e\delta^\nu_\alpha & \frac{\mathrm{i}}{2} h\varepsilon^{deb}\bar{\psi}^e_k\gamma_\alpha \\ \frac{\mathrm{i}}{2} g\varepsilon^{deb}\varphi^e g_{\alpha\lambda} & 0 & -\frac{1}{2} g\varepsilon^{deb}\bar{\psi}^e_k\gamma_\lambda\gamma_\alpha \\ 0 & 0 & -\frac{1}{2} h\varepsilon^{deb}\varphi^e\gamma_\alpha \end{pmatrix}
$$

$$
= \begin{pmatrix} -\frac{1}{4} g^2 g_{\alpha\beta}(\varphi^a\varphi^b - \delta^{ab}\varphi^c\varphi^c) & 0 & \mathrm{i}g^2(\delta^{ab}\bar{\psi}^c_k\varphi^c - \bar{\psi}^a_k\varphi^b)\gamma_\beta\gamma_a + \mathrm{i}h^2(\delta^{ab}\bar{\psi}^c_k\varphi^c - \bar{\psi}^b_k\varphi^a)\gamma_\beta\gamma_\alpha \\ 0 & -\frac{1}{4} g^2 g_{\mu\beta}\delta^\nu_\alpha(\varphi^a\varphi^b - \delta^{ab}\varphi^c\varphi^c) & \frac{1}{4} gh(\delta^{ab}\bar{\psi}^c_k\varphi^c - \bar{\psi}^a_k\varphi^b)\gamma_\alpha g_{\mu\beta} - \frac{1}{4} gh(\delta^{ab}\bar{\psi}^c_k\varphi^c - \bar{\psi}^b_k\varphi^a)\gamma_\mu\gamma_\beta\gamma\alpha \\ 0 & 0 & \frac{1}{4} h^2\gamma_\beta\gamma_\alpha(\varphi^a\varphi^b - \delta^{ab}\varphi^c\varphi^c) \end{pmatrix}. \tag{3.104}
$$

Note also that

$$
\hat{\mathbb{1}} = \begin{pmatrix} \delta^{ab} & 0 & 0 \\ 0 & \delta^{ab}\delta^\nu_\mu & 0 \\ 0 & 0 & 1\delta^{ab} \end{pmatrix}
$$

$$
(\nabla_\alpha\nabla_\beta - \nabla_\alpha\nabla_\beta)\hat{\mathbb{1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^{ab} R^\nu_{\ \mu\alpha\beta} & 0 \\ 0 & 0 & \frac{1}{4}\delta^{ab} R_{\alpha\beta\lambda\sigma}\gamma^\lambda\gamma^\sigma \end{pmatrix}. \tag{3.105}
$$

Using (3.102), (3.104) and (3.105) it is easy to find $\hat{P}$ and $\hat{S}$(3.96). The result of this is that the one-loop divergences of effective action are

$$
\begin{aligned}
\Gamma^{(1)}_{\mathrm{div}} = \frac{1}{(4\pi)^2(n-4)} \int d^n x\sqrt{-g}\Big\{ & \tfrac{1}{2}(8h^2 - 8g^2) g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^a \\
& + \tfrac{1}{2} R\varphi^a\varphi^a \left[(\tfrac{1}{6} - \xi)(\tfrac{5}{3} f - 4g^2) - \tfrac{4}{3} g^2 + \tfrac{4}{3} h^2\right] \\
& + \tfrac{1}{4!}(\tfrac{11}{3} f^2 - 8g^2 f + 72 g^4 - 96 h^4)(\varphi^a\varphi^a)^2 \\
& + \tfrac{1}{2}\varphi^a\varphi^a \left[-48 h^2 M^2 + (\tfrac{5}{3} f - 4g^2) m^2\right] \\
& + \mathrm{i}\sum_{k=1}^{\infty} \bar{\psi}^a_k \Big[2\delta^{ab}(h^2 + 2g^2)\gamma^\alpha\nabla_\alpha + 2h(h^2 - 6g^2)\varepsilon^{acb}\varphi^c \\
& - 4\mathrm{i}(h^2 - 4g^2) M\delta^{ab}\Big]\psi^b_k \\
& + (\text{vacuum divergences})\Big\}.
\end{aligned} \tag{3.106}
$$

To calculate the vacuum contributions to the one-loop divergences it is necessary to consider the free theory, i.e. to put $g = h = f = 0$ in (3.103). Such calculations are well-known in the literature [17–19] and we will not discuss them here.

From (3.106) we obtain the following one-loop renormalization transformations for φ^a, ψ^a and f, h, ξ

$$
\begin{aligned}
&\varphi_0 = \mu^{(n-4)/2} Z_1^{1/2} \varphi \qquad Z_1 = 1 + \frac{8}{(4\pi)^2(n-4)}(h^2 - g^2) \\
&\xi_0 = Z_2 \xi + Z_3 \qquad Z_2 = 1 + \frac{1}{(4\pi)^2(n-4)}(-\tfrac{5}{3} f + 12g^2 - 8h^2) \\
&Z_3 = -\frac{1}{6(4\pi)^2(n-4)}(-\tfrac{5}{3} f + 12g^2 - 8h^2) \\
&\psi_0 = \mu^{(n-4)/2} \bar{Z}_1^{1/2} \qquad \bar{Z}_1 = 1 + \frac{1}{(4\pi)^2(n-4)}(2h^2 + 24g^2) \\
&h_0 = \mu^{(4-n)/2} Z_h h \qquad Z_h = 1 + \frac{1}{(4\pi)^2(n-4)}(12g^2 + 8h^2) \\
&f_0 = \mu^{4-n} Z_f f \\
&Z_f f = f - \frac{1}{(4\pi)^2(n-4)}(\tfrac{11}{3} f^2 - 24fg^2 + 72g^4 + 16fh^2 - 96h^4) \\
&M_0 = \tilde{Z}_2 M \qquad \tilde{Z}_2 = 1 + \frac{1}{(4\pi)^2(n-4)}(12g^2 - 6h^2) \\
&m_0^2 = Z_2 m^2 + \tilde{Z}_3 M^2 \\
&\tilde{Z}_3 = \frac{1}{(4\pi)^2(n-4)} 48h^2.
\end{aligned}
\tag{3.107}
$$

We note that

$$
\xi_0 = \xi + \frac{1}{(4\pi)^2(n-4)}(-\tfrac{5}{3} f + 12h^2 - 8g^2)(\xi - \tfrac{1}{6})
$$

in accordance with the general result.

So, the above consideration illustrates the efficiency of the general algorithms (3.95) and (3.96) for the one-loop divergences calculation in an external gravitational field.

3.7 The calculation of one-loop β-functions

Let us consider briefly the calculation of one-loop β-functions. The structure of one-loop renormalization in massless theory is

$$
\begin{aligned}
f_0 &= \mu^{4-n}\Big(f + \frac{a(f,g,h)}{n-4}\Big) \\
g_0 &= \mu^{(4-n)/2}\Big(g + \frac{b(f,g,h)}{n-4}\Big) \\
h_0 &= \mu^{(4-n)/2}\Big(h + \frac{c(f,g,h)}{n-4}\Big) \\
\xi_0 &= \xi + \frac{d(f,g,h)}{n-4}(\xi - \tfrac{1}{6}) \\
a_{0i} &= \mu^{n-4}\Big(a_i + \frac{A_i(\xi)}{n-4}\Big).
\end{aligned}
\tag{3.108}
$$

We consider only the massless theory. From the discussion in section 3.5 we know that in an asymptotic free theory the effective masses tend to zero in the high-energy limit. That is why all non-trivial asymptotics correspond to the massless theory.

Let us start with β_f. According to (3.66)

$$
\beta_f = \mu \frac{\mathrm{d}f}{\mathrm{d}\mu}_{f_0}.
$$

It is obvious that $(\mu df_0/d\mu)_{f_0} = 0$. Then, from the first relation from (3.108) it follows that

$$
\begin{aligned}
0 = (4-n)\Big(f + \frac{a}{n-4}\Big) + \mu\frac{\mathrm{d}f}{\mathrm{d}\mu} \\
+ \frac{1}{n-4}\Big(\frac{\partial a}{\partial f}\mu\frac{\mathrm{d}f}{\mathrm{d}\mu} + \frac{\partial a}{\partial g}\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} + \frac{\partial a}{\partial h}\mu\frac{\mathrm{d}h}{\mathrm{d}\mu}\Big).
\end{aligned}
\tag{3.109}
$$

In the same way

$$
\begin{aligned}
&\frac{4-n}{2}\Big(g + \frac{b}{n-4}\Big) + \mu\frac{\mathrm{d}g}{\mathrm{d}\mu} \\
&\quad + \frac{1}{n-4}\Big(\frac{\partial b}{\partial f}\mu\frac{\mathrm{d}f}{\mathrm{d}\mu} + \frac{\partial b}{\partial g}\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} + \frac{\partial b}{\partial h}\mu\frac{\mathrm{d}h}{\mathrm{d}\mu}\Big) = 0. \\
&\frac{4-n}{2}\Big(h + \frac{c}{n-4}\Big) + \mu\frac{\mathrm{d}h}{\mathrm{d}\mu} \\
&\quad + \frac{1}{n-4}\Big(\frac{\partial c}{\partial f}\mu\frac{\mathrm{d}f}{\mathrm{d}\mu} + \frac{\partial c}{\partial g}\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} + \frac{\partial c}{\partial h}\mu\frac{\mathrm{d}h}{\mathrm{d}\mu}\Big) = 0.
\end{aligned}
\tag{3.110}
$$

Note that the finiteness of β_f, β_g and β_h is caused by the finiteness of f, g, h at $n \to 4$. Then from (3.109) and (3.110) it follows that

$$\beta_f = a - f\frac{\partial a}{\partial f} - \tfrac{1}{2}g\frac{\partial b}{\partial g} - \tfrac{1}{2}h\frac{\partial c}{\partial h}$$
$$\beta_g = \tfrac{1}{2}b - \tfrac{1}{2}g\frac{\partial b}{\partial g} - f\frac{\partial a}{\partial f} - \tfrac{1}{2}h\frac{\partial c}{\partial h}$$
$$\beta_h = \tfrac{1}{2}c - \tfrac{1}{2}h\frac{\partial c}{\partial h} - f\frac{\partial a}{\partial f} - \tfrac{1}{2}g\frac{\partial b}{\partial g}.$$

For ξ we have

$$\mu\frac{\mathrm{d}\xi}{\mathrm{d}\mu}\Big(1 + \frac{d}{n-4}\Big) + \frac{1}{n-4}\big(\xi - \tfrac{1}{6}\big) \times \Big(\frac{\partial d}{\partial f}\mu\frac{\mathrm{d}f}{\mathrm{d}\mu} + \frac{\partial d}{\partial g}\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} + \frac{\partial d}{\partial h}\mu\frac{\mathrm{d}h}{\mathrm{d}\mu}\Big) = 0. \tag{3.111}$$

The expession $\beta_\xi = \mu\mathrm{d}\xi/\mathrm{d}\mu$ is finite. Then from (3.109) and (3.110) we obtain

$$\beta_\xi = -(\xi - \tfrac{1}{6})\Big(f\frac{\partial d}{\partial f} + \tfrac{1}{2}g\frac{\partial d}{\partial g} + \tfrac{1}{2}h\frac{\partial d}{\partial h}\Big). \tag{3.112}$$

Finally, for β_{a_i} we obtain

$$(n-4)\Big(a_i + \frac{A_i}{n-4}\Big) + \mu\frac{da_i}{\mathrm{d}\mu} + \frac{1}{n-4}\frac{\partial A_i}{\partial \xi}\mu\frac{\mathrm{d}\xi}{\mathrm{d}\mu} = 0$$

which leads to

$$\beta_{a_i} = -A_i. \tag{3.113}$$

Relations (3.111)–(3.113) solve the problem of finding one-loop β-functions. These β-functions are connected with the coefficients at the poles $1/(n-4)$ in renormalization transformations. In the next section we will write the expressions for β-functions without any details.

3.8 Behaviour of the effective coupling constants and asymptotic conformal invariance in quantum field theory models

The general analysis given in section 3.5 shows that there are the models of field theory where $\xi(t) \to 1/6$ when $t \to \infty$ (asymptotic conformal invariance). Now we will start the construction of such models.

Let us start the investigation of the model based on gauge group $SU(2)$. The Lagrangian is given by (3.97) where $s = 1, 2$ (otherwise asymptotic freedom is lost [85]). In flat space this model has been investigated in [85]. The equation for $\xi(t)$ follows from the results of sections 3.6 and 3.7 (see (3.107))

$$(4\pi)^2\frac{d\xi(t)}{dt} = \left(\xi(t) - \tfrac{1}{6}\right)\left(\tfrac{5}{3}f(t) + 8h^2(t) - 12g^2(t)\right). \tag{3.114}$$

In an asymptotically free regime $f(t) = k_1 g^2(t), h^2(t) = k_2 g^2(t)$ (the constants k_1 and k_2 are found in [85]) the solution of (3.114) is

$$\xi(t) = \tfrac{1}{6} + (\xi - \tfrac{1}{6})\left(1 + \frac{B^2 g^2 t}{(4\pi)^2}\right)^{\lambda/b^2}. \tag{3.115}$$

Here b^2 = constant, $\lambda = -(12 - \frac{5}{3}k_1 - 8k_2)$, $\lambda < 0$ when $s = 1$ and $\lambda > 0$ when $s = 2$ [54]. Thus, if $s = 1$, case (1) of section 3.5 is realized (asymptotic conformal invariance). If $s = 2$, case (2) of section 3.5 is realized. Thus, the model with action (3.97) is asymptotically conformally invariant if this model contains one spinor multiplet.

Let us consider a more complicated model based also on the group SU(2). The action is

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big\{ &- \tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a \\ &+ \tfrac{1}{2}\xi_1 R\varphi^a\varphi^a + \tfrac{1}{2}g^{\mu\nu}(D_\mu\chi)^a(D_\nu\chi)^a + \tfrac{1}{2}\xi_2 R\chi^a\chi^a \\ &+ i\bar\psi^a\gamma^\mu(D_\mu\psi)^a - ih_1\bar\psi^a\varepsilon^{acb}\varphi^c\psi^b + h_2\bar\psi^a\gamma_5\varepsilon^{acb}\psi^b\chi^c \\ &- \tfrac{1}{8}f_1(\varphi^a\varphi^a)^2 - \tfrac{1}{8}f_2(\chi^a\chi^a)^2 - \tfrac{1}{4}f_3\varphi^a\varphi^a\chi^b\chi^b \\ &- \tfrac{1}{2}f_4(\varphi^a\varphi^a\chi^b\chi^b - \varphi^a\chi^a\varphi^b\chi^b)\Big\}. \end{aligned} \tag{3.116}$$

Here φ^a and χ^a are scalars. This model is interesting because in flat space when $f_1 = f_2 = f_3 = 0$, $h_1 = h_2 = g$, $f_4 = g^2$ it corresponds to $N = 2$ supersymmetric theory.

The equations for the effective coupling constants $\xi(t)$ are [63]

$$\begin{aligned} (4\pi)^2\frac{d\xi_1(t)}{dt} &= \left(5f_1(t) + 8h_1^2(t) - 12g^2(t)\right)\left(\xi_1(t) - \tfrac{1}{6}\right) \\ &\quad + \left(3f_3(t) + 4f_4(t)\right)\left(\xi_2(t) - \tfrac{1}{6}\right) \\ (4\pi)^2\frac{d\xi_2(t)}{dt} &= \left(3f_3(t) + 4f_4(t)\right)\left(\xi_1(t) - \tfrac{1}{6}\right) \\ &\quad + \left(5f_1(t) + 8h_2^2(t) - 12g^2(t)\right)\left(\xi_2(t) - \tfrac{1}{6}\right). \end{aligned} \tag{3.117}$$

Asymptotic freedom is realized when $h_1^2 = h_2^2 = g^2$, $f_i = k_i g^2$, $k_i =$ constant, $i = 1, \ldots, 4$, and

$$g^2(t) = g^2\left(1 + 8g^2 t(4\pi)^{-2}\right)^{-1}$$

[85]. We consider two cases.

(I). $k_1 = k_2 = k_3 = 0$, $k_4 = 1$. It corresponds to the supersymmetric model in flat space. The solution of (3.117) is

$$\begin{aligned} \xi_1(t) &= \tfrac{1}{2}(\xi_1 + \xi_2) - \tfrac{1}{2}(\xi_2 - \xi_1)\frac{g^2(t)}{g^2} \\ \xi_2(t) &= \tfrac{1}{2}(\xi_1 + \xi_2) + \tfrac{1}{2}(\xi_2 - \xi_1)\frac{g^2(t)}{g^2}. \end{aligned} \tag{3.118}$$

When $t \to \infty$, $\xi_1(t) \to \frac{1}{2}(\xi_1 + \xi_2)$, $\xi_2(t) \to \frac{1}{2}(\xi_1 + \xi_2)$. The theory does not dictate the asymptotic behaviour of $\xi_1(t), \xi_2(t)$ and does not prohibit asymptotic conformal invariance. It is evident that when $\xi_1 + \xi_2 = 1/3$ asymptotic conformal invariance occurs.

(II). $k_1 = k_2 = k_3 = \frac{4}{7}\sqrt{7/15}$, $k_4 = \sqrt{7/15}$ [85]. The solution of (3.117) is

$$\begin{aligned} \xi_1(t) &= \tfrac{1}{6} + \tfrac{1}{2}\left[(\xi_1 + \xi_2 - \tfrac{1}{3})\left(\frac{g^2(t)}{g^2}\right)^{-a} + (\xi_1 - \xi_2)\left(\frac{g^2(t)}{g^2}\right)^{-b}\right] \\ \xi_1(t) &= \tfrac{1}{6} + \tfrac{1}{2}\left[(\xi_1 + \xi_2 - \tfrac{1}{3})\left(\frac{g^2(t)}{g^2}\right)^{-a} + (\xi_2 - \xi_1)\left(\frac{g^2(t)}{g^2}\right)^{-b}\right] \end{aligned} \tag{3.119}$$

where $a = \frac{1}{2}\left(\sqrt{15/7} - 1\right)$, $b = -\frac{1}{2}\left(\frac{1}{3}\sqrt{15/7} + 1\right)$. Let $\xi_1 + \xi_2 \neq 1/3$, then when $t \to \infty$ $\mid \xi_{1,2}(t) \mid \to \infty$. Let $\xi_1 + \xi_2 = 1/3$, then when $t \to \infty$, $\xi_1(t) \to 1/6$, $\xi_2(t) \to 1/6$ (asymptotic conformal invariance).

In references [57, 60–62] the more complicated theories (grand unification theories) have been considered. In [57] SU(5) GUT based on the asymptotically free regime found in [88] with two effective coupling constants $\xi_i(t)$ has been discussed and in [60] the SU(5) GUT based on the asymptotically free regime found in [86] with three effective coupling constants $\xi_i(t)$ has been considered. In these models $\mid \xi_i(t) \mid \to \infty$ when $t \to \infty$. The GUT based on the exceptional group E_6 with the asymptotically free regime found in [87] with three effective coupling constants $\xi_i(t)$ has been considered in [61]. The behaviour of non-minimal coupling constants corresponds to case 2). of section 3.5 as above. The question now is: do GUTs admit asymptotic conformal invariance?

To answer this question we will consider the theory with the action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a + \tfrac{1}{2}\xi R\varphi^a\varphi^a - \tfrac{1}{8}f_1(\varphi^a\varphi^a)^2 - \tfrac{1}{8}f_2(\varphi^a D_{rab}\varphi^b)^2 + \mathrm{i}\sum_{k=1}^{m}\bar{\psi}^a_k\Big[\gamma^\mu D^{ab}_\mu - hf^{acb}\varphi^c\Big]\psi^b_k\Big\}. \tag{3.120}$$

Here the gauge group is SU(N) with scalars φ^a and spinors ψ^a in adjoint representation, $a, b, c = 1, \ldots, N^2-1$, f^{abc} are group structure constants, the properties of matrices D_{rab} are given in Chapter 9 (see also [62]). This theory is asymptotically free when [62]

$$g^2(t) = \frac{g^2}{1+b^2g^2t} \qquad b^2 = 7N - \tfrac{2}{3}m$$
$$h^2(t) = kg^2(t) \qquad f_1(t) = k_1g^2(t) \qquad f_2(t) = k_2g^2(t) \tag{3.121}$$

and $m = 2,3,4,5, N = 5,\ldots,10$. For example, if $m = 2$ then numerical analysis gives: for SU(10) $k = 0.64$, $k_1 = 0.06$, $k_2 = 0.09$; for SU(9) $k = 0.63$, $k_1 = 0.06$, $k_2 = 0.09$ and so on. The β-functions for g, h, f are given in Chapter 9.

The RGE for $\xi(t)$ is

$$(4\pi)^2\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = \Big(\xi(t) - \tfrac{1}{6}\Big)\lambda g^2(t) \tag{3.122}$$

where $\lambda = (N^2+1)k_1 + 2k_2(N^2-4)/N - 6N + 4mNk$. It was shown in [62] that $\xi(t) \to 1/6$ when $m = 2$, $N = 5,\ldots,10$. Thus, for GUTs based on gauge group SU(5), ... , SU(10) asymptotic conformal invariance occurs. If $m = 3,4,5, N = 5,\ldots,10$ then $|\xi(t)| \to \infty$ when $t \to \infty$. Thus, the asymptotically conformally invariant GUTs do exist!

We have also discussed the theories which are asymptotically free for special solutions of the RGEs. What about the asymptotic freedom for general solution of RGEs? Let us consider the theory with the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a + \tfrac{1}{2}\xi R\varphi^a\varphi^a - \tfrac{1}{8}f_1(\varphi^a\varphi^a)^2 - \tfrac{1}{8}f_2(\varphi^a D_{rab}\varphi^b)^2 + \mathrm{i}\sum_{k=1}^{m}\bar{\psi}^i_k\Big[\gamma^\mu D^{ij}_\mu - \mathrm{i}h\Big(\lambda^a/2\Big)^{ij}\varphi^a\Big]\psi^j_k\Big\}. \tag{3.123}$$

Here $(\lambda^a/2)^{ij}$ are the gauge generators in the fundamental representation of SU(N), i, j = 1, 2, ... , N, scalars φ^a are in the ajoint and spinors ψ^i_k are in the fundamental representation of gauge group SU(N). It was shown in [62] that asymptotic freedom in general solutions of the RGEs is possible when N and m are equal to

SU(N)	SU(6)	SU(7)	SU(8)	SU(9)	SU(10)
m	63	$71-73$	$80-84$	$89-94$	$97-105$.

In all these cases $\xi(t) \to 1/6$. Thus asymptotic conformal invariance occurs for general solutions of RGEs. Generally speaking, the asymptotic behaviour of the non-minimal effective coupling depends on a specific model under consideration. The asymptotic conformal invariance is not restricted by the models with gauge group SU(2). In this regard it is interesting to find more realistic GUTs to investigate the real behaviour of $\xi(t)$ in the early Universe.

Here we do not discuss the behaviour of the effective constants which correspond to the vacuum energy couplings. This behaviour is described in section 3.5 for arbitrary theory. It is interesting to note that in field theory in curved space–time with boundary (see, for example, [109, 110]) some new vacuum effective coupling constants (boundary coupling constants) are possible [111]. These terms can be important in quantum cosmology (see, for example, [112, 113]).

To conclude this section we will discuss the asymptotic superconformal invariance in external supergravity. The action of $N = 1$ supergauge theory with gauge group SU(N) is

$$S = \int \mathrm{d}^8 ZE\left\{\frac{-1}{R}W^\alpha W_\alpha - \frac{1}{\bar{R}}\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha} + \bar\phi \exp[gV]\phi + \tfrac{1}{2}\xi(\bar\phi^2 + \phi^2)\right\} \tag{3.124}$$

where V is the gauge superfield, W_α and $\bar{W}_{\dot\alpha}$ are the corresponding strengths [114], ϕ is the chiral superfield, g is the coupling constant, the adjoint representation of SU(N) is used, ξ is the constant of non-minimal coupling of chiral matter with external supergravity, R is one of the external superfields which are used for writing the tensors of curvature and torsion. If $\xi \neq 0$ superconformal invariance is absent but supersymmetry is present. As is known from [114]

$$g^2(t) = g^2\left(1 + \frac{Ng^2t}{4\pi^2}\right)^{-1}$$

Direct calculation gives

$$\xi(t) = \xi\left(1 + \frac{Ng^2t}{4\pi^2}\right)^{-2} \tag{3.125}$$

Hence, $g^2(t) \to 0$, $\xi(t) \to 0$ if $t \to \infty$. Thus, in the high energy limit the quantum corrections restore broken superconformal invariance. The asymptotically free theory also becomes asymptotically superconformally invariant.

3.9 Behaviour of the effective coupling constants in 'finite' theories in curved space–time

Recently great interest has been drawn to the study of finite theories, (see, for example, [90–99, 493]), i.e. theories where the β-functions for coupling constants, masses and fields vanish (a good example is $N = 4$ supersymmetric Yang–Mills theory which is finite to all orders of perturbation theory). Not long ago realistic finite grand unification theories based on gauge group SU(5) were constructed (see, for example, [91, 93–99]).

One of the most attractive physical features of the finite gauge theories is the possibility of overcoming the zero-charge problem. The values of the coupling constants in those theories do not depend on the renormalization group parameter. Therefore if we take small enough bare values of the couplings then a consistent use of perturbation theory is ensured [106–108]. Since finiteness restricts the values of the Yukawa and scalar coupling constants, only the additional condition $g^2 \ll 0$ has to be imposed. Then the finite gauge models in flat space–time are free of the zero-charge problem and are alternatives to the asymptotically free ones.

Now we will discuss the behaviour of effective coupling constants in curved space–time for theories which are finite in flat space–time. It should be noted from the beginning that a theory finite in flat space will obviously not be finite in curved space–time because of one-loop vacuum divergences. It is known that these divergences already appear in free theory. Therefore, we call such theories 'finite' ones.

We will show that in 'finite' theories in curved space–time the following situations are possible when $t \to \infty$:

(*a*) $\xi(t) \to 1/6$ (asymptotic conformal invariance)
(*b*) $|\,\xi(t)\,| \to \infty$
(*c*) $\xi(t) = \xi$.

However, $\xi(t)$ behaves in an exponential fashion (not as in asymptotically free theories). We follow [100, 101] (see also, review [102]).

Let the action in flat space correspond to a theory finite for all coupling constants at least at the one-loop level. This means that g, h and f are connected by

$$h^2 = k_1 g^2 \qquad f = k_2 g^2 \tag{3.126}$$

where k_1 and k_2 are constants. The β-functions for g^2, h^2 and f are equal to zero. The effective coupling constants $g^2(t)$, $h^2(t)$ and $f(t)$ are

$$g^2(t) = g^2 \qquad h^2(t) = k_1 g^2 \qquad f(t) = k_2 g^2 \tag{3.127}$$

and $g^2 \ll 1$. We see that both limits $t \to \infty$ and $t \to -\infty$ (infrared limit) exist in the theory.

In curved space–time the theory is no longer finite (there are vacuum divergences and, possibly, the divergences connected with non-minimal coupling constants).

Let us now consider the behaviour of the effective coupling constants in a theory of the type described above (with single coupling constant ξ). The RGE for $\xi(t)$ has the following form in the one-loop approximation (see section 3.3)

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = \left(\xi(t) - \tfrac{1}{6}\right) C g^2 \qquad \xi(0) = \xi \tag{3.128}$$

where the sign of the constant C is defined by specific features of the theory, C can sometimes be equal to zero. The solution of (3.128) is

$$\xi(t) = \tfrac{1}{6} + \left(\xi - \tfrac{1}{6}\right) \mathrm{e}^{C g^2 t} \qquad C \neq 0 \tag{3.129}$$

and $\xi(t) = \xi$ if $C = 0$. Thus, $t \to \infty$, $\xi(t) \to 1/6$ if $C < 0$ (asymptotic conformal invariance). However, in comparison with asymptotically free theories, $\xi(t)$ tends to $1/6$ much faster (exponential growth). If $C > 0$, then for $t \to \infty$, $|\xi(t)| \to \infty$. Here again $\xi(t)$ grows exponentially. In the weak gravitational field limit ($t \to -\infty$) we have asymptotic conformal invariance in this case.

As an example we consider SU(2) gauge theory with SU(4) global invariance. The action is (see [90])

$$\begin{aligned} S = \int \mathrm{d}^4x \sqrt{-g} \Big\{ & - \tfrac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} - \mathrm{i}\bar{\psi}^s_a \bar{\sigma}^\mu D_\mu \psi^s_a + \tfrac{1}{2} g^{\mu\nu} (D_\mu \varphi)^a_\sigma (D_\nu \varphi)^a_\sigma \\ & - \frac{\mathrm{i}}{2} C_1 g \varepsilon^{acb} (\psi^a_s \psi^b_\tau h^\sigma_{s\tau} \varphi^c_\sigma - \bar{\psi}^a_s \bar{\psi}^b_\tau h^{\sigma *}_{s\tau} \varphi^c_\sigma) + \tfrac{1}{2} \xi R \varphi^a_\sigma \varphi^a_\sigma \\ & - \tfrac{1}{4} C_2 g^2 \varphi^a_\sigma \varphi^a_\sigma \varphi^b_\tau \varphi^b_\tau + \tfrac{1}{4} C_3 g^2 \varphi^a_\sigma \varphi^b_\sigma \varphi^a_\tau \varphi^b_\tau \Big\}. \end{aligned} \tag{3.130}$$

Here the indices a, b and c correspond to the gauge group SU(2), D_μ is a general covariant and gauge invariant derivative, Weyl spinors ψ^a_s[90] and scalars are taken in the adjoint representation of SU(2).

With respect to global group SU(4) the spinors and scalars are taken in fundamental and six-dimensional antisymmetric adjoint representations respectively, $h^\sigma h^{\tau+} + h^\tau h^{\sigma+} = 2\delta^{\sigma\tau}$. The direct calculation gives C in (3.129) in the form [100]

$$C = \frac{1}{(4\pi)^2}(40C_2 + 16C_1^2 - 20C_3 - 12).$$

If $C_1 = C_2 = C_3 = 1$ then the theory under consideration is finite to all orders of perturbation theory in flat space (it corresponds to $N = 4$ extended supersymmetric gauge theory). If $C_1 = 1$, $C_2 = 0.758$ and $C_3 = 0.352$ [90] then the theory is one-loop finite in flat space. In both cases $C > 0$. Therefore, when $t \to \infty$, $|\xi(t)| \to \infty$ and when $t \to -\infty$, $\xi(t) \to 1/6$. Thus, in the early Universe (at high energies) the particles which are contained in the theory under consideration can be described by free field equations (with non-minimal coupling for scalars). Asymptotic conformal invariance is realized in the weak gravitational field limit.

It should be noted that at the one-loop level vacuum effective coupling constants, for which the β-functions do not include ξ, are the same as in asymptotically free theories. Vacuum effective coupling constants (for which the β-functions contain ξ) change and behave exponentially when t grows (see [100]).

As a second example let us discuss SU(2) gauge theory with global SU(6)× SU(6) symmetry [90]. The field content of the model consists of Weyl spinors λ^a, $\chi_{i\sigma}$, ψ^i_α, gauge field A^a_μ and complex scalars C^i_α and $B_{i\sigma}$, $a = 1, 2, 3$; $\alpha, \sigma = 1, \ldots, 6$; $i = 1, 2$. With respect to SU(2)× SU(6)× SU(6), ψ^i_α, C^i_α transform like $(\bar{2}, 6, 1)$, λ^a like (3, 1, 1), $\chi_{i\sigma}$, $B_{i\sigma}$ like (2, 1, 6). The action is (see [90])

$$\begin{aligned}
S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ &- \tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + (D_\mu B_{i\sigma})^*(D^\mu B_{i\sigma}) \\
&+ (D_\mu C^i_\alpha)^* D^\mu C^i_\alpha + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a + \mathrm{i}\bar{\chi}^{i\sigma}\bar{\sigma}^\mu D_\mu\chi_{i\sigma} \\
&+ \mathrm{i}\bar{\psi}^\alpha_i\bar{\sigma}^\mu D_\mu\psi^i_\alpha - \sqrt{2}k_1 T^{aj}_i\big(B^{i\sigma*}\lambda^a\chi_{j\sigma} + B_{j\sigma}\bar{\lambda}^a\bar{\chi}^{i\sigma}\big) \\
&+ \sqrt{2}k_2 T^{aj}_i(C^{\alpha*}_j\lambda^a\psi^i_\alpha + C^i_\alpha\bar{\lambda}^a\bar{\psi}^\alpha_j) + \tfrac{1}{8}G_1 B_{i\sigma}B_{j\tau}B^{i\sigma*}B^{j\tau*} \\
&- \tfrac{1}{4}G_2 B_{i\sigma}B_{j\tau}B^{i\tau*}B^{j\sigma*} + \tfrac{1}{8}G_3 C^i_\alpha C^j_\beta C^{\alpha*}_i C^{\beta*}_j \\
&- \tfrac{1}{4}G_4 C^i_\alpha C^j_\beta C^{\beta*}_i C^{\alpha*}_j - \tfrac{1}{4}G_5 C^i_\alpha C^{\alpha*}_i B_{j\sigma}B^{j\sigma*} \\
&+ \tfrac{1}{2}G_6 C^i_\alpha C^{\alpha*}_j B_{i\sigma}B^{j\sigma*} + R\Big(\xi_B B^*_{i\sigma}B^{i\sigma} + \xi_C C^{i*}_\alpha C^\alpha_i\Big). \qquad (3.131)
\end{aligned}$$

Here T^{aj}_i are SU(2) generators in the fundamental representation, for more details of action (3.131) see [90]. This theory leads to three one-loop finite models in flat space–time. For the first one-loop finite theory [90]

$$g^2(t) = g^2 \qquad k_1(t) = k_2(t) = g \tag{3.132}$$

$$G_1(t) = \ldots = G_6(t) = g^2. \tag{3.133}$$

This is the $N = 1$ supersymmetric gauge theory in flat space.

For the second and third non-supersymmetric one-loop finite models [90]

$$\begin{aligned} G_1(t) &= G_3(t) = 0.919g^2 \\ G_2(t) &= G_4(t) = G_6(t) = 0.989g^2 \\ G_5(t) &= 1.058g^2 \end{aligned} \tag{3.134}$$

or

$$\begin{aligned} G_1(t) &= G_3(t) = -1.373g^2 \\ G_2(t) &= G_4(t) = -0.492g^2 \\ G_5(t) &= -1.094g^2 \\ G_6(t) &= -1.050g^2. \end{aligned} \tag{3.135}$$

The condition (3.132) is not changed.

The RGEs for $\xi_B(t)$, $\xi_C(t)$ are (see [101])

$$\begin{aligned} (4\pi)^2\frac{d\xi_B(t)}{dt} &= \Big(\xi_B(t) - \tfrac{1}{6}\Big) \\ &\quad\times\big[-\tfrac{9}{2}g^2(t) + 3k_1^2(t) - \tfrac{13}{2}G_1(t) + 8G_2(t)\big] \\ &\quad + 6[G_5(t) - G_6(t)]\Big(\xi_C(t) - \tfrac{1}{6}\Big) \\ (4\pi)^2\frac{d\xi_C(t)}{dt} &= 6[G_5(t) - G_6(t)]\Big(\xi_B(t) - \tfrac{1}{6}\Big) \\ &\quad + \Big(\xi_C(t) - \tfrac{1}{6}\Big)\big[\tfrac{9}{2}g^2(t) + 3k_2^2(t) \\ &\quad - \tfrac{13}{2}G_3(t) + 8G_4(t)\big]. \end{aligned} \tag{3.136}$$

For the first 'finite' model which is $N = 1$ supersymmetric in flat space we obtain from (3.132) and (3.133)

$$\xi_B(t) = \xi_B \qquad \xi_C(t) = \xi_C.$$

Thus, the model is one-loop finite in the matter sector in curved space–time for arbitrary ξ_B and ξ_C (residual supersymmetry).

For the second model ((3.132) and (3.134)) we can show that when $t \to +\infty$, $\xi_B(t) \to 1/6$, $\xi_C(t) \to 1/6$ and when $t \to -\infty$, $|\xi_B(t)| \to \infty$, $|\xi_C(t)| \to \infty$. Thus, the 'exponential' asymptotic conformal invariance is taking place.

For the third model ((3.132) and (3.135)) we have: if $t \to \infty$, $|\xi_B(t)| \to \infty$, $|\xi_C(t)| \to \infty$ and if $t \to -\infty$, asymptotic conformal invariance is taking place.

Thus, asymptotically conformally invariant 'finite' theories may be constructed.

To conclude this section we note that our considerations are just one-loop. Beyond the one-loop level some problems appear (if the theory is not finite in all orders of perturbation theory in flat space) which are not essential in asymptotically free theories. Note also that asymptotic superconformal invariance can be realized in 'finite' theories in external supergravity (see [100]).

3.10 Asymptotic finiteness and asymptotic supersymmetry

In the previous section we described the features of 'finite' theories in curved space–time. Here we want to consider some interesting and non-trivial aspects of the finite theories in flat space–time. One can regard this section as a comment on the previous one.

The finiteness of the theory (as well as asymptotic freedom) is possible only if some conditions on the field content and the parameters of the theory are satisfied. If restrictions on the coupling constants are absent then the theory is not finite. However, the theory may be multiplicative renormalizable. In this case, we can use the RG method for the investigation of the asymptotic behaviour of the effective coupling constants.

Let us study in the one-loop approximation the asymptotic behaviour of effective coupling constants in massless theories containing one gauge coupling g, some Yukawa coupling constants h and some scalar coupling constants f(all indices are omitted). When some restrictions on the field content of scalars and spinors are chosen then the theory is one-loop finite on g. In this case $g(t) = g$. The RGE for the Yukawa effective coupling constants is

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = a_1 h(t)\Big(h^2(t) - a_2 g^2\Big) \tag{3.137}$$

where a_1 and a_2 are constant matrices. As a rule, there are two fixed points of equation (3.137): $h_1 = 0$ and $h_2 \neq 0$. When $t \to \infty$, the point $h_1 = 0$ is stable [106] and in the infrared limit ($t \to -\infty$) the point $h_2 \neq 0$ is stable. If the theory under consideration can be supersymmetric (in particular this means $h^2 \propto g^2$, $f \propto g^2$) then h_2 can correspond to a value corresponding to supersymmetry.

The conditions for one-loop finiteness of scalar coupling constants (corresponding to β-functions) are

$$K_1 f^2 + K_2 f g^2 + K_3 g^4 + K_4 f h^2 + K_5 h^4 + K_6 h^2 g^2 = 0 \qquad (3.138)$$

where K_i and $i = 1, \ldots, 6$ are the constants. It is known that for realistic gauge groups this equation has real solutions for $h = h_2$.

There are two possibilities when $t \to \infty$ [107]. If $h(0)$ is arbitrary then if $t \to \infty$, $h(t) \to 0$ [106]. In this case there are no real solutions of (3.138) and $|f(t)| \to \infty$. However, we can fix the initial conditions demanding $h(0) = h_2$. Then $h(t) = h_2$ and (3.138) has real solutions.

If the field content of the theory admits the supersymmetry, then from the solutions of (3.138), one solution, f_1, corresponds to the supersymmetric theory. Let f_2 be all other solutions of (3.138). Usually, one of the non-supersymmetric points f_2 is stable when $t \to \infty$ (asymptotic finiteness). The supersymmetric fixed point f_1 is stable when $t \to -\infty$ (asymptotic supersymmetry).

We can assume that for any gauge finite model the supersymmetric fixed point (with the coupling values h_2 and f_1) is stable in the UV or in the IR limit. Of course, the IR limit looks more realistic, because the value h_2, as a rule, is stable in the $t \to -\infty$ limit. If asymptotic supersymmetry takes place, then the corresponding theory is asymptotically finite and asymptotically supersymmetric in the IR limit.

Let the theory under consideration be finite to all orders of perturbation theory when $h = h_2$ and $f = f_1$. It means that the field content of the theory leads to $\beta_h = \beta_g = \beta_f = 0$ to all orders of perturbation theory when $h = h_2$ and $f = f_1$. The β-functions have fixed points $h = h_2$ and $f = f_1$ to all orders of perturbation theory. The one-loop consideration shows that points $h = h_2$ and $f = f_1$ are probably stable to all orders of perturbation theory. Thus, we can suppose that the theory is asymptotically finite and asymptotically supersymmetric when $t \to -\infty$ also in higher loops.

Recently a gauge theory based upon a direct product $G_1 \times G_2$ of two gauge theories has been investigated [115], where the theory based on G_1 is asymptotically finite and the theory based on G_2 is asymptotically free. It has been shown that this product theory can be effectively asymptotically finite.

In conclusion,we note that asymptotic finiteness can also occur in theories in curved space–time (however, only in the matter sector [102]). In this case we should analyse RGE for ξ as above for h and f. It is easy to perform this task [102]. For example, an asymptotically finite theory can be asymptotically conformally invariant.

4 The Renormalization Group Method in Curved Space–time with Torsion

4.1 Introduction

One of the possible developments in general relativity is connected with the torsion field which is used for the description of gravity with a metric. It is well-known that in general relativity matter interacts with the gravitational field only through the energy–momentum tensor. Spin tensors which are independent of energy–momentum tensors are not connected with gravity. On the other hand, spin (together with the mass) is a fundamental characteristic of elementary particles. It seems natural, that a theory where matter and gravitational field interaction is considered on a microscopic level should take into account both fundamental characteristics. The assumption that the sources of the gravitational field are energy–momentum tensors and spin tensors leads to a theory where the gravitational field is described by two geometrical objects, namely, a metric and torsion. The description of the gravitational field by a metric and torsion follows from the gauge approach to gravity. One can find a review of classical gravity theory with torsion in papers [124–126].

The source of the torsion is spin of the the elementary particles, which is an intrinsically quantum object. Therefore it is natural to suppose that the theory which takes torsion into account is just quantum theory. The simple estimation shows, for example, that the torsion effects are important when the density of spin matter is about the value $\rho_c = 4\pi^2 m^2 c^2/\hbar L_{\mathrm{p}}^2$ [125], where L_{p} is the Planck length and m is the mass of a spinning particle.

This chapter is devoted to the problems of quantum field theory renormalization in curved space–time with torsion. This investigation represents a necessary stage in the construction of quantum gravity.

Some aspects of quantum field theory in curved space–time with torsion have been investigated (see, for example, [127–134]). The problem of particle creation from the vacuum in curved space–time with torsion is studied in papers [127, 128]. Some aspects of the quantum theory of free fields in an external gravitational field with torsion have been considered in [129–131]. The initial investigation of interacting theories has been reported in papers [132–134].

The general analysis and one-loop counterterm calculations for several gauge models lead us to the following conclusion. The theory is multiplicatively renormalizable in curved space–time with torsion only in the presence of non-minimal interaction of matter fields with torsion. The non-minimal interaction with torsion and curvature has the same significance in metric-torsion theory as the non-minimal interaction of $\xi R\varphi^2$ type in pure metric theory.

In this chapter we will consider the general aspects of quantum field theory renormalization in curved space–time with torsion. It is organized in the following way. First of all we present the necessary information about the metric-torsion gravity. Then the structure of minimal and non-minimal interaction with torsion is established.

In section 4.4 the renormalization structure is considered. We shall separate all the non-minimal parameters into sets of essential and non-essential parameters (from the renormalization point of view). Then we demonstrate the validity of our consideration for several gauge theories. In the last section the renormalization group equations for non-minimal effective couplings are investigated.

4.2 Preliminary discussion of gravity with torsion

Let us consider the manifold with the metric $g_{\mu\nu}$ and connection $\Gamma^\lambda{}_{\alpha\beta}$. In general relativity the connection is supposed to be symmetrical: $\Gamma^\lambda{}_{\alpha\beta} = \Gamma^\lambda{}_{\beta\alpha}$. In this case the condition on the metric

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\mu\alpha} g_{\lambda\nu} - \Gamma^\lambda{}_{\nu\alpha} g_{\mu\lambda} = 0 \tag{4.1}$$

allows us to connect the connection coefficients with the metric by means of the ordinary relation

$$\Gamma^\lambda{}_{\alpha\beta} = \tfrac{1}{2} g^{\lambda\tau}(\partial_\alpha g_{\tau\beta} + \partial_\beta g_{\tau\alpha} - \partial_\tau g_{\alpha\beta}). \tag{4.2}$$

In the metric-torsion theory the connection $\tilde{\Gamma}^\lambda{}_{\alpha\beta}$ is not assumed to be symmetrical:

$$\tilde{\Gamma}^\alpha{}_{\beta\gamma} - \tilde{\Gamma}^\alpha{}_{\gamma\beta} = T^\alpha{}_{\beta\gamma} \neq 0. \tag{4.3}$$

In the following consideration we will mark with the tilde sign the non-symmetrical connection $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ as well as the corresponding covariant derivative $\tilde{\nabla}_{\alpha}$ and the curvature tensor $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}$. We will use the notation without the tilde sign for connection (4.2) and for the covariant derivative ∇_{α} and curvature tensor built on the basis of (4.2).

We will call such objects which are built from the metric Riemannian. Let us suppose that the connection $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ satisfies the metric condition: $\tilde{\nabla}_{\alpha} g_{\mu\nu} = 0$. Then the coefficients $T^{\alpha}{}_{\beta\gamma}$ compose the torsion tensor. From the metric condition and (4.3) it follows that

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + K^{\alpha}{}_{\beta\gamma} \tag{4.4}$$

where

$$K^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}(T^{\alpha}{}_{\beta\gamma} - T_{\beta}{}^{\alpha}{}_{\gamma} - T_{\gamma}{}^{\alpha}{}_{\beta}) \tag{4.5}$$

is the contorsion tensor.

Let $S_m(\phi)$ be the action of matter fields ϕ in an external gravitational field with torsion. The energy–momentum tensor $\tilde{T}_{\mu\nu}(x)$, and the spin tensor $\tilde{\Sigma}^{\alpha\beta\gamma}(x)$ are defined dynamically in the following way [124–126]:

$$\begin{aligned}
\tilde{T}_{\mu\nu}(x) &= -\frac{2}{\sqrt{-g(x)}}\,\frac{\delta S_m}{\delta g^{\mu\nu}(x)} \\
\tilde{\Sigma}^{\alpha\beta\gamma}(x) &= -\frac{1}{\sqrt{-g(x)}}\,\frac{\delta S_m}{\delta K_{\alpha\beta\gamma}(x)}.
\end{aligned} \tag{4.6}$$

Let us introduce some other objects. The torsion tensor $T^{\alpha}{}_{\beta\gamma}$ is separated into three independent parts: the trace $T_{\mu} = T^{\alpha}{}_{\mu\alpha}$, the pseudotrace $S^{\nu} = \mathrm{i}\epsilon^{\alpha\beta\mu\nu}T_{\alpha\beta\mu}$ and the tensor $q_{\alpha\beta\gamma}$ which satisfies the relations

$$q^{\alpha}{}_{\beta\alpha} = 0 \qquad \epsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0.$$

The values of $T^{\alpha}{}_{\beta\gamma}$, T_{α}, S_{α} and $q_{\alpha\beta\gamma}$ are connected by the following relation

$$T_{\alpha\beta\mu} = q_{\alpha\beta\mu} + \tfrac{1}{3}(T_{\beta}g_{\alpha\mu} - T_{\mu}g_{\alpha\beta}) + \frac{\mathrm{i}}{6}S^{\nu}\epsilon_{\alpha\beta\mu\nu}. \tag{4.7}$$

4.3 Minimal and non-minimal interaction of matter fields with gravity

Let us describe the interaction of free matter fields with gravity. For our purposes it is necessary to consider fields of spin 0, 1/2, 1.

4.3.1 Scalar field

In flat space the action of a real free scalar field is written in the form

$$S_0 = \tfrac{1}{2}\int \mathrm{d}^4x(\eta^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - m^2\varphi^2).$$

There are two different ways to write the action of matter fields in curved space–time with torsion. The first way leads to the minimal generalization of the theory. One must substitute $\eta^{\alpha\beta}$ by $g^{\alpha\beta}$, and d^4x by $\mathrm{d}^4x\sqrt{-g}$, and all the derivatives ∂_α by the covariant ones $\tilde{\nabla}_\alpha$. Since $\tilde{\nabla}_\alpha\varphi = \partial_\alpha\varphi$ the minimal action of scalar field has the form

$$S_0 = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - \varphi^2 m^2). \tag{4.8}$$

To obtain the non-minimal action one must add to the minimal form (4.8) terms that are equal to zero in flat space–time. So, as in the metric theory (see Chapter 3) those terms must satisfy the following requirements: *a*) general covariance, *b*) new terms must have the same symmetry properties as the minimal action and must not contain new dimensional parameters.

Generally speaking, all possible additional terms are divided into the two classes. The terms which are built only with the help of gravitational fields $g_{\mu\nu}$, $T^\alpha_{\ \beta\gamma}$ are subsumed under the first class. This is the action of external fields. The second class consists of terms which contain the gravitational fields together with the matter fields. The general form of the vacuum action in curved space–time with torsion have been considered in [135]. This action includes all the invariants of dimension four which are built from $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ and $T^\alpha_{\ \beta\gamma}$. Some aspects of renormalization in the vacuum sector will be considered below.

Let us return to the non-minimal generalization of action (4.8). The action of the scalar field together with all permissible additions is written in the form

$$S_0 = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\{g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - m^2\varphi^2 + \xi_i P_i\varphi^2\}. \tag{4.9}$$

Here

$$\begin{array}{lll} P_1 = R & P_2 = \nabla_\alpha T^\alpha & P_3 = T_\alpha T^\alpha \\ P_4 = S_\alpha S^\alpha & P_5 = q_{\alpha\beta\gamma}q^{\alpha\beta\gamma} & \end{array} \tag{4.10}$$

where ξ_i with $i = 1, 2, .., 5$ are the parameters of non-minimal coupling. The parameter ξ_1 corresponds to the analogous parameter in metric theory (see Chapter 3). On the contrary, parameters $\xi_{2,..,5}$ do not have analogies in the metric theory.

In more complicated cases it is possible to have the action (4.9) completed by additional terms. For example, in the case of complex scalar field φ the expression $\int d^4x\sqrt{-g}\xi_0 T^\mu(\varphi^+\partial_\mu\varphi - \partial_\mu\varphi^+ \cdot \varphi)$ is possible [132]. If the initial action describes a scalar φ, together with the pseudoscalar Λ fields then an additional term of the following form is possible

$$\xi_\sigma \int d^4x\sqrt{-g}S^\mu(\Lambda\partial_\mu\varphi - \varphi\partial_\mu\Lambda).$$

4.3.2 The Dirac field

The action of the free Dirac field in flat space–time has the following Hermitian form:

$$S_{1/2} = \frac{i}{2}\int d^4x(\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^\mu\Psi + 2iM\bar{\Psi}\Psi). \tag{4.11}$$

Let us consider the minimal generalization of (4.11) in curved space–time with torsion. This is written in the form:

$$S_{1/2} = \frac{i}{2}\int d^4x\sqrt{-g}(\bar{\Psi}\gamma^\mu\tilde{\nabla}_\mu\Psi - \tilde{\nabla}_\mu\bar{\Psi}\gamma^\mu\Psi + 2iM\bar{\Psi}\Psi). \tag{4.12}$$

The covariant derivatives of spinor fields are built on the basis of spinor connection coefficients ω_μ^{ab} in the following way

$$\begin{aligned}\tilde{\nabla}_\mu\Psi &= \partial_\mu\Psi + \frac{i}{2}\tilde{\omega}_\mu^{ab}\sigma_{ab}\Psi \\ \tilde{\nabla}_\mu\bar{\Psi} &= \partial_\mu\bar{\Psi} - \frac{i}{2}\tilde{\omega}_\mu^{ab}\bar{\Psi}\sigma_{ab}\end{aligned} \tag{4.13}$$

where $\sigma_{ab} = \frac{i}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$. To obtain the obvious form of $\tilde{\omega}_\mu^{ab}$ we will consider the expression $\bar{\Psi}\gamma^\alpha\Psi$. Under general coordinate transformations this combination transforms as a typical vector field. Therefore

$$\tilde{\nabla}_\mu(\bar{\Psi}\gamma^\alpha\Psi) = \partial_\mu(\bar{\Psi}\gamma^\alpha\Psi) + \tilde{\Gamma}^\alpha_{\lambda\mu}(\bar{\Psi}\gamma^\lambda\Psi). \tag{4.14}$$

With the help of the metric condition one can obtain from (4.14) the following relation

$$\partial_\mu\gamma^\alpha + \tilde{\Gamma}^\alpha_{\lambda\mu}\gamma^\lambda + \frac{i}{2}\tilde{\omega}_\mu^{ab}(\sigma_{ab}\gamma^\alpha - \gamma^\alpha\sigma_{ab}) = 0. \tag{4.15}$$

After multiplying (4.15) by γ^β and taking the trace we obtain finally

$$\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} + \tfrac{1}{4}K^{\alpha}{}_{\lambda\mu}(e^{\lambda a}e^b_\alpha - e^{\lambda b}e^a_\alpha) \tag{4.16}$$

where the coefficients of

$$\omega_{\mu ab} = \tfrac{1}{4}(e_{b\alpha}\partial_\mu e^\alpha_a - e_{a\alpha}\partial_\mu e^\alpha_b) + \tfrac{1}{4}\Gamma^\alpha_{\lambda\mu}(e^\lambda_a e_{b\alpha} - e^\lambda_b e_{a\alpha})$$

compose the Riemannian spinor connection.

One can rewrite the action (4.12) with the help of (4.13), (4.16) and (4.5) in the following form:

$$\begin{aligned} S_{1/2} &= \mathrm{i}\int \mathrm{d}^4x\sqrt{-g}\bar{\Psi}(\gamma^\mu\tilde{\nabla}_\mu - \tfrac{1}{2}\gamma^\mu T_\mu + \mathrm{i}M)\Psi \\ &= \mathrm{i}\int \mathrm{d}^4x\sqrt{-g}\bar{\Psi}(\gamma^\mu\nabla_\mu - \tfrac{1}{8}\gamma_5\gamma^\mu S_\mu + \mathrm{i}M)\Psi. \end{aligned} \qquad (4.17)$$

From the final form of $S_{1/2}$ it follows that the minimal Dirac action allows the spinor field to interact only with S_μ, but not with T_μ and $q_{\mu\nu\alpha}$.

The non-minimal action of the Dirac field is obtained in the same way as for a scalar field. This action has the form:

$$S_{1/2} = \mathrm{i}\int \mathrm{d}^4x\sqrt{-g}\bar{\Psi}(\gamma^\mu\nabla_\mu + \eta_j Q_j + \mathrm{i}M)\Psi. \qquad (4.18)$$

Here $Q_1 = \gamma_5\gamma^\mu S_\mu$ and $Q_2 = \gamma^\mu T_\mu$. $\eta_{1,2}$ are the parameters of non-minimal coupling of the spinor field with torsion. One can obtain the minimal action (4.17) from (4.18) if $\eta_1 = -1/8, \eta_2 = 0$.

4.3.3 Gauge vector field

For Abelian gauge field A_μ from the relation

$$\tilde{\nabla}_\mu A_\nu = \nabla_\mu A_\nu - K^\lambda{}_{\nu\mu}A_\lambda$$

it follows that

$$\tilde{F}_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu = F_{\mu\nu} - A_\lambda(K^\lambda{}_{\nu\mu} - K^\lambda{}_{\mu\nu}). \qquad (4.19)$$

It is obvious that the expression $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ is not invariant under typical gauge transformations $\delta A_\mu(x) = \partial_\mu\varepsilon(x)$. Therefore the minimal interaction with torsion contradicts gauge invariance (see also [125, 126]). Hence the requirement of gauge invariance leads us to only a non-minimal interaction with torsion for a gauge field. For the Yang–Mills fields a non-minimal interaction with torsion is forbidden as well as a minimal one. However, for an Abelian gauge field a non-minimal interaction is possible. The action of an Abelian gauge field with non-minimal additional terms has the form:

$$S_1 = \int d^4x\sqrt{-g}(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + K^{\mu\nu}F_{\mu\nu}). \tag{4.20}$$

Here

$$\begin{aligned} K^{\mu\nu} &= \alpha_1\nabla^\mu T^\nu + \alpha_2\nabla_\rho S_\sigma\epsilon^{\mu\nu\rho\sigma} + \alpha_3 T_\rho S_\sigma\epsilon^{\rho\sigma\mu\nu} \\ &\quad + \alpha_4 T^\rho q_\rho{}^{\mu\nu} + \alpha_5 S^\sigma q_{\sigma\kappa\omega}\epsilon^{\mu\nu\kappa\omega} + \alpha_6\nabla_\alpha q^{\alpha\mu\nu}. \end{aligned}$$

Let us now establish the conformal transformations form for metric, torsion and matter fields and find the values of non-minimal parameters ξ_i, η_j, α_k which allow the actions (4.9), (4.18) and (4.20) to be conformally invariant. Since we shall use dimensional regularization it is useful to write the conformal transformations for an n-dimensional manifold with metric and torsion. In n-dimensional space the expressions for non-minimal Lagrangians have the same form as in 4-dimensional space.

In the purely metric theory the conformal transformations of matter fields and metric have the following infinitesimal form (3.49)

$$\begin{aligned} \delta\varphi &= \frac{2-n}{n}\cdot\varphi\sigma(x) \qquad & \delta g_{\mu\nu} &= g_{\mu\nu}\cdot 2\sigma(x) \\ \delta A_\mu &= 0 \qquad & \delta\Psi &= \frac{1-n}{2}\Psi\sigma(x). \end{aligned} \tag{4.21}$$

The conformal invariance of metric theory requires $\xi_1 = (n-2)/(4(n-1))$ (see Chapter 3).

It is evident that the transition to the metric theory in the non-minimal actions (4.9), (4.18) and (4.20) means the rejection of all the non-minimal parameters ξ, η, α, except ξ_1. Let those parameters not equal zero. Then it is necessary to add to (4.21) the conformal transformation of the torsion field

$$\delta T^\lambda_{\alpha\beta} = \kappa T^\lambda{}_{\alpha\beta}\cdot\sigma(x) + \omega(\delta^\alpha_\gamma\partial_\beta - \delta^\alpha_\beta\partial_\gamma)\sigma(x)$$

where κ and ω are any real constants. It is easy to see that for $\kappa \neq 0$ conformal invariance contradicts the possibility of interaction with torsion. Therefore, the conformal transformation of torsion has the form

$$\delta T^\alpha_{\beta\gamma} = \omega(\delta^\alpha_\gamma\partial_\beta - \delta^\alpha_\beta\partial_\gamma)\sigma(x). \tag{4.22}$$

From (4.22) it follows that

$$\delta T_\alpha = \omega(n-1)\partial_\alpha\sigma(x) \qquad \delta S_\alpha = 0 \qquad \delta q^\alpha_{\beta\gamma} = 0.$$

For $\omega \neq 0$ the conformal invariance of the non-minimal expressions (4.9), (3.18) and (3.20) requires

$$\xi_1 = \frac{n-2}{4(n-1)} \qquad \xi_2 = \xi_3 = \xi_5 = 0$$
$$\eta_2 = 0 \qquad \alpha_3 = \alpha_4 = \alpha_6 = 0 \qquad m = M = 0. \tag{4.23}$$

In the case $\omega = 0$ the requirements are less severe

$$\xi_1 = \frac{n-2}{4(n-1)} \qquad \xi_2 = 0 \qquad m = M = 0. \tag{4.24}$$

For the metric theory conformal invariance of the matter fields' action $S_m(\Phi)$ leads to the well-known identity

$$-\frac{1}{2\sqrt{-g}} g_{\mu\nu} \frac{\delta S_m(\Phi)}{\delta g_{\mu\nu}} = T_\mu{}^\mu = 0. \tag{4.25}$$

It is interesting to find the same identity for gravity with torsion. Let us make the conformal transformation

$$\delta S_m(\Phi) = \int d^4x \sqrt{-g} \left\{ \frac{\delta S_m}{\delta \Phi} \delta \Phi + \frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S_m}{\delta T^\lambda{}_{\alpha\beta}} \delta T^\lambda{}_{\alpha\beta} \right\}. \tag{4.26}$$

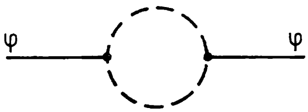
From the extremum of equations and relations (4.26), (4.22) and (4.6) the conditions of conformal invariance follow

$$\tilde{T}^\mu{}_\mu = 0 \qquad \nabla_\mu \tilde{\Sigma}_\nu{}^{\mu\nu} = 0. \tag{4.27}$$

So the presence of the free parameter ω in the conformal transformation (4.22) leads to a new identity $\nabla_\mu \tilde{\Sigma}_\nu{}^{\mu\nu} = 0$. This identity has no analogies in the metric theory.

4.4 Renormalization of quantum field theory in curved space–time with torsion

We shall consider the renormalization of the interacting theory in an external gravitational field with torsion. The analysis given in Chapter 3 shows that two criteria are important for the field theory renormalization in an external gravitational field. Firstly, one should expect the appearance of vacuum divergences to illustrate which it is necessary to introduce counterterms depending only on the external field. Secondly, one should expect the appearance of new divergences in the matter field sector. Let us consider the theory which is multiplicatively renormalizable in flat space–time. From dimensional considerations, general and gauge invariance in the matter sector one can expect counterterms of the same types, as the additional, non-minimal terms in the actions (4.9), (4.18) and (4.20). Then to achieve multiplicative renormalizability it is necessary to include into the bare Lagrangian the corresponding non-minimal terms. Hence it follows that the classical Lagrangian which is the basis for building the quantum theory contains non-minimal interaction of the gravitational field with matter. Direct interaction of the scalar field with torsion seems surprising because minimally only the spinor field interacts with torsion. The necessity of such interaction is accounted for by the Yukawa coupling of scalar and spinor fields. Consider, for example, the diagram making a contribution into the scalar field self-energy:



(4.28)

Here a dashed line is the propagator of the spinor field. Let the spinor field be minimally coupled with torsion, then the spinor field propagator depends on torsion. Therefore the divergences of the diagram under consideration depend on torsion and in the scalar field sector counterterms of the type $P_i \times \varphi^2$ are necessary. Thus, a direct non-minimal interaction of the scalar field with torsion must appear.

We shall make further considerations taking as an examples the concrete models based on the gauge groups SU(2), U(1) and SU(N).

(a) Let us start from the asymptotically free model based on the gauge group SU(2). The Lagrangian of the model has the form

$$\begin{aligned} L = &-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a \\ &+ \tfrac{1}{2}(\xi_i P_i - m^2)\varphi^a\varphi^a - \tfrac{1}{4!}f(\varphi^a\varphi^a)^2 \\ &+ \mathrm{i}\bar{\Psi}^a[\gamma^\mu D^{ab}_\mu + (\eta_j Q_j + \mathrm{i}M)\delta^{ab}]\Psi^b \\ &- \mathrm{i}h\varepsilon^{acb}\bar{\Psi}^a\varphi^c\Psi^b. \end{aligned} \tag{4.29}$$

If the torsion tensor equals zero we obtain the model with Lagrangian (3.97). Also

$$(D_\mu\varphi)^a = \partial_\mu\varphi^a + g\varepsilon^{acb}A^c_\mu\varphi^b$$
$$D^{ab}_\mu\Psi^b = \nabla_\mu\Psi^a + g\varepsilon^{acb}A^c_\mu\varphi^b$$

where ∇_μ is a covariant spinor derivative without torsion.

One-loop counterterms for the theory with the Lagrangian (4.29) can be calculated on the basis of the background field method and Schwinger–De Witt technique for finding the effective action divergences (see Chapter 3, for details and references). Note that the general method for the calculation of one-loop divergences does not demand any modifications in the presence of the external torsion field. Omitting the cumbersome algebraic transformations and limiting ourselves to the matter field sector let us give the final result. The relations between the bare $\xi_i^{(0)}, \eta_j^{(0)}$ and renormalized ξ_i, η_j parameters have the form (see also [158] for details)

$$\xi_1^{(0)} = \xi_1 + (Z_2 - 1)(\xi_1 - \tfrac{1}{6}) \qquad \xi_{2,3,5}^{(0)} = Z_2 \xi_{2,3,5}$$
$$\xi_4^{(0)} = Z_2\xi_4 + Z_4\eta_1^2 \qquad \eta_1^{(0)} = Z_\eta \eta_1 \qquad \eta_2^{(0)} = \eta_2$$
$$Z_2 = 1 + \frac{1}{\varepsilon}(12g^2 - 8h^2 - \tfrac{5}{3}f) \qquad Z_4 = \frac{32}{\varepsilon}h^2 \qquad Z_\eta = 1 + 4h^2/\varepsilon. \tag{4.30}$$

Here $\varepsilon = (4\pi)^2(n-4)$ is the parameter of dimensional regularization. Relations (4.30) are of some interest. From (4.30) it follows that the minimal theory is not multiplicatively renormalized. In fact for the minimal theory $\eta_1 = -1/8, \eta_2 = \xi_i = 0$. It is obvious that if one chooses the bare action to be minimally coupled to torsion then the one-loop counterterms would not repeat the structure of this action. To achieve multiplicative renormalizability it is necessary to introduce a non-minimal interaction with the coupling parameters η_1, ξ_4. These two parameters connect the spinor and scalar fields with the torsion tensor pseudotrace S_μ (4.7). The parameter ξ_1 is renormalized in the same way as in the gravitational field without torsion. We call the parameters η_1, ξ_4 and ξ_1 essential. The parameter η_2 is not renormalized at all and one can put $\eta_2 = 0$ without breaking renormalization multiplicativity. The parameters ξ_2, ξ_3, ξ_5 are renormalized in such a way that they were not introduced into the bare Lagrangian, the corresponding counterterms would not be necessary. These parameters can also be put equal to zero without breaking the renormalization multiplicativity.

We shall investigate the renormalization group equations for non-minimal effective coupling constants in the next section.

(*b*) The most important feature from the renormalization group point of view is connected with the renormalization of the parameter η_1. To research the general peculiarities of the η_1 renormalization we shall consider the one-loop counterterms in an Abelian gauge theory. Moreover this theory allows us to understand the significance of the non-minimal terms (4.20).

The action of Abelian gauge theory has the form

$$S = \int d^4x\sqrt{-g}\{L_0 + L_{1/2} \\ + L_1 + h\varphi\bar{\Psi}\Psi - \tfrac{1}{24}f\varphi^4 + eA_\mu\bar{\Psi}\gamma^\mu\Psi\}. \tag{4.31}$$

Here φ, Ψ and A_μ are scalar, spinor and electromagnetic fields. $L_{0,1/2,1}$ are defined by (4.9), (4.18) and (4.20). e, h, f are the constants of gauge, Yukawa and scalar couplings.

We shall use the standard algorithm to obtain the one-loop counterterms. The decomposition of the fields into classical fields φ, A_μ, $\bar{\Psi}$, Ψ and quantum fields , $\bar{\sigma}$, a_μ, $\bar{\eta}$ and χ is performed in the following way

$$\varphi \to \varphi + i\bar{\sigma} \qquad A_\mu \to A_\mu + a_\mu \qquad \bar{\Psi} \to \bar{\Psi} + \bar{\eta} \\ \Psi \to \Psi + \eta \qquad \eta = -\frac{i}{2}\gamma^\alpha\nabla_\alpha\chi. \tag{4.32}$$

For our purposes it is convenient to choose the gauge condition for the vector field in the form $\nabla_\mu a^\mu$. The corresponding ghosts contribute to the vacuum counterterms, as well as the Jacobian of transformation (4.32).

One-loop divergences of the effective action are defined by the bilinear (with respect to the quantum fields) part $S^{(2)}$ of the action (4.31) (together with the gauge fixing action). Then $S^{(2)}$ has the form

$$S^{(2)} = \tfrac{1}{2}\int d^4x\sqrt{-g}(a^\alpha|\bar{\sigma}|\bar{\eta})(\widehat{H})(a^\beta|\bar{\sigma}|\chi)^{\mathrm{T}}. \tag{4.33}$$

The operator $\widehat{H}$ has the following structure

$$\widehat{H} = \hat{1}\Box + \widehat{E}^\mu\nabla_\mu + \widehat{D}.$$

In the sector of matter fields the one-loop counterterms in theory (4.3) are defined by expression (3.95)

$$\Delta S = \frac{1}{\varepsilon}\int d^4x\sqrt{-g}\,\mathrm{Tr}[\tfrac{1}{2}\widehat{P}\widehat{P} + \tfrac{1}{12}\widehat{S}_{\mu\nu}\widehat{S}^{\mu\nu}] \tag{4.34}$$

where

$$\widehat{P} = \widehat{D} - \tfrac{1}{2}\nabla_\mu\widehat{E}^\mu - \tfrac{1}{4}\widehat{E}_\mu\widehat{E}^\mu + \frac{\hat{1}}{6}R$$
$$\widehat{S}_{\mu\nu} = \nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu + \tfrac{1}{2}(\nabla_\nu\widehat{E}_\mu - \nabla_\mu\widehat{E}_\nu) + \tfrac{1}{4}(\widehat{E}_\nu\widehat{E}_\mu - \widehat{E}_\mu\widehat{E}_\nu).$$

The expressions for $\widehat{E}^\mu$ and $\widehat{D}$ have the following matrix form

$$\widehat{E}^\mu = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \qquad \widehat{D} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}.$$

For our purposes it is necessary to consider only those parts of $\widehat{P}$ and $\widehat{S}_{\mu\nu}$ which give a contribution to the renormalization of parameter η_1. The corresponding sectors of $\widehat{P}$, $\widehat{S}_{\mu\nu}$ have the form

$$\begin{aligned}
&P_{31} = 2e\gamma_\beta \Psi \qquad P_{32} = 2h\Psi \\
&P_{13} = \frac{\mathrm{i}}{2}e\nabla_\mu \bar{\Psi}\gamma_\alpha\gamma^\mu - \mathrm{i}eh\bar{\Psi}\gamma_\alpha\varphi - \frac{1}{2}e^2\bar{\Psi}A_\mu\gamma_\alpha\gamma^\mu + \frac{\mathrm{i}}{4}e\bar{\Psi}\gamma_\alpha\gamma^\mu\eta_j Q_j\gamma_\mu \\
&P_{23} = \frac{\mathrm{i}}{2}h\nabla_\mu\bar{\Psi}\gamma^\mu - \mathrm{i}h^2\varphi\bar{\Psi} - \frac{1}{2}eh\bar{\Psi}A_\mu\gamma^\mu + \frac{\mathrm{i}}{4}h\bar{\Psi}\gamma^\mu\eta_j Q_j\gamma_\mu \\
&S_{31} = S_{32} = 0.
\end{aligned} \tag{4.35}$$

Note that non-minimal terms $K^{\mu\nu}F_{\mu\nu}$ (4.20) and the vacuum action do not contribute to the one-loop counterterms. The final expression for one-loop counterterms has the form

$$\begin{aligned}
\Delta S = \frac{1}{\varepsilon}\int \mathrm{d}^4x\sqrt{-g}\Big\{&2h^2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - (2h^4 - \tfrac{1}{8}f^2)\varphi^4 - \frac{2e^2}{3}F^2_{\mu\nu} \\
&- \tfrac{8}{3}e\eta_2 F_{\mu\nu}\nabla^\mu T^\nu + \tfrac{1}{2}\varphi^2[(\tfrac{1}{6}R - \xi_i P_i)f + \tfrac{2}{3}Rh^2 - 16h^2\eta_1 S_\mu S^\mu] \\
&+ (2e^2 + h^2)\mathrm{i}\bar{\Psi}\gamma^\mu(\nabla_\mu - \mathrm{i}eA_\mu - \mathrm{i}\eta_2 T_\mu)\Psi \\
&+ (2e^2 - h^2)\mathrm{i}\bar{\Psi}\gamma_5\gamma^\mu\eta_1 S_\mu\Psi + (8e^2 - 2h^2)h\bar{\Psi}\varphi\Psi\Big\} + \ldots
\end{aligned} \tag{4.36}$$

Here the dots stand for vacuum and mass terms.

Let us pass to the renormalization structure analysis. In accordance with the previous result the multiplicativity of one-loop renormalization requires the non-minimal interaction of matter fields with torsion.

If we start from the minimal interaction with torsion $\eta_1 = 1/8$, $\eta_2 = \xi_{2,\ldots,5} = \alpha_k = 0$ then the counterterms (4.36) have a different structure compared with the bare action (4.31). To achieve multiplicativity it is necessary to introduce the non-minimal coupling constants: η_1 in spinor and ξ_4 in scalar sectors. Besides η_1 and ξ_4 one must introduce the non-minimal parameter α_2 in the vector sector, because the counterterm

$$\int \mathrm{d}^4x\sqrt{-g}F_{\alpha\omega}\nabla_\mu S_\nu\varepsilon^{\alpha\omega\mu\nu}$$

can appear in higher loops. Therefore in the Abelian theory the essential parameters are η_1, ξ_4 and α_2. These parameters connect the matter fields with the torsion pseudotrace S_μ. The other constants η_2, $\xi_{2,3,5}$, $\alpha_{1,3,\ldots,6}$ are pure non-minimal parameters. If one put them simultaneously equal to zero, the multiplicativity of renormalization is not violated. Then matter fields interact only with the pseudotrace S_μ.

Let us suppose that not only S_μ but also the torsion trace T_μ is present in the bare action. Then the multiplicativity of renormalization requires the inclusion of the non-minimal parameters η_2, $\xi_{2,3}$ and $\alpha_{1,3}$. The counterterms connected with α_1 arise even on a one-loop level (4.36). Therefore the principle of multiplicative renormalization requires us to add the bare action by including the non-minimal terms (4.20) which have no analogies in the non-Abelian theory. The situation where the renormalizability requires the linear terms to be included in the action is well-known. As an example one can remember the massive scalar field theory with $\lambda\varphi^3$ interaction [1].

It is interesting to note that the traceless and pseudotraceless part of torsion tensor $q_{\alpha\beta\mu}$ is not sufficient from the renormalization point of view. The corresponding terms do not give any contribution to the counterterms. Therefore the renormalization of non-minimal coupling ξ_5 never includes other non-minimal parameters. Moreover, ξ_5 never contributes to the renormalization of $\alpha_{4,5,6}$, and none of these parameters are necessary for renormalization multiplicativity even in higher loops.

Let us now consider the details of the derivation of counterterms which are important not only with respect to Abelian but also to non-Abelian theories. One-loop renormalization of $\eta_{1,2}$ is defined by expressions (4.34) and (4.35). Since $S_{31} = S_{32} = 0$ the term $\frac{1}{2}\mathrm{Tr}S^2_{\mu\nu}$ does not contribute to the renormalization in the spinor sector. The expressions P_{31} and P_{32} contain the torsion trace T_μ in the combination $eA_\mu - \eta_2 T_\mu$. From the gauge invariance it follows that the coefficients for the counterterms $\mathrm{i}\bar{\Psi}\gamma^\mu\nabla_\mu\Psi$ and $e\bar{\Psi}\gamma^\mu A_\mu\Psi$ coincide. Hence, the parameter η_2 is not renormalized on the one-loop level.

It is easy to see, that the sign difference for e^2 and h^2 coefficients in the $\mathrm{i}\bar{\Psi}\eta_1 Q_1\Psi$ counterterm is explained by the difference of γ-matrix arrangement in P_{13} and P_{23} expressions. Note, that the general peculiarities of bilinear form (4.33) as well as its matrix structure, the arrangement of γ-matrices etc. do not depend on the gauge theory under consideration. Therefore, all the above-mentioned arguments are correct in non-Abelian gauge theories as well as in the Abelian theory (4.31). The relation between the bare and the renormalizable values of $\eta_{1,2}$ have the form

$$\eta_2^0 = \eta_2 \qquad \eta_1^0 = \eta_1 + \frac{1}{\varepsilon} ch^2 \eta_1. \tag{4.37}$$

Here c is a constant which depends on the model under consideration. For example, in the gauge model with action (4.29) $c = 4$, and for Abelian theory with action (4.31) $c = 2$. The detailed analysis of expression (4.35) shows that in all cases $c > 0$. We shall verify this fact for SU(N) gauge theory, including one scalar multiplet Φ^a in adjoint and m spinor multiplets $\psi^i_{(k)}$ in the vector representation of the gauge group. The matter Lagrangian has the form

$$\begin{aligned} L_m = & -\tfrac{1}{4}(G^a_{\mu\nu})^2 + \tfrac{1}{2} g^{\mu\nu}(D_\mu \Phi)^2 (D_\nu \Phi)^2 + \tfrac{1}{2}\xi_i P_i \Phi^a \Phi^a \\ & - \tfrac{1}{8} f_1 (\Phi^a \Phi^a)^2 - \tfrac{1}{8} f_2 (\Phi^a D_{rab} \Phi^b)^2 \\ & + \mathrm{i} \sum_{k=1}^{m} \bar{\Psi}^i_{(k)} (\gamma^\mu D^{ij}_\mu + \eta_j Q_j + h(\frac{\lambda}{2}^a)^{ij} \Phi^a) \Psi^j_{(k)}. \end{aligned} \tag{4.38}$$

We used this model (without torsion) in Chapter 3. We shall not present the details of the derivation of one-loop counterterms for the theory (4.38) or even their the final expression. Let us only note that the resulting value $c = 1$ in relation (4.37) confirms our consideration. The renormalization group equations for the effective couplings of theory with action (4.38) will be presented in the next section together with equations for SU(N)-theories.

4.5 Renormalization group equations and the UV limit in external gravitational field with torsion

Now we will use the renormalization group method to investigate the asymptotic behaviour of the non-minimal effective couplings in the UV limit. In the presence of the external torsion field the renormalization group equations are constructed according to the common scheme as described in Chapter 3.

We consider equations for effective couplings which correspond to non-minimal parameters in the framework of concrete quantum field theory models in curved space–time with torsion.

(a) Consider the model with action (4.29) based on the gauge group SU(2). The equations for non-minimal effective couplings $\eta_j(t)$, $\xi_i(t)$ are defined by relations (4.30)

$$
\begin{aligned}
(4\pi)^2 \frac{d\xi_1}{dt} &= (\xi_1 - \tfrac{1}{6})(\tfrac{5}{3}f + 8h^2 - 12g^2) \\
(4\pi)^2 \frac{d\xi_i}{dt} &= \xi_i(\tfrac{5}{3}f + 8h^2 - 12g^2) - 32\delta_{i4}h^2\eta_1 \qquad i \neq 1 \\
(4\pi)^2 \frac{d\eta_1}{dt} &= 4h^2\eta_1 \qquad \frac{d\eta_2}{dt} = 0.
\end{aligned}
\tag{4.39}
$$

The equations for $f(t)$, $h^2(t)$, $g^2(t)$ are the same as in flat space. If we use the special solutions method for $h^2(t)$ and $f(t)$ these couplings behave according to

$$
\begin{aligned}
f(t) &= k_2 \cdot g^2(t) \qquad h^2(t) = k_1 \cdot g^2(t) \\
g^2(t) &= g^2(1 + l^2 t)^{-1}
\end{aligned}
\tag{4.40}
$$

where $k_1 = 23/24$, $k_2 = 97/22$ and $l^2 = g^2b^2/(4\pi)^2$. Then the solution of (4.39) is written in the form

$$
\begin{aligned}
\eta_2(t) &= \eta_2 \\
\eta_1(t) &= \eta_1(1 + l^2 t)^{4k_1^2/b^2} \\
\xi_i(t) &= (\xi_i + 32\delta_{i4}\frac{\eta_1^2}{8k_2 + b^2})(1 + l^2 t)^{-a^2/b^2} \\
&\quad - 32\delta_{i4}\frac{\eta_1^2}{8k_2 + b^2}(1 + l^2 t)^{8k_1/b^2} \qquad i \neq 1 \\
a^2 &= 12 - \tfrac{5}{3}k_1 - 8k_2 \\
\xi_1(t) &= \tfrac{1}{6} + (\xi_1 - \tfrac{1}{6})(1 + l^2 t)^{-a^2/b^2}.
\end{aligned}
\tag{4.41}
$$

When $t \to +\infty$ ('UV' limit or small distances or strong curvature limit) the effective non-minimal couplings have the following asymptotic values

$$
\begin{aligned}
&\eta_2(t) \equiv \eta_2 \qquad \eta_1(t) \to \infty \,\mathrm{sgn}\, \eta_1 \qquad \xi_1(t) \to \tfrac{1}{6} \\
&|\xi_4(t)| \to \infty \qquad \xi_{2,3,5}(t) \to 0.
\end{aligned}
\tag{4.42}
$$

It is now useful to discuss the conformal invariance of the theory (4.29). From relations (4.30) it follows that if theory (4.29) is invariant under the conformal transformations (4.21) and (4.22) with $\omega \neq 0$ then the one-loop counterterms are also invariant. On the other hand, even if the conditions of conformal invariance (4.23) are violated, it follows from equations (4.41) that the asymptotic values of η_j and ξ_i correspond to a conformal theory. Therefore the asymptotic conformal invariance phenomenon takes place in curved space–time with torsion as well as in pure metric theories.

(*b*) Let us consider another model based on gauge group SU(2). The Lagrangian is written in the form [85]

$$
\begin{aligned}
L = {} & -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + g^{\mu\nu}(\partial_\mu\varphi^+ + \frac{\mathrm{i}g}{2}\varphi^+\tau^a A^a_\mu)(\partial_\nu\varphi - \frac{\mathrm{i}g}{2}\tau^a A^a_\nu\varphi) \\
& + \xi_i P_i \varphi^+\varphi + \mathrm{i}\xi_0 T^\alpha(\varphi^+\partial_\alpha\varphi - \partial_\alpha\varphi^+\varphi) \\
& + \mathrm{i}\sum_{k=1}^{m+n} \bar{\psi}^{(k)}(\gamma^\mu\nabla_\mu - \frac{\mathrm{i}g}{2}\gamma^\mu\tau^a A^a_\mu + \eta_j Q_j)\psi^{(k)} \\
& + \mathrm{i}\sum_{k=1}^{m} \bar{\chi}^{(k)}(\gamma^\mu\nabla_\mu + \delta_j Q_j)\chi^{(k)} \\
& - h\sum_{k=1}^{m}(\bar{\psi}^{(k)}\chi^{(k)}\varphi + \varphi^+\bar{\chi}^{(k)}\psi^{(k)}) - \tfrac{1}{8}f(\varphi^+\varphi)^2.
\end{aligned}
\tag{4.43}
$$

Renormalization group equations for the effective parameters $\xi_i(t), \xi_0(t)$, $\eta_j(t)$, $\delta_j(t)$ can be written in the form

$$
\begin{aligned}
(4\pi)^2\frac{\mathrm{d}\xi_0}{\mathrm{d}t} &= 4h^2(t)(\delta_2 - \eta_2)m^2 \\
(4\pi)^2\frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= (\xi_1 - \tfrac{1}{6})A(t) \\
(4\pi)^2\frac{\mathrm{d}\xi_i}{\mathrm{d}t} &= \xi_i A(t) - mh^2(t)\delta_{i4}(\eta_1 + \delta_1)^2 \\
&\quad + 4mh^2(t)(\eta_2 - \delta_2)^2 \qquad i \neq 1 \\
(4\pi)^2\frac{\mathrm{d}\eta_2}{\mathrm{d}t} &= h^2(t)(\eta_2 - \delta_2 - \tfrac{1}{2}\xi_0) \\
A(t) &= \tfrac{3}{2}f(t) - \tfrac{9}{2}g^2(t) + 4h^2(t)m \\
(4\pi)^2\frac{\mathrm{d}\delta_2}{\mathrm{d}t} &= h^2(t)(-2\eta_2 + \xi_0) \\
(4\pi)^2\frac{\mathrm{d}\eta_1}{\mathrm{d}t} &= h^2(t)(\eta_1 + \delta_1) \\
(4\pi)^2\frac{\mathrm{d}\delta_1}{\mathrm{d}t} &= 2h^2(t)\eta_1.
\end{aligned}
\tag{4.44}
$$

Here we do not show explicitly the t-dependence of the non-minimal coupling constants. One can find the solution of equations (4.44) in the regime of asymptotic freedom when special solutions of renormalization group equations ($m = 1, n = 10$)

$$
\begin{aligned}
& h^2(t) = \tfrac{1}{2}g^2(t) \qquad f(t) = k_2 g^2(t) \qquad k_2 > 0 \\
& g^2(t) = g^2(1 + l^2 t)^{-1} \qquad l^2 = g^2 b^2/(4\pi)^2.
\end{aligned}
$$

are used. If $t \to \infty$ this solution leads to the following asymptotic behaviour of effective couplings

$$\begin{gathered}\xi_1(t) \to 1/6 \qquad \xi_0(t) \to 0 \qquad \xi_{2,3,5}(t) \to 0 \\ \delta_1(t),\ \eta_1(t) \to \infty\,\mathrm{sgn}(\delta_1 + 2\eta_1) \\ \eta_2(t), \delta_2(t) \to 0 \qquad \xi_4(t) \to \infty.\end{gathered}$$

Therefore one can see that asymptotic conformal invariance takes place for the model (4.43).

(*c*) The next SU(2) model contains scalar and pseudoscalar fields. The Lagrangian is written in the form

$$\begin{aligned}L = &-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^a(D_\nu\varphi)^a + \tfrac{1}{2}g^{\mu\nu}(D_\mu\Lambda)^a(D_\nu\Lambda)^a \\ &+ \tfrac{1}{2}\xi_i P_i \varphi^a\varphi^a + \tfrac{1}{2}\delta_i P_i \Lambda^a\Lambda^a - \tfrac{1}{8}f_1(\varphi^a\varphi^a)^2 - \tfrac{1}{8}f_2(\Lambda^a\Lambda^a)^2 \\ &- \tfrac{1}{4}f_3\varphi^a\varphi^a\Lambda^b\Lambda^b - \tfrac{1}{2}f_4[\varphi^a\varphi^a\Lambda^b\Lambda^b - (\varphi^a\Lambda^a)^2] + \mathrm{i}\xi_0 S^\mu(\Lambda^a\partial_\mu\varphi^a \\ &- \varphi^a\partial_\mu\Lambda^a) + \mathrm{i}\bar{\psi}^a(\gamma^\mu D^{ab}_\mu + h_1\varepsilon^{abc}\varphi^c \\ &- \mathrm{i}\varepsilon^{acb}h_2\gamma_5\Lambda^c + \eta_j Q_j)\psi^a.\end{aligned} \tag{4.45}$$

The renormalization group equations for the non-minimal effective couplings have the form

$$\begin{aligned}(4\pi)^2\frac{\mathrm{d}\xi_0}{\mathrm{d}t} &= (4h_1^2(t) + 4h_2^2(t) - 15g^2(t))\xi_0 \qquad \frac{\mathrm{d}\eta_2}{\mathrm{d}t} = 0 \\ (4\pi)^2\frac{\mathrm{d}\eta_1}{\mathrm{d}t} &= 4[h_1^2(t) + h_2^2(t)]\eta_1 + 2h_1(t)h_2(t)\xi_0 \\ (4\pi)^2\frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= A_1(t)(\xi_1 - \tfrac{1}{6}) + A(t)(\delta_1 - \tfrac{1}{6}) \\ (4\pi)^2\frac{\mathrm{d}\delta_1}{\mathrm{d}t} &= A_2(t)(\delta_1 - \tfrac{1}{6}) + A(t)(\xi_1 - \tfrac{1}{6}) \\ (4\pi)^2\frac{\mathrm{d}\xi_i}{\mathrm{d}t} &= A_1(t)\xi_i + A(t)\delta_i + B_1\xi_0^2\delta_{i4} \\ &\quad - 32\delta_{i4}\eta_1^2h_1^2(t) \qquad i \neq 1 \\ (4\pi)^2\frac{\mathrm{d}\delta_i}{\mathrm{d}t} &= A_2(t)\delta_i + A(t)\xi_i + B_2\xi_0^2\delta_{i4} \\ &\quad - 32\delta_{i4}\eta_1^2h_1^2(t) \qquad i \neq 1 \\ A(t) &= 3f_3(t) + 4f_4(t) \\ A_{1,2}(t) &= 5f_{1,2}(t) + 8h_{1,2}^2(t) - 12g^2(t) \\ B_{1,2}(t) &= \tfrac{5}{2}f_{1,2}(t) - \tfrac{3}{2}f_3(t) - 2f_4(t) + 4g^2(t).\end{aligned} \tag{4.46}$$

We have not written the general solution of (4.46). Let us consider the asymptotics of the effective couplings. In the asymptotically free regime two cases are possible.

(c_1) $h_1^2(t) = h_2^2(t) = g^2(t)$, $f_1(t) = f_2(t) = f_3(t) = 0$, $f_4(t) = g^2(t)$ — these relations for couplings are required by the supersymmetry [85]. If $t \to \infty$ then the effective couplings behave in the following way

$$\begin{array}{ll} \xi_{1,2,3,5}(t) \qquad \delta_{1,2,3,5}(t) \to \frac{1}{2}(\xi_{1,2,3,5} + \delta_{1,2,3,5}) & \\ \xi_4(t), \delta_4(t) \to \infty \qquad \eta_1(t) \to \infty & \\ \eta_2(t) = \eta_2 \qquad \xi_0(t) \to 0. & \end{array} \tag{4.47}$$

Here asymptotic conformal invariance is possible, if the physical values of the couplings satisfy the conditions

$$\xi_1 + \delta_1 = 1/3 \qquad \xi_{2,3,5} + \delta_{2,3,5} = 0. \tag{4.48}$$

(c_2) $h_1^2(t) = h_2^2(t) = g^2(t)$, $f_1(t) = f_2(t) = f_3(t) = 4g^2(t)/\sqrt{105}$, $f_4(t) = \sqrt{7/15}g^2(t)$. It is easy to see that

$$\begin{array}{ll} \eta_1(t) \to \infty & \eta_2(t) = \eta_2 \\ \xi_0(t) \to \infty & \xi_4(t), \delta_4(t) \to \infty. \end{array} \tag{4.49}$$

If the values of $\xi_{1,2,3,5}$ and $\delta_{1,2,3,5}$ satisfy the condition (4.48), then asymptotic conformal invariance takes place. On the other hand, if (4.48) is violated, then

$$\xi_{1,2,3,5}(t) \to \infty \text{ and } \delta_{1,2,3,5}(t) \to \infty. \tag{4.50}$$

Let us proceed to a preliminary discussion of the results. Relations (4.42), (4.45), (4.47) and (4.49) show that the effective couplings, corresponding to the non-minimal interaction of matter with the completely antisymmetric part of the torsion tensor, increase indefinitely. (We have called such couplings essential in the previous section.) As a consequence of asymptotic freedom all the effective masses and effective constants of quantum interactions tend to zero at $t \to \infty$. Thus, in a strong gravitational field scalar, spinor and gauge fields in the mentioned models are described by free massless field equations which are non-minimally coupled with the torsion.

The described behaviour of the essential non-minimal couplings is based on the special form of the corresponding renormalization group equations. In particular, the equation for the non-minimal effective coupling $\eta_1(t)$ has the form

$$\frac{\mathrm{d}\eta_1(t)}{\mathrm{d}t} = c \cdot h^2(t) \cdot \eta_1(t) \tag{4.51}$$

From relation (4.37) it follows that $c > 0$. As was pointed out earlier the renormalization formula (4.37) has a universal form and does not depend on the choice of gauge group. Of course in concrete models (see, for example, model (4.45)) relation (4.37) and equation (4.51) require some modifications, but the sign of c is constant.

(d) Let us now consider the renormalization group equations in a SU(N) gauge model (4.38) which is asymptotically free for general solutions of renormalization group equations (see Chapter 3).

The renormalization group equations for the effective couplings $\eta_{1,2}(t); \xi_{1,\ldots,5}(t)$ in the gauge model (4.38) have the form

$$\begin{aligned}
&(4\pi)^2\frac{\mathrm{d}\eta_1}{\mathrm{d}t} = h^2(t)\eta_1 \qquad \frac{\mathrm{d}\eta_2}{\mathrm{d}t} = 0 \qquad (4\pi)^2\frac{\mathrm{d}\xi_1}{\mathrm{d}t} = (\xi_1 - \tfrac{1}{6})A(t)\\
&(4\pi)^2\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \xi_i A(t) + 8\eta_1 h^2(t)\delta_{i4} \qquad i \neq 1\\
&\qquad A(t) = 2mh^2(t) - 6Ng^2(t) + (N^2+1)f_1(t)\\
&\qquad\qquad + \frac{2(N^2-4)}{N}\cdot f_2(t).
\end{aligned} \tag{4.52}$$

The solutions of the equations for $g^2(t)$ and $h^2(t)$ have the form

$$\begin{aligned}
&g^2(t) = g^2 z^{-1} \qquad h^2(t) = (Pz^d + Qz)^{-1}\\
&z = 1 + \frac{b^2 g^2}{(4\pi)^2}t
\end{aligned} \tag{4.53}$$

where

$$d = \tfrac{2}{3}m - N - \frac{6}{N} \qquad Q = \frac{N^2 + 2Nm + 1}{g^2 d^2} \qquad P = 1/h^2 - Q.$$

If one takes the physical value $h > Q^{-2}$ then from (4.53) it follows that the effective coupling constant $h(t)$ tends to zero when $t \to \infty$ and $h(t) \ll g(t)$. Therefore, it is possible to investigate the equation for $f_{1,2}(t)$ and $\xi_i(t)$ taking $h(t)$ to be equal to zero. The numerical calculations show that $\bar{f}_{1,2}(t) = f_{1,2}(t)/g^2(t)$ have a UV stable fixed point for $N = 6$ and $m = 63$, for $N = 7$, $71 \leqslant m \leqslant 73$, for $N = 8$

and $80 \leqslant m \leqslant 84$, for $N = 9$, $89 \leqslant m \leqslant 94$, for $N = 10$, $97 \leqslant m \leqslant 105$ [134]. When $t \to \infty$ then the effective scalar couplings behave as $f_{1,2}(t) = f^*_{1,2} \cdot g^2(t)$, where $f^*_{1,2}$=constant. Note that the models $N = 6, m = 63$; $N = 8, m = 84$; $N = 10, m = 105$ are finite (or asymptotically finite in the case $\bar{f}_{1,2}(t) \neq f^*_{1,2}$) on the one-loop level. For asymptotically free (AF) and finite models the asymptotic behaviour of $\xi_{1,2,3,5}(t)$ is determined by the sign of the expression.

$$B^* = (N^2+1)f_1^* + \frac{2(N^2+4)}{N} f_2^* - 6N. \tag{4.54}$$

For the gauge model (4.38) $B^* < 0$. Therefore at $t \to \infty$ the effective couplings are

$$\xi_1(t) \to 1/6 \qquad \xi_{2,3,5} \to 0. \tag{4.55}$$

With the help of (4.53) the equation for $\eta_1(t)$ is written in the form

$$\frac{\mathrm{d}\eta_1}{\eta_1} = b^{-2} g^{-2} (Qz + Pz^d)^{-1}\, \mathrm{d}z. \tag{4.56}$$

It is possible to integrate (4.56) in the obvious way

$$\eta_1(t) = \eta_1 \exp\left\{ Q^{-1}(d-1)^{-1} b^{-2} g^{-2} \ln \left| \frac{(Q+P)z^{d-1}}{Q+Pz^{d-1}} \right| \right\}. \tag{4.57}$$

From (4.57) the asymptotic relation follows

$$\begin{aligned} \eta_1(\infty) = \eta_1 \exp \Bigg\{ & - \frac{N}{N^2 + 2Nm + 1} \\ & \times \ln \left| 1 - \bar{h} \frac{N^3 + 2mN^2 + N}{\frac{2}{3}mN - N^2 - 6} \right| \Bigg\} \end{aligned} \tag{4.58}$$

where $\bar{h}$ is the initial value of the effective constant $\bar{h}(t) = h(t)/g(t)$.

The effective coupling $\eta_1(t)$ has a UV stable fixed point, but the asymptotical value $\eta_1(t)$ depends on the values of h and g . The power in (4.58) has a value near 0.01 (for example, in the case of $N = 7$, $m = 72$ it is equal to 0.006 604...). If we make the choice of the value $\bar{h}$ in an appropriate way, then $\eta_1(\infty) \gg \eta_1$. In another case $\eta_1(\infty) \simeq \eta_1$.

To study $\xi_4(t)$ asymptotical behaviour we can use: (a) the existence of a stable fixed point for $\eta_1(t)$; (b) the asymptotic relation $\bar{h}(t) \ll g(t)$; (c) the existence of stable fixed points for $\bar{f}_{1,2}(t)$. Omitting the terms containing $h(t)$ we obtain for $\xi_4(t)$ the same equation as for $\xi_{2,3,5}(t)$. Consequently $\xi_4(t) \to 0$ when $t \to \infty$.

Let us make one remark concerning $\eta_1(t)$ behaviour in the framework of special solutions of renormalization group equations. One special solution is $h^2(t) = kg^2(t) = kz^{-1}$, where k=constant (see Chapter 3 and [85] for a review). Then from (4.51) it follows that $\eta_1(t) \to \infty$ when $t \to \infty$. The effective coupling $\xi_1(t) \to \infty$ in the UV limit. This remark is in a good accordance with the results obtained in the models (a), (b), (c).

The additional interesting example is the finite model. In such theories the divergences are absent at the one-loop or multi-loop level (see Chapter 3, section 3.9). In an external gravitational field with torsion there are divergences concerned with the non-minimal interactions. The renormalization group equation for $\eta_1(t)$ have the form (4.51) where $h(t) \equiv h$. It is obvious, that $\eta_1(t) \propto \eta_1 \cdot e^{hct}$, $\xi_4(t) \propto e^{hct}$ when $t \to \infty$.

And so we have shown that the effective value $\eta_1(t)$ increases in the UV limit. For the gauge models, which are AF on the general solutions of renormalization group equations $\eta_1(t)$ tends to the stable fixed point $\eta_1(\infty)$. If we choose the value for h in some special way, then $\eta_1(\infty) \gg \eta_1$. If asymptotic freedom takes place for the special solutions, $\eta_1(t) \to \infty$ in the UV limit as well as in the finite models.

We see that the effective value of essential non-minimal matter–torsion couplings is caused by the radiative corrections. One can suppose that the interaction of matter fields with torsion is very strong at small distances when $t \to \infty$. Then, owing to the radiative corrections, such an interaction disappears when the parameter t decreases. As a result we never observe torsion at long distances.

5 Renormalization Group Method and Effective Action in Curved Space–time

5.1 Introduction

The effective action (EA) is the main object of interest in quantum field theory on curved space–time. The form of the EA also determines such quantum phenomena as axial, conformal and gravitational anomalies, quantum corrections to Einstein equations and curvature induced phase transitions.

In cosmology the EA is of particular interest in constructing new scenarios of the evolution of the early universe. Firstly, effective equations (Einstein equations with quantum corrections) allow us to find singularity-free cosmological models [222–228]. Secondly, some assumptions concerning the EA form the basis of one of the models of the inflationary universe (for a review, see the book [229]).

In papers [191, 230] on the basis of effective equations, gravitational collapse has been investigated. It was noted that quantum effects can eliminate the singularity which appears at the classical level.

The authors of papers [231, 232] (see also review [17]) have shown that interaction with the external gravitational field may lead to spontaneous symmetry breaking. From their results it follows that there is the possibility of giving masses for gauge fields without the introduction of a 'negative square' mass in the scalar sector. The natural question arises: how do quantum corrections affect the spontaneous symmetry breaking in the external gravitational field?

The reasons mentioned above stimulated numerous investigations of EP (effective potential) in quantum field theory in curved space–time.

In papers [233,234] the one-loop effective potential for $f\varphi^4$-theory and scalar electrodynamics in an open and closed Robertson–Walker universe is calculated. The effective potential is calculated as the mean value of the canonical Hamiltonian over states with a definite scalar field value. The spectrum of the Laplace operator in a space with constant curvature has been used for this purpose.

In papers [235] the one-loop effective potential in a space of constant curvature at absolute zero and non-zero temperature was found and the role of curvature in the thermal phase transition was investigated. The authors considered $f\varphi^4$-theory and scalar electrodynamics.

Symmetry breaking and vacuum stability in curved space–time have been studied in papers [236–240]. The authors have found one-loop quantum corrections to the mean value of scalar field square in a Robertson–Walker universe. They have also shown that the curvature can be a symmetry-breaking factor.

In paper [218] the technique of ζ-regularization [164, 165] has been used for calculating one-loop effective potentials in $f\varphi^4$-theory and scalar electrodynamics on the de Sitter background. This technique has recently been used for the calculation of the effective potential in more complicated theories on de Sitter space or static space [219, 241–246].

The Schwinger–De Witt method has been applied to calculate the one-loop EA in paper [247] as far as terms to linear in the curvature terms. In papers [248, 249] this has been carried out including also geometrical invariants of fourth order (see also [250, 251]). The authors of paper [252] carried out partial summation of the series for the Schwinger–De Witt heat kernel. Using results [252] the authors of [76] obtained effective Einstein equations.

In papers [253–256] the one-loop effective potential in Wess–Zumino (supersymmetric) models on the background of anti-de Sitter space has been calculated.

The authors of [215] have developed the general technique of one-loop EA calculation in de Sitter space on the basis of regularization by means of a generalized ζ-function. They also discussed the Λ-term problem and calculated the one-loop potential in supergravity (SG) on the (anti) de Sitter background.

It must be noted that the methods which have been used in the EA calculations in all the papers mentioned above are not quite feasible in the general case of gauge field theory on an arbitrary curved background. On the contrary, a special choice of background metric or some resrictions on quantum gauge theory have always been used. Now we shall consider the approach to the problem of EA calculations in curved space–time which is based on the application of RGEs. The

RG method of EA calculation was first developed in the papers [14, 56, 58, 257, 258]. Our approach has the following advantages. Firstly, it becomes possible to construct an approximate scheme of EA calculation where the specific details of the Lagrangian and external gravitational field are not necessary. The specific information about the given theory is included into RG functions β and γ. Secondly, the same approach can be used for the study of EA behaviour in the strong gravitational field limit or the high energy limit. Thirdly, this approach can be easily generalized for the case where the theory is considered in curved space–time with torsion [132–134]. We will present results which have been obtained using this technique. In this chapter curvature induced phase transitions in gauge theories in curved space–time are discussed. We also consider the problem of the calculation of asymptotics of the effective action in strong fields.

In the last section of this chapter we consider a cosmological model in gravity with torsion on the basis of the effective action caused by vacuum quantum effects. We consider quantum fields in an external gravitational field with torsion and calculate the one-loop corrections to the external-field action. Then we write the equation of motion resulting from the effective action. From this action a system of equations for finding the metric and torsion arises. In a gravitational field without torsion this self-consistent approach follows on from attempts to build non-singular cosmological models (see, for example, [222, 223]). We shall follow [133] to construct a non-singular cosmological model with torsion.

5.2 The renormalization group equation and effective action

Consider an arbitrary quantum field theory containing scalar (φ), spinor (ψ) and vector (A_μ) fields (we do not write internal indices). Let us introduce $\Gamma[g_{\mu\nu}, \varphi]$ — the renormalized effective action, where spinor and vector fields are 'switched off'. In this part we will describe the method of construction of the approximate expression of $\Gamma[g_{\mu\nu}, \varphi]$. This method is based on RG equations, and its general advantage is that it does not depend on the details of the Lagrangian and external gravitational field.

Generally speaking, one can expect the EA to be a non-local functional. At the same time we can search for the EA as a power series in the background curvature and matter field derivatives. These series are evidently constructed of the local elements and therefore we can formally assume the locality of the EA.

Assuming the locality of functional $\Gamma[g_{\mu\nu}, \varphi]$ let us introduce effective Lagrangian $L_{\text{eff}}(g_{\mu\nu}, \varphi)$

$$\Gamma[g_{\mu\nu}, \varphi] = \int \mathrm{d}^4x\sqrt{-g}L_{\text{eff}}(g_{\mu\nu}, \varphi). \tag{5.1}$$

First we consider the simple case of a general EA in curved space–time, that is the effective potential in the one-loop approximation up to terms linear over scalar curvature. In this approximation the EA can be easily found by a direct solution of the renormalization group equation. We will then study the effective Lagrangian in the form of the expansion in scalar field derivatives. We shall limit ourselves to the second-order derivative terms and to geometrical invariants containing four metric derivatives.

5.2.1 *Simple example of EA derivation*

The renormalization group equations (3.67) lead directly to finding the effective potential in curved space. Consider the arbitrary massless theory with the classical potential of the form

$$V_0 = af\varphi^4 - b\xi R\varphi^2. \tag{5.2}$$

Here a and b are numerical factors (group indices are not explicitly calculated) and f is a scalar coupling constant. The effective potential V is obtained by expanding the renormalized effective action Γ in a series over scalar field derivatives (spinor and vector fields are switched off).

$$\Gamma = \Gamma^{(0)} - \int \mathrm{d}^4x\sqrt{-g}\,V + \ldots. \tag{5.3}$$

From equation (3.67) we obtain

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \delta\frac{\partial}{\partial\alpha} + \beta_\xi\frac{\partial}{\partial\xi} + \gamma\varphi\frac{\partial}{\partial\varphi}\right)V = 0. \tag{5.4}$$

Remember that λ is a set of interaction constants characterizing the theory in plane space, δ is a renormalization group function corresponding to the gauge parameter α, and $V = V_1 + V_2$, where V_1 is the effective potential in the absence of the external field and $V_2 = R\tilde{V}_2$. Both V_1 and V_2 must satisfy equation (5.4). Then from equation (5.4) we obtain

$$\begin{gathered}
(D - 4\gamma)V_1^{(4)} = 0 \qquad (D - 2\gamma)\tilde{V}_2^{(2)} = 0 \\
D = -(1-\gamma)\frac{\partial}{\partial t} + \beta_\lambda\frac{\partial}{\partial\lambda} + \delta\frac{\partial}{\partial\alpha} + \beta_\xi\frac{\partial}{\partial\xi} \\
t = \tfrac{1}{2}\ln(\varphi^2/\mu^2) \qquad V_1^{(4)} = \frac{\mathrm{d}^4V_1}{\mathrm{d}\varphi^4} \qquad \tilde{V}_2^{(2)} = \frac{\mathrm{d}^2\tilde{V}_2}{\mathrm{d}\varphi^2}.
\end{gathered} \tag{5.5}$$

Add to equations (5.5) the normalization conditions

$$\left.\frac{\mathrm{d}^2V_1}{\mathrm{d}\varphi^2}\right|_{\varphi=0} = 0 \qquad \left.\frac{\mathrm{d}^4V_1}{\mathrm{d}\varphi^4}\right|_{\varphi=\mu} = 4!f$$
$$\left.\frac{\mathrm{d}^2V_2}{d\varphi^2}\right|_{\varphi=\mu} = -2b\xi R. \tag{5.6}$$

Then from equations (5.5) it follows that

$$\frac{\mathrm{d}^4V_1(t)}{\mathrm{d}\varphi^4} = 4!af(t)\sigma^4(t) \qquad \frac{\mathrm{d}^2V_2}{\mathrm{d}\varphi^2} = -2b\xi(t)R\sigma^2(t) \tag{5.7}$$

where

$$\sigma^2(t) = \exp\left\{-2\int_0^t \bar{\gamma}_\varphi(f(t'),\alpha(t'))\,\mathrm{d}t'\right\}.$$

Here $f(t)$, $\xi(t)$ and $\alpha(t)$ are the effective coupling constants which satisfy the equations

$$\dot{f}(t) = \bar{\beta}_f(t) \qquad \dot{\xi}(t) = \bar{\beta}_\xi(t) \qquad \dot{\alpha}(t) = \bar{\delta}(t)$$
$$f(0) = f \qquad \xi(0) = \xi \qquad \alpha(0) = \alpha \tag{5.8}$$
$$(\bar{\beta},\bar{\delta},\bar{\gamma}_\varphi) = \frac{1}{1+\gamma_\varphi}(\beta,\delta,\gamma_\varphi).$$

Now we shall limit ourselves to the one-loop approximation in which

$$\bar{f}(t) = f + \beta_f t \qquad \bar{\xi}(t) = \xi + \beta_\xi t \qquad \bar{\alpha}(t) = \alpha + \delta t.$$

Then equations (5.7) take the form:

$$\frac{\mathrm{d}^4V_1}{\mathrm{d}\varphi^4} = 4!a[f + \tfrac{1}{2}(\beta_f - 4f\gamma_\varphi)\ln(\varphi^2/\mu^2)]$$
$$\frac{\mathrm{d}^2V_2}{\mathrm{d}\varphi^2} = -2bR[\xi + \tfrac{1}{2}(\beta_\xi - 2\xi\gamma_\varphi)\ln(\varphi^2/\mu^2)]. \tag{5.9}$$

Integrating equations (5.9) with the initial conditions (5.6) we obtain

$$V = V_1 + V_2 = af\varphi^4 + A\varphi^4[\ln(\varphi^2/\mu^2) - 25/6]$$
$$\qquad - b\xi R\varphi^2 - BR\varphi^2[\ln(\varphi^2/\mu^2) - 3] \tag{5.10}$$
$$A = \frac{a}{2}(\beta_f + 4f\gamma) \qquad B = \frac{b}{2}(\beta_\xi + 2\xi\gamma).$$

Equation (5.10) is the final expression for the effective potential of an arbitrary massless theory in the one-loop approximation, accurate up to terms linear in the scalar curvature. Here the first two terms give the effective potential in flat space at an arbitrary value of the gauge parameter α. The last two terms are the curvature contribution. The specific character of the theory only manifests itself in the coefficients a, b and the functions β_f, β_ξ, γ. The dependence of the effective potential on the gauge parameter α enters only via the function γ. It is also easy to obtain an expression for V in the case of gravity with torsion [259]. In this case only the expressions for A and B(5.10) need change, but not the structure of the formula. Note that it is not difficult to calculate the functions A, B in any theory for which one-loop renormalization constants are known.

We now present the expression for the functions A, B (5.10), i.e. for the effective potential, in some specific models.

1. *Scalar electrodynamics* Here $a = 1/4!$ and $b = 1/2$. We easily obtain

$$\begin{aligned} A &= (16\pi)^{-2}[\tfrac{10}{9}f^2 + 12e^4 - \tfrac{4}{3}\alpha e^2 f] \\ B &= (4\pi)^{-2}[\tfrac{1}{3}f(\xi - \tfrac{1}{6}) + \tfrac{1}{4}e^2 - \tfrac{1}{2}e^2\alpha\xi]. \end{aligned} \tag{5.11a}$$

2. *The model with action* (3.97) Here $a = 1/4!$, $b = 1/2$, $\varphi^2 = \varphi^a\varphi^a$. The functions β_f, γ are known from the results of Chapter 3 (see also [54]). Using relations (5.10) we get

$$\begin{aligned} A &= (4\pi)^{-2}[\tfrac{11}{9}f^2 + 24g^4 - 32h^4 - \tfrac{8}{3}\alpha f g^2] \\ B &= (8\pi)^{-2}[\xi(\tfrac{5}{3}f - 4\alpha g^2) - \tfrac{1}{6}(\tfrac{5}{3}f + 8h^2 - 12g^2)] \end{aligned} \tag{5.11b}$$

Note that in the asymptotically free regime $f \propto g^2$, $h^2 \propto g^2$.

3. *Models with gauge group $SU(5)$* The general Lagrangian is written in the form

$$\begin{aligned} L = &-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \tfrac{1}{2}\operatorname{Tr}(\partial_\mu\phi - ig[A_\mu, \phi])^2 \\ &- |\partial_\mu H - igA_\mu H|^2 - \frac{1}{2\alpha}(\nabla_\mu A^{a\mu})^2 - \tfrac{1}{4}\lambda_1(\operatorname{Tr}\phi^2)^2 - \tfrac{1}{2}\lambda_2\operatorname{Tr}\phi^4 \\ &- \tfrac{1}{4}\lambda_3(H^+H)^2 - \tfrac{1}{2}\lambda_4 H^+H\operatorname{Tr}\phi^2 - \tfrac{1}{2}\lambda_5 H^+_\alpha(\phi^2)^\alpha_\beta H^\beta \\ &+ \sum^{N_F}[-\bar\psi_{\mathrm{R}} i\gamma^\mu(\nabla_\mu - igA_\mu)\psi_{\mathrm{R}} - \bar\psi_{\mathrm{L}\alpha\beta} i\gamma^\mu \\ &\times (\nabla_\mu\psi^{\alpha\beta}_{\mathrm{L}} - igA_\mu{}^\alpha{}_\gamma\psi^{\gamma\beta}_{\mathrm{L}} - igA_\mu{}^\beta{}_\gamma\psi^{\alpha\gamma}_{\mathrm{L}}) \\ &- (\sqrt{2}h\bar\psi_{\mathrm{L}\alpha\beta}\psi^\alpha_{\mathrm{R}}H^\beta + \text{H.C.}) \end{aligned}$$

$$
\begin{aligned}
&+ (\tfrac{1}{4} h' \varepsilon_{\alpha\beta\rho\tau\sigma} \psi_{\mathrm{L}}^{\alpha\beta} C^{-1} \psi_{\mathrm{L}}^{\rho\tau} H^{\sigma} + \text{H.C.})] \\
&+ \sum^{n_1} [-\bar{\chi}_\alpha \mathrm{i}\gamma^\mu (\nabla_\mu \chi^\alpha - \mathrm{i} g A_\mu{}^\alpha{}_\beta \chi^\beta) \\
&- \bar{B}^\alpha_\beta \mathrm{i}\gamma^\mu (\nabla_\mu B^\beta_\alpha - \mathrm{i} g [A_\mu, B]^\beta_\alpha) - k_2 \bar{\chi}_\alpha \chi^\beta \phi^\alpha_\beta \\
&- (k_4 \bar{B}^\alpha_\beta \chi^\beta H^+_\alpha + \text{H.C.}) - k_5 \bar{B}^\alpha_\beta B^\beta_\gamma \phi^\gamma_\alpha - k_6 \bar{B}^\beta_\gamma B^\alpha_\beta \phi^\gamma_\alpha] \\
&+ \tfrac{1}{2} \xi_\phi R \cdot \mathrm{Tr}\, \phi^2 + \xi_{\mathrm{H}} R H^+ H
\end{aligned}
\tag{5.12}
$$

All the notations used above are given in [88]. The Lagrangian (5.12) in the boson sector has a 24-plet of gauge vector bosons and a 24-plet of Higgs bosons ϕ that are transforming under the adjoint representation of the group SU(5), and also a 5-plet of Higgs bosons transforming under the fundamental representation of the group SU(5). In flat space when heavy fermions are absent ($n_1 = 0$) and the number of generations of light fermions $N_F = 3$, the Lagrangian (5.12) leads to the SU(5) GUT [260]. Inclusion of one heavy fermion generation leads to the second model with asymptotical freedom [88]. When $n_1 = 1, N_F = 7$ the Lagrangian (5.12) is the first asymptotically free model of paper [88]. Note that asymptotically free models with gauge group SU(5) were first given by Fradkin and Kalashnikov [86] and studied by Chang *et al* [88] and Zhelonkin *et al* [89].

The 24-plet φ is responsible for the breaking of SU(5) symmetry. We assume that breaking SU(5) into SU(3) SU(2) U(1) takes place. Then we may consider that $\phi = \varphi\,\mathrm{diag}(1,1,1,-3/2,-3/2)$. Here $a = 15/16$, $b = 15/4$ and $15\lambda_1 + 7\lambda_2$ plays the role of f . Let $n_1 = 1$ and N_F be arbitrary. The functions $\beta_{\lambda 1}$, $\beta_{\lambda 2}$, γ_1, γ_2 are known [88]. Using relations (5.10) and the functions $\beta_{\lambda 1}$, $\beta_{\lambda 2}$, γ_1 and γ_2 we get

$$
\begin{aligned}
A = \frac{15}{32(4\pi)^2} &[15 \cdot 64 \lambda_1^2 + 1296 \lambda_1 \lambda_2 + 32 \cdot \tfrac{91}{5} \lambda_2^2 \\
&+ 75 \lambda_4^2 + 30 \lambda_4 \lambda_5 + \tfrac{375}{2} g^4 + \tfrac{7}{2} \lambda_5^2 - 300 a g^2 \lambda_1 \\
&- 140 a g^2 \lambda_2 - 28 k_2^4 + \tfrac{256}{5} (k_5^3 k_6 + k_5 k_6^3)
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&- \tfrac{48}{5} \cdot 67 k_5^2 k_6^2 - \tfrac{636}{5} (k_5^4 + k_6^4)] \\
B = \frac{15}{8(4\pi)^2} &[\xi_\phi (-10 a g^2 + 52 \lambda_1 + \tfrac{188}{5} \lambda_2) \\
&+ \tfrac{1}{6} (30 g^2 - 52 \lambda_1 - \tfrac{188}{5} \lambda_2 - 4 k_2^2 - \tfrac{92}{5} (k_5^2 + k_6^2) \\
&+ 16 k_5 k_6) + (\xi_{\mathrm{H}} - \tfrac{1}{6})(10 \lambda_4 + 2 \lambda_5)].
\end{aligned}
$$

In the asymptotically free regime [86, 88, 89], $\lambda_1 \propto g^2$, $k_1^2 \propto g^2$. The effective potential (EP) in the Georgi–Glashow model is obtained by using relations (5.13) and (5.10).

From relations (5.11)–(5.13) it follows, generally speaking, that the effective potential depends on the gauge parameter α. However when the conditions $A \propto g^4$, $B \propto g^2$ occur, as is usual in GUTs, the one-loop effective potential does not depend on α because all the terms containing α appear in the two-loop approximation.

5.2.2 *General consideration of the renormalization group based method of EA derivation*

We will obtain the effective Lagrangian in the form of an expansion in field derivatives accurate up to second-order derivatives on the matter fields and fourth-order derivatives of the metric.

In the tree approximation L_{eff} coincides with the classical Lagrangian which could be written in the form

$$L_0 = \Lambda + \kappa R + a_1 R^2 + a_2 C^2 + a_3 G + a_4 \Box R - a f \varphi^4 + b(\xi R - m^2)\varphi^2 + \rho \Box \varphi^2 - b\varphi\Box\varphi. \tag{5.14}$$

Here $a > 0$, $b > 0$ and ρ are the values defined by the basic theory, m is the scalar field mass and f is the scalar coupling constant. The constants Λ, κ, $a_1, \ldots, a_4$ are parameters of the Lagrangian of the external field, which should be introduced in order to make the theory multiplicatively renormalizable (see discussion in Chapter 3). There is also the term $\rho\Box\varphi^2$ with parameter ρ introduced into the classical Lagrangian to take account of all possible $\Box\varphi^2$-like terms in the effective Lagrangian using RGEs. In the final expression one can put $\rho = 0$, $\Lambda = \kappa = 0$, $a_1 = \ldots = a_4 = 0$. If these parameters were 'switched off' from the very beginning, the multiplicative renormalizability would be broken and in this case RGEs could not be applied.

Within the renormalization group method, we deal only with the multiplicatively renormalized theories. Therefore the effective action $\Gamma[g_{\mu\nu}, \varphi]$ does not depend on the dimensional parameter μ. This fact leads to the RGE

$$\begin{aligned} &D L_{\text{eff}}(g_{\mu\nu}, \varphi) = 0 \\ &D = \mu\frac{\partial}{\partial\mu} + \beta_\xi \frac{\partial}{\partial\xi} + \beta_p \frac{\partial}{\partial p} + \delta\frac{\partial}{\partial\alpha} \\ &\qquad + (\gamma_2 m^2 + \tilde{\gamma}_2 M^2)\frac{\partial}{\partial m^2} + \gamma_M \frac{\partial}{\partial M} + \gamma_\varphi \frac{\partial}{\partial\varphi}. \end{aligned} \tag{5.15}$$

Here γ_2 and $\tilde{\gamma}_2$ are RG functions for parameters m^2 and M^2. γ_M, γ, β_ξ, β_p are RG functions for the spinor field of mass M, for the field φ, for the non-minimal coupling constants ξ and for the set of coupling constants $p = \{\Lambda, \kappa, a_i, \rho, f, h, g, \ldots\}$ respectively; h and g are Yukawa and gauge coupling constants (h and M contain internal indices). The parameter α corresponds to the gauge breaking term $(1/2\alpha)(\nabla_\mu A^\mu)^2$ in the Lagrangian, δ is the corresponding RG function.

Although the approach given below could be easily generalized to higher order loops, we will solve equation (5.15) for L in the one-loop approximation. Within the background field method one can show that the only thing that should be done in the one-loop approximation is to calculate the expression $\ln \mathrm{Ber}\, \delta^2 S/\delta\phi\delta\phi$, where S is the renormalized action with the fixing gauge and ghost field terms taken into account, $\phi = \{\varphi, \psi, A_\mu, \ldots\}$ (dots imply the fact that there are ghost fields). $\mathrm{Ber}\, X$ is the Berezinian of supermatrix X. The Berezinian is the generalization of an ordinary determinant for the case when the matrix contains elements with different Grassman parities. For our case we find

$$L_{\text{eff}} = \sum_s L_{\text{eff}}^{(s)} \qquad s = 0, 1/2, 1. \tag{5.16}$$

Here $L_{\text{eff}}^{(s)}$ is the contribution of the field with spin s to the effective Lagrangian. We will study $L_{\text{eff}}^{(s)}$ in the form

$$\begin{aligned} L_{\text{eff}}^{(s)} &= \Lambda^{(s)} - \kappa^{(s)} R + a_1^{(s)} R^2 + a_2^{(s)} C^2 + a_3^{(s)} G + a_4^{(s)} \Box R \\ &\quad - af^{(s)}\varphi^4 + b(\xi^{(s)} R - m_{(s)}^2)\varphi^2 + \rho^{(s)}\Box\varphi^2 - b\omega^{(s)}\varphi\Box\varphi. \end{aligned} \tag{5.17}$$

Here $\Lambda^{(s)}$, $\kappa^{(s)}$, $a_1^{(s)}$, ..., $a_4^{(s)}$, $f^{(s)}$, $\xi^{(s)}$, $m_{(s)}^2$, $\rho^{(s)}$ are functions depending only on coupling constants and φ.

From Chapter 3, for these functions it follows that

$$\begin{aligned} D\Lambda^{(s)} = D\kappa^{(s)} = Da_i^{(s)} &= 0 \qquad (D + 4\gamma) f^{(s)} = 0 \\ (D + 2\gamma)\xi^{(s)} = (D + 2\gamma) m_{(s)}^2 & \\ = (D + 2\gamma)\rho^{(s)} &= (D + 2\gamma)\omega^{(s)} = 0. \end{aligned} \tag{5.18}$$

We will solve equations (5.18) for each $s = 0, 1/2, 1$. Considering the structure of expression $\ln \mathrm{Ber}(\delta^2 S/\delta\phi\delta\phi)$ within the general model [14], one can note that each term in L_{eff} (5.1) will contain special combinations of $\tau^{(s)}$-dimensional parameters, where

$$\begin{aligned} \tau^{(0)} &= 2b[m^2 - (\xi + \gamma_3/\gamma_2)R] + 12af^2\varphi^2 \\ \tau^{(1/2)} &= (M + h\varphi)^2 \qquad \tau^{(1)} = g^2\varphi^2. \end{aligned} \tag{5.19}$$

Here the relations which define the renormalization of parameters m^2 and ξ (see Chapter 3) are used. Note also, that $\beta_\xi = \gamma_2\xi + \gamma_3$ (obtained in Chapter 3).

In equation (5.18) instead of μ for each s we introduce the corresponding variable $t^{(s)}$ according to the rule

$$t^{(s)} = \tfrac{1}{2}\ln\left(\frac{\tau^{(s)}}{\mu^2}\right). \tag{5.20}$$

Then equations (5.18) become

$$\begin{aligned} \bar{D}\Lambda^{(s)} = \bar{D}\kappa^{(s)} = \bar{D}a_i^{(s)} = 0 \qquad (\bar{D} - 4\bar{\gamma})f^{(s)} = 0 \\ (\bar{D} - 2\bar{\gamma})\xi^{(s)} = (\bar{D} - 2\bar{\gamma})m^2_{(s)} = (\bar{D} - 2\bar{\gamma})\rho^{(s)} \\ = (\bar{D} - 2\bar{\gamma})\omega^{(s)} = 0 \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} \bar{D} &= \frac{\partial}{\partial t^{(s)}} - \bar{\beta}_m\frac{\partial}{\partial m^2} - \bar{\beta}_M\frac{\partial}{\partial M} - \bar{\beta}_\varphi\frac{\partial}{\partial\varphi} \\ &\quad - \bar{\beta}_\xi\frac{\partial}{\partial\xi} - \bar{\beta}_\rho\frac{\partial}{\partial\rho} - \bar{\delta}\frac{\partial}{\partial\alpha} \\ &\{\bar{\beta}_m, \bar{\beta}_M, \bar{\beta}_\varphi, \bar{\beta}_\xi, \bar{\beta}_\rho, \bar{\delta}, \bar{\gamma}\} = (1+Q)^{-1} \\ &\quad \times\{\gamma_2 m^2 + \tilde{\gamma}_2 M^2, \gamma_M M, \gamma_\varphi, \beta_\xi, \beta_\rho, \delta, \gamma\} \\ Q &= \Big\{(\gamma_2 m^2 + \tilde{\gamma}_2 M^2)\frac{\partial}{\partial m^2} + \gamma_M M\frac{\partial}{\partial M} - \gamma_\varphi\frac{\partial}{\partial\varphi} \\ &\quad - \beta_\xi\frac{\partial}{\partial\xi} - \beta_\rho\frac{\partial}{\partial\rho} - \delta\frac{\partial}{\partial\alpha}\Big\}t^{(s)}. \end{aligned} \tag{5.22}$$

We do not need the explicit form for Q . The only thing that matters, is that in the tree approximation $Q = 0$.

The general solution of equations (5.21) is written in the following form

$$\begin{aligned} &\Lambda^{(s)} = F_\Lambda^{(s)} \qquad a_i^{(s)} = F_{a_i}^{(s)} \qquad f^{(s)} = F_f^{(s)}\sigma^4(t^{(s)}) \\ &\kappa^{(s)} = F_\kappa^{(s)} \qquad \xi^{(s)} = F_\xi^{(s)}\sigma^2(t^{(s)}) \qquad m^2_{(s)} = F_m^{(s)}\sigma^2(t^{(s)}) \\ &\rho^{(s)} = F_\rho^{(s)}\sigma^2(t^{(s)}) \qquad \omega^{(s)} = F_\omega^{(s)}\sigma^2(t^{(s)}) \\ &\sigma(t^{(s)}) = \exp\Big\{-\int_0^{t^{(s)}} d\tilde{t}^{(s)}\bar{\gamma}\Big(m^2(\tilde{t}^{(s)}), M(\tilde{t}^{(s)}), \\ &\qquad \varphi(\tilde{t}^{(s)}), \xi(\tilde{t}^{(s)}), \rho(\tilde{t}^{(s)}), \alpha(\tilde{t}^{(s)})\Big)\Big\}. \end{aligned} \tag{5.23}$$

$F_\Lambda^{(s)}$, $F_\kappa^{(s)}, ..., F_\omega^{(s)}$ are arbitrary functions of the effective couplings $m^2(t^{(s)})$, $M(t^{(s)})$, $\varphi(t^{(s)})$, $\xi(t^{(s)})$, $p(t^{(s)})$, which satisfy the system of equations

$$\begin{aligned}
&\dot{m}^2(t^{(s)}) = \bar{\beta}_m(t^{(s)}) \qquad m^2(0) = m^2 \\
&\dot{M}(t^{(s)}) = \bar{\beta}_M(t^{(s)}) \qquad M(0) = M \\
&\dot{\varphi}(t^{(s)}) = \bar{\beta}_\varphi(t^{(s)}) \qquad \varphi(0) = \varphi \qquad \dot{\xi}(t^{(s)}) = \bar{\beta}_\xi(t^{(s)}) \\
&\xi(0) = \xi \qquad \dot{p}(t^{(s)}) = \bar{\beta}_p(t^{(s)}) \qquad p(0) = p \\
&\dot{\alpha}(t^{(s)}) = \bar{\delta}(t^{(s)}) \qquad \alpha(0) = \alpha.
\end{aligned} \tag{5.24}$$

To find functions $F_\Lambda^{(s)}$, $F_\kappa^{(s)}, \ldots, F_\omega^{(s)}$ we require the following boundary conditions at $t^{(s)} = 0$

$$\begin{aligned}
&b_i^{(s)}(0) = b_i^{(s)} + \delta_{s,0} C_{b_i} \qquad b_i = \{\Lambda, \kappa, a_i, f, \xi, m^2, \rho\} \\
&\omega^{(s)}(0) = 1 + \delta_{s,0} \cdot C_\omega
\end{aligned} \tag{5.25}$$

where $C_\Lambda, C_\kappa, C_\omega$ are arbitrary constants, which are zero in the tree approximation.

These constants will be determined later from the additional normalization conditions, imposed on the effective Lagrangian. It follows from relations (5.14) and (5.17) that conditions (5.25) imply that when $t = 0$, the effective Lagrangian L_{eff} becomes equal to the classical one L_{cl} accurate up to terms vanishing in the tree approximation. From (5.23–5.25) it follows that

$$b_i^{(s)} = b_i(t^{(s)}) + \delta_{s,0} C_{b_i} \qquad \omega^{(s)} = (1 + \delta_{s,0} \cdot C_\omega)\sigma^2(t^{(s)}). \tag{5.26}$$

Limiting ourselves to the one-loop approximation in the equations for the effective charges (5.24) we have

$$b_i(t^{(s)}) = b_i + \beta_{b_i} t^{(s)} \qquad \sigma(t^{(s)}) = 1 + \gamma t^{(s)}. \tag{5.27}$$

Here all RG functions β_{b_i} are calculated in the one-loop approximation. Taking into account that according to the results of Chapter 3, $\beta_\xi = \gamma_2(\xi - 1/6)$ and $\gamma_3/\gamma_2 = -1/6$ we have (see 5.19)

$$\tau^{(0)} = 2b[m^2 - (\xi - 1/6)R] + 12af\varphi^4. \tag{5.28}$$

Let us now substitute relations (5.26) and (5.27) into (5.17) and take into account (5.19) and (5.28). This leads to the following expression for the effective Lagrangian

$$
\begin{aligned}
L_{\text{eff}} = {} & \Lambda + C_\Lambda + m^4 b_\Lambda^{(0)} t^{(0)} + M^4 b_\Lambda^{(1/2)} t^{(1/2)} \\
& + (\kappa + C_\kappa + m^2 b_\kappa^{(0)} t^{(0)} + M^2 b_\Lambda^{(1/2)} t^{(1/2)}) R \\
& + (a_1 + C_{a_1} + \sum_s \beta_{a_1}^{(s)} t^{(s)}) R^2 + (a_2 + C_{a_2} + \sum_s \beta_{a_2}^{(s)} t^{(s)}) C^2_{\mu\nu\alpha\beta} \\
& + (a_3 + C_{a_3} + \sum_s \beta_{a_3}^{(s)} t^{(s)}) G + (a_4 + C_{a_4} + \sum_s \beta_{a_4}^{(s)} t^{(s)}) \Box R \\
& - a(f + C_f + \sum_s E_f^{(s)} t^{(s)}) \varphi^4 + b(\xi + C_\xi + \sum_s E_\xi^{(s)} t^{(s)}) R\varphi^2 \\
& - b(m^2 + C_m + \sum_s E_m^{(s)} t^{(s)}) \varphi^2 + (\rho + C_\rho + \sum_s E_\rho^{(s)} t^{(s)}) \Box\varphi^2 \\
& - b(1 + C_\omega + 2 \sum_s \gamma^{(s)} t^{(s)}) \varphi\Box\varphi.
\end{aligned}
\tag{5.29}
$$

Here

$$
\begin{aligned}
& E_f^{(s)} = \beta_f^{(s)} + 4f\gamma^{(s)} \qquad E_\xi^{(s)} = \gamma_2^{(s)}(\xi - \tfrac{1}{6}) + 2\xi\gamma^{(s)} \\
& E_m^{(s)} = m^2(\gamma_2^{(s)} + 2\gamma^{(s)}) + M^2\tilde{\gamma}_2^{(s)} \qquad E_\rho^{(s)} = \beta_\rho^{(s)} + 2\rho\gamma^{(s)}.
\end{aligned}
\tag{5.30}
$$

We take into account that

$$
\begin{aligned}
& \beta_\Lambda^{(0)} = b_\Lambda^{(0)} m^4 \qquad \beta_\Lambda^{(1/2)} = b_\Lambda^{(1/2)} M^4 \\
& \beta_\kappa^{(0)} = m^2 b_\kappa^{(0)} \qquad \beta_\kappa^{(1/2)} = M^2 b_\kappa^{(1/2)} \\
& \beta_\Lambda^{(1)} = 0 \qquad \beta_\kappa^{(1)} = 0.
\end{aligned}
\tag{5.31}
$$

The parameters $b_\Lambda^{(s)}$, $b_\kappa^{(s)}$, $s = 0, 1/2$ are dimensionless. The values $\beta_\Lambda^{(s)}$, $\beta_\kappa^{(s)}, \ldots, \gamma^{(s)}$ in (5.29)–(5.31) are additive contributions corresponding to its spin s into the respective RG functions. Let us now find the values of the constants $C_\Lambda, C_\kappa, \ldots, C_\omega$ in (5.29). For this purpose we impose the following normalization conditions on the Lagrangian

$$\left.\frac{\partial^4 L_{\text{eff}}}{\partial\varphi^4}\right|_{\varphi=\varphi_f,R=0} = -4!af \qquad \left.\frac{\partial^2 L_{\text{eff}}}{\partial\varphi^2}\right|_{\varphi=\varphi_m,R=0} = -2bm^2$$

$$\left.\frac{\partial^3 L_{\text{eff}}}{\partial\varphi^2\partial R}\right|_{\varphi=\varphi_0,R=R_0,C^2=C_0^2,G=G_0,\Box R=(\Box R)_0} = 2b\xi$$

$$\left.\frac{\partial L_{\text{eff}}}{\partial R}\right|_{R=0,\varphi=0} = \kappa$$

$$\left.\frac{\partial^2 L_{\text{eff}}}{\partial R^2}\right|_{\varphi=\varphi_1,R=R_1,C^2=C_1^2,G=G_1,\Box R=(\Box R)_1} = 2a_1 \tag{5.32}$$

$$\left.\frac{\partial L_{\text{eff}}}{\partial C^2}\right|_{R=R_2,\varphi=\varphi_2} = a_2 \qquad \left.\frac{\partial L_{\text{eff}}}{\partial G}\right|_{R=R_3,\varphi=\varphi_3} = a_3$$

$$\left.\frac{\partial L_{\text{eff}}}{\partial(\Box R)}\right|_{R=R_4,\varphi=\varphi_4} = a_4 \qquad \left.\frac{\partial L_{\text{eff}}}{\partial(\Box\varphi^2)}\right|_{R=0,\varphi=\varphi_\rho} = \rho$$

$$\left.\frac{\partial L_{\text{eff}}}{\partial(\varphi\Box\varphi)}\right|_{R=0,\varphi=\varphi_\omega} = -b \qquad L^{\text{eff}}|_{R=0,\varphi=0} = \Lambda.$$

Here $R = 0$ implies the transition to the flat space; φ_0, φ_1, φ_2, φ_3, φ_4, φ_f, φ_m, φ_ρ, φ_ω are some fixed values of the scalar field, R_0, R_1, R_2, R_3, R_4 are some fixed values of scalar curvature, C_0^2, C_1^2, G_0, G_1 are some fixed values of the corresponding square of the Weyl tensor and the Gauss–Bonnet density invariant. Using (5.32) in (5.29) we finally obtain

$$\begin{aligned}
L_{\text{eff}}
&= \Lambda + \tfrac{1}{2}m^4 b_\Lambda^{(0)} \ln\frac{\tau^{(0)}}{2bm^2} + \tfrac{1}{2}M^4 b_\Lambda^{(1/2)} \ln\frac{\tau^{(1/2)}}{M^2} \\
&+ \left[\kappa + \tfrac{1}{2}m^2(\xi - \tfrac{1}{6})b_\Lambda^{(0)} + \tfrac{1}{2}m^2 b_\kappa^{(0)} \ln\frac{\tau^{(0)}}{2bm^2}\right. \\
&\left. + \tfrac{1}{2}M^2 b_\kappa^{(1/2)} \ln\frac{\tau^{(1/2)}}{M^2}\right] R \\
&+ \left[a_1 + \tilde{C}_{a_1} + \tfrac{1}{2}\beta_{a_1}^{(0)} \ln\frac{\tau^{(0)}}{m_1^2} + \tfrac{1}{2}\beta_{a_1}^{(1/2)} \ln\frac{\tau^{(1/2)}}{M_1^2} + \tfrac{1}{2}\beta_{a_1}^{(1)} \ln\frac{\tau^{(1)}}{g^2\varphi_1^2}\right] R^2 \\
&+ \left[a_2 + \tfrac{1}{2}\beta_{a_2}^{(0)} \ln\frac{\tau^{(0)}}{m_2^2} + \tfrac{1}{2}\beta_{a_2}^{(1/2)} \ln\frac{\tau^{(1/2)}}{M_2^2} + \tfrac{1}{2}\beta_{a_2}^{(1)} \ln\frac{\tau^{(1)}}{g^2\varphi_2^2}\right] C^2_{\mu\nu\alpha\beta} \\
&+ \left[a_3 + \tfrac{1}{2}\beta_{a_3}^{(0)} \ln\frac{\tau^{(0)}}{m_3^2} + \tfrac{1}{2}\beta_{a_3}^{(1/2)} \ln\frac{\tau^{(1/2)}}{M_3^2} + \tfrac{1}{2}\beta_{a_3}^{(1)} \ln\frac{\tau^{(1)}}{g^2\varphi_3^2}\right] G \\
&+ \left[a_4 + \tfrac{1}{2}\beta_{a_4}^{(0)} \ln\frac{\tau^{(0)}}{m_4^2} + \tfrac{1}{2}\beta_{a_4}^{(1/2)} \ln\frac{\tau^{(1/2)}}{M_4^2} + \tfrac{1}{2}\beta_{a_4}^{(1)} \ln\frac{\tau^{(1)}}{g^2\varphi_4^2}\right] \Box R
\end{aligned}$$

$$
\begin{aligned}
&- a\Big[f + \tilde{C}_f + \tfrac{1}{2}E_f^{(0)} \ln \frac{\tau^{(0)}}{m_f^2} + \tfrac{1}{2}E_f^{(1/2)} \ln \frac{\tau^{(1/2)}}{M_f^2} + \tfrac{1}{2}E_f^{(1)} \ln \frac{\tau^{(1)}}{g^2\varphi_f^2}\Big]\varphi^4 \\
&+ b\Big[\xi + \tilde{C}_\xi + \tfrac{1}{2}E_\xi^{(0)} \ln \frac{\tau^{(0)}}{m_0^2} + \tfrac{1}{2}E_\xi^{(1/2)} \ln \frac{\tau^{(1/2)}}{M_2^2} + \tfrac{1}{2}E_\xi^{(1)} \ln \frac{\tau^{(1)}}{g^2\varphi_0^2}\Big] R\varphi^2 \\
&- b\Big[m^2 + \tilde{C}_m + \tfrac{1}{2}E_m^{(0)} \ln \frac{\tau^{(0)}}{m_m^2} + \tfrac{1}{2}E_m^{(1/2)} \ln \frac{\tau^{(1/2)}}{M_m^2} + \tfrac{1}{2}E_m^{(1)} \ln \frac{\tau^{(1)}}{g^2\varphi_m^2}\Big]\varphi^2 \\
&+ \Big[\rho + \tfrac{1}{2}E_\rho^{(0)} \ln \frac{\tau^{(0)}}{m_\rho^2} + \tfrac{1}{2}E_\rho^{(1/2)} \ln \frac{\tau^{(1/2)}}{M_\rho^2} + \tfrac{1}{2}E_\rho^{(1)} \ln \frac{\tau^{(1)}}{g^2\varphi_\rho^2}\Big]\Box\varphi^2 \\
&- b\Big[1 + \gamma^{(0)} \ln \frac{\tau^{(0)}}{m_\omega^2} + \gamma^{(1/2)} \ln \frac{\tau^{(1/2)}}{M_\omega^2} + \gamma^{(1)} \ln \frac{\tau^{(1)}}{g^2\varphi_\omega^2}\Big]\varphi\Box\varphi. \qquad (5.33)
\end{aligned}
$$

Here

$$
\begin{aligned}
m_i^2 &= 2b[m^2 - (\xi - \tfrac{1}{6})R] + 12af\varphi_i^2 \\
M_i &= M + h\varphi_i \qquad i = 0, \ldots, 4. \\
m_l^2 &= 2bm^2 + 12af\varphi_l^2 \\
M_l &= M + h\varphi_l \qquad l = (f, m, \rho, \omega).
\end{aligned} \qquad (5.34)
$$

Constants $\tilde{C}_{a_1}$, $\tilde{C}_f$, $\tilde{C}_\xi$, $\tilde{C}_m$ have the following form

$$
\begin{aligned}
\tilde{C}_{a_1} &= b(\xi - \tfrac{1}{6})[b(\xi - \tfrac{1}{6})(u_1 - aE_f^{(0)}\varphi_1^4 + k_1\varphi_1^2 \\
&\quad + R_1V_1)m_1^{-2} + W_1]m_1^{-2} \\
\tilde{C}_f &= \sum_s (A_f^{(s)}E_f^{(s)} + A_m^{(s)}E_m^{(s)} + A_\Lambda^{(s)}b_\Lambda^{(s)}) \\
\tilde{C}_\xi &= \Big\{b(\xi - \tfrac{1}{6})[(2u^2 - 5u + 2)k_0 - a\varphi_0^2(2u^2 - 9u + 12)E_f^{(0)}] \\
&\quad + 12af[b\varphi_0^2(u - 5)E_\xi^{(0)} + 2b(\xi - \tfrac{1}{6})(2u - 1)(R_0V_0 + U_0)m_0^{-2} \\
&\quad + (u - 1)W_0]\Big\} \cdot (2bm^2)^{-1} + \tfrac{1}{2}v(v - 4)E_\xi^{(1/2)} \\
&\quad + v^2(M^2b_\kappa^{(1/2)} + 2\beta_{a_1}^{(1/2)}R_0)(2b\varphi_0^2)^{-1} - \tfrac{3}{2}E_\xi^{(1)} + \beta_{a_1}^{(1)}R_0(b\varphi_0^2)^{-1} \\
\tilde{C}_m &= \sum_s (B_f^{(s)}E_f^{(s)} + B_m^{(s)}E_m^{(s)} + B_\Lambda^{(s)}b_\Lambda^{(s)}) - \frac{6a}{b}\varphi_m^2\tilde{C}_f
\end{aligned} \qquad (5.35)
$$

where the following notations are introduced

$$\begin{aligned}
A_f^{(0)} &= -q(84p^3 + 129p^2q + 94pq^2 + 25q^3)/12(p+q)^4 \\
A_m^{(0)} &= bf(-6p^3 + 21p^2q + 4pq^2 + q^3)/(p+q)^4 \\
A_\Lambda^{(0)} &= -36af^2m^4(p^2 - 6pq + q^2)/(p+q)^4 \\
A_f^{(1/2)} &= -r(48M^3 + 108M^2r + 88Mr^2 + 25r^3)/12(M+r)^4 \\
A_m^{(1/2)} &= bh^2(6M^2 + 4Mr + r^2)/12a(M+r)^4 \\
A_\Lambda^{(1/2)} &= -(hM)^4/4a(M+r)^4 \\
A_f^{(1)} &= -25/12 \qquad A_m^{(1)} = b/12a\varphi_f^2 \qquad A_\Lambda^{(1)} = 0 \\
B_f^{(0)} &= -\frac{a}{2b}\varphi_m^2[6\ln(m_m^2/m_f^2) + \omega(9p+7\omega)(p+\omega)^{-2}] \\
B_m^{(0)} &= -\omega(5p+3\omega)/2(p+\omega)^2 \\
B_\Lambda^{(0)} &= 6afm^4(p-\omega)/b(p+\omega)^2 \\
B_f^{(1/2)} &= -\frac{a}{2b}\varphi_m^2[6\ln(M_m^2/M_f^2) + Z(8M+7Z)/(M+Z)^2] \\
B_m^{(1/2)} &= -Z(4M+3Z)/2(M+Z)^2] \\
B_\Lambda^{(1/2)} &= -h^2M^4/2b(M+Z)^2] \\
B_f^{(1)} &= -\frac{a}{2b}\varphi_m^2[6\ln(\varphi_m^2/\varphi_f^2) + 7] \qquad B_m^{(1)} = -\tfrac{3}{2} \\
B_\Lambda^{(1)} &= 0 \qquad W_j = V_j + R_j\beta_{a_1}^{(0)} \\
U_j &= m^4b_\Lambda^{(0)} + \beta_{a_2}^{(0)}C_j^2 + \beta_{a_3}^{(0)}G_j + \beta_{a_4}^{(0)}(\Box R) \\
V_j &= m^2b_\kappa^{(0)} + R_j\beta_{a_1}^{(0)} \qquad k_j = b(R_jE_\xi^{(0)} - E_m^{(0)}) \qquad j = 0,1 \\
p &= 2bm^2 \qquad q = 12af\varphi_f^2 \qquad r = h\varphi_f \qquad u = 24af\varphi_0^2m_0^{-2} \\
v &= h\varphi_0M^{-1} \qquad \omega = 12af\varphi_m^2 \qquad Z = h\varphi_m.
\end{aligned} \tag{5.36}$$

Relations (5.33)–(5.36) solve the problem tackled in the beginning of this section. They present a general expression for the one-loop effective Lagrangian L_{eff} for a general theory accurate up to the second scalar field derivatives and linear on geometrical invariants $R^2, C^2, G, \Box R$ terms. The effective Lagrangian is calculated in terms of RG functions, which together with the numerical coefficients a and b depend on a specific theory. In the final expression one can put $\Lambda = \kappa = a_1 = \ldots = a_4 = \rho = 0$. All RG functions for specific theories are, as a rule, known. Therefore, to calculate effective action for a specific theory within the approximations that we use, there is no need to carry out any special investigations. The result will always be given by relations (5.33–5.36) with the specific values of coefficients a and b and specific RG functions. Note also that the relations (5.33)–

(5.36) define the effective Lagrangian for an arbitrary value of gauge parameter. The Lagrangian depends on this parameter only through the functions $\gamma^{(s)}$.

The effective Lagrangian is written in a most simple way in scalar field theory with the interaction $f\varphi^4$. Here $a = 1/4!, b = 1/2, M = 0, h = g = 0$. All RG functions are known (see, for example, [79–89]).

If we put $\varphi_i = \varphi_m = 0, i = 0, \ldots, 4, C_j^2 = G_j = (\Box R)_j = 0, j = 1, 2$, then relation (5.36) is reduced to the effective Lagrangian, obtained in [248] using rather cumbersome calculations. Moreover, the method of calculation used by the authors of [248] when applied to more complicated theories meets great calculational difficulties. Our approach gives the effective Lagrangian for a vast variety of theories.

As an example of a non-trivial application we consider scalar electrodynamics. Here $a = 1/4!, b = 1/2, g = e$, where e is electrical charge. Using the result of RG functions calculations (see, for example, [257, 261, 262]) and also calculated by us

$$\beta_\rho^{(0)} = -\frac{f}{9(4\pi)^2}$$

$$\beta_\rho^{(1)} = \frac{e^2}{(4\pi)^2}[2\rho(\alpha - 3) - \tfrac{1}{3}]$$

we obtain from (5.33)–(5.36)

$$\begin{aligned} L_{\text{eff}} = &-\frac{m^4}{2(4\pi)^2}\ln\frac{\tau^{(0)}}{m^2} - \frac{m^2}{(4\pi)^2}(\xi - \tfrac{1}{6})\left(\tfrac{1}{2} - \ln\frac{\tau^{(0)}}{m^2}\right)R \\ &+ \left[\tilde{C}_{a_1} - \frac{1}{2(4\pi)^2}(\xi - \tfrac{1}{6})^2\ln\frac{\tau^{(0)}}{m_1^2}\right]R^2 \\ &- \frac{1}{20(4\pi)^2}\left[\tfrac{1}{6}\ln\frac{\tau^{(0)}}{m_2^2} + \ln\frac{\tau^{(1)}}{e^2\varphi_2^2}\right]C^2 \\ &+ \frac{1}{360(4\pi)^2}\left[\ln\frac{\tau^{(0)}}{m_3^2} + 31\ln\frac{\tau^{(1)}}{e^2\varphi_3^2}\right]G \\ &+ \frac{1}{6(4\pi)^2}\left[(\xi - \tfrac{1}{5})^2\ln\frac{\tau^{(0)}}{m_4^2} - \tfrac{1}{5}\ln\frac{\tau^{(1)}}{e^2\varphi_4^2}\right]\Box R \end{aligned}$$

$$-\tfrac{1}{4!}\left[f + \tilde{C}_f + \frac{5f^2}{3(4\pi)^2}\ln\frac{\tau^{(0)}}{m_f^2} + \frac{1}{8\pi^2}(9e^4 - \alpha f e^2)\ln\frac{\tau^{(1)}}{e^2\varphi_f^2}\right]\varphi^4$$

$$
\begin{aligned}
&+ \tfrac{1}{2}\Big[\xi + \tilde{C}_\xi + \frac{f}{24\pi^2}(\xi - \tfrac{1}{6})\ln\frac{\tau^{(0)}}{m_0^2} \\
&+ \frac{e^2}{(4\pi)^2}(\tfrac{1}{2} - \alpha\xi)\ln\frac{\tau^{(1)}}{e^2\varphi_0^2}\Big] R\varphi^2 \\
&- \tfrac{1}{2}\Big[m^2 + \tilde{C}_m + \frac{m^2 f}{24\pi^2}\ln\frac{\tau^{(0)}}{m_m^2} - \frac{m^2 e^2\alpha}{(4\pi)^2}\ln\frac{\tau^{(1)}}{e^2\varphi_m^2}\Big]\varphi^2 \\
&- \frac{1}{6(4\pi)^2}\Big[\frac{f}{3}\ln\frac{\tau^{(0)}}{m_\rho^2} + e^2\ln\frac{\tau^{(1)}}{e^2\varphi_\rho^2}\Big]\Box\varphi^2 \qquad (5.37) \\
&- \tfrac{1}{2}\Big[1 - \frac{e^2(\alpha-3)}{(4\pi)^2}\ln\frac{\tau^{(1)}}{e^2\varphi_\omega^2}\Big]\varphi\Box\varphi \\
m_i^2 &= m^2 - (\xi - \tfrac{1}{6})R_i + f\varphi_i^2/2 \qquad i = 0,\ldots,4 \\
m_l^2 &= m^2 + f\varphi_l^2/2 \qquad l = \{f, m, \rho, \omega\} \\
\tau^{(0)} &= m^2 - (\xi - \tfrac{1}{6})R + f\varphi^2/2 \qquad \tau^{(1)} = e^2\varphi^2.
\end{aligned}
$$

The constants $\tilde{C}_{a_1}$, $\tilde{C}_f$, $\tilde{C}_\xi$, $\tilde{C}_m$ are given by relations (5.35)–(5.36) where $a = 1/4!$, $b = 1/2$.

Note, that the expression for the effective potential (5.10) is just a particular case of the general expression (5.36). Let us suppose that the scalar field is slowly varying, so all derivatives of scalar fields are omitted. We also suppose that the gravitational field is slowly varying, so we need only linear curvature terms in (5.10). If we consider the massless theory, L has the form

$$
L_{\text{eff}} = -(V_1 + RV_2) = -V \qquad (5.38)
$$

where V_1 and V_2 do not depend on gravitational field. If we use the massless renormalization conditions (5.6), expression (5.36) gives the effective potential of the form (5.10).

And so, we have developed a RG-based method derivation of the EA in an external gravitational field. One can use this method to obtain the next (in comparison with (5.37)) terms in the series for effective action. This method is really universal and can be used in the case of an arbitrary gauge model on an arbitrary background.

5.3 Curvature-induced phase transition in curved space–time

At present it is known that phase transitions take place in quantum field theories with scalar fields [263]. These phase transitions are caused by external parameters of the theory, such as temperature, external electrical fields, Higgs boson masses, etc. In some papers [218, 219, 247, 257] it is noted that phase transitions can also be induced by the external gravitational field. It this section the possibility of curvature induced phase transitions is investigated. We are only interested in the first-order phase transitions when the order parameter $\langle\varphi\rangle$ at some critical curvature R_c is quickly changed.

We write the effective potential with the help of the dimensionless variables $x = \varphi^2/\mu^2$, $y = |R|/\mu^2$

$$\frac{V}{\mu^4} = afx^2 + Ax^2(\ln x - \tfrac{25}{6}) - b\xi\varepsilon xy - B\varepsilon xy(\ln x - 3). \qquad (5.39)$$

Here $\varepsilon = \operatorname{sgn} R$. The critical parameters x_c and y_c corresponding to the first-order phase transitions are found from the condition [263, 495]

$$V(x_c, y_c) = 0 \qquad \left.\frac{\partial V}{\partial x}\right|_{x_c,y_c} = 0 \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_c,y_c} > 0. \qquad (5.40)$$

Relation (5.39) and the first two terms from (5.40) lead to the equalities

$$\begin{aligned} x_c &= qy_c \qquad q = \frac{-D \pm \sqrt{D^2 - 4A^2B^2}}{2A^2} \\ D &= (Ab\xi - afB - \frac{5AB}{6})\varepsilon. \end{aligned} \qquad (5.41)$$

Using equalities (5.41) and the first two terms from (5.40) we obtain the expression for critical curvature

$$y_c = q^{-1} \exp\left(\frac{(af - \frac{25}{6}A)q + (3B - b\xi)\varepsilon}{B\varepsilon - Aq}\right). \qquad (5.42)$$

The third relation in (5.41) leads to the condition

$$af - \tfrac{8}{3}A + A\ln x_c - \frac{B\varepsilon}{2q} > 0. \qquad (5.43)$$

It is necessary to add the condition $q > 0$ to relations (5.41) and (5.42) because in equality (5.42) x_c and y_c are positive. Besides, we must assume $x_c \gg y_c$ because otherwise it cannot limit itself by the linear curvature terms in the EP. This leads to $q \gg 1$. Then in expressions (5.41)–(5.43) we must keep only the terms which are not beyond the scope of the one-loop approximation.

Consider first relations (5.41)–(5.43) for asymptotically free theories [79–89]. From the results of section 2 it follows that in this case

$A \propto g^4$ and $B \propto g^2$. Then in the one-loop approximation $D = Ab\xi - afB \propto g^2$, $q \propto g^{-4}$, $x_c = \exp(-afA^{-1}) \propto \exp(-g^{-2})$. From these relations and condition (5.43) we find

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_c, y_c} = 0 + O(g^4).$$

But the terms $\propto g^4$ in this relation can be modified by two-loop corrections. Therefore, in the one-loop approximation

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_c, y_c} = 0.$$

Thus, the one-loop approximation appears not to be enough to study the question about the first-order phase transition in asymptotically free theories. However, an absolute minimum of the EP leading to spontaneous symmetry breaking can exist. For example, if $\xi R > 0$ then at the tree approximation the value $\varphi_0^2 = \xi R/2af$ is the potential minimum.

Consider now non-asymptotically free theories. Suppose that the condition $af = \frac{11}{3}A$ takes place where terms such as f^2, fg^2, etc, are absent in the expression for A so that $f \propto g^4$. The introduced condition is the analogue of the condition which is often used for estimates in flat space (see [261]). Note that A must be positive, otherwise this condition is contradictory. (We consider that $f > 0$; otherwise the classical flat potential is not stable.) Let $\xi \neq 0$ and $|\xi| \gg g^2$ which allows us to neglect all the terms besides the first in the expression for D. Then $D \approx Ab\xi\varepsilon$. In the same approximation $q \approx -b\xi\varepsilon A^{-1}$, $x_c \approx \exp(-0.5)$, $y_c \approx -Ax_c/b\xi\varepsilon$ and condition (5.43) is fulfilled. The condition $q \gg 1$ results in the inequality $|\xi| \ll g^{-4}$. Thus, when $g^2 \ll |\xi| \ll g^{-4}$ the theory admits the first-order phase transition induced by curvature. Note that, formally, the critical curvature y_c depends upon the gauge parameter α through the dependence of A on α. However, as was shown in section 5.2 when $f \propto g^4$ the dependence of A on α is beyond the scope of the one-loop approximation. Therefore, in the one-loop approximation the critical curvature does not depend on α. From the results obtained, as a simple particular case, the statement given in [247] in the gauge $\alpha = 0$ follows, and when $R < 0$, $\xi = 1/6$ the scalar electrodynamics admits the first-order curvature induced phase transition.

Consider the case $|\xi| \ll g^2$ or $\xi = 0$. Then A and B do not depend on α (in the one-loop approximation). From (5.41) and (5.42) we get $D \approx -\frac{9}{2}AB\varepsilon$, $q \approx 4.3B\varepsilon A^{-1}$ (in the expression for q (5.41) one should take the negative sign, otherwise the condition $q \gg 1$ will not be fulfilled), $x_c \approx \exp(-0.27)$ and $y_c \approx Ax_c/4.3B\varepsilon$. Condition (5.43) is then fulfilled. Thus, when $|\xi| \ll g^2$, $\varepsilon > 0$ a first-order curvature-induced phase transition takes place. In particular, for scalar electrodynamics when $\xi = 0$ and $R > 0$ the results of [247] are obtained from this.

We now investigate the GUT with the Lagrangian (5.10). Let $\xi_\varphi \simeq 1$ and the critical curvature $y_c \propto A \propto g^4$. It is known that $g^2 \propto 1/3$ and then $y_c \simeq 10^{-1}$. On the other hand, estimates (see [247]) show that in the GUT epoch the values y satisfy the condition $10^{-7} \leqslant |y| \leqslant 10^{-5}$. Then $y_c \simeq 10^{-1}$ does not seem to be realistic. But, as shown before [54], every time t in the universe corresponds to an effective constant $g^2(t)$. GUTs are asymptotically free as far as the constant g^2. Then $g^2(t) = g^2/(1+c\ln a(t))$ where c is a constant, g^2 is the coupling in the present epoch and $a(t)$ is the scale factor. When studying curvature-induced phase transitions in the early universe, we must use the charge $g^2(t)$ which is less than g^2. Let us estimate it for the inflationary universe where $a(t) \propto e^{Ht}$. Then $g^2(t) \propto g^2/(1+Ht)$. By the beginning of inflation $Ht > 70$ [229]. Therefore, when $g^2 \simeq 1/3$ we obtain $g^2(t) \simeq 10^{-3}$. The critical curvature adequate to this moment $y_c \simeq 10^{-6}$ does not contradict the estimate $10^{-7} \leqslant |y| \leqslant 10^{-5}$.

Now let us investigate the analogue of the dimensional transmutation [261] in curved space–time. Suppose that the theory parameters and scalar curvature are such that at $\varphi^2 - \varphi_0^2 \neq 0$ EP (5.39) has a minimum. Then the condition

$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=\varphi_0} = 0$$

results in

$$2\varphi_0^2(af - \tfrac{11}{3}A) - (b\xi - 2B)R = 0. \tag{5.44}$$

In flat space, $af = \frac{11}{3}A$ follows from this, which leads to the fact that the dimensionless parameter φ_0^2 becomes redundant (dimensional transmutation). In an external gravitational field the equality $af = \frac{11}{3}A$ does not follow from above relation. Therefore, if we suppose $af \neq \frac{11}{3}A$ the above relation allows us to define the minimum

$$\varphi_0^2 = (b\xi - 2B)R/2(af - \tfrac{11}{3}A). \tag{5.45}$$

Dimensional transmutation does not then take place, but if we suppose $af = \frac{11}{3}A$ then $\xi = 2B/b$ follows from above relation. Thus, the dimensionless parameters f and ξ are expressed through other parameters, and at the same time the independent parameter φ_0^2 appears. Dimensional transmutation takes place.

5.4 Asymptotic behaviour of the effective action in an external gravitational field

This section deals with the investigation of the asymptotics of EA and with the question of stability of the quantum field theory in curved space–time (we follow [265]). As the basis of our approach we take renormalization group equations. In flat space this approach was introduced by Voronov and Tyutin [264]. We shall study the behaviour of the EA in the following limits: (a) strong gravitational field (SGF), (b) Strong scalar fields (SSF), (c) both SGF and SSF.

Let us consider an arbitrary asymptotically free theory containing scalars φ, spinors ψ and gauge fields A^a_μ. This theory is characterized by the gauge coupling g, the Yukawa coupling h and the scalar coupling f .

The condition of multiplicative renormalizability is given in the form written in section 3.4.

If we choose the Landau gauge, there is no gauge parameter derivative in the RGE for effective action (5.15). The RGE has the form

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_p\frac{\partial}{\partial p}+m\gamma_m\frac{\partial}{\partial m}+\beta_\xi\frac{\partial}{\partial\xi}+\gamma_\varphi\varphi\frac{\partial}{\partial\varphi}\right)\Gamma=0. \tag{5.46}$$

The homogeneity condition of effective action in an external gravitational field is written in the following form

$$\Gamma[\varphi e^t, me^t, g_{\alpha\beta}e^{-2t}, \mu e^t]=\Gamma[\varphi, m, g_{\alpha\beta}, \mu] \tag{5.47}$$

From (5.47) we obtain

$$\begin{aligned}&\left\{\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\mu}+d_p p\frac{\partial}{\partial p}+d_\varphi\int \mathrm{d}^4x\,\varphi(x)\frac{\delta}{\delta\varphi(x)}\right\}\\ &\quad\times\Gamma[\varphi,p,e^{-2t}g_{\alpha\beta},\mu]=0\end{aligned} \tag{5.48}$$

$$\begin{aligned}&\left\{\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\mu}+d_p p\frac{\partial}{\partial p}+d_{\bar{\bar\varphi}}\int \mathrm{d}^4x\,\bar{\bar\varphi}(x)\frac{\delta}{\delta\bar{\bar\varphi}(x)}\right\}\\ &\quad\times\Gamma[\exp(d_{\bar\varphi}t)\bar\varphi,\bar{\bar\varphi},p,e^{-2t}g_{\alpha\beta},\mu]=0\end{aligned} \tag{5.49}$$

$$\begin{aligned}&\left\{\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\mu}+d_p p\frac{\partial}{\partial p}+d_{\bar{\bar\varphi}}\int \mathrm{d}^4x\,\bar{\bar\varphi}(x)\frac{\delta}{\delta\bar{\bar\varphi}(x)}\right.\\ &\quad\left.-2\int \mathrm{d}^4x g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}\right\}\\ &\quad\times\Gamma[\exp(\mathrm{d}_{\bar\varphi}t)\bar\varphi,\bar{\bar\varphi},p,g_{\alpha\beta},\mu]=0\end{aligned} \tag{5.50}$$

where $\varphi\equiv(\bar\varphi,\bar{\bar\varphi})$, and the division of φ into $\bar\varphi$ and $\bar{\bar\varphi}$ is arbitrary. First we consider equations (5.48) and (5.49). Combining them with

(5.46) we obtain

$$\Big\{D_1 - (\gamma - d_\varphi)\int \mathrm{d}^4x\varphi(x)\frac{\delta}{\delta\varphi(x)}\Big\}$$
$$\times\,\Gamma[\varphi, p, e^{-2t}g_{\alpha\beta}, \mu] = 0 \tag{5.51}$$
$$\Big\{D_1 - \int \mathrm{d}^4x\Big[(\bar{\bar{\gamma}} - d_{\bar{\bar{\varphi}}})\bar{\bar{\varphi}}(x)\frac{\delta}{\delta\bar{\bar{\varphi}}(x)} + \bar{\gamma}\bar{\varphi}(x)\frac{\delta}{\delta\bar{\varphi}(x)}\Big]\Big\}$$
$$\times\,\Gamma(\exp(d_{\bar{\varphi}}t)\bar{\varphi},\ \bar{\bar{\varphi}}, p, e^{-2t}g_{\alpha\beta}, \mu) = 0 \tag{5.52}$$
$$D_1 \equiv \frac{\partial}{\partial t} - (\beta_p - d_p p)\frac{\partial}{\partial p}$$

Equation (5.51) gives us the possibility of studying the asymptotics of Γ in the SGF limit. By means of equation (5.52) we can explore the behaviour of Γ in the SGF and SSF limit. Let us divide $\bar{\varphi}$ into two parts: $\bar{\varphi}_1$ and $\bar{\varphi}_2$. The choice of $\bar{\varphi}_1$ will be discussed later. It is obvious that

$$\Big\{\frac{\partial}{\partial t} + \int \mathrm{d}^4x\Big[2g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)} - \bar{\varphi}_1\frac{\delta}{\delta\bar{\varphi}_1} - \bar{\varphi}_2\frac{\delta}{\delta\bar{\varphi}_2}\Big]\Big\}$$
$$\times\,\Gamma[\exp(d_{\bar{\varphi}_1}t)\bar{\varphi}_1, \exp(d_{\bar{\varphi}_2}t)\bar{\varphi}_2, \bar{\bar{\varphi}}, p, e^{-2t}g_{\alpha\beta}, \mu] = 0 \tag{5.53}$$
$$\Big\{\frac{\partial}{\partial t} - \int \mathrm{d}^4x\Big[\bar{\varphi}_1\frac{\delta}{\delta\bar{\varphi}_1} + \bar{\varphi}_2\frac{\delta}{\delta\bar{\varphi}_2}\Big]\Big\}$$
$$\times\,\Gamma[\exp(d_{\bar{\varphi}_1}t)\bar{\varphi}_1, \exp(d_{\bar{\varphi}_2}t)\bar{\varphi}_2, \bar{\bar{\varphi}}, p, \mathrm{e}^{-2t}g_{\alpha\beta}, \mu] = 0. \tag{5.54}$$

Combining (2.2), (5.49), (5.53), and (2.2), (5.50), (5.54) we obtain

$$\Big\{D_2 - 2\frac{\bar{\gamma}_1}{1-\bar{\gamma}_1}\int \mathrm{d}^4x\, g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}\Big\}$$
$$\times\,\Gamma[\exp(d_{\bar{\varphi}}t)\bar{\varphi}, \bar{\bar{\varphi}}, p, e^{-2t}g_{\alpha\beta}, \mu] = 0 \tag{5.55}$$
$$\Big\{D_2 - \frac{2}{1-\bar{\gamma}_1}\int \mathrm{d}^4x\, g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}\Big\}$$
$$\times\,\Gamma[\exp(d_{\bar{\varphi}}t)\bar{\varphi}, \bar{\bar{\varphi}}, p, g_{\alpha\beta}, \mu] = 0 \tag{5.56}$$
$$D_2 = \frac{\partial}{\partial t} + \frac{1}{1-\bar{\gamma}_1}\Big[-(\beta_p - d_p p)\frac{\partial}{\partial p} + \int \mathrm{d}^4x(\bar{\gamma}_1 - \bar{\gamma}_2)\bar{\varphi}_2(x)\frac{\delta}{\delta\bar{\varphi}_2(x)}$$
$$-(\bar{\bar{\gamma}} - d_{\bar{\bar{\varphi}}})\bar{\bar{\varphi}}(x)\frac{\delta}{\delta\bar{\bar{\varphi}}(x)}\Big].$$

By means of equations (5.55) and (5.56) we can study the asymptotics of the EA ($t \to \infty$) in the SSF limit and both SSF and SGF limits. It is not difficult to solve equations (5.51), (5.52) and (5.56)

$$\Gamma[\varphi, p, e^{-2t}g_{\alpha\beta}, \mu] = \Gamma[\varphi(t), p(t), g_{\alpha\beta}, \mu] \tag{5.57}$$

$$\Gamma[\bar{\varphi}\exp(d_{\bar{\varphi}}t),\bar{\bar{\varphi}},p,e^{-2t}g_{\alpha\beta},\mu]=\Gamma[\bar{\varphi}(t),\bar{\bar{\varphi}}(t),p(t),g_{\alpha\beta},\mu] \quad (5.58)$$

$$\begin{aligned}
&\dot{p}(t)=\beta_p(t)-d_p p(t) && p(0)=p\\
&\dot{\varphi}(t)=(\gamma(t)-d_\varphi)\varphi(t) && \varphi(0)=\varphi\\
&\dot{\bar{\varphi}}(t)=\bar{\gamma}(t)\bar{\varphi}(t) && \bar{\varphi}(0)=\bar{\varphi}\\
&\dot{\bar{\bar{\varphi}}}(t)=\bar{\bar{\gamma}}(t)\bar{\bar{\varphi}}(t) && \bar{\bar{\varphi}}(0)=\bar{\bar{\varphi}}
\end{aligned} \quad (5.59)$$

$$\Gamma[\bar{\varphi}\exp(d_{\bar{\bar{\varphi}}}t),\bar{\varphi},p,\mathrm{e}^{-2t}g_{\alpha\beta},\mu]=\Gamma[\tilde{\bar{\varphi}}(t),\tilde{\bar{\bar{\varphi}}}(t),\tilde{p}(t),g_{\alpha\beta},\mu] \quad (5.60)$$

$$\Gamma[\bar{\varphi}\exp(d_{\bar{\varphi}}t),\bar{\bar{\varphi}},p,g_{\alpha\beta},\mu]=\Gamma[\tilde{\bar{\varphi}}(t),\tilde{\bar{\bar{\varphi}}}(t),\tilde{p}(t),\tilde{g}_{\alpha\beta}(t),\mu] \quad (5.61)$$

$$\begin{aligned}
&\dot{\tilde{p}}(t)=\frac{1}{1-\bar{\gamma}_1(t)}(\beta_p(t)-d_p\tilde{p}(t)) \qquad \tilde{p}(0)=p\\
&\tilde{\bar{\varphi}}_1(t)=\bar{\varphi}_1 \qquad \dot{\tilde{\bar{\varphi}}}_2(t)=[1-\bar{\gamma}_1(t)]^{-1}[\bar{\gamma}_2(t)-\bar{\gamma}_1(t)]\tilde{\bar{\varphi}}_2(t)\\
&\tilde{\bar{\varphi}}_2(0)=\bar{\varphi}_2\\
&\dot{\bar{\bar{\varphi}}}(t)=[1-\bar{\gamma}_1(t)]^{-1}[\bar{\bar{\gamma}}(t)-d_\varphi]\bar{\bar{\varphi}}(t) \qquad \bar{\bar{\varphi}}(0)=\bar{\bar{\varphi}}
\end{aligned} \quad (5.62)$$

$$\begin{aligned}
&\frac{\mathrm{d}g_{\alpha\beta}(t)}{\mathrm{d}t}=\frac{2\bar{\gamma}_1(t)}{1-\bar{\gamma}_1(t)}g_{\alpha\beta}(t) \qquad g_{\alpha\beta}(0,x)=g_{\alpha\beta}(x)\\
&\frac{\mathrm{d}\tilde{g}_{\alpha\beta}(t)}{\mathrm{d}t}=\frac{2}{1-\bar{\gamma}_1(t)}\tilde{g}_{\alpha\beta}(t) \qquad \tilde{g}_{\alpha\beta}(0,x)=g_{\alpha\beta}(x).
\end{aligned}$$

Note that there is the effective coupling constant $\xi(t)$ among the effective couplings. $\xi(t)$ has no analogue in flat space–time. It is interesting to emphasize the fact that among the effective couplings, which determine the asymptotic behaviour of the EA, there appears the metric of the effective gravitational field.

Equations for the effective couplings $\tilde{p}(t)$ differ from the corresponding equations for $p(t)$ by the presence of the factor $(1-\bar{\gamma}_1)^{-1}$ on the right-hand side. Using the arguments given in [264] one can easily prove that this factor does not affect the asymptotics of effective coupling constants. We will not take this factor into account in any of the subsequent discussions. Now let us explore the asymptotic behaviour of the EA. We shall limit ourselves to the case of asymptotically free theories (Chapter 3) where $g^2(t)\propto b/t$, when $t\to\infty$, $b>0$. Moreover $h^2(t)g^{-2}(t)$ and $f(t)g^{-2}(t)$ tend to constant values or zero, when $t \to \infty$. Then one can show that [264] (see also Chapter 3)

$$\gamma_\varphi \propto -a_\varphi b/t \qquad \gamma_2 \propto -a_2 b/t. \tag{5.63}$$

From equations (5.57)–(5.62) it becomes clear that the EA behaviour when $t \to \infty$ is determined by the asymptotic behaviour of the effective couplings. Since in asymptotically free theories the effective couplings $g(t)$, $h(t)$ and $f(t)$ tend to zero, the asymptotic behaviour of the EA is determined by the low order of perturbation theory with parameters which are effective couplings. From equation (5.62) one can see that $m(t) \propto e^{-t}$ and the effective masses decrease much faster than the other effective charges and do not affect the asymptotic behaviour of the EA. We have considered the behaviour of vacuum energy when $t \to \infty$ in Chapter 3. Here we investigate the EA behaviour in the matter field sector when scalars are slowly varying. The EA reduces to the EP in this case.

5.4.1 Asymptotic form of the effective potential in the SGF limit

From equations (5.57), (5.59) and (5.61) it follows that

$$\begin{aligned} V(\varphi, p, \mathrm{e}^{-2t} g_{\alpha\beta}) &\propto \mathrm{e}^{4t} V_{cl}(\varphi(t), p(t), g_{\alpha\beta}) \\ &\propto e^{4t}\xi(t) R\varphi^2(t) \\ &\propto e^{2t} t^{-2a_\varphi b} R\varphi^2 \xi(t). \end{aligned} \tag{5.64}$$

The asymptotic form of the EP in the SGF limit is determined by the non-minimal coupling of the scalar and gravitational field. The general analysis and direct calculations (Chapter 3) show that when $t \to \infty$, then

$$\xi(t) \propto \tfrac{1}{6} + (\xi - \tfrac{1}{6}) t^{-a_2 b} \text{ or } \xi(t) \to \text{constant}.$$

5.4.2 Asymptotic form of the EP in the SGF and SSF limit

From equations (5.58), (5.59) and (5.61) we obtain

$$\begin{aligned} \bar\varphi(t) \propto \bar\varphi t^{2a_\varphi} \bar\varphi^b \qquad & \bar{\bar\varphi}(t) \propto e^{-2t} \bar{\bar\varphi} \\ V(e^t\bar\varphi, \bar{\bar\varphi}, p, e^{-2t} g_{\alpha\beta}) &\propto e^{4t} V_{cl}(\bar\varphi(t), \bar{\bar\varphi}(t), p(t), g_{\alpha\beta}) \\ &\propto e^{4t}(\xi(t) R\bar\varphi^2(t) + f(t)\bar\varphi^4(t)). \end{aligned} \tag{5.65}$$

For further investigation we divide the fields $\bar\varphi$ into two sets $\bar\varphi_1$ and $\bar\varphi_2$, where $\bar\varphi_1$ are the fields with the smallest value $|a_\varphi|$. $\bar\varphi_2$ are the other fields. It is evident that when $a_{\varphi_1} > 0$, $\bar\varphi_1$ decreases more slowly than $\bar\varphi_2$ and when $a_{\bar\varphi_1} < 0$, $\bar\varphi_1$ increases faster than $\bar\varphi_2$. Thus $\bar\varphi_1$ are the only fields which determine the asymptotics of V.

If $a_{\bar{\varphi}_1} > 0$, then it is evident that the asymptotic behaviour of the EP is given by the term $\xi(t)R\bar{\varphi}_1^2(t)$. If $a_{\bar{\varphi}_1} < 0$, then, if $\xi(t)$ =constant or if $a_2 > 0$ and $a_{\bar{\varphi}_1} > -1/2b$, the asymptotic of EP is determined by the non-minimal coupling as before. If $a_{\bar{\varphi}_1} < 0$, $a_2 > 0$ or $\xi(t)$ =constant, $a_{\bar{\varphi}_1} < -1/2b$, then the main contribution to the asymptotic behaviour of the EP is $f(t)\bar{\varphi}_1^4(t)$. If $a_{\bar{\varphi}_1} < 0$ and $a_2 < 0$ then, when $a_2 - 2a_{\bar{\varphi}_1} > -1/2b$ the asymptotic behaviour of the EP is determined by the term $-f(t)\bar{\varphi}_1^4(t)$ and when $a_{\bar{\varphi}_1} < 0$, $a_2 < 0$, $a_2 - 2a_{\bar{\varphi}_1} < -1/2b$ the main contribution to the asymptotic of V is $f(t)\bar{\varphi}_1^4(t)$.

5.4.3 *Asymptotic form of the* EP *in the* SSF *limit*

From equations (5.61)–(5.63) we obtain

$$\tilde{g}_{\alpha\beta}(t,x) \propto g_{\alpha\beta}(x)e^{2t} \tag{5.66}$$

$$V(e^t\bar{\varphi},\bar{\bar{\varphi}},p,g_{\alpha\beta}) \propto e^{4t}[\xi(t)\tilde{R}(t)\bar{\varphi}_1^2(t) + f(t)\bar{\varphi}_1^4(t)] \tag{5.67}$$

where $\tilde{R}(t)$ is the effective scalar curvature. In this case the effective gravitational field is always weak ($\tilde{R}(t) \to 0$). That is why the asymptotic form of V is determined by the term $f\varphi^4$ as in flat space–time [264].

Let us consider the stability conditions of quantum field theory in curved space–time with slowly changing curvature. The stability conditions in flat space–time were obtained in [264] and are of the form

$$V_{cl}^{(0)}(\varphi(t),f(t))|_{m=0} > 0 \qquad \left.\frac{\partial^2 V_{cl}^{(0)}(\varphi(t),f(t))}{\partial\varphi^2}\right|_{m=0} \geqslant 0. \tag{5.68}$$

Hence, in particular, it follows that $f(t) > 0$, which is why the classically unstable theory can be stable owing to quantum corrections. In curved space–time, where the theory is characterized by the additional parameters ξ, the stability conditions slightly vary

$$\begin{gathered} V_{cl}^{(0)}(\varphi(t),\xi(t),f(t))|_{m=0} > 0 \\ \left.\frac{\partial^2 V_{cl}^{(0)}(\varphi(t),\xi(t),f(t))}{\partial\varphi^2}\right|_{m=0} \geqslant 0. \end{gathered} \tag{5.69}$$

From the above, taking into account the asymptotic analysis, we get $f(t) > 0$, $\xi(t)R < 0$ which differ from the classical stability conditions $f < 0$, $\xi R < 0$ (the former conditions means that the classical potential in the SGF is positive). That is why a classical unstable theory can be stable because of quantum corrections. On the other hand, even a stable theory in flat space–time can be unstable in curved space–time because when $f(t) > 0$ positive $\xi(t)R$ can be found.

As an example, we can consider the asymptotically free SU(2) model with Lagrangian (3.97). It can be shown (Chapter 3) that $\xi(t) \propto \frac{1}{6} + (\xi - \frac{1}{6}) \cdot t^{-k}$, $k > 0$. Therefore, $\xi(t) \to \frac{1}{6}$ independently of the initial conditions (asymptotic conformal invariance).

Let the scalar curvature R vary slowly and $R < 0$, $\xi R > 0$. The classical theory is unstable. The quantum stability condition yields $R < 0$, which is fulfilled. The condition $f(t) > 0$ is also fulfilled. The theory is stable because of quantum corrections.

5.5 Vacuum quantum effects and non-singular cosmological model with torsion

In this section the effective action is used to construct the non-singular cosmological model with torsion. Of course the main problem is to derive the effective action. Since the problem of obtaining the full expression for the EA is not a real one, we must look for some appropriate approximation. Here we shall follow the method of [133, 191, 266, 267] to solve this problem. The theory of matter fields conformally coupled with gravity leads to the conformal trace anomaly which is closely related to the EA. The use of the RG method enables us to reconstruct the EA up to terms which are not essential from the cosmological viewpoint. The remaining part is constructed only from torsion and is not essential for our purposes. Consider an arbitrary theory containing the scalar φ, the spinor ψ and vector field A_μ (we will not write internal indices). The behaviour of the parameters of the theory, such as the masses and coupling constants in the strong gravitational field, are determined by the RGEs. We limit ourselves to asymptotically free theories, where the effective coupling constants of quantum-field interactions and effective masses vanish. In the framework of such theories, the approximation of massless free fields is good enough for the description of quantum effects in the early universe. Then it can be shown that multiplicative renormalization of such theories in curved space–time with torsion requires the introduction of quantitatively new counterterms in comparison with plane space. These counterterms correspond to non-minimal coupling of scalars and spinors with the metric and with the pseudotrace of the torsion tensor (see Chapter 4). Suppose, in addition, that the theory under consideration is asymptotically conformally invariant. This means that the effective couplings corresponding to non-minimal coupling parameters in the Lagrangian of the theory tend to values typical of conformally invariant theories. In accordance with the above remarks the action should be chosen in the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\{\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + \tfrac{1}{12}R\varphi^2 + \tfrac{1}{2}\zeta_1 S_\mu S^\mu\varphi^2 + \mathrm{i}\bar{\psi}\gamma^\mu\nabla_\mu\psi - \zeta_2\bar{\psi}\gamma_5\gamma^\mu S_\mu\psi\}. \tag{5.70}$$

Here $S^\nu = \varepsilon^{\alpha\beta\mu\nu}T_{\alpha\beta\mu}$ is the pseudotrace of the torsion tensor; R and ∇_μ are the scalar (without torsion) curvature and the covariant (without torsion) derivative respectively; ζ_1 and ζ_2 are the non-minimal coupling constants of scalars and spinors with torsion. $\zeta_1 = 0$ and $\zeta_2 = 1/8$ correspond to minimal interaction with torsion. The action (5.70) is invariant with respect to the conformal transformations $g'_{\mu\nu} = \mathrm{e}^{2\sigma}g_{\mu\nu}$, $\varphi' = \mathrm{e}^{-\sigma}\varphi$ and $\psi' = \mathrm{e}^{-3\sigma/2}\psi$ ((5.21) and (5.22)), for any values of ζ_1 and ζ_2. The action of the free vector field A_μ does not interact with torsion even in a non-minimal way, the consideration of the vector field is of no interest for our purposes.

The only divergences arising in the theory with action (5.70) are one-loop divergences of vacuum energy. It is not difficult to establish the form of the counterterms by removing the divergences, taking into account dimension and general covariance. The renormalized action in the framework of dimensional regularization is written in the form

$$S_{\mathrm{R}} = \mu^{n-4}\Big(\int \mathrm{d}^n x\sqrt{-g}(a_i - Z_{a_i})I_i + S\Big). \tag{5.71}$$

Here S is the action (5.70) in n-dimensional space, where the coefficient in front of $\frac{1}{2}R\varphi^2$ is $(n-2)/4(n-1)$; $I_i = C^2_{\mu\nu\alpha\beta}, G, R^2, \Box R, F^2_{\mu\nu}$, $(S_\mu S^\mu)^2$, $\nabla_\mu(S_\nu\nabla^\nu S^\mu - S^\mu\nabla_\nu S^\nu)$; G is the Gauss–Bonnet invariant; $F_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$ and a_i are the parameters of the external field Lagrangian. Z_{a_i} are the renormalization constants. As in the theory there are only one-loop divergences, in the framework of minimal subtractions $Z_{a_i} = C_i/(n-4)$, where C_i depend, generally speaking, on ζ_1 and ζ_2; μ is an arbitrary mass parameter. We also introduce the bare action of the external field

$$S = \int \mathrm{d}^4x\sqrt{-g}a_i^{(0)}I_i.$$

Then the renormalized action (5.71) is obtained from the bare action $S + S_{\mathrm{ext}}$ with the help of the renormalization transformation

$$a_i^{(0)} = \mu^{n-4}(a_i + Z_{a_i}).$$

Let $\Gamma_0[a_i^{(0)}, g_{\alpha\beta}, S_\alpha]$ be the vacuum energy in the theory with the action $S+S_{\text{ext}}$ and $\Gamma[a_i, g_{\alpha\beta}, S_\alpha, \mu]$ is the vacuum energy in the theory with action (5.71). The multiplicative renormalizability condition of the theory gives

$$\Gamma_0[a_i^{(0)}, g_{\alpha\beta}, S_\alpha] = \Gamma[a_i, g_{\alpha\beta}, S_\alpha, \mu]. \tag{5.72}$$

From relations (5.72) we obtain

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_i\frac{\partial}{\partial a_i}\right)\Gamma = 0 \qquad \beta_{a_i} = \lim_{n\to 4}\mu\frac{\mathrm{d}a_i}{\mathrm{d}\mu} = -c_i.$$

Let us consider the obvious equality

$$S_{\mathrm{R}}(a_i, \mathrm{e}^{-2t}g_{\alpha\beta}, S_\alpha, \mathrm{e}^t\mu) = S_{\mathrm{R}}(a_i, g_{\alpha\beta}, S_\alpha, \mu).$$

Hence it follows that

$$\Gamma[a_i, \mathrm{e}^{-2t}g_{\alpha\beta}, S_\alpha, \mathrm{e}^t\mu] = \Gamma[a_i, g_{\alpha\beta}, S_\alpha, \mu]. \tag{5.73}$$

Differentiating equation (5.73) with respect to t and taking $t = 0$ we find

$$\mu\frac{\partial\Gamma}{\partial\mu} = 2\int \mathrm{d}^n x\, g_{\alpha\beta}(x)\frac{\delta\Gamma}{\delta g_{\alpha\beta}(x)}. \tag{5.74}$$

Substituting equation (5.74) into RG equation (5.73) and taking into account that

$$g_{\alpha\beta}(x)\frac{\delta\Gamma}{\delta g_{\alpha\beta}(x)} = -\tfrac{1}{2}\sqrt{-g(x)}g_{\alpha\beta}(x)\langle T^{\alpha\beta}(x)\rangle$$

where $\langle T^{\alpha\beta}(x)\rangle$ is the renormalized stress-energy tensor (SET), we obtain

$$\int \mathrm{d}^4x\sqrt{-g}\langle T^\alpha_\alpha(x)\rangle = \beta_i\frac{\partial\Gamma}{\partial a_i}. \tag{5.75}$$

From the structure of the renormalized action it is evident that

$$\Gamma = \int \mathrm{d}^4x\sqrt{-g}\mu^{n-4}(a_i + Z_{a_i})\cdot I_i + W$$

where W does not depend on a_i. Using this and equation (5.75) we obtain

$$\langle T^\alpha_\alpha(x)\rangle = \beta_i I_i. \tag{5.76}$$

Relation (5.76) is the general expression for the anomalous trace of the SET. The finding of conformal anomalies reduces to the calculation of the function β_i which is equivalent to the calculation of the constants Z_{a_i}.

The divergences in the theory with action (5.70) are determined by the divergences of the expression $\frac{i}{2}\,\mathrm{Tr}\ln\Delta_0 - \mathrm{i}\,\mathrm{Tr}\ln\Delta_{1/2}$, where

$$\Delta_0 = \Box - \frac{n-2}{4(n-1)}R - \zeta_1 S_\mu S^\mu \qquad \Delta_{1/2} = \gamma^\mu\nabla_\mu + \mathrm{i}\zeta_2\gamma_5\gamma^\mu S_\mu.$$

The divergences of the given expression can be found using the general algorithm [176]. Let us give the expression for the anomalous trace of the SET

$$\begin{aligned}\langle T^\alpha_\alpha(x)\rangle = (4\pi)^{-2}\Big\{ &-\frac{7}{120}C^2_{\mu\nu\alpha\beta} + \frac{1}{30}G - \frac{7}{180}\Box R \\ &+ \tfrac{2}{3}\zeta_2^2 F_{\mu\nu}(S)F^{\mu\nu}(S) - \tfrac{1}{2}\zeta_1^2(S_\mu S^\mu)^2 + \tfrac{1}{3}\left(\tfrac{1}{2}\zeta_1^2 - 2\zeta_2^2\right) \\ &\times \Box(S_\mu S^\mu) + \tfrac{2}{3}\zeta_2^2\nabla_\mu(S_\nu\nabla^\nu S^\mu - S^\mu\nabla_\nu S^\nu)\Big\}. \end{aligned}\tag{5.77}$$

Relation (5.77) shows that the anomalous trace of the SET contains a torsion contribution. When $\zeta_1 = \zeta_2 = 0$ equation (5.77) leads to the known result. From the definition of the renormalized SET and equation (5.76) we have

$$\begin{aligned}\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} &= aC^2_{\mu\nu\alpha\beta} + bG + c\Box R + dF^2 + e(S_\mu S^\mu)^2 + f\Box(S_\mu S^\mu) \\ &\quad + g\nabla_\mu(S_\nu\nabla^\nu S^\mu - S^\mu\nabla_\nu S^\nu). \end{aligned}\tag{5.78}$$

The explicit form of the coefficients a, b, c, d, e, f, g for the theory with the action (5.70) follows from equation (5.77). We shall consider these coefficients to be arbitrary parameters therefore further description is valid for an arbitrary theory.

Equation (5.78) can be considered as the equation for finding the effective action Γ. The reconstruction of the effective action using the conformal anomalies was given in [191, 266, 267]. In our case the anomalous trace contains, in comparison with [191, 266, 267], additional terms caused by torsion. Therefore, it is also necessary to study additionally the contribution of the torsion to the effective action. Let the metric have the form $g_{\mu\nu}(x) = e^{2\sigma(x)}\bar{g}_{\mu\nu}(x)$ where $\bar{g}_{\mu\nu}(x)$ is a fixed metric. Then equation (5.78) takes the form

$$\frac{\delta\Gamma}{\delta\sigma(x)} = \sqrt{-g}\Big\{a\bar{C}^2_{\mu\nu\alpha\beta} + b(\bar{G} - \tfrac{2}{3}\bar{\Box}\bar{R}) + d\bar{F}^2 + e\bar{S}^4$$

$$
\begin{aligned}
&+ f\bar{\Box}\bar{S}^2 + g\bar{\nabla}_\mu(\bar{S}_\nu\bar{\nabla}^\nu\bar{S}^\mu - \bar{S}^\mu\nabla_\nu\bar{S}^\nu) + 4b\bar{\Delta}\sigma \\
&+ (c + \tfrac{2}{3}b)(\bar{\Box}\bar{R} + 12\bar{R}^{\alpha\beta}\bar{\nabla}_\alpha\sigma\bar{\nabla}_\beta\sigma \\
&+ 2\bar{R}\bar{\Box}\sigma + 2\bar{\nabla}_\alpha\bar{R}\bar{\nabla}^\alpha\sigma + \sigma\bar{\Box}^2\sigma - 12\bar{\Box}\sigma)^2 \\
&- 12\bar{\Box}\sigma\bar{\nabla}^\alpha\sigma\bar{\nabla}_\alpha\sigma - 24\bar{\nabla}^\alpha\bar{\nabla}^\beta\sigma\bar{\nabla}_\alpha\sigma\bar{\nabla}_\beta\sigma \\
&+ 12\bar{\nabla}^\alpha\bar{\nabla}^\beta\sigma\bar{\nabla}_\alpha\bar{\nabla}_\beta\sigma] - (2f+g)\bar{\nabla}^\mu(S^2\nabla_\mu\sigma) \\
&- 2g\bar{\nabla}_\mu(\bar{S}^\mu\bar{S}^\nu\bar{\nabla}_\nu\sigma)\Big\}.
\end{aligned}
\tag{5.79}
$$

Here $\bar{S}_\mu = S_\mu$, $\bar{S}^\mu = \bar{g}^{\mu\nu}\bar{S}_\nu$, $S^2 = S_\mu S^\mu$ and

$$
\Delta = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{2}{3}R\Box + \tfrac{1}{3}\nabla_\mu R\nabla^\mu
$$

(the operator Δ was introduced in [180]). The solution of equation (5.79) is written in the form

$$
\begin{aligned}
\Gamma[\sigma, \bar{g}_{\mu\nu}, \bar{S}_\nu] = &\int d^4x\sqrt{-\bar{g}}\{\sigma[a\bar{C}^2 + b(\bar{G} - \tfrac{2}{3}\bar{\Box}\bar{R}) + d\bar{F}^2 + e\bar{S}^4] \\
&+ 2b\sigma\bar{\Delta}\sigma - f\bar{\nabla}_\mu\sigma\bar{\nabla}^\mu\bar{S}^2 - g\bar{\nabla}_\mu\sigma(\bar{S}_\nu\bar{\nabla}^\nu\bar{S}^\mu - \bar{S}^\mu\bar{\nabla}_\nu\bar{S}^\nu) \\
&+ (f + \tfrac{1}{2}g)\bar{S}^2\bar{\nabla}^\mu\sigma\bar{\nabla}_\mu\sigma + g\bar{S}^\mu\bar{S}^\nu\bar{\nabla}_\mu\sigma\bar{\nabla}_\nu\sigma \\
&- \tfrac{1}{12}(c + \tfrac{2}{3}b)[R - 6(\bar{\Box}\sigma + \bar{\nabla}_\alpha\sigma\bar{\nabla}^\alpha\sigma)]^2\} + \bar{\Gamma}[\bar{g}_{\mu\nu}, \bar{S}_\mu].
\end{aligned}
\tag{5.80}
$$

Equation (5.80) defines the dependence of the effective action Γ on the scale factor $\sigma(x)$. $\bar{\Gamma}[\bar{g}_{\mu\nu}, \bar{S}_\mu]$ is an arbitrary conformally-invariant functional. We shall also consider only conformally-plane metrics for which $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Then $\bar{\Gamma}$ depends only on S_μ and is the effective action in plane space with torsion. We shall show that in the strong gravitational field the functional $\bar{\Gamma}$ can be omitted. Let some effect be defined by a typical length l . Then the typical values of curvature invariants defining the density of the gravitational field will be l^{-4}. Passing to decreasing values of l we just pass to stronger gravitational fields. The transition to small distances is performed by the transformation $g_{\mu\nu} \to e^{-2t}g_{\mu\nu}$, $t \to \infty$ where the curvature invariants (R^2, C^2, G) grow as e^{4t}. But the limit $t \to -\infty$ is equivalent to $\sigma \to -\infty$ in equation (5.80). Therefore, for the considerations of the quantum phenomena in the strong gravitational field it is enough to leave in the effective action (5.80) only the terms which grow together with $|\sigma|$. It is obvious that the term $\bar{\Gamma}$, which does not depend on σ, is not essential here.

Let us suppose that the metric has the form $g_{\mu\nu} = a^2\eta_{\mu\nu}$, $\sigma = \ln a$, where the scale factor a depends only on the conformal time η. In accordance with the conditions of homogeneity and isotropy we shall consider that $S_\mu = (T(\eta), 0, 0, 0)$. In this notation effective action (5.80) (without the term $\bar{\Gamma}$) is written in the form

$$\Gamma = V \int d\eta [k_1 (\ln a)''^2 + k_2 (\ln a)'^4 + k_3 (\ln a)'' + k_4 T^2 (\ln a)'^2 + k_5 T^4 \ln a]. \tag{5.81}$$

Here $a' = da/d\eta$, V is the three-dimensional volume, $k_1 = -3c$, $k_2 = -(3c + 2b)$, $k_3 = f$, $k_4 = f + \frac{3}{2}g$ and $k_5 = e$. Equation (5.81) is the final expression for the effective action.

The effective action (5.81) can be considered as the quantum correction caused by vacuum effects on the classical gravitational action. The classical gravitational action depending on the metric and torsion, which is chosen to be totally antisymmetric, can be written in the form

$$S = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} (R + hS^2). \tag{5.82}$$

We shall consider h as the theory parameter. When $h = -1/24$, expression (5.82) is the action of the Einstein–Cartan theory, for arbitrary h the action (5.82) describes the non-minimal coupling of the metric and torsion. The equations of motion resulting from the action $S + \Gamma$ (5.81) and (5.82) have the following form

$$\begin{aligned} &12aa''/\kappa^2 - 4k_2[(\ln a)'^3]' + 2k_1 (\ln a)^{(4)} \\ &-2hT^2a^2/\kappa^2 + k_3 T^{2''} - 2k_4[T^2(\ln a)']' + k_5 T^4 = 0 \\ &-(h/\kappa^2)a^2 + k_3 (\ln a)''^2 + k_4 (\ln a)'^2 + 2k_5 T^2 \ln a = 0. \end{aligned} \tag{5.83}$$

We shall look for the special solution of the system (5.83) leading to the non-singular functions $a(t)$, $T(t)$. (The physical time t is connected with the conformal time η by the relation $dt = a(\eta)d\eta$.) Let us find, first of all, the solution of equation (5.83) when $k_5 = 0$. We shall use this solution as an approximation to the general case (5.83) with $k_5 \neq 0$.

The solution of the second equation of (5.83) is written in the form

$$\eta - C_1 = \pm \int da\, a^{(k_4 - k_3)/k_3} \left(C_2 + \frac{h}{\kappa^2 (k_3 + k_4)} a^l \right)^{-1/2} \tag{5.84}$$

where $l = 2 + 2k_3/k_4, C_1, C_2$ are arbitrary constants and we shall choose them equal to zero. We shall assume that $h(k_3 + k_4) > 0$. Then from equation (5.84) we find

$$a^2(\eta) = \frac{1}{H^2 \eta^2} \qquad H^2 = \frac{h}{\kappa^2 (k_3 + k_4)}. \tag{5.85}$$

In terms of the physical time we have $a(t) \propto e^{Ht}$. Thus we obtained the metric of the inflationary universe. The parameter H^2 is determined by the values h, k_3 and k_4 characterizing the torsion field. Let us proceed to the first equation (5.83) which, taking into account (5.85), has the form

$$k_3 T^{2''} + 2k_4\left(\frac{1}{\eta}T^2\right)' - \frac{2h}{\kappa^2 H^2 \eta^2}T^2 + \frac{12}{\eta^4}\left(k_1 - k_2 + \frac{2}{\kappa^2 H^2}\right) = 0. \quad (5.86)$$

The solution of this equation has the form

$$T^2(t) = d_1 e^{-2Ht} + d_2 e^{(2k_4+k_3)Ht/k_3} + Ce^{2Ht}. \qquad (5.87)$$

Here d_1 and d_2 are arbitrary constants, $C = 3(k_1 - k_2 + 2/\kappa^2 H^2)/(2k_4 - k_3)$ if $2k_4 - k_3 \neq 0$ and C is arbitrary if $2k_4 - k_3 = 0$; in the latter case it is necessary to consider that $k_1 - k_2 + 2/\kappa^2 H^2 = 0$.

Thus, we have found a non-singular solution of equations (5.83). The metric is determined by the scale factor $a(t) \propto e^{Ht}$ and the torsion is determined by equation (5.87) (see also [116]).

Consider in detail expression (5.87). The last term in (5.87) always grows when t grows. The second term grows if $(2k_4+k_3)/k_3 > 0$. It is natural to consider that torsion should vanish when time increases. Otherwise because of exponential inflation, the torsion $T^2(t)$ can become so great that it contradicts astronomical data. It is natural to suppose that $C = 0$ and if $(2k_4 + k_3)/k_3 > 0$ then $d_2 = 0$. In this case $T^2(t) \propto e^{-2Ht}$ and this is the only case we shall consider further. The relation $C = 0$ leads to $\kappa^2 H^2 = (k_2 - k_1)/2$. This relation literally coincides with the corresponding relation obtained in [222]. On the other hand, from equation (5.85) we find $\kappa^2 H^2 = h/(k_3 + k_4)$. As a result for $\kappa^2 H^2$ we have two expressions leading to the conditions $h = 2(k_3 + k_4)/(k_2 - k_1)$. The coefficients k_3 and k_4 depend on ζ_1 and ζ_2. Therefore, for example, for the theory with action (5.70) $h = 360(2\zeta_2^2 - 2\zeta_1)/13$. Thus, for the expression $T^2(t)$ (5.87) not to contain growing terms the parameters of non-minimal coupling should not all be arbitrary. The theory dictates a relation between them

$$h = 2(k_2 - k_1)^{-1}[k_3(\zeta_1, \zeta_2) + k_4(\zeta_1, \zeta_2)].$$

Let us show that the solutions $a(t) \propto e^{Ht}$ and $T^2(t) \propto e^{-2Ht}$ obtained when $k_5 \neq 0$ are approximate solutions of equations (5.83) when $k_5 \neq 0$. Substitute the functions $a(t) \propto e^{Ht}$, $T^2(t) \propto e^{-2Ht}$ into the left-hand sides of equations (5.83). Then there are the terms $k_5 e^{-4Ht}$ and $k_5 Hte^{-2Ht}$ on the left-hand side of the first and second equation (5.83) respectively. It is evident that when $Ht \geqslant 3$ these terms are numerically too small and they can approximately be put equal to zero. The condition $Ht \geqslant 3$ in terms of values means that $t > t_p$, which is the condition for the application of the whole approach. The considerations given above can be applied to the evaluation of the contribution $\bar{\Gamma}$ into self-consistent equations for a and T^2. This consideration shows that the functions $a(t) \propto e^{Ht}$ and $T^2(t) \propto e^{-2Ht}$ are approximate solutions when k_5 is present.

6 Unique Effective Action in Quantum Field Theory

6.1 Gauge invariant and gauge fixing independent effective action

The standard convenient effective action was introduced in Chapter 2, where its general properties were considered. By definition, the standard effective action is the generating functional of vertex functions and therefore it must depend on the parametrization of the quantum fields. In gauge theories it leads to dependence of the effective action on the choice of gauge conditions which are used for quantization. From the geometric point of view, this gauge dependence of the standard effective action is due to the fact that the functional integral for effective action depends on the parametrization of the space of group orbits. However, the effective action on the shell or the S-matrix does not depend on the parametrization or the choice of gauge conditions.

In [190] (see also [191]) Vilkovisky proposed a new definition of effective action which leads to the same S-matrix and is gauge-invariant and gauge-fixing-independent (see below). It was called the unique effective action. Recently it has been realized that it is possible to modify Vilkovisky's definition and therefore to introduce an infinite class of parametrization invariant effective actions which contain, as a special case, the Vilkovisky unique effective action.

Here we give a short introduction to the formalism of gauge-invariant and gauge-fixing-independent effective actions [190–195]. First of all let us consider the non-gauge bosonic theory of fields ϕ^i and construct a parametrization invariant effective action. Suppose that the field space $\{\phi^i\}$ contains a Christoffel connection $\Gamma^i_{jk}[\phi]$. (Usually, for theories of physical interest there is a natural choice of metric on the space of fields $\{\phi^i\}$.) The geodesic equation is

$$\frac{\mathrm{d}^2\phi(s)}{\mathrm{d}s^2} + \Gamma^i_{jk}[\phi(s)]\frac{\mathrm{d}\phi^j(s)}{\mathrm{d}s}\frac{\mathrm{d}\phi^k(s)}{\mathrm{d}s} = 0. \tag{6.1}$$

Let us introduce a two-point functional

$$\sigma^i(\phi(s), \phi(0)) = s\frac{\mathrm{d}\phi^i(s)}{\mathrm{d}s} \tag{6.2}$$

with boundary conditions

$$\sigma^i(\phi, \phi) = 0 \qquad \sigma^i_{;j}(\phi, \phi) = \delta^i_j \qquad \left.\frac{\delta\sigma^i(\phi_*, \phi)}{\delta\phi^j}\right|_{\phi_*=\phi} = -\delta^i_j$$

where a semicolon denotes covariant differentiation.

This functional $\sigma^i(\phi_*, \phi)$ defines the Gaussian normal coordinate of ϕ where ϕ_* is an arbitrary reference point (background field). It is important that $\sigma^i(\phi_*, \phi)$ are scalar functions of the field ϕ and transform as vectors under coordinate transformations of ϕ_*. One can show that $\sigma^i(\phi_*, \phi)$ has a Taylor expansion

$$-\sigma^i(\phi_*, \phi) = (\phi^i - \phi^i_*) + \tfrac{1}{2}\Gamma^i_{jk}(\phi^j - \phi^j_*)(\phi^k - \phi^k_*) + \dots \tag{6.3}$$

where the connection Γ^i_{jk} is calculated at the point ϕ^i_*.

Now a parametrization-invariant generating functional $W[J, \phi_*]$ can be defined [190–195] (the parametrization invariance follows from the fact that the integrand in (6.4) is a scalar with respect to field ϕ^i transformations)

$$\exp\frac{\mathrm{i}}{\hbar}W[J, \phi_*] = \int \mathrm{D}\phi \, \exp\frac{i}{\hbar}(S[\phi] - J_i\sigma^i(\phi_*, \phi)). \tag{6.4}$$

If J_i is taken to lie in the cotangent space at ϕ_*, $W[J, \phi_*]$ is invariant under coordinate transformations of ϕ_*. Note that for the conventional generating functional we use $J_i\phi^i$ (which is not scalar) instead of $-J_i\sigma^i$. It leads to the parametrization non-invariance of the conventional generating functional.

Let us perform the Legendre transform of W

$$\hat{\Gamma}[v^i; \phi_*] \equiv W[J, \phi_*] - J_i v^i \qquad v^i \equiv \frac{\delta W[J, \phi_*]}{\delta J_i}. \tag{6.5}$$

Using expressions (6.4) and (6.5) we obtain

$$\exp\frac{\mathrm{i}}{\hbar}\hat{\Gamma}[v^i;\phi_*] = \int \mathrm{D}\phi \exp\frac{\mathrm{i}}{\hbar}\left(S[\phi] + \frac{\delta\hat{\Gamma}[v^i;\phi_*]}{\delta v^i}\right)$$
$$\times\,(\sigma^i(\phi_*,\phi) + v^i). \tag{6.6}$$

From equations (6.3)–(6.6) it follows that

$$\frac{\delta\hat{\Gamma}[v^i;\phi_*]}{\delta v^i} = -J_i \qquad v^i = -\langle\sigma^i(\phi_*,\phi)\rangle_J. \tag{6.7}$$

It is easy to see that $\hat{\Gamma}[v^i;\phi_*]$ does not depend on the quantum field parametrization. (This follows from (6.5) and the parametrization invariance of W.) However, it does depend on ϕ_*^i. It means that equation (6.6) defines an infinite class of parametrization invariant effective actions. Moreover, one can show [190–195] that $\hat{\Gamma}$ is the generating functional of one-particle irreducible correlation functions for the operator $\sigma^i(\phi_*,\phi)$. It also has the usual physical interpretation as the minimal expectation value of the corresponding Hamiltonian [194].

Differentiating both sides of equation (6.6) with respect to ϕ_*^i one can obtain the identity [192, 193] which contains the dependence of $\hat{\Gamma}$ upon ϕ_*

$$\frac{\delta\hat{\Gamma}[v^i;\phi_*]}{\delta\phi_*^i} - C_i^j[\phi_*]\frac{\delta\hat{\Gamma}[v^i;\phi_*]}{\delta v^j} = 0. \tag{6.8}$$

Here $C_i^j[\phi_*] \equiv \langle\sigma^j_{;i}(\phi_*,\phi)\rangle_J$. It is possible to find $\sigma^j_{;i}$ as a Taylor expansion in terms of $\sigma^k(\phi_*,\phi)$ [192, 193].

Let us define the average field $\bar{\phi}^i$ by the following equation

$$-\sigma^i(\phi_*,\bar{\phi}) = v^i. \tag{6.9}$$

Then equation (6.6) can be rewritten in the form

$$\exp\frac{\mathrm{i}}{\hbar}\hat{\Gamma}[\bar{\phi};\phi_*] = \int \mathrm{D}\phi \exp\frac{\mathrm{i}}{\hbar}\Bigg\{S[\phi] - \frac{\delta\hat{\Gamma}[\bar{\phi};\phi_*]}{\delta\bar{\phi}^j}$$
$$\times\,(D^{-1})_i^j(\sigma^i(\phi_*,\phi)) - \sigma^i(\phi_*,\bar{\phi}))\Bigg\} \tag{6.10}$$

where $D_j^i[\bar{\phi};\phi_*] = \delta\sigma^i(\phi_*,\bar{\phi})/\delta\bar{\phi}^j$ [194]. Sometimes it is convenient to have a single representative of the infinite class of effective actions

$\hat{\Gamma}$. In particular, we can consider the unique effective action [190, 191] at $\phi_* = \bar{\phi}$ (sometimes called the Vilkovisky effective action)

$$\Gamma_{\mathrm{V}}[\bar{\phi}] \equiv \hat{\Gamma}[\bar{\phi};\bar{\phi}]. \tag{6.11}$$

Then we obtain

$$\exp\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{V}}[\bar{\phi}] = \int \mathrm{D}\phi \, \exp\frac{\mathrm{i}}{\hbar}\left\{S[\phi] + \frac{\delta\Gamma_{\mathrm{V}}[\bar{\phi}]}{\delta\bar{\phi}^j}\,\sigma^j(\bar{\phi},\phi)\right\}. \tag{6.12}$$

The Vilkovisky effective action was analysed by De Witt [192, 193] who proposed some modifications of the original formulation. The modified definition (sometimes called the Vilkovisky–De Witt effective action) coincides with (6.12) at the one-loop level. However, it was shown [196] that as compared with effective action (6.11), the Vilkovisky–De Witt effective action does not lead to non-local divergences in the Yang–Mills theory at the two-loop level. Moreover, effective action (6.11) does not generate one-particle irreducible Green's function [197]. The Vilkovisky–De Witt effective action is defined in the following way: we must calculate $\hat{\Gamma}$ according to equation (6.6) or (6.10), then let ϕ_* tend to $\bar{\phi}$. Then we obtain [192, 193]

$$\begin{aligned}\exp\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{VD}}[\bar{\phi}] &= \hat{\Gamma}[0;\bar{\phi}] = \lim_{\phi_*\to\bar{\phi}} \exp\frac{\mathrm{i}}{\hbar}\hat{\Gamma}[\bar{\phi};\phi_*] \\ &= \int \mathrm{D}\phi \, \exp\frac{\mathrm{i}}{\hbar}\left\{S[\phi] + \frac{\delta\Gamma_{\mathrm{VD}}[\bar{\phi}]}{\delta\bar{\phi}^j}(C^{-1})^j_i\,\sigma^i(\bar{\phi},\phi)\right\}\end{aligned} \tag{6.13}$$

where $(C^{-1})^i_j C^j_k = \delta^i_k$ and $\langle\sigma^i(\phi_* = \bar{\phi},\phi)\rangle_J = 0$. $\Gamma_{\mathrm{VD}}[\bar{\phi}]$ admits an energy interpretation [194] and is calculable perturbatively from one-particle irreducible graphs [197]. However, $\Gamma_{\mathrm{VD}}[\bar{\phi}]$ does not generate amputated one-particle irreducible correlation functions of any operator [195] and the argument of Γ_{VD} has no direct interpretation as the expectation value of a well-defined operator (see, for example, [195]).

In the one-loop approximation, the unique effective action (6.1) coincides with the Vilkovisky–De Witt effective action

$$\Gamma^{(1)}_{\mathrm{V}}[\bar{\phi}] = \Gamma^{(1)}_{\mathrm{VD}}[\bar{\phi}] = S[\bar{\phi}] + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln S_{;ij}[\bar{\phi}]. \tag{6.14}$$

In this book some examples of the calculation of one-loop Vilkovisky (Vilkovsky–De Witt) effective action in quantum gravity are given (see, references [154, 198–200, 202–210, 220, 221]).

As has already been shown, the definition of a parametrization invariant effective action requires the choice of the connection on the space of fields. The great progress made by Vilkovisky [190, 191] was in finding a suitable connection for gauge theories. The explicit construction of a connection on the space of gauge fields made it possible to construct off-shell gauge-fixing independent, gauge invariant and parametrization independent effective actions.

Let us present the Vilkovisky–De Witt construction (6.13) in gauge theories with closed gauge algebra (for a review, see [195, 201, 202, 211]). The criterion that determines the connection is [190]

$$\begin{aligned} \sigma^i_{;k}(\bar\phi,\phi)R^k_\alpha[\bar\phi] &\propto R^i_\alpha[\phi] \\ \frac{\delta\sigma^i(\bar\phi,\phi)}{\delta\phi^k}R^k_\alpha[\phi] &\propto R^i_\alpha[\bar\phi]. \end{aligned} \tag{6.15}$$

Here $R^i_\alpha[\phi]$ are the generators of gauge group, $(\delta S/\delta\phi^i)R^i_\alpha[\phi] \equiv 0$. The conditions (6.15) can be solved by the affine connection [190, 191]

$$\begin{aligned} \mathcal{T}^i_{mn} = \Gamma^i_{mn} &- 2R^i_{\alpha(m}\gamma_{n)k}R^k_\beta N^{-1\alpha\beta} \\ &+ N^{-1\alpha\delta}R^j_\delta\gamma_{jm}N^{-1\beta\gamma}R^l_\gamma\gamma_{ln}R^k_{(\beta}R^i_{\alpha),i} \end{aligned} \tag{6.16}$$

where $\gamma_{ij}[\phi]$ is a local metric satisfying $\gamma_{mn,i}R^i_\alpha + R^i_{\alpha,m}\gamma_{in} + R^i_{\alpha,n}\gamma_{mi} = 0$, $N_{\alpha\beta} = R^i_\alpha\gamma_{ij}R^j_\beta$, $N^{\alpha\beta}N_{\beta\gamma} = \delta^\alpha_\gamma$, [190] and $\gamma_{mn,i}$ corresponds to the covariant derivative associated with Γ^i_{mn}, where Γ^i_{mn} is the connection caused by γ_{mn}. The covariant derivative associated with $\mathcal{T}^i_{mn}$ has been abbreviated now as a semicolon.

The gauge-invariant and gauge fixed independent Vilkovisky–De Witt effective action has the form

$$\begin{aligned} \exp\frac{\mathrm{i}}{\hbar}\Gamma_{\mathrm{VD}}[\bar\phi] = \int \mathrm{D}\phi \exp\frac{\mathrm{i}}{\hbar}\Big\{ & S[\phi] + \tfrac{1}{2}\eta_{\alpha\beta}\chi^\alpha[\phi;\bar\phi]\chi^\beta[\phi;\bar\phi] \\ & - \mathrm{i}\,\mathrm{Tr}\ln\theta^\alpha_\beta[\phi;\bar\phi] + \frac{\delta\Gamma_{\mathrm{VD}}[\bar\phi]}{\delta\bar\phi^j}(C^{-1})^i_j(\bar\phi)\sigma^j(\bar\phi,\phi)\Big\} \end{aligned} \tag{6.17}$$

where the Faddeev–Popov operator is

$$\theta^\alpha_\beta[\phi;\bar\phi] \equiv \frac{\delta\chi^\alpha[\phi;\bar\phi]}{\delta\phi^i}R^i_\beta[\phi].$$

It is possible to prove that this effective action is indeed gauge fixing independent and gauge invariant as follows from [196] (see also [190–195]). The original Vilkovisky effective action can be obtained from

(6.17) substituting $(C^{-1})^i_j \to \delta^i_j$. Note that in (6.17) $\sigma^i(\bar{\phi}, \phi)$ is defined with the help of $\mathcal{T}^i_{mn}$ (not with the help of Γ^i_{mn} as in (6.3)).

Finally we write the family of gauge invariant and gauge fixing independent effective actions as a generalized form of (6.6)

$$\exp \frac{\mathrm{i}}{\hbar}\hat{\Gamma}[v^i; \phi_*] = \int D\phi \exp \frac{\mathrm{i}}{\hbar}\Big\{ S[\phi] + \tfrac{1}{2}\eta_{\alpha\beta}\chi^\alpha \chi^\beta - \mathrm{i}\,\mathrm{Tr}\ln \theta^\alpha_\beta + \frac{\delta\hat{\Gamma}[v^i; \phi_*]}{\delta v^i}\left(\sigma^i(\phi_*, \phi) + v^i\right)\Big\}. \tag{6.18}$$

Gauge invariance and gauge fixing independence have been proved in [195]. In fact, if a suitable local metric which leads to an affine connection is known then the effective action (6.18) can be constructed. The choice of the suitable metric in the space of fields and the uniqueness of this metric are now very important points in gauge invariant effective action formalism. (It is possible to construct different metrics, for a more detailed discussion see reviews [192–195].) We assume that this metric (and the corresponding connection) is known. We adopt a pragmatic point of view and investigate one-loop gauge invariant and gauge-fixing independent effective action choosing Vilkovisky or Vilkovisky–De Witt effective action as representative of such actions in quantum gravity and calling it unique effective action.

Let us discuss the relation of the one-loop unique effective action to the standard one. The unique (Vilkovisky or Vilkovisky–De Witt) effective action is gauge invariant and gauge fixing independent. So, we can use any gauge for the calculation of it. The Landau–De Witt gauge appears to be most useful

$$R_{i\alpha}[\phi](\bar{\phi}^i - \phi^i) = 0. \tag{6.19}$$

Then, using (6.19) the corresponding Taylor expansion for $\sigma^i(\bar{\phi}, \phi)$ has the form (cf. with (6.3))

$$\begin{aligned}\sigma^i(\bar{\phi}, \phi) &= (\bar{\phi}^i - \phi^i) - \tfrac{1}{2}\mathcal{T}^i_{jk}(\bar{\phi}^j - \phi^j)(\bar{\phi}^k - \phi^k) + O(\mathcal{T}^3) \\ &\equiv (\bar{\phi}^i - \phi^i) - \tfrac{1}{2}\Gamma^i_{jk}(\bar{\phi}^j - \phi^j)(\bar{\phi}^k - \phi^k) + \ldots.\end{aligned} \tag{6.20}$$

As noted by Fradkin and Tseytlin [198] (see also [196]), if there exists a parametrization in which γ_{ij} is field independent then $\Gamma^i_{jk} = 0$. Thus, in this parametrization $\sigma^i(\bar{\phi}, \phi) = \bar{\phi}^i - \phi^i$ and the one-loop unique effective action coincides with the standard one calculated in the Landau–De Witt gauge. Therefore to calculate the one-loop unique effective action in the Yang–Mills theory (with matter) it is sufficient to calculate the standard effective action in the Landau–De Witt gauge (there is a parametrization of the Yang–Mills theory where γ_{ij} is field independent [190]). For theories with a non-trivial metric (such as gravity), the unique effective action does not coincide with the standard effective action calculated in Landau–De Witt gauge already at one-loop. However, in the case of spherically symmetric backgrounds (de Sitter), and also for quantum gravity [198], arguments such as those above lead to the same conclusion. Thus, the one-loop unique effective action in quantum gravity on a de Sitter background coincides with the standard one calculated in the Landau–De Witt gauge. Explicit calculations will be presented in section 2. Note also that discussion of one-loop unique effective action will be given in Chapter 10 for quantum Kaluza–Klein theory.

6.2 Unique effective action in de Sitter space

Let us discuss one-loop effective action in Euclidean de Sitter space (S^4) (such a discussion is also useful in connection with Coleman's wormhole proposal [498])

$$\begin{aligned} &R_{\lambda\mu\nu\rho} = \tfrac{1}{3}\Lambda(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}) \qquad R_{\mu\nu} = \Lambda g_{\mu\nu} \\ &\int \mathrm{d}^4x\sqrt{g} = 24\pi^2/\Lambda^2. \end{aligned} \tag{6.21}$$

In this discussion we will follow the paper [215] (see, also [216–219]) and will consider the scalar electrodynamics and Einstein gravity as an example.

Let us present the discussion of ζ-regularization in de Sitter space. Introducing the decompositions of vectors and tensors:

$$\begin{aligned} A_\mu &= A_\mu^\perp + \nabla_\mu\chi \qquad \nabla^\mu A_\mu^\perp = 0 \\ h_{\mu\nu} &= \bar{h}_{\mu\nu} + \tfrac{1}{4}g_{\mu\nu}h \qquad \bar{h}_{\mu\nu}g^{\mu\nu} = 0 \\ \bar{h}_{\mu\nu} &= \bar{h}_{\mu\nu}^\perp + \nabla_\mu\xi_\nu^\perp + \nabla_\nu\xi_\mu^\perp + \nabla_\mu\nabla_\nu\sigma - \tfrac{1}{4}g_{\mu\nu}\Box\sigma \end{aligned} \tag{6.22}$$

where χ, h, σ are the scalars. We define the operators Δ_S

$$\begin{aligned} \Delta_0\phi &= (-\Box + X)\phi \\ \Delta_1^{\mu\nu}A_\nu^\perp &= (-g^{\mu\nu}\Box + g^{\mu\nu}X)A_\nu^\perp \\ \Delta_{2\alpha\beta}^{\mu\nu}\bar{h}_{\mu\nu}^\perp &= (-\delta_{\alpha\beta}^{\mu\nu}\Box + \delta_\alpha^\mu\delta_\beta^\nu X)\bar{h}_{\mu\nu}^\perp. \end{aligned} \tag{6.23}$$

The spectra of these operators is [215–219]

$$
\begin{aligned}
&s=0;\ (n,0) \text{ representation of SO(5)}\\
&\qquad \bar\lambda_n \equiv 3\lambda_n/\Lambda = n^2+3n+3X/\Lambda\\
&\qquad\qquad d_n = \tfrac{1}{6}(n+1)(n+2)(n+3) \qquad n=0,1,\ldots
\end{aligned}
\tag{6.24}
$$

$$
\begin{aligned}
&s=1;\ (n,1);\ A^{\perp}_{\mu}\\
&\qquad \bar\lambda_n = n^2+3n-1+3X/\Lambda\\
&\qquad\qquad d_n = \tfrac{1}{2}n(n+3)(2n+3) \qquad n=1,2,\ldots
\end{aligned}
\tag{6.25}
$$

$$
\begin{aligned}
&s=2;\ (n,2);\ h^{\perp}_{\mu\nu}\\
&\qquad \bar\lambda_n = n^2+3n-2+3X/\Lambda\\
&\qquad\qquad d_n = \tfrac{5}{6}(n-1)(n+4)(2n+3) \qquad n=2,3,\ldots
\end{aligned}
\tag{6.26}
$$

The general structure of the one-loop effective action on a de Sitter background has the form

$$\Gamma^{(1)} \propto \ln\det(\Delta_s/\mu^2). \tag{6.27}$$

The parameter $\rho^2=\frac{1}{3}\Lambda$ is the scale of the eigenvalues of Δ_s

$$\Delta_s\varphi_n = \lambda_n\varphi_n \qquad \lambda_n = \Lambda\bar\lambda_n/3. \tag{6.28}$$

Then, the generalized ζ-function [164, 165] is defined as (d_n is the multiplicity of λ_n)

$$\zeta(p)=\sum_n d_n/\bar\lambda_n^p \qquad \zeta'=\frac{\mathrm{d}\zeta}{\mathrm{d}p} \tag{6.29}$$

for $\mathrm{Re}p>2$ and by analytic continuation in all other points. For the constrained operators (6.23) we have [215]

$$\zeta_s(p)=\tfrac{1}{3}(2s+1)F(p,2s+1,(s+1/2)^2,b_s) \tag{6.30}$$

where

$$F(p,k,a,b)=\sum_{\nu=\frac{1}{2}k+1}^{\infty}\frac{\nu(\nu^2-a)}{(\nu^2-b)^p} \qquad \mathrm{Re}p>2$$

$$b_0=\tfrac{9}{4}-\frac{3X}{\Lambda} \qquad b_1=\tfrac{13}{4}-\frac{3X}{\Lambda} \qquad b_2=\tfrac{17}{4}-\frac{3X}{\Lambda}.$$

It is not difficult to show [215] that

$$
\begin{aligned}
F'(p=0) &= \tfrac{1}{4}b^2-\tfrac{1}{12}b-\tfrac{1}{8}bk(k+2)\\
&\quad -\tfrac{1}{2}\int_0^b \mathrm{d}z(z-a)\psi(\tfrac{1}{2}k+1\pm\sqrt{z})+\text{constant}.
\end{aligned}
\tag{6.31}
$$

Here $\psi(x \pm y) \equiv \psi(x+y) + \psi(x-y)$, $\psi(x)$ is a logarithmic derivative of the Γ-function. Remember, that in the ζ-function prescription the regularized one-loop effective action is

$$\Gamma(\Delta_s) \propto \tfrac{1}{2} B_4 \ln(\rho^2/\mu^2) - \tfrac{1}{2}\zeta'(0). \tag{6.32}$$

Here

$$B_4 = (16\pi^2)^{-1} \int \bar{b}_4 \sqrt{g} \mathrm{d}^4 x$$

$\bar{b}_4$ is the well-known De Witt coefficient (or a_2 coefficient) (see Chapter 3). The B_4 coefficient is well-known for all theories which are of physical interest, so expressions (6.31) and (6.32) are enough to calculate one-loop effective action in the form proposed by Fradkin and Tseytlin [215]. We note that in equation (6.29) it was assumed that the sum goes over all modes of Δ including negative and zero ones (for more details see [215]). Other regularization in de Sitter space has been discussed in [214].

Let us calculate the one-loop unique effective action in scalar electrodynamics with the Lagrangian

$$L = \tfrac{1}{4} F_{\mu\nu}^2 + \tfrac{1}{2}(\nabla_\mu \varphi^a + g\varepsilon^{ab}\varphi^b A_\mu)^2 + \frac{\lambda}{4!}(\varphi^a\varphi^a)^2 + \tfrac{1}{2}\xi R \varphi^a \varphi^a \tag{6.33}$$

on the classical de Sitter background. Note that the standard effective action in scalar electrodynamics in de Sitter space has been investigated in [215, 218, 219].

The Landau–De Witt gauge for the theory with Lagrangian (6.33) has the form

$$\nabla^\mu A_\mu - g\varepsilon^{ab}\phi^a\varphi^b = 0 \tag{6.34}$$

where ϕ^a is the constant background scalar and A_μ, φ^b are quantum fields. Then the gauge-fixing Lagrangian is

$$L_{\mathrm{GF}} = \frac{1}{2\alpha}(\nabla_\mu A^\mu - g\varepsilon^{ab}\phi^a\varphi^b)^2 \tag{6.35}$$

(at the end of the calculations α tends to zero).

We must now rewrite the quadratic part of the Lagrangian $L + L_{\mathrm{GF}}$ on quantum fields using the decomposition (6.22). Using the following relations from [215]

$$\begin{aligned} \det(-\Box + X)_{A_\mu} &= \det \Delta_1 \det \Delta_0(X - \Lambda) \\ \mathrm{d}A_\mu &\to \mathrm{d}A_\mu^\perp \mathrm{d}\chi [\det \Delta_0(0)]^{1/2} \end{aligned} \tag{6.36}$$

and subsequent Gaussian integration over $A_\mu^\perp, \chi$ we can find the final result in the limit $\alpha \to 0$

$$\begin{aligned}\Gamma^{(1)} &= \tfrac{1}{2}\ln\det\Delta_1(\Lambda + g^2\phi^2) - \ln\det\Delta_0(g^2\phi^2) \\ &\quad + \tfrac{1}{2}\ln\det\Delta_0(4\xi\Lambda + \lambda\phi^2/2) \\ &\quad + \tfrac{1}{2}\Bigg\{\ln\det\Delta_0\left(\frac{a+\sqrt{a^2-4g^4\phi^4}}{2}\right) \\ &\quad + \ln\det\Delta_0\left(\frac{a-\sqrt{a^2-4g^4\phi^4}}{2}\right)\Bigg\}. \end{aligned} \tag{6.37}$$

where $a = 4\xi\Lambda + (\lambda\phi^2/6) + 2g^2\phi^2$ and $\phi^2 = \phi^a\phi^a$. We must note that decompositions like (6.22) introduce additional zero modes not present for an initially unconstrained operator. In the Fradkin and Tseytlin approach we adopt the prescription [215]

$$\det\Delta = (\det J)^{-1}\Big(\prod_s \det\Delta_s \det' J\Big)$$

where the prime means that zero modes are to be omitted. Then

$$B_4(\Delta) = \sum_s \zeta_s(0) - \bar{n}(J)$$

where $\bar{n}(J)$ is the number of zero modes of the Jacobian.

Using (6.32) in (6.37) we get the final expression for the unique effective action for scalar electrodynamics in de Sitter space

$$\begin{aligned}\Gamma &= 24\pi^2[\tfrac{1}{24}\lambda x^2 + 2\xi x] + \tfrac{1}{2}B_4\ln(\Lambda/3\mu^2) \\ &\quad - \tfrac{1}{6}\Bigg\{3\left[\tfrac{1}{4}b_1^2 - \tfrac{47}{24}b_1 - \tfrac{1}{2}\int_0^{b_1} dz(z-\tfrac{9}{4})\psi(\tfrac{5}{2}\pm\sqrt{z})\right] \\ &\quad - 2\left[\tfrac{1}{4}b_0^{(1)2} - \tfrac{11}{24}b_0^{(1)} - \tfrac{1}{2}\int_0^{b_0^{(1)}} dz(z-\tfrac{1}{4})\psi(\tfrac{3}{2}\pm\sqrt{z})\right] \\ &\quad + \sum_{i=2,3,4}\left[\tfrac{1}{4}b_0^{(i)2} - \tfrac{11}{24}b_0^{(i)} - \tfrac{1}{2}\int_0^{b_0^{(i)}} dz(z-\tfrac{1}{4})\psi(\tfrac{3}{2}\pm\sqrt{z})\right]\Bigg\} \end{aligned} \tag{6.38}$$

where

$$\begin{aligned}B_4 &= -\tfrac{2}{45} + 24\xi^2 - 8\xi + \tfrac{1}{4}x^2(9g^4 + \tfrac{5}{6}\lambda^2 + 2\lambda g^2) \\ &\quad + x(\tfrac{3}{2}g^2 + 12\xi g^2 + 4(\xi - 1/6)\lambda) \\ x &= \phi^2/\Lambda \\ b_0^{(1)} &= \tfrac{9}{4} - 3g^2x \qquad b_0^{(2)} = \tfrac{9}{4} - 12\xi - \frac{3\lambda}{2}x \\ b_0^{(3,4)} &= \tfrac{9}{4} - \frac{3}{\Lambda}\left(\frac{a\pm\sqrt{a^2-4g^4\phi^4}}{2}\right).\end{aligned}$$

It is difficult to analyse effective equations obtained from this effective action so we do not present this analysis here.

As the second example we consider the Einstein quantum gravity on de Sitter background

$$L = -(R - 2\Lambda_0)/K^2. \tag{6.39}$$

The gauge-fixing Lagrangian corresponding to the Landau–De Witt gauge has the form

$$L_{\mathrm{GF}} = \frac{1}{2\alpha K^2}(\nabla^\mu \bar{h}_{\mu\nu} - \tfrac{1}{4}\nabla_\nu h)^2. \tag{6.40}$$

Let us write the bilinear part of the Einstein Lagrangian on S^4 using expressions (6.22) and L_{GF} (6.40)

$$\begin{aligned} L_2 = \frac{1}{2K^2}\Big\{ & \tfrac{1}{2}\bar{h}^\perp \Delta_2(\tfrac{8}{3}\Lambda - 2\Lambda_0)\bar{h}^\perp + 2(\Lambda - \Lambda_0)\xi^\perp \Delta_1(-\Lambda)\xi^\perp \\ & - \tfrac{3}{16}\Big[\sigma\Box(\Box + \tfrac{4}{3}\Lambda)(-\Box + 4\Lambda_0 - 4\Lambda + \frac{3\Box}{\alpha} + \frac{4\Lambda}{\alpha})\sigma \\ & + 2(1 - 1/\alpha)\sigma\Box(\Box + \tfrac{4}{3}\Lambda)h + h(-\Box - \tfrac{4}{3}\Lambda_0 + \frac{1}{3\alpha}\Box)h\Big]\Big\}. \end{aligned} \tag{6.41}$$

We use the fact that the one-loop standard effective action in the Landau–De Witt gauge ($\alpha \to 0$) for quantum gravity on the de Sitter background coincides with the unique effective action (see [198, 215]). Then, after integrating over quantum fields, taking into account the ghost contribution and Jacobians (as in the case of scalar electrodynamics) [215], we obtain

$$\begin{aligned} \Gamma^{(1)} = {} & \tfrac{1}{2}\ln\det\Delta_2(\tfrac{8}{3}\Lambda - 2\Lambda_0) - \tfrac{1}{2}\ln\det\Delta_1(-\Lambda) \\ & - \tfrac{1}{2}\ln\det\Delta_0(-2\Lambda) + \tfrac{1}{2}\ln\det\Delta_0(-2\Lambda_0). \end{aligned} \tag{6.42}$$

This is our second example of a calculation of the one-loop unique effective action. The analysis of this expression in a slightly modified regularization and also taking account of the number of zero Killing modes of ghost action is given in [214]. Note that attempts to investigate the unique effective action in de Sitter space have been presented for quantum gravity with matter in [212] and for $N = 1$ supergravity in [213].

Finally, let us make some remarks concerning the problems which appear in the Vilkovisky–De Witt formalism. First of all, one can see that we have a one-parameter dependent family of unique effective actions. In this sense, the 'unique effective action' is not unique. Moreover, there are other possibilities for off-shell extensions of the S-matrix. Of course, the effective action has no real physical meaning (only the S-matrix has) and we can use any of these effective actions for real calculations.

The other remark is connected with the local metric γ_{mn}. This metric was introduced in [190] where the conditions for this metric were presented.

1. γ_{mn} should be ultralocal (no derivatives).
2. γ_{mn} should satisfy Killing's equation (see (6.16)).
3. The matrix $N_{\alpha\beta}$ should be non-degenerate.

Also, an additional requirement has been given in [190]. It has been suggested that the metric should be fixed in the space of fields in accordance with the coefficients of the highest space–time derivatives contained in the classical action. This requirement is not necessary. Without it, the local metric in quantum gravity has the following form

$$\gamma_{mn} = \frac{\sqrt{g}}{4}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - ag^{\alpha\beta}g^{\mu\nu}). \tag{6.43}$$

The additional requirement fixes a. However, for different models of quantum theory (Einstein, R^2-gravity), it gives different values of a. For example, for Einstein gravity it gives $a = 1$. (This value of a was used in this section as well as in chapter 10.)

The calculation of the unique effective action in quantum gravity with the metric (6.43) leads to an explicit a-dependence of the unique effective action [499] (configuration-space metric dependence). That is why in real calculations one must have some guiding principle to determine a single value of a. One can take Vilkovisky's additional requirement as such a principle. However, the question is still open: is this the fundamental physical principle? If it is not we have a dilemma: which is better the gauge dependence of the standard effective action or the configuration-space metric dependence of the unique effective action? It seems, finally, that the choice of formalism (gauge dependent or gauge independent) used in direct calculations is only a question of convenience. The S-matrix and other physical quantities should be the same for both approaches.

7 Effective Action of Composite Fields in Curved Space–time

7.1 Introduction

Composite fields play a very important role in many applications of quantum field theory. For example, if the elementary background field is equal to zero and the composite field is not equal to zero then dynamical generation of particle masses is possible (for a review, see [268–270]).

In order to study the Green's functions containing composite fields the effective action of composite fields which generates these Green's functions was introduced in [268]. This effective action for a scalar field has the well-known form [268].

$$\exp\frac{\mathrm{i}}{\hbar}W[J,K] = \int \mathrm{D}\varphi\,\exp\frac{\mathrm{i}}{\hbar}\left(S[\varphi] + \varphi J + \tfrac{1}{2}\varphi K\varphi\right)$$

$$\Gamma[\Phi,G] = W[J,K] - J\frac{\delta W[J,K]}{\delta J} - K\frac{\delta W[J,K]}{\delta K}.$$

Here $\varphi J \equiv \int \mathrm{d}^4x\,\varphi(x)J(x)$ and $\varphi K\varphi \equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\,\varphi(x)K(x,y)\varphi(y)$. The effective equations (when sources are equal to zero) are

$$\frac{\delta\Gamma[\Phi,G]}{\delta\Phi} = 0 \qquad \frac{\delta\Gamma[\Phi,G]}{\delta G} = 0.$$

In comparison with the effective action of elementary fields an additional effective equation appears and this leads to some new consequences. For example, we can look at the solutions of the above equations like: $\Phi = 0$, $G \neq 0$ [268]. Such solutions can indicate dynamical symmetry breaking. Note that the standard effective action is $\Gamma[\Phi,G]$ at $K = 0$.

The effective action of composite fields has the well-known loop expansion in the form [268]

$$\Gamma[\Phi, G] = S[\Phi] + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln DG^{-1} + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\,D^{-1}(\Phi)G$$
$$+\,\Gamma_2[\Phi, G] + \text{ constant.}$$

Here

$$\mathrm{i}D^{-1}(\Phi) = \frac{\delta^2 S}{\delta\Phi^2} = \mathrm{i}D^{-1} + \frac{\delta^2 S_{\mathrm{int}}}{\delta\Phi^2} \qquad \Gamma_2[\Phi, G]$$

is the sum of two-particle-irreducible vacuum diagrams.

In the usual treatment of the inflationary universe based on the scalar field sector of a GUT, it is assumed that the expectation value of some component of the Higgs scalar field has a non-zero value corresponding to symmetry breaking. However, as was discussed in [271], the expectation value of Φ will usually be zero because Φ and $-\Phi$ are equally probable. To find the solution of this problem, the effective action of composite fields in curved space–time is very useful [271] (see, also [239]). Thus, the effective action of composite fields appears very naturally in connection with the early universe.

In this chapter we discuss some results connected with the effective action of composite fields in curved space–time. We consider the $O(N)$-model, the Gross–Neveu model, and quantum electrodynamics with composite fermions in curved space–time. Also, we formulate renormalization group equations for the effective action of composite fields and construct gauge and the parameterization invariant effective action of composite fields. We must mention that the general structure of the effective action of composite fields is not so clear as the structure of EA of elementary fields. Since there are different formulations of the EA of composite fields (see, for example, [272, 273, 295]).

7.2 $O(N)$-Model in curved space–time

Let us consider now the $O(N)$-model in curved space–time with slowly-varying curvature R. The Lagrangian has the form

$$L = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^a - \tfrac{1}{2}(m^2 + \xi R)\varphi^a\varphi^a - \frac{\lambda}{4!\,N}(\varphi^a\varphi^a) \qquad (7.1)$$

where $a = 1, \ldots, N$. We want to find the EA of composite fields in the theory with Lagrangian (7.1) for $N \to \infty$. The loop expansion for the EA has the same structure as in flat space. It is easy to see that formal calculations are the same as in flat space [268, 274]. Putting $G^{ab}(x,y) = \delta^{ab} g(x,y)$[268] and repeating the calculations of [268] we obtain

$$\begin{aligned}\Gamma[\Phi,\chi] = \int & \mathrm{d}^4x\sqrt{-g}\big(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\Phi^a\partial_\nu\Phi^a - \tfrac{1}{2}(m^2+\xi R)\Phi^2 \\ & - \frac{\lambda}{4!N}\Phi^4 + \frac{3N}{2\lambda}(\chi - m^2 - \xi R - \frac{\lambda}{6N}\Phi^2)^2\big) \\ & + \frac{\mathrm{i}}{2}N\,\mathrm{Tr}\ln(\Box + \chi). \end{aligned} \tag{7.2}$$

Here $\Phi^2 = \Phi^a\Phi^a$, $\chi = m^2 + \xi R + (\lambda\Phi^2/6N) + (\lambda/6)g(x,x)$ and $g(x,y)$ is defined by the self-consistent equation

$$\begin{aligned} g^{-1}(x,y) = \mathrm{i}\Big(\Box + m^2 + \xi R + \frac{\lambda\Phi^2}{6N} + \frac{\lambda}{6}g(x,x)\Big) \\ \times \frac{\delta(x-y)}{\sqrt{-g}}. \end{aligned} \tag{7.3}$$

In order to calculate $\mathrm{Tr}\ln(\Box + \chi)$ we will use the local-momentum representation of propagators [28]. Then calculating $\mathrm{Tr}\ln(\Box + \chi)$ to the accuracy of terms linear in the curvature it is necessary to use the propagator $(k^2 - m^2 - (\xi - 1/6)R)^{-1}$ [28] instead of the standard one. Then for constant fields Φ, χ we obtain [275]

$$\begin{aligned} V[\Phi,\chi] = -\Big(\Gamma[\Phi,\chi]\Big/\int \mathrm{d}^4x\sqrt{-g}\Big) &= -\frac{3N}{2\lambda}\chi^2 + \frac{3N}{\lambda}\chi(m^2+\xi R) \\ & + \tfrac{1}{2}\chi\Phi^2 + \frac{N(\chi - R/6)}{64\pi^2}[\ln(\chi - R/6)\mu^{-2} - \tfrac{1}{2}] \\ & + C_1R\Phi^2 + C_2R\chi. \end{aligned} \tag{7.4}$$

When obtaining the renormalized potential (7.4) the natural normalization condition is used

$$\frac{\partial V_{\text{quantum}}[\Phi,\chi]}{\partial\chi}\Big|_{\chi=\mu^2, R=0} = 0$$

where μ is the normalization point. However, to find the constants C_1 and C_2 it is necessary to add normalization conditions which have no flat analogues. Let us choose these conditions in a form leading to vanishing C_1 and C_2

$$\begin{aligned} \frac{\partial^2 V}{\partial\Phi\partial R}\Big|_{R=0,\Phi=\Phi_0} &= 0 \\ \frac{\partial^2 V}{\partial\chi\partial R}\Big|_{\chi=\mu^2,R=0} &= -\frac{3N\xi}{\lambda} + \frac{N}{192\pi^2}. \end{aligned} \tag{7.5}$$

Now we write the effective equations for effective potential (7.4)

$$\frac{\partial V[\Phi,\chi]}{\partial \Phi} = 0 \qquad \frac{\partial V[\Phi,\chi]}{\partial \chi} = 0. \tag{7.6}$$

The first of equations (7.6) has the solution $\Phi = 0$ or $\chi = 0$. If $\chi = 0$, then the solution of the second equation (7.6) has the form

$$\Phi^2 = -\frac{6N}{\lambda}(m^2 + \xi R) + \frac{2NR}{192\pi^2} \ln\left(-\frac{R}{6\mu^2}\right). \tag{7.7}$$

When $\xi R < 0$, $R < 0$, $m^2 < |\xi R|$ expression (7.7) is real. If $\Phi = 0$ then the second equation of (7.6) has the form

$$-\frac{3\chi}{\lambda} + \frac{3}{\lambda}(m^2 + \xi R) + \frac{(\chi - R/6)}{32\pi^2} \ln \frac{(\chi - R/6)}{\mu^2} = 0 \tag{7.8}$$

One can solve equation (7.8) by an iterative method. However, it seems that the solution of equation (7.8) is outside the range of validity of our approximation (see discussion in [274]). Thus, solution (7.8) corresponds to spontaneous O(N)-symmetry breaking.

In this section we have presented a simple example of the theory where the possibility of dynamical symmetry breaking without elementary fields ($\Phi = 0$) exists. Note also that it is possible to find EA in O(N)-model beyond the linear curvature approximation (for example, in de Sitter space [275]) or in anisotropic space–time [276].

7.3 Gross–Neveu model in curved space–time

As has already been discussed, the restoration of a broken symmetry in quantum field theory can occur as a phase transition caused by external fields. However, usually phase transitiona are investigated in field models containing scalars (section 5.3, for example). For example, in such models a phase transition induced by an external gravitational field is possible.

This section is devoted to the study of the possibility of the phase transition induced by curvature in a field model without scalars. Our investigation has been given in the framework of the Gross–Neveu model [277] in the external gravitational field. This model is rather simple from a calculation point of view and it reflects some features of a realistic field theory such as renormalizability, asymptotic freedom and dynamical breaking of chiral symmetry. Some authors [278–282] have considered phase transitions in the Gross–Neveu model.

They have shown that broken chiral symmetry can be restored at high temperature, increase in fermion-number density, change of the electromagnetic field or increase of the external gravitational field. Note that the composite field $\langle\bar{\psi}\psi\rangle$ has been considered here as an order parameter. We will closely follow [282] in this section.

The Gross–Neveu model in an external gravitational field is described by the action

$$S = \int \mathrm{d}^2x\sqrt{-g}\Big(\bar{\psi}\mathrm{i}\gamma^\mu(x)\nabla_\mu\psi + \sqrt{\lambda/N}\bar{\psi}\psi\sigma - \tfrac{1}{2}\sigma^2\Big) \tag{7.9}$$

where $\gamma^\mu(x)$ are two-dimensional Dirac matrices in curved space–time, $\nabla_\mu\psi$ is the covariant derivative of the spinor ψ, σ is the auxiliary field. If we eliminate σ from the action (7.9) with the help of the equation of motion $\sigma = \sqrt{\lambda/N}\bar{\psi}\psi$ then we obtain the standard form of the Gross–Neveu model action [277].

Consider the generating functional of connected Green's functions of σ

$$\mathrm{e}^{iW[J]} = \int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi\,\mathrm{D}\sigma\,\exp\left[\mathrm{i}\left(S + \int \mathrm{d}^2x\sigma J\right)\right]. \tag{7.10}$$

Then we define the generating functional of 1PI Green's functions $\Gamma(\Sigma)$ according to the rule

$$\Gamma(\Sigma) = W[J] - \int \mathrm{d}^2x\,\Sigma J$$

where the source J is expressed in terms of the Σ field from the equation $\delta W/\delta J = \Sigma$. From expression (7.10) and from the definition of $\Gamma(\Sigma)$ we find

$$\mathrm{e}^{\mathrm{i}\Gamma(\Sigma)} = \int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi\,\mathrm{D}\sigma\,\exp\left[\mathrm{i}\left(S + \int \mathrm{d}^2x(\Sigma-\sigma)\delta\Gamma(\Sigma)/\delta\Sigma\right)\right]. \tag{7.11}$$

We shall substitute the variable for $\sigma - \Sigma \to \sigma$ in the path-integral and keep only one-loop terms

$$\Gamma(\Sigma) = -\tfrac{1}{2}\Sigma^2 - \mathrm{i}\,\mathrm{Sp}\ln\left(\mathrm{i}\gamma^\mu(x)\nabla_\mu + \sqrt{\lambda/N}\Sigma\right). \tag{7.12}$$

Equation (7.12) is the formal expression for Gross–Neveu effective action in the one-loop approximation

Notice that from equation (7.10) and from the definition $\Gamma(\Sigma)$ the equality

$$\Sigma \frac{\delta\Gamma}{\delta\Sigma}\frac{1}{\sqrt{-g}} = \sqrt{\lambda/N}\langle\bar{\psi}\psi\rangle_J$$

follows. Here the right-hand side is the mean value for a non-zero source. It follows from this equation that for the stationary point Σ_0 of Γ (i.e. $\delta\Gamma(\Sigma_0)/\delta\Sigma_0 = 0$) $\langle\sigma\rangle \equiv \Sigma_0 = \sqrt{\lambda/N}\langle\bar{\psi}\psi\rangle$ is satisfied. Thus, the use of the mean value of the composite field $\bar{\psi}\psi$ as the order parameter is equivalent to the use of the order parameter $\Sigma_0 = \langle\sigma\rangle$.

We can write $\Gamma(\sigma)$ (the argument of the effective action is designated by σ) in the functional series

$$\Gamma(\sigma) = \int \mathrm{d}^2x\sqrt{-g}\Big(- V(\sigma) + \tfrac{1}{2}Z(\sigma)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \ldots\Big).$$

Assume that the gravitational field and σ field change slowly so that all derivatives of σ may be neglected. Then the effective action $\Gamma(\sigma)$ is determined only by the effective potential $V(\sigma)$. It is enough to calculate $V(\sigma)$ for a constant-curvature space. From equation (7.12) we have

$$V(\sigma) = \tfrac{1}{2}\sigma^2 + \mathrm{i}N\,\mathrm{Tr}\ln\langle x \mid \mathrm{i}\gamma^\mu\nabla_\mu + m \mid x\rangle \tag{7.13}$$

where m is $\sqrt{\lambda/N}\sigma$ and Tr includes summation over spinor indices.

For the calculation of the effective potential (7.13) it is convenient to find first the derivative $V' = \partial V/\partial\sigma$. From (7.13) we obtain

$$V'(\sigma) = \sigma + \mathrm{i}\sqrt{\lambda N}\,\mathrm{Tr}\,S(x, x/\sigma) \tag{7.14}$$

where $S(x, y/\sigma)$ is the Green's function of the Dirac equation in two-dimensional space of constant curvature

$$(\mathrm{i}\gamma^\mu\nabla_\mu - m)S(x, y/\sigma) = \frac{\delta(x-y)}{\sqrt{-g(x)}}. \tag{7.15}$$

Let us calculate now the Green's function in terms of Schwinger's method [283]. We introduce a proper-time representation for Green's function

$$S(x, y/\sigma) = -\mathrm{i}\int_0^\infty \mathrm{d}s\exp[-\mathrm{i}s(m^2 + R/4)]\mathrm{e}^{\mathrm{i}s\pi^2}\langle x \mid \gamma^\mu\pi_\mu + m \mid y\rangle. \tag{7.16}$$

Here $\pi_\mu = \mathrm{i}\nabla_\mu$, $\pi^2 = g^{\mu\nu}\pi_\mu\pi_\nu$.

Consider a quantum-mechanical system with coordinates x^μ, momentum π_ν and the Hamiltonian $H = -\pi^2$. We define the matrix element of the evolution operator as $\langle x|U(s)|y\rangle \equiv \langle x(s)|y\rangle$. Then, from the equation for the evolution operator we obtain

$$\mathrm{i}\frac{\partial}{\partial s}\langle x(s)|y\rangle = \langle x(s)|H \mid y\rangle. \tag{7.17}$$

The operators x^μ, π_ν satisfy the following commutation relations

$$[x^\mu, \pi_\nu] = -\mathrm{i}\delta^\mu_\nu \qquad [\pi_\mu, \pi_\nu] = R\sigma_{\mu\nu}/4. \tag{7.18}$$

Here $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, the second equation is held in a constant-curvature space. Introduce the operators in the Heisenberg representation $x^\mu(s) = U^{-1}(s)x^\mu U(s), \pi_\mu(s) = U^{-1}(s)\pi_\mu U(s)$, which satisfy the equations

$$\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s} = 2\pi^\mu(s) \qquad \frac{\mathrm{d}\pi_\mu(s)}{\mathrm{d}s} = \mathrm{i}\frac{R}{2}\sigma_\mu{}^\nu\pi_\nu(s). \tag{7.19}$$

The solution of equation (7.19) is

$$\begin{aligned}\pi_\mu(s) &= (\mathrm{e}^{\mathrm{i}(R/2)s\sigma})_\mu{}^\nu\pi_\nu \\ x^\mu(s) &= x^\mu + \mathrm{i}\frac{4}{R}(\sigma^{-1})^\mu{}_\nu[1 - \exp[\mathrm{i}(R/2)s\sigma]]^\nu{}_\alpha\pi^\alpha.\end{aligned} \tag{7.20}$$

Using

$$[x^\mu, x^\nu(s)] = -\mathrm{i}\frac{8}{R}\Big(\sigma^{-1}\sin(Rs\sigma/4)\cos(Rs\sigma/4)\Big)^{\mu\nu}$$

allows us to write

$$\begin{aligned}H &= -x(s)Kx(s) - xKx + 2x(s)Kx - \mathrm{i}\frac{R}{8}\operatorname{Tr}(\sigma\cot(Rs\sigma/4)) \\ K_{\mu\nu} &= \frac{R^2}{64}\Big[\sigma\big(\sin(Rs\sigma/4)\big)^{-1}\Big]^2_{\mu\nu}.\end{aligned} \tag{7.21}$$

Equation (7.21) enables us to calculate the matrix element

$$\begin{aligned}\langle x(s) \mid H \mid y\rangle = \Big[&-(x-y)K(x-y) \\ &- \mathrm{i}\frac{R}{8}\operatorname{Tr}(\sigma\cot(Rs\sigma/4))\Big]\langle x(s) \mid y\rangle\end{aligned}$$

and, then, to find the solution of equation (7.17) in the form

$$\begin{aligned}\langle x(s) \mid y\rangle = G(x,y)\exp\Big[&-\frac{\mathrm{i}}{4}(x-y)\frac{R\sigma}{4}\cot\frac{Rs\sigma}{4}(x-y) \\ &- \tfrac{1}{2}\operatorname{Tr}\ln\sin\frac{Rs\sigma}{4}\Big].\end{aligned} \tag{7.22}$$

It is not difficult to show that

$$-\tfrac{1}{2}\,\mathrm{Tr}\ln\sin\frac{Rs\sigma}{4} = \ln[\sin(Rs/4)]^{-1}.$$

To find the function $G(x,y)$ encountered in (7.22) we write the equations for the matrix element of π_μ

$$\begin{aligned} &\mathrm{i}\Big(\frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)\Big)\langle x(s) \mid y\rangle = \langle x(s) \mid \pi_\mu(s) \mid y\rangle \\ &\mathrm{i}\Big(-\frac{\partial}{\partial y^\mu} + \Gamma_\mu(y)\Big)\langle x(s) \mid y\rangle = \langle x(s) \mid \pi_\mu \mid y\rangle. \end{aligned} \tag{7.23}$$

Substituting (7.20) and (7.22) into (7.23) after using the boundary condition $\langle x(s) \mid y\rangle \underset{s\to 0}{\longrightarrow} \delta^2(x-y)/\sqrt{-g(x)}$ we then use

$$\begin{aligned} \langle x(s) \mid y\rangle = {}& \frac{R}{16\pi} P\exp\left[-\int_y^x \mathrm{d}x^\mu \Gamma_\mu(x)\right]\big(\sin(Rs/4)\big)^{-1} \\ & \times\exp\left[-\frac{\mathrm{i}}{4}(x-y)\frac{R\sigma}{4}\cot\frac{R\sigma s}{4}(x-y)\right]. \end{aligned} \tag{7.24}$$

Here $\Gamma_\mu = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab}$, ω_μ^{ab} is the spinor connection. The integration in (7.24) is along a geodesic which connects x- and y-points. The symbol P designates an ordering with respect to a parameter along a geodesic. As the final expression for the Green's function we obtain

$$\begin{aligned} S(x,y/\sigma) = {}& -\mathrm{i}\frac{R}{16\pi}\int_0^\infty \frac{\mathrm{d}s}{\sin(Rs/4)}\exp[-\mathrm{i}s(\lambda\sigma^2/N + R/4)] \\ & \times\left[\frac{R}{4}\gamma^\mu\left(\cot\frac{Rs\sigma}{4} - \mathrm{i}\right)_\mu^{\;\alpha}\sigma_\alpha^\nu(x-y)_\nu + \sqrt{\lambda/N}\sigma\right] \\ & \times P\exp\left(-\int_y^x \mathrm{d}x^\mu\Gamma_\mu(x)\right)\exp\left[-\frac{\mathrm{i}R}{16}(x-y)^\mu\right. \\ & \left.\times\sigma_\mu^{\;\alpha}\left(\cot\frac{Rs\sigma}{4}\right)_\alpha^{\;\nu}(x-y)_\nu\right]. \end{aligned} \tag{7.25}$$

We now turn to the calculation of the effective potential of composite fields. From relations (7.14) and (7.25) we find

$$V'(\sigma) = \sigma\left[1 - \frac{\lambda}{2\pi}\int_0^\infty \frac{\mathrm{d}sR/4}{\sin Rs/4}\exp(-\mathrm{i}s(R/4 + \lambda\sigma^2/N))\right] + \mathrm{CT}. \tag{7.26}$$

Here CT are counterterms which have to be added for the expression for $V'(\sigma)$ to be finite. To find the CT we used the renormalization condition

$$V''(\sigma)\mid_{\sigma=\mu, R=0} = 1 \tag{7.27}$$

where μ is an arbitrary renormalization point. We have determined the CT from (7.26) and (7.27) and have calculated the integral in (7.26), whereby

$$V'(\sigma) = \sigma\left\{1 + \frac{\lambda}{2\pi}\left[\Psi\left(\frac{2\lambda\sigma^2}{NR} + 1\right) - \ln\frac{2\lambda\mu^2}{NR} - 2\right]\right\}. \tag{7.28}$$

Here $\Psi(z)$ is the digamma-function. The effective potential is found by the integration of relation (7.28)

$$\begin{aligned} V(\sigma) = {} & \frac{\sigma^2}{2}\left[1 - \frac{\lambda}{2\pi}\left(2 + \ln\frac{2\lambda\mu^2}{NR}\right)\right] \\ & + \frac{NR}{8\pi}\ln\Gamma\left(1 + \frac{2\lambda\sigma^2}{NR}\right). \end{aligned} \tag{7.29}$$

Here $\Gamma(z)$ is the usual gamma-function. Relation (7.29) yields a final expression for the effective potential of σ field in the one-loop approximation. Notice that as $R \to 0$ expression (7.29) turns into the effective potential of the Gross–Neveu model in flat space–time [277].

Consider some particular cases which follow from equation (7.29). We determine the form of the effective potential in a weak gravitational field when consideration may be restricted to the first non-vanishing correction involving the curvature. Using expression

$$\ln\Gamma(z+1) = z(\ln z - 1) - \tfrac{1}{2}\ln\frac{z}{2\pi} - O\left(\frac{1}{z}\right)$$

we obtain

$$V(\sigma) = \frac{\sigma^2}{2}\left[1 + \frac{\lambda}{2\pi}\left(\ln\frac{\sigma^2}{\mu^2} - 3\right)\right] - \frac{NR}{16\pi}\ln\left(\frac{\lambda\sigma^2}{\pi NR}\right). \tag{7.30}$$

On the contrary, in a strong gravitational field when $R \gg \lambda\sigma^2/N$ and $R \gg \lambda\mu^2/N$ from (7.29) we find

$$V(\sigma) = \frac{\sigma^2}{2}\left\{1 - \frac{\lambda}{2\pi}\left[2 + \ln\left(\frac{2\lambda\mu^2}{NR}\right)\right]\right\}. \tag{7.31}$$

Subsequently we also need the second derivative of the effective potential. From (7.28)

$$V''(\sigma) = 1 + \frac{\lambda}{2\pi}\left[\Psi\left(\frac{2\lambda\sigma^2}{NR}+1\right) - \ln\frac{2\lambda\mu^2}{NR} - 2 \right. \\ \left. + \frac{4\lambda\sigma^2}{NR}\zeta\left(2, \frac{2\lambda\sigma^2}{NR}+1\right)\right]. \tag{7.32}$$

Here $\zeta(2, z)$ is the generalized Riemann ζ function.

Now we will consider the problem of symmetry breaking in the Gross–Neveu model in an external gravitational field. Suppose that the curvature is small ($R \ll \lambda\sigma^2/N$, $R \ll \lambda\mu^2/N$). Then, using (7.30) we conclude that the effective potential has a minimum at $\sigma_0^2 = \bar{\sigma}_0^2(1 - RN/(4\lambda\bar{\sigma}_0^2))$, where $\bar{\sigma}_0^2 = \mu^2\exp(2 - 2\pi/\lambda)$. Thus, the symmetry is broken at sufficiently small R and moreover with an increasing value of R the value of the order parameter tends to decrease. Consider the case when the curvature is large ($R \gg \lambda\sigma^2/N$, $R \gg \lambda\mu^2/N$). Then, using (7.31) shows that the effective potential has a minimum at $\sigma_0 = 0$. Hence, the symmetry is restored. Thus, by an increase of the curvature of gravitational field a spontaneously broken symmetry is restored.

Consider the process of the symmetry restoration in more detail. The condition of a minimum of the effective potential at arbitrary value R is written in the form

$$\sigma\left\{1 + \frac{\lambda}{2\pi}\left[\Psi\left(\frac{2\lambda\sigma^2}{NR}+1\right) - \ln\frac{2\lambda\mu^2}{NR} - 2\right]\right\} = 0$$

$$1 + \frac{\lambda}{2\pi}\left[\Psi\left(\frac{2\lambda\sigma^2}{NR}+1\right) - \ln\frac{2\lambda\mu^2}{NR} - 2 \right. \\ \left. - \frac{4\lambda\sigma^2}{NR}\zeta\left(2, \frac{2\lambda\sigma^2}{NR}+1\right)\right] > 0. \tag{7.33}$$

The first equation (7.33) defines the extremum of the effective potential, one of which is $\sigma_0 = 0$, another (if it exists), generally speaking, depends on R. The behaviour of the effective potential at small R testifies to the existence of the second point of the extremum. We have numerically investigated equation system (7.33) and have found the dependence of the minimum point σ_0 of the effective potential on R . This investigation has shown that the behaviour of the minimum of V as a function of R is typically that of a second-order phase transition with some critical curvature R_c. Thus, restoration of the broken symmetry in the Gross–Neveu model with an external gravitational field occurs in the form of the second-order phase transition. The critical curvature remains to be evaluated. It may be done by taking into account that at the critical point the transition occurs ranging from the case when $\sigma_0 = 0$ is the minimum of V and $V''(0) > 0$ to the case when $\sigma_0 = 0$ is not the minimum of V and $V''(0) < 0$ whereby at $R = R_c$, $V''(0) = 0$ holds, and we have $R_c = (2\lambda/N)\sigma_0^{-2}\mathrm{e}\gamma$ where γ is the Euler number.

7.4 Schwinger–Dyson equations in QED

A more realistic model with composite fermion fields is quantum electrodynamics (QED). It is well-known [284,285] that in QED there is a phase with spontaneously broken chiral symmetry for a sufficiently strong bare coupling constant exists (for a review, see [269]). This fact has been established by direct solution of the Schwinger–Dyson equations [284,285] which are effective equations for the effective action of composite fields.

Let us present a short discussion of the Schwinger–Dyson equations in QED [284,285]. Let

$$S_0(p) = \hat{p}^{-1} \qquad D^0_{\mu\nu}(k) = -\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4} \tag{7.34}$$

be the free fermion and photon (in the Landau gauge) propagators. Here $\hat{p} = \gamma^\mu p_\mu$.

The exact fermion propagator is

$$G^{-1}(p) = \hat{p} - B(p^2) \tag{7.35}$$

where $B(p^2)$ is the mass function.

The loop expansion for EA of composite fields in QED has the following well-known form [268]

$$\Gamma[G] = \mathrm{i}\,\mathrm{Sp}[\ln S_0^{-1}G - S_0^{-1}G + 1] + \Gamma_2(G). \tag{7.36}$$

Note that in this case the elementary background field is equal to zero from the very beginning. In (7.36) $\Gamma_2(G)$ is given by all two-loop two-particle irreducible vacuum graphs with the fermion propagators set equal to $G(p)$. Just one such two-loop vacuum graph exists in QED. After explicit calculation of effective action (7.36) (see [268]), varying this EA with respect to G_0 we can obtain

$$B(u) = \lambda \int_0^\infty \frac{vB(v)\mathrm{d}v}{v + B^2(v)}\left(\frac{\theta(u-v)}{u} + \frac{\theta(v-u)}{u}\right) \tag{7.37}$$

where $\lambda = 3e^2/16\pi^2$, $u = p^2$, $v = q^2$. This equation is well-known in flat space [268, 269, 284–285]. The analysis of solutions of equation (7.37) shows the existence of the critical coupling constant α_c corresponding to a second-order phase transition [288] and separating massless and massive QED phases [269, 270] (see also [286–287]). It is interesting to note that equation (7.37) can be written in the form of a non-linear differential equation of second order with some boundary conditions.

Let us try to generalize the above picture for curved space–time. It is well-known that momentum representation is not valid in a general curved space–time. So we limit ourselves to the linear curvature approximation. Then in local momentum representation of propagators [28, 57] we obtain for free fermion and photon propagators

$$\begin{aligned} S(p) &= S_0(p) - \frac{R}{24}\hat{p}^{-3} \\ D_{\mu\nu}(k) &= D^0_{\mu\nu}(k) - \frac{R}{12}\left(\frac{\eta_{\mu\nu}}{k^4} + \frac{2k_\mu k_\nu}{k^6}\right). \end{aligned} \tag{7.38}$$

Here we consider the spaces for which $R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$.

The natural continuation of exact fermion propagator (7.35) in curved space–time is [289]

$$G^{-1}(p) = \hat{p} - B(p^2) + \frac{R}{24}\left(\frac{C(p^2)}{\hat{p} - B(p^2)} + \frac{\hat{p}F(p^2)}{(\hat{p} - B(p^2))^2}\right) \tag{7.39}$$

where $C(p^2)$ and $F(p^2)$ are unknown functions.

The expression for EA has the same form as in flat space (7.36) with the substitution $S_0(p) \to S(p)$.Then the same steps as above lead to two new Schwinger–Dyson equations [289] (equation (6.37) is unchanged)

$$\begin{aligned} &\{B^2(u)[u + B^2(u)] - u[C(u) - 1][u + B^2(u)] - uF(u)[u - B^2(u)]\} \\ &\quad \times u^{-1}(u + B^2(u))^{-2} \\ &\quad = 2\lambda\left[\frac{1}{u + B^2(u)} + \int_0^\infty \frac{dv}{v + B^2(v)}\right. \\ &\quad \left.\times\left(\frac{v^2}{u^2}\frac{\theta(u - v)}{u - v} + \frac{\theta(v - u)}{v - u}\right)\right] \end{aligned} \tag{7.40}$$

$$-B(u)\frac{[C(u)(u + B^2(u)) + 2uF(u)]}{[u + B^2(u)]^2}$$

$$
\begin{aligned}
&= \lambda \Big\{ \frac{10B(u)}{u + B^2(u)} \\
&+ 4 \int_0^\infty dv \frac{B(v)}{v + B^2(v)} \left(\frac{v}{u} \frac{\theta(u-v)}{(u-v)} + \frac{\theta(v-u)}{(v-u)} \right) \\
&- \int_0^\infty dv B(v) \left(C(v) \frac{(3v - B^2(v))}{[v + B^2(v)]^3} + 4vF(v) \right. \\
&\left. \times \frac{(v - B^2(v))}{(v + B^2(v))^4} \right) \left(\frac{v}{u} \theta(u-v) + \theta(v-u) \right) \Big\}
\end{aligned}
\tag{7.41}
$$

The system of the linear integral equations (7.40) and (7.41) for the functions $C(u)$ and $F(u)$ cannot be written in the same differential form as (7.37) because of the terms involving $(u-v)^{-1}$. The presence of such terms means that the infrared divergence, connected with the zero photon mass appears in the local momentum representation.

The analytical solution of equation (7.37) for $B(p^2)$ is unknown. Hence we cannot find solutions of equations (7.40) and (7.41). For a known asymptotic form of $B(p^2)$ equations (7.40) and (7.41) are also quite complicated. Only the chiral symmetric solution ($B = 0$) of equations (7.40) and (7.41) has been found in [289]. These results probably mean that expansion in the curvature is not good in this model (or must be modified). Also it is interesting to investigate the Schwinger–Dyson equations on the same background (for example, on the de Sitter background), not in terms of expansion on curvature.

7.5 Renormalization group equations for composite fields

We have already discussed the renormalization group equations enabling one to investigate the asymptotic behaviour of the EA of elementary fields. Now we will formulate the renormalization group equations for EA of composite fields in curved space–time [291, 292]. Note that these equations have been formulated in [290] for flat space–time (see also [500]). It was found that stability conditions of the theory in the strong composite field limit impose some restrictions on the multiplet structure of the theory [290]. As will be shown here, the curved space–time considerations [291, 292] lead to some new restrictions on the multiplet structure of the theory.

Let us consider an arbitrary theory containing scalars φ^i, spinors ψ^j and gauge fields A^a_μ which will be noted as ϕ. We also define composite fields $\tau \equiv \{\varphi^i\varphi^i,\ A^{a\mu}A_{a\mu}\}$ and introduce the notation

$\Omega \equiv \{\phi, \tau\}$. Let us write the generating functional of the Green's functions of the fields Ω

$$\exp\left(\frac{\mathrm{i}}{\hbar}W_0[I,K]\right) = \int \mathrm{D}\mu[\phi] \exp\frac{\mathrm{i}}{\hbar}\left(S + \int \mathrm{d}^4x\sqrt{-g}(I\phi + K\tau)\right). \tag{7.42}$$

Here S is the renormalized action of the theory in curved space–time which takes into account gauge fixing and ghosts. The integration over the ghosts is included into the notation $D\mu[\phi]$. I, K are the sources for ϕ and τ, respectively. Let $N \equiv \{I, K\}$. The functional $W_0[I,K]$ leads to a finite theory when $K = 0$. The Green's functions containing insertions of composite fields diverge. To find the structure of the corresponding divergences, consider the terms $K\tau$ as additional vertices. Then, if we look at the divergence degree we shall see that only diagrams of first and second degrees of K diverge. Hence, the generating functional W_R which leads to the finite functions of elementary and composite fields (and vacuum energy, which is ensured by the presence of vacuum counterterms in S) has the form

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}W_R\right) = \int \mathrm{D}\mu[\phi] \exp\Big[\frac{\mathrm{i}}{\hbar}\Big(S + \int \mathrm{d}^4x\sqrt{-g}\\ \times (I\phi + KZ_{K\tau}\tau + KZ_{K\phi}\phi\\ + Z^{(1)}K + KZ^{(2)}K + Z^{(3)}KR\Big)\Big].\end{aligned} \tag{7.43}$$

Here $Z_{K\tau}$, $Z_{K\phi}$, $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ are renormalization constants, $Z^{(1)}$, $Z_{K\phi}$ have non-zero dimensions. The constants $Z_{K\tau}$, $Z_{K\phi}$, $Z^{(1)}$, $Z^{(2)}$ are the same as in flat space [290]. $Z^{(3)}$ is a new constant having no flat-space analogue. The functional W_R depends on the sources N and the set of charges f and normalization point μ. Note that $f \equiv (f_0, \lambda)$ where f_0 are the parameters of the external field Lagrangian, λ are the parameters characterizing the matter Lagrangian. We shall assume that the curvature and the sources are slowly changing functions, so their derivatives can be ignored. Then $W_R[N, f, \mu, g_{\alpha\beta}]$ can be written in the form

$$W_R[N, f, \mu, g_{\alpha\beta}] = \Gamma_0[f_0, \mu, g_{\alpha\beta}] + \int \mathrm{d}^4x\sqrt{-g}W[N, \lambda, \mu, g_{\alpha\beta}].$$

Here Γ_0 is the vacuum energy. Using arguments given in [290], it is easy to show that W satisfies the RGE

$$(\mu\partial_\mu + \beta_\lambda\partial_\lambda + N\gamma\partial_N)W = \gamma^{(1)}K + K\gamma^{(2)}K + R\gamma^{(3)}K. \tag{7.44}$$

The functions γ, $\gamma^{(1)}$ and $\gamma^{(2)}$ are defined in [290], β_λ are the RG functions for all the parameters λ and

$$\gamma^{(3)} = \left[S_{K_T} \mu \frac{\partial}{\partial \mu} \left(S_{K_T}^{-1} Z^{(3)} \right) \right]_{\lambda_0}$$

where $S_{K_T} = Z_{K_T} Z^{-1/2} Z^{-1/2}$, Z are the renormalization constants of the elementary fields, λ_0 are the bare parameters.

Le us define $V[\Omega]$ by the relation $-V[\Omega] = W - \Omega N$, where $\Omega = \partial W / \partial N$, $N = -\partial V / \partial \Omega$. Then the RGE follows from equation (7.44)

$$\left(\mu \partial_\mu + \beta_\lambda \partial_\lambda - \Omega \gamma \frac{\partial}{\partial \Omega} \right) V$$
$$= -\gamma^{(1)} \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial \tau} \gamma^{(2)} \frac{\partial V}{\partial \tau} - \gamma^{(3)} R \frac{\partial V}{\partial \tau}. \qquad (7.45a)$$

Equations (7.44) and (7.45a) are, in fact, a consequence of the multiplicative renormalizability of the theory with composite fields.

We now study the asymptotic form of the functionals W and V. Let us assume for the sake of simplicity that $I = 0$, $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$ are diagonal matrices. Let d_λ be the dimensions of the parameters λ. It is easy to show that the homogeneity condition for W is (see also Chapter 3)

$$W(p^2 K, p^{d_\lambda} \lambda, p\mu, p^{-2} g_{\alpha\beta}) = p^4 W(K, \lambda, \mu, g_{\alpha\beta}) \qquad p = \text{constant}. \qquad (7.45b)$$

Equations (7.45) yield the following relations

$$\left(\frac{\partial}{\partial t} + \mathrm{d}_\lambda \lambda \frac{\partial}{\partial \lambda} + \mu \partial_\mu - \int \mathrm{d}^4 x \, 2 g_{\alpha\beta}(x) \frac{\delta}{\delta g_{\alpha\beta}(x)} - 4 \right)$$
$$\times W[\exp[2t] K, \lambda, \mu, g_{\alpha\beta}] = 0 \qquad (7.46a)$$

$$\left(\frac{\partial}{\partial t} + 2K \frac{\partial}{\partial K} + \mathrm{d}_\lambda \lambda \frac{\partial}{\partial \lambda} + \mu \partial_\mu - 4 \right)$$
$$\times W[K, \lambda, \mu, \mathrm{e}^{-2t} g_{\alpha\beta}] = 0 \qquad (7.46b)$$

$$\left(\frac{\partial}{\partial t} + \mathrm{d}_\lambda \lambda \frac{\partial}{\partial \lambda} + \mu \partial_\mu - 4 \right) W[\mathrm{e}^{2t} K, \lambda, \mu, \mathrm{e}^{-2t} g_{\alpha\beta}] = 0. \qquad (7.46c)$$

Eliminating $\mu(\partial W / \partial \mu)$ from equations (7.46a–c) and substituting the result in equation (7.44), after some transformations we obtain

$$\left\{\frac{\partial}{\partial t}-\left(1-\frac{\gamma}{2}\right)^{-1}\left(\beta_\lambda-d_\lambda\lambda)\frac{\partial}{\partial\lambda}\right.\right.$$
$$\left.\left.+2\int \mathrm{d}^4x g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}+4\right)\right\}W[\mathrm{e}^{2t}K,\lambda,\mu,g_{\alpha\beta}]$$
$$=-\left(1-\frac{\gamma}{2}\right)^{-1}[\gamma^{(1)}\mathrm{e}^{2t}K+\gamma^{(2)}\mathrm{e}^{4t}K^2+\gamma^{(3)}\mathrm{e}^{-2t}KR] \quad (7.47a)$$

$$\left\{\frac{\partial}{\partial t}-(\beta_\lambda-\mathrm{d}_\lambda\lambda)\frac{\partial}{\partial\lambda}-K(\gamma-2)\frac{\partial}{\partial K}-4\right\}W[K,\lambda,\mathrm{e}^{-2t}g_{\alpha\beta}]$$
$$=-\gamma^{(1)}K-\gamma^{(2)}K^2-\gamma^{(3)}KR\mathrm{e}^{2t} \quad (7.47b)$$

$$\left\{\frac{\partial}{\partial t}-\left(1-\frac{\gamma}{2}\right)^{-1}\left((\beta_\lambda-\mathrm{d}_\lambda\lambda)\frac{\partial}{\partial\lambda}\right.\right.$$
$$\left.\left.+\gamma\int \mathrm{d}^4x g_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}+4\right)\right\}W[\mathrm{e}^{2t}K,\lambda,\mathrm{e}^{-2t}g_{\alpha\beta}]$$
$$=-\left(1-\frac{\gamma}{2}\right)^{-1}[\mathrm{e}^{2t}\gamma^{(1)}K+\mathrm{e}^{4t}\gamma^{(2)}K^2+\mathrm{e}^{4t}\gamma^{(3)}KR] \quad (7.47c)$$

Equations (7.47) make it possible to find the asymptotic form of W when $t\to\infty$. Then equation (7.47a) enables us to study W at large values of K. This equation differs from the corresponding equation in flat space [290] by the presence of the parameter ξ, the term KR and also the term $g_{\alpha\beta}(\delta/\delta g_{\alpha\beta})$. Equation (7.47$b$) enables us to study W in a strong gravitational field (SGF) limit (as from $g_{\alpha\beta}\to \mathrm{e}^{-2t}g_{\alpha\beta}$ it follows that $R^2\to \mathrm{e}^{4t}R^2$). Equation (6.47$c$) makes it possible to study the behaviour of W in both SGF and strong composite field (SCF) limit. These equations can be rewritten in the form of equations for V, but we are not interested in it here.

Let us consider the asymptotic behaviour of W following from equations (7.47a–c) for asymptotically free theories.

(a) *The asymptotic form of W in the strong composite field limit.* It follows from equation (7.47a) that when $t\to\infty$, W increases either as e^{4t} or faster. In the first case, its asymptotic form is given by the expression aK^2, where a satisfies equation (7.51) (given in [290]). In the second case, the right-hand part of equation (7.47a) can be ignored. Then this equation is solved with the aid of effective coupling constants among which there is an effective metric satisfying the equation

$$\frac{\mathrm{d}g_{\alpha\beta}(t,x)}{\mathrm{d}t}=\left(2+\frac{2\gamma}{2-\gamma}\right)g_{\alpha\beta}(t,x) \quad (7.48)$$
$$g_{\alpha\beta}(0,x)=g_{\alpha\beta}(x).$$

Equation (7.48) gives $g_{\alpha\beta}(t,x) \propto \mathrm{e}^{2t}t^q g_{\alpha\beta}(x)$ where q = constant. Hence, the effective curvature behaves as e^{-2t} and when $t \to \infty$, it is always small. Therefore, the SCF limit is the limit of small curvature. Then at strong K, all the curvature effects can be ignored and the asymptotic form of W is defined by the relation given in [290] (compare with section 5.4).

(*b*) *The asymptotic form of W in the* SGF *limit.* It follows from equation (7.47*b*) that $W[K, \lambda, \mathrm{e}^{-2t}g_{\alpha\beta}]$ increases either as e^{2t} or faster when $t \to \infty$. In the first case, the asymptotic form is given by the expression bRK, where b satisfies the equation

$$\left(\beta_\lambda \frac{\partial}{\partial\lambda} + \gamma\right) b = \gamma^{(3)} \tag{7.49}$$

and depends only on dimensionless parameters. In this case V tends to zero. In the second case, the right-hand part of equation (7.49) can be ignored. Then its solution can be written in the form

$$\begin{aligned} W[K,\lambda,\mathrm{e}^{-2t}g_{\alpha\beta}] &= \mathrm{e}^{4t}W[K(t),\lambda(t),g_{\alpha\beta}] \\ \frac{dK(t)}{\mathrm{d}t} &= (\gamma(t)-2)K(t) \\ \frac{d\lambda(t)}{\mathrm{d}t} &= (\beta_\lambda(t) - \mathrm{d}_\lambda\lambda(t))\lambda(t) \\ K(0) = K \qquad & \lambda(0) = \lambda. \end{aligned} \tag{7.50}$$

It follows from equation (7.50) that $K(t) \propto \mathrm{e}^{-2t}t^l K$, l = constant. Hence, when $t \to \infty$, $K(t) \to 0$. Only dimensionless constants remain when $t \to \infty$. Let their limits be λ_0. Then

$$W[K,\lambda,\mathrm{e}^{-2t}g_{\alpha\beta}] \propto \mathrm{e}^{4t}W[0,\lambda_0,g_{\alpha\beta}].$$

But as vacuum energy Γ_0 is subtracted from W_R, then $W[0,\lambda_0,g_{\alpha\beta}] = 0$. Thus in the SGF limit $W[K,\lambda,g_{\alpha\beta}] \propto bRK$, where b satisfies equation (7.49).

(*c*) *The asymptotic form of W in both the* SCF *and* SGF *limit.* It follows from equation (7.47*c*) that $W[\mathrm{e}^{2t}K,\lambda,\mathrm{e}^{-2t}g_{\alpha\beta}]$ increases either as e^{4t} or faster. In the first case the asymptote of W is $aK^2 + bRK$, where a satisfies the equation

$$(\beta_\lambda\partial_\lambda + 2\gamma)a = \gamma^{(2)} \tag{7.51}$$

and depends only on dimensionless parameters λ. b satisfies equation (7.49). Here the effective potential has the following asymptote:

$V \propto (\tau - bR)^2/4a$. In the second case the right-hand part of equation (7.47c) can be ignored. Then its solution has the form

$$\begin{aligned} & W[e^{2t}K,\lambda, e^{-2t}g_{\alpha\beta}] \\ & \quad = e^{4t}\exp\left(\int_0^t \frac{2\gamma}{1-\gamma/2}dt'\right) W[K,\lambda(t), g_{\alpha\beta}(t)] \\ & \frac{d\lambda(t)}{dt} = \frac{\beta_\lambda(t) - d_\lambda \lambda(t)}{1-\gamma(t)/2} \qquad \frac{dg_{\alpha\beta}(t,x)}{dt} = \frac{\gamma(t)g_{\alpha\beta}(t,x)}{1-\gamma(t)/2} \\ & \lambda(0) = 0 \qquad g_{\alpha\beta}(0,x) = g_{\alpha\beta}(x). \end{aligned} \tag{7.52}$$

The effective metric appears here as in case (a). We shall further analyse asymptotically free theories: where $g^2(t) \propto (2/c^2 t)$, $\gamma \propto \gamma_\varepsilon g^2(t)$, $\xi(t) \to \bar{\xi}$, where $\bar{\xi} = 1/6$ or $\bar{\xi} = \pm\infty$ (Chapter 3). Then

$$\begin{aligned} & W[e^{2t}K,\lambda, e^{-2t}g_{\alpha\beta}] \\ & \qquad \propto e^{4t} t^{\frac{4\gamma_\varepsilon}{c^2}} W(K, g^2 = 0, \xi = \bar{\xi}, t^{\frac{2\gamma_\varepsilon}{c^2}} g_{\alpha\beta}(x)). \end{aligned} \tag{7.53}$$

It follows that in the first case $W \propto e^{4t}$ is realized when $\gamma_\varepsilon \leqslant 0$. When $\gamma_\varepsilon > 0$, the second case is realized and the effective curvature turns out to be small and the leading behaviour of W is the same as in the case (a). When $\gamma_\varepsilon < 0$ the effective curvature is strong. Then to analyse W one can use the case (b) according to which

$$W[K, g^2 = 0, \xi = \bar{\xi}, t^{2\gamma_\varepsilon/c^2} g_{\alpha\beta}] \propto b(0) K R t^{-2\gamma_\varepsilon/c^2}$$

Here $b(0)$ is the solution b of equation (7.49) when $g^2 = 0$. As a result

$$W[e^{2t}K, \lambda, e^{-2t}g_{\alpha\beta}] \propto e^{4t} t^{2\gamma_\varepsilon/c^2} b(0) R K.$$

Let us turn to the consideration of the stability conditions for the theory. As W is the energy density in the presence of the external field $g_{\alpha\beta}$ and sources K, then this quantity must be non-negative at all the values of its arguments. In particular, the asymptotic form of W must be non-negative. We shall consider the stability conditions in the cases (b) and (c) for Yang–Mills theory interacting with fermions (case (a) is investigated in [290] and a general review of QCD is given in [501]).

In the SGF limit and fixed K(case (b)) the asymptotic form has the behaviour $W \propto bRK$, where b satisfies (7.49). The RG functions included in it are

$$
\begin{aligned}
\beta_g &= \frac{g^3}{(4\pi)^2}\left(\tfrac{11}{3}C - \tfrac{4}{3}T(R)\right) \\
\gamma &= -\frac{g^2}{(4\pi)^2}\left(\tfrac{35}{6}C + \tfrac{1}{2}C\alpha - \tfrac{8}{3}T(R)\right) \qquad (7.54)\\
\gamma^{(3)} &= \frac{n}{(4\pi)^2}\left(\tfrac{1}{3}\alpha + \tfrac{1}{2}\right).
\end{aligned}
$$

Here $f^{abc}f^{dbc} = C\delta^{ad}$, $\mathrm{Sp}\,\Gamma^a\Gamma^b = T(R)\delta^{ab}$, Γ^a are the gauge group generators, n is the dimension of the group, α is the gauge parameter. The solution of equation (7.49) taking into account functions (7.54) gives (in the $\alpha = 0$ gauge)

$$
b^{-1}(g) = \frac{3Cg^2}{n}\left[1 + Bg^2\frac{9C}{4(11C - 4T(R))}\right]^{-1}. \qquad (7.55)
$$

B is an arbitrary constant. At sufficiently small g^2 the value $b(g)$ is always positive (independent of B), if the condition for asymptotic freedom is satisfied. Therefore, the sign of the asymptotic $W \propto bRK$ depends only on the signs of K and R, the theory structure does not influence it.

In both the SCF and SGF limit (case (c)) it is necessary to know $W[K, 0, g_{\alpha\beta}]$ to find the asymptotic form. We have calculated $W[K, 0, g_{\alpha\beta}]$ for large slowly changing K and R [291].

$$
\begin{aligned}
W[K, 0, g_{\alpha\beta}] = \frac{-n}{(4\pi)^2}&\left[K^2(\alpha^2 + 3)\right. \\
&\left.+KR(\tfrac{1}{6}\alpha + \tfrac{1}{4})\right]\left(\ln\frac{K}{\mu^2} + \ln\frac{|R|}{\mu^2}\right). \qquad (7.56)
\end{aligned}
$$

If $\gamma_\varepsilon > 0$ then, according to the above results, the external gravitational field will not alter the results of [290]. The theory is not stable. The case $\gamma_\varepsilon \leqslant 0$ is of more interest. But in this case according to the results obtained from (7.47c) the asymptotic form is $W \propto aK^2 + bRK$. The explicit form of $a(g)$ is found in [290]. When g^2 is small, the function $b(g) > 0$ and $a(g) > 0$, for $(4/3)T(R) - (13/6)C > 0$. There are no additional restrictions on the multiplet structure of the theory.

Let as analyse the conditions of the stability for the same model with composite fields $\sigma = \bar{\psi}^i\psi^i$. Let $\tilde{K}$ be the sources for σ, $K \equiv \tilde{K}^2$. Then it is easy to show that renormalization group equations are the same as for composite bosonic fields (7.47).

The one-loop renormalization group functions in equations (7.47) for the composite fields $\sigma = \bar{\psi}^i\psi^i$ are [292]

$$\gamma = -\frac{g^2 C_2(G)}{(4\pi)^2}(3+\alpha) \qquad \gamma^{(2)} = -\frac{2d(G)}{(4\pi)^2}$$
$$\gamma^{(3)} = -\frac{d(G)}{(3\pi)^2}. \tag{7.57}$$

We will consider the gauge $\alpha = 0$. Here $d(G)$ is the dimension of the representation of G and $C_2(G)$ is the quadratic Casimir operator for the group G .

(a) For strong composite fields we obtain $W \propto aK^2$, where [292]

$$\begin{aligned} a^{-1}(g) &= d^{-1}(G)[\tfrac{4}{3}T(R) - \tfrac{11}{3}C + 12C_2(G)](1 + Ag^B)^{-1} \\ B &= 2[\tfrac{4}{3}T(R) - \tfrac{11}{3}C + 12C_2(G)](\tfrac{4}{3}T(R) - \tfrac{11}{3}C)^{-1} \end{aligned} \tag{7.58}$$

A is arbitrary. It is easy to show that new restrictions (in comparison with the condition of asymptotic freedom $\frac{11}{3}C > T(R)$) do not appear.

(b) The asymptotic form of W in the strong gravitational field limit. Here $W \propto bRK$, where the solution of equation (7.49) yields

$$b^{-1}(g) = 6d^{-1}(G)\Big(\tfrac{4}{3}T(R) - \tfrac{11}{3}C + 6C_2(G)\Big)(1 + A_1 g^D)^{-1}. \tag{7.59}$$

Here

$$D = 2\Big(\tfrac{4}{3}T(R) - \tfrac{11}{3}C + 6C_2(G)\Big)\Big(\tfrac{4}{3}T(R) - \tfrac{11}{3}C\Big)^{-1}$$

A_1 is arbitrary; $b(g)$ is positive if $\frac{4}{3}T(R) - \frac{11}{3}C + 6C_2(G) > 0$ and $A_1 > 0$ or $A_1 < 0$, $|A_1| \propto g^2$. This condition leads to new restrictions on the multiplet structure of fermions. For example, for SU(N) this condition does not permit the presence of one family of fermions in fundamental representation. Thus, we have shown that curved space–time can lead to new restrictions on the multiplet structure of the theory. These restrictions are absent in flat space.

7.6 The parameterization and gauge invariant effective action of composite fields

As was mentioned above, the effective action of composite fields depends on the gauge condition as well as the effective action of elementary fields (see, for example, [294, 296–298]). It is interesting to construct the parameterization invariant and gauge-fixing independent formulation of the effective action of composite fields. Here we will consider this problem which has been discussed in [293].

Let us consider an arbitrary non-gauge theory of fields ϕ. We define the generating functional W in the following way. The sources are coupled not to the quantum fields but to the fluctuations with respect to the arbitrary fixed background fields

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}W[J,K;\phi_*,\phi_{**}]\right) & \\ \equiv \int \mathrm{D}\phi \exp\Bigg(\frac{\mathrm{i}}{\hbar}\Big[S[\phi] & \\ + J_i(\phi^i-\phi^i_*) + K_{i_1\dots i_n} & \\ \times(\phi^{i_1}-\phi^{i_1}_{**})\times\dots\times(\phi^{i_n}-\phi^{i_n}_{**})\Big]\Bigg). & \qquad (7.60)\end{aligned}$$

Here ϕ^i are the quantum fields, $\mathrm{D}\phi$ is an invariant measure on the space of all fields, and ϕ^i_*, ϕ^i_{**} are arbitrary, fixed background fields. Let us define now the fields v^i and $\Sigma^{i_1\dots i_n}$

$$\begin{aligned}v^i &\equiv \frac{\delta W[\,]}{\delta J_i} = \langle\phi^i\rangle_{J,K} - \phi^i_* \equiv \bar{\phi}^i - \phi^i_* \\ \Sigma^{i_1\dots i_n} &\equiv \frac{\delta W[\,]}{\delta K_{i_1\dots i_n}} - (\bar{\phi}^{i_1}-\phi^{i_1}_{**})\times\dots\times(\bar{\phi}^{i_n}-\phi^{i_n}_{**}). \qquad (7.61)\end{aligned}$$

Here

$$\begin{aligned}\langle\phi^i\rangle_{J,K} = \exp\left(-\frac{\mathrm{i}}{\hbar}W[\,]\right)\int \mathrm{D}\phi\,\phi^i \exp\Big(\frac{\mathrm{i}}{\hbar}\big[\,S[\phi] & \\ + J_i(\phi^i-\phi^i_*) + K_{i_1\dots i_n}(\phi^{i_1}-\phi^{i_1}_{**})\times\dots\times(\phi^{i_n}-\phi^{i_n}_{**})\,\big]\Big). &\end{aligned}$$

For example, if $n=2$ then

$$\Sigma^{i_1 i_2} = \langle\phi^{i_1}\phi^{i_2}\rangle_{J,K} - \bar{\phi}^{i_1}\bar{\phi}^{i_2}.$$

We now introduce the EA as the Legendre transform of the generating functional W

$$\begin{aligned}\Gamma[v,\Sigma;\phi_*,\phi_{**}] = W[J,K;\phi_*,\phi_{**}] - J_i v^i - K_{i_1\dots i_n} & \\ \times\left[\Sigma^{i_1\dots i_n} + (\bar{\phi}^{i_1}-\phi^{i_1}_{**})\times\dots\times(\bar{\phi}^{i_n}-\phi^{i_n}_{**})\right] & \\ (7.62)\end{aligned}$$

where

$$\frac{\delta\Gamma[\,]}{\delta v^i} \equiv -J_i - K_{i_1\ldots i_n}\frac{\delta}{\delta v^i}\Big[(\bar{\phi}^{i_1} - \phi^{i_1}_{**}) \times \ldots \times (\bar{\phi}^{i_n} - \phi^{i_n}_{**})\Big]$$
$$\frac{\delta\Gamma[\,]}{\delta\Sigma^{i_1\ldots i_n}} = -K_{i_1\ldots i_n}. \tag{7.63}$$

Taking into account expressions (7.60)–(7.63) the functional integral for the EA $\Gamma[\,]$ has the form

$$\Gamma[v,\Sigma;\phi_*,\phi_{**}] = \int \mathrm{D}\phi \exp\Big\{\frac{\mathrm{i}}{\hbar}\Big[S[\phi] - \frac{\delta\Gamma[\,]}{\delta v^i}(\phi^i - \phi^i_* - v^i)$$
$$- \frac{\delta\Gamma[\,]}{\delta\Sigma^{i_1\ldots i_n}}\Big((\phi^{i_1} - \phi^{i_1}_{**}) \times \ldots \times (\phi^{i_n} - \phi^{i_n}_{**}) - \Sigma^{i_1\ldots i_n}$$
$$- (\bar{\phi}^{i_1} - \phi^{i_1}_{**}) \times \ldots \times (\bar{\phi}^{i_n} - \phi^{i_n}_{**})$$
$$- \frac{\delta}{\delta v^i}\Big[(\bar{\phi}^{i_1} - \phi^{i_1}_{**}) \times \ldots \times (\bar{\phi}^{i_n} - \phi^{i_n}_{**})\Big](\phi^i - \phi^i_* - v^i)\Big)\Big]\Big\}. \tag{7.64}$$

Differentiating both sides of equation (7.64) with respect to ϕ^i and ϕ^i_* one derives the identities

$$\frac{\delta\Gamma[v,\Sigma;\phi_*,\phi_{**}]}{\delta\phi^i_*} = 0 \qquad \frac{\delta\Gamma[v,\Sigma;\phi_*,\phi_{**}]}{\delta\phi^i_{**}} = 0. \tag{7.65}$$

Thus, the EA does not depend on the choice of the background fields ϕ^i_*, ϕ^i_{**} . However, W and Γ are not scalar functions of the configuration space coordinates. This results in the off-shell parameterization dependence of the ordinary EA. As a result, in gauge theories the EA depends upon the choice of the gauge condition. (The structure of the gauge dependence of the convenient EA of composite fields for general gauge theories has been investigated in [294].)

Let us proceed to the construction of the parameterization-invariant EA of composite fields for the non-gauge theory using the approach of Vilkovisky [190] (see Chapter 6). Let $\Gamma^i_{jk}[\phi]$ be the connection for the space of fields $\{\phi^i\}$, s is the parameter. We define the two-point functional written in Chapter 6

$$\sigma^i(\phi(s),\phi(0)) = s\frac{\mathrm{d}\phi^i(s)}{\mathrm{d}s}. \tag{7.66}$$

This functional $\sigma^i(\phi_*,\phi)$ is a vector function of the field ϕ_* and a scalar function of ϕ.

Let us introduce the parameterization invariant generating functional W of composite fields analogous to the parameterization invariant generating functional for elementary fields (Chapter 6)

$$\exp\left(\frac{\mathrm{i}}{\hbar}W[J,K;\phi_*,\phi_{**}]\right) = \int \mathrm{D}\phi \, \exp\left[\frac{\mathrm{i}}{\hbar}\Big(S[\phi] - J_i\sigma^i(\phi_*,\phi) - K_{i_1\ldots i_n}\sigma^{i_1}(\phi_{**},\phi)\times\ldots\times\sigma^{i_n}(\phi_{**},\phi)\Big)\right]. \tag{7.67}$$

It is evident that the integrand is a scalar functional of the field ϕ. If J_i and $K_{i_1\ldots i_n}$ lie in the cotangent space at ϕ_* and ϕ_{**} respectively, $W[\ ]$ is invariant under coordinate transformations ϕ. As in the above, we define the new fields v^i, $\Sigma^{i_1\ldots i_n}$

$$v^i \equiv \frac{\delta W[\ ]}{\delta J_i} = -\langle\sigma^i(\phi_*,\phi)\rangle_{J,K} \equiv -\sigma^i(\phi_*,\bar\phi) \tag{7.68}$$

$$\Sigma^{i_1\ldots i_n} \equiv \frac{\delta W[\ ]}{\delta K_{i_1\ldots i_n}} + \langle\sigma^{i_1}(\phi_{**},\phi)\rangle_{J,K}\times\ldots\times\langle\sigma^{i_n}(\phi_{**},\phi)\rangle_{J,K}. \tag{7.69}$$

Then the parameterization invariant EA is the Legendre transformation of $W[\ ]$(compare with [190–194] and Chapter 6)

$$\hat\Gamma[v,\Sigma;\phi_*,\phi_{**}] \equiv W[J,K;\phi_*,\phi_{**}] - J_iv^i - K_{i_1\ldots i_n}[\Sigma^{i_1\ldots i_n} - \sigma^{i_1}(\phi_{**},\bar\phi)\times\ldots\times\sigma^{i_n}(\phi_{**},\bar\phi)]. \tag{7.70}$$

Using expressions (7.67)–(7.70) it is easy to obtain the single functional differential equation for $\hat\Gamma$

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}\hat\Gamma[v,\Sigma;\phi_*,\phi_{**}]\right) &= \int \mathrm{D}\phi\,\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\Bigg[S[\phi] + \frac{\delta\hat\Gamma[\ ]}{\delta v^i}[\sigma^i(\phi_*,\phi)+v^i] \\ &+ \frac{\delta\Gamma[\ ]}{\delta\Sigma^{i_1\ldots i_n}}(\sigma^{i_1}(\phi_{**},\phi)\times\ldots\times\sigma^{i_n}(\phi_{**},\phi) + \Sigma^{i_1\ldots i_n} \\ &- \sigma^{i_1}(\phi_{**},\bar\phi)\times\ldots\times\sigma^{i_n}(\phi_{**},\bar\phi) \\ &+ \frac{\delta}{\delta v^i}[\sigma^{i_1}(\phi_{**},\bar\phi)\times\ldots\times\sigma^{i_n}(\phi_{**},\bar\phi)][\sigma^i(\phi_*,\phi)+v^i]\Big)\Bigg]\Bigg\}.\end{aligned} \tag{7.71}$$

Thus, we have obtained an infinite class of parameterization invariant EAs of composite fields. These EAs depend on ϕ_* and ϕ_{**} . One can show that $\hat{\Gamma}$ generates the one-particle-irreducible correlation functions for the operators $\sigma^i(\phi_*, \phi)$ and $\sigma^{i_1}(\phi_{**}, \phi) \times \ldots \times \sigma^{i_n}(\phi_{**}, \phi)$. Sometimes, the following representation for the EA is more convenient (compare with [194] and section 6.1)

$$\begin{aligned}
&\exp\left(\frac{\mathrm{i}}{\hbar}\hat{\Gamma}[\bar{\phi}, \Sigma; \phi_*, \phi_{**}]\right) \\
&\quad = \int \mathrm{D}\phi \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S[\phi] - \frac{\delta\hat{\Gamma}[\,]}{\delta\bar{\phi}^j}(D^{-1})^j{}_i\right.\right. \\
&\quad \times [\sigma^i(\phi_*, \phi) - \sigma^i(\phi_*, \bar{\phi})] + \frac{\delta\hat{\Gamma}[\,]}{\delta\Sigma^{i_1\ldots i_n}} \\
&\quad \times \Big(\sigma^{i_1}(\phi_{**}, \phi) \times \ldots \times \sigma^{i_n}(\phi_{**}, \phi) \\
&\quad + \Sigma^{i_1\ldots i_n} - \sigma^{i_1}(\phi_{**}, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_{**}, \bar{\phi}) \\
&\quad - \frac{\delta}{\delta\bar{\phi}^j}[\sigma^{i_1}(\phi_{**}, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_{**}, \bar{\phi})](D^{-1})^j{}_i \\
&\quad \left.\left. \times [\sigma^i(\phi_*, \phi) - \sigma^i(\phi_*, \bar{\phi})]\Big)\right]\right\} \qquad (7.72)
\end{aligned}$$

where

$$D^i{}_j(\bar{\phi}, \phi_*) = \delta\sigma^i(\phi_*, \bar{\phi})/\delta\bar{\phi}^j .$$

Note that chosing in equation (7.72) $\phi^i_* = \phi^i_{**} = \bar{\phi}^i$, the EA (7.72) of the composite fields is practically a direct generalization of the Vilkovisky EA [190,191].

As compared with the standard EA the parameterization invariant EA depends upon the background field and ϕ_{**}. This dependence is governed by the following identities which are obtained by differentiating equation (7.72) with respect to ϕ_* and ϕ_{**} (see also [192–194])

$$\begin{aligned}
&\frac{\delta\hat{\Gamma}[\,]}{\delta\phi^k_*} + \frac{\delta\hat{\Gamma}[\,]}{\delta\bar{\phi}^j}(D^{-1})^j{}_i[\langle\sigma^i_{;k}(\phi_*, \phi)\rangle_{JK} - \sigma^i_{;k}(\phi_*, \bar{\phi})] \\
&\quad - \frac{\delta\hat{\Gamma}[\,]}{\delta\Sigma^{i_1\ldots i_n}}\Big(\Sigma^{i_1\ldots i_n}_{;k} - \frac{\delta}{\delta\bar{\phi}^j}[\sigma^{i_1}(\phi_{**}, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_{**}, \bar{\phi})] \\
&\quad \times (D^{-1})^j{}_i[\langle\sigma^i_{;k}(\phi_*, \phi)\rangle_{JK} - \sigma^i_{;k}(\phi_*, \bar{\phi})]\Big) = 0 \qquad (7.73)
\end{aligned}$$

$$\begin{aligned}
&\frac{\delta\hat{\Gamma}[\,]}{\delta\phi^k_{**}} - \frac{\delta\hat{\Gamma}[\,]}{\delta\bar{\phi}^j}(D^{-1})^j{}_i\frac{\delta\sigma^i(\phi_*, \bar{\phi})}{\delta\phi^k_{**}} - \frac{\delta\hat{\Gamma}[\,]}{\delta\Sigma^{i_1\ldots i_n}} \\
&\quad \times \Big(\langle[\sigma^{i_1}(\phi_{**}, \phi) \times \ldots \times \sigma^{i_n}(\phi_{**}, \phi)]_{,k}\rangle_{JK} + \Sigma^{i_1\ldots i_n}_{,k}
\end{aligned}$$

$$- [\sigma^{i_1}(\phi_{**}, \phi) \times \ldots \times \sigma^{i_n}(\phi_{**}, \phi)]_{,k}$$
$$+ \frac{\delta}{\delta \bar{\phi}^j} [\sigma^{i_1}(\phi_{**}, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_{**}, \bar{\phi})](D^{-1})^j_i$$
$$\times \left. \frac{\delta \sigma^i(\phi_*, \bar{\phi})}{\delta \phi^k_{**}} \right) = 0 \tag{7.74}$$

where colon and semicolon denote the covariant derivative on ϕ_{**} and ϕ_* respectively.

For simplicity in further considerations we will set $\phi_* \equiv \phi_{**}$ in EA (7.72). In this case only one identity appears

$$\frac{\delta \hat{\Gamma}[\,]}{\delta \phi^k_*} + \frac{\delta \hat{\Gamma}[\,]}{\delta \bar{\phi}^j} (D^{-1})^j{}_i [\langle \sigma^i_{;k}(\phi_*, \phi) \rangle_{JK} - \sigma^i_{;k}(\phi_*, \bar{\phi})]$$
$$- \frac{\delta \hat{\Gamma}[\,]}{\delta \Sigma^{i_1 \ldots i_n}} \Big(\langle [\sigma^{i_1}(\phi_*, \phi) \times \ldots \times \sigma^{i_n}(\phi_*, \phi)]_{;k} \rangle_{JK}$$
$$+ \Sigma^{i_1 \ldots i_n}_{;k} - [\sigma^{i_1}(\phi_*, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_*, \bar{\phi})]_{;k}$$
$$- \frac{\delta}{\delta \bar{\phi}^j} [\sigma^{i_1}(\phi_*, \bar{\phi}) \times \ldots \times \sigma^{i_n}(\phi_*, \bar{\phi})]$$
$$\times (D^{-1})^j{}_i [\langle \sigma^i_{;k}(\phi_*, \phi) \rangle_{JK} - \sigma^i_{;k}(\phi_*, \bar{\phi})] \Big) = 0 \tag{7.75}$$

where $\hat{\Gamma}[\,] \equiv \hat{\Gamma}[\bar{\phi}, \Sigma; \phi_*]$. We calculate the EA $\hat{\Gamma}[\,]$, then take the limit $\phi_* \to \bar{\phi}$, use the identity (7.75) and finally obtain

$$\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma[\bar{\phi}, \Sigma]\right) = \int \mathrm{D}\phi \exp\left\{ \frac{\mathrm{i}}{\hbar} \left[S[\phi] + \frac{\delta \Gamma[\,]}{\delta \bar{\phi}^k} (C^{-1})^k{}_j \sigma^j(\bar{\phi}, \phi) \right.\right.$$
$$- \frac{\delta \Gamma[\,]}{\delta \Sigma^{i_1 \ldots i_n}} \left(C^{i_1 \ldots i_n}_k + \frac{\delta \Sigma^{i_1 \ldots i_n}}{\delta \bar{\phi}^k} \right) (C^{-1})^k{}_j \sigma^j(\bar{\phi}, \phi)$$
$$\left.\left. + \frac{\delta \Gamma[\,]}{\delta \Sigma^{i_1 \ldots i_n}} [\sigma^{i_1}(\bar{\phi}, \phi) \times \ldots \times \sigma^{i_n}(\bar{\phi}, \phi) + \Sigma^{i_1 \ldots i_n}] \right] \right\}. \tag{7.76}$$

Here

$$C^{i_1 \ldots i_n}_k = \langle [\sigma^{i_1}(\bar{\phi}, \phi) \times \ldots \times \sigma^{i_n}(\bar{\phi}, \phi)]_{;k} \rangle .$$

We have obtained the parameterization invariant EA of composite fields. It can be called the unique EA of composite fields. It is convenient to consider this EA as the representative of the infinite class of parameterization invariant EAs (as in Chapter 6).

In [190] it was shown that in order to generalize the parameterization invariant EA for the case of gauge theories one needs to construct a connection with definite properties in field space. Using the reduction of the gauge theory to the non-gauge theory this connection has been constructed in [190]. One can then construct the gauge invariant EA which does not depend upon the choice of the gauge condition and parameterization of quantum fields (Chapter 6). Using the results of [190–195] and Chapter 6 along with the explicit form of the EA (7.71) and (7.72) one can construct the gauge and parameterization invariant EA of composite fields in the form

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma[v,\Sigma;\phi_*]\right) = \int \mathrm{D}\phi\ \mathrm{Det}\,\theta^\alpha_\beta \exp\Bigg\{\frac{\mathrm{i}}{\hbar}\Bigg[&S[\phi] \\ &+ \tfrac{1}{2}\eta_{\mu\nu}\chi^\mu\chi^\nu + \frac{\delta\hat{\Gamma}[v,\Sigma;\phi_*]}{\delta v^i}[\sigma^i(\phi_*,\phi)+v^i] \\ &+ \frac{\delta\hat{\Gamma}[v,\Sigma;\phi_*]}{\delta\Sigma^{i_1\ldots i_n}} \\ &\times\Big(\sigma^{i_1}(\phi_*,\phi)\times\ldots\times\sigma^{i_n}(\phi_*,\phi)+\Sigma^{i_1\ldots i_n} \\ &-(-1)^n v^{i_1}\times\ldots\times v^{i_n} \\ &+\frac{\delta}{\delta v^i}[(-1)^n v^{i_1}\times\ldots\times v^{i_n}][\sigma^i(\phi_*,\phi)+v^i]\Big)\Bigg]\Bigg\}\end{aligned} \tag{7.77}$$

where θ^α_β is a non-singular Faddeev–Popov operator, $\eta_{\mu\nu}$ represents an arbitrary chosen, constant metric on the orbits and χ^α is the gauge fixing condition.

The fact that the infinite class of the EAs (7.77) does not depend upon the choice of the gauge condition and is gauge invariant can be proved as was done in [190–196]. The EA (7.76) is generalized for the case of gauge theories in a similar way.

Let us discuss one important aspect of this problem. Consider the EA (7.77) in the one-loop approximation. As is well-known, in calculating the functional integral it is enough to limit ourselves to terms with ϕ^2 only (where ϕ is a quantum field). In this case, we can omit all the terms containing powers of σ^i of more than second order (and taking into account that $\sigma^i(\phi_*,\phi) = -(\phi^i_* - \phi^i)+\ldots$. We suppose also that the source J_i is equal to zero and that $v^i = 0$. If these restrictions are valid equation (7.76) defines the one-loop convenient EA of the composite field $(\phi^{i_1}_* - \phi^{i_1})(\phi^{i_2}_* - \phi^{i_2})$. This EA is parameterization and gauge invariant. From this follows the rule for the calculation of the one-loop EA of composite fields (like ϕ^2). This EA does not depend on the gauge condition and parameterization. It is necessary to write the unique EA of the composite field $\sigma^{i_1}(\phi_*,\phi)\sigma^{i_2}(\phi_*,\phi)$ and calculate this EA in the one-loop approximation.

As a simple example let us consider Yang–Mills theory and calculate the one-loop Vilkovisky generating functional $W[0, K]$ (background field $\bar{A}^a_\mu = 0$)

$$\exp\left(\frac{\mathrm{i}}{\hbar}W[0,K]\right) = \int \mathrm{D}A^a_\mu \; \mathrm{Det}\,\theta^\alpha_\beta \exp\Big(\frac{\mathrm{i}}{\hbar} \times \Big[-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}\chi^\mu\chi_\mu - K\sigma^{i_1}(0, A^a_\mu)\sigma^{i_1}(0, A^b_\nu)\Big]\Big) \tag{7.78}$$

It is known that in this parameterization the connection is equal to zero. Then, according to [198] if we choose the gauge condition in the form $R_{i\alpha}\phi^i = 0$, where $R_{i\alpha}$ are the gauge generators, we obtain $\sigma^{i_1} = A^{a\perp}_\mu$. Then the convenient generating functional $W[0, K]$ which is calculated in the Landau gauge coincides with the gauge and parameterization invariant generating functional. The explicit expression for $W[0, K]$ in the Landau gauge is calculated in [294] (see also [290])

$$\frac{W[0,K]}{\int \mathrm{d}^4x} = -\frac{3n}{16\pi^2}K^2(\ln K + C) \tag{7.79}$$

where $n = \delta^{aa}$, C is the normalization constant. From equation (7.79) $\Sigma = -(3n/16\pi^2)2K(\ln K + C + \frac{1}{2})$. Then the gauge- and parameterization-invariant unique EA is found to be

$$\Gamma[\Sigma] = \frac{3nK^2(\Sigma)}{16\pi^2}\left[\ln K(\Sigma) + C + \tfrac{1}{2}\right] \tag{7.80}$$

where $K(\Sigma)$ is defined from the equation which connects Σ and K (see above). Equation (7.80) is the parameterization- and gauge-invariant unique EA of the composite field $A^a_\mu A^{a\mu}$ in the one-loop approximation.

In this chapter we have discussed some properties of the effective action of composite fields in curved space–time. However, there are many questions for effective action of composite fields which are still open. We can say that we understand the structure of effective action of elementary fields better than the structure of effective action of composite fields.

PART 3

SELECTED PROBLEMS OF QUANTUM GRAVITY

8 Higher-derivative Quantum Gravity

8.1 Introduction

General covariance is the most fundamental principle of general relativity. The Einstein action which is constructed on the basis of this principle contains second derivatives of the metric. The corresponding equations of motion are covariant and moreover have a sensible Newtonian limit. These equations are in good accord with known experimental data at the classical level.

However, the Einstein theory does not give the framework for any quantum theory which is free of contradictions. It is well-known that the quantum theory is non-renormalizable off the mass shell. The presence of matter fields violates the renormalizability even on mass shell at one-loop level [146, 147]. The interest in quantum gravity leads to the necessity to construct theories which differ from general relativity. There are many possible ways of doing this. We can mention (without discussion) supergravity theories and the theory of superstrings.

During the last twenty years there has been considerable interest in higher-derivative gravity theory. This theory is obtained from general relativity by the addition of new terms to the Einstein action. As usual, the additional terms are only of second degree in the curvature tensor and therefore higher-derivative theory is also called R^2-gravity. The corresponding dynamical equations contain the fourth-order derivatives. The action of R^2-gravity is written in the following general form

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ -\frac{1}{\kappa^2}R + \Lambda + aR_{\mu\nu}R^{\mu\nu} + bR^2$$

$$+ cR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathrm{d}\Box R \Big\}. \tag{8.1}$$

Higher-derivative gravity theory (8.1) is multiplicatively renormalizable [136,137] and asymptotically free [138, 141], but it is not unitary within usual perturbation theory. One can (possibly) overcome the problem of non-unitarity by using non-perturbative approach (see, for example [140,141,145,148–150]). As the canonical quantization analysis shows, five quantitatively different versions of R^2-gravity are possible [150]. Those versions differ by the values of the parameters κ, Λ, $\alpha_1, \ldots \alpha_5$ in action (8.1). Note, that only the general version of the action (8.1) undoubtedly possesses multiplicative renormalizability. The renormalization properties of the conformal version are not clear (owing to the problem of conformal anomalies). The only well-known fact is that at the one-loop level conformal (Weyl) theory looks multiplicatively renormalized in the framework of the special conformal regularization [138, 151]. This regularization includes a non-local reparametrization of the metric field. The status of this renormalization will be discussed below.

At present a point of view exists that superstring theory gives a good possibility for a consistent discussion of quantum gravity effects. The description of 'usual' physical phenomena which occur at energy scale of less than the Planck energy corresponds to a low-energy approximation in superstring theory. From this point of view R^2-gravity is an effective theory and its applicability is restricted only by the energy scale. Generally speaking the effective theory does not have to satisfy all the requirements imposed on the fundamental theory. Therefore, even if R^2-gravity is really non-unitary it does not mean it is not applicable in the quantum region. At the same time the effective R^2-gravity may be useful itself since it allows us to study a number of special questions. For example, this gravitational theory is a useful base for unified models with good physical properties (see Chapter 9). Just the same reasons may help to explain the investigation of Weyl theory. One can ignore the problem of conformal anomalies because it is essential only in higher loops. Then we can understand our theory as a purely one-loop model and we suppose that at higher energy levels new degrees of freedom are switched on. Then all the phenomena are described by some fundamental theory but not just by Weyl gravity.

As was shown in Chapter 3, equation (8.1) is the general form of the vacuum action for GUT models in curved space–time. Consequently the theory (8.1) gives a possible basis for the construction of a unified quantum theory of all the fundamental interactions. This unified theory is multiplicatively renormalized as well as 'pure' R^2-gravity.

In this chapter we will consider a number of questions connected with the higher-derivative gravity. First of all we must investigate the propagator structure and point to the formal essence of the unitarity problem. Then we shall use the method described in Chapter 2 to write BRST transformations and corresponding identities for the generating functional of Green's functions and for the vertex functional. In sections 8.3, 8.4 and 8.5 we shall establish (following Stelle [136], Voronov and Tyutin [137]) the multiplicative renormalizability of R^2-gravity. In the next two sections the derivation of the one-loop counterterms is described (following Julve and Tonin [139], Fradkin and Tseytlin [138], Avramidy [153], Avramidy and Barvinsky [154]). The section 8.9 is devoted to asymptotic freedom in R^2-gravity. The features of the conformal theory and the theory with torsion are discussed in sections 8.8 and 8.10. The last sections 8.11 and 8.12 are devoted to the canonical quantization of R^2-gravity.

8.2 The structure of propagator and unphysical ghosts

Expression (8.1) is the general form of Lagrangian of R^2-gravity, but this Lagrangian contains some terms which are not important for our purposes. First of all we have the term $\Box R$ which is a surface term. This term does not give any contribution to the dynamical equations as well as to the propagator. The next non-important term is concerned with the Gauss–Bonnet topological invariant.

$$\begin{aligned} G &= \tfrac{1}{4}\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\kappa\omega}R_{\mu\nu\rho\sigma}R_{\kappa\omega\alpha\beta} \\ &= R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned} \tag{8.2}$$

In the four-dimensional space–time the integral

$$\int \mathrm{d}^4x\sqrt{-g}G \tag{8.3}$$

does not contribute to the dynamical equations. Note that in n-dimensional space–time ($n \neq 4$) the term (8.3) is essential because G does not form an n-dimensional divergence. Therefore, if we use dimensional regularization, expression (8.3) gives a contribution to the finite parts of the one-loop diagrams of the quantum theory. The k-loop approximation (8.3) gives the contribution to the poles $1/(n-4)^{k-i}$ where $i \geqslant 1$. One can begin the investigation of the propagator structure in four-dimensional space–time by neglecting the terms which contains $\Box R$ and G . We shall mention those terms below as necessary.

Let us now absorb the conformally invariant part of the action (8.1). For this purpose we may use the Weyl tensor

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \tfrac{1}{2}(g_{\mu\beta}R_{\nu\alpha} - g_{\mu\alpha}R_{\nu\beta} + g_{\nu\alpha}R_{\mu\beta} - g_{\nu\beta}R_{\mu\alpha}) + \tfrac{1}{6}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \tag{8.4}$$

It is easy to verify the conformal invariance of the Weyl tensor. The conformal invariance of $C_{\mu\nu\alpha\beta}$ allows us to construct the conformal invariant which has a structure like (8.1). This invariant is usually called the Weyl action and has the following form

$$S_W = \frac{1}{\lambda}\int \mathrm{d}^4x\sqrt{-g}C^2 = \frac{1}{\lambda}\int \mathrm{d}^4x\sqrt{-g}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}. \tag{8.5}$$

Since (8.5) is the single conformal invariant of the type found in (8.1) it is useful to rewrite (8.1) in the following form

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ -\frac{1}{\kappa^2}R + \Lambda + \frac{1}{\lambda}C^2 - \frac{\omega}{3\lambda}R^2 + (\text{surface terms})\Big\}. \tag{8.6}$$

Here κ, λ, Λ and ω are constants which characterize the gravitational interaction. Note that $C^2_{\mu\nu\alpha\beta}$ differs from the expression $W = R^2_{\mu\nu} - \frac{1}{3}R^2$ by the topological term $G = C^2 - 2W$. For further consideration we shall use action (8.6) to investigate the quantum higher-derivative gravity.

The dynamical equations have the form

$$\frac{1}{2\lambda}\Big\{g^{\alpha\beta}R^2_{\mu\nu} - \frac{\omega+1}{2}g^{\alpha\beta}R^2 + 2(\omega+1)RR^{\alpha\beta} - 2R_{\mu\nu}R^{\mu\alpha\nu\beta} + (2\omega+1)g^{\alpha\beta}\Box R - \Box R^{\alpha\beta} - (2\omega+1)\nabla^\alpha\nabla^\beta R\Big\} + \frac{1}{\kappa^2}\Big(R^{\alpha\beta} - \tfrac{1}{2}Rg^{\alpha\beta}\Big) + \tfrac{1}{2}\Lambda g^{\alpha\beta} = 0. \tag{8.7}$$

In the case of static fields the Newtonian limit of the theory (8.6) has the form [136]

$$h^{00} = \sqrt{-g}g^{00} - \eta^{00} \propto \frac{1}{r} - \frac{4}{3}\frac{e^{-m_2 r}}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r} \tag{8.8}$$

where

$$m_2 = -\left(\frac{\kappa^2}{2\lambda}\right)^{-1/2}$$

and

$$m_0 = -\left(\frac{\kappa^2\omega}{\lambda}\right)^{-1/2}.$$

If m_0 and m_2 are large enough then the additional terms in (8.8) are not essential at large distances and theory (8.1) has a good Newtonian limit. It is not true, however, in the theory without the Einstein term. Therefore this term is necessary to obtain a good classical potential.

If we want to investigate the theory (8.6) at a quantum level it is necessary to carry out the usual perturbative approach. First of all, one needs the propagator and vertices. The analysis of the propagator enables us to establish the particle content of the theory that is a first step to a quantum theory. To obtain the propagator one must present the metric as some expansion on the flat background. In the case of theory (8.6) it is impossible because the flat metric $\eta_{\mu\nu}$ is not the solution of the equations (8.7). To obtain the propagator one must put the cosmological constant Λ equal to zero. Later on we shall discuss possible consequences of this decision. To calculate the propagator one must invert the kinetic matrix of the classical action. Since the theory (8.6) is a gauge — the kinetic terms are degenerate — it is necessary to introduce some gauge fixing condition. A useful way of taking the gauge invariance of the action into account is the Faddeev–Popov method which was discussed in Chapter 2. We shall make some modifications here due to the features of higher-derivative theory.

Let us firstly define the quantum variables $h_{\mu\nu}$ by the relations

$$h_{\mu\nu} = \kappa^{-1}(g_{\mu\nu} - \eta_{\mu\nu}). \tag{8.9}$$

The general coordinate transformations of the fields $h_{\mu\nu}$ have the form

$$\begin{aligned}\delta h_{\mu\nu} &= R_{\mu\nu,\alpha}\xi^\alpha(x) = h_{\mu\nu,\alpha}\xi^\alpha(x) + \kappa^{-1}(\eta_{\mu\alpha}\partial_\nu\xi^\alpha + \eta_{\nu\alpha}\partial_\mu\xi^\alpha)\\ &\quad + h_{\mu\alpha}\partial_\nu\xi^\alpha + h_{\nu\alpha}\partial_\mu\xi^\alpha.\end{aligned} \tag{8.10}$$

To construct the generating functional by the Faddeev–Popov method we must choose some gauge fixing condition $\chi_\mu(h_{\alpha\beta}) = l_\mu$ and the weight operator $G_{\mu\nu}$. Then the generating functional of the Green's functions have the form

$$\begin{aligned}Z[J^{\mu\nu}] = (\mathrm{Det}\, G^{\mu\nu})^{1/2}\int \mathrm{D}h_{\mu\nu}\,\mathrm{D}\bar{C}_\alpha\,\mathrm{D}C^\beta\, \exp\mathrm{i}\{S(\eta_{\mu\nu} + \kappa h_{\mu\nu})\\ + S_{\mathrm{GF}}(\chi_\mu, G^{\mu\nu}) + S_{\mathrm{gh}}(h_{\mu\nu}, \bar{C}_\alpha, C^\beta)\\ + \int \mathrm{d}^4x\sqrt{-g}h_{\mu\nu}J^{\mu\nu}\}\end{aligned} \tag{8.11}$$

where the gauge term and the Faddeev–Popov ghost action have the form

$$S_{\mathrm{GF}} = \int \mathrm{d}^4x\sqrt{-g}\tfrac{1}{2}\chi_\alpha G^{\alpha\beta}\chi_\beta$$
$$S_{\mathrm{gh}} = \int \mathrm{d}^4x\sqrt{-g}\bar{C}_\alpha M^\alpha_\beta C^\beta \tag{8.12}$$

where

$$M^\alpha_\beta = \frac{\delta\chi^\alpha}{\delta h_{\mu\nu}} R_{\mu\nu,\beta}. \tag{8.13}$$

Generally speaking the operator M^α_β in (8.12) should possibly be multiplied by some power of the operator $G^{\mu\nu}$. Then we must change the power of the compensating expression ($\mathrm{Det}\, G^{\mu\nu}$) in (8.11).

To investigate the propagator structure it is reasonable to choose the forms of χ_μ and $G^{\mu\nu}$ which would not violate the locality of the action and do not lead to new dimensional parameters in S_{GF} and S_{gh}. We shall use the linear gauge condition of the following general form

$$\chi_\nu = \kappa\Big(\partial_\mu h^\mu_\nu - [\beta + \tfrac{1}{4}]\partial_\nu[h^{\alpha\beta}\eta_{\alpha\beta}]\Big) \tag{8.14}$$
$$G^{\mu\nu} = (1/\alpha)(-\eta^{\mu\nu}\Box + [\gamma - 1]\partial^\mu\partial^\nu). \tag{8.15}$$

Here α, β and γ are independent gauge parameters. To calculate the propagator of the field $h_{\mu\nu}$ we must inverse the kinetic form

$$\Omega^{\mu\nu,\alpha\beta}(x,y) = \tfrac{1}{2}\,\frac{\delta^2(S + S_{\mathrm{GF}})}{\delta h_{\mu\nu}(x)\delta h_{\alpha\beta}(y)}\bigg|_{h_{\mu\nu}=0}. \tag{8.16}$$

The bilinear part of the action $S + S_{\mathrm{GF}}$ with respect to quantum field $h_{\mu\nu}$ has the form

$$\begin{aligned}(S + S_{\mathrm{GF}})^{(2)} = \frac{\kappa^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,h_{\mu\nu}(x)\Big\{&\delta^{\mu\nu,\alpha\beta}\Big[\frac{1}{2\lambda}\Box^2 \\ &- \frac{1}{2\kappa^2}\Box\Big] + \eta^{\alpha\beta}\eta^{\mu\nu}\Big[\Big(-\frac{1+4\omega}{6\lambda} + \frac{\gamma C^2}{\alpha}\Big)\Box^2 + \frac{1}{2\kappa^2}\Box\Big] \\ &+ \Big[\frac{1-2\omega}{3\lambda} + \frac{\gamma - 1}{\alpha}\Big]\partial^\alpha\partial^\beta\partial^\mu\partial^\nu + \Big[\Big(\frac{1+4\omega}{3\lambda} - \frac{2\gamma C}{\alpha}\Big)\Box - \frac{1}{\kappa^2}\Big] \\ &\times \eta^{\alpha\beta}\partial^\mu\partial^\nu + \Big[\Big(\frac{1}{\alpha} - \frac{1}{\lambda}\Big)\Box + \frac{1}{\kappa^2}\Big]\eta^{\mu\alpha}\partial^\nu\partial^\beta\Big\}_x h_{\alpha\beta}(x).\end{aligned} \tag{8.17}$$

Here $C = \beta + \frac{1}{4}$. Let us note by the way that for the values

$$\alpha = 1 \qquad \beta = \frac{3\omega}{4(\omega+1)} \qquad \gamma = \tfrac{2}{3}(\omega + 1) \tag{8.18}$$

the bilinear form in (8.17) is minimal. For these values of gauge parameters expression (8.17) contains the derivatives only in the combination $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. To investigate the propagator one must go over to the momentum space and, following Stelle [136], rewrite the expressions $\Omega^{\mu\nu,\alpha\beta}(k)$ with the help of transverse and longitudinal projectors for vector quantities.

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{8.19}$$

Let us now use (8.19) to introduce the set of projectors for symmetric second-rank tensors in momentum space.

$$\begin{aligned} P^{(2)}_{\mu\nu,\rho\sigma} &= \tfrac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \tfrac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \\ P^{(1)}_{\mu\nu,\rho\sigma} &= \tfrac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}) \\ P^{(0-s)}_{\mu\nu,\rho\sigma} &= \tfrac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \\ P^{(0-\omega)}_{\mu\nu,\rho\sigma} &= \omega_{\mu\nu}\omega_{\rho\sigma}. \end{aligned} \tag{8.20}$$

It is easy to verify that the projectors (8.20) do not form a complete basis in the corresponding space. In order to have a complete basis we must also add to (8.20) two transfer operators

$$\begin{aligned} P^{(0-s\omega)}_{\mu\nu,\rho\sigma} &= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma} \\ P^{(0-\omega s)}_{\mu\nu,\rho\sigma} &= \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma}. \end{aligned} \tag{8.21}$$

The projectors and transfer operators satisfy the following orthogonality relations

$$\begin{aligned} P^{i-a}P^{j-b} &= \delta^{ij}\delta^{ab}P^{j-b} \\ P^{i-ab}P^{j-cd} &= \delta^{ij}\delta^{bc}P^{j-a} \\ P^{i-a}P^{j-bc} &= \delta^{ij}\delta^{ab}P^{j-ac} \\ P^{i-ab}P^{j-c} &= \delta^{ij}\delta^{bc}P^{j-ac}. \end{aligned} \tag{8.22}$$

It is easy to find the relations between the operators (8.20) and (8.21) and the second-rank tensors which appear in bilinear form (8.17) in the momentum space

$$\begin{aligned} \delta_{\mu\nu,\rho\sigma} &= P^{(2)}_{\mu\nu,\rho\sigma} + P^{(1)}_{\mu\nu,\rho\sigma} + P^{(0-s)}_{\mu\nu,\rho\sigma} + P^{(0-\omega)}_{\mu\nu,\rho\sigma} \\ \eta_{\mu\nu}\eta_{\rho\sigma} &= 3P^{(0-s)}_{\mu\nu,\rho\sigma} + \sqrt{3}P^{(0-s\omega)}_{\mu\nu,\rho\sigma} + \sqrt{3}P^{(0-\omega s)}_{\mu\nu,\rho\sigma} + P^{(0-\omega)}_{\mu\nu,\rho\sigma} \\ k_\mu k_\nu\eta_{\rho\sigma} + k_\rho k_\sigma\eta_{\mu\nu} &= \sqrt{3}k^2(P^{(0-s\omega)}_{\mu\nu,\rho\sigma} + P^{(0-\omega s)}_{\mu\nu,\rho\sigma}) + 2k^2P^{(0-\omega)}_{\mu\nu,\rho\sigma} \\ k_\mu k_\rho\eta_{\nu\sigma} + k_\mu k_\sigma\eta_{\nu\rho} + k_\nu k_\rho\eta_{\mu\sigma} + k_\nu k_\sigma\eta_{\mu\rho} &= 2k^2P^{(1)}_{\mu\nu,\rho\sigma} + 4k^2P^{(0-\omega)}_{\mu\nu,\rho\sigma} \\ k_\mu k_\nu k_\rho k_\sigma &= k^4P^{(0-\omega)}_{\mu\nu,\rho\sigma}. \end{aligned} \tag{8.23}$$

At last we can use (8.20)–(8.22) to invert the matrix $\Omega^{\mu\nu,\alpha\beta}(k)$. Let $\Omega(k)$ be written in the form (the indices are not essential and may be omitted)

$$\Omega(k) = a_1 P^{(2)} + a_2 P^{(1)} + a_3 P^{(0-s)} + a_4 P^{(0-\omega)} + a_5 P^{(0-s\omega)} + a_6 P^{(0-\omega s)}. \tag{8.24}$$

Then the propagator $G_0 = -[\mathrm{i}/(2\pi)^4]\Omega^{-1}$ has the form

$$G_0 = -\frac{\mathrm{i}}{(2\pi)^4}\left\{\frac{1}{a_1}P^{(2)} + \frac{1}{a_2}P^{(1)} + \frac{a_4}{\Delta}P^{(0-s)} + \frac{a_3}{\Delta}P^{(0-\omega)} - \frac{a_5}{\Delta}P^{(0-s\omega)} - \frac{a_6}{\Delta}P^{(0-\omega s)}\right\} \tag{8.25}$$

where $\Delta = a_3 a_4 - a_5 a_6$. Let us now rewrite the expression for $\Omega_{\mu\nu,\alpha\beta}$ in momentum space using (8.20) and (8.21)

$$\begin{aligned}\Omega(k) = \kappa^2\Bigg\{&\frac{k^2}{2\lambda}\left(k^2 + \frac{\lambda}{\kappa^2}\right)P^{(2)} + k^4\left(\frac{2}{\alpha} - \frac{3}{2\lambda}\right)P^{(1)}\\ &+ \frac{k^2}{\lambda}\left[\left(-2\omega + \frac{3\lambda\gamma C^2}{\alpha}\right)gk^2 - \frac{\lambda}{\kappa^2}\right]P^{(0-s)}\\ &+ \frac{k^2}{\lambda}\left[k^2\left(\frac{-\gamma C^2 - 2\gamma C + \gamma - 1}{\alpha}\lambda + \frac{4\omega}{3}\right) + \frac{\lambda}{\kappa^2}\right]P^{(0-\omega)}\\ &+ \sqrt{3}(P^{(0-\omega s)} + P^{(0-s\omega)})\left[\frac{-\gamma C(C+1)}{\alpha}k^4\right]\Bigg\}.\end{aligned} \tag{8.26}$$

Before using (8.25) to invert $\Omega(k)$, it is convenient to choose the special values for gauge parameters α, $\beta = C - 1/4$ and γ. For example, the values

$$\alpha = \frac{4\lambda}{3} \qquad C = \beta + \tfrac{1}{4} = 0 \qquad \gamma = \frac{9 - 16\omega}{9}$$

simplify expression (8.26). Hence we obtain the propagator $G(k)$ in the following form

$$\begin{aligned}G_0(k) = -\frac{\mathrm{i}}{(2\pi)^4}\Bigg\{&\frac{2\lambda}{k^2[k^2 + (\lambda/\kappa^2)]}P^{(2)}\\ &- \frac{\lambda}{2\omega\kappa^2}\,\frac{1}{k^2[k^2 + (\lambda/2\omega\kappa^2)]}P^{(0-s)} + \frac{1}{k^2}P^{(0-\omega)}\Bigg\}.\end{aligned} \tag{8.27}$$

The expressions (8.26) and (8.27) allow us to make a classification of the different versions of the theory (8.6). We can separate, for example, the general case (8.6), the theory without dimensional parameters $1/\kappa^2 = 0$, the conformal theory $\omega = 0$, $\Lambda = 1/\kappa^2 = 0$ and so on. An analogous classification will be given below in the framework of the canonical quantization. Now we will consider only the general case.

From expression (8.27) it follows that the propagator of the field $h_{\mu\nu}$ describes the propagation of the following particles. There are massless transverse fields of spin-two particles called gravitons, one transverse field of spin-two particles with mass $m_2 = (-\lambda\kappa^{-2})^{1/2}$, one massless scalar field and one scalar field of particles with mass $m_0 = (\lambda(2\kappa^2\omega)^{-1})^{1/2}$. Note that the coefficient in front of $P^{(2)}$ in (8.26) does not depend on the choice of gauge parameters. Therefore the value of m_2 is gauge-independent. On the contrary the coefficient in front of $P^{(0-s)}$ and the value of m_0 is gauge-dependent. In particular, one can choose the parameters β, γ so, that the mass in the $P^{(0-s)}$ term vanishes. Then the $P^{(0-\omega)}$ part of the propagator acquires a non-zero mass.

The form of the propagator depends not only on the choice of the gauge parameters but also on the choice of the quantum gravitational field parametrization. In particular, one can parametrize the gravitational field by $\Phi^{\mu\nu}$, which are defined by the relation

$$\kappa\Phi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}. \tag{8.28}$$

The gauge transformation for $\Phi^{\mu\nu}$ has the form

$$\begin{aligned}\delta\Phi^{\mu\nu} &= \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial_\alpha\xi^\alpha \\ &\quad + \kappa\Big(\partial_\alpha\xi^\mu\Phi^{\alpha\nu} + \partial_\alpha\xi^\nu\Phi^{\alpha\mu} - \xi^\alpha\partial_\alpha\Phi^{\mu\nu} - \Phi^{\mu\nu}\partial_\alpha\xi^\alpha\Big) \\ &= R^{\mu\nu,}_{\alpha}\xi^\alpha. \end{aligned} \tag{8.29}$$

Let us choose the gauge fixing condition and the weight operator in the following form

$$\chi^\tau = F^\tau_{\mu\nu}\Phi^{\mu\nu} \qquad F^\tau_{\mu\nu} = \delta^\tau_\mu\partial_\nu \qquad G_{\tau\sigma} = -\frac{1}{\alpha}\kappa^2\Box\eta_{\tau\sigma}. \tag{8.30}$$

Then the propagator of the field $\Phi^{\mu\nu}$ has the following form [136]

$$G_0(k) = -\frac{\mathrm{i}}{(2\pi)^4}\left\{\frac{2\lambda}{k^2(k^2+\lambda\kappa^{-2})}P^{(2)} - \frac{2\alpha}{\kappa^2k^4}P^{(1)}\right.$$

$$- \frac{2\lambda}{\omega\kappa^2 k^2 (k^2 + \lambda/2\omega\kappa^2)} P^{(0-s)}$$
$$- \frac{\alpha}{\kappa^2 k^4} (3P^{(0-s)} - \sqrt{3}P^{(0-s\omega)} - \sqrt{3}P^{(0-\omega s)}$$
$$+ P^{(0-\omega)}) \Big\}. \tag{8.31}$$

Note that the transition from the fields $h_{\mu\nu}$(8.9) to the fields $\Phi^{\mu\nu}$(8.28) is connected with a non-polynomial change of variables in the functional integral (8.11).

Nevertheless, the $P^{(2)}$ part of the field $\Phi^{\mu\nu}$ propagator (8.31) coincides with the $P^{(2)}$ part of the $h_{\mu\nu}$ field propagator (8.27). At the same time there is a remarkable difference between (8.27) and (8.31). If one put $\alpha \to 0$ in (8.31) then the propagator contains only transversal projectors $P^{(2)}$ and $P^{(0-s)}$. On the contrary, there is no combination of the gauge parameters which can remove the longitudinal components in propagator (8.27). We must conclude that this property of the propagator (8.31) concerns the choice of field variables as well as the choice of gauge fixing conditions.

Higher-derivative gravity leads to a gravitational field propagator which behaves like k^{-4} for large momenta. This property of the propagator solves the renormalization problem, but the unitarity problem arises instead. The transverse spin-two part of the propagator may be rewritten in the form

$$G_0(k) \propto P^{(2)} \frac{1}{m_2^2} \left(\frac{1}{k^2} - \frac{1}{k^2 + m_2^2} \right). \tag{8.32}$$

It follows from (8.32) that the physical content of higher-derivative gravity includes gravitons, i.e., the massless spin-two particles and the massive spin-two particles, which are usually called massive ghosts. The sign of the 'massive' term in (8.32) testifies to the negative energy of the corresponding state. Of course, we can reformulate the theory (to change the state of vectors) to provide positive energy for the ghosts, but then the massive pole will have a negative residue and therefore the corresponding norm in the state vector space will be negative. Hence, the massive spin-2 particles have negative energy or a negative norm and are therefore non-physical. The negative-norm states cannot be removed from the physical spectrum and hence unitarity is violated. Unlike the Faddeev–Popov ghost, which cancels with non-physical components of the gauge fields and hence preserves unitarity, the appearance of the massive spin-two ghosts contradicts unitarity. Note that the problem of unitarity is the most difficult in higher-derivative gravity. Lach of unitarity is the general reason why we do not consider R^2-gravity as the first and best candidate for the quantum gravity description. At the same time, this theory is a very attractive 'toy model' for the 'theory of everything'.

Of course, the problem of non-unitarity takes place for all higher-derivative theories. Possible ways of overcoming this obstacle have been discussed by a number of authors (see, e.g., [140, 141, 148]).

In particular, the following modification of perturbation theory has been proposed in [141,148]. The bare propagator $G_0(k^2)$ is replaced by the complete propagator $G(k^2)$

$$G^{-1}(k^2) = G_0^{-1}(k^2) - \Pi(k^2)$$

where $\Pi(k^2)$ is the sum of all the one-particle irreducible self-energy parts. Then one must consider only diagrams without self-energy parts. It is supposed that in some approximation the complete propagator $G(k^2)$ has a pair of complex conjugate poles instead of one real massive pole. In this case the ghosts disappear in asymptotic states and the unitarity is satisfied. An example of an approximation which satisfies this expectation is given by the $1/N$ expansion in the limit $N \to \infty$ [141]. Note that the corresponding consideration does not give the complete proof of unitarity.

We will not discuss the unitarity problem in more detail because it is not yet clear. One can find a detailed discussion of this problem in the papers mentioned above.

8.3 Properties of the effective action

We have chosen the gauge condition in higher-derivative gravity. It contains a differential operator of non-zero weight and consequently differs from the similar condition which we considered in Chapter 2. Therefore, one can expect that the BRST transformations and Ward equations for effective action also differ from the same objects in usual field theories. We need the modified version of BRST transformations and Zinn–Justin equations to investigate the renormalization structure in quantum R^2-gravity. A full consideration of this subject was given in the papers by Stelle [136] and Voronov and Tyutin [137]. To follow these papers it is useful to choose the field parametrization (8.28) and linear gauge condition like (8.30).

Let us rewrite the generating functional of Green's functions taking into account the gauge condition (8.30)

$$Z[J_{\mu\nu}] = \int \mathrm{D}\Phi^{\mu\nu}\,\mathrm{D}\bar{C}_\alpha\,\mathrm{D}C^\beta\,\exp \mathrm{i}\left\{ S + \int \mathrm{d}^4x \bar{C}_\tau \boldsymbol{F}^\tau_{\mu\nu} D^{\mu\nu}_\alpha C^\alpha\right.$$

$$- \frac{1}{2\alpha}\kappa^2 \int d^4x \chi_\tau \Box \chi^\tau + \int d^4x \Phi^{\mu\nu} J_{\mu\nu} \Big\} . \tag{8.33}$$

The action $S + S_{\text{GF}} + S_{\text{gh}}$ is not invariant under the local coordinate transformation, but it is invariant under the global BRST-transformations which have the following form

$$\delta_{\text{BRST}} \Phi^{\mu\nu} = \kappa R^{\mu}_{,\alpha} C^{\alpha} \delta\lambda \tag{8.34}$$

$$\delta_{\text{BRST}} C^{\alpha} = -\kappa^2 \partial_\beta C^\alpha C^\beta \delta\lambda$$

$$\delta_{\text{BRST}} \bar{C}_\tau = -\kappa^2 \alpha^{-1} \Box^2 \chi^\tau \delta\lambda. \tag{8.35}$$

Here $\delta\lambda$ is an infinitesimal anticommuting constant parameter. The difference between (8.35) and (2.167) is due to the choice of the weight operator $G_{\tau\sigma}$ (8.30). As in the general case the BRST-variations of S is zero, and the BRST-variation of S_{GF} and S_{gh} cancel each other. Therefore, the sum is BRST-invariant

$$\delta_{\text{BRST}}(S + S_{\text{GF}} + S_{\text{gh}}) = 0.$$

We need also the nilpotent property of BRST-transformations. Let us first establish the commutation relation for the generators (8.29)

$$\frac{\delta R^{\mu\nu}_{,\alpha}}{\delta \Phi^{\rho\sigma}} R^{\rho\sigma}_{,\beta} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta) = \kappa R^{\mu\nu}_{,\beta} (\partial_\alpha \xi^\beta \eta^\alpha - \partial_\alpha \eta^\beta \xi^\alpha). \tag{8.36}$$

Here and later we use condensed summation conventions: repeated indexes denote both summation over the discrete values and integration over the space–time arguments. To remove the ambiguities let us repeat (8.36) in the obvious form

$$\int dy\, du\, dv \left\{ \frac{\delta R^{\mu\nu}_{,\alpha}(x,y)}{\delta \Phi^{\rho\sigma}(y)} R^{\rho\sigma}_{,\beta}(y,v) [\xi^\alpha(u) \eta^\beta(v) - \eta^\alpha(v) \xi^\beta(u)] \right\}$$

$$= \kappa \int dy R^{\mu\nu}_{,\lambda}(x,y) \left[\frac{\partial}{\partial y^\alpha} \xi^\lambda(y) \eta^\alpha(y) - \xi^\alpha(y) \frac{\partial}{\partial y^\alpha} \eta^\lambda(y) \right].$$

To obtain (8.36) one can use the well-known Lie theorem on the representations of the general coordinate transformations group. All these representations have the same structure constants (2.134). Then the commutator satisfies the usual relation

$$\frac{\delta R^{\mu\nu}_{,\alpha}}{\delta \Phi^{\rho\sigma}} R^{\rho\sigma}_{,\beta} - \frac{\delta R^{\mu\nu}_{,\beta}}{\delta \Phi^{\rho\sigma}} R^{\rho\sigma}_{,\alpha} = R^{\mu\nu}_{,\lambda} f^{\lambda}_{\beta\alpha} \eta^\beta \xi^\alpha$$

and (8.36) follows. The nilpotency of BRST-transformations (8.34) and (8.35) follows from commutation relation (8.36) and the anti-commutating nature of C^α and $\delta\lambda$

$$\begin{aligned} \delta_{\mathrm{BRST}}(\partial_\beta C^\alpha C^\beta) &= 0 \\ \delta_{\mathrm{BRST}}(R^{\mu\nu}_{,\lambda} C^\alpha) &= 0. \end{aligned} \tag{8.37}$$

The generating functional of the Green's functions has the form

$$\begin{aligned} Z[J_{\mu\nu}, \bar{\beta}_\sigma, \beta^\tau, K_{\mu\nu}, L_\sigma] = & \int \mathrm{D}\Phi^{\mu\nu}\, \mathrm{D}C^\sigma\, \mathrm{D}\bar{C}^\tau \\ & \exp\left\{\mathrm{i}\left[\bar{S}(\Phi^{\mu\nu}, C^\sigma, \bar{C}_\tau, K_{\mu\nu}, L_\sigma) + \bar{\beta}_\sigma C^\sigma + \bar{C}_\tau \beta^\tau \right.\right. \\ & \left.\left. + \kappa J_{\mu\nu}\Phi^{\mu\nu}\right]\right\}. \end{aligned} \tag{8.38}$$

Here $\bar{S}$ is the general form of BRST-invariant action

$$\bar{S} = S + S_{\mathrm{GF}} + S_{\mathrm{gh}} + \kappa K_{\mu\nu} R^{\mu\nu}_{,\alpha} C^\alpha + \kappa^2 L_\sigma \partial_\beta C^\sigma C^\beta. \tag{8.39}$$

The sources $\bar{\beta}_\sigma$, β^τ, $K_{\mu\nu}$ and L_σ have been included for the ghost and antighost fields as well as for additional BRST-invariants. BRST-invariance of $\bar{S}$ is reflected in the following identity

$$\frac{\delta \bar{S}}{\delta K_{\mu\nu}} \frac{\delta \bar{S}}{\delta \Phi^{\mu\nu}} + \frac{\delta \bar{S}}{\delta L_\sigma} \frac{\delta \bar{S}}{\delta C^\sigma} + \kappa^3 \frac{1}{\alpha} \Box \chi_\tau \frac{\delta \bar{S}}{\delta \bar{C}_\tau} = 0. \tag{8.40}$$

The BRST-identity for the generating functional (8.38) is obtained in the usual way (see Chapter 2). The infinitesimal transformation (8.34)–(8.35) does not change the value of functional Z. The corresponding identity is written in the form

$$\begin{aligned} \int & \mathrm{D}\Phi^{\mu\nu}\, \mathrm{D}C^\sigma\, \mathrm{D}\bar{C}_\tau \left\{\kappa^2 J_{\mu\nu} R^{\mu\nu}_{,\alpha} C^\alpha \right. \\ & \left. - \kappa^2 \bar{\beta}_\sigma \partial_\beta C^\sigma C^\beta + \kappa^3 \alpha^{-1} \beta^\tau \Box F_{\tau,\mu\nu} \Phi^{\mu\nu}\right\} \\ & \times \exp\left\{\mathrm{i}\left[\bar{S} + \kappa J_{\mu\nu}\Phi^{\mu\nu} + \bar{\beta}_\sigma C^\sigma + \bar{C}_\tau \beta^\tau\right]\right\}. \end{aligned} \tag{8.41}$$

The supplementary identity (the equation of motion for ghosts) is a consequence of the invariance of the generating functional (8.38) under the transformation $\bar{C}_\tau \to \bar{C}_\tau + \delta \bar{C}_\tau$

$$\begin{aligned} \int & \mathrm{D}\Phi^{\mu\nu}\, \mathrm{D}C^\sigma\, \mathrm{D}\bar{C}_\tau \left(\frac{\delta \bar{S}}{\delta \bar{C}_\tau} + \beta_\tau\right) \\ & \exp \mathrm{i}\left\{\bar{S} + \kappa J_{\mu\nu}\Phi^{\mu\nu} + \bar{\beta}_\sigma C^\sigma + \bar{C}_\tau \beta^\tau\right\} = 0. \end{aligned} \tag{8.42}$$

The generating functional of connected Green's functions is defined as usual by

$$W = -\mathrm{i}\ln Z.$$

We can rewrite the BRST-identities (8.41) and (8.42) for W (see Chapter 2). Now we can define the generating functional of proper vertices $\Gamma[\Phi^{\mu\nu}, C^\sigma, \bar{C}_\tau, K_{\mu\nu}, L_\sigma]$ as a result of a Legendre transformation

$$\begin{aligned}\Gamma[\Phi^{\mu\nu}, C^\sigma, \bar{C}_\tau, K_{\mu\nu}, L_\sigma] = W[J_{\mu\nu}, \bar{\beta}_\sigma, \beta^\tau, K_{\mu\nu}, L_\sigma]\\ - \kappa J_{\mu\nu}\Phi^{\mu\nu} - \bar{\beta}_\sigma C^\sigma - \bar{C}_\tau \beta^\tau \qquad (8.43)\end{aligned}$$

where $\Phi^{\mu\nu}(x)$, $C^\sigma(x)$ and $\bar{C}_\tau(x)$ are the expectation values of the fields, which are the averages in the functional interval

$$\begin{aligned}\Phi^{\mu\nu}(x) &= \frac{\delta W}{\kappa\delta J_{\mu\nu}(x)}\\ C^\sigma(x) &= \frac{\delta W}{\delta\bar{\beta}_\sigma(x)} \qquad (8.44)\\ \bar{C}_\tau(x) &= -\frac{\delta W}{\delta\beta_\tau(x)}.\end{aligned}$$

The identities (8.41), (8.42) are easily rewritten in terms of Γ (see Chapter 2)

$$\frac{\delta\Gamma}{\delta K_{\mu\nu}}\frac{\delta\Gamma}{\delta\Phi^{\mu\nu}} + \frac{\delta\Gamma}{\delta L_\sigma}\frac{\delta\Gamma}{\delta C^\sigma} + \kappa^3\alpha^{-1}\Box \boldsymbol{F}_{\tau,\mu\nu}\Phi^{\mu\nu}\frac{\delta\Gamma}{\delta\bar{C}_\tau} = 0 \qquad (8.45)$$

$$\kappa^{-1}\boldsymbol{F}_{\tau,\mu\nu}\frac{\delta\Gamma}{\delta K_{\mu\nu}} - \frac{\delta\Gamma}{\delta\bar{C}_\tau} = 0. \qquad (8.46)$$

The last equation has an obvious analogy for the action $\bar{S}$

$$\kappa^{-1}\boldsymbol{F}^\tau_{\mu\nu}\frac{\delta\bar{S}}{\delta K_{\mu\nu}} - \frac{\delta\bar{S}}{\delta\bar{C}_\tau} = 0. \qquad (8.47)$$

Note that equation (8.45) also has the same form as the equation (8.40) for $\bar{S}$. The equations (8.40) and (8.45) can be simplified with the help of reduced quantities $\tilde{S}$ and $\tilde{\Gamma}$

$$\begin{aligned}\tilde{S} &= \bar{S} + \tfrac{1}{2}\kappa^2\alpha^{-1}\chi_\tau\Box\chi^\tau\\ \tilde{\Gamma} &= \Gamma + \tfrac{1}{2}\kappa^2\alpha^{-1}(\boldsymbol{F}_{\tau,\mu\nu}\Phi^{\mu\nu})\Box(\boldsymbol{F}^\tau_{,\mu\nu}\Phi^{\rho\sigma}).\end{aligned} \qquad (8.48)$$

In the last equation $\boldsymbol{F}_{\tau,\mu\nu}\Phi^{\mu\nu}$ does not coincide with χ_τ because the gauge condition (8.30) concerns only classical quantities.

Substitution of (8.48) into (8.40) and (8.45) (with the help of (8.47)) gives the usual form of Ward identities for $\tilde{\Gamma}$ and their classical analogy

$$\frac{\delta\tilde{S}}{\delta K_{\mu\nu}}\frac{\delta\tilde{S}}{\delta\Phi^{\mu\nu}}+\frac{\delta\tilde{S}}{\delta L_{\sigma}}\frac{\delta\tilde{S}}{\delta C^{\sigma}}=0 \tag{8.49}$$

$$\frac{\delta\tilde{\Gamma}}{\delta K_{\mu\nu}}\frac{\delta\tilde{\Gamma}}{\delta\Phi^{\mu\nu}}+\frac{\delta\tilde{\Gamma}}{\delta L_{\sigma}}\frac{\delta\tilde{\Gamma}}{\delta C^{\sigma}}=0. \tag{8.50}$$

(Note that sometimes this is called the Zinn–Justin equation.) The ghost equation (8.46) does not change under the substitution $\Gamma\to\tilde{\Gamma}$, and we obtain

$$\kappa^{-1}\boldsymbol{F}^{\tau}_{,\mu\nu}\frac{\delta\tilde{\Gamma}}{\delta K_{\mu\nu}}-\frac{\delta\tilde{\Gamma}}{\delta\bar{C}_{\tau}}=0. \tag{8.51}$$

The similarities between the form of (8.49) and (8.50) as well as between (8.47) and (8.51) are explained by the fact that on the tree level $\Gamma=\bar{S}$, $\tilde{\Gamma}=\tilde{S}$.

8.4 The renormalization structure

Equations (8.50) and (8.51) give a basis for the analysis of the effective action divergences and so to the research of the renormalization structure. The first proof of multiplicative renormalizability of the R^2-gravity was made by Stelle [136] with the help of the locality hypothesis. A more complete version including also the verification of this hypothesis was given in [137] by Voronov and Tyutin with the use of the previous papers [142–144].

Thus, we have a choice of what method to use. The complete approach of the paper [137] assumes prior knowledge of the modern and powerful methods of renormalization of gauge theories. Of course consideration of such methods does not have a place in the framework of this book. On the other hand, it is not necessary to repeat the paper [136] entirely because we would not improve it. As a result of long deliberations we have decided to write only the main ideas of both approaches. The reader can find a more complete exposition in the original papers.

We will use the following notation. The reduced effective action which is calculated in the k-loop order approximation is indicated as $\tilde{\Gamma}^{(k)}$. The renormalized action which leads to the finite effective action up to k-loop order is indicated as $\tilde{S}^{(k)}$. Of course, $\tilde{\Gamma}^{(k)}$ has a finite, $\tilde{\Gamma}^{(k)}_{\text{fin}}$, and a divergent, $\tilde{\Gamma}^{(k)}_{\text{div}}$ part, and

$$\tilde{\Gamma}^{(k)}=\tilde{\Gamma}^{(k)}_{\text{div}}+\tilde{\Gamma}^{(k)}_{\text{fin}}. \tag{8.52}$$

Suppose that we have renormalized the reduced action $\tilde{\Gamma}$ up to $n-1$ loop order. Then we have constructed renormalized reduced action $\tilde{S}^{(n)}$ which leads to the finite reduced effective action $\tilde{\Gamma}^{(i)}$ for $1 \leqslant i \leqslant n-1$. Hence, the first non-zero divergent term in (8.52) is $\tilde{\Gamma}^{(n)}_{\text{div}}$. All subdiagrams in every n-loop diagrams give the finite contributions and only the last n-loop integration gives the divergences. Let us now use Ward identities (8.50) to obtain the equation for $\tilde{\Gamma}^{(n)}$. If we put $\tilde{\Gamma} = \sum_{i=0}^{n} \tilde{\Gamma}^{(i)}$ and keep only the n-loop term then the following equation is valid:

$$\sum_{i=0}^{n} \left\{ \frac{\delta \tilde{\Gamma}^{(n-i)}}{\delta K_{\mu\nu}} \frac{\delta \tilde{\Gamma}^{(i)}}{\delta \Phi^{\mu\nu}} + \frac{\delta \tilde{\Gamma}^{(n-i)}}{\delta L_{\sigma}} \frac{\delta \tilde{\Gamma}^{(i)}}{\delta C^{\sigma}} \right\} = 0.$$

If we separate the terms which contain $\tilde{\Gamma}^{(n)}$ and take into account the relation $\tilde{\Gamma}^{(0)} = \tilde{S}$ we obtain

$$\mathbf{E}\left\{\tilde{\Gamma}^{(n)}_{\text{div}}\right\} = -\sum_{i=0}^{n} \left\{ \frac{\delta \tilde{\Gamma}^{(n-i)}_{\text{fin}}}{\delta K_{\mu\nu}} \frac{\delta \tilde{\Gamma}^{(i)}_{\text{fin}}}{\delta \Phi^{\mu\nu}} + \frac{\delta \tilde{\Gamma}^{(n-i)}_{\text{fin}}}{\delta L_{\sigma}} \frac{\delta \tilde{\Gamma}^{(i)}_{\text{fin}}}{\delta C^{\sigma}} \right\} \tag{8.53}$$

where

$$\mathbf{E} = \frac{\delta \tilde{S}}{\delta \Phi^{\mu\nu}} \frac{\delta}{\delta K_{\mu\nu}} + \frac{\delta \tilde{S}}{\delta K_{\mu\nu}} \frac{\delta}{\delta \Phi^{\mu\nu}} + \frac{\delta \tilde{S}}{\delta L_{\sigma}} \frac{\delta}{\delta C^{\sigma}} + \frac{\delta \tilde{S}}{\delta C_{\sigma}} \frac{\delta}{\delta L^{\sigma}}. \tag{8.54}$$

It is obvious that the right-hand side of (8.53) is not essential and may be omitted. If we use dimensional regularization then the left-hand side contains a factor $1/\varepsilon$ at least while the right-hand side remains finite at $\varepsilon \to 0$. Therefore both sides of equation (8.53) are equal to zero separately and we obtain a linear equation for $\tilde{\Gamma}^{(n)}$:

$$\mathbf{E}\left\{\tilde{\Gamma}^{(n)}_{\text{div}}\right\} = 0. \tag{8.55}$$

Moreover it is easy to use equation (8.51) in the same manner. The additional equation for $\tilde{\Gamma}^{(n)}_{\text{div}}$ is written in the form

$$\kappa^{-1} \boldsymbol{F}^{\tau}_{,\mu\nu} \frac{\delta \tilde{\Gamma}^{(n)}_{\text{div}}}{\delta K_{\mu\nu}} - \frac{\delta \tilde{\Gamma}^{(n)}_{\text{div}}}{\delta \bar{C}_{\tau}} = 0. \tag{8.56}$$

The last two equations have the same form for all linear gauges and do not depend on the explicit form of $\boldsymbol{F}^{\tau}_{,\mu\nu}$. Now we will try to construct the local solutions of equations (8.55) and (8.56). Let us first note that the operator $\mathbf{E}$(8.54) is nilpotent: $\mathbf{E}^2 = 0$. The cumbersome proof of this fact requires only operations with anticommuting variables, and hence we omit this proof. Since the operator (8.54) is nilpotent the general solution of equation (8.55) is

$$\tilde{\Gamma}^{(n)}_{\text{div}} = A[\Phi^{\mu\nu}] + \mathbf{E}\left\{X[\Phi^{\mu\nu}, C^\sigma, \bar{C}^\tau, K_{\mu\nu}, L_\sigma]\right\} \tag{8.57}$$

where A is an arbitrary local gauge-invariant functional of $\Phi^{\mu\nu}$ and its derivatives and X is an arbitrary local functional of $\Phi^{\mu\nu}$, C^σ, $\bar{C}_\tau$, $K_{\mu\nu}$, L_σ and their derivatives. In fact, (8.57) is the general local solution of the Ward identities (8.55) [136, 137].

To investigate the structure of expression (8.57) we need the quantity called the ghost number N_{g}. This quantity is defined in the following way

$$\begin{array}{lll} N_{\text{g}}[\Phi^{\mu\nu}] = 0 & N_{\text{g}}[C^\sigma] = 1 & N_{\text{g}}[\bar{C}_\tau] = -1 \\ N_{\text{g}}[K_{\mu\nu}] = -1 & N_{\text{g}}[L_\sigma] = -2. & \end{array} \tag{8.58}$$

All the terms in expressions (8.38) and (8.39) have zero ghost number and all the terms in operator $\mathbf{E}$(8.55) have ghost number equal to 1. We define N_{g} as the conserved quantity and therefore

$$N_{\text{g}}[\tilde{S}] = N_{\text{g}}[\tilde{\Gamma}] = 0 \qquad N_{\text{g}}[\mathbf{E}] = 0. \tag{8.59}$$

The consistency of conditions (8.59) requires

$$N_{\text{g}}[X] = -1 \tag{8.60}$$

To investigate the structure of $\tilde{\Gamma}^{(n)}_{\text{div}}$ it is necessary to use the power counting to obtain restrictions on the divergences. The power counting is the particular point for higher-derivative gravity which makes this theory different from general relativity [136]. The following notations will be used: n_E — the number of graviton vertices with two derivatives, n_{EE} — the number of graviton vertices with four derivatives, n_G — the number of graviton–ghost–antighost vertices, n_K — the number of K-graviton–ghost vertices, n_L - the number of L-ghost–ghost vertices, I_G — the number of internal ghost propagators, E_C — the number of external ghosts, $E_{\bar{C}}$ — the number of external antighosts.

The general relation for the degree of the divergence for an arbitrary diagram has the form

$$D = \sum_{l_{\text{int}}}(4 - r_l) - 4(n-1) + \sum_\nu K_\nu \tag{8.61}$$

where r_l is the momentum degree of the propagator denominator, n is the general number of the vertices, K_ν is the number of the derivatives of the vertex ν. The first sum is taken on all the internal lines. Since for the gravitons $r_l = 4$ and for the ghosts $r_l = 2$ we obtain

$$\begin{aligned} D &= 2I_G - 4(n_{EE} + n_E + n_G + n_K + n_L - 1) + 4n_{EE} + 2n_E \\ &\quad + 2n_G + n_K + n_L - E_{\bar{C}} \\ &= 4 + 2I_G - 2n_E - 2n_G - 3n_K - 3n_L - E_{\bar{C}}. \end{aligned} \tag{8.62}$$

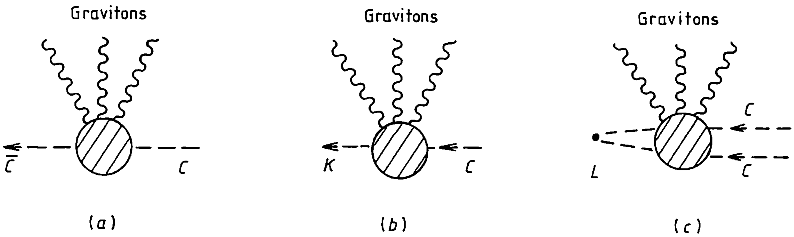
The appearance of $E_{\bar{C}}$ in the last expression is explained by the fact that the $\bar{C}\phi C$ vertex really has only one derivative (the second one acts on the antighost field and does not contribute to the degree of the divergence). Let us write the obvious relation between the number of internal ghost lines, the number of external ghost lines and the number of ghost tails in all the diagram vertices

$$2I_G - 2n_G = 2n_L + n_K - E_C - E_{\bar{C}}. \tag{8.63}$$

Inserting (8.63) into (8.62) we obtain the final expression

$$D = 4 - 2n_E - n_L - 2n_K - E_C - 2E_{\bar{C}} \tag{8.64}$$

The last relation together with the condition $N_G[\Gamma^{(n)}_{\text{div}}] = 0$ allows us to separate only three types of divergent diagrams which contain the internal ghost lines. It is illustrated in the figure below.



The number of external graviton lines is arbitrary. Each of the diagrams have degree of divergence $D = 1 - 2n_E$. Hence the divergent diagrams have the values of $n_E = 0$ and $D = 1$. The corresponding terms in $\tilde{\Gamma}^{(n)}_{\text{div}}$ contain one derivative which acts on one of the fields.

From the ghost equation (8.56) it follows that the field $\bar{C}_\tau$ and the source $K_{\mu\nu}$ appear in $\tilde{\Gamma}^{(n)}_{\text{div}}$ only within the structure $K_{\mu\nu} - \kappa^{-1}\bar{C}_\tau \boldsymbol{F}^\tau_{\mu\nu}$ and therefore (8.57) is rewritten in the form

$$\tilde{\Gamma}^{(n)}_{\text{div}} = A[\Phi^{\mu\nu}] + \mathbf{E}\left\{X(\Phi^{\mu\nu}, C^\sigma, K_{\mu\nu} - \kappa^{-1}\bar{C}_\tau \boldsymbol{F}^\tau_{\mu\nu}, L_\sigma)\right\}. \tag{8.65}$$

Consequently we can limit ourselves to the diagrams without internal $\bar{C}_\tau$ lines. In particular the structure (a) may be omitted in further considerations. The diagrams without internal ghost lines have the degree of divergence $D = 4 - 2n_E$. The permissible values of n_E are $0, 1, 2$ and therefore the gauge-invariant functional $A(\Phi^{\mu\nu})$ in (8.57) contains terms with four and two derivatives and terms without derivatives.

The restrictions imposed by the ghost number, by formula (8.65) and by the power counting allow us to establish the following general expression for $\tilde{\Gamma}^{(n)}_{\text{div}}$

$$\begin{aligned}\tilde{\Gamma}^{(n)}_{\text{div}} = A(\Phi^{\mu\nu}) + \mathbf{E}\left\{(K_{\mu\nu} - \kappa^{-1}\bar{C}_\tau \boldsymbol{F}^\tau_{,\mu\nu})P^{\mu\nu}(\Phi^{\alpha\beta})\right.\\ \left.+L_\sigma Q^\sigma_\tau(\Phi^{\alpha\beta})C^\tau\right\} \qquad (8.66)\end{aligned}$$

where $A[\Phi^{\mu\nu}]$ is a local gauge-invariant functional which contains terms with four, two and zero derivatives, $P^{\mu\nu}(\Phi^{\alpha\beta})$ and $Q^\sigma_\tau(\Phi^{\alpha\beta})$ are arbitrary Lorentz-covariant functions of the gravitational field $\Phi^{\mu\nu}$, but not of its derivatives. $P^{\mu\nu}$ and Q^σ_τ reflect the sums of all the diagrams of types (b) and (c), respectively.

Now we should use the form of the action (8.39) and the gauge condition (8.30) to obtain the explicit form of the operator $\mathbf{E}$ (8.54). Let us write the dynamical equation for $\tilde{S}$

$$\begin{aligned}\frac{\delta\tilde{S}}{\delta\Phi^{\mu\nu}} &= (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{\rho\sigma})\frac{\delta R^{\rho\sigma}_{,\alpha}}{\delta\Phi^{\mu\nu}}C^\alpha + \frac{\delta S}{\delta\Phi^{\mu\nu}}\\ \frac{\delta\tilde{S}}{\delta C^\sigma} &= -(\kappa K_{\mu\nu} - \bar{C}_\tau \boldsymbol{F}^\tau_{,\mu\nu})R^{\mu\nu}_\sigma + \kappa^2 L_\sigma C^\beta \overleftarrow{\partial}_\beta - \kappa^2 L_\tau(\partial_\sigma C^\tau)\\ \frac{\delta\tilde{S}}{\delta K_{\mu\nu}} &= \kappa R^{\mu\nu}_{,\alpha}C^\alpha \qquad \frac{\delta\tilde{S}}{\delta C^\sigma} = \kappa^2\partial_\beta C^\sigma C^\beta\end{aligned} \qquad (8.67)$$

and put (8.67) into expression (8.54) and (8.66). As a result, we obtain the final form of $\tilde{\Gamma}^{(n)}_{\text{div}}$ which is useful for the renormalization analysis.

$$\tilde{\Gamma}^{(n)}_{\text{div}} = A[(\phi^{\mu\nu}] \qquad (8.68)$$

$$+ \frac{\delta S}{\delta\Phi^{\mu\nu}}P^{\mu\nu} + (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\rho\sigma})\left(\frac{\delta R^{\rho\sigma}_{,\alpha}}{\delta\Phi^{\mu\nu}}C^\alpha\right)P^{\mu\nu} \qquad (8.68a)$$

$$- (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\rho\sigma})\frac{\delta P^{\rho\sigma}}{\delta\Phi^{\mu\nu}}R^{\mu\nu}_{,\alpha}C^\alpha \qquad (8.68b)$$

$$- (\kappa K_{\mu\nu} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\mu\nu})R^{\mu\nu}_{,\sigma}(Q^\sigma_\varepsilon C^\varepsilon) - \kappa^2 L_\sigma\partial_\beta(Q^\sigma_\tau C^\tau)C^\beta$$

where r_l is the momentum degree of the propagator denominator, n is the general number of the vertices, K_ν is the number of the derivatives of the vertex ν. The first sum is taken on all the internal lines. Since for the gravitons $r_l = 4$ and for the ghosts $r_l = 2$ we obtain

$$\begin{aligned} D &= 2I_G - 4(n_{EE} + n_E + n_G + n_K + n_L - 1) + 4n_{EE} + 2n_E \\ &\quad + 2n_G + n_K + n_L - E_{\bar{C}} \\ &= 4 + 2I_G - 2n_E - 2n_G - 3n_K - 3n_L - E_{\bar{C}}. \end{aligned} \tag{8.62}$$

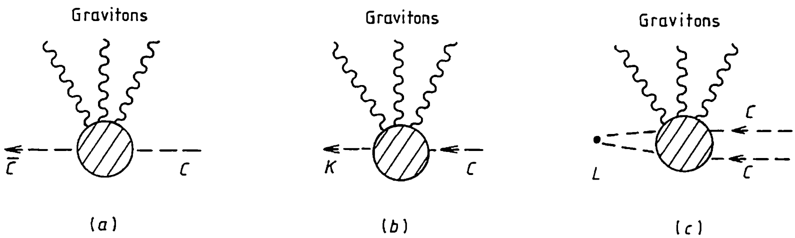
The appearance of $E_{\bar{C}}$ in the last expression is explained by the fact that the $\bar{C}\phi C$ vertex really has only one derivative (the second one acts on the antighost field and does not contribute to the degree of the divergence). Let us write the obvious relation between the number of internal ghost lines, the number of external ghost lines and the number of ghost tails in all the diagram vertices

$$2I_G - 2n_G = 2n_L + n_K - E_C - E_{\bar{C}}. \tag{8.63}$$

Inserting (8.63) into (8.62) we obtain the final expression

$$D = 4 - 2n_E - n_L - 2n_K - E_C - 2E_{\bar{C}} \tag{8.64}$$

The last relation together with the condition $N_G[\Gamma^{(n)}_{\text{div}}] = 0$ allows us to separate only three types of divergent diagrams which contain the internal ghost lines. It is illustrated in the figure below.



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Consequently we can limit ourselves to the diagrams without internal $\bar{C}_\tau$ lines. In particular the structure (a) may be omitted in further considerations. The diagrams without internal ghost lines have the degree of divergence $D = 4 - 2n_E$. The permissible values of n_E are $0, 1, 2$ and therefore the gauge-invariant functional $A(\Phi^{\mu\nu})$ in (8.57) contains terms with four and two derivatives and terms without derivatives.

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$$\tilde{\Gamma}^{(n)}_{\text{div}} = A(\Phi^{\mu\nu}) + \mathbf{E}\left\{(K_{\mu\nu} - \kappa^{-1}\bar{C}_\tau \boldsymbol{F}^\tau_{,\mu\nu})P^{\mu\nu}(\Phi^{\alpha\beta}) \right. \\ \left. + L_\sigma Q^\sigma_\tau(\Phi^{\alpha\beta})C^\tau\right\} \tag{8.66}$$

where $A[\Phi^{\mu\nu}]$ is a local gauge-invariant functional which contains terms with four, two and zero derivatives, $P^{\mu\nu}(\Phi^{\alpha\beta})$ and $Q^\sigma_\tau(\Phi^{\alpha\beta})$ are arbitrary Lorentz-covariant functions of the gravitational field $\Phi^{\mu\nu}$, but not of its derivatives. $P^{\mu\nu}$ and Q^σ_τ reflect the sums of all the diagrams of types (b) and (c), respectively.

Now we should use the form of the action (8.39) and the gauge condition (8.30) to obtain the explicit form of the operator $\mathbf{E}$ (8.54). Let us write the dynamical equation for $\tilde{S}$

$$\begin{aligned} \frac{\delta\tilde{S}}{\delta\Phi^{\mu\nu}} &= (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{\rho\sigma})\frac{\delta R^{\rho\sigma}_{,\alpha}}{\delta\Phi^{\mu\nu}}C^\alpha + \frac{\delta S}{\delta\Phi^{\mu\nu}} \\ \frac{\delta\tilde{S}}{\delta C^\sigma} &= -(\kappa K_{\mu\nu} - \bar{C}_\tau \boldsymbol{F}^\tau_{,\mu\nu})R^{\mu\nu}_\sigma + \kappa^2 L_\sigma C^\beta \overleftarrow{\partial}_\beta - \kappa^2 L_\tau(\partial_\sigma C^\tau) \\ \frac{\delta\tilde{S}}{\delta K_{\mu\nu}} &= \kappa R^{\mu\nu}_{,\alpha}C^\alpha \qquad \frac{\delta\tilde{S}}{\delta C^\sigma} = \kappa^2 \partial_\beta C^\sigma C^\beta \end{aligned} \tag{8.67}$$

and put (8.67) into expression (8.54) and (8.66). As a result, we obtain the final form of $\tilde{\Gamma}^{(n)}_{\text{div}}$ which is useful for the renormalization analysis.

$$\tilde{\Gamma}^{(n)}_{\text{div}} = A[(\phi^{\mu\nu}] \tag{8.68}$$

$$+ \frac{\delta S}{\delta\Phi^{\mu\nu}}P^{\mu\nu} + (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\rho\sigma})\left(\frac{\delta R^{\rho\sigma}_{,\alpha}}{\delta\Phi^{\mu\nu}}C^\alpha\right)P^{\mu\nu} \tag{8.68a}$$

$$- (\kappa K_{\rho\sigma} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\rho\sigma})\frac{\delta P^{\rho\sigma}}{\delta\Phi^{\mu\nu}}R^{\mu\nu}_{,\alpha}C^\alpha \tag{8.68b}$$

$$- (\kappa K_{\mu\nu} - \bar{C}_\tau \overleftarrow{F}^\tau_{,\mu\nu})R^{\mu\nu}_{,\sigma}(Q^\sigma_\varepsilon C^\varepsilon) - \kappa^2 L_\sigma \partial_\beta(Q^\sigma_\tau C^\tau)C^\beta$$

$$- \kappa^2 L_\sigma \partial_\beta C^\sigma Q^\beta_\tau C^\tau \tag{8.68c}$$

$$- \kappa L_\sigma \frac{\delta Q^\sigma_\tau}{\delta \Phi^{\mu\nu}} C^\tau R^{\mu\nu}_{,\alpha} C^\alpha + \kappa^2 L_\sigma Q^\sigma_\tau \partial_\beta C^\tau C^\beta. \tag{8.68d}$$

To remove divergences of the effective action (8.68) one must renormalize the coupling constants of the bare action (8.6) and the fields $\Phi^{\mu\nu}$ and C^σ as well as the gauge and BRST-transformation. Let us first consider the elimination of gauge-non-invariant divergences (8.68*a*–*d*). The necessary renormalization includes a non-linear reparametrization of the fields together with a corresponding gauge transformations and a non-linear reparametrization of the ghost field together with a corresponding BRST-transformations. We shall present the explicit form of this transformation without the details. To remove (8.68*a*) we must make the substitution

$$\Phi^{\mu\nu} \to \Phi^{\mu\nu} - P^{\mu\nu}(\Phi^{\alpha\beta}) \tag{8.69}$$

The elimination of (8.68*b*) requires the renormalization of the gauge transformations connected with (8.69). The terms (8.68*c*) are removed by the substitution

$$C^\sigma \to C^\sigma + Q^\sigma_\tau(\Phi^{\alpha\beta})C^\tau. \tag{8.70}$$

Finally, terms (8.68*d*) vanish when the BRST-transformations are modified in accordance with (8.70).

We will also consider a more simple version of the renorma- lization treatment. This version is considered in [136] and is used in [137]. Note that before (8.67) we had not used the explicit form of the gauge condition (8.30). All the above reasons do not depend on the choice of the linear operator $F^\tau_{,\mu\nu}$. The special feature of the variables (8.38) and the gauge condition (8.30) is that when $\alpha \to 0$ the propagator of the gravitational field contains only the transverse components. This choice of the gauge condition is equivalent to the direct use of the gauge condition

$$\partial_\mu \Phi^{\mu\nu} = 0. \tag{8.71}$$

The vertex of the antighost–graviton–ghost interaction (8.13) which corresponds to the gauge condition (8.30) and to the generator (8.29) has the form

$$V_{\bar{C}\Phi C} = \partial_\sigma \bar{C}_\alpha \partial_\rho \Phi^{\rho\sigma} C^\alpha + \partial_\alpha \bar{C}_\sigma \partial_\rho \Phi^{\rho\sigma} C^\alpha - \partial_\rho \partial_\sigma \bar{C}_\alpha \Phi^{\rho\sigma} C^\alpha. \tag{8.72}$$

The first two terms in (8.72) are equal to zero owing to (8.71). If we do not use (8.71) immediately then these two terms vanish under multiplication with the propagator. After integration by parts condition (8.71) is also used in the last term and we obtain the following expression

$$V_{\bar{C}\Phi C} = \bar{C}_\alpha \Phi^{\rho\sigma} \partial_\rho \partial_\sigma C^\alpha + \ldots . \tag{8.73}$$

Here the dots stand for terms which vanish due to (8.71). Every ghost and antighost line is connected with the vertex (8.73) in any one-particle irreducible diagram. Therefore each of these lines carries with it two factors of external momentum. Then the degree of divergence for any diagram is defined by the new (with respect to (8.62)) expression

$$D = 4 - 2n_E - n_L - 2n_K - 3E_C - 3E_{\bar{C}}. \tag{8.74}$$

Then one can easily test that the three types of diagrams considered above do not give divergent contributions to the gauge non-invariant part of $\tilde{\Gamma}^{(n)}_{\text{div}}$. Consequently $\tilde{\Gamma}^{(n)}_{\text{div}}$ is a gauge invariant functional of $\Phi^{\mu\nu}$. This functional contains terms with four and two derivatives and a term without derivatives. Therefore, the counterterms which are necessary to remove the divergences of the effective action at n-loop order have the following general form

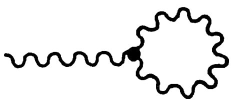
$$\Delta\tilde{S}^{(n)} = \int \mathrm{d}^4x\sqrt{-g}\Big\{\Big(-\frac{1}{\kappa^2}\Big)^{(n)} R + \Lambda^{(n)} + c^{(n)} R^2_{\mu\nu\alpha\beta} + a^{(n)} R^2_{\mu\nu} + b^{(n)} R^2 + \mathrm{d}^{(n)} \Box R^2\Big\}. \tag{8.75}$$

It is obvious that these counterterms may be cancelled by multiplicative renormalization of the parameters $\kappa, \Lambda, \alpha_1, \ldots, \alpha_4$. In the case when the bare action is taken in the form (8.6) we can use the Gauss–Bonnet identity in expression (8.75) to renormalize the surface and topological terms in a conventional manner (see Chapter 3 for a previous discussion). Then all the remaining counterterms have the same structure as the action (8.6) and we can remove them by corresponding multiplicative renormalizations.

Before the discussion of the problems which arise for the renormalization of the gauge-invariant terms we shall consider the elimination of the gauge-non-invariant counterterms (8.68*a*–*d*). First of all one can note that the analysis based on the gauge condition (8.71) does not use the locality hypothesis. That is why the corresponding field parametrization (8.28) together with the gauge condition lead to gauge-invariant functional form of $\tilde{\Gamma}^{(n)}$. Moreover, the general form of the gauge-invariant functional $A[\Phi^{\mu\nu}]$ (8.57) is fixed by the power counting. Of course, for another choice of quantum variables the disappearance of the gauge non-invariant part of the effective actionmay no longer hold. For example, the propagator of the field $h_{\mu\nu}$ (8.9) contains longitudinal components for every choice of the gauge parameters and hence the corresponding counterterms are not purely gauge-invariant.

In accordance with the main result of [137], if we want to prove local renormalizability it is enough to find one concrete particular parametrization like (8.28) which allows us to use local and multiplicative renormalization. The transition to another quantum field parametrization and to another gauge condition is realized by some canonical transformation of the quantum variables. Simultaneously the generators of the gauge group and the BRST-transformations are changed. The multiplicative renormalization of theory (8.6) was consistently proved (without the use of the locality hypothesis) in the framework of this approach.

Let us now consider gauge-invariant counterterms. The greatest problem here is connected with the cosmological constant. To apply the usual quantum field theory approach to the theory (8.6) we need the expansion of the metric on a flat background. Hence we must put $\Lambda = 0$ in the bare action. On the other hand a divergent counterterm which requires Λ to be renormalized has to appear if the Einstein term is present in the bare action. Really the propagator of the gravitational field contains a massive pole and therefore the tadpole diagram



(8.76)

is not proportional to $\delta^4(0)$ as it is in general relativity. We will also see that divergences like $\int \mathrm{d}^4x\sqrt{-g}\Lambda$ really appear when the counterterms are calculated in the framework of the background field method at the one-loop level.

Hence the cosmological constant problem is serious enough for the theory (8.6). The possible solution of this problem is connected with the use of the theory without dimensional parameters. Then the action has the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{\lambda}C^2_{\mu\nu\alpha\beta} - \frac{\omega}{3\lambda}R^2\right\} \tag{8.77}$$

instead of (8.6).

If we consider theory (8.77) as part of some unified theory including the massless matter fields then the Einstein and cosmological terms may appear together with matter field masses as a consequence of some symmetry breaking (see, for example, [156, 157]).

8.5 Interaction with matter fields

As was mentioned above, interaction with matter is important for R^2-gravity. In this section we will briefly describe the method of construction of a unified theory including R^2- gravity and a GUT model. We should like such a model to be multiplicatively renormalizable. Note that the use of interaction with matter allows us to deduce the asymptotic freedom of higher-derivative gravity (in the framework of $1/N$ expansion [141]). On the other hand, the interaction with R^2-gravity provides asymptotic freedom within a new GUT models with better physical properties [121—123]. The last question will be considered in the next chapter.

To obtain the action of a multiplicatively renormalized unified theory including R^2-gravity we must take the non-minimal action of matter fields in an external gravitational field and add this action to (8.6). All the surface and topological terms must also be included.

The GUT model usually contains Yang–Mills A_μ, spinor Ψ and scalar φ fields. Therefore, the action of the corresponding unified theory has the form

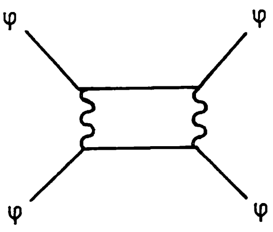
$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ & \frac{1}{\lambda}C^2_{\mu\nu\alpha\beta} - \frac{\omega}{3\lambda}R^2 - \frac{1}{\kappa^2}R + \Lambda + L_{\mathrm{YM}} \\ & + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)(D_\nu\varphi) + \tfrac{1}{2}\xi R\varphi^2 - V(\varphi) + \mathrm{i}\bar{\Psi}(D - \mathrm{i}h\varphi)\Psi \\ & + \tfrac{1}{2}m^2\varphi^2 - M\bar{\Psi}\Psi + \gamma G + \delta\Box R\Big\}. \end{aligned} \tag{8.78}$$

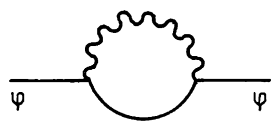
Here λ, ω, γ, δ and ξ are constants connected with the gravitational interaction, L_{YM} is the action of gauge fields.

Action (8.78) has the same structure as the action of a multiplicatively renormalized gauge theory in an external gravitational field. We will not give the formal proof but only some arguments about the multiplicative renormalizability in theory (8.78). Firstly, the gauge non-invariant divergences in theory (8.78) vanish when the appropriate quantum field parametrization and gauge conditions are used. For example, the gauge condition which is required for the metric field is (8.30). The transition to another gauge condition is produced by canonical transformation, which preserves the locality of the effective action [137, 143, 144]. Secondly, the theory (8.78) is renormalizable in the sense of the power counting. Then gauge invariance requires the divergences of the effective action to repeat the structure of the classical action. These divergences may then be removed by multiplicative renormalization of the theory. Note, that we can find a formal proof of the renormalizability of the R^2-gravity with a scalar field in [136].

From a well-known theorem it follows that dimensional parameters like m, M, κ, Λ do not influence the renormalization of the dimensionless parameters λ, ω, g etc. Therefore, a theory without massive, Einstein and cosmological terms is renormalizable as well as the general version. If we take the action without the dimensional parameters as the basis for quantum theory, then the Einstein and cosmological terms and mass terms may appear in the low-energy limit [136, 137]. These terms may be induced, for example, as a result of spontaneous symmetry breaking.

Although the action (8.78) coincides with the general action of a renormalizable theory in curved space–time, the renormalization structure is different. For example, let the fields A_μ and Ψ be absent from the action (8.78). Then multiplicative renormalizability requires terms like $\xi R\varphi^2$ and $f\varphi^4$ to be included to the action. If $\xi = f = 0$, then corresponding divergences arise due to the contribution of the diagrams





and renormalizability is lost.

8.6 Effective action of R^2-gravity in the background gauge

The one-loop approximation is of special interest in quantum field theory. One of the reasons is that investigation of higher loops is much more difficult. On the other hand, any physical properties can usually be found at the one-loop level. One example of such a property is asymptotic freedom (see, for example, [492]). The asymptotic freedom of the theory is usually established with the help of an investigation of β-functions. It is well-known that β-functions are closely connected with the renormalization constants. Hence, we must obtain the values of $\alpha_i^{(n)}$, $(1/\kappa^2)^{(n)}$ and $\Lambda^{(n)}$ (8.75) to investigate the asymptotic behaviour of the corresponding effective couplings. The values $\alpha_i^{(n)}$ do not depend on the choice of gauge condition [137]. Therefore, to calculate renormalization constants we can use arbitrary gauge conditions.

The most useful technique to obtain the one-loop divergences is the Schwinger–De Witt technique generalized by Barvinsky and Vilkovisky [105]. The application of this method is connected with the use of the background field method with the specific background gauge (see Chapter 2). The renormalization structure for R^2-theory in these gauges has been investigated in [137]. Since the theory (8.6) has a very complicated structure any quantum calculations here are quite difficult. The attempt to obtain one-loop renormalization constants was first made by Julve and Tonin [139]. Fradkin and Tseytlin [138] corrected the calculational method of [139]. The final correct result has been obtained in [153, 154]. We will follow these papers to demonstrate the method of calculations and also to obtain the β-functions.

In accordance with the usual background field method we must make the splitting of the metric $g_{\mu\nu}$ into the classical $g_{\mu\nu}$ and quantum fields $h_{\mu\nu}$. Let us use the following choice of quantum variables

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}.$$

To quantize the theory we choose the background gauge χ_μ and the weight functional $G^{\mu\nu}$ in the form

$$\chi_\mu = \chi_\mu(g_{\alpha\beta}, h_{\alpha\beta}) \qquad G^{\mu\nu} = G^{\mu\nu}(g_{\alpha\beta}). \tag{8.79}$$

Then the effective action is written in the form

$$\begin{aligned} Z[g_{\mu\nu}] = (\mathrm{Det}\, G^{\mu\nu})^{-1/2} \int & \mathrm{D}h_{\mu\nu}\, \mathrm{D}\bar{C}_\alpha\, \mathrm{D}C^\beta \\ & \times \exp\left\{ \mathrm{i}\Big[S(g+h) - \frac{\delta S}{\delta g_{\mu\nu}} h_{\mu\nu} + \tfrac{1}{2}\chi_\mu G^{\mu\nu}\chi_\nu \right. \\ & \left. + \bar{C}_\alpha M^\alpha_\beta C^\beta \Big] \right\} \end{aligned} \tag{8.80}$$

where

$$M^\alpha_\beta = G^{\alpha\lambda} \frac{\delta \chi_\lambda}{\delta g_{\rho\sigma}} R_{\rho\sigma,\beta}. \tag{8.81}$$

All the integrations and variational derivatives are performed in a covariant manner with respect to the background metric $g_{\mu\nu}$ (see, for example, [2]). The degree of the factor $(\operatorname{Det} G^{\mu\nu})$ in (8.80) is explained by the form of the ghost action (see section 8.2, for discussion). It is useful for us now to choose a ghost action containing four derivatives. This is the reason for (8.81).

To investigate the one-loop approximation (8.80) we must remember that only the bilinear terms of the action give a contribution to this perturbation order (Chapter 2). Let us present the formulae which are necessary to calculate the bilinear form of the action (8.6). Let

$$\bar{R}_{\mu\nu} = \bar{R}_{\mu\nu}(\bar{g}_{\alpha\beta}) \qquad \bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}.$$

Then

$$\begin{aligned}
\bar{g}^{\mu\nu} &= g^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_{\lambda}^{\ \nu} - \dots \\
\sqrt{-\bar{g}} &= \sqrt{-g}(1 + \tfrac{1}{2}h^* + \tfrac{1}{8}h^{*2} - \tfrac{1}{4}h_{\alpha\beta}h^{\alpha\beta}) + \dots \\
\bar{R}_{\mu\nu} &= R_{\mu\nu} + \tfrac{1}{2}(\nabla_\lambda\nabla_\mu h^\lambda_{\ \nu} + \nabla_\lambda\nabla_\nu h^\lambda_{\ \mu} - \nabla_\mu\nabla_\nu h^* - \Box h_{\mu\nu}) \\
&\quad + \tfrac{1}{2}h^{\lambda\tau}(\nabla_\nu\nabla_\mu h_{\lambda\tau} + \nabla_\lambda\nabla_\tau h_{\mu\nu} - \nabla_\lambda\nabla_\mu h_{\nu\tau} - \nabla_\lambda\nabla_\nu h_{\mu\tau}) \\
&\quad + \tfrac{1}{2}\nabla_\lambda h^{\lambda\tau}(\nabla_\tau h_{\mu\nu} - \nabla_\mu h_{\nu\tau} - \nabla_\nu h_{\mu\tau}) \\
&\quad + \tfrac{1}{2}(\tfrac{1}{2}\nabla_\mu h_{\lambda\tau}\nabla_\nu h^{\lambda\tau} + \nabla_\lambda h_\mu^{\ \tau}\nabla^\lambda h_{\nu\tau} - \nabla_\lambda h_\mu^{\ \tau}\nabla_\tau h^\lambda_\nu \\
&\quad + \tfrac{1}{2}\nabla_\lambda h^*\nabla_\mu h^\lambda_{\ \nu} + \tfrac{1}{2}\nabla_\lambda h^*\nabla_\nu h^\lambda_{\ \mu} - \tfrac{1}{2}\nabla_\lambda h^*\nabla^\lambda h_{\mu\nu}) + \dots \\
\bar{R} &= R + \nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box h^* - h^{\alpha\beta}R_{\alpha\beta} - h^{\mu\lambda}h^\nu_{\ \lambda}R_{\mu\nu} \\
&\quad + h^{\mu\nu}\nabla_\mu\nabla_\nu h^* + h^{\mu\nu}\Box h_{\mu\nu} - h^{\mu\nu}\nabla_\lambda\nabla_\mu h^\lambda_\nu - h^{\mu\nu}\nabla_\mu\nabla_\lambda h_\nu^{\ \lambda} \\
&\quad + \nabla_\lambda h^{\lambda\tau}\nabla_\tau h^* - \nabla_\lambda h^{\lambda\tau}\nabla_\sigma h^\sigma_{\ \tau} - \tfrac{1}{2}\nabla_\lambda h_{\tau\sigma}\nabla^\tau h^{\lambda\sigma} \\
&\quad + \tfrac{3}{4}\nabla_\sigma h_{\lambda\tau}\nabla^\sigma h^{\lambda\tau} - \tfrac{1}{4}\nabla_\lambda h^*\nabla^\lambda h^* + \dots .
\end{aligned} \tag{8.82}$$

Here $h^* = g^{\mu\nu}h_{\mu\nu}$. The dots stand for higher orders in an expansion on $h_{\mu\nu}$.

To obtain the bilinear form of the action (8.6) it is useful to substitute the Weyl term by the expression $W = R^2_{\mu\nu} - \frac{1}{2}R^2$. From now on we will not keep the surface terms. Of course, then we will not be able to obtain surface counterterms but we can greatly simplify our calculations. After substituting (8.82) into (8.6) and integrating by parts we obtain the second-order term in the $h_{\mu\nu}$-expansion in the form

$$
\begin{aligned}
&\left(\sqrt{-\bar g}\bar R_{\mu\nu}\bar R^{\mu\nu}\right)^{(2)}\\
&\quad = \tfrac{1}{2}\sqrt{-g}h_{\mu\nu}\Big(\tfrac{1}{2}\delta^{\mu\nu,\rho\sigma}\Box^2\\
&\quad + \tfrac{1}{2}g^{\rho\sigma}g^{\mu\nu}\Box^2 - \Box\nabla^\mu\nabla^\rho g^{\nu\sigma} + \nabla^\sigma\nabla^\rho\nabla^\mu\nabla^\nu - \nabla^\rho\nabla^\sigma\Box g^{\mu\nu}\\
&\quad + (R^{\mu\rho\nu\sigma} + R^{\mu\rho}g^{\nu\sigma})\Box + (\delta^{\mu\nu,\rho\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma})R^{\lambda\tau}\nabla_\lambda\nabla_\tau\\
&\quad - \tfrac{3}{2}R^{\rho\lambda}g^{\nu\sigma}(\nabla^\mu\nabla_\lambda + \nabla_\lambda\nabla^\mu) + \tfrac{1}{2}R^{\mu\rho}(\nabla^\sigma\nabla^\nu + \nabla^\nu\nabla^\sigma)\\
&\quad - R^{\rho\mu\sigma\lambda}(\nabla^\nu\nabla_\lambda + \nabla_\lambda\nabla^\nu) + g^{\mu\nu}R^{\rho\lambda}(\nabla_\lambda\nabla^\sigma + \nabla^\sigma\nabla_\lambda)\\
&\quad + \tfrac{5}{2}g^{\mu\rho}R^{\nu\lambda}R_\lambda{}^\sigma + 2R^\sigma{}_\lambda R^{\mu\rho\nu\lambda} + R^{\rho\lambda\sigma\tau}R^\mu{}_\lambda{}^\nu{}_\tau\\
&\quad - 2R^{\mu\lambda}R_\lambda{}^\nu g^{\rho\sigma} + \tfrac{1}{2}R^{\mu\rho}R^{\nu\sigma} - \tfrac{1}{2}R_{\lambda\tau}R^{\lambda\tau}(\delta^{\mu\nu,\rho\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma})\Big)h_{\rho\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\sqrt{-\bar g}\bar R^2\right)^{(2)}\\
&\quad = \tfrac{1}{2}\sqrt{-g}h_{\rho\sigma}\Big[2\nabla^\rho\nabla^\sigma\nabla^\mu\nabla^\nu\\
&\quad - 2g^{\mu\nu}\nabla^\rho\nabla^\sigma\Box - 2\Box\nabla^\mu\nabla^\nu g^{\rho\sigma} + 2g^{\mu\nu}g^{\rho\sigma}\Box^2 + R(g^{\rho\sigma}\nabla^\mu\nabla^\nu\\
&\quad + g^{\mu\nu}\nabla^\rho\nabla^\sigma + \delta^{\mu\nu,\rho\sigma}\Box - g^{\mu\nu}g^{\rho\sigma}\Box - g^{\mu\rho}\nabla^\nu\nabla^\sigma\\
&\quad - g^{\mu\rho}\nabla^\sigma\nabla^\nu) + 2R^{\rho\sigma}(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu)\\
&\quad + 2R^{\mu\nu}(g^{\rho\sigma}\Box - \nabla^\rho\nabla^\sigma) - \tfrac{1}{2}R^2(\delta^{\mu\nu,\rho\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma})\\
&\quad + R(5R^{\mu\rho}g^{\nu\sigma} - R^{\mu\rho\nu\sigma} - R^{\mu\nu}g^{\rho\sigma}) + 2R^{\mu\nu}R^{\rho\sigma}\Big]h_{\mu\nu}.
\end{aligned}
\tag{8.83}
$$

$$
\begin{aligned}
\left(\sqrt{-\bar g}\bar R\right)^{(2)} &= \tfrac{1}{2}\sqrt{-g}h_{\rho\sigma}\Big(\tfrac{1}{2}\delta^{\mu\nu,\rho\sigma}\Box - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma}\Box\\
&+ g^{\rho\sigma}\nabla^\mu\nabla^\nu - g^{\nu\sigma}\nabla^\mu\nabla^\rho + 2R^{\mu\rho}g^{\nu\sigma}\\
&- R^{\mu\nu}g^{\rho\sigma} - \tfrac{1}{2}R(\delta^{\mu\nu,\rho\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma})\Big)h_{\mu\nu}
\end{aligned}
$$

$$
\left(\sqrt{-\bar g}\Lambda\right)^{(2)} = \tfrac{1}{2}\sqrt{-g}h_{\rho\sigma}\Big(-\tfrac{1}{2}\Lambda(\delta^{\mu\nu,\rho\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma})\Big)h_{\mu\nu}.
$$

A useful form of the background gauge and weight functional (8.79) is the following

$$
\begin{aligned}
\chi_\mu &= \nabla_\alpha h^\alpha_\mu - (\beta + \tfrac{1}{4})\nabla_\mu(h^{\alpha\beta}g_{\alpha\beta})\\
G^{\mu\nu} &= \alpha^{-1}(-g^{\mu\nu}\Box - \gamma\nabla^\mu\nabla^\nu + \nabla^\nu\nabla^\mu).
\end{aligned}
\tag{8.84}
$$

Here α, β and γ are the gauge parameters. Note that (8.84) is the generalization of (8.13) to the case of a non-trivial background metric. As was pointed earlier, the values of the gauge parameters (8.18)

$$\alpha = 1 \qquad \beta = \frac{3\omega}{4(\omega+1)} \qquad \gamma = \tfrac{2}{3}(\omega+1)$$

provide the four derivative terms to be minimal. Really, the gauge-fixing term has the form (after some integration by parts)

$$\begin{aligned} L_{\mathrm{GF}} = \frac{1}{2\alpha} h_{\mu\nu} \Big\{ & \gamma C^2 g^{\mu\nu} g^{\alpha\beta} \Box^2 + g^{\nu\beta} \Box \nabla^\alpha \nabla^\mu + (\gamma - 1) \nabla^\mu \nabla^\nu \nabla^\alpha \nabla^\beta \\ & - 2\gamma C g^{\alpha\beta} \nabla^\mu \nabla^\nu \Box - R^{\alpha\mu} \nabla^\nu \nabla^\beta - R^{\alpha\mu} g^{\nu\beta} \Box + R^{\mu\alpha\nu\beta} \Box \\ & + 2 R^{\mu\alpha\nu\tau} \nabla_\tau \nabla^\beta - R^{\mu\tau} g^{\nu\beta} \nabla_\tau \nabla^\alpha \Big\} h_{\alpha\beta} \end{aligned} \tag{8.85}$$

where $C = \beta + 1/4$. We can easily verify that the non-minimal terms are cancelled. We shall use the notation

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{2\lambda}} (h_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} h^*) \qquad h = \sqrt{-\frac{\beta}{2\lambda}} h^* \tag{8.86}$$

where $h^* = h_{\mu\nu} g^{\mu\nu}$. The final expression for the bilinear part of the action has the form:

$$(S + S_{\mathrm{GF}})^{(2)} = \tfrac{1}{2} \int d^4x \sqrt{-g} (\bar{h}_{\rho\sigma} | h) (\hat{H}) \begin{pmatrix} \bar{h}_{\mu\nu} \\ h \end{pmatrix}. \tag{8.87}$$

The operator $\hat{H}$ has the following minimal structure

$$\hat{H} = \hat{1} \Box^2 + \hat{V}^{\rho\omega} \nabla_\rho \nabla_\omega + \hat{N}^\rho \nabla_\rho + \hat{U}. \tag{8.88}$$

Here

$$\hat{1} = \begin{pmatrix} \delta^{\mu\nu,\rho\sigma} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} & 0 \\ 0 & 1 \end{pmatrix}$$

Let us now use the notation of background dimension which was introduced in [105]. That is a value of dimension with respect only to background fields. It is easy to see that the background dimension of the $\hat{H}$ elements is $[\hat{V}^{\rho\omega}] = 1/l^2$, $[\hat{N}^\rho] = 1/l^3$, $[\hat{U}] = 1/l^3$. Since the theory (8.6) is multiplicatively renormalizable, the counterterms have the same background dimension as the classical action, that is $1/l^4$. Therefore the $\hat{N}^\rho$ matrix can appear in the counterterms only in the combination $\nabla_\rho \hat{N}^\rho$. Hence this matrix gives a contribution only to the surface counterterm $\Box R$ and we will omit $\hat{N}^\rho$ below. The expressions for $\hat{V}^{\rho\omega}$ and $\hat{U}$ have the form

$$
\begin{aligned}
\hat{V}^{\rho\omega} &= \begin{pmatrix} \hat{V}_{\bar{h}\bar{h}} & \hat{V}_{\bar{h}h} \\ \hat{V}_{h\bar{h}} & \hat{V}_{hh} \end{pmatrix} \qquad \hat{U} = \begin{pmatrix} \hat{U}_{\bar{h}\bar{h}} & \hat{U}_{\bar{h}h} \\ \hat{U}_{h\bar{h}} & \hat{U}_{hh} \end{pmatrix} \\
\hat{V}_{\bar{h}\bar{h}}^{\tau\omega,\mu\nu,\rho\sigma} &= 4g^{\tau\omega}R^{\mu\rho\nu\sigma} + 2\delta^{\mu\nu,\rho\sigma}(R^{\tau\omega} - \tfrac{1}{3}(1+\omega)Rg^{\tau\omega}) \\
&\quad - 4g^{\mu\rho}(R^{\nu(\tau}g^{\omega)\sigma} - R^{\sigma(\tau}g^{\omega)\nu}) \\
&\quad + \tfrac{4}{3}(1+\omega)(R^{\rho\sigma}\delta^{\mu\nu,\tau\omega} + R^{\mu\nu}\delta^{\rho\sigma,\tau\omega} + Rg^{\mu\rho}\delta^{\tau\omega,\nu\sigma}) \\
\hat{V}_{\bar{h}h}^{\tau\omega,\rho\sigma} &= -\frac{1}{\sqrt{-\beta}}\omega R^{\rho\sigma}g^{\tau\omega} \\
\hat{V}_{h\bar{h}}^{\tau\omega,\mu\nu} &= -\frac{1}{\sqrt{-\beta}}\omega R^{\mu\nu}g^{\tau\omega} \\
\hat{V}_{hh}^{\tau\omega} &= \frac{\omega}{4\beta}Rg^{\tau\omega} \\
U_{\bar{h}\bar{h}}^{\rho\sigma,\mu\nu} &= -\delta^{\mu\nu,\rho\sigma}\left(R_{\lambda\tau}R^{\lambda\tau} - \frac{1+\omega}{3}R^2 - \frac{\lambda}{\kappa^2}R + \lambda\Lambda\right) \\
&\quad + 5R^{\mu\lambda}R^{\rho}_{\lambda}g^{\nu\sigma} + 2R^{\rho}_{\lambda}R^{\mu\sigma\nu\lambda} + 2R^{\mu}_{\lambda}R^{\rho\nu\sigma\lambda} \\
&\quad + 2R^{\rho\lambda\sigma\tau}R^{\mu}{}_{\lambda}{}^{\nu}{}_{\tau} + R^{\mu\rho}R^{\nu\sigma} - \tfrac{4}{3}(1+\omega)R^{\mu\nu}R^{\rho\sigma} \\
&\quad - \frac{1+\omega}{3}R(10R^{\mu\rho}g^{\nu\sigma} - 2R^{\mu\rho\nu\sigma}) - \frac{4\lambda}{\kappa^2}R^{\mu\rho}g^{\nu\sigma} \\
U_{hh} &= \frac{8(\omega+1)}{3\beta}R^2 - \frac{4}{\beta}\lambda\Lambda.
\end{aligned}
\tag{8.89}
$$

We need also the ghost operator to obtain the explicit form of one-loop background functional. The gauge generator of quantum field $h_{\rho\sigma}$ has the form

$$R_{\rho\sigma,\beta} = g_{\rho\beta}\nabla_\sigma + g_{\sigma\beta}\nabla_\rho. \tag{8.90}$$

Then from (8.81) and (8.90) it follows that

$$M^\alpha_\beta = \delta^\alpha_\beta\Box^2 + \tfrac{2}{3}(2\omega-1)R_\beta{}^\lambda\nabla^\alpha\nabla_\lambda + R^{\lambda\alpha}R_{\lambda\beta}. \tag{8.91}$$

Note that the operator $\hat{M}$(8.91) has the same structure as $\hat{H}$ (8.88). We can now write the expression for the one-loop effective action in the form

$$\Gamma^{(1)} = \tfrac{1}{2}\,\mathrm{Tr}\ln\hat{H} - \mathrm{Tr}\ln(M^\alpha_\beta) - \tfrac{1}{2}\,\mathrm{Tr}\ln(G^{\mu\nu}) \tag{8.92}$$

where $\hat{H}$, M^α_β and $G^{\mu\nu}$ are defined by (8.88), (8.91) and (8.84).

8.7 Computation of one-loop divergences

To calculate the divergent part of the one-loop effective action (8.92) we shall use the universal functional traces which are the important part of the generalized Schwinger–De Witt technique [105]. The reader can easily find the details of this technique in the original paper [105]. That is why we will not describe the general method as well as the way of obtaining the universal traces. Instead of this we shall present the final expressions for these traces and demonstrate their application to the one-loop divergences calculation. The table of universal functional traces has the form [105]

$$\mathrm{Tr}\ln(\hat{1}\Box)\,|^{\mathrm{div}} = \frac{2}{\varepsilon}\int d^4x\sqrt{-g}\,\mathrm{Tr}\Big\{\tfrac{1}{180}(R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu}) + \tfrac{1}{72}R^2 + \tfrac{1}{30}\Box R + \tfrac{1}{12}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}\Big\}. \tag{8.93}$$

Here $\hat{1}$ is the unit matrix in the space of some field Φ and $\hat{R}_{\mu\nu}$ is the commutator of covariant derivatives

$$\hat{R}_{\mu\nu}\Phi = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\Phi$$

$$\begin{aligned}
&\nabla_\mu\nabla_\nu\frac{\hat{1}}{\Box}\delta(x,y)\Big|^{\mathrm{div}}_{y=x} \\
&= \frac{2}{\varepsilon}\sqrt{-g}\tfrac{1}{2}\Big\{\hat{1}\Big[-g_{\mu\nu}\Big(\tfrac{1}{180}R^2_{\alpha\beta\rho\sigma} - \tfrac{1}{180}R^2_{\rho\sigma} \\
&+ \tfrac{1}{72}R^2 + \tfrac{1}{30}\Box R\Big) + \tfrac{1}{45}R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \tfrac{1}{45}R_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda}{}_{\nu} \\
&- \tfrac{2}{45}R_{\mu\alpha}R^{\alpha}_{\nu} + \tfrac{1}{18}RR_{\mu\nu} + \tfrac{1}{30}\Box R_{\mu\nu} + \tfrac{1}{10}\nabla_\mu\nabla_\nu R\Big] \\
&- \tfrac{1}{12}g_{\mu\nu}\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} + \tfrac{1}{6}RR_{\mu\nu} + \tfrac{1}{3}\hat{R}_{(\mu\alpha}\hat{R}^{\alpha}_{\nu)} - \tfrac{1}{3}\nabla_{(\mu}\nabla^{\alpha}\hat{R}_{\alpha\nu)}\Big\}
\end{aligned} \tag{8.94}$$

$$\frac{\hat{1}}{\Box}\delta(x,y)\Big|^{\mathrm{div}}_{y=x} = \frac{2}{\varepsilon}\sqrt{-g}\Big[\tfrac{1}{6}R\hat{1}\Big] \tag{8.95}$$

$$\nabla_\mu\nabla_\nu\frac{\hat{1}}{\Box^2}\delta(x,y)\Big|^{\mathrm{div}}_{y=x} = -\frac{2}{\varepsilon}\sqrt{-g} \times\Big[\tfrac{1}{6}\Big(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\Big)\hat{1} + \tfrac{1}{2}\hat{R}_{\mu\nu}\Big] \tag{8.96}$$

$$\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_{2n-3}}\frac{\hat{1}}{\Box^n}\delta(x,y)\Big|^{\text{div}}_{y=x} = 0 \tag{8.97}$$

$$\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_{2n-4}}\frac{\hat{1}}{\Box^n}\delta(x,y)\Big|^{\text{div}}_{y=x} = -\frac{2}{\varepsilon}\sqrt{-g}\times\frac{\hat{1}g^{(n-2)}_{\mu_1\mu_2\dots\mu_{2n-4}}}{2^{n-2}(n-1)!} \tag{8.98}$$

where

$$g^{(0)} = 1 \qquad g^{(1)}_{\mu\nu} = g_{\mu\nu}$$
$$g_{(\mu\nu\alpha\beta)} = g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}$$
$$g^{(n)}_{\mu_1\dots\mu_{2n}} = g_{\mu_1\mu_2}g_{\mu_3\mu_4}\dots g_{\mu_{2n-1}\mu_{2n}} + \dots$$
$$= \frac{(2n)!}{2^n n!}g_{\mu_1\mu_2}g_{\mu_3\mu_4}\dots g_{(\mu_{2n-1}\mu_{2n})}$$

We have not presented the cumbersome expressions for the traces $\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta(\hat{1}/\Box^3)$ and $\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\nabla_\rho\nabla_\sigma(\hat{1}/\Box^4)$ because they will not be used below.

Now we can apply universal traces (8.93)–(8.98) to separate the divergences of the expression $\operatorname{Tr}\ln(\hat{H})$. The necessary transformations are not complicated.

$$\operatorname{Tr}\ln(\hat{H}) = \operatorname{Tr}\ln(\hat{1}\Box^2 + \hat{V}^{\rho\omega}\nabla_\rho\nabla_\omega + \hat{N}^\rho\nabla_\rho + \hat{U})\Big|^{\text{div}}$$
$$= \operatorname{Tr}\ln\left[\hat{1} + \hat{V}^{\rho\omega}\nabla_\rho\nabla_\omega\frac{1}{\Box^2} + \hat{N}^\rho\nabla_\rho\frac{1}{\Box^2} + \hat{U}\frac{1}{\Box^2}\right]\Bigg|^{\text{div}}$$
$$+ \operatorname{Tr}\ln(\hat{1}\Box^2)\,|^{\text{div}}$$
$$= 2\operatorname{Tr}\ln(\hat{1}\Box)\Big|^{\text{div}} + \operatorname{Tr}\left[\hat{V}^{\rho\omega}\nabla_\rho\nabla_\omega\frac{1}{\Box^2} + \hat{N}^\rho\nabla_\rho\frac{1}{\Box^2}\right.$$
$$\left.+ \hat{U}\frac{1}{\Box^2} - \tfrac{1}{2}\hat{V}^{\tau\omega}\nabla_\tau\nabla_\omega\frac{1}{\Box^2}\hat{V}^{\rho\sigma}\nabla_\rho\nabla_\sigma\frac{1}{\Box^2}\right]\Bigg|^{\text{div}} + \dots. \tag{8.99}$$

The omitted terms have a background dimension of more than $1/l^4$ and therefore do not contribute to divergences. All the terms except the last one admit the direct use of the universal traces (8.93)–(8.98). Since the background dimension of the last term in (8.99) is $1/l^4$, the additional terms which arise under the commutations, have a background dimension of more than $1/l^4$, and do not contribute to divergences. We can rewrite (8.99) with the use of the universal traces (8.93)–(8.98) in the form

$$\operatorname{Tr}\ln(\hat{H})|^{\text{div}} = \frac{2}{\varepsilon}\operatorname{tr}\left(\tfrac{1}{6}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} - \hat{U} - \tfrac{1}{6}\hat{V}^{\mu\nu}R_{\mu\nu} + \tfrac{1}{48}\hat{V}^2\right.$$

$$+ \tfrac{1}{12}\hat{V}^{\mu}_{\mu}R + \tfrac{1}{24}\hat{V}_{\mu\nu}\hat{V}^{\mu\nu} + \tfrac{1}{3}\nabla_{\mu}\nabla_{\nu}\hat{V}^{\mu\nu} + \tfrac{1}{12}\Box(\hat{V}^{\mu}_{\mu})^2$$
$$+ \tfrac{\hat{1}}{60}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + \tfrac{\hat{1}}{36}R^2 - \tfrac{\hat{1}}{180}G\Big). \tag{8.100}$$

We have presented the complete result including the surface and topological terms although these terms are not necessary for the renormalization of theory (8.6). The formula is also useful for the calculation of the ghost contribution.

To obtain the contribution of the weight functional we need the general formula for the non-minimal second-order operator which acts in vector space [105]

$$-\tfrac{1}{2}\,\mathrm{Tr}\ln\Big(\delta^{\mu}_{\nu}\Box - \frac{\zeta}{1+\zeta}\nabla^{\mu}\nabla_{\nu} + P^{\mu}_{\nu}\Big)\,\Big|^{\mathrm{div}}$$
$$= -\tfrac{1}{3}\frac{1}{\varepsilon}\int !\mathrm{d}^4x\sqrt{-g}\Big\{-\tfrac{11}{60}G + \Big(\tfrac{1}{8}\zeta^2 + \tfrac{1}{4}\zeta - \tfrac{4}{5}\Big)R_{\mu\nu}R^{\mu\nu}$$
$$+ \Big(\tfrac{1}{16}\zeta^2 + \tfrac{1}{4}\zeta + \tfrac{7}{20}\Big)R^2 + \Big(\tfrac{1}{4}\zeta^2 + \zeta\Big)P_{\mu\nu}R^{\mu\nu} + \Big(\tfrac{1}{8}\zeta^2 + \tfrac{3}{4}\zeta + \tfrac{3}{2}\Big)$$
$$\times P_{\mu\nu}P^{\mu\nu} + \tfrac{1}{16}\zeta^2 P^{\mu}_{\mu}P^{\nu}_{\nu} + \Big(\tfrac{1}{8}\zeta^2 + \tfrac{1}{4}\zeta + \tfrac{1}{2}\Big)RP^{\mu}_{\mu}\Big\}. \tag{8.101}$$

Substituting expressions (8.89), (8.91), (8.84) and (8.85) into (8.100), (8.101) and (8.92) we can find the final result for the one-loop divergences of the effective action in the theory (8.6) [138, 153, 154]

$$\Gamma^{(1)}_{\mathrm{div}} = \frac{1}{\varepsilon}\int \mathrm{d}^4x\sqrt{-g}\Big\{\beta_1 G + \tfrac{1}{2}\beta_2 C^2$$
$$+ \tfrac{1}{3}\beta_3 R^2 + \beta_4\frac{1}{\kappa^4} + \gamma\frac{1}{\kappa^2}(R - 2\Lambda\kappa^2)\Big\} \tag{8.102}$$

where

$$\beta_1 = -\tfrac{196}{45} \qquad \beta_2 = \tfrac{133}{10} \qquad \beta_3 = \tfrac{10}{3}\omega^2 + 5\omega + \tfrac{5}{12}$$
$$\beta_4 = \frac{\lambda^2}{2}\Big(5 + \frac{1}{4\omega^2}\Big) + \frac{\lambda}{3}\kappa^4\Lambda\Big(20\omega + 15 - \frac{1}{2\omega}\Big) \tag{8.103}$$
$$\gamma = \lambda\Big(\tfrac{10}{3}\omega - \tfrac{13}{6} - \frac{1}{4\omega}\Big).$$

Let us briefly comment on expressions (8.103). Since $\beta_4 \neq 0$ the cosmological constant problem is essential in R^2-gravity even at one-loop order. If we reject the cosmological term in the bare action, the corresponding counterterm cannot be removed by the renormalization of the bare parameters of theory (8.6).

8.8 Effective action in Weyl gravity

This section is devoted to the consideration of the features of the conformal (Weyl) version of higher-derivative gravity. The action of the Weyl theory (8.5) is invariant under conformal transformation of the metric field

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \mathrm{e}^{2\sigma(x)} g_{\mu\nu}(x). \tag{8.104}$$

Note that the conformal invariance of the action (8.5) is violated when an n-dimensional space with $n \neq 4$ is under consideration. This fact has a remarkable consequence. The point is that dimensional regularization, which is usually used in gauge theories breaks the conformal invariance. We do not know a regularization which can preserve both general covariance and conformal invariance in the quantum region. Therefore this theory possibly gives rise to anomalies. Note that there is no clear proof of the existence of the conformal anomaly in theory (8.5). The only known fact is that this anomaly can arise. Of course, if the anomaly does exist, then the conformally non-invariant counterterms arise and renomalization is no longer possible.

As will be checked below, theory (8.5) is very useful for the investigation of some GUT features. This is why we need the method to consider Weyl gravity as a multiplicatively renormalized theory. Note that there really is a way of avoiding the investigation of the anomaly problem in this theory. It is connected with the use of a trick which was proposed in a number of papers [138, 151, 152]. Within this method, one has to make an appropriate change of the background fields after all the quantum calculations. Then conformal invariance is restored and the anomaly is formally absent. Of course, the change of variables may turn out to be just the origin of the anomaly, but formally we obtain conformal invariant counterterms. We shall consider here an example of the change of variables, which was found in [138] as an element of regularization. We also apply this regularization to one-loop divergence calculation. The generalization of this procedure in the case of conformal gravity with matter will be considered in Chapter 9.

First let us consider the origin of the conformal invariance violation at one-loop order in the framework of the background field method. Since general covariance is not the only symmetry in Weyl theory the background gauge condition (8.84) leaves the four derivative terms in the bilinear form (8.88) degenerate. To remove this degeneracy we must introduce a condition, which fixes the conformal symmetry. A useful condition of such type is $h = 0$. Then the unit matrix in the space of gravitational fields is the projector

$$\hat{1} = \delta^{\mu\nu,\rho\sigma} - \tfrac{1}{4} g^{\mu\nu} g^{\rho\sigma}. \qquad (8.105)$$

The conformal symmetry fixing condition $h = 0$ does not contain derivatives. Therefore, this condition does not require the use of the Faddeev–Popov method and does not need gauge ghosts to preserve unitarity of the S-matrix. We can easily modify expressions (8.89) to obtain the conformal ones. It is necessary only to reject the $H_{\bar{h}h}$, $H_{h\bar{h}}$ and H_{hh} parts of the bilinear form $(\hat{H})$ which are concerned with the field h. Of course, we must use also the conditions $\omega = 0$, $1/\kappa^2 = 0$, $\Lambda = 0$ in the $H_{\bar{h}\bar{h}}$ part.

Hence, we have to put restrictions on the quantum field which remove the degeneracy of the bilinear form of the classical action. Before proceeding to the analysis of the divergence structure we must answer the following question. Do we satisfy all the requirements of the gauge fixing conditions which are necessary in the framework of the background field method or not? The answer is negative. The trouble is that the general covariance fixing condition (8.84) violate the conformal covariance of the bilinear form (without removing the degeneracy). As a result the expressions $\hat{V}^{\rho\omega}$ and $\hat{U}$ are not conformally covariant operators. The one-loop divergences of the effective action are built with the help of these expressions and therefore $\Gamma^{(1)}_{\text{div}}$ is possibly a conformally non-invariant functional.

The corresponding computations at one-loop order have been presented in [138]. The β-functions in $\Gamma^{(1)}_{\text{div}}(8.102)$ which were obtained in the way described above have the values

$$\beta_1 = -\tfrac{261}{60} \qquad \beta_2 = \tfrac{199}{15} \qquad \beta_3 = -\tfrac{1}{2}. \qquad (8.106)$$

Since $\beta_3 \neq 0$ violation of conformal invariance really takes place. It is most likely that this fact is not a manifestation of conformal anomaly. The existence of non-zero β_3 at one-loop order only means the absence of a conformally covariant calculation technique. Simple considerations show that we cannot construct a conformally covariant technique with the help of the choice of gauge fixing conditions only. For example, the gauge fixing condition (8.84) leads to a gauge fixing term which has a four derivative structure for any values of gauge parameters. At the same time a single conformally covariant operator of this type is connected with the Weyl action. Of course, such a gauge fixing term is impossible.

To remove the conformally non-covariant counterterms we will follow [138, 151]. Let us introduce the new quantity $P(g_{\mu\nu})$ which satisfies the equation

$$\Box P(g_{\mu\nu}) = \tfrac{1}{6} R P(g_{\mu\nu}). \qquad (8.107)$$

We can write the solution of equation (8.107) in a following form

$$P = 1 + \tfrac{1}{6}\hat{F}R$$

where $\hat{F} = (\Box - \frac{1}{6}R)^{-1}$ is the corresponding Green's function with zero boundary conditions at infinity (we mean that the background metric is asymptotically flat).

The value of $P(g_{\mu\nu})$ changes under conformal transformation (8.104) as a usual scalar field

$$P \to P' = P\exp\left[\frac{2-n}{2}\sigma(x)\right]. \tag{8.108}$$

If the space–time dimension is four then

$$\tilde{g}_{\mu\nu} = P^2(g_{\alpha\beta})g_{\mu\nu} \tag{8.109}$$

is conformally invariant. The basic idea is to substitute the background metric $g_{\mu\nu}$ by the new conformally invariant one $\tilde{g}_{\mu\nu}$. The metric $\tilde{g}_{\mu\nu}$ possesses some interesting properties. All the metric field manifolds may be separated into classes of conformally equivalent metrics. Every two metrics in such a class are connected by conformal transformations like (8.104). Every class of conformally equivalent metrics contains a single metric $\tilde{g}_{\mu\nu}$(8.109). Since (8.109) is the conformal transformation, this transformation does not change the action (8.5). The straightforward calculation shows that $R(\tilde{g}_{\mu\nu}) = 0$. Therfore the conformal transformation (8.109) removes the non-conformal counterterms and we obtain

$$\beta_1 = -\tfrac{261}{60} \qquad \beta_2 = \tfrac{199}{15} \tag{8.110}$$

instead of (8.106). The substitution of $\tilde{g}_{\mu\nu}$ instead of $g_{\mu\nu}$ is an element of special conformal regularization which supplements the background field method. This regularization corrects the shortcomings of the calculation method. The described substitution is really a non-local change of the background variables. As was pointed out in [151] this regularization does not break the S-matrix unitarity. Thus, the special conformal regularization can help to avoid difficulties related with the anomaly. Of course, if the anomaly does exist, then this method as a whole is not completely consistent. At the same time, we believe that this method is interesting in itself. Note that we can consider (8.5) as a pure one-loop quantum theory. Then we use the arguments which were considered in the introduction of this chapter.

8.9 Renormalization group equations and the asymptotic behaviour of the effective couplings

We will consider the asymptotic behaviour of the effective coupling constants for the general R^2-theory (8.6) as well as for the Weyl theory (8.5). The form of the renormalization group equations for the effective constants $\lambda(t), \omega(t), \kappa(t), \Lambda(t)$ follows from expressions (8.103) [138,153,154]

$$\frac{d\lambda}{dt'} = -a^2\lambda^2 \qquad t' = \frac{t}{(4\pi)^2} \qquad a^2 = \tfrac{133}{10} = \beta_2 \tag{8.111a}$$

$$\frac{d\omega}{dt'} = -\lambda(\omega\beta_2 + \beta_3) = -\lambda\left[\tfrac{10}{3}\omega^2 + (5+a^2)\omega + \tfrac{5}{12}\right] \tag{8.111b}$$

$$\frac{d\kappa^2}{dt'} = \kappa^2\gamma = \lambda\kappa^2\left[\tfrac{20}{3}\omega - \tfrac{13}{3} - \frac{1}{2\omega}\right] \tag{8.111c}$$

$$\frac{d\Lambda}{dt'} = \kappa^{-4}\beta_4 - 2\gamma\Lambda = \frac{\lambda^2}{\kappa^4}\left(\tfrac{5}{2} + \frac{1}{8\omega^2}\right) + \lambda\Lambda\left(\tfrac{28}{3} + \frac{1}{3\omega}\right) \tag{8.111d}$$

The form of the equations (8.111c) and (8.111d) is explained by the fact that γ is really the anomalous dimension for the non-essential coupling $\kappa^2(t)$ (see [138, 153, 154, 494] for a discussion). The structure of the equations (8.111*a*–*d*) allows us to investigate the asymptotic behaviour of the couplings $\lambda(t)$, $\omega(t)$, $\kappa^2(t)$, $\Lambda(t)$ one after another. Equation (8.111*a*) leads to the asymptotically free behaviour of the coupling $\lambda(t)$

$$\lambda(t') = \lambda(1 + \lambda a^2 t')^{-1} \tag{8.112}$$

where $\lambda = \lambda(0)$. Note that the positive value of λ is necessary for a positive contribution of the Weyl term to the Euclidean action. Equation (8.111*b*) have two fixed points

$$\omega_1 \approx -0.02 \qquad \omega_2 \approx -5.47. \tag{8.113}$$

Hence this equation has two special solutions

$$\omega(t) \equiv \omega_1 \text{ and } \omega(t) \equiv \omega_2.$$

The investigation of the general solution shows that $\omega(t) \to \omega_1$ when $\omega > \omega_2$ and $\omega(t) \to -\infty$ when $\omega < \omega_2$ in the UV limit $t \to +\infty$. Here $\omega = \omega(0)$. Moreover in the case $\omega > \omega_2$ when $t \to +\infty$

$$\omega(t) \approx \omega_1 + c(1 + \lambda a^2 t')^{-p} \tag{8.114}$$

where $p = (10/3a^2)(\omega_1 - \omega_2) \approx 1.36$ [153, 154]. We can reject the case $\omega < \omega_2$ because the unphysical zero-charge behaviour of $\omega(t)$ leads to the same behaviour of the effective coupling $\lambda(t)/\omega(t)$. Then the asymptotic value $\omega(\infty) = \omega_1$ and we can investigate the equations (8.111c), (8.111d) for this constant value of $\omega(t)$. γ-function (8.103) depends on the choice of gauge condition [153, 154]. A possible way of investigating the asymptotic behaviour of the couplings $\kappa^2(t)$ and $\Lambda(t)$ is connected with the use of unique effective action [154] (see also Chapter 6 for a discussion of this approach). An alternative way is connected with the separation of the essential dimensionless coupling $\tilde{\Lambda} = \kappa^4\Lambda$ [138, 153, 154, 494]. The renormalization group equation for $\tilde{\Lambda}(t)$ does not depend on the choice of gauge fixing condition and has the form

$$\frac{d\tilde{\Lambda}}{dt'} = \tfrac{1}{2}\beta_4 = \lambda\tilde{\Lambda}\left(\frac{20}{3}\omega + 5 - \frac{1}{6\omega}\right) + \tfrac{1}{2}\lambda^2\left(\tfrac{5}{2} + \frac{1}{8\omega^2}\right). \tag{8.115}$$

We can easily obtain the solution of the last equation for a fixed value of $\omega(t)$

$$\tilde{\Lambda}(t) = c(1 + \lambda a^2 t')^{W_1/a^2} - \frac{W_2\lambda}{W_1 + a^2}(1 + \lambda a^2 t')^{-1} \tag{8.116}$$

where $W_1 = \frac{20}{3}\omega + 5 - 1/6\omega$, $W_2 = \frac{5}{4} + 1/16\omega^2$ and $c =$ constant. Let us consider two cases.

(a) $\omega > \omega_2$. Then we can use the value $\omega(t) \equiv \omega_1$ in (8.16) and when $t \to \infty$ [153, 154]

$$\tilde{\Lambda}(t) \propto c(1 + \lambda a^2 t')^q \qquad q \approx 0.91. \tag{8.117}$$

We can see that in this case flat space–time is not stable even in the UV limit. Note that the version of the renormalization group equation we have used here is based on a typical quantum field theory which requires the validity of the expansion for a flat background. The result we have obtained emphasizes the shortcomings of all the approaches based on action (8.6) with $\Lambda \neq 0$.

(b) $\omega = \omega_2$. Here we use the special solution for the coupling $\omega(t)$. In this case

$$\tilde{\Lambda}(t) \to 0 \text{ when } t \to \infty \tag{8.118}$$

and flat space–time is stable in the UV limit.

8.10 Renormalized quantum gravity model with torsion

One of the reasons for investigating the higher-derivative gravity theory (8.1) is connected with the desirability of constructing a unified model of all the fundamental interactions. Then the form of gravitational actions is connected with the form of the possible vacuum counterterms in an external gravitational field. The considerations presented above have convinced us that the corresponding quantum gravity theory is multiplicatively renormalized.

As was pointed out in Chapter 4, the renormalization structure of a quantum field theory in an external gravitational field with torsion looks like the same structure for a pure metric theory. In particular, the pressure of an external field is necessary for the renormalizability. Then it is interesting to construct the action of the gravitational field with torsion which allows us to build a unified renormalizable theory.

Usual power counting shows that the action of renormalized gravity with torsion contains invariants of dimension 4 which are built with the help of curvature and torsion tensors and covariant derivatives. The full list of these invariants was presented in [135]. The general action also contains invariants of dimension 2 and 0 with the corresponding dimensional parameters. The action of [135] contains 168 independent structures. This action leads to a renormalized quantum gravity with torsion. At the same time this action is too cumbersome and the corresponding theory is too complicated. It seems reasonable to consider the simplified version of the theory which deals only with the purely antisymmetric part of the torsion tensor. In the same way we use the fact that for matter fields of spin 0, 1/2, 1 only the interaction with the antisymmetric part of the torsion tensor is essential from the renormalization point of view.

Let us write the general action of higher-derivative gravity with torsion. We can use the curvature $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}$ and covariant derivative $\tilde{\nabla}_{\alpha}$ (see Chapter 4) as well as purely metric quantities $R^{\alpha}{}_{\beta\gamma\delta}$ and ∇_{α}. It is natural to require the action for the gravitational field to preserve parity. Then the action contains only even degrees of the pseudovector field S_{μ}(the value of S_{μ} contains all information on the purely antisymmetric part of $T^{\alpha}_{\beta\gamma}$). We can construct two groups of invariants which satisfy the conditions described above

$$(a)\ \ C^2_{\mu\nu\alpha\beta},\ R^2,\ F_{\mu\nu}(S)F^{\mu\nu}(S),\ (S_{\mu}S^{\mu})^2,\ (\nabla_{\mu}S^{\mu})^2, \\ R_{\mu\nu}S^{\mu}S^{\nu},\ -\frac{1}{\kappa^2}R,\ -\frac{1}{\kappa^2}\xi S_{\mu}S^{\mu}, \Lambda. \tag{8.119}$$

where $F_{\mu\nu}(S) = \partial_{\mu}S_{\nu} - \partial_{\nu}S_{\mu}$ and

$$(b)\ \ G, \Box R,\ \Box(S_{\mu}S^{\mu}),\ \nabla_{\mu}(S^{\mu}\nabla_{\nu}S^{\nu}),\ \nabla_{\mu}(S^{\nu}\nabla_{\nu}S^{\mu}). \tag{8.120}$$

The invariants (8.120) are surface and topological terms of the action. These terms do not influence the properties of the quantum perturbative theory. In what follows we will consider the action which is constructed with the help of terms in (8.119) only. The action of the renormalized theory has the form

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2\lambda}C^2_{\mu\nu\alpha\beta} - \frac{\omega}{3\lambda}R^2 - \frac{1}{\kappa^2}R + \Lambda + \alpha_1 F^2_{\mu\nu}(S) + \alpha_2(\nabla_\mu S^\mu)^2 + \alpha_3(S_\mu S^\mu)^2 + \alpha_4 R^{\mu\nu}S_\mu S_\nu + \alpha_5 R S_\mu S^\mu - \frac{1}{\kappa^2}\xi S_\mu S^\mu\Big) + (\text{surface and topological terms}). \quad (8.121)$$

The action (8.121) gives the appropriate basis for the construction of the unified field theory with torsion. Note that the cosmological constant problem exists in the theory (8.121) as well as in the pure metric theory (8.6). To solve this problem we can use the scale invariant action (that is the action without dimensional parameters). On the other hand, the Einstein–Cartan theory with antisymmetrical torsion may be obtained from (8.121) by omitting the scale-invariant terms.

The quantum properties of the Einstein–Cartan action with external spinor current have been investigated, for example, in [497]. This theory is renormalizable and asymptotically free 'on mass shell'.

The next attempt to simplify the metric-torsion gravity action is connected with the conformal version of higher-derivative gravity. The conformal invariants put rigid restrictions on the parameters of action (8.121). The metric-torsion generalization of the Weyl theory (8.5) is connected with the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\lambda}C^2_{\mu\nu\alpha\beta} + \alpha_1 F^2_{\mu\nu}(S) + \alpha_3(S_\mu S^\mu)^2\right\} \quad (8.122)$$

Note that the theory (8.122) has unusual quantum properties. Let the parameter $\alpha_1 = -\frac{1}{4}$. If we make the substitution $S_\mu \to S_\mu + \sigma_\mu$, $g_{\mu\nu} = \eta_{\mu\nu}$ then the bilinear form of action (8.122) has the form

$$\frac{1}{2}\frac{\delta^2 S}{\delta\sigma_\alpha(x)\delta\sigma_\beta(y)} = (\eta^{\alpha\beta}\Box - \partial^\alpha\partial^\beta + \Pi^{\alpha\beta})\delta(x-y) \quad (8.123)$$

where $\Pi^{\alpha\beta} = 4\alpha_3(2S^\alpha S^\beta + \eta^{\alpha\beta}S_\lambda S^\lambda)$. The bilinear form (8.123) has a degenerate second derivative part. When the field S_μ is switched off expression (8.123) becomes degenerate. At the same time theory (8.122) does not possess a gauge invariance. Therefore, we cannot construct the main element of perturbation theory, i.e. a propagator in theory (8.122). It is possible that this theory does not allow the usual perturbative approach at all, and other methods may have to be used for its investigation.

8.11 On the canonical quantization of higher-derivative gauge theories

The next two sections of this chapter are devoted to the foundation of the covariant perturbation technique which we have used in R^2-gravity.

It is well-known that the basis of covariant methods in quantum field theory is canonical quantization. The R^2-gravity theory represents an example of the theory with higher (fourth) order derivatives in the equations of motion. The canonical quantization of such theories is not well-known. That is why we begin with a description of the general canonical quantization method in higher-derivative gauge theories [150]. Then we will apply this method directly to R^2-gravity [150].

The canonical quantization is based on the classical Hamiltonian formulation of the theory. The Hamiltonian formulation of higher-derivative theories was first developed by Ostrogradsky in the last century (see, for example, [472]). Ostrogradsky's method is not as useful in gauge theories as the original Hamiltonian formalism. The canonical formulation of higher-derivative gauge theories is explained in [473] (see also [159]). This formulation was obtained by using the unification of the Ostrogradsky method with the Dirac method of Hamiltonization of the gauge theories. We will consider a more general method [150] which allows us to take into account specific features of the gauge theory.

Consider a dynamical system with generalized coordinates x_a, $a = 1, 2, \ldots, N$ with the Lagrangian

$$L(x_a, \dot{x}_a, \ldots, x_a^{(n_a)}) \qquad x_a^{(s)} = \frac{\mathrm{d}^s x_a}{\mathrm{d}t^s}. \tag{8.124}$$

Here n_a is the highest order of time derivative of the coordinates x_a in the Lagrangian.

The Ostrogradsky method is based on the introduction of additional coordinates by the following rule

$$x_a = q_a^1, \dot{x}_a = q_a^2, \ldots, \; x_a^{(n_a - 1)} = q_a^{n_a}. \tag{8.125}$$

Then the Lagrangian (8.124) has the form

$$L\left(q_a^1, q_a^2, \ldots, q_a^{n_a}, \dot{q}_a^{n_a}\right). \tag{8.126}$$

The expression (8.126) formally corresponds to a theory without higher derivatives. However, the usual formulation of the gauge theories is given in proper geometrical terms like the covariant derivatives and curvature tensors. The Ostrogradsky method of the introduction of additional coordinates (8.125) does not take into account these peculiarities of the theory. As a result the Hamiltonization procedure is not adequate to deal with the structure of the theory. We will show that in higher-derivative gauge theories there is a remarkable arbitrariness which is connected with the introduction of the additional coordinates. This fact allows us to carry out the Hamiltonization in an appropriate way.

Let us denote $x_a = k_a^1$ and introduce the additional coordinates $k_a^2, k_a^3, \ldots, k_a^{n_a}$ according to the following rule

$$\begin{aligned} k_a^{s_a+1} &= K_a^{s_a}(x_b, \dot{x}_b, \ldots, x_b^{(\theta_{ab})}) \qquad s_a = 1, 2, \ldots, n_a - 1 \\ \theta_{ab} &= \min(s_a, n_b - 1). \end{aligned} \tag{8.127}$$

Here $K_a^{s_a}$ are arbitrary functions and

$$\begin{gathered} \Delta_l \equiv \det \frac{\partial K_a^l}{\partial x_b^{(l)}} \neq 0 \\ l = 1, 2, \ldots, \nu - 1 \qquad \nu = \max\{n_a\}. \end{gathered} \tag{8.128}$$

Relations (8.127) and (8.128) show that the function $K_a^{s_a}$ may depend on $x_b^{(r)}$ only in the case of $r \leqslant s_a$. That is, the additional coordinates $k_a^{s_a+1}$ replace those x_b derivatives which have order less than n_b. The matrices of the determinants Δ_l contained in expressions (8.128) are constructed with the help of the elements $\partial K_a^l / \partial x_b^{(l)}$ where the requirements $l < n_a$ and $l < n_b$ are satisfied. Then the conditions (8.128) ensure the non-degeneracy of the transformation $x^{(l)} \to k^{l+1}$.

Let us consider an example which illustrates the relations (8.127) and (8.128). Let $L = L(x_1, \dot{x}_1, \ddot{x}_1, x_2, \dot{x}_2, \ddot{x}_2, x_2^{(3)}, x_2^{(4)})$. Then from (8.127) and (8.128) it follows that the additional coordinates must be introduced according to the rule

$$\begin{aligned} &k_1^1 = x_1 \qquad k_2^1 = x_2 \\ &k_1^2 = K_1^1(x_1, \dot{x}_1, x_2, \dot{x}_2) \qquad k_2^2 = K_2^1(x_1, \dot{x}_1, x_2, \dot{x}_2) \\ &k_2^3 = K_2^2(x_1, \dot{x}_1, \dot{x}_2, \ddot{x}_2) \qquad k_2^4 = K_2^3(x_1, \dot{x}_1, x_2, \dot{x}_2, \ddot{x}_2, x_2^{(3)}). \end{aligned} \tag{8.129}$$

Then

$$\Delta_1 = \frac{\mathrm{D}(K_1^1, K_2^1)}{\mathrm{D}(\dot{x}_1, \dot{x}_2)} \qquad \Delta_2 = \frac{\partial K_2^2}{\partial \ddot{x}_2} \qquad \Delta_3 = \frac{\partial K_2^3}{\partial x_2^{(3)}}. \tag{8.130}$$

Note that the additional coordinates $k_a^{s_a+1}$ are introduced according to Ostrogradsky's procedure (8.125) and their definition (8.129) contains the arbitrary functions $K_a^{s_a}$. As a result there is freedom in the choice of these functions. Therefore we can take into account the specific features of the theory under consideration. We could expect that the different choice of the functions $K_a^{s_a}$ leads to the different Hamiltonian formulations but, as we will see, all these formulations are canonically equivalent.

Relations (8.128) allow us to express the derivatives of $x_a^{(s_a)}$ as a function of additional coordinates $k_a^{s_a+1}$ in the form

$$x_a^{(s_a)} = G_a^{s_a}(k_b^1, k_b^2, \dots, k_b^{\theta_{ab}+1}) \qquad \det \frac{\partial G_a^l}{\partial k_b^{l+1}} \neq 0. \tag{8.131}$$

Differentiating equalities (8.131) with respect to time we can obtain

$$\dot{k}_a^{s_a} = \varphi_a^{s_a}(k_b^1, \dots, k_b^{\theta_{ab}+1}) \qquad s_a = 1, \dots, n_a - 1 \tag{8.132}$$

where functions $\varphi_a^{s_a}$ are determined by

$$\varphi_a^1 = G_a^1 \qquad \sum_{b=1}^{N} \sum_{l=1}^{\theta_{ab}} \frac{\partial G_a^{p_a}}{\partial k_b^l} \varphi_b^l = G_a^{p_a+1} \qquad p_a = 1, 2, \dots, n_a - 2. \tag{8.133}$$

Let us introduce new variables v_a according to the rule

$$v_a = \dot{k}_a^{n_a}. \tag{8.134}$$

Then relations (8.129)–(8.132) and (8.134) show that $x_a^{(n_a)}$ are functions of $k_b^1, k_b^2, \dots, k_b^{n_b}, v$

$$x_a^{(n_a)} = G_a^{n_a}\left(k_b^1, \dots, k_b^{n_b}, v\right) \qquad \det \frac{\partial G_a^{n_a}(k, v)}{\partial v_b} \neq 0. \tag{8.135}$$

The equations of motion follow from the action principle with the original Lagrangian. However, the extremum problem for the action with Lagrangian (8.124) can be replaced by the equivalent problem of a conditional extremum of action with Lagrangian L_k

$$L_k = L_k(k, v) = L\big|_{x_a = k_a^1, x_a^{(r_a)} = G_a^{r_a}, r_a = 1, 2, \dots, n_a} \tag{8.136}$$

and the additional conditions (8.132) and (8.134). Both problems lead to the same equations of motion. The last problem, in its turn, can be replaced by the problem of an unconditional extremum of the action

$$\int dt\Big\{\pi_a^{s_a}\Big(\dot{k}_a^{s_a} - \varphi_a^{s_a}(k)\Big) + \pi_a^{n_a}\Big(\dot{k}_a^{n_a} - v_a\Big) + L_k(k,v)\Big\}. \tag{8.137}$$

Here $\pi_a^{r_a}$, $r_a = 1,2,\ldots,n_a$ are the Lagrangian multipliers which have the sense of momenta, canonically conjugate to the coordinates $k_a^{r_a}$ (we imply summation on all the repeated indices in (8.137)).

Let us define the function

$$H^* = \pi_a^{s_a}\varphi_a^{s_a} + \pi_a^{n_a}v_a - L_k. \tag{8.138}$$

Then we can rewrite the equation of motion which follows from the action (8.137) in the form

$$\dot{k}_a^{r_a} = \frac{\partial H^*}{\partial \pi_a^{r_a}} \qquad \dot{\pi}_a^{r_a} = \frac{\partial H^*}{\partial k_a^{r_a}} \qquad \left(\pi_a^{r_a} = \frac{\partial L_k}{\partial v_a}\right) \tag{8.139}$$

$$\frac{\partial H^*}{\partial v_a} = 0. \tag{8.140}$$

Let us denote

$$\operatorname{rank}\left(\frac{\partial^2 L(x,\dot{x},\ldots,x^{(n)})}{\partial x_a^{(n_a)}\partial x_b^{(n_b)}}\right) = \rho. \tag{8.141}$$

If $\rho = N$ for all $a,b = 1,\ldots,N$ then such a theory is usually said to be non-singular. Ostrogradsky (1850) considered only this case (and the choice of the additional coordinates (8.126)). If $\rho < N$ the corresponding theory is usually called singular. The Hamiltonization of the singular higher-derivative theories has been studied in [473] (see also [159]). In those papers the additional coordinates were introduced in the spirit of Ostrogradsky. Here the general case is studied where $\rho \leqslant N$ and the additional coordinates are introduced with the help of the arbitrary functions $K_a^{s_a}$ (8.127).

Let us consider equation (8.140) for some values of ρ for the equations (8.140) can be solved with respect to some variables v_a which are denoted as V_i, $i = 1,\ldots,\rho$ in the form

$$V_i = \bar{V}_i(k,\pi^n,\lambda). \tag{8.142}$$

Here λ_α are the remaining variables v_a, $\alpha = 1, \ldots, m$; $m = N - \rho$. After the substitution of V_i (in the form (8.142)) into the remaining m equations (8.140) the last will contain only $k_a^{r_a}$ and $\pi_a^{n_a}$

$$\Phi_\alpha^{(1)}(k, \pi^n) = 0 \qquad \alpha = 1, \ldots, m. \tag{8.143}$$

Equations (8.143) are constraints on the canonical variables k and π; it is natural to refer to them as primary constraints, in the spirit of the Dirac method. If V is replaced by $\bar{V}$ (8.142) then the Hamiltonian equations will have the form

$$\dot{k} = \{k, H_1\} \qquad \dot{\pi} = \{\pi, H_1\}$$
$$\{A, B\} = \sum_{a=1}^{N} \sum_{r=1}^{N} \left(\frac{\partial A}{\partial k_a^r} \frac{\partial B}{\partial \pi_a^r} - \frac{\partial A}{\partial \pi_a^r} \frac{\partial B}{\partial k_a^r} \right). \tag{8.144}$$

The Hamiltonian H_1 is constructed according to the rule

$$H_1 = \bar{H}^* \qquad \bar{F} = F(k, \pi, v) \Big|_{V_i = \bar{V}_i(k, \pi^n, \lambda)} . \tag{8.145}$$

Here F is an arbitrary function. We can show that the Hamiltonian H_1 has the following structure

$$H_1 = H + \lambda_\alpha \Phi_\alpha^{(1)}. \tag{8.146}$$

The function H is defined by (8.146). Thus, the described theory is formulated in phase space with the coordinates $k_a^{r_a}$, $\pi_a^{n_a}$. The equations of motion have a Hamiltonian form, and the primary constraints have the form (8.143). The theory with higher derivatives is transformed to the form which allows us to use the Dirac method [177] and to carry out the canonical quantization of the theory.

It is evident that the various choice of the functions $K_a^{s_a}$ in relations (8.127) leads us to different Hamiltonian formulations of the theory. However, there is a theorem [150, 475] which shows that all these formulations are canonically equivalent.

Theorem The Hamiltonian formulations constructed on the basis of one and the same Lagrangian with the help of different ways of introducing additional coordinates (i.e. different functions $K_a^{s_a}$) are connected by a point canonical transformation under which the initial generalized coordinates $x_a = k_a^1$ are unchanged.

Proof Let the additional coordinates $k_a^{s_a+1}$ be introduced with help of the arbitrary functions $K_a^{s_a}$. We shall show that the corresponding Hamiltonian formulations are canonically equivalent to the Hamiltonian formulation, corresponding to the choice of the additional coordinates in accordance with Ostrogradsky's method.

Let $p_a^{r_a}$, $r_a = 1, 2, \ldots, n_a$ be the momenta canonically conjugated to the coordinates $q_a^{r_a}$. We introduce the generating function

$$F(k,p) = p_a^1 k_a^1 + p_a^{s_a+1} G_a^{s_a}(k). \tag{8.147}$$

The functions $G_a^{s_a}(k)$ are determined by relations (8.135). The function (8.147) generates the canonical transformation

$$q_a^{r_a} = \frac{\partial F}{\partial p_a^{r_a}} \qquad \pi_a^{r_a} = \frac{\partial F}{\partial k_a^{r_a}}. \tag{8.148}$$

We can show that the canonical transformation (8.148) together with (8.133) transforms the constraints $\Phi_\alpha^{(1)}$(8.143) into the constraints

$$\begin{aligned} \Psi_\alpha^{(1)} &= p_\alpha^{n_\alpha} - f_\alpha(q, p^n) \\ f_\alpha &= \frac{\partial L_q}{\partial \lambda_\alpha} \qquad L_q = L\big|_{x^{(s)}=q^{s+1}, x^{(n)}=v}. \end{aligned} \tag{8.149}$$

Note that only $\Psi_\alpha^{(1)}$ are the primary constraints when the additional coordinates are choosen following Ostrogradsky [159, 473]. In the same way we can show that the function $H_1(k, \pi)$ (8.145) transforms into the function

$$\begin{aligned} H_1(q,p) &= \bar{H}^*(q,p,v) \\ \bar{H}^*(q,p,v) &= p_a^{s_a} q_a^{s_a+1} + p_a^{n_a} v_a \\ &\quad - L(q_a^1, q_a^2, \ldots, q_a^{n_a}, v) \\ \bar{H}^* &= H^*\big|_{V_i = \bar{V}_i(q, p_n)}. \end{aligned} \tag{8.150}$$

Here $V_i = \bar{V}_i(q, p^{(n)}, \lambda)$ are the solutions of the equations $p_i^{n_i} = \partial L_q / \partial V_i$. But $H_1(q,p)$ (8.150) is the Hamiltonian of the theory when the additional coordinates are chosen in accordance with Ostrogradsky [159, 473]. Thus, the Hamiltonian formulations which correspond to the various choices of the introduction of additional coordinates are canonically equivalent.

From the form of the generating function (8.147) it follows that

$$q_a^1 = \frac{\partial F}{\partial p_a^1} = k_a^1 = x_a. \tag{8.151}$$

Therefore the canonical transformation which is generated by the function (8.147) does not affect the initial coordinates. This statement completes the proof of the theorem.

Let us now pass to the canonical quantization. It is well-known that canonical quantization of constrained theories leads to the following generating functional of Green's functions [178, 179]

$$Z[J] = \int \mathrm{D}k\, \mathrm{D}\pi\, \delta[\Phi] \det{}^{1/2}\{\Phi, \Phi\}$$
$$\times \exp\left[\mathrm{i} \int \mathrm{d}t \left(\pi_a^{r_a} \dot{k}_a^{r_a} - H + J_a x_a\right)\right]. \qquad (8.152)$$

Here Φ is the complete set of constraints including the constraints and the additional conditions (gauges). From the theorem proved above it follows that the transition from one method of definiting the additional coordinates-to another is just a canonical transformation which does not affect the initial coordinates x_a. Since the integrand in (8.152) does not change under this transformation and the measure $\mathrm{D}\pi\, \mathrm{D}k$ is canonically invariant, expression (8.152) does not depend on the choice of additional coordinates.

If we integrate over the momenta $\pi_a^{r_a}$ and over additional coordinates k_a^2, $k_a^3, \ldots, k_a^{n_a}$ in expression (8.152), then the generating functional takes the form of the integral over the initial coordinates

$$Z[J] = \int \mathrm{D}x_a\, \mu[x] \exp\left[\mathrm{i}S[x] + \int \mathrm{d}t\, x_a J_a\right]. \qquad (8.153)$$

Here $\mu[x]$ is the local measure which has the form depending on the Lagrangian structure. The formalism given above enables us to take into account the structure of Lagrangian under the Hamiltonization of the theory.

8.12 Hamiltonian formulation and canonical quantization of higher-derivative gravity

Let us consider the problem of the Hamiltonian formulation construction and the canonical quantization of the theory with the action

$$S = \int \mathrm{d}^4x \mathcal{L}$$
$$\mathcal{L} = \sqrt{-g}(\Lambda - \frac{1}{\kappa^2} R + a R_{\mu\nu} R^{\mu\nu} + b R^2). \qquad (8.154)$$

Here Λ, κ^2, a and b are parameters. When $a = b = 0$, Lagrangian (8.154) corresponds to the Einstein theory with the cosmological term.

The canonical formulation of the Einstein theory was developed in the classical papers of Dirac [476] and Arnowitt–Deser–Misner (ADM)

[477]. The various aspects of the Hamiltonian formalism for theory (8.154) have been discussed in a large number of papers [150, 478–489]. A complete analysis of the problem was presented in [487, 489]. It has been established that the theory has five essentially different variants

$$\begin{array}{llll} (1) & a \neq 0 & b \neq -\frac{1}{3}a & \\ (2) & a = 0 & b \neq 0 & \\ (3a) & a = -3b & 1/\kappa^2 = \Lambda = 0 & \\ (3b) & a = -3b & 1/\kappa^2 \neq 0 & \\ (3c) & a = -3b & 1/\kappa^2 = 0 & \Lambda \neq 0. \end{array} \tag{8.155}$$

These five variants have essentially different dynamics and, in particular, a different numbers of physical degrees of freedom.

For the construction of the Hamiltonian formalism it is convenient to use as initial variables the ADM parametrization α, β_i, g_{ij} of the metric $g_{\mu\nu}$

$$\alpha = 1/\sqrt{g_{00}} \qquad \beta_i = g_{0i}$$
$$g^{\mu\nu} = \begin{pmatrix} \alpha^{-2} & -\beta^j \alpha^{-2} \\ -\beta^i \alpha^{-2} & e^{ij} + \alpha^{-2}\beta^i\beta^j \end{pmatrix}. \tag{8.156}$$

Here g_{ij} is the metric on the three-dimensional surface $x^0 =$ constant. The following abbreviations are used: metric with signature $(+---)$, $g = \det g_{\mu\nu}$, ${}^3g = \det g_{ij}$, $e^{ik}g_{kj} = \delta^i_j$, e^{ij} is the inverse matrix for g_{ij}. The three-dimensional indices $i, j, k, l = 1, 2, 3$ are raised (lowered) with the help of $e^{ij}(g_{ij})$.

Lagrangian (8.154) is expressed in terms of ADM coordinates as

$$\begin{aligned} \mathcal{L} = \alpha(-{}^3g)^{1/2}\Big\{ & \Lambda - \frac{1}{\kappa^2}(2F^i_i + k + {}^3R) \\ & + a\Big[(F^i_i + k)^2 + 2\kappa^k_{ik}e^{ij}\kappa^l_{jl} \\ & + (F^j_i + {}^3R^j_i)(F^i_j + {}^3R^i_j)\Big] + b(F^i_i + k + {}^3R)^2\Big\}. \end{aligned} \tag{8.157}$$

Here ${}^3R_{ij}$ and ${}^3R$ are the Ricci tensor and the scalar curvature constructed from the metric g_{ij}.

$$\begin{aligned} k &= K_i{}^i K_j{}^j - K_{ij}K^{ij} \\ \kappa^k_{ij} &= {}^3\nabla_i K_j{}^k - {}^3\nabla_j K_i{}^k \end{aligned} \tag{8.158}$$

${}^3\nabla_i$ is a covariant derivative constructed from the metric g_{ij} and the corresponding Christoffel symbols ${}^3\Gamma^k_{ij}$. K_{ij} are the extrinsic curvature tensors of the surface $x^0 =$ constant.

$$\begin{aligned} K_{ij} &= \alpha \Gamma^0_{ij} = \frac{1}{2\alpha}(\dot{g}_{ij} - {}^3\nabla_i \beta_j - {}^3\nabla_j \beta_i) \\ F_{ij} &= \frac{1}{\alpha}\Big(\dot{K}_{ij} + 2\alpha K_{ik} K_j{}^k - \alpha K_{ij} K_k{}^k - {}^3\nabla^3_i \nabla_j \alpha \\ &\quad - K_{ik}{}^3\nabla_j \beta^k - K_{jk}{}^3\nabla_i \beta^k - \beta^k\, {}^3\nabla_k K_{ij}\Big). \end{aligned} \tag{8.159}$$

Relations (8.157)–(8.159) show that the Lagrangian (8.154) contains the second derivative $\ddot{g}_{ij}$ with respect to x^0.

According to the general approach stated above it is necessary to introduce additional canonical coordinates, 'absorbing' $\dot{g}_{ij}$. As was pointed in the preceding section (see also [150, 159, 487]) we can introduce the additional coordinates in various ways. In this case it is convenient to choose K_{ij} as such additional coordinates. So we shall consider α, β, g_{ij}, K_{ij} as the coordinates and introduce the corresponding canonically conjugated momenta π^0, π^i, π^{ij}, P^{ij}.

Lagrangian (8.154) does not depend on the velocities $\dot{\alpha}$ and $\dot{\beta}_i$. Hence there are primary constraints of the form

$$\Phi^{(1)\mu} \equiv \pi^\mu = 0. \tag{8.160}$$

Momenta P^{ij} are introduced according to the general rule (8.140)

$$P^{ij} = \frac{\partial L_k}{\partial \dot{K}_{ij}} = 2\alpha G^{ijkl} F_{kl} + 2A^{ij}. \tag{8.161}$$

Here

$$\begin{aligned} G^{ijkl} &= \frac{1}{\alpha}\sqrt{-{}^3g}\left[\tfrac{1}{2}a\left(e^{ik}e^{jl} + e^{il}e^{jk}\right) + (a+4b)e^{ij}e^{kl}\right] \\ A_{ij} &= \sqrt{-{}^3g}\left[a^3 R_{ij} + (2b^3 R + (a+2b)k - 1/\kappa^2) g_{ij}\right]. \end{aligned} \tag{8.162}$$

Note that the matrix G^{ijkl} (8.162) is an analogy of the De Witt 'supermetric' in Einstein gravity [103].

Further analysis demands a separate consideration of each of the variants (8.154).

Variant 1 Let $a \neq 0$, $b \neq 1/3$. In this case matrix G^{ijkl} (8.162) has the inverse matrix G_{ijmn}

$$\begin{aligned} G_{ijmn} &= \frac{\alpha}{a\sqrt{-{}^3g}}\left[\tfrac{1}{2}(g_{im}g_{jn} + g_{in}g_{jm}) - \frac{a+4b}{4(a+3b)} g_{ij} g_{mn}\right] \\ G_{ijmn}G^{mnkl} &= \delta^{kl}_{ij,} \qquad G^{ijkl}G_{klmn} = \delta^{ij,}_{mn}. \end{aligned} \tag{8.163}$$

Then we can express the velocity $\dot{K}_{ij}$ by the canonical variables

$$\begin{aligned} \bar{\dot{K}}_{ij} &= G_{ijkl}\Big(\tfrac{1}{2}P^{kl} - A^{kl}\Big) - \varphi_{ij} \\ \varphi_{ij} &= 2\alpha K_{ik}K_j{}^k - \alpha K_{ij}K_k{}^k - {}^3\nabla_i^3\nabla_j\alpha \\ &\quad - K_{ik}{}^3\nabla_j\beta^k - K_{jk}{}^3\nabla_i\beta^k - \beta^{k3}\nabla_k K_{ij}. \end{aligned} \tag{8.164}$$

The total Hamiltonian is constructed according to the general rule (8.145) and has the form

$$\begin{aligned} \mathcal{H}_1 &= \pi^{ij}\Big({}^3\nabla_i\beta_j + {}^3\nabla_j\beta_i - 2\alpha K_{ij}\Big) + P^{ij}\bar{\dot{K}}_{ij} + \pi^\mu\lambda_\mu - \bar{\mathcal{L}}_k \\ &= \pi^{ij}\Big({}^3\nabla_i\beta_j + {}^3\nabla_j\beta_i - 2\alpha K_{ij}\Big) \\ &\quad + \Big(\tfrac{1}{2}P^{ij} - A^{ij}\Big)G_{ijkl}\Big(\tfrac{1}{2}P^{kl} - A^{kl}\Big) - P^{ij}\varphi_{ij} \\ &\quad + \alpha\sqrt{-{}^3g}\Big[-\Lambda - \frac{1}{\kappa^2}(k + {}^3R) - a\Big(k^2 + \kappa_{ik}^k e^{ij}\kappa_{jl}^l + {}^3R_{ij}{}^3R^{ij}\Big) \\ &\quad - b(k +^3 R)^2\Big] + \lambda_\mu\pi^\mu \equiv \mathcal{H} + \lambda_\mu\Phi^{(1)\mu} \end{aligned} \tag{8.165}$$

Here λ_μ are the Lagrange multipliers related to the constraints $\Phi^{(1)\mu}$ (8.165). The non-vanishing Poisson brackets among the canonical variables have the form

$$\begin{aligned} \{g_{ij}(x), \pi^{mn}(y)\} &= \delta_{ij}^{mn}\delta(x-y) \\ \{K_{ij}(x), P^{mn}(y)\} &= \delta_{ij}^{mn}\delta(x-y) \\ \{\alpha(x), \pi^0(y)\} &= \delta(x-y) \\ \{\beta_i(x), \pi^j(y)\} &= \delta_i^j\delta(x-y). \end{aligned} \tag{8.166}$$

The requirement of the conservation of the primary constraints (8.160) leads to the secondary constraints

$$\dot{\Phi}^{(1)\mu} = \{\pi^\mu, H_1\} \equiv T^\mu = 0$$

where

$$\begin{aligned} T^0 &= -\Big(\tfrac{1}{2}P^{ij} - A^{ij}\Big)G_{ijkl}\Big(\tfrac{1}{2}P^{kl} - A^{kl}\Big) + 2\pi^{ij}K_{ij} \\ &\quad + \Big(2K_{ik}K_j{}^k - K_{ij}K_k{}^k - {}^3\nabla_i{}^3\nabla_j\Big)P^{ij} \\ &\quad - \sqrt{-{}^3g}\Big[-\Lambda - \frac{1}{\kappa^2}(k +^3 R) + a\Big(k^2 + \kappa_{ik}^k e^{ij}\kappa_{jl}^l + {}^3R_{ij}{}^3R^{ij}\Big) \\ &\quad + b\Big(k + {}^3R\Big)^2\Big] \end{aligned} \tag{8.167}$$

$$T^i = 2^3\nabla_j\pi^{ij} + 2^3\nabla_k\Big(P^{kj}K_j^i\Big) - P^{jk3}\nabla^i K_{jk}. \tag{8.168}$$

Note that momenta π^{ij}, P^{ij} are three-dimensional tensor densities, and therefore

$$ {}^3\nabla_j P^{ij} = \partial_j P^{ij} + {}^3\Gamma^i{}_{jk} P^{jk}. $$

Using the constraints T^0, T^i (8.167) and (8.168) we can to rewrite the Hamiltonian in the form

$$ \begin{aligned} \mathcal{H} &= -\alpha T^0 - \beta_i T^i + \partial_i Q^i \\ Q^i &= P^{ij}\partial_j\alpha - \alpha\,{}^3\nabla_j P^{ij} + 2\beta_k K^k{}_j P^{ji} + 2\beta_j \pi^{ij}. \end{aligned} \tag{8.169} $$

We see that the Hamiltonian (8.169) is reduced to a total divergence on the constraints (as well as in the Einstein theory).

It is not hard to show that the constraints $\Phi^{(1)\mu}$, T^μ are first-order ones and the algebra of constraints T^μ coincides with that of constraints within Einstein gravity. As in Einstein theory the constraints T^i are the generators of general coordinate transformation of x^i on the surface x^0 = constant. Let us now find the number of physical degrees of freedom n in every point . It is well-known that $n = N - m_1 - \frac{1}{2}m_2$, where N is the number of canonical variable pairs, m_1 is the first-class and m_2 is the second-class constraint number (see, for example, [159]). In our case $N = 16$, $m_1 = 8$, $m_2 = 0$. Hence the number of physical degrees of freedom is $n = 8$.

Let us now consider canonical quantization. Since the theory contains primary constraints we must introduce the additional conditions for gauge fixing. We shall choose these conditions in the form

$$ \chi_\mu(g_{ij}) = 0 \qquad \operatorname{rank}\frac{\partial\chi_\mu}{\partial g_{ij}} = 4 \tag{8.170} $$

where

$$ \sigma_\mu = 0 \qquad \sigma_0 \equiv \alpha - 1 \qquad \sigma_i \equiv \beta_i. \tag{8.171} $$

Let us denote the complete system of constraints $\Phi \equiv (\Phi^{(1)\mu}, T^\mu, \chi_\mu, \sigma_\mu)$ the components of which are now second-order. The generating functional for Green's functions is written in the form

$$ \begin{aligned} Z[J] = \int & \mathrm{D}g_{ij}\,\mathrm{D}\sigma_\mu\,\mathrm{D}K_{ij}\,\mathrm{D}P^{ij}\,\mathrm{D}\pi^{ij}\,\mathrm{D}\pi^\mu\,\delta(\pi^\mu)\delta(\sigma_\mu) \\ & \times \delta(T^\mu)\delta(\chi^\mu)\det\{\chi_\mu, T^\nu\}\exp\Big\{\mathrm{i}\int \mathrm{d}x\Big(\pi^\mu\dot{\sigma}_\mu \\ & + \pi^{ij}\dot{g}_{ij} + P^{ij}\dot{K}_{ij} + T^0 - \partial_i Q^i + g_{ij}J^{ij}\Big)\Big\}. \end{aligned} \tag{8.172} $$

Here J^{ij} are the sources of the field g_{ij}. Let us first integrate (8.172) over π^μ and σ_μ taking into account the corresponding δ-functions.

Then we represent

$$\delta(T^\mu) = \int \mathrm{D}\sigma_\mu \, \exp \mathrm{i}\Big(\int \mathrm{d}x \, \sigma_\mu T^\mu \Big).$$

Now we replace the integration over α, β_i, g_{ij} by integration over $g_{\mu\nu}$. The Jacobian of the corresponding replacement is $(g^{00})^{1/2}$. The integral over P^{ij} is a Gaussian one and the integral over π^{ij} leads to

$$\delta\left(K_{ij}/\sqrt{g^{00}} - \Gamma^0_{ij}/g^{00}\right).$$

Now we can integrate over K_{ij}. The resulting expression is

$$Z[J] = \int \mathrm{D}g_{\mu\nu} \, \mu[g] \Delta \delta(\chi_\mu) \exp\left[\mathrm{i} \int \mathrm{d}x \Big(\mathcal{L} + g_{ij} J^{ij}\Big)\right]. \qquad (8.173)$$

Here $\mathcal{L}$ is the original Lagrangian (8.154), Δ is the Faddeev–Popov determinant corresponding to the additional conditions $\chi_\mu(g_{ij}) = 0$. $\mu[g]$ is the non-trivial functional

$$\mu[g] = (g^{00})^4 (-g)^{-3/2} \qquad (8.174)$$

Relations (8.173) and (8.174) represent the final result of canonical quantization. Equality (8.173) shows that the generating functional of Green's functions corresponds to the Faddeev–Popov ansatz, except for the local measure. Of course the derivation of the local measure is not achieved in the framework of this ansatz. Let us note that the expression for the local measure (8.174) differs from the local measure in general relativity [490].

Variant 2 Let $a = 0$, $b \neq 0$. In this case the supermetric G^{ijkl}(8.162) is degenerate. Really, if $a = 0$ then

$$G^{ijkl} = \frac{4b}{\alpha} \sqrt{-{}^3g} e^{ij} e^{kl}.$$

Let us denote the arbitrary traceless matrix by $\tilde{b}_{kl}$, $\tilde{b}_{kl} e^{kl} \equiv 0$. It is obvious that $G^{ijkl} \tilde{b}_{kl} \equiv 0$. Therefore, we can use relations (8.161) to find the only $\dot{K}_i{}^i$. We denote the traceless part of the arbitrary tensor B_{ij} by $\tilde{B}_{ij}$ where

$$\tilde{B}_{ij} = B_{ij} - \tfrac{1}{3} g_{ij} B_k{}^k \qquad \tilde{B}^{kl} = B^{kl} - \tfrac{1}{3} e^{ij} B^k_k.$$

Then it is easy to see that the velocities $\dot{\tilde{K}}_{ij}$ are not contained in the Lagrangian. Hence, there is an additional primary constraints $\Phi^{(1)ij}$ as well as $\Phi^{(1)\mu}$ (8.160)

$$\widetilde{\Phi}^{(1)ij} \equiv \widetilde{P}^{ij} = 0. \tag{8.175}$$

The total Hamiltonian $\mathcal{H}_1$ is constructed according to the general rule (8.145) and has the form

$$\begin{aligned}\mathcal{H}_1 = &\frac{\alpha}{144b\sqrt{-^3g}}P_i{}^i P_j{}^j + \pi^{ij}\left({}^3\nabla_j\beta_i + {}^3\nabla_i\beta_j - 2\alpha K_{ij}\right)\\ &+ \alpha\Big(\frac{1}{12b\kappa^2} - \frac{1}{3\alpha}\varphi_i{}^i P_j^j - \tfrac{1}{6}(k + {}^3R)P_i{}^i\\ &- \alpha\sqrt{-^3g}\left(\Lambda - \frac{1}{4b\kappa^4}\right) + \lambda_\mu\pi^\mu + \tilde{\lambda}_{ij}\tilde{P}^{ij}\\ = &\,\mathcal{H} + \lambda_\mu\Phi^{(1)\mu} + \tilde{\lambda}_{ij}\widetilde{\Phi}^{(1)ij}.\end{aligned} \tag{8.176}$$

Here λ_μ, $\tilde{\lambda}_{ij}$ are the Lagrange multipliers which correspond to the primary constraints $\Phi^{(1)\mu}$, $\widetilde{\Phi}^{(1)ij}$.

Conservation of the constraints $\Phi^{(1)\mu}$ leads to the following secondary constraints

$$\begin{aligned}\tau^0 \equiv \{\pi^0, \mathcal{H}_1\} = &-\frac{1}{144b\sqrt{-^3g}}P_i{}^i P_j{}^j\\ &+ \tfrac{1}{6}\left[3K_{ij}K^{ij} - K_i{}^i K_j{}^j + {}^3R - \frac{1}{2b\kappa^2} - 2^3\nabla^{i3}\nabla_i\right]P_k{}^k\\ &+ 2\pi^{ij}K_{ij} + \sqrt{-^3g}\left(\Lambda - \frac{1}{4b\kappa^4}\right) = 0\end{aligned} \tag{8.177}$$

$$\begin{aligned}\tau^i \equiv \{\pi^i, \mathcal{H}_1\} = &\,2^3\nabla_j\pi^{ij} + \tfrac{2}{3}{}^3\nabla_j\left(K^{ij}P_k{}^k\right)\\ &- \tfrac{1}{3}P_j{}^{j3}\nabla^i K_k{}^k = 0.\end{aligned} \tag{8.178}$$

In terms of secondary constraints τ^μ (8.177) and (8.178) we can write the Hamiltonian in the form

$$\begin{aligned}&\mathcal{H} = -\alpha\tau^0 - \beta_i\tau^i + \partial_i Q^i\\ &Q^i = 2\pi^{ij}\beta_j + \tfrac{2}{3}\beta_j K^{ji}P_k^k + \tfrac{1}{3}e^{ij}(P_k^k\partial_j\alpha - \alpha\partial_j P_k{}^k).\end{aligned} \tag{8.179}$$

Taking into account the primary constraints $\widetilde{\Phi}^{(1)ij} = \tilde{P}^{ij}$ we can replace the constraints τ^μ by the equivalent constraints T^μ, where

$$\begin{aligned}&T^0 = \tau^0 + \left({}^3R_{ij} + 2K_{ik}K^k_j - {}^3\nabla_i{}^3\nabla_j\right)\tilde{P}^{ij}\\ &T^i = \tau^i + 2^3\nabla_j\left(\tilde{P}^j{}_k K^{ki}\right) - \tilde{P}^{kj}\,{}^3\nabla^i K_{kj}.\end{aligned} \tag{8.180}$$

It is possible to show that the algebra of the constraints T^μ coincides with the constraint algebra in Einstein gravity.

The conservation condition for the constraints $\tilde{\Phi}^{(1)ij}$ also leads to secondary constraints

$$\left\{\tilde{P}^{ij}, \mathcal{H}_1\right\} = 2\alpha\Phi^{(2)ij} = 0 \qquad \Phi^{(2)ij} = \tilde{\pi}^{ij} + \tfrac{1}{6}\tilde{K}^{ij}P^k_k. \tag{8.181}$$

It is possible to show that the following relations take place

$$\begin{aligned} &\left\{\Phi^{(1)ij}, \Phi^{(1)\mu}\right\} = 0 \qquad &&\left\{\Phi^{(1)ij}, T^\mu\right\} = 0 \\ &\left\{\Phi^{(2)ij}, \Phi^{(1)\mu}\right\} = 0 \qquad &&\left\{\Phi^{(2)ij}, T^\mu\right\} = 0 \\ &\left\{\tilde{\Phi}^{(1)ij}, \Phi^{(2)}_{kl}\right\} = \tilde{\delta}^{ij}_{,kl} \qquad &&\tilde{\delta}^{ij}_{,kl} = \delta^{ij}_{kl} - \tfrac{1}{3}e^{ij}g_{kl}. \end{aligned} \tag{8.182}$$

The conditions for the conservation of constraints $\Phi^{(2)ij}$ (8.181) lead to equations for the Lagrange multipliers $\tilde{\lambda}_{ij}$. Since these multipliers are fixed, further constraints do not arise. Thus, $\Phi^{(1)\mu}$ and T^μ are constraints of the first class while $\tilde{\Phi}^{(1)}$ and $\Phi^{(2)}$ are second class and there are no other constraints. Let us find the number of physical degrees of freedom n. In this variant $N = 16$, $m_1 = 8$, $m_2 = 10$. Therefore, the number of physical degrees of freedom is $n = 3$.

Let us now consider canonical quantization. Since the theory contains the first class constraints $\Phi^{(1)\mu}$ and T^μ one must introduce the corresponding number of additional conditions. It is useful to take the additional conditions in the form

$$\chi^\mu(g_{ij}) = 0 \qquad \operatorname{rank}\frac{\partial\chi^\mu}{\partial g_{ij}} = 4$$
$$\sigma_\mu = 0$$

where σ_μ has been given by (8.171). The generating functional of Green's functions is written in the form

$$\begin{aligned} Z[J] = \int &\mathrm{D}g_{ij}\,\mathrm{D}\sigma_\mu\,\mathrm{D}\pi^{ij}\,\mathrm{D}\pi^\mu\,\mathrm{D}K_{ij}\,\mathrm{D}P^{ij}\,\delta(\Phi^{(1)\mu})\delta(\tilde{\Phi}^{(1)ij}) \\ &\times \delta(\Phi^{(2)}_{ij})\delta(\sigma_\mu)\delta(\tau_\mu)\delta(\chi^\nu)\det\{\chi_\mu, \tau^\nu\} \\ &\times \exp\left\{\mathrm{i}\int \mathrm{d}x\left(\pi^{ij}\dot{g}_{ij} + \pi^\mu\dot{\sigma}_\mu + P^{ij}\dot{K}_{ij} + \sigma_\mu\tau^\mu\right.\right. \\ &\left.\left. +\tau^0 - \partial_i Q^i + J^{ij}g_{ij}\right)\right\}. \end{aligned} \tag{8.183}$$

The integration may be performed as in variant 1. The final result has the form

$$Z[J] = \int \mathrm{D}g_{\mu\nu}\, \mu[g] \Delta\delta(\chi_\mu) \exp\left\{ \mathrm{i} \int \mathrm{d}^4x \left(\mathcal{L} + g_{ij} J^{ij} \right\} \right. \tag{8.184}$$

Here $\mathcal{L}$ is the original Lagrangian (8.154) with $a = 0$, $b \neq 0$. Δ is the Faddeev–Popov determinant corresponding to the additional conditions $\chi^\mu(g_{ij}) = 0$. $\mu[g]$ is the non-trivial functional measure

$$\mu[g] = (g^{00})^{3/2}(-g)^{-3/2}. \tag{8.185}$$

Note that the measure (8.185) differs from that in the previous variant (8.174) and in general relativity. Excepting the local measure, the generating functional corresponds to the Faddeev–Popov ansatz.

Variant 3 Let $b = -\frac{1}{3}a$. In this case the supermetric G^{ijkl} (8.63) is degenerate. In fact, if $b = -a/3$ then G^{ijkl} has the form

$$G^{ijkl} = \frac{a}{\alpha}\sqrt{-{}^3g}\left[\tfrac{1}{2}(e^{ik}e^{jl} + e^{il}e^{jk}) - \tfrac{1}{3}e^{ij}e^{kl}\right] \equiv \widetilde{G}^{ijkl}. \tag{8.186}$$

Therefore $\widetilde{G}^{ijkl}g_{kl} = 0$. However, in the linear vector space of symmetric traceless matrices $\widetilde{G}^{ijkl}$ has an inverse matrix of the form

$$\begin{aligned} \widetilde{G}_{ijkl} &= \frac{\alpha}{a\sqrt{-{}^3g}}\left[\tfrac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \tfrac{1}{3}g_{ij}g_{kl}\right] \\ \widetilde{G}_{ijkl}\widetilde{G}^{klmn} &= \tilde{\delta}^{mn}_{ij}, \qquad \widetilde{G}^{ijkl}\widetilde{G}_{klmn} = \tilde{\delta}^{ij}_{mn}. \end{aligned} \tag{8.187}$$

From relation (8.161) it follows that only the traceless part of velocities $\dot{K}_{ij}$ may be expressed by the canonical variables

$$\overline{\widetilde{\dot{K}}}_{ij} = G_{ijmn}\left(\tfrac{1}{2}P^{mn} - a\sqrt{-{}^3g}\,{}^3R^{mn}\right) - \widetilde{\varphi}_{ij}. \tag{8.188}$$

Equations (8.161) do not contain $\dot{K}_i{}^i$ so there is an additional primary constraint among the canonical variables

$$C^{(1)} = P_i{}^i + b\sqrt{-{}^3g}\left[\frac{1}{\kappa^2} - \frac{a}{3}(k - {}^3R)\right]. \tag{8.189}$$

The Hamiltonian is constructed according to the general rule and has the form

$$\mathcal{H}_1 = \widetilde{G}_{ijmn}\left(\tfrac{1}{2}\tilde{P}^{ij} - a\sqrt{-{}^3g}\tilde{R}^{ij}\right)\left(\tfrac{1}{2}\tilde{P}^{mn} - a\sqrt{-{}^3g}\tilde{R}^{mn}\right)$$

$$
\begin{aligned}
&+\pi^{ij}\left({}^3\nabla_j\beta_i + {}^3\nabla_i\beta_j - 2\alpha K_{ij}\right) + \tilde{P}^{ij}\tilde{\varphi}_{ij} \\
&-\tfrac{2}{3}\sqrt{-{}^3g}\left[a(k-{}^3R) - \frac{3}{\kappa^2}\right]\varphi_i{}^i \\
&-\alpha\sqrt{-{}^3g}\Big[\Lambda - \frac{1}{\kappa^2}(k+{}^3R) + a\left(2\kappa^k_{ik}e^{ij}\kappa^l_{jl} + {}^3R_{ij}{}^3R^{ij}\right) \\
&-\tfrac{1}{3}(k+{}^3R)^2\Big] + \lambda_\mu\pi^\mu + \lambda_c C^{(1)} \\
\equiv\ &\mathcal{H} + \lambda_\mu\Phi^{(1)\mu} + \lambda_c C^{(1)}.
\end{aligned} \tag{8.190}
$$

Now it is useful to make a canonical transformation which does not concern the variables α, β_i, π^μ. The generating function of this transformation is

$$
\begin{aligned}
&\mathcal{F}\left(g_{ij}, \pi'^{ij}, K_{ij}, P'^{ij}\right) = \pi'^{ij}g_{ij} + P'^{ij}K_{ij} \\
&\quad + \sqrt{-{}^3g}\left(-\frac{2}{\kappa^2}K_i{}^i - \frac{10a}{27}(K_i{}^i)^3 - \frac{2a}{3}K_i{}^i\left(K_{jl}K^{jl} + {}^3R\right)\right. \\
&P^{ij} = \frac{\delta\mathcal{F}}{\delta K_{ij}} \qquad \pi^{ij} = \frac{\delta\mathcal{F}}{\delta g_{ij}} \\
&K'_{ij} = \frac{\delta\mathcal{F}}{\delta P'^{ij}} \qquad g'_{ij} = \frac{\delta\mathcal{F}}{\delta\pi'^{ij}}.
\end{aligned} \tag{8.191}
$$

Let us introduce new variables $P'^i{}_j$, $K'^i{}_j$ instead of the variables P'^{ij}, K_j using the rule $P'^i_j = P'^{ik}g_{kj}$, $K'^i{}_j = e^{ik}K'_{kj}$ (see [480]). Note that we choose to ignore the symmetries of variables and hence increased the number of variables. At the same time the momenta π'^{ij} must be transformed to

$$
\pi'^{ij} = \pi^{ij} + \tfrac{1}{2}(P'^{ik}K'_k{}^j + P'^{jk}K'_k{}^i).
$$

In order to conserve the original number of variables we introduce the constraints in the following form (we shall omit the primes from now on):

$$
\begin{aligned}
S^i_j &\equiv \frac{1}{\sqrt{2}}\left(P^i_j - g_{jk}P^k_l e^{li}\right) = 0 \\
\sigma^i{}_j &\equiv \frac{1}{\sqrt{2}}\left(K^i{}_j - g_{il}K^l{}_k e^{ki}\right) = 0.
\end{aligned} \tag{8.192}
$$

Now it is necessary to replace Poisson brackets by Dirac brackets with respect to constraints (8.192). In particular

$$
\{K^i{}_j, P^k_l\}_D = \tfrac{1}{2}\left(\delta^i_l\delta^k_j + e^{ik}g_{jl}\right) = \{e^{in}K_{nj}, P^{km}g_{ml}\}.
$$

Using these new variables we can rewrite constraints (8.189) and the Hamiltonian (8.190) as

$$
\begin{aligned}
C^{(1)} &\equiv P^i = 0 \\
\mathcal{H} &= -\alpha T^0 - \beta_i T^i + \alpha K^i{}_i C^{(2)} + \partial_i Q^i.
\end{aligned}
\tag{8.193}
$$

Here

$$
\begin{aligned}
T^0 = &-\frac{1}{4a\sqrt{-{}^3g}}\tilde{P}^j{}_i\tilde{P}^i{}_j + 2\pi^{ij}K_{ij} + \left({}^3R^j_i - {}^3\nabla_i{}^3\nabla^j\right)\tilde{P}^i{}_j \\
&- \sqrt{-{}^3g}\Big[-\Lambda + \frac{1}{\kappa^2}\left({}^3R - \tilde{K}^j{}_i\tilde{K}^i{}_j - \tfrac{2}{3}(K^i{}_i)^2\right) \\
&- \frac{a}{3}\Big(2(\tilde{K}^j{}_i\tilde{K}^i{}_j)^2 + 2{}^3R\tilde{K}^j{}_i\tilde{K}^i{}_j + 2{}^3\nabla^{i3}\nabla_i\left(\tilde{K}^j_n\tilde{K}^n_j\right) \\
&+ 6\left({}^3\nabla_j\tilde{K}^j{}_i\right)\left({}^3\nabla^k\tilde{K}^i{}_k\right) + 2{}^3\nabla_i{}^3\nabla^i R\Big)\Big].
\end{aligned}
\tag{8.194}
$$

$$
\begin{aligned}
T^i &= 2{}^3\nabla_j\pi^{ij} - \tilde{P}^{k3}_l\nabla^i K^l{}_k - {}^3\nabla_j\left(\tilde{P}^i_k\tilde{K}^{kj} - \tilde{P}^j_k\tilde{K}^{ki}\right) \\
C^{(2)} &= \tfrac{2}{3}\pi^i{}_i + \tilde{P}^i{}_j\tilde{K}^j{}_i - \sqrt{-{}^3g}\left({}^3\nabla_i{}^3\nabla_j\tilde{K}^{ij} - \frac{4}{3\kappa^2}K^i{}_i\right).
\end{aligned}
\tag{8.195}
$$

The condition for the conservation of the constraint $C^{(1)}$ leads to the constraint $C^{(2)} = 0$ (8.195)

$$
\dot{C}^{(1)} \equiv \{C^{(1)}, \mathcal{H}_1\} = \alpha C^{(2)} = 0. \tag{8.196}
$$

Conservation of the primary constraints $\Phi^{(1)\mu}$ (8.160) gives the secondary constraints $T^\mu = 0$ (8.194). The algebra of constraints T^μ(8.194) coincides with the algebra of the secondary constraints in Einstein gravity.

Now we must elaborate on the theory in greater details.

Variant 3a Let $1/\kappa^2 = 0$, $\Lambda = 0$. In this case the constraint $C^{(2)}$ commutes with the constraints $\Phi^{(1)\mu}$, T^μ and $C^{(1)}$. Therefore, the theory contains only first-class constraints. The appearance of additional first-class constraints (in comparison with previous variants) is the expression of an additional gauge (conformal) invariance. It is evident that the constraint $C^{(2)}$ is a generator of conformal transformation of the metric $g_{ij}(x)$

$$
\left\{g_{ij}(x), \int d^3y\sigma(y)C^{(2)}(y)\right\} = \sigma(x)g_{ij}(x).
$$

It is not difficult to find the number n of physical degrees of freedom in conformal gravity. Here $N = 16$, $m_1 = 10$. Then $n = 6$.

For canonical quantization we must impose additional (gauge) conditions, the number of which is equal to number of first number constraints. Let us take these conditions in the form

$$\chi^\mu(g_{ij}) = 0 \qquad \operatorname{rank}\frac{\partial\chi^\mu}{\partial g_{ij}} = 4$$
$$\sigma_\mu = 0 \qquad \chi^5 \equiv h^i{}_i = 0 \qquad K^i{}_i = 0. \tag{8.197}$$

Here σ_μ is defined by equalities (8.171), and

$$h_{ij} = g_{ij} - \delta_{ij} \qquad h^i{}_i = \delta^{ij} h_{ij}. \tag{8.198}$$

Now the complete system of constraints $\Phi^{(1)\mu}$, T^μ, $C^{(1)}$, $C^{(2)}$ and the additional conditions χ^μ, σ_μ, χ^5, $K^i{}_i$ is a system of the second class. Let

$$\Phi \equiv (\Phi^{(1)\mu}, T^\mu, C^{(1)}, C^{(2)}, \chi^\mu, \sigma_\mu, \chi^5, K^i{}_i)$$

The generating functional of Green's functions is written in the form

$$Z[J] = \int \mathrm{D}g_{ij}\, \mathrm{D}\sigma_\mu\, \mathrm{D}K^j_i\, \mathrm{D}\pi^{ij}\, \mathrm{D}\pi^\mu\, \mathrm{D}P^j{}_i\, \det{}^{1/2}\{\Phi,\Phi\}$$
$$\times\, \delta[\Phi] \exp\left(\mathrm{i}\int \mathrm{d}^4x \left(\pi^{ij}\dot g_{ij} + \pi^\mu\dot\sigma_\mu + P^j{}_i \dot K^i{}_j\right.\right.$$
$$\left.\left. + \sigma_\mu T^\mu + T^0 + \alpha K^i{}_i C^{(2)} + \partial_i Q^i + g_{ij}J^{ij}\right)\right). \tag{8.199}$$

Integral (8.199) may be transformed by means of the same methods as given in variant 1. The final result has the form

$$Z[J] = \int \mathrm{d}g_{\mu\nu}\, \mu[g] \Delta \delta(\chi) \exp\left(\mathrm{i}\int \mathrm{d}x \left(\mathcal{L} + g_{ij}J^{ij}\right)\right). \tag{8.200}$$

Here $\chi = (\chi^\mu, \chi^5)$, Δ is the Faddeev–Popov determinant which corresponds to the additional conditions χ, $\mathcal{L}$ in the Lagrangian (8.154) with $b = a/3$, $1/\kappa^2 = \Lambda = 0$. The functional measure $\mu[g]$ has the form

$$\mu[g] = (g^{00})^3(-g)^{-7/4} \tag{8.201}$$

which differs from measures in Einstein gravity as well as from the measure in previous variants.

Variant 3*b* Let $1/\kappa^2 \neq 0$. In this case the constraints $C^{(1)}$ and $C^{(2)}$ do not commute with each other but commute with T^μ. Thus, $\Phi^{(1)\mu}$ and T^μ are of the first class, but $C^{(1)}$, $C^{(2)}$ are of the second class. Now conditions (8.170) are sufficient for gauge fixing. The number of physical degrees of freedom is equal to seven. The generating functional of Green's functions may be transformed to the form

$$Z[J] = \int \mathrm{D}g_{\mu\nu}\, \mu[g] \Delta \delta(\chi^\mu) \exp\left(\mathrm{i} \int \mathrm{d}x \left(\mathcal{L} + g_{ij} J^{ij}\right)\right). \quad (8.202)$$

Here Δ is a Faddeev–Popov determinant which corresponds to the additional conditions χ^μ. $\mathcal{L}$ is the initial Lagrangian with $b = -a/3$, $1/\kappa^2 \neq 0$. The local measure is written in the form

$$\mu[g] = (g^{00})^{7/2} (-g)^{3/2}. \quad (8.203)$$

Variant 3c Let $1/\kappa^2 = 0$, $\Lambda \neq 0$. The requirement of the conservation of the constraints $C^{(2)}$ leads in this case to a new constraint $C^{(3)}$

$$\{C^{(2)}, \mathcal{H}_1\} = \alpha C^{(3)} \qquad C^{(3)} \equiv \tfrac{3}{2}\Lambda(-{}^3g)^{1/2} + \ldots = 0$$

where non-essential terms are omitted. Conservation of a new constraint $C^{(3)}$ gives a further constraint

$$C^{(4)} = \Lambda(-{}^3g)^{1/2} K^i{}_i + \ldots = 0$$

which does not commute with $C^{(1)}$. After this, new constraints do not appear. Thus, $\Phi^{(1)}$ and T^μ are constraints of the first class and $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ are of the second class. The conditions (8.170) are sufficient for gauge fixing. The number of physical degrees of freedom in this variant is equal to six. Canonical quantization leads to the generating functional in the form (8.202) where the measure is

$$\mu[g] = (g^{00})^3 (-g)^{-3/2} \quad (8.204)$$

and $\mathcal{L}$ is the initial Lagrangian at $b = -a/3$, $1/\kappa^2 = 0$, $\Lambda \neq 0$.

Let us now make some general remarks. We showed that there are five qualitatively various variants of the theory (8.154) within the canonical approach. In each of these variants the constraints and the Hamiltonian have been found, and canonical quantization has been carried out. In all cases the generating functional of Green's functions corresponds to the Faddeev–Popov ansatz, except for the local measure. In every variant of the theory the local measure differs from the measures in the other variants and from the measure in Einstein gravity. The general structure of the local measure in any gravity theory was investigated in [150, 487]. There it was proved that the general expression for the functional measure is

$$\begin{aligned} \mu[g] &= (g^{00})^{n/2} (-g)^M \\ M &= \tfrac{1}{8}\left[d_\varphi(r_\varphi + 2) - d_\xi(r_\xi + 2)\right]. \end{aligned} \quad (8.205)$$

Here n is the number of physical degrees of freedom (the degree of g^{00} was first found by the authors of [490]). d_φ, d_ξ are the number of the field φ components (φ includes the metric $g_{\mu\nu}$) and the number of gauge parameters ξ respectively. r_φ and r_ξ denote the tensor dimensions of the field φ and of the parameter ξ (e.g. tensor dimension of a scalar is zero, tensor dimension of a covariant vector is minus one, tensor dimension of contravariant vector is one and so on). In expression (8.205) summation should be carried out over all kinds of fields φ and parameters ξ. It is easy to verify that expressions (8.174), (8.185), (8.201), (8.203), (8.204) correspond to the general formula (8.205).

Since the explicit form of the functional measure has been established for all the variants of the theory, we have obtained the possibility of performing quantum calculations in the framework of arbitrary regularization schemes. Let us make some remarks. It is well-known that $\delta^4(0) = 0$ in dimensional regularization and hence the functional measure is trivial here. However, this regularization assumes a formulation of the theory in the space of arbitrary dimensions. But then the term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ must be added to the Lagrangian (8.154). In the four-dimensional case this term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ may be transformed to the linear combinations of $R_{\mu\nu}R^{\mu\nu}$ and R^2 using the Gauss–Bonnet invariant. But in the other dimensions $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is an independent invariant (see [503] for a discussion of R^2-gravity quantization in higher dimensions). We see that in the case under consideration dimensional regularization demands essential modification of the initial Lagrangian. This aspect has been studied in detail in [491].

As a result we have the following alternative. If we use dimensional regularization with trivial functional measure then the term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ must be added to the Lagrangian. On the other hand, if we use Lagrangian (8.154) and another regularization scheme then the functional measure must be taken into account. As a conclusion let us note that this alternative is not essential at one-loop order, where we can use the dimensional regularization, the trivial functional measure and the initial Lagrangian (8.154).

9 Asymptotically Free Models of Grand Unification with Quantum R^2 Gravity

9.1 Introduction

The attractive features of higher-derivative quantum gravity lead to the attempt to use this theory as the possible basis for a unified theory of all the fundamental interactions. In this chapter we shall try to investigate GUTs with quantum R^2-gravity. It is natural to expect that the unification with gravity takes place at the Planck scale of energies. From the GUT point of view those energies are typical for the region of asymptotic freedom.

The discovery of the AF phenomenon by Gross and Wilczek, and Politzer [79, 80] has stimulated numerous investigations of the renormalization group equations (RGEs) in the theory of Yang–Mills fields coupled with scalar and spinor fields (see, for example [81–89]). The problem was to contruct unified models of elementary particles which would be asymptotically free in all the coupling constants and possess an appropriate physical content. It was established in the course of the development of this problem that one should put a number of restrictions on the structure of the theory to achieve AF. A number of realistic GUTs has been constructed with the help of so-called special solutions of the RGEs (see Chapter 3), but the possibility of choosing arbitrarily either the Yukawa and scalar coupling constants or the multiplet content or both has been lost.

Since the requirement of AF puts some restrictions on the GUT structure it is interesting to investigate whether quantum gravity changes these restrictions or not. A consistent quantum gravity has not yet been formulated. In such a situation any consideration of quantum gravity effects is necessarily based on some model. Since the AF phenomenon is investigated with the help of RGEs then it is natural to choose for the description of gravitational interactions a model which leads to a multiplicatively renormalizable quantum theory and admits a renormalization group analysis. As was shown in the preceding chapter, higher-derivative quantum gravity satisfies these requirements.

The problem of the asymptotic freedom in higher-derivative gravity with matter was investigated in [117, 118, 121–123]. We will follow these papers to consider the properties of such unified theories.

9.2 Structure of the one-loop renormalization

Let us consider a grand unified theory coupled to the quantized gravitational field. The classical action of such a theory has the following form (see the preceding chapter)

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\Big\{ & \frac{1}{\lambda}C^2_{\mu\nu\alpha\beta} - \frac{\omega}{3\lambda}R^2 - \tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \\ & + \tfrac{1}{2}g^{\mu\nu}(D_\mu\varphi)^i(D_\nu\varphi)^i + \tfrac{1}{2}R\xi\varphi^i\varphi^i - V(\varphi)\Big\} \\ & + S_F(n_1, n_2). \end{aligned} \tag{9.1}$$

Here φ is a set of real scalar fields

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu\varphi)^i &= \partial_\mu\varphi^i + \mathrm{i}g(\theta^a)^{ij}A^a_\mu\varphi^j \end{aligned}$$

where A^a_μ is the gauge field, f^{abc} are the structural constants of the gauge group, θ^a are the generators of gauge transformations taken in the corresponding representation. The potential $V(\varphi)$ depends on the scalar coupling constants f_{ijkl} in a general manner

$$V(\varphi) = \tfrac{1}{24}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l. \tag{9.2}$$

The given form of the potential makes it possible to generalize our consideration to the case of several scalar fields, complex ones inclusive. In this case multiplicative renormalizability of the theory requires each scalar multiplet to have its own parameter of non-minimal coupling ξ.

The action (9.1) may be supplemented with the Einstein term, the cosmological constant, and also with mass terms of the scalar and spinor fields. We do not violate multiplicative renormalizability of the theory by taking into account all of these terms, but it is not necessary when studying the question of AF. As is generally known, these dimensional parameters cannot contribute to the renormalization of the dimensionless coupling constants and to the corresponding RGES.

The action for the spinor fields has the form

$$S_F(n_1, n_2) = \mathrm{i} \int \mathrm{d}^4x\sqrt{-g}\bar{\psi}_p(\gamma^\mu D_\mu^{pq} - h_i^{pq}\varphi^i)\psi_q \qquad (9.3)$$

with n_1, n_2 being the numbers of the spinor multiplets belonging to the adjoint and the fundamental representations of the gauge group respectively; D_μ^{pq} is the spinor covariant derivative, and h_i^{pq} are the Yukawa constants. Action (9.1) is a variant of the general form (8.78).

In section 9.6 we will find that the effective Yukawa couplings approach zero at a faster rate than the gauge coupling constant, and hence one may consider the possibility where these couplings are excluded from consideration when analysing the AF in the scalar couplings. In that case we can neglect the Yukawa interaction of the spinor fields, and these fields will contribute to the renormalization group equations only for the effective couplings $\lambda(t)$, $g(t)$ and $\omega(t)$. Note that the spinor fields' contributions to these equations are well-known, therefore we do not need to calculate the counterterms arising due to the spinor fields. Instead of that we can use the general expressions for the spinor contributions to the renormalization group equations for the couplings $\lambda(t)$, $\omega(t)$ and $g(t)$. Thus, we may consider S_F to be zero when calculating the one-loop counterterms.

To derive the one-loop counterterms the background field method is used. We divide the fields in action (9.1) into background φ, $A_\mu^a, g_{\mu\nu}$ and quantum fields σ^i, B_μ^a, $\bar{h}_{\mu\nu}$, h part as follows (see equations (8.85) and (8.86))

$$\begin{gathered} g_{\mu\nu} \to g_{\mu\nu} + \sqrt{2\lambda}\bar{h}_{\mu\nu} + \tfrac{1}{4}\sqrt{\frac{2\lambda}{-\beta}}g_{\mu\nu}h \\ \beta = 3\omega/[4(\omega+1)] \qquad \bar{h}_{\mu\nu}g^{\mu\nu} = 0 \\ A_\mu^a = A_\mu^a + B_\mu^a \qquad \varphi^i = \varphi^i + (-\mathrm{i})\tilde{\sigma}^i \end{gathered} \qquad (9.4)$$

and now, when carrying out the quantization of the theory, one should introduce gauges to fix general-coordinate and gauge symmetries. Let us choose for $\bar{h}_{\mu\nu}$, h the same gauge condition as in section 8.6 and for B_μ^a the gauge $\nabla_\mu B^{a\mu}$. For these gauges one can easily write down the corresponding ghost terms.

It is known, that one-loop renormalization of the gauge coupling constant does not depend on the higher-derivative quantum gravity contributions [138]. Therefore we do not need to calculate specially the counterterms $G^a_{\mu\nu}G^{a\mu\nu}$. These counterterms will be the same as in the case when the gravitational field is switched off. Hence we can set the background field A^a_μ to be zero throughout the calculation of the counterterms, and then restore the final expression with the aid of the gauge invariance.

The quadratic part of the action (9.1) (which leads to the one-loop counterterms dependent on φ and $g_{\mu\nu}$) has (by taking account of gauges) the following structure:

$$S^{(2)} = \frac{1}{2}\int d^4x\sqrt{-g}\begin{pmatrix} h_{\rho\sigma} & h & B^{a\alpha} & \tilde{\sigma}^i \end{pmatrix}(\hat{\mathcal{H}})\begin{pmatrix} \bar{h}_{\mu\nu} \\ h \\ B^{b\beta} \\ \tilde{\sigma}^j \end{pmatrix}. \tag{9.5}$$

The one-loop effective action is given by the expession $\frac{1}{2}\operatorname{Tr}\ln(\hat{\mathcal{H}})$. The matrix $(\hat{\mathcal{H}})$ has the following block structure, that is, a generalization of equations (3.93) and (8.88)

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{1}\Box^2 + \hat{V}^{\rho\omega}\nabla_\rho\nabla_\omega + \hat{U} & \hat{Q}_1^{\rho\omega}\nabla_\rho\nabla_\omega + \hat{Q}_2^\rho\nabla_\rho + \hat{Q}_3 \\ \hat{P}_1^{\rho\omega}\nabla_\rho\nabla_\omega + \hat{P}_2^\rho\nabla_\rho + \hat{P}_3 & \hat{1\!\!1}\Box + \hat{\mathrm{E}}^\rho\nabla_\rho + \hat{\mathrm{D}} \end{pmatrix} \tag{9.6}$$

where all operators have matrix form

$$\hat{1} = \begin{pmatrix} \delta_{\mu\nu,\rho\sigma} - \frac{1}{4}g^{\mu\nu}g^{\rho\sigma} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\delta^{\mu\nu,\rho\sigma} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \qquad \hat{1\!\!1} = \begin{pmatrix} \delta^{ab}g_{\alpha\beta} & 0 \\ 0 & \delta^{ij} \end{pmatrix}$$

$$\hat{P}_1^{\rho\omega} = i\xi\varphi^i\sqrt{2\lambda}\begin{pmatrix} 0 & 0 \\ -\delta^{\mu\nu,\rho\omega} & \frac{3}{4\sqrt{-\beta}}\,g^{\rho\omega} \end{pmatrix}$$

$$\hat{Q}_1^{\rho\omega} = i\xi\sqrt{2\lambda}\varphi^j\begin{pmatrix} 0 & -\delta^{\tau\sigma,\rho\omega} \\ 0 & \frac{3}{4\sqrt{-\beta}}\,g^{\rho\omega} \end{pmatrix}$$

$$\hat{P}_2^\rho = i\sqrt{2\lambda}\begin{pmatrix} 0 & 0 \\ (\xi-1)g^{\rho\nu}\nabla^\mu\varphi^i & \frac{1}{4\sqrt{-\beta}}(1-3\xi)\nabla^\rho\varphi^i \end{pmatrix}$$

$$\hat{Q}_2^\rho = i\sqrt{2\lambda}\begin{pmatrix} 0 & (1-\xi)g^{\rho\sigma}\nabla^\tau\varphi^j \\ 0 & \frac{1}{4\sqrt{-\beta}}(1-3\xi)\nabla^\rho\varphi^j \end{pmatrix}$$

$$\hat{P}_3 = i\sqrt{2\lambda}\begin{pmatrix} g(\theta^a)^{ij}\varphi^j\delta^\nu_\alpha\nabla^\mu\varphi^i & \frac{1}{4\sqrt{-\beta}}g(\theta^a)^{ij}\varphi^j\partial_\alpha\varphi^i \\ -\nabla^\mu\nabla^\nu\varphi^i + \xi R^{\mu\nu}\varphi^i & \frac{1}{4\sqrt{-\beta}}(\Box\varphi^i - \xi R\varphi^i) \\ & +\frac{1}{48\sqrt{-\beta}}\varphi^n\varphi^k\varphi^l(f_{inkl} \\ & +f_{nikl} + f_{nkil} + f_{nkli}) \end{pmatrix}$$

$$\hat{Q}_3 = i\sqrt{2\lambda}\begin{pmatrix} g(\theta^b)^{ij}\varphi^j\nabla^\rho\varphi^i\delta^\alpha_\beta & \xi\varphi^j R^{\rho\sigma} \\ & -\frac{1}{4\sqrt{-\beta}}\xi\varphi^j R^{\rho\sigma} \\ \frac{1}{4\sqrt{-\beta}}g(\theta^\beta)^{ij}\varphi^j\partial_\beta\varphi^i & +\frac{1}{48\sqrt{-\beta}}\varphi^n\varphi^k\varphi^l(f_{inkl} \\ & +f_{nikl} + f_{nkil} + f_{nkli}) \end{pmatrix}$$

$$\hat{E}^\rho = \begin{pmatrix} 0 & g(\theta^a)^{jl}\varphi^l\delta^\rho_\alpha \\ -g(\theta^b)^{ik}\varphi^k\delta^\rho_\beta & 0 \end{pmatrix}$$

$$\hat{D} = \begin{pmatrix} -\delta^{ab}R_{\alpha\beta} & \\ -g^2 g_{\alpha\beta}(\theta^a)^{lm}(\theta^b)^{lk}\varphi^m\varphi^k & g(\theta^a)^{kj}\partial_\alpha\varphi^k \\ g[(\theta^b)^{ki} - (\theta^b)^{ik}]\partial_\beta\varphi^k & -\xi R\delta^{ij} + \frac{1}{12}\varphi^l\varphi^k(f_{ijkl} \\ & +f_{ikjl} + f_{iklj} + f_{kilj} \\ & +f_{kijl} + f_{klij}) \end{pmatrix}$$

$$\hat{V}^{\tau\omega} = \begin{pmatrix} V_{\bar{h}\bar{h}} & V_{\bar{h}h} \\ V_{h\bar{h}} & V_{hh} \end{pmatrix} \qquad \hat{U} = \begin{pmatrix} U_{\bar{h}\bar{h}} & U_{\bar{h}h} \\ U_{h\bar{h}} & U_{hh} \end{pmatrix}$$

$$\begin{aligned}
\left[V_{\bar{h}\bar{h}}\right]^{\tau\omega} &= \lambda\xi\varphi^k\varphi^k\left(\tfrac{1}{2}g^{\tau\omega}\delta^{\mu\nu,\rho\sigma} - g^{\mu\rho}\delta^{\tau\omega,\nu\sigma}\right) + \left[V_{\bar{h}\bar{h}}(8.89)\right]^{\tau\omega} \\
\left[V_{h\bar{h}}\right]^{\tau\omega} &= \frac{1}{4\sqrt{-\beta}}\lambda\xi\varphi^k\varphi^k\delta^{\mu\nu,\tau\omega} + \left[V_{h\bar{h}}(8.89)\right]^{\tau\omega} \\
\left[V_{\bar{h}h}\right]^{\tau\omega} &= \frac{1}{4\sqrt{-\beta}}\lambda\xi\varphi^k\varphi^k\delta^{\rho\sigma,\tau\omega} + \left[V_{\bar{h}h}(8.89)\right]^{\tau\omega} \\
\left[V_{hh}\right]^{\tau\omega} &= \frac{3}{16\beta}\lambda\xi\varphi^k\varphi^k g^{\tau\omega} + \left[V_{hh}(8.89)\right]^{\tau\omega} \\
\left[U_{\bar{h}\bar{h}}\right] &= 2\lambda\Big(-\tfrac{1}{4}\delta^{\mu\nu,\rho\sigma}\nabla^\omega\varphi^k\nabla_\omega\varphi^k + g^{\mu\rho}\nabla^\nu\varphi^k\nabla^\sigma\varphi^k \\
&\quad - \tfrac{1}{4}\xi R\varphi^k\varphi^k\delta^{\mu\nu,\rho\sigma} + \xi R^{\mu\sigma}g^{\nu\rho}\varphi^k\varphi^k + \tfrac{1}{48}\delta^{\mu\nu,\rho\sigma} \\
&\quad \times f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l\Big) + \left[U_{\bar{h}\bar{h}}(8.89)\right] \\
\left[U_{hh}\right] &= \frac{1}{192\beta}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l + \left[U_{hh}(8.89)\right].
\end{aligned} \tag{9.7}$$

Here the omitted terms stand for the pure 'gravitational' part of the operators V, U which have been written as (8.89). We will also see that the blocks $U_{h\bar{h}}$ and $U_{\bar{h}h}$ do not contribute to the counterterms.

The matrices (9.7) are given, for simplicity, in non-symmetric form with respect to the transposition of the indices μ and ν, ρ and σ, τ

and ω. Such symmetry is automatically restored in the expression (9.5) for the quadratic part of the action when multiplying by quantum fields. Of course, one should take this symmetry into account when carrying out further calculations. Similarly, those blocks of the matrix $\hat{\mathcal{H}}$ which in the action $S^{(2)}$ in (9.5) are contracted with the field $\bar{h}_{\mu\nu}$ should be multiplied by the corresponding projector (see the expression for $\hat{1}$ in (9.7)).

Note that the coupling constants g and f enter the bilinear form (9.6) of the action being multiplied by the matter background fields and therefore the one-loop counterterms for C^2 and R^2 do not depend on the constants g and f. Hence these terms can be obtained as a sum of counterterms arising in 'pure' quantum gravity and also of the additive contributions of the free scalar, spinor and vector fields in an external gravitational field. All these counterterms are well-known. With the help of equations (3.91) and (3.92) we can write the renormalization group equations for the effective couplings $\lambda(t)$ and $\omega(t)$ in their final form

$$\begin{aligned} (4\pi)^2\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= -\alpha^2\lambda^2 \\ (4\pi)^2\frac{d\omega}{\mathrm{d}t} &= -\lambda\omega\alpha^2 - \lambda\left[\tfrac{10}{3}\omega^2 + 5\omega + \tfrac{5}{12} + \tfrac{3N_0}{2}(\xi - \tfrac{1}{6})^2\right] \\ \alpha^2 &= \tfrac{133}{10} + \tfrac{1}{5}N_1 + \tfrac{1}{10}N_{1/2} + \tfrac{1}{60}N_0. \end{aligned} \tag{9.8}$$

Here $N_0, N_{1/2}$, N_1 are the numbers of the real scalar, spinor and vector fields, respectively, described by the theory (9.1).

Now we can ignore all counterterms which do not depend on the background scalar field since all such counterterms are already taken into account in equations (9.8). At the same time we cannot, for instance, regard the background metric as being flat, since it would then be impossible to calculate the non-minimal counterterms like $\int \mathrm{d}^4x\sqrt{-g}R\varphi^i\varphi^i$.

When the conformal version of the theory (9.1) is considered we find the same problem as in the pure Weyl theory. The possible solutions of the conformal anomaly problem were discussed in section 8.8.

To quantize the conformal gravity it is necessary to fix conformal invariance. To do so we shall use the condition $h = 0$. In this case we must omit all the terms in (9.5) which should be multipied by the field h from expression (9.6) for $\mathcal{H}$. (Instead of that we can eliminate all the terms containing the parameter β from expressions (9.7)). The use of the conformal gauge condition $h = 0$ does not violate the conformal invariance of the background functional, but the naive application of the background field method does not allow us to preserve conformal invariance even in the one-loop approximation. The reason for this is that the general coordinate gauge condition is not conformally covariant. So, we see, that the standard quantization scheme (supplemented by the background field method) leads to the appearance of counterterms of $R\varphi^2$ or R^2-type, which break the conformal invariance of the effective action. To remove these counterterms we shall use a special conformal regularization scheme. Within this scheme the theory under consideration is renormalizable and allows the use of RG methods (see the discussion in section 8.8)). In the presence of matter fields the regularization is carried out by the following substitution

$$\begin{aligned} g_{\mu\nu} &\to \tilde{g}_{\mu\nu} = P^2 \circ g_{\mu\nu} \qquad & \varphi &\to \tilde{\varphi} = P^{-1} \circ \varphi \\ A_\mu &\to \tilde{A}_\mu = A_\mu \qquad & \psi &\to \tilde{\psi} = P^{-3/2} \circ \psi \end{aligned} \tag{9.9}$$

where the function $P = P(g_{\mu\nu})$ is determined by (8.107). The new metric $\tilde{g}_{\mu\nu}$ is conformally invariant and possesses an important property $R(\tilde{g}_{\mu\nu}) = 0$. Since the conformal invariance is violated only due to $R\varphi^2$- and R^2-terms, the above substitution makes the one-loop counterterms conformally invariant. For example, let the counterterm (without special regularization) be

$$\sqrt{-g}(\Delta Z_1 g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \Delta Z_3 R\varphi^2).$$

Then, after the substitution $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$, $\varphi \to \tilde{\varphi}$ (which should be regarded as an element of the renormalization procedure) is made, it takes (in terms of $g_{\mu\nu}$, φ) the form

$$\sqrt{-g}\Delta Z_1(g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \tfrac{1}{6}R\varphi^2).$$

The calculation of the one-loop divergences of the effective action gives the value of α^2 in equation (9.33) below. The second equation in (9.8) is absent, of course.

To close this section we present the renormalization group equations for the effective gauge constant. This equation has the same form as in the case when the gravitational field is switched off

$$\begin{aligned} (4\pi)^2\frac{\mathrm{d}g^2}{\mathrm{d}t} &= -b^2 g^4 \\ b^2 &= \tfrac{11}{3}N + \tfrac{4}{3}T_{1/2} + \tfrac{1}{3}T_0 \end{aligned} \tag{9.10}$$

where N is the gauge group dimension (for O(N) and SU(N) groups), and $T_{1/2}$, T_0 are the eigenvalues of the Casimir operators for the spinor and scalar fields, respectively.

9.3 Calculation of one-loop counterterms

To calculate the divergences of the expression $\mathrm{Tr}\ln(\hat{\mathcal{H}})$ we will use the method of the universal traces [105] described in section 8.7. First of all we rewrite $\mathrm{Tr}\ln(\hat{\mathcal{H}})$ in a more convenient form

$$\mathrm{Tr}\ln(\hat{\mathcal{H}}) = \mathrm{Tr}\ln\begin{pmatrix} \hat{1}\Box^2 & 0 \\ 0 & \hat{1\!\!1}\Box \end{pmatrix} + \mathrm{Tr}\ln\left[\begin{pmatrix} \hat{1} & 0 \\ 0 & \hat{1\!\!1} \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} \hat{V}^{\tau\omega}\nabla_\tau\nabla_\omega\frac{1}{\Box^2} + \hat{U}\frac{1}{\Box^2} & \left(\hat{Q}_1^{\tau\omega}\nabla_\tau\nabla_\omega + \hat{Q}_2^\tau\nabla_\tau + \hat{Q}_3\right)\frac{1}{\Box} \\ \left(\hat{P}^{\tau\omega}\nabla_\tau\nabla_\omega + \hat{P}_2^\tau\nabla_\tau + \hat{P}_3\right)\frac{1}{\Box^2} & \left(\hat{E}^\tau\nabla_\tau + \hat{D}\right)\frac{1}{\Box} \end{pmatrix}\right] \tag{9.11}$$

where all abbreviations are the same as in (9.7). The first term in expression (9.11) contributes only to the purely gravitational divergences, and subsequently we disregard it. Futher, in the second term we expand the logarithm in a series and keep only the terms whose background dimension does not exceed four. It is convenient now to introduce the following notations

$$\begin{array}{llll} \hat{V}^{\mu\nu}\nabla_\mu\nabla_\nu\dfrac{1}{\Box^2} = V & \hat{U}\dfrac{1}{\Box^2} = U & \hat{Q}^{\tau\omega}\nabla_\tau\nabla_\omega\dfrac{1}{\Box} = Q_1 & \\ \hat{Q}_2^\nu\nabla_\nu\dfrac{1}{\Box} = Q_2 & \hat{Q}_3\dfrac{1}{\Box} = Q_3 & \hat{P}_1^{\tau\omega}\nabla_\tau\nabla_\omega\dfrac{1}{\Box^2} = P_1 & \\ \hat{P}_2^\nu\nabla_\nu\dfrac{1}{\Box^2} = P_2 & \hat{P}_3\dfrac{1}{\Box^2} = P_3 & \hat{E}^\tau\nabla_\tau\dfrac{1}{\Box} = E & \hat{D}\dfrac{1}{\Box} = D \end{array} \tag{9.12}$$

and, by using them, rewrite the second term in (9.11) in a more simple form

$$\begin{aligned} \mathrm{Tr}\ln(\hat{\mathcal{H}}) = {} & \mathrm{Tr}\ln(\hat{1\!\!1}\Box + \hat{\mathrm{E}}^\tau\nabla_\tau + \hat{\mathrm{D}}) \\ & + \mathrm{Tr}\ln(-P_1Q_1 - P_1Q_2 - P_2Q_1 - P_2Q_2 - P_1Q_3 \\ & - P_3Q_1 + VQ_1P_1 + Q_1EP_1 + Q_2EP_1 + Q_1EP_2 \\ & + Q_1DP_1 - \tfrac{1}{2}Q_1P_1Q_1P_1 - Q_1E^2P_1 \\ & + U + V - \tfrac{1}{2}V^2) + \ldots . \end{aligned} \tag{9.13}$$

Here we omitted terms in expression (9.11) and depending on the field $g_{\mu\nu}$ only, and also terms which do not contribute to divergences of the effective action.

The first term in expression (9.13) contributes to the counterterms, arising in the theory (9.1) when the gravitational field is purely classical and one can calculate the corresponding divergences in the standard way which was described in Chapter 3.

It is now necessary to demonstrate the method of commutation which allows us to use the universal traces (8.93)–(8.98).

$$
\begin{aligned}
-\operatorname{Tr} Q_2 P_1 &= -\operatorname{Tr} \hat{Q}_2^{\alpha} \nabla_\alpha \frac{1}{\Box} \hat{P}_1^{\tau\omega} \nabla_\tau \nabla_\omega \frac{1}{\Box^2} \\
&= -\operatorname{Tr} \hat{Q}_2^{\alpha} \nabla_\alpha \Big\{ \hat{P}_1^{\tau\omega} \nabla_\tau \nabla_\omega \frac{1}{\Box^3} + \frac{1}{\Box} \Big[\hat{P}_1^{\tau\omega}, \Box \Big] \frac{1}{\Box} \nabla_\tau \nabla_\omega \frac{1}{\Box^2} \Big\} \\
&= -\operatorname{Tr} \hat{Q}_2^{\alpha} \nabla_\alpha \Big\{ \hat{P}_1^{\tau\omega} \nabla_\tau \nabla_\omega \frac{1}{\Box^3} - 2\nabla^\lambda \hat{P}_1^{\tau\omega} \nabla_\lambda \nabla_\tau \nabla_\omega \frac{1}{\Box^4} \Big\} \\
&= \frac{2i}{\varepsilon} \tfrac{1}{6} \operatorname{tr} \Big\{ \hat{Q}_2^{\alpha} \nabla_\alpha \hat{P}_{1\beta}^{\beta} - \hat{Q}_2^{\alpha} \nabla^\beta \hat{P}_{1\alpha\beta} \Big\}.
\end{aligned}
\tag{9.14}
$$

Using the same method one can obtain the divergences in the following final form (as well as expression (8.100))

$$
\begin{aligned}
-\operatorname{Tr} P_1 Q_1\big|_{\mathrm{div}} &= \frac{2i}{\varepsilon} \operatorname{tr} \Big\{ \hat{Q}_1^{\alpha\beta} \Big(\tfrac{1}{8} \nabla_\alpha \nabla_\beta \hat{P}_1{}^{\tau}{}_{\tau} - \tfrac{1}{48} g_{\alpha\beta} \Box \hat{P}_1{}^{\tau}{}_{\tau} \\
&\quad - \tfrac{1}{24} \Box \hat{P}_{1\alpha\beta} + \tfrac{1}{24} g_{\alpha\beta} \nabla_\mu \nabla_\nu \hat{P}_1^{\mu\nu} - \tfrac{1}{6} \nabla_\mu \nabla_\alpha \hat{P}_{1\beta}^{\mu} \\
&\quad + \tfrac{1}{24} R_{\alpha\beta} \hat{P}_{1\tau}^{\tau} + \tfrac{1}{24} g_{\alpha\beta} R_{\mu\nu} \hat{P}_1^{\mu\nu} - \tfrac{1}{48} g_{\alpha\beta} R \hat{P}_1{}^{\tau}{}_{\tau} \\
&\quad - \tfrac{1}{24} R \hat{P}_{1\alpha\beta} - \tfrac{1}{6} R_{\mu\alpha\nu\beta} \hat{P}_1^{\mu\nu} + \tfrac{1}{6} R_{\alpha\tau} R \hat{P}_1{}^{\tau}{}_{\beta} \Big) \Big\} \\
&= \frac{2i}{\varepsilon} \Big\{ -\frac{3\lambda\xi^2}{2} g^{\mu\nu} \partial_\mu \varphi^i \, \partial_\nu \varphi^i + \tfrac{1}{2} \Big(-\tfrac{3}{8} \frac{\lambda\xi^2}{\beta} - \tfrac{1}{2} \lambda \xi^2 \Big) \\
&\quad \times R \varphi^i \varphi^i \Big\}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr} P_2 Q_2\big|_{\mathrm{div}} &= \frac{2i}{\varepsilon} \operatorname{tr} \Big(\tfrac{1}{4} \hat{Q}_2^{\tau} \hat{P}_{2\tau} \Big) \\
&= \frac{2i}{\varepsilon} \Big\{ \frac{9\lambda}{4} (\xi - 1)^2 - \frac{\lambda}{16\beta} (3\xi - 1)^2 \Big\} \tfrac{1}{2} g^{\mu\nu} (\partial_\mu \varphi)^i (\partial_\nu \varphi)^i
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr} (Q_2 P_1 + Q_1 P_2)\big|_{\mathrm{div}} &= \frac{2i}{\varepsilon} \operatorname{tr} \Big(\tfrac{1}{6} \hat{Q}_2^{\alpha} \nabla_\alpha \hat{P}_1{}^{\beta}{}_{\beta} - \tfrac{1}{6} \hat{Q}_2^{\alpha} \nabla^\beta \hat{P}_{1\alpha\beta} \\
&\quad + \tfrac{1}{3} \hat{Q}_1^{\alpha\beta} \nabla_\alpha \hat{P}_{2\beta} - \tfrac{1}{12} \hat{Q}_1{}^{\alpha}{}_{\alpha} \nabla_\beta \hat{P}_2^{\beta} \Big) \\
&= \frac{2i}{\varepsilon} \Big\{ 3\lambda\xi \Big(\frac{1-\xi}{2} + \frac{3\xi - 1}{8\beta} \Big) \tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \, \partial_\nu \varphi^i \Big\}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}(Q_3P_1+Q_1P_3)|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}\tfrac{1}{4}\Big(\hat{Q}_3\hat{P}_1^{\alpha}{}_{\alpha}+\hat{Q}_1^{\alpha}{}_{\alpha}\hat{P}_3\Big)\\
&= \frac{2\mathrm{i}}{\varepsilon}\Big\{\frac{-3\lambda\xi}{4\beta}\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi^i\,\partial_\nu\varphi^i+\Big(-\tfrac{3}{2}\frac{\lambda\xi^2}{\beta}\Big)\\
&\qquad\times\tfrac{1}{2}R\varphi^i\varphi^i+\frac{3\lambda\xi}{\beta}\tfrac{1}{24}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l\Big\}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}VP_1Q_1|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}(-\tfrac{1}{192}\hat{V}^{\tau\omega}Q_1^{\alpha\beta}P_1^{\mu\nu}g^{(3)}_{\tau\omega\alpha\beta\mu\nu})\\
&= \frac{2\mathrm{i}}{\varepsilon}\Big\{\Big(-3\lambda\xi^2+\lambda\omega\xi^2-\frac{9\lambda\xi^2\omega}{16\beta}\Big)\tfrac{1}{2}R\varphi^i\varphi^i\\
&\qquad+\Big(-\tfrac{3}{8}+\frac{9}{16\beta}-\frac{27}{128\beta^2}\Big)\lambda^2\xi^3(\varphi^i\varphi^i)^2\Big\}
\end{aligned}
$$

$$
\operatorname{Tr}Q_1EP_1=\operatorname{Tr}Q_1EP_2=\operatorname{Tr}Q_2EP_1=\operatorname{Tr}Q_1E^2P_1=0
$$

$$
\begin{aligned}
\operatorname{Tr}Q_1DP_1|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}\Big(-\tfrac{1}{24}\hat{Q}_1^{\alpha\beta}\hat{D}\hat{P}_1^{\mu\nu}g^{(2)}_{\mu\nu\alpha\beta}\Big)\\
&= \frac{2\mathrm{i}}{\varepsilon}\Big\{\Big(-3\lambda\xi^3+\frac{9}{4\beta}\lambda\xi^3\Big)\tfrac{1}{2}R\varphi^i\varphi^i\\
&\qquad+\Big(18-\frac{27}{2\beta}\Big)\lambda\xi^2\tfrac{1}{24}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l\Big\}
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{1}{2}\operatorname{Tr}Q_1P_1Q_1P_1|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}\tfrac{1}{3840}\hat{Q}_1^{\alpha\beta}\hat{P}_1^{\mu\nu}\hat{Q}_1^{\rho\omega}\hat{P}_1^{\epsilon\tau}g^{(4)}_{\alpha\beta\mu\nu\rho\omega\epsilon\tau}\\
&= \frac{2\mathrm{i}}{\varepsilon}\Big(\tfrac{9}{8}-\frac{27}{16\beta}+\frac{81}{128\beta^2}\Big)\lambda^2\xi^4(\varphi^i\varphi^i)^2
\end{aligned}
$$

$$
\operatorname{Tr}U|_{\rm div}=\frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}(-\hat{U})=\frac{2\mathrm{i}}{\varepsilon}(-9\lambda-\frac{\lambda}{4\beta})\tfrac{1}{24}f_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l
$$

$$
\begin{aligned}
\operatorname{Tr}V|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}\Big(\tfrac{1}{12}R\hat{V}^{\alpha}_{\alpha}-\tfrac{1}{6}R_{\alpha\beta}\hat{V}^{\alpha\beta}\Big)\\
&= \frac{2\mathrm{i}}{\varepsilon}\lambda\xi\Big(\tfrac{3}{4}+\frac{1}{16\beta}\Big)R\varphi^i\varphi^i+\dots
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{1}{2}\operatorname{Tr}V^2|_{\rm div} &= \frac{2\mathrm{i}}{\varepsilon}\operatorname{tr}\Big(\tfrac{1}{24}\hat{V}_{\alpha\beta}\hat{V}^{\alpha\beta}+\tfrac{1}{48}\hat{V}^{\alpha}_{\alpha}\hat{V}^{\beta}_{\beta}\Big)\\
&= \frac{2\mathrm{i}}{\varepsilon}\Big\{\Big(\tfrac{63}{4}-\frac{9}{8\beta}+\frac{27}{64\beta^2}\Big)\lambda^2\xi^2\tfrac{1}{24}\Big(\varphi^i\varphi^i\Big)^2\\
&\qquad+\Big(-\tfrac{7}{2}\omega-2+\frac{3\omega}{32\beta^2}\Big)\lambda\xi R\varphi^i\varphi^i\Big\}+\dots. \qquad (9.15)
\end{aligned}
$$

Throughout formulae (9.15) terms depending only on the field $g_{\mu\nu}$ are omitted.

In what follows we have to show how expressions (9.15) change when the theory describes a number of different scalar multiplets and, in accordance with that, a number of different non-minimal coupling parameters ξ are embedded. Let us consider the case when there is one real scalar field ϕ^a belonging to the adjoint representation, and one complex scalar field φ_i belonging to the fundamental representation The theory is based on the gauge group SU(N). In that case the Lagrangian of the scalar fields has the known form

$$\begin{aligned} L_0 = {} & \tfrac{1}{2} g^{\mu\nu}(D_\mu\phi)^a(D_\nu\phi)^a + g^{\mu\nu}(D_\mu\varphi^+)_i(D_\nu\varphi)^i \\ & + \tfrac{1}{2}\xi_1 R\phi^a\phi^a + \xi_2 R\varphi_i^+\varphi^i - \tfrac{1}{8} f_1(\phi^a\phi^a) \\ & - \tfrac{1}{8} f_2(\phi^a D_{rab}\phi^b)(\phi^c D_{rcd}\phi^d) - \tfrac{1}{2} f_3\phi^a\phi^a\varphi_i^+\varphi^i \\ & - \tfrac{1}{2} f_4(\phi^a D_{arb}\phi^b)\varphi_i^+\left(\frac{\lambda^r}{2}\right)^i{}_j\varphi^j - \tfrac{1}{2} f_5(\varphi_i^+\varphi^i)^2 . \end{aligned} \tag{9.16}$$

Here the generators of the SU(N) group $(\lambda^a/2)^i_j$, the structural constants f^{abc} and the elements of symmetrical matrix D_{abc} obey the following relations (we also used those relations in Chapter 3).

$$\begin{gathered} \operatorname{tr}\left(\frac{\lambda^a}{2}\right)\left(\frac{\lambda^b}{2}\right) = \tfrac{1}{2}\delta^{ab} \qquad \left[\left(\frac{\lambda^a}{2}\right),\left(\frac{\lambda^b}{2}\right)\right] = \mathrm{i} f^{abc}\left(\frac{\lambda^c}{2}\right) \\ \left\{\left(\frac{\lambda^a}{2}\right),\left(\frac{\lambda^b}{2}\right)\right\} = D_{abc}\left(\frac{\lambda^c}{2}\right) + \tfrac{1}{2}\delta^{ab} \\ D_{abc} = D_{acb} = D_{cab} \end{gathered} \tag{9.17}$$

with $a, b, c = 1, 2, \ldots, N^2 - 1$ and $i, j = 1, 2, \ldots, N$. The following relations connecting the generators $(\lambda^a/2)^j_i$ with the structure constants f^{abc} and with the matrices D_{abc} are valid:

$$\begin{aligned} & f^{abr} f^{bpa} = N\delta^{rp} \qquad D_{arb}D_{bpa} = \frac{N^2-4}{N}\delta_{rp} \\ & f^{apb} f^{brc} f^{cqd} f^{dsa} = \frac{N}{4}(D_{apr}D_{aqs} + D_{aps}D_{aqr} - D_{apq}D_{ars}) \\ & \qquad + (\delta_{rp}\delta_{qs} + \delta_{ps}\delta_{rq}). \end{aligned} \tag{9.18}$$

The blocks (9.7) of the matrix $(\hat{\mathcal{H}})$ depend on all constants ξ_i in different ways. These constants appear either in combination with their 'own' background scalar field or in that block of the matrix $(\hat{\mathcal{H}})$, which in expression (9.5) is multiplied by their 'own' quantum scalar field. Hence the terms containing the constants ξ_1, ξ_2 additively enter the counterterms like (9.16). This means that the structure of the renormalization of every parameter ξ_1, ξ_2 is determined by relations (9.15). In particular, these parameters are renormalized independently of each other.

The question of how the parameters ξ_1, ξ_2 enter the renormalization relations for the scalar coupling constants seems to be more complicated. To derive the corresponding counterterms one should represent the complex scalar field as a sum of two real scalar ones. Then the contribution of every trace in (9.15) is calculated by analysing the block structure of the multiplied matrices. For instance, the divergent part of the trace $\mathrm{Tr}(Q_1 P_1)^2$ in the case of the model (9.16) is

$$-\tfrac{1}{2}\,\mathrm{Tr}(Q_1 P_1)^2\big|_{\mathrm{div}} = \frac{2\mathrm{i}}{\varepsilon}\left(\tfrac{9}{8} - \frac{27}{16\beta} + \frac{81}{128\beta^2}\right)\lambda^2 \times\left(\xi_1^2\phi^a\phi^a + 2\xi_2^2\varphi_i^+\varphi^i\right)^2. \qquad (9.19)$$

Now, with the aid of the method described above, one can get (without carrying out special calculations) the one-loop counterterms for the theory, containing an arbitrary number of scalar multiplets. Let us make one more observation concerning the conformal version of the theory (9.1). The splitting (9.4) of the metric into background and quantum parts we have made above is singular for $\omega = 0$, so that we cannot proceed to the conformal limit in further calculations by putting $\omega = 0$, $\xi = 1/6$. At the same time it follows from (9.4), that contributions of the field h to the counterterms (9.15) always contain $1/\beta$ as a coefficient. Since the gauge $h = 0$ is usually used in the conformal theory, and these contributions do not arise at all, one should pass to the conformal limit in the counterterms (9.15) as follows: $1/\beta = 0$, $\xi = 1/6$. It is easy to check that this ansatz yields correct results. Note, that after the explicit expression (9.4) for β is used, one loses the possibility of passing to the conformal theory.

Note, finally, that the gravitational contributions to the counterterms (9.15) do not depend on the gauge group. In particular, for the models based on the gauge groups $\mathrm{SU}(N)$ and $\mathrm{O}(N)$, which are under consideration here, these counterterms do not depend on the group dimension N. Of course, all this is also valid for the conformal theory. The universal nature of the counterterms (9.15) is due to the fact that the diagrams with the external matter fields lines and the internal graviton lines do not contain any summation over the group indices. An exception is here the diagram corresponding to the expression $\mathrm{Tr}\,P_1 E^2 Q_1$. However, a straightforward calculation shows that this trace vanishes for the gauge groups O(N) and SU(N) in which we are interested.

9.4 Asymptotic behaviour of the scalar effective coupling constants

Now we turn to studying the asymptotic properties of the effective coupling for the simplest O(N) and SU(N) models. It follows from expression (9.15) for the counterterms that in the theory (9.1) (we include the case of several scalar fields) the renormalization group equations for the effective constants $\xi(t)$ and $f(t)$ are

$$
\begin{aligned}
&(4\pi)^2\frac{d\xi}{dt} = \beta_\xi + \Delta\beta_\xi \\
&(4\pi)^2\frac{df}{dt} = \beta_f + \Delta\beta_f \qquad \Delta\beta_f = 3\Delta\beta_f^{(1)} + \Delta\beta_f^{(2)}.
\end{aligned}
\tag{9.20}
$$

Here the renormalization group function β_f corresponds to flat space–time, the function β_ξ arises in the theory (9.1) when the gravitational field is purely external, while the additional terms $\Delta\beta_\xi$ and $\Delta\beta_f$ take account of the quantum gravity effects. The functions β_ξ and β_f depend essentially on the choice of the gauge group and on the compostion of the model multiplet. The other terms in (9.20) have a universal nature for every model like (9.1) and we can find them independently of the chosen matter Lagrangian. The method of calculation of the renormalization group functions β_f, β_ξ is carried out in Chapter 3. Our calculations give the following expressions for the $\Delta\beta_\xi$, $\Delta\beta_f^{(1)}$ and $\Delta\beta_f^{(2)}$ structures

$$
\begin{aligned}
\Delta\beta_f^{(1)} &= \lambda^2\xi^2\Big(5 + \frac{1}{4\omega^2} - \frac{3\xi}{\omega^2} + \frac{9\xi^2}{\omega^2}\Big) \\
\Delta\beta_f^{(2)} &= -\lambda f\Big(5 + 3\xi^2 + \frac{33}{2\omega}\xi^2 - \frac{6}{\omega}\xi + \frac{1}{2\omega}\Big) \\
\Delta\beta_\xi &= \lambda\xi\Big(- \tfrac{3}{2}\xi^2 + 4\xi + 3 + \frac{10}{3}\omega - \frac{9}{4\omega}\xi^2 + \frac{5}{\omega}\xi + \frac{1}{\omega}\Big)
\end{aligned}
\tag{9.21}
$$

which are determined only by quantum gravity.

Our next step is to consider a number of the simplest O(N) and SU(N) models of GUT and to use the formalism presented in detail.

(a) *The theory with the gauge group O(N)* which describes one real scalar multiplet belonging to the fundamental representation. In this case the potential $V(\varphi)$ has the simplest form

$$
V(\varphi) = \tfrac{1}{24}f(\varphi_i\varphi_i)^2 \tag{9.22}
$$

which is defined by only one coupling constant.

The generators $(t^a)_{ij}$ taken in the fundamental representation obey the relations

$$\operatorname{tr} t^a t^b = \tfrac{1}{2}\delta^{ab} \qquad (t^a)_{ij} = -(t^a)_{ji}$$

with $a, b = 1, 2, \ldots, N(N-1)/2$ and $i, j = 1, 2, \ldots, N$. The generator's algebra is determined by the structural constants which have the following properties

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c \qquad f^{abd} f^{bcd} = \frac{N-2}{2}\delta^{bc}$$
$$f^{bdl} f^{dec} f^{ema} f^{acb} = \frac{(N-2)^2}{8}\delta^{ml}.$$

The renormalization group equations for the effective couplings $\lambda(t)$, $\omega(t)$ and $g^2(t)$ have the form of equations (9.8) and (9.10) with the coefficients as follows

$$\begin{aligned} a^2 &= \frac{798 + 6N^2 - 5N}{60} + \frac{1}{10} N \left(n_2 + \frac{N-1}{2} n_1\right) \\ b^2 &= \frac{22N - 45}{6} - \tfrac{4}{3}\Big(n_2 + n_1(N-2)\Big). \end{aligned} \tag{9.23}$$

The equations for $\xi(t)$ and $f(t)$, after all contributions have been gathered, have the form

$$\begin{aligned} (4\pi)^2 \frac{\mathrm{d}\xi}{\mathrm{d}t} &= \left(\xi - \tfrac{1}{6}\right)\left(\frac{N+2}{3} f - \frac{3N-3}{2} g^2\right) + \Delta\beta_\xi \\ (4\pi)^2 \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{N+8}{3} f^2 - 3(N-1) f g^2 + \tfrac{9}{4}(N-1) g^4 \\ &\quad + 3\Delta\beta_f^{(1)} + \Delta\beta_f^{(2)}. \end{aligned} \tag{9.24}$$

We are going to solve all the equations found step by step. More precisely, when the question of AF in the set of differential equations (9.8), (9.10), (9.23) and (9.24) is studied we consider that the ratio $\lambda(t)/g^2(t) \to k = b^2/a^2$ as $t \to \infty$. Hence one can put $\lambda(t) = kg^2(t)$ at large t and rewrite the equations (9.8) and (9.24) in terms of a new effective coupling $\bar{f}(t) = f(t)/g^2(t)$ and a new variable τ defined by the relation $\mathrm{d}\tau = (4\pi)^{-2} g^2(t)\mathrm{d}t$. The question of whether AF takes place in that case or not is determined by the existence of stable real fixed points of the system of equations for $\omega(\tau)$, $\xi(\tau)$ and $f(\tau)$. Note that we do not demand that the coupling constants $\omega(t)$ and $\xi(t)$ behave as asymptotically free (i.e. decrease to zero) in the theory with R^2-gravity, and so these effective couplings have non-zero limiting values. The numerical calculations show, that the AF takes place only when b^2 is small enough: $b^2 \leqslant b^2_{\rm max}$. Here $b^2_{\rm max}$ is the greatest value of b^2 still providing the AF in the scalar coupling. On the other hand, we need to have $b^2 > 0$ for the AF in the effective coupling $g(t)$. Thus AF in all the coupling constants yields the restriction $0 < b^2 \leqslant b^2_{\rm max}$. Since $b^2 = b^2(n_1, n_2)$, there arises a restriction on the permissible number of the spinor multiplets. Moreover, the existence of the real fixed points imposes a restriction from below on the permissible group dimension.

The situation described is typical for the model (9.1) in flat space–time. Here our goal is to find the restriction on n_1, n_2 and N which arises in general R^2-gravity for the model (9.22). This analysis is a very complicated task and analytical calculation produces only preliminary restrictions on n_1 and n_2 under the condition that $g(t)$ tends to zero. To obtain $N_{\rm min}$ and more precise values of n_1 and n_2 one can use numerical calculations for solving the algebraic system of non-linear equations by using a computer [121, 122]. The main result is that AF for the constant f takes place only if $N \geqslant 7$ as well as in flat space–time and in conformal theory which will be considered below. Also the parameters n_1 and n_2 should have definite values. In table 9.1 we present, as an example, the values of n_1 and n_2 giving AF for $N = 7, 8, 9$. One can see from this table, that for each given value of N and n_1 quantum gravity effects decrease by one the minimal number of the spinor fields n_2 belonging to the fundamental representation, which still provides AF. In the table we underline these new values of n_2, which correspond to new versions of asymptotically free unified models. Note, that within the conformal theory such an effect becomes apparent only for $N \geqslant 8$. So, we find that general R^2-gravity is more suitable for building realistic GUTs models with a more 'pure' spinor multiplet composition. This tendency is likely to be model-independent and it is interesting to verify it by investigating other models.

(*b*) *The model with the gauge group SU(N) described two scalar multiplets within (9.16).* The expressions for the coefficients a^2 and b^2 are determined by equations (9.8) and (9.10) and after rearranging the simple algebra the result has the form

$$\begin{aligned} a^2 &= \frac{785 + 13N^2 + 2N}{60} + \frac{1}{10}\left[n_1(N^2 - 1) + Nn_2\right] \\ b^2 &= \tfrac{1}{3}(21N - 1) - \tfrac{4}{3}\left(n_1 N + \tfrac{1}{2}n_2\right). \end{aligned} \qquad (9.25)$$

The set of renormalization group equations for the effective couplings

Table 9.1 Permissible values of n_2 for O(N).

		$N\backslash n_1$	0	1	2
Without taking account of gravitation		7	13	8	3
		8	15, 16	9, 10	3, 4
		9	17,18,19	10,11,12	3,4,5
With taking account of gravitation	Conformal theory	7	8	3	3
		8	14,15,16	8,9,10	2,3,4
		9	16,...,19	9,...,12	2,...5
	General R^2-theory	7	12,13	7,8	2,3
		8	14,15,16	8,9,10	2,3,4
		9	16,...,19	9,...,12	2,...,5

within the considered SU(N) model is complicated enough and involves ten equations, five of which are generalized equations for the flat SU(N) case. Our analytical calculations finally yield the whole set of these equations as follows

$$
\begin{aligned}
(4\pi)^2\frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= \left(\xi_1 - \tfrac{1}{6}\right)\Big[(N^2+1)f_1 + \frac{2}{N}(N^2-4)f_2 \\
&\quad - 2Nf_3 - 6Ng^2\Big] + \Delta\beta_{\xi_1} \\
(4\pi)^2\frac{\mathrm{d}\xi_2}{\mathrm{d}t} &= \left(\xi_2 - \tfrac{1}{6}\right)\Big[(N^2-1)f_3 + 2(N+1)f_5 \\
&\quad - \frac{3}{N}(N^2-1)g^2\Big] + \Delta\beta_{\xi_2}
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2\frac{\mathrm{d}f_1}{\mathrm{d}t} &= (N^2+7)f_1^2 + 24g^4 - 12Nf_1g^2 + 2Nf_3^2 \\
&\quad + \frac{4}{N}(N^2-4)f_2\left(f_1 + \frac{2}{N}f_2\right) + \Delta\beta_{f_1}^{(1)}(\xi_1) + \Delta\beta_{f_1}^{(2)}(\xi_1) \\
(4\pi)^2\frac{\mathrm{d}f_2}{\mathrm{d}t} &= \frac{4}{N}(N^2-15)f_2^2 - 12Nf_2g^2 + 3Ng^4 \\
&\quad + 12f_1f_2 + f_4^2 + \Delta\beta_{f_2}^{(2)}(\xi_1)
\end{aligned}
$$

$$\begin{aligned}
(4\pi)^2\frac{\mathrm{d}f_3}{\mathrm{d}t} &= 4f_3^2 - \frac{3}{N}(N^2-1)f_3g^2 + 6g^4 + (N^2+1)f_1f_3 \\
&\quad + \frac{2}{N}(N^2-4)f_2f_3 + 2(N+1)f_3f_5 + \frac{2}{N^2}(N^2-4)f_4^2 \\
&\quad + \lambda^2\xi_1\xi_2\Big(5 + \frac{1}{3\omega^2} - \frac{3}{2\omega^2}\xi_1 - \frac{3}{2\omega^2}\xi_2 + \frac{9}{\omega^2}\xi_1\xi_2\Big) \\
&\quad + \Delta\beta_{f_3}^{(3)}(\xi_1,\xi_2) \\
(4\pi)^2\frac{\mathrm{d}f_4}{\mathrm{d}t} &= \frac{N^2-12}{N}f_4^2 - \frac{3(3N^2-1)}{N}g^2f_4 + 3Ng^4 + 2f_1f_4 \\
&\quad + \frac{2(N^2-8)}{N}f_2f_4 + 2f_4f_5 + 8f_3f_4 + \Delta\beta_{f_4}^{(3)}(\xi_1,\xi_2) \\
\Delta\beta_f^{(3)}(\xi_1,\xi_2) &= \lambda f\Big[-5 - \frac{1}{2\omega} + \frac{3(\xi_1+\xi_2)}{\omega} \\
&\quad - \Big(\tfrac{3}{2} + \frac{9}{4\omega}\Big)\Big(\xi_1^2+\xi_2^2\Big) - 12\frac{\xi_1\xi_2}{\omega}\Big] \\
(4\pi)^2\frac{\mathrm{d}f_5}{\mathrm{d}t} &= 2(N+4)f_5^2 - \left[\frac{6}{N}(N^2-1)g^2\right]f_5 \\
&\quad + \frac{3(N-1)(N^2+2N-2)}{2N^2}g^4 + (N^2-1)f_3^2 \\
&\quad + \frac{(N^2-4)(N-1)}{2N^2}f_4^2 + \Delta\beta_{f_5}^{(1)}(\xi_2) + \Delta\beta_{f_5}^{(2)}(\xi_2)
\end{aligned} \tag{9.26}$$

Numerical analysis shows that in this case AF takes place for $N \geqslant 8$. Account of quantum gravity does not change this threshold value of N, but for $N \geqslant 13$ it relaxes the restriction put on the spinor composition of the theory (see table 9.2). For this version of the SU(N) model the tendency proposed by us before concerning a realistic GUT with R^2-gravity is also correct. The minimal value of n_2 (which still provides the AF) again decreases by one for $N \geqslant 13$, $n_1 = 1,2,3,4$ when R^2-gravity is embedded. This positive effect is more pronounced in comparison with the conformal theory and takes place also for $N = 15, n_1 = 0,1$. In the table we underline the values of n_2, which provide new versions of asymptotically free grand unified models.

(*c*) *The model with the gauge group SU(N), which contains one real scalar multiplet belonging to the adjoint representation.* The scalar field action for this model is described by expression (9.16) where one should put $\varphi^i = \varphi_i^+ = 0$ and, at the same time $f_3 = f_4 = f_5 = \xi_2 = 0$. We obtain the renormalization group equations here by eliminating all the terms containing f_3, f_4, f_5, ξ_2 from equations (9.26). For the model studied the parameters a^2 and b^2 have the

Table 9.2 Permissible values of n_2 for SU(N) model containing two scalar multiplets.

		$N \backslash n_1$	0	1	2	3	4	5
Without taking account of gravitation		13	129–136	103–110	77–84	51–58	25–32	0–6
		14	138–146	110–118	82–90	54–62	26–34	0–6
		15	147–157	117–127	87–97	57–67	27–37	0–7
With taking account of gravitation	Conformal theory	13	$\underline{128}$–136	$\underline{102}$–110	$\underline{76}$–84	$\underline{50}$–58	$\underline{24}$–32	0–6
		14	$\underline{137}$–146	$\underline{109}$–118	$\underline{81}$–90	$\underline{53}$–62	$\underline{25}$–34	0–6
		15	147–157	117–127	$\underline{86}$–97	$\underline{56}$–67	$\underline{26}$–37	0–7
	General R^2-theory	13	$\underline{128}$–136	$\underline{102}$–110	$\underline{76}$–84	$\underline{50}$–58	$\underline{24}$–32	0–6
		14	$\underline{137}$–146	$\underline{109}$–118	$\underline{81}$–90	$\underline{53}$–62	$\underline{25}$–34	0–6
		15	$\underline{146}$–157	$\underline{116}$–127	$\underline{86}$–97	$\underline{56}$–67	$\underline{26}$–37	0–7

Table 9.3 Permissible values of n_2 for SU(N) model containing one real scalar multiplet belonging to the adjoint representation.

	$N \backslash n_1$	0	1	2	3	4	5
Without taking	8	80–84	64–68	48–52	32–36	16–20	0–4
account of	9	89–94	71–76	53–58	35–40	17–22	0–4
gravitation	10	97–105	77–85	57–65	37–45	17–25	0–5
With taking	8	$\underline{79}$–84	$\underline{63}$–68	$\underline{47}$–52	$\underline{31}$–36	$\underline{15}$–20	0–4
account of	9	$\underline{87}$–94	$\underline{69}$–76	$\underline{51}$–58	$\underline{33}$–40	$\underline{15}$–22	0–4
gravitation	10	$\underline{96}$–105	$\underline{76}$–85	$\underline{56}$–65	$\underline{36}$–45	$\underline{16}$–25	0–5

following form

$$\begin{aligned} a^2 &= \frac{785 + 13N^2}{60} + \frac{1}{10}[n_1(N^2-1) + Nn_2] \\ b^2 &= 7N - \left(\tfrac{4}{3}\right)\left(Nn_1 + \frac{n_2}{2}\right). \end{aligned} \tag{9.27}$$

Here AF takes place for $N \geqslant 6$. Account of gravitation does not change the threshold value of $N = 6$, but for $N \geqslant 8$ it relaxes (see table 9.3) the restriction imposed on the spinor composition of the theory. Again one can see that the minimal number of the spinor multiplets decreases by one for $N = 8$ or 10 and by two for $N = 9$.

(*d*) *The model with the gauge group SU(N), which contains one complex scalar multiplet belonging to the fundamental representation.* We obtain the scalar field action from (9.16) by putting $\phi^a = 0$. The renormalization group equations for this model are obtained by eliminating all the terms containing f_1, f_2, f_3, f_4 and ξ_1 from equations (9.2) with the parameters a^2 and b^2 as follows

$$\begin{aligned} a^2 &= \frac{786 + 12N^2 + 2N}{60} + \frac{1}{10}\Big[n_1(N^2-1) + Nn_2\Big] \\ b^2 &= \tfrac{1}{3}(22N - 1) - \tfrac{4}{3}\Big(n_1 N + \tfrac{1}{2}n_2\Big). \end{aligned} \tag{9.28}$$

AF takes place for $N \geqslant 3$. Account of gravitation does not change the threshold value of N, as well, but for $N \geqslant 4$ it allays the restriction imposed on the spinor composition of the theory (see table 9.4).

9.5 Asymptotic behaviour of matter effective coupling constants in the conformal theory

To make the scenario discussed here more complete, we will now briefly present the results for GUT models with conformal gravity

Table 9.4 Permissible values of n_2 for SU(N) model containing one complex scalar multiplet belonging to the fundamental representation.

	$N\backslash n_1$	0	1	2	3	4	5
Without taking account of gravitation	4	40–43	32–35	24–27	16–19	8–11	0–3
	5	48–54	38–44	28–34	18–24	8–14	0–4
	6	55–65	43–53	31–41	19–29	7–17	0–5
With taking account of gravitation	4	<u>39</u>–43	<u>30</u>–35	<u>22</u>–27	<u>14</u>–19	<u>6</u>–11	0–3
	5	<u>45</u>–54	<u>35</u>–44	<u>25</u>–34	<u>15</u>–24	<u>5</u>–14	0–4
	6	<u>52</u>–65	<u>40</u>–53	<u>28</u>–41	<u>16</u>–29	<u>3</u>–17	0–5

since this is, in principle, a good way of trying to formulate a realistic theory. As was mentioned in section 9.2, we pass to the conformal theory by putting $\xi = 1/6, \omega = 0$ in the general expression (9.1). That is why there are no effective constants $\xi(t)$ and $\omega(t)$ in the conformal theory. Equation (9.10) for the effective charge $g^2(t)$ does not contain, as in the general R^2-gravity, quantum gravity contributions. The equation for the scalar constants has, as before, the general structure of (9.20), the expressions for β_f remain, naturally, unchanged. The only structures subject to change are $\Delta\beta_f^{(1)}$ and $\Delta\beta_f^{(2)}$, which in the conformal theory have the form

$$\Delta\beta_f^{(1)} = -\tfrac{41}{8}\lambda f \qquad \Delta\beta_f^{(2)} = \tfrac{5}{12}\lambda^2. \tag{9.29}$$

Calculation of the counterterms containing spinor fields is considered in [117] for the case of conformal theory (see, also, section 9.6). Now let us present the equation for the Yukawa effective couplings

$$(4\pi)^2\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \beta_h + \Delta\beta_h. \tag{9.30}$$

Here the renormalization group function β_h corresponds to flat space–time, while the additional term $\Delta\beta_h$ takes account of the quantum gravity effects. The explicit expression for $\Delta\beta_h$ is [117, 123]

$$\Delta\beta_h = -\tfrac{61}{32}h\lambda. \tag{9.31}$$

Now we consider a theory with only a single Yukawa constant to start with. We can obtain the following RGE by using the general expression for β_h

$$(4\pi)\frac{\mathrm{d}h(t)}{\mathrm{d}t} = Ah^3 - Bhg^2 - Ch\lambda \tag{9.32}$$

where A, B and C are constants, $C = 61/32$. As was mentioned in section 9.4, we can put $\lambda(t) = kg^2(t)$ with $k = b^2/\alpha^2$ at large t. Here b^2 is determined by relation (9.10), and the expression for α^2 has, in contrast to (9.8), the form

$$\alpha^2 = \tfrac{199}{15} + \frac{1}{5N} N_1 + \tfrac{1}{10} N_{1/2} + \tfrac{1}{60} N_0. \tag{9.33}$$

We now rewrite equation (9.32) in terms of a new effective constant $\bar{h}(t)$ and a new variable τ; $h = \bar{h}g$, $\mathrm{d}\tau = (4\pi)^{-2} g^2(t)\mathrm{d}t$

$$\frac{\mathrm{d}\bar{h}}{d\tau} = \bar{h}[A\bar{h}^2 - (B - \tfrac{1}{2}b^2 + Ck)]. \tag{9.34}$$

Equation (9.34) differs from the corresponding equation in flat space–time by the substitution $B - \frac{1}{2}b^2 \to B - \frac{1}{2}b^2 + Ck$. Standard arguments show that equation (9.34) only has a stable fixed point $\bar{h} = 0$ under the condition $B - \frac{1}{2}b^2 + Ck > 0$. Since $Ck > 0$, then account of the gravitational interaction favours AF. Indeed, if AF in the Yukawa constant takes place when the gravitational interaction is not taken into account (i.e. $C = 0$), and $B - \frac{1}{2}b^2 > 0$, then account of the gravitational interaction makes $\bar{h}(t)$ vanish even more rapidly. Note, finally, that one can solve equation (9.32) in quadratures and all the speculations are confirmed by this.

Now let us consider the case of several Yukawa couplings. Suppose the Yukawa interaction has the structure investigated in [81]. Then, after the gravitational interaction is taken into account, the renormalization group equation takes the form

$$(4\pi)^2 \frac{\mathrm{d}h_{\alpha\beta}}{\mathrm{d}t} = A(hhh)_{\alpha\beta} + \mathrm{Tr}(hh)h_{\alpha\beta} - Bg^2 h_{\alpha\beta} - C\lambda h_{\alpha\beta}. \tag{9.35}$$

To study the behaviour of $h_{\alpha\beta}(t)$ at large t we make, as before, the replacement $\lambda(t) = kg^2(t)$. Equation (9.35) reduces then, up to the substitution $B \to B + Ck$, to the equation investigated in [81]. Therefore, when $B^2 - \frac{1}{2}b^2 + Ck > 0$, all the Yukawa effective couplings approach zero, and they do it more rapidly than $g(t)$ does. If $C > 0$, then account of the gravitational interaction makes the Yukawa constants vanish even more rapidly.

In the $\mathrm{O}(N)$ and $\mathrm{SU}(N)$ gauge models the Yukawa constants are asymptotically free without taking account of the gravitational interaction, and they approach zero at a faster rate than $g(t)$. If the Yukawa interaction term has the structure $h_{\alpha\beta}\bar{\psi}_{(\alpha)}\varphi\psi_{(\beta)}$, then one can show that $C = 61/32$. The coefficient C does not depend on the group dimension N. We can use here the same reasons as those which make the gravity contributions to $\Delta\beta_f^{(1)}$ and $\Delta\beta_f^{(2)}$ independent of N. The latter takes place both in conformal and in general R^2-gravity. The asymptotic behaviour of scalar coupling constants is determined by equation (9.20), with $\Delta\beta_f$ taken from (9.29) in the conformal case, and from (9.21) in the non-conformal one. Since the Yukawa coupling constants approach zero at a faster rate than $g(t)$, they may be omitted when studying the asymptotic behaviour of the renormalization group equation for the scalar constants. We can carry out this study just as in section 9.4, and its results are given in the tables of section 9.4. Not only are the equations arising in the conformal theory considerably simpler than those in the general R^2-gravity but, in a number cases (when a single scalar multiplet is present in the theory), they admit analytic investigation. In all the cases the results of such investigation coincide with the results of numerical calculation carried out by using a computer.

9.6 Asymptotic behaviour of Yukawa effective couplings

In preceding sections we have studied the asymptotic behaviour of the effective coupling constants for the theory (9.1) on the supposition that the Yukawa effective couplings vanish rapidly as the renormalization group parameter t increases, so that they are of no importance when studying the asymptotic behaviour of other couplings. Now let us verify this supposition. When quantum gravity is switched off, the equation for $h(t)$ has the form (9.32) with $C = 0$. The coefficients A and B are positive, so that $h(t)$ is driven to zero at a faster rate than $g(t)$ as $t \to \infty$. In the conformal gravity we have $C > 0$. For this reason $h(t)$ in the conformal theory vanishes even more rapidly.

Now let us turn to calculating one-loop counterterms for the general (non-conformal) R^2-gravity. We have seen that in the scalar sector these counterterms differ qualitatively from the corresponding counterterms in the conformal theory. The analysis of equation (9.15) shows that the nature of the dependence of the counterterms on the parameters λ, g, f, h is the same both for the conformal and for the general R^2-gravity. At the same time, an additional dependence on the parameters ω, ξ arises in the non-conformal theory, this dependence cannot take place in the Weyl gravity. Of considerable importance here is the fact that we can already carry out the general analysis of the dependence of the scalar sector counterterms on the coupling constants with the aid of formula (9.13), i.e., before explicit calculation of the counterterms is done.

An analogous investigation of the dependence of the spinor sector counterterms on the theory coupling constants gives the following equation for the Yukawa effective couplings

$$(4\pi)^2 \frac{\mathrm{d}h}{\mathrm{d}t} = Ah^3 - Bhg^2 - C(\xi,\omega)\lambda h \tag{9.36}$$

where the function $C(\xi,\omega)$ is determined by the structure of the counterterms. In the conformal theory this function reduces, as follows from the results of section 9.5 (see, also [117]), to the constant $C = 61/32$. The value C in the conformal theory does not depend on the choice of the gauge group and the field theory model. The expressions for the counterterms in conformal quantum gravity is determined, just as in the general (non-conformal) theory, with the aid of formula (9.13). It is important that the structure of the dependence of the matrix blocks of $(\hat{\mathcal{H}})$ on the constants λ, g, f, h is the same both for the conformal and for the non-conformal theory. It follows from this fact that the nature of the dependence of the counterterms on these parameters in the non-conformal theory must be just the same as in the Weyl gravity. Thus, we can assert that the function $C(\xi,\omega)$ does not depend on the parameters λ, g, f, h. For the same reason $C(\xi,\omega)$ does not depend on the choice of gauge group, so it has (in this sense) a universal nature. An analogous situation also takes place in the case of conformal quantum gravity.

The objective of this section is to derive the explicit expression for $C(\xi,\omega)$. If the function $C(\xi,\omega)$ proves to be positive at the asymptotic values of parameters ξ and ω (which, as numerical calculations show, for the above-mentioned GUT models are: $\xi \approx 0.16$, $\omega \approx -0.01$), then the coupling $h(t)$ will be driven to zero at a faster rate than $g(t)$ as $t \to \infty$.

Now we turn to the calculation of the counterterms. Since $C(\xi,\omega)$ does not depend, as already discussed above, on the choice of the gauge group, we can choose any given group, e.g. the SU(2) group, to carry out the calculation. Thus, let us consider the theory with action (9.1) and with the gauge group SU(2). Action (9.3) in this case has the form

$$S_F = \mathrm{i} \int \mathrm{d}^4x \sqrt{-g}\bar{\psi}_a \Big(\gamma^\mu D_\mu^{ab} + h\varepsilon^{abc}\varphi^c\Big)\psi_b. \tag{9.37}$$

We divide the fields $g_{\mu\nu}$, A_μ^a, φ^a, ψ_a, $\bar{\psi}_b$ into background and quantum ones, $\bar{\psi}_a$, ψ_b being added to the background fields and $\bar{\eta}_a$, η_b to the quantum ones

$$\bar{\psi}_a \to \bar{\psi}_a + \bar{\eta}_a \qquad \psi_b \to \psi_b + \eta_b. \tag{9.38}$$

Let us make the change of variables $\eta_b = -\frac{i}{2}\gamma^\alpha \partial_\alpha \chi_b$. Since the counterterms in the spinor sector in which we are interested do not contain curvature, we can regard the background metric as being flat: $g_{\mu\nu} = h_{\mu\nu}$. In addition, we can regard the background gauge field as being equal to zero. Further, since $C(\xi, \omega)$ does not depend, as was mentioned above, on the constants f and g, one can put them equal to zero when calculating the counterterms. The quadratic part of action (9.37) will have the structure (9.5) and (9.6), and one should also calculate one-loop counterterms with the aid of formulae (9.13) and (9.15).

Since action (9.37) contains the vierbein e^μ_a (in the combination $\gamma^\mu = e^\mu_a \gamma^a$) the calculation of the quadratic part of the action with respect to the quantum fields $h_{\mu\nu}$ and η contains the non-trivial moment. To distinquish the bilinear part one must use the following expansion for the vierbein

$$
\begin{aligned}
e^\mu_a &\to \bar{e}^\mu_a = e^\mu_a - \tfrac{1}{2} h^{\mu\alpha} e_{\alpha a} + \tfrac{3}{8} h^{\mu\alpha} h^\beta_\alpha e_{\beta a} + \dots \\
e^a_\mu &\to \bar{e}^a_\mu = e^a_\mu + \tfrac{1}{2} h^\alpha_\mu e^a_\alpha - \tfrac{1}{8} h^\alpha_\mu h^\beta_\alpha e^a_\beta + \dots
\end{aligned}
\tag{9.39}
$$

Now we present the expressions for the blocks of the matrix $(\hat{\mathcal{H}})$:

$$
\begin{aligned}
\hat{U} &= \hat{U}((9.7)) \\
&+ 2\lambda \begin{pmatrix}
\begin{matrix} -\frac{1}{2}\delta^{\rho\sigma,\mu\nu} i\bar{\psi}\gamma^\lambda \partial_\lambda \psi \\ +\frac{3}{4} g^{\mu\rho} i\bar{\psi}\gamma^\nu \partial^\sigma \psi \\ -\frac{1}{2} h \delta^{\mu\nu,\rho\sigma} i\bar{\psi}^a \varepsilon^{abc} \varphi^c \psi^b \end{matrix} & \Delta U_{\bar{h}h} \\
\Delta U_{\bar{h}h} & \begin{matrix} -\frac{3}{64\beta} i\bar{\psi}\gamma^\lambda \partial_\lambda \psi \\ -\frac{ih}{8\beta} \bar{\psi}^a \varepsilon^{abc} \varphi^c \psi^b \end{matrix}
\end{pmatrix}
\end{aligned}
$$

$$
\hat{V} = \hat{V}((9.7))
$$

$$
\hat{P}_1^{\chi\omega} = \begin{pmatrix} 0 & 0 \\ -i\xi\sqrt{2\lambda}\varphi^a \delta^{\mu\nu,\chi\omega} & \frac{3i}{4}\sqrt{\frac{2\lambda}{-\beta}}\xi\varphi^a g^{\chi\omega} \\ 0 & 0 \end{pmatrix}
$$

$$
\hat{Q}_1^{\chi\omega} = \begin{pmatrix} 0 & -i\xi\sqrt{2\lambda}\varphi^b \delta^{\rho\sigma,\xi\omega} & -\frac{1}{4}\bar{\psi}^b \gamma^\rho \gamma^{(\chi} g^{\omega)\sigma}\sqrt{2\lambda} \\ 0 & \frac{3\lambda i}{4}\sqrt{\frac{2\lambda}{-\beta}}\xi\varphi^b g^{\xi\omega} & \frac{3}{16}\sqrt{\frac{2\lambda}{-\beta}}\bar{\psi}^b g^{\xi\omega} \end{pmatrix}
$$

$$
\begin{aligned}
\hat{P}_2^{\chi} &= \sqrt{2\lambda}\begin{pmatrix} 0 & 0 \\ \mathrm{i}(\xi-1)g^{\chi\nu}\partial^{\mu}\varphi^{a} & \frac{\mathrm{i}}{4\sqrt{-\beta}}(1-3\xi)\partial^{\chi}\varphi^{a} \\ \frac{-\mathrm{i}}{2}\gamma^{\mu}\psi^{a}g^{\chi\nu} & \frac{3\mathrm{i}}{8\sqrt{-\beta}}\gamma^{\chi}\psi^{a} \end{pmatrix} \\
\hat{Q}_2^{\chi} &= \sqrt{2\lambda}\begin{pmatrix} 0 & \mathrm{i}(1-\xi)g^{\sigma\chi}\partial^{\rho}\varphi^{b} & \frac{1}{4}\partial^{\sigma}\bar{\psi}^{b}\gamma^{\rho}\gamma^{\chi} \\ 0 & \frac{\mathrm{i}}{4\sqrt{-\beta}}(3\xi-1)\partial^{\chi}\varphi^{b} & -\frac{3}{16\sqrt{-\beta}}\partial_{\lambda}\bar{\psi}^{b}\gamma^{\lambda}\gamma^{\chi} \\ & & +\frac{1}{2\sqrt{-\beta}}h\bar{\psi}^{a}\varepsilon^{abc}\varphi^{c}\gamma^{\chi} \end{pmatrix} \\
\hat{P}_3 &= \sqrt{2\lambda}\begin{pmatrix} 0 & 0 \\ -\mathrm{i}\partial^{\mu}\partial^{\nu}\varphi^{a} & \frac{\mathrm{i}}{4\sqrt{-\beta}}\Box\varphi^{a}+\frac{1}{2\sqrt{-\beta}}h\varepsilon^{abc}\bar{\psi}^{b}\psi^{c} \\ -\mathrm{i}\gamma^{\mu}\partial^{\nu}\psi^{a} & \frac{3\mathrm{i}}{4\sqrt{-\beta}}\gamma^{\lambda}\partial_{\lambda}\psi^{a}+\frac{\mathrm{i}}{\sqrt{-\beta}}h\varepsilon^{abc}\varphi^{c}\psi^{b} \end{pmatrix} \\
\hat{Q}_3 &= \sqrt{2\lambda}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{-\beta}}h\varepsilon^{abc}\bar{\psi}^{a}\psi^{c} & 0 \end{pmatrix} \\
\hat{E}^{\chi} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}h\varepsilon^{abc}\bar{\psi}^{c}\gamma^{\chi} \\ 0 & 0 & h\varepsilon^{abc}\varphi^{c}\gamma^{\chi} \end{pmatrix} \\
\hat{D} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2h\varepsilon^{abc}\psi^{c} & 0 \end{pmatrix}
\end{aligned}
\tag{9.40}
$$

Omitting bulky expressions for the counterterms (which are computed like (9.15)), we present only the final result

$$C(\xi,\omega) = \tfrac{15}{8} + \tfrac{9}{8}\xi^2 - \frac{9}{32\beta}\xi^2. \tag{9.41}$$

The two first terms make it possible to proceed to the conformal limit. At $1/\beta = 0, \xi = 1/6$ we obtain $C = 61/32$.

Since $\beta < 0$ for the asymptotic values of ξ and ω then the 'new' term $-9\xi^2/32\beta$ is positive. Thus, in R^2-gravity we have $C(\xi,\omega) > 0$, hence $h(t)$ vanishes more rapidly than $g(t)$.

To conclude, we can say that for the models studied in section 9.4 we have found stable fixed points, these points being zero for the couplings $g(t)$, $f(t)$, $\bar{h}(t) = h(t)/g(t)$, $\lambda(t)$ and non-zero for $\omega(t)$ and $\xi(t)$.

We may conclude therefore that AF of the GUT models does not suffer from their unification with quantum gravity. On the contrary, new restrictions (which arise when one solves the RGEs and takes gravitation into account) are less rigid and admit some decrease (although it is small) of the spinor field multiplet composition of the theory. Thus, there arises in GUT models with gravity a new promising opportunity for constructing asymptotically free theories of grand unification with a more 'pure' set of spinor multiplets, which is highly desirable for asymptotically free GUTs always possessing 'a surplus' of spinor fields. It is important, that including quantum gravity in GUT models only improves their physical content and does not conflict with their AF.

10 Effective Action in Multidimensional Quantum Gravity

10.1 Introduction

The Kaluza–Klein approach has been considered as one of the ways of constructing a unified theory of fundamental interactions. One of the reason for the interest in investigation of the field theory in multidimensional spaces was due to obtaining $d = 4$ supergravity (SG) as a consequence of a reduction from $N = 1, d = 11$ SG. Also, popular string models are formulated in $d > 4$ ($d = 10$, $d = 26$).

The fact that the additional dimensions are not observable means that there must be a mechanism leading to small characteristic sizes for these dimensions. This mechanism is spontaneous compactification due to which the additional dimensions correspond to a compact manifold with a characteristic size of the order of the Planck length $L_{\rm P}$. Numerous investigations of spontaneous compactification resulting from the solutions of the classical field equations have been carried out (we can find the corresponding references in the reviews [301–305]). However, as the compactification scale can be compared with the Planck length, classical considerations must be complemented by an analysis of quantum gravity effects. This makes the investigation of the EA in Kaluza–Klein theories necessary.

The contribution of matter fields to the quantum EA on the multidimensional compactified background has been calculated by many authors (see, for example, [306–333] and references therein). The influence of quantum effects on spontaneous compactification has also been considered. However, the investigation of the matter fields in spaces with $d > 4$, does in fact contradict the Kaluza–Klein approach according to which $d > 4$ contains only those objects which characterize geometry (metric and, possibly, torsion) and matter fields appear only after the dimensional reduction from $d > 4$ to $d = 4$. (In our opinion the consideration of the matter fields in $d > 4$ is consistent with the Kaluza–Klein approach only in the case where these fields are included in the multiplet of some supergravity model.) Hence, quantum effects in the Kaluza–Klein approach must be considered in a theory where the metric (and, possibly, torsion) is the quantum field, i.e. in (super)gravity theory.

Thus, in this chapter we will discuss the effective action in multidimensional gravity and supergravity on a compactified background.

10.2 Classical five-dimensional gravity

Let us give some basic remarks concerning five-dimensional Kaluza–Klein gravity which unifies electromagnetism with gravitation. The theory has five-dimensional general coordinate invariance. However, we assume (as in [301, 303, 334]) that one of the spatial dimensions compactifies so as to have the geometry of a one-dimensional sphere S_1 with very small radius. This means that there is a residual four-dimensional general coordinate invariance and Abelian gauge invariance associated with the transformations of S_1. Thus, the resulting theory is ordinary gravity in four dimensions combined with Abelian gauge theory.

The five-dimensional metric has the form

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -g_{55} \end{pmatrix} \tag{10.1}$$

where five-dimensional coordinates $x^A = \{x^\mu, x^5\}$, $\mu = 0,1,2,3$ and $g_{\mu\nu} = \mathrm{sgn}(+,-,-,-)$ is the metric of four-dimensional space R_4, coordinate x^5 usually being an angle to parametrize the one-dimensional sphere geometry.

We may parametrize the metric in the form

$$g_{AB}(x) = g_{55}\begin{pmatrix} g_{\mu\nu}(x) - \xi^2 A_\mu(x) A_\nu(x) & \xi A_\mu(x) \\ \xi A_\nu(x) & -1 \end{pmatrix} \tag{10.2}$$

where $x \equiv \{x^\mu\}$, ξ is a scale parameter and $A_\mu(x)$ is a conventionally normalized vector field.

It is not difficult to show that general coordinate transformation induces an Abelian gauge transformation on A_μ (see, for example, [301]).

Now we will consider the action of five-dimensional gravity

$$S = -\frac{1}{\kappa^2}\int \mathrm{d}^5x\sqrt{|g|}R \tag{10.3}$$

where R is five-dimensional curvature and κ^2 is the five-dimensional gravitational constant. Substituting the ansatz (10.2) in (10.3) and integrating over the extra coordinate x^5 we obtain the reduced action

$$S = -\frac{2\pi\rho}{\kappa^2}\int \mathrm{d}^4x\sqrt{|g|}R - \frac{\xi^2 g_{55}}{4}\frac{2\pi\rho}{\kappa^2}\int \mathrm{d}^4x\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} \tag{10.4}$$

where now g is $\det g_{\mu\nu}$ and R is four-dimensional curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ρ is the radius of S_1, $2\pi\rho$ is the 'volume' of S_1.

To obtain the standard normalization for the gauge field we must choose

$$\xi^2 = \kappa^2/(2\pi\rho g_{55}). \tag{10.5}$$

Then the effective reduced four-dimensional action is

$$S = -\frac{1}{\kappa^2_{\text{ind}}}\int \mathrm{d}^4x\sqrt{|g|}R - \tfrac{1}{4}\int \mathrm{d}^4x\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} \tag{10.6}$$

where the four-dimensional gravitational constant is

$$1/\kappa^2_{\text{ind}} = 2\pi\rho/\kappa^2. \tag{10.7}$$

In this chapter we will consider the calculation of the EA on a compactified background where non-diagonal elements are equal to zero ($A_\mu = 0$) in the parametrization (10.2). However, it is possible to include $A_\mu \neq 0$ in g_{AB} and calculate the EA in this case.

In conclusion we also note that if $d > 5$ then the reduced action contains non-Abelian gauge fields (see [301, 303, 334] for more details).

10.3 One-loop effective action in multidimensional quantum gravity and spontaneous compactification

Let us discuss the calculation of the EA in multidimensional gravity on a torus as a compactified background. This calculation gives the possibility of checking the quantum stability of the considered background with the help of spontaneous compactification conditions. Of course, it would be more interesting to find the EA on an arbitrary background and then find the vacuum state as the stable solution of effective equations. Unfortunately, it is impossible to calculate EA on an arbitrary background in modern quantum field theory. That is why we consider the space $R_4^0 \times T_{d-4}$, where R_4^0 is flat four-dimensional space and T_d is d-dimensional torus as the proposed ground state and calculate the EA on this background. Note that $R_4^0 \times T_{d-4}$ is not the solution of the classical equations in Einstein gravity with the cosmological constant (classical instability). However, one can expect that the situation can be changed by taking account of quantum corrections. Anyway, the results of such considerations are useful because we can extract some qualitative consequences which probably take place on a more realistic background. It should be also noted that the effective action in multidimensional quantum gravity has been considered in [335–342, 352, 355, 343–351, 353, 354] for Einstein gravity without the cosmological term, Einstein gravity with the cosmological term and higher-derivative gravity, respectively.

The action of d-dimensional Einstein gravity under consideration is

$$S = -\frac{1}{\kappa^2} \int \mathrm{d}^d x \sqrt{g} (R - 2\Lambda) . \tag{10.8}$$

where κ^2 and Λ are d-dimensional gravitational and cosmological constants and we use here (and in the next two sections) Euclidean notations for one-loop calculations.

As the background we choose the flat space $R_4^0 \times T_{d-4}$ and the gauge fixing action is chosen in the form

$$S_{\mathrm{GF}} = \frac{1}{2\alpha\kappa^2} \int \mathrm{d}^d x \sqrt{g} \Big(\nabla_B h_A^B + \gamma \nabla_A h \Big) \Big(\nabla_C h^{AC} + \gamma \nabla^A h \Big) \tag{10.9}$$

where h_{AB} is the quantum field, α and γ are gauge parameters (the choice $\alpha = 1$, $\gamma = -1/2$ corresponds to Fock–De Donder gauge).

The general expression for the effective action is

$$\begin{aligned} \Gamma = S[g_{AB}] - \ln \int \mathrm{D} h_{AB} \exp \Big[- \tfrac{1}{2} h_{AB} \Big(\frac{\delta^2}{\delta h_{AB} \delta h_{CD}} \Big(S[g+h] \\ + S_{\mathrm{GF}}[g,h] \Big) \Big|_{h_{AB}=0} h_{CD} \Big] \det \Delta^{AB} . \end{aligned} \tag{10.10}$$

Here g_{AB} is the background metric corresponding to space $R_4^0 \times T_{d-4}$,

$$\Delta^{AB} = \frac{\delta \chi^A}{\delta h^{CD}} \frac{\delta h^{CD}}{\delta \xi_B} \Big|_{h_{AB}=0}$$

χ^A is the gauge condition and ξ^A is the parameter of general coordinate transformations. The second variation of action (10.8) has the form

$$\delta^2 S = -\frac{1}{2\kappa^2}\int d^d x\sqrt{g}\Big(\tfrac{1}{2}h_{AB}(\Box+\Lambda)h^{AB} - \tfrac{1}{2}h(\Box+2\Lambda)h + (\nabla^A h_{AB} - \tfrac{1}{2}\nabla_B h)^2\Big). \qquad (10.11)$$

Making use of (10.11) and (10.9) in (10.10) we easily obtain

$$\Gamma = S + \tfrac{1}{2}\Big(\operatorname{Sp}\ln H^{AB,CD} - 2\operatorname{Sp}\ln\Delta^{AB}\Big) \qquad (10.12)$$

where the ghost operator is

$$\begin{aligned}
\Delta^{AB} &= g^{AB}\Box + (1+2\gamma)\nabla^A\nabla^B\\
H^{AB,CD} &\equiv \tfrac{1}{4}(g^{AC}g^{BD} + g^{AD}g^{BC})(\Box+2\Lambda)\\
&+ \Big[\Big(\frac{\gamma^2}{\alpha} - \tfrac{1}{2}\Big)\Box - \frac{\Lambda}{2}\Big]g^{AB}g^{CD}\\
&+ \tfrac{1}{2}\Big(1+\frac{2\gamma}{\alpha}\Big)\Big(g^{AB}\nabla^C\nabla^D + g^{CD}\nabla^A\nabla^B\Big)\\
&+ \tfrac{1}{4}\Big(\frac{1}{\alpha} - 1\Big)\Big(g^{AC}\nabla^B\nabla^D + g^{AD}\nabla^B\nabla^C\\
&+ g^{BC}\nabla^A\nabla^D + g^{BD}\nabla^A\nabla^C\Big).
\end{aligned}$$

Here we have taken into account that $\nabla_A\nabla_B = \nabla_B\nabla_A$ on the background $R_4^0 \times T_{d-4}$.

As we can see the operator $\Delta \equiv \operatorname{Sp}\ln H^{AB,CD}$ is of non-minimal structure. It means that this operator contains not only terms involving $\Box$ but also others like $\nabla\nabla$. To present the operator Δ as a minimal operator the method developed in [207] is very useful. This method is based on the solution of the equation

$$\frac{\partial\Delta}{\partial\xi} = \operatorname{Sp} H^{-1}\frac{\partial H}{\partial\xi} \qquad (10.13)$$

with the corresponding boundary condition $\Delta_0 = \Delta|_{\xi=\xi_0}$, where ξ is a parameter. The main problem in (10.13) is to find H^{-1}. For this purpose, we present H as

$$\begin{aligned}
H^{AB,CD} &= A_0 L^{AB,CD} + A_1 P^{AB,CD} + A_2 M^{AB,CD}\\
&+ A_3(L^{AB}P^{CD} + P^{AB}L^{CD}) + A_4 L^{AB}L^{CD} \qquad (10.14)
\end{aligned}$$

where $L^{AB} = g^{AB} - \nabla^A\nabla^B/\Box$, $L^{AB,CD} = \frac{1}{2}(L^{AC}L^{BD} + L^{AD}L^{BC})$, $P^{AB} = \nabla^A\nabla^B/\Box$, $P^{AB,CD} = \frac{1}{2}(P^{AC}P^{BD} + P^{AD}P^{BC})$, $M^{AB,CD} = \frac{1}{2}(L^{AC}P^{BD} + P^{AC}L^{BD} + L^{AD}P^{BC} + P^{AD}L^{BC})$, all operators are given on $R_4^0 \times T_{d-4}$ background, and $A_0, ..., A_4$ are coefficients.

The algebra of operators $L^{AB}, \ldots, M^{AB,CD}$ is given by

$$\begin{aligned}
&L^{AB}L_B^C = L^{AC} \qquad P^{AB}P_B^C = P^{AC} \qquad L^{AB}P_{BC} = 0\\
&L^{AB,CD}L_{CD,}^{KL} = L^{AB,KL} \qquad P^{AB,CD}P_{CD,}^{KL} = P^{AB,KL}\\
&M^{AB,CD}M_{CD,}^{KL} = M^{AB,KL} \qquad L^{AB,CD}L_{CD} = L^{AB}\\
&P^{AB,CD}P_{CD} = P^{AB}\\
&L^{AB,CD}P_{CD,}^{KL} = L^{AB,CD}M_{CD,}^{KL} = P^{AB,CD}M_{CD,}^{KL} = 0\\
&M^{AB,CD}L_{CD} = M^{AB,CD}P_{CD} = L^{AB,CD}P_{CD} = P^{AB,CD}L_{CD} = 0.
\end{aligned} \tag{10.15}$$

Let us represent the operator H^{-1} as

$$\begin{aligned}
(H^{-1})_{AB,CD} &= \alpha_0 L_{AB,CD} + \alpha_1 P_{AB,CD} + \alpha_2 M_{AB,CD}\\
&\quad + \alpha_3(L_{AB}P_{CD} + P_{AB}L_{CD}) + \alpha_4 L_{AB}L_{CD}
\end{aligned} \tag{10.16}$$

We can show (using (10.15)) that

$$\begin{aligned}
&\alpha_0 = A_0^{-1} \qquad \alpha_1 = -\Big(A_0 + (d-1)A_4\Big)\delta^{-1}\\
&\alpha_2 = A_2^{-1} \qquad \alpha_3 = A_3\delta^{-1}\\
&\alpha_4 = -\frac{1}{(d-1)A_0} - \frac{A_1}{\delta} \qquad \delta = (d-1)A_3^2 - A_1\Big[A_0 + (d-1)A_4\Big].
\end{aligned} \tag{10.17}$$

Then

$$\begin{aligned}
\frac{\partial\Delta}{\partial\xi} = \mathrm{Sp}\Big\{&\Big[\alpha_0\frac{d(d-1)}{2} + \alpha_4(d-1)\Big]\frac{\partial A_0}{\partial\xi}\\
&+ \Big[\alpha_0(d-1) + \alpha_4(d-1)^2\Big]\frac{\partial A_4}{\partial\xi}\\
&+ \alpha_1\frac{\partial A_1}{\partial\xi} + \alpha_2\frac{\partial A_2}{\partial\xi} + 2\alpha_3(d-1)\frac{\partial A_3}{\partial\xi}\Big\}.
\end{aligned}$$

Hence finally we obtain

$$\frac{\partial\Delta}{\partial\xi} = \frac{\partial}{\partial\xi}\mathrm{Sp}\left\{\left[\frac{d(d-1)}{2} - 1\right]\ln A_0 + (d-1)\ln A_2 + \ln\delta\right\}. \tag{10.18}$$

This is the basic equation for the calculation of $\mathrm{Sp}\ln H$.

The reader can easily find Δ using (10.18). However, the final expression is quite complicated. That is why we give the answer when $\gamma = -1/2$ in expression (10.9)

$$\begin{aligned} \operatorname{Sp}\ln H^{AB,CD} &= \frac{d^2-d}{2}\operatorname{Sp}\ln(\Box + 2\Lambda) + d\operatorname{Sp}\ln(\Box + 2\alpha\Lambda) \\ \operatorname{Sp}\ln \Delta^{AB} &= d\operatorname{Sp}\ln\Box. \end{aligned} \tag{10.19}$$

Thus, in gauge (10.9) with $\gamma = -1/2$ we obtain the one-loop effective potential:

$$\begin{aligned} V = \Gamma\Big/\int \mathrm{d}^4x = \frac{2\Lambda}{\kappa^2}\prod_{i=1}^{d-4} 2\pi\rho_i + \tfrac{1}{2}\Big\{\frac{d^2-d}{2}\operatorname{Sp}\ln(\Box + 2\Lambda) \\ + d\operatorname{Sp}\ln(\Box + 2\alpha\Lambda) - 2d\operatorname{Sp}\ln\Box\Big\}\Big/\int \mathrm{d}^4x \end{aligned} \tag{10.20}$$

where ρ_i are the radii of the torus.

Now, we must calculate $\operatorname{Sp}\ln(\Box + X)$ on $R_4^0 \times T_{d-4}$. We will give here calculations for $d = 5$ and will present only the final result for arbitrary d (the reader can repeat this calculation himself).

Let us write

$$\frac{\operatorname{Sp}\ln(\Box + X)}{2\int \mathrm{d}^4x} = \zeta(0)\ln\mu^2 + \zeta'(0) \tag{10.21}$$

where $\zeta(s) = \sum_n \lambda_n^{-s}$ and λ_n are the eigenvalues of the operator $\Box + X$ on $R_4^0 \times T_1$ and μ is a dimensional parameter. As is known from the investigation of divergences of EA in multidimensional space [356–358] $\zeta(0) = 0$ for odd-dimensional spaces. To calculate $\zeta'(0)$ the eigenvalues and multiplicity D_l of $\Box$ on S_1 are required: $\lambda_l = -l^2/\rho^2$, l is an integer, $D_l = 2$.

We put the dimension of space R_n^0 equal to n (at the end of calculations $n = 4$). Then after trivial integration over n-dimensional momentum we obtain

$$\zeta'(0) = -\frac{\Gamma(-n/2)}{(4\pi)^{n/2}\rho^4}\sum_{l=-\infty}^{\infty}[l^2 + \gamma]^{n/2} \tag{10.22}$$

where $\gamma = -\rho^2 X$ and ρ is the radius of a 1-sphere. Using the well-known formulae

$$[l^2+\gamma]^{n/2} = \frac{1}{\Gamma(-n/2)}\int_0^\infty \mathrm{d}s s^{-n/2-1}\exp[-s(l^2+\gamma)]$$

$$\sum_{l=-\infty}^{\infty} \mathrm{e}^{-l^2 s} = \sqrt{\pi/s}\sum_{l=-\infty}^{\infty}\exp(-\pi^2 l^2/s)$$

in (10.22) we can obtain

$$\zeta'(0) = -\frac{1}{32\pi^2\rho^4}\mathrm{d}_2(-\rho^2 X) \tag{10.23}$$

where

$$d_2(y) = -\tfrac{8}{15}\pi y^{5/2} + \sum_{n=1}^{\infty}\left(\frac{3}{2\pi^4 n^5} + \frac{3\sqrt{y}}{\pi^3 n^4} + \frac{2y}{\pi^2 n^3}\right)\exp(-2\pi n\sqrt{y}).$$

When $X > 0$, an imaginary part appears in (10.23).

As an example of the application of (10.23) we write the one-loop effective potential in $d = 5$ gravity in gauge (10.9) with arbitrary α and γ [207]

$$\begin{aligned} V = \frac{4\pi\rho\Lambda}{\kappa^2} - \frac{1}{32\pi^2\rho^4}&[9d_2(-2\lambda) + 4d_2(-2\alpha\lambda) \\ &+ \mathrm{d}_2\Big[\Big(-b_1 - \sqrt{b_1^2 - \alpha/(\gamma+1)^2}\Big)\lambda\Big] \\ &+ \mathrm{d}_2\Big[\Big(-b_1 + \sqrt{b_1^2 - \alpha/(\gamma+1)^2}\Big)\lambda - 10d_2(0)\Big]. \end{aligned} \tag{10.24}$$

where $\lambda = \Lambda\rho^2$ and

$$b_1 = (\gamma+1)^{-2}\left[\frac{\alpha}{4} + \left(\frac{1 - 2\gamma - 5\gamma^2}{3}\right)\right].$$

Note that the gauge dependence of the EA in multidimensional quantum gravity has also been investigated in [350, 351].

Making use of the same technique as above we can find $\mathrm{Sp}\ln(\Box + X)$ on the $R_4^0 \times T_{d-4}$ background

$$\mathrm{Sp}\ln(\Box + X)/\left(2\int \mathrm{d}^4x\right)^{-1} \equiv F_{d-4}(-X).$$

For odd d

$$\begin{aligned} F_{d-4}(y) = -\frac{(\pi y)^{d/2}}{(2\pi)^d}\prod_{i=1}^{d-4}(2\pi\rho_i)&\Bigg[\tfrac{1}{2}\Gamma(-d/2) \\ &+ \sum_{n_1,\ldots,n_d=-\infty}^{\infty\prime}\frac{K_{d/2}\left(\sqrt{y}\left[\sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\right]^{1/2}\right)}{\left(\frac{1}{2}\sqrt{y}\left[\sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\right]^{1/2}\right)^{d/2}}\Bigg] \end{aligned} \tag{10.25}$$

where $K_a(z)$ is the modified Bessel (McDonald's) function, the prime means that the term with $n_1 = \ldots = n_{d-4} = 0$ is omitted.

For even d

$$F_{d-4}(y) = (\pi y)^{d/2}\left(\prod_{i=1}^{d-4}(2\pi\rho_i)\right)(2\pi)^{-d}\Big[\frac{1}{2}\frac{(-1)^{d/2}}{(d/2)!}$$
$$\times\left[\left(\ln\frac{2\pi\mu^2}{y}\right)+\gamma+\psi\left(3+\frac{d-4}{2}\right)\right]$$
$$+\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}' \frac{K_{d/2}\left(\sqrt{y}\left[\sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\right]^{1/2}\right)}{\left(\frac{1}{2}\sqrt{y}\left[\sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\right]^{1/2}\right)^{d/2}}\Big] \tag{10.26}$$

where γ is the Euler constant and $\psi(x)$ is the di-gamma function.

Making use of (10.24)–(10.26) we can rewrite (10.20) in the following form

$$V = \frac{2\Lambda}{\kappa^2}\prod_{i=1}^{d-4}2\pi\rho_i+\left\{\frac{d^2-d}{2}F_{d-4}(-2\Lambda)+dF_{d-4}(-2\alpha\Lambda)-2dF_{d-4}(0)\right\}. \tag{10.27}$$

This is the final result for a one-loop effective potential with gauge-fixing action (10.9) ($\gamma = -1/2$). This effective potential depends on the gauge parameter α. It is interesting to note that when $\Lambda = 0$ the effective potential (10.27) becomes independent of the gauge. However, the one-loop effective potential $V \neq 0$ at $\Lambda = 0$.

Now let us discuss the spontaneous compactification taking account of quantum gravitational effects. For simplicity we limit ourselves to $d = 5$. Self-consistent conditions corresponding to spontaneous compactification are written in the form

$$V = 0 \qquad \frac{\partial V}{\partial \rho} = 0. \tag{10.28}$$

The first condition (10.28) is the condition for the vanishing of the four-dimensional cosmological constant, the second is the equation of motion. It is evident that the solution of (10.28) for effective potential (10.24) leads to gauge dependence of the found radius of spontaneous compactification: $\rho = \rho(\alpha, \gamma)$. This situation seems to be unnatural. We will present one way of solving this problem in section 10.5.

We have calculated the EA in Einstein quantum gravity. But since there is no consistent quantum gravity, any consideration of quantum gravity effects depends on the choice of model. Let us calculate the EA in multidimensional R^2-gravity (see [354]). What accounts for this choice of model for quantum gravity? Firstly, it is known that in $d = 4$ higher-derivative gravity results in a multiplicatively renormalized and asymptotically free quantum theory (Chapters 8 and 9). Moreover, there is a point of view according to which Einstein gravity is an effective theory, which is a low-energy approximation to the quantum gravity (R^2-gravity). Secondly, multidimensional R^2-gravity is induced from the superstring theory in $d = 10$. Thirdly, R^2-gravity is consistent with the Kaluza–Klein approach as it is formulated completely in terms of the metric.

Let us calculate the EA for some variants of the theory with the action

$$\begin{aligned} S = \int \mathrm{d}^d x\sqrt{g}\Big(&\alpha_1 R^2 + \alpha_2 R_{AB}R^{AB} + \alpha_3 R_{ABCD}R^{ABCD} \\ &+ \frac{1}{K}(R - \Lambda)\Big) \end{aligned} \tag{10.29}$$

on the background $R_4^0 \times T_{d-4}$. Here α_1, α_2, α_3, K, Λ are dimensional constants. Consider the case when action (10.29) has no Einstein term ($1/K = 0$). Let us write down the second variation of action (10.29)

$$\begin{aligned} \delta^2 S = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{g}\Big\{&\frac{\Lambda}{K}\Big(\tfrac{1}{2}h_{AB}h^{AB} - \tfrac{1}{4}h^2\Big) \\ &+ (2\alpha_1 + \alpha_2 + 2\alpha_3)\nabla_A\nabla_B h^{AB}\nabla_C\nabla_D h^{CD} \\ &+ \Big(\tfrac{1}{2}\alpha_2 + 2\alpha_3\Big)h_{AB}\Box^2 h^{AB} + \Big(2\alpha_1 + \tfrac{1}{2}\alpha_2\Big)h\Box^2 h \\ &- (4\alpha_1 + \alpha_2)\Box h\nabla_A\nabla_B h^{AB} \\ &+ (\alpha_2 + 4\alpha_3)\nabla^A h_{AC}\Box(\nabla_B h^{BC})\Big\}. \end{aligned} \tag{10.30}$$

The conditions $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq -4\alpha_3$ must be fulfilled, otherwise the higher-derivative terms form a total divergence. The gauge-fixing action will have the form

$$\begin{aligned} &S_{\mathrm{GF}} = \tfrac{1}{2}\beta_1\int \mathrm{d}^4x\sqrt{g}\chi_A H^{AB}\chi_B \\ &\chi_A = \nabla_B h_A^B - \beta_2\nabla_A h \qquad H^{AB} = g^{AB}\Box + \beta_3\nabla^A\nabla^B. \end{aligned} \tag{10.31}$$

Here β_1, β_2, β_3 are the gauge parameters. Let us choose these parameters from the condition of eliminating non-minimal terms with higher derivatives in (10.31) (compare with Chapter 8). Then

$$\begin{aligned}
\beta_1 &= -(\alpha_2 + 4\alpha_3) \equiv \bar{\beta}_1 \\
\beta_2 &= \frac{4\alpha_1 + \alpha_2}{4(\alpha_1 - \alpha_3)} \equiv \bar{\beta}_2 \\
\beta_3 &= -\frac{(2\alpha_1 + \alpha_2 + 2\alpha_3)}{\alpha_2 + 4\alpha_3} \equiv \bar{\beta}_3 .
\end{aligned} \tag{10.32}$$

The one-loop EA is written in the form (see also Chapters 8 and 9)

$$\begin{aligned}
\Gamma = S + \tfrac{1}{2}\Big\{ &\operatorname{Sp}\ln\left[\frac{\delta^2}{\delta h_{AB}\delta h_{CD}}(S[g+h] + S_{\mathrm{GF}}[g,h])\right]_{h_{AB}=0} \\
&- \ln\det H^{AB} - 2\ln\det \Delta^A_B \Big\}.
\end{aligned} \tag{10.33}$$

Here $\Delta^A_B = \delta^A_B \Box + (1 - 2\bar{\beta}_2)\nabla^A\nabla_B$.

Taking into account (10.33) and (10.30) we can find the effective action in the theory. For simplicity, we give the result for the case when $1/K = 0$, $\Lambda/K \neq 0$, $d = 5$, $\beta_3 = 0$ [354]

$$\begin{aligned}
\Gamma = S + \tfrac{1}{2}\Big\{ &9\operatorname{Sp}\ln\left(\Box^2 + \frac{\lambda_2}{\alpha_2 + 4\alpha_3}\right) + 4\operatorname{Sp}\ln\left(\Box^2 - \frac{\lambda_2}{\beta_1}\right) \\
&+ \operatorname{Sp}\ln\Big\{\tfrac{3}{8}\lambda_2^2 + \lambda_2[\beta_1(2\beta_2^2 - 2\beta_2 - 1) - 2\alpha_1 - \tfrac{5}{8}\alpha_2 - \tfrac{1}{2}\alpha_3]\Box^2 \\
&+ [4\beta_1^2\beta_2^2(3 - 2\beta_2) + \beta_1(\beta_2 - 1)^2(8\alpha_1 + \tfrac{5}{2}\alpha_2 + 2\alpha_3)]\Box^4\Big\} \\
&- 15\operatorname{Sp}\ln\Box.
\end{aligned} \tag{10.34}$$

Here $\lambda_2 = \Lambda/K$, β_1, β_2 are arbitrary. Again, we see that the EA depends on the gauge parameters. It leads to gauge dependence on the radius of spontaneous compactification as well as on Einstein gravity. Of course, it is possible to find the EA in this theory when $1/K \neq 0$ or $d > 5$, or in other gauges (see [354]).

It is possible to find the EA in other models of multidimensional gravity, for example, in gravity with torsion or in conformal gravity, etc. We have thus shown that the effective action in the class of linear gauges depending on some parameters also depends on these. This leads to a gauge dependence of the self-consistent spontaneous compactification radius found from conditions (10.28) (see also [207, 350, 351]).

10.4 Effective action in multidimensional supergravity

The status of the multidimensional supergravity has greatly changed over the last few years. During the 1980s, the multidimensional supergravities (SG)s and, especially, $d = 10$ and 11 supergravities were the candidates for the 'theory of everything'. Now, these theories are a mostly good theoretical 'laboratory' for physicists. Of course, multidimensional SGs are also interesting as field limits of superstrings and supermembranes.

In this section we investigate the EA in multidimensional supergravities on a torus compactified background. Let us note that vacuum energy in multidimensional supergravities has been investigated in [359, 360, 362–364] for $d = 11$ and in [361] for $d = 5$ supergravity. In papers [365, 366] the general method for the calculation of EA for arbitrary multidimensional supergravity on the background $R_n \times T_{d-n}$ has been given. The EA has been presented as an expansion in the curvature and its derivatives. In [208, 370] the EA in multidimensional supergravity at non-zero temperature on torus compactified background has been obtained. The gauge dependence of the EA in multidimensional supergravity has been studied in [367, 368]. The review of the quantum aspects of multidimensional (super)gravity has been presented in paper [200].

Let us investigate the EA in supersymmetric Kaluza–Klein theory on a $R_n \times T_{d-n}$ background, where R_n is curved n-dimensional space. Each of these theories is characterized by a supermultiplet consisting of tetrad e^A_M, some spin-vectors (gravitino) ψ_M, some spinors χ and some antisymmetric tensor fields $A_{M_1 \ldots M_n} (0 \leqslant n < d)$ on mass shell. We choose periodic boundary conditions on the torus for the part of fields of supermultiplet and antiperiodic boundary conditions in some compactified dimensions for rest fields (twisted fields [369]) of supermultiplet. Note that twisted supermultiplets were introduced in [299].

The one-loop effective action has the following general structure

$$\Gamma = S + \Gamma_1 \qquad \Gamma_1 = \tfrac{1}{2}\,\mathrm{Sp}\ln\frac{\delta^2 S}{\delta\Phi^2} + \Gamma_{\mathrm{gh}} = \sum_p N_p \Gamma_1^{(p)} \tag{10.35}$$

where S is the classical action of the theory on $R_n \times T_{d-n}$ background (for theories like $d = 10$ super Yang–Mills theory [371, 372] $S = 0$), Φ is a complete supermultiplet, when calculating $\delta^2 S/\delta\Phi^2$ we take into account the necessary gauge terms, Γ_{gh} is the ghost contribution and Sp is calculated with positive sign for bosons and with negative sign for fermions. In (10.35) we take all types of fields to be included in the supermultiplet, N_p is the number of fields of given type, $\Gamma_1^{(p)}$

is the one-loop EA for p-type field. Using proper-time representation (Schwinger–De Witt technique) and (10.35) we can easily get (see also [365, 366])

$$\Gamma = -\frac{\Omega_{d-n}}{\kappa^2}\int d^n x\,\sqrt{g}R - \tfrac{1}{2}\int d^n x\,\sqrt{g}\lim_{k\to n}\int_0^\infty \frac{ds}{(4\pi)^{k/2}s^{k/2+1}}$$
$$\times \sum_{l_1,\ldots,l_{d-k}=-\infty}^{\infty} \exp\left(-s\sum_{r=1}^{d-k}\frac{(l_r+g_r)^2}{\rho_r^2}\right)\sum_p\sum_{j=0}^{\infty} s^j N_p\,\mathrm{tr}\,a_j^{(p)}. \quad (10.36)$$

Here R is the n-dimensional curvature

$$\Omega_{d-n} = (2\pi)^{d-n}\prod_{i=1}^{d-n}\rho_i$$

is the volume of torus $T_{d-n} = S_{1(1)}\times\ldots\times S_{1(i)}\times\ldots\times S_{1(d-n)}$, ρ_i is the radius of $S_{1(i)}$ and $a_j^{(p)}$ are the De Witt coefficients for a p-type field. If for the p-type field only periodic boundary conditions along $1,\ldots,k$ compactified dimensions are chosen then $g_1=\ldots=g_k=0$ and the compactified rth component of momentum is $p_r = (l_r + g_r)/\rho_r$.

Using the well-known values for De Witt coefficients $a_j^{(p)}$ for fields with spins 0, 1/2, 1, 3/2, 2 in d-dimensional spaces [356, 357, 365] we can show that

$$\sum_p N_p\,\mathrm{tr}\,a_0^{(p)} \equiv \sum_m N_m C_{d-2}^m - N_{1/2}\frac{\nu}{\gamma} - N_{3/2}\frac{\nu}{\gamma}(d-3)$$
$$+ N_2\frac{d(d-3)}{2} \quad (10.37)$$

$$\sum_p N_p\,\mathrm{tr}\,a_1^{(p)} \equiv R\sum_p N_p\tilde{a}_1^{(p)} \equiv R\Big[\sum_m N_m\left(\tfrac{1}{6}C_{d-2}^m - C_{d-4}^{m-1}\right)$$
$$+ N_{1/2}\frac{\nu}{12\gamma} + N_{3/2}\frac{\nu(d-3)}{12\gamma}$$
$$+ N_2\left(-\tfrac{5}{12}d^2 + \tfrac{3}{12}d - 2\right)\Big] \quad (10.38)$$

$$\sum_p N_p\,\mathrm{tr}\,a_2^{(p)} = R_{MNPQ}^2\Big[\sum_m N_m\Big(\frac{1}{180}C_{d-2}^m - \tfrac{1}{12}C_{d-4}^{m-1}$$
$$+ \tfrac{1}{2}C_{d-6}^{m-2}\Big) + N_{1/2}\frac{\nu}{\gamma}\left(\tfrac{1}{96} - \tfrac{1}{180}\right)$$

$$+ N_{3/2}\frac{\nu}{\gamma}\Big(\frac{3-d}{180}+\frac{d-19}{96}\Big)+N_2\Big(\frac{d(d-3)}{360}-\frac{d-18}{12}\Big)\Big]$$
$$+ R^2_{MN}\Big[\sum_m N_m\Big(-\tfrac{1}{180}C^m_{d-2}+\tfrac{1}{2}C^{m-1}_{d-4}-2C^{m-2}_{d-6}\Big)+N_{1/2}\frac{\nu}{180\gamma}$$
$$- N_{3/2}\frac{\nu(3-d)}{180\gamma}+N_2\Big(-\frac{d^2}{360}+\frac{183d}{360}-4+\frac{2}{1-d/2}\Big)\Big]$$
$$+ R^2\Big[\sum_m N_m\Big(\tfrac{1}{72}C^m_{d-2}-\tfrac{1}{6}C^{m-1}_{d-4}+\tfrac{1}{2}C^{m-2}_{d-6}\Big)-N_{1/2}\frac{\nu}{288\gamma}$$
$$+ N_{3/2}\frac{\nu(3-d)}{288\gamma}+N_2\Big(\frac{25d^2-99d+24}{144}-\frac{1}{1-d/2}\Big)\Big)$$
$$+ \Box R\Big[\sum_m N_m\Big(\tfrac{1}{30}C^m_{d-2}-\tfrac{1}{6}C^{m-1}_{d-4}\Big)+N_{1/2}\frac{\nu}{120\gamma}$$
$$- N_{3/2}\frac{\nu(3-d)}{120\gamma}+N_2\frac{-4d^2+2d-20}{69}\Big]. \qquad (10.39)$$

Here N_2, $N_{3/2}$, $N_{1/2}$, N_m are the numbers of gravitons, gravitino, spinors and antisymmetric tensors $A_{M_1,\dots,M_m}(0 \leqslant m < d)$ respectively. Antisymmetrical tensor fields include the scalars ($m = 0$) and vectors ($m = 1$), $\nu = 2^{[d/2]}$, $\gamma = 1, 2, 4$ for Dirac, Majorana and Majorana–Weyl spinors, respectively, $C^m_d = d(d-1)\dots(d-m+1)/m!$, $C^0_r = 1$, $C^{-1}_r = 0$, $C^k_{-1} = (-1)^k$, $C^k_{-2} = (-1)^k(k+1)$, $C^{n-m}_{d-2-p} = \sum_{k=0}^{n-m} C^{n-m-k}_{d-p} C^k_{-2}$.

We now introduce the general Epstein Z-function of pth order [373]

$$Z\begin{vmatrix} g_1 & \dots, & g_p \\ h_1, & \dots, & h_p \end{vmatrix}(s)_\varphi = \sum_{l_1,\dots,l_p}^{\infty}{}' [\varphi(l+g)]^{-ps/2} \times \exp[2\pi \mathrm{i}(m,h)]. \qquad (10.40)$$

Here $(g,h) = \sum_{k=1}^p g_k h_k$, $\varphi(x) = \sum_{k,n=1}^p a_{kn}x_k x_n$. A prime means that there are no terms in the sum with $l = -g$ (when all g are integer). Relation (10.40) defines Z when $\operatorname{Re} s > 1$. For all the other values of s, Z is defined with help of the analytical continuation. Using the functional equation [373]

$$\Gamma(\tfrac{1}{2}ps)Z\begin{vmatrix} g \\ h \end{vmatrix}(s)_\varphi = \det a^{-1/2}\pi^{-p(1-2s)/2} \times \Gamma\Big(\tfrac{1}{2}p(1-s)\Big)e^{2\pi\mathrm{i}(g,h)}Z\begin{vmatrix} h \\ -g \end{vmatrix}(1-s)_\varphi. \qquad (10.41)$$

where $\varphi^*(x) = \sum_{k,n=1}^{p} a_{kn}^{-1} x_k x_n$. We can represent the EA (10.36) by the following expression

$$\Gamma = -\frac{\Omega_{d-n}}{\kappa^2}\int \mathrm{d}^n x\sqrt{g}R - \int \mathrm{d}^n x\sqrt{g}\sum_p \sum_{j=0}^{\infty} N_p \operatorname{tr} a_j^{(p)} V_j^{(p)}(\rho_1,\dots,\rho_{d-n}) \quad (10.42)$$

$$V_j^{(p)} = \begin{cases} \dfrac{\Gamma(\frac{d}{2}-j)}{2(4\pi)^{n/2}\pi^{(d+n)/2-2j}} \displaystyle\prod_{i=1}^{d-n}\rho_i Z\begin{vmatrix} 0, & \dots, & 0 \\ -g_1, & \dots, & -g_{d-n}\end{vmatrix}\left(\frac{d-2j}{d-n}\right)_{\varphi^*} \\ \text{for } j \leqslant n/2 \\ \dfrac{\Gamma(j-\frac{n}{2})}{2(4\pi)^{n/2}} Z\begin{vmatrix} g_1, & \dots, & g_{d-n} \\ 0, & \dots, & 0\end{vmatrix}\left(\frac{2j-n}{d-n}\right)_{\varphi} \quad j > n/2 \end{cases}$$

$a_{km} = \delta_{km}/\rho_m^2$ and $g_1,\dots,g_{d-n}$ are 0 or 1/2.

Expression (10.42) has the same structure in the case of even or odd-dimensional theories. We can show that this fact is caused by the choice of the torus as internal manifold. Using relation (10.42) it is easy to get an explicit expression for EA for an arbitrary SG on the background $R_n \times T_{d-n}$. Let us give some examples. We will find a one-loop EA in $d = 4$SG on $R_3 \times T_1$ (see also [502] for topologically trivial curved space R_4). We will consider curved space R_3 and present the EA up to linear curvature terms. On the gravitational background $R_3 \times T_1$ we obtain

$$\Gamma = -\int \mathrm{d}^3 x\,\sqrt{g}\sum_p N_p V_0^{(p)}(\rho)\operatorname{tr} a_0 - \int \mathrm{d}^3 x\,\sqrt{g}R\left[\frac{2\pi\rho}{\kappa^2} + \sum_p N_p \operatorname{tr}\tilde{a}_1 V_1^{(p)}(\rho)\right]. \quad (10.43)$$

Here

$$\operatorname{tr} a_1 \equiv R\operatorname{tr}\tilde{a}_1 \qquad V_0(\rho) = \frac{\pi}{90(2\pi\rho)^3}\begin{cases} 1 & g=0 \\ -7/8 & g=1/2 \end{cases}$$

$$V_1(\rho) = \frac{1}{48\pi\rho}\begin{cases} 1 & g=0 \\ -7/8 & g=1/2 \end{cases}$$

We can rewrite EA (10.43) in the form

$$\Gamma = -\int \mathrm{d}^3 x\,\sqrt{g}A - \int \mathrm{d}^3 x\sqrt{g}R\left[\frac{2\pi\rho}{\kappa^2} + B\right]. \quad (10.44)$$

Taking into account the explicit expressions for a_0 and a_1, (10.37)–(10.39), and expression (10.44) we find

(*a*) $N = 1, 2$ SG (all fields of supermultiplet satisfy the same boundary conditions): $A = 0$, $B/V_1(\rho) = -15/2, -49/6$ for $N = 1, 2$ respectively SG;

(*b*) $N = 1$ SG: $A = 15\pi^2/360(2\pi\rho)^3$, $B = -31/192\pi\rho$ ($g = 0$ for bosons, $g = -1/2$ for fermions). It means that in this SG the periodic boundary conditions on the sphere are chosen for bosons and antiperiodic for fermions.

Let us write the explicit expressions for the EA in multidimensional SGs on the background $R_4 \times T_{d-4}$. For $N = 2$, $d = 5$ SG (see [374]) on the background $R_4 \times T_1$ we obtain

$$\Gamma = -\int d^4x\,\sqrt{g}A - \int d^4x\,\sqrt{g}R\left[\frac{2\pi\rho}{\kappa^2} + B\right]$$

where $A = 93\zeta(5)/2(2\pi)^6\rho^4$, $B = -73\zeta(3)/12(2\pi)^4\rho^2$, if $g = 0$ for bosons, $g = +1/2$ for fermions. In the discussed example a large cosmological constant is induced (since $\rho \propto L_p$, then $\Lambda_{\text{ind}} \propto L_p^{-4}$).

We can also find the EA on the background $R_4^0 \times T_6$ for $d = 10$ conformal SG [375]. Taking into account the supermultiplet of the theory and supposing periodical boundary conditions for bosons and antiperiodical boundary conditions along some compactified dimensions for fermions, we obtain by using (10.42) (see [409])

$$\Gamma\Big/\int d^4x = -\frac{96}{\pi^2}\prod_{i=1}^{6}\rho_i\left(Z\begin{vmatrix}0, & \ldots, & 0\\ 0, & \ldots, & 0\end{vmatrix}\left(\tfrac{10}{6}\right)_{\varphi}\right.$$
$$\left.-\,Z\begin{vmatrix}0, & \ldots, & 0\\ -g_1, & \ldots, & -g_6\end{vmatrix}\left(\tfrac{10}{6}\right)_{\varphi}\right)$$

where

$$\varphi(l) = \sum_{k=1}^{6}\rho_k^2 l_k^2$$

and $g_1, \ldots, g_6$ are 0 or 1/2. For example, if periodical boundary conditions along 5, 6 and 7 dimensions and antiperiodical boundary conditions along 8, 9 and 10 dimensions are chosen for the gravitino, then $g_1 = g_2 = g_3 = 0$ and $g_4 = g_5 = g_6 = 1/2$.

Let us now calculate EA in $d = 5$, $d = 7$ and $d = 10$ SGs. The supermultiplets of $d = 5$ SGs have the following structure [374]

$$\begin{aligned} N &= 2(e_M^A, 2\psi_M, A_M) \\ N &= 4(e_M^A, 4\psi_M, 6A_M, 4\chi, \varphi) \\ N &= 6(e_M^A, 6\psi_M, 15A_M, 20\chi, 14\varphi) \\ N &= 8(e_M^A, 8\psi_M, 27A_M, 48\chi, 42\varphi) \end{aligned} \tag{10.45}$$

where internal indices are dropped and χ; ψ_M are generalized Majorana spinors ($\gamma = 2$). Using relations (10.42) and (10.45) we can find the EA up to linear curvature terms for $d = 5$ SGs on $R_4 \times S_1$ ($g_1 = 0$)

$$\Gamma = -\left[\frac{2\pi\rho}{\kappa^2} + bV_1(\rho)\right] \int \mathrm{d}^4x\sqrt{g}R \tag{10.46}$$

where $V_1(\rho) = \zeta(3)/2(2\pi)^4\rho^2$, $b = -11, -12, -11, -7$ for $d = 5$, $N = 2, 4, 6, 8$ SGs respectively. Comparing relation (10.46) with the action of 4-dimensional gravity, we obtain

$$\Lambda_{\text{ind}} = 0 \qquad \frac{1}{\kappa_{\text{ind}}^2} = \frac{2\pi\rho}{\kappa^2} + bV_1(\rho). \tag{10.47}$$

If we suppose ρ and κ^2 to be independent parameters of the theory then if $\rho^3/\kappa^2 > -b\zeta(3)/2(2\pi)^5$ the condition $\kappa_{\text{ind}}^2 > 0$ is valid. It seems much more natural to obtain ρ from the requirement that the EA should be minimal. Then from $\partial\Gamma/\partial\rho = 0$ it follows that

$$\rho = \left(b\zeta(3)\kappa^2/(2\pi)^5\right)^{1/3}. \tag{10.48}$$

We can show now that the condition $\rho > 0$ is fulfilled only if $\kappa^2 < 0$ (antigravity). Substituting (10.48) into (10.47) we obtain

$$\frac{1}{\kappa_{\text{ind}}^2} = 3\pi\left(b\zeta(3)\kappa^{-4}/(2\pi)^5\right)^{1/3}.$$

We can see that from this relation it follows that the induced gravitational constant in all 5-dimensional SGs has the wrong sign. Note that this fact should be regarded as the result of periodic boundary conditions upon which the Laplacian spectrum of the torus has been found.

Consider the case when for all the fields of supermultiplet antiperiodic boundary conditions are chosen ($g_1 = 1/2$). Using (10.42) it is easy to show that

$$\Gamma = -\left(\frac{2\pi\rho}{\kappa^2} - \tfrac{3}{4}bV_1(\rho)\right) \int \mathrm{d}^4x\sqrt{g}R. \tag{10.49}$$

From the condition of the minimum of the EA it follows that $\rho = (-3b\zeta(3)\kappa^2(2\pi)^{-5}/4)^{1/3}$. This does not contradict the fact that $\rho > 0$ and κ^2 are positive. For $\kappa^2_{\rm ind}$ we obtain

$$\frac{1}{\kappa^2_{\rm ind}} = 3\pi\left(-\frac{3b\zeta(3)}{4(2\pi)^5\kappa^4}\right)^{1/3}.$$

Thus, when antiperiodic boundary conditions are chosen and $\kappa^{4/3} \sim L_p^2$, the EA of 5-dimensional SGs induces 4-dimensional gravity with the true gravitational constant and $\Lambda_{\rm ind} = 0$. Note that $\kappa^2_{\rm ind} > 0$.

Let us now write the EA for $d = 7$, $N = 4$ SG. According to paper [376] the supermultiplet includes: e^A_M, $4\psi_M$, $5A_{MN}$, $10A_M$, 16χ and 14φ. From (10.42) we obtain ($g_r = 0$, $r = 1, 2, 3$):

$$\Gamma = -\left(\frac{(2\pi)^3\rho_1\rho_2\rho_3}{\kappa^2} - \tfrac{112}{3}V_1(\rho_1,\rho_2,\rho_3)\right)\int d^4x\sqrt{g}R.$$

where

$$V_1(\rho_1,\rho_2,\rho_3) = \frac{3}{128\pi^5}\left\{2\zeta(5)\left[\frac{\rho_2\rho_3}{\rho_1^4} + \frac{\rho_1\rho_3}{\rho_2^4} + \frac{\rho_1\rho_2}{\rho_3^4}\right]\right.$$
$$+ 4\left[\frac{\rho_2\rho_3}{\rho_1^4}\zeta\left(\frac{\rho_2}{\rho_1};5\right) + \frac{\rho_1\rho_3}{\rho_2^4}\zeta\left(\frac{\rho_3}{\rho_2};5\right) + \frac{\rho_1\rho_2}{\rho_3^4}\zeta\left(\frac{\rho_1}{\rho_3};5\right)\right]$$
$$\left. + \frac{8\rho_2\rho_3}{\rho_1^4}\zeta\left(\frac{\rho_2}{\rho_1},\frac{\rho_3}{\rho_1};5\right)\right\}$$

$$\zeta(x;r) = \sum_{l_1,l_2=1}^{\infty}(l_1^2 + x^2l_2^2)^{-r/2}$$

$$\zeta(x,y;r) = \sum_{l_1,l_2,l_3=1}^{\infty}(l_1^2 + x^2l_2^2 + y^2l_3^2)^{-r/2}.$$

When $\rho_1 \propto \rho_2 \gg \rho_3$, for example, $V_1(\rho_1,\rho_2,\rho_3) \approx 3\zeta(5)\rho_1\rho_2/64\pi^5\rho_3^4$. If ρ_1, ρ_2, ρ_3, κ^2 are free parameters, then it is not difficult to ensure the positivity of induced Newton constant. We will not analyse the sign of $\kappa^2_{\rm ind}$ using conditions $\partial\Gamma/\partial\rho_i = 0$, $i = 1, 2, 3$ for it leads to cumbersome calculations.

Let us now write the EA up to R^2-terms for $N = 1$, $d = 10$ SG interacting with $N = 1$, $d = 10$ super-Yang–Mills theory. (Such a model arises in the low-energy limit of heterotic string [377].) From (10.42) we obtain ($g_r = 0$)

$$\Gamma = -\int d^4x\sqrt{g}R\left\{\frac{\Omega_6}{\kappa^2} - 36V_1(\rho_1,\ldots,\rho_6)\right\}$$
$$-\int d^4x\sqrt{g}\left(2R^2_{AB} + \tfrac{21}{2}R^2 - \tfrac{98}{15}\Box R\right)V_2(\rho_1,\ldots,\rho_6).$$

Here

$$\Omega_6 = \prod_{i=1}^{6}(2\pi\rho_i)$$

$$V_j = \frac{\Gamma(5-j)}{2(4\pi)^2\pi^{7-2j}} \prod_{i=1}^{6}\rho_i \sum_{l_1,\ldots,l_6=-\infty}^{\infty\,\prime} \left(\sum_{i=1}^{6}\rho_i^2 l_i^2\right)^{j-5}$$

$$j = 1, 2.$$

The situation with the induced gravitational constant is the same as in $d = 7$ SG above.

Note that κ^{-2} has an especially simple form when $n = 9$ (for induced 9-dimensional Einstein gravity) [200, 408]

$$\frac{1}{\kappa_{\rm ind}^2} = \frac{2\pi\rho}{\kappa^2} - 36\left(\frac{3\zeta(8)}{2^8\pi^{12}\rho^7}\right).$$

In this case compactification should be made in two steps. Obtaining induced $d = 9$ gravity with zero Λ-term is the first step. For the second step compactification $R_9 \to R_4 \times K_5$ should be carried out. Here K_5 is a compact manifold with a Planck scale. The compactification $R_9 \to R_4 \times K_5$ can be caused, for example, by non-trivial solutions of the classical field equations. Thus, we have discussed the effective action in multidimensional supergravity.

When investigating multidimensional quantum field theory the natural question arises: is it possible to find evidence of the physical acceptability of the Kaluza–Klein approach? According to estimations the typical sizes of additional $(d-4)$-dimensions fall within the $(10-100)L_p$ interval. At present typical energies it is difficult to find the effect which would indicate their existence. If non-zero sizes of additional $(d-4)$-dimensions are relevant to physics then it seems to be natural that it should be possible to find some phenomena indicating their existence in the early hot universe. That is why the investigation of temperature effects in quantum Kaluza–Klein theories is of a great interest [310, 313, 323, 324, 328–330, 336, 370].

Let us now find the EA in $d = 5$ supergravity on $R_4 \times S_1$ background at non-zero temperature. This means that the topology of space R_4 is chosen in the form S_1(temperature) $\times R_3$, where R_3 is 3-dimensional curved space with trivial topology and slow-changing curvature. Using (10.36) and the method of calculating $\mathrm{Sp}\ln(\Box + X)$ on S_1(temperature) $\times R_3 \times S_1$ we obtain for the EA, taking account of linear curvature terms

$$\Gamma = -\beta \int d^3x\sqrt{g}\left[\frac{A}{\rho^4} + \frac{B}{\rho^2}R + \ldots\right] \qquad (10.50)$$

where

$$A = \frac{1}{32\pi^2}\Big[\Big(5 + N_\varphi + 3N_{A_M}\Big)d_2(0,\beta) - \Big(2N_\chi + 4N_{\psi_M}\Big)d_2^F(0,\beta)\Big]$$

$$B = \frac{1}{32\pi^2}\Big[\Big(-\tfrac{67}{6} + \tfrac{1}{6}N_\varphi - \tfrac{1}{2}N_{A_M}\Big)d_1(0,\beta)$$
$$+ \Big(\tfrac{1}{6}N_\chi + \tfrac{1}{3}N_{\psi_M}\Big)d_1^F(0,\beta)\Big]$$

and β is the inverse temperature. For periodical boundary conditions for bosons and fermions we have [200, 370]

$$d_2(0,\beta) = \frac{3}{4\pi^4}\sum_{n,m=-\infty}^{\infty}{}' \frac{1}{\left(n^2 + \left(\frac{\beta m}{2\pi\rho}\right)^2\right)^{5/2}}$$

$$d_1(0,\beta) = \frac{1}{2\pi^2}\sum_{n,m=-\infty}^{\infty}{}' \frac{1}{\left(n^2 + \left(\frac{\beta m}{2\pi\rho}\right)^2\right)^{3/2}}$$

$$d_2^F(0,\beta) = \frac{3}{4\pi^4}\sum_{n,m=-\infty}^{\infty}{}' \left[\frac{2}{\left(n^2 + \left(\frac{\beta m}{\pi\rho}\right)^2\right)^{5/2}} - \frac{1}{\left(n^2 + \left(\frac{\beta m}{2\pi\rho}\right)^2\right)^{5/2}}\right]$$

$$d_1^F(0,\beta) = \frac{1}{2\pi^2}\sum_{n,m=-\infty}^{\infty}{}' \left[\frac{2}{\left(n^2 + \left(\frac{\beta m}{\pi\rho}\right)^2\right)^{3/2}} - \frac{1}{\left(n^2 + \left(\frac{\beta m}{2\pi\rho}\right)^2\right)^{3/2}}\right] \tag{10.51}$$

and for antiperiodic boundary conditions for bosons and fermions we can find these functions in [370] (see also [200]).

Using (10.50) we can find the induced gravitational and cosmological constant

$$\frac{\Lambda_{\text{ind}}}{\kappa_{\text{ind}}^2} = -\frac{A}{\rho^4} \qquad \frac{1}{\kappa_{\text{ind}}^2} = \frac{B}{\rho^2} + \frac{2\pi\rho}{\kappa^2}.$$

Turning off the temperature ($\beta \to \infty$) we have $A = 0$. This case was discussed above.

An especially simple form is the effective potential on flat background $R_4^0 \times T_{d-4}$ at non-zero temperature [209]

$$V = \operatorname{tr} a_0^B \Gamma(d/2)\pi^{-d/2} \prod_{i=1}^{d-4}(2\pi\rho_i)\Big\{ - \sum_{m,n_i=-\infty}^{\infty} \Big[(\beta m)^2 + \sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\Big]^{-d/2} + \sum_{m,n_i=-\infty}^{\infty} \Big[(2\beta m)^2 + \sum_{i=1}^{d-4}(2\pi\rho_i n_i)^2\Big]^{-d/2}\Big\}. \tag{10.52}$$

Here a_0^B is a bosonic a_0-coefficient. We can find the asymptotics of V. If $\beta \to +\infty$ $(T \sim 0)$, $2\pi\rho_i \ll \beta$, then $V = 0$, i.e. supersymmetry is restored. If $\beta \sim 0 (T \sim \infty)$, $2\pi\rho_i \gg \beta$, then $V \sim -\infty$ and spontaneous compactification is impossible.

To conclude this section we will discuss the effective action in $N = 2$, $d = 5$ gauged supergravity on the background $R_4^0 \times S_1$. The Lagrangian has the form [378]

$$\begin{aligned} e^{-1}L = & -\frac{1}{\kappa^2}R + \tfrac{1}{2}\bar{\psi}_M^i \Gamma^{MNP} D_N \psi_P^i + \tfrac{1}{4}\dot{a}_{IJ} F_{AB}^I F^{JAB} \\ & + \tfrac{1}{2}\bar{\lambda}^{ia}\hat{D}\lambda^{ia} + \frac{1}{\kappa^2} g_{xy}\partial_M \varphi^x \partial^M \varphi^y + \frac{4g^2}{\kappa^2}P(\varphi) \\ & + \frac{\sqrt{3}g}{4\kappa^2}\bar{\psi}_M^i \Gamma^{MN}\psi_N^i P_0(\varphi) + \frac{g}{\kappa}\bar{\lambda}^{ia}\gamma^M \psi_M^i P_a(\varphi) \\ & - \frac{g}{2\kappa\sqrt{3}}\bar{\lambda}^{ia}\lambda^{ib}P_{ab}(\varphi) + \dots . \end{aligned} \tag{10.53}$$

Here $\hat{D} = \gamma^M D_M$, $\Gamma^{MN} = \frac{1}{2}[\gamma^M, \gamma^N]$, $\Gamma^{MNP} = \gamma^{[M}\gamma^N\gamma^{P]}$, ψ_M is the gravitino, which transforms as a doublet of SU(2) automorphism group of the supersymmetry algebra. n-Maxwell supermultiplets $(A_M^a, \bar{\lambda}^{ia}, \varphi^a)$, $a = 1, \dots, n$ are coupled to $d = 5$, $N = 2$ SG $(e_M^A, \bar{\psi}_M^i, A_M^j)$, $j = 1, 2, 3$, $i = 1, 2$. Scalar $\{\varphi^x\}$, $x = 1, \dots, n$ parametrizes n-dimensional Riemann manifold N with metric $g_{xy}(\varphi) = f_x^a f_y^b \delta_{ab}$ and veilbein f_x^a. The $(n+1)$-vector fields $A_M^I, I = 1, \dots, n+1$ are inert under the action of the tangent space group SO(n). The quantities $\dot{a}_{IJ}$, $P(\varphi)$, $P_0(\varphi)$, $P_a(\varphi)$, $P_{ab}(\varphi)$ are functions of the scalar fields. Lagrangian (10.53) corresponds to U(1) gauging of the SU(2) automorphism group of the supersymmetry algebra. Only kinetic and scalar interaction terms are written explicitly.

We take the gravitational background $R_4^0 \times S_1$ with constant scalar field as the vacuum state of the theory. (Note that such a vacuum state is not supersymmetric on the classical field equations.) Let the background constant scalar field correspond to the critical point $\phi_c^0(maximum)$ of the classical potential $P(\varphi)$, then according to [378]

$$\begin{gathered}
P_a(\phi_c) = 0 \qquad P_{ab}(\phi_c) = \tfrac{1}{2}\delta_{ab}P_0(\phi_c) \qquad P(\phi_c) = -P_0^2(\phi_c) \\
\left.\frac{\partial P(\phi)}{\partial \phi^x}\right|_{\phi_c} = 0 \qquad P_0(\phi_c) = \frac{2}{\beta_1} \qquad \beta_1 = \sqrt{2/3} \\
\left.\frac{\partial^2 P(\phi)}{\partial \phi^x \partial \phi^y}\right|_{\phi_c} = \beta_1^2 g_{xy}(\phi_c)P(\phi_c).
\end{gathered} \tag{10.54}$$

This choice of background scalar field is caused by the fact that the structure of scalar potential $P(\varphi)$ has not been investigated in detail [378].

The contributions of vector and scalar fields to the EA are found in the standard way

$$\begin{aligned}
Z_{A_M} &= \exp(-\Gamma_{1A_M}) = \det{}^{-3(n+1)/2}\Box \\
Z_\varphi &= \det{}^{-n/2}\left[\Box - \frac{4g^2\beta_1^2 P(\phi_c)}{\kappa^2}\right].
\end{aligned}$$

(Since $\partial P/\partial\phi^x|_{\phi_c} = 0$, linear terms on scalar quantum fields in $\delta^2 S$ are absent.)

The gravitational contribution to the EA was found in section 10.3 (expression (10.24)) with the substitution $\Lambda \to 2g^2P(\phi_c)/\kappa^2$. For the fermionic sector we obtain

$$\begin{aligned}
Z_\lambda &= \det{}^{2n}\left[\Box - \frac{g^2P_0^2(\phi_c)}{12\kappa^2}\right] \\
Z_\psi &= \frac{\det^6(\Box - m^2)\det^2(\Box - \gamma_1)}{\det^4(\Box - 25m^2/9)}.
\end{aligned}$$

Here $m = g\sqrt{3}P_0(\phi_c)/2\kappa$, $\gamma_1 = 25m^2\gamma_0/(4m^2+3\gamma_0)^2$, and the gauge-breaking term $L_{\mathrm{GF}} = \frac{1}{2}\bar{\psi}^i(\gamma_0/\hat{D})\psi^i$ $(\psi^i = \gamma^M\psi^i_M)$ is used.

The resulting expression for one-loop EA $\Gamma = S + \Gamma_1(\Gamma_1 = -\ln Z)$ is given by the product of the above expressions. For simplicity we write this expression in the gauge $\alpha = 0$, $\gamma = -1/2$, $\gamma_1 = 0$

$$\begin{aligned}
V = &-\frac{48\pi g^2\rho}{\kappa^2} - \frac{1}{16\pi^2\rho^4}\left\{\left(\frac{3n}{2} - \tfrac{11}{2}\right)d_2(0) + \frac{n}{2}d_2\left(\frac{-16g^2\rho^2}{\kappa^2}\right)\right. \\
&+ \tfrac{15}{2}d_2\left(\frac{24g^2\rho^2}{\kappa^2}\right) - 2nd_2\left(\frac{g^2\rho^2}{2\kappa^2}\right) \\
&\left. - 6d_2\left(\frac{9g^2\rho^2}{2\kappa^2}\right) + 4d_2\left(\frac{25g^2\rho^2}{2\kappa^2}\right)\right\}.
\end{aligned} \tag{10.55}$$

It follows from (10.55) that the effective potential in $d = 5$, $N = 2$ gauged supergravity has an imaginary part in any gauge under consideration. This can be indicated by the instability of the proposed vacuum state.

Thus, we have considered the effective action in multidimensional supergravities on a torus compactified background. We have shown that it is possible to calculate the EA in some simple examples. However, for cosmological applications it would be necessary to find the EA on more realistic backgrounds (with time-dependent scale factors, for example). Of course, such calculations are very tedious.

10.5 Unique effective action in Kaluza–Klein quantum gravity

In the previous sections we obtained the EA in multidimensional (super)gravity on a torus compactified background. It was shown that the EA depends on the gauge under consideration. This leads to gauge dependence of the self-consistent radius of spontaneous compactification. In [203] and, independently, in [204] the use of gauge- and parametrization-invariant and gauge-fixing-independent effective action (see Chapter 6) was suggested as a solution to this problem. This EA has been considered in [203–210] as the true off-shell EA in gauge theories. (The general discussion of gauge fixing independent and parametrization independent effective action was presented in Chapter 6.)

Let us now consider the calculation of one-loop gauge and parametrization independent effective action (unique EA) in multidimensional gravity. (Remember that the unique EA by Vilkovisky coincides with the Vilkovisky–De Witt EA in the one-loop approximation.)

The unique EA for one-loop quantum gravity has the following general form (Chapter 6)

$$\begin{aligned}
\Gamma^{(1)} &= \tfrac{1}{2}\,\mathrm{Sp}\ln F_{mn} - \mathrm{Sp}\ln\left(\nabla^m_A \frac{\delta\chi^B}{\delta g^m}\right) \\
F_{mn} &= \left(\frac{\delta^2 S}{\delta g^m \delta g^n} - \Gamma^i_{mn}\frac{\delta S}{\delta g^i}\right) - \frac{\delta\chi^A}{\delta g^m} H_{AB} \frac{\delta\chi^B}{\delta g^n}.
\end{aligned} \tag{10.56}$$

Here

$$
\begin{aligned}
g^m &\equiv g_{AB} \\
\Gamma^i_{mn} &= T^i_{mn} - 2\gamma_{(mk}\nabla^k_A N^{AB} D_{n)}\nabla^i_B + \gamma_{(mr}\nabla^r_A N^{AB}\nabla^p_B(D_p\nabla^i_C) \\
&\quad \times N^{CD}\nabla^k_D\gamma_{kn)} \\
T^i_{mn} &= \tfrac{1}{2}\gamma^{ik}\left(\frac{\delta\gamma_{kn}}{\delta g^m} + \frac{\delta\gamma_{km}}{\delta g^n} - \frac{\delta\gamma_{mn}}{\delta g^k}\right) \\
N_{AB} &= \nabla^m_A\gamma_{mn}\nabla^n_B \qquad D_n\nabla^i_A = \frac{\delta\nabla^i_A}{\delta g^n} + T^i_{mn}\nabla^m_A \\
N^{AC}N_{CB} &= \delta^A_B \qquad \gamma^{ik}\gamma_{kn} = \delta^i_n \qquad \delta^i_n \equiv \delta^{AB}_{CD}.
\end{aligned}
$$

∇^n_A are the generators of general coordinate transformations, χ^A is the gauge condition, γ_{mn} is the configuration space metric satisfying several conditions [190] and S is the classical action. This expression differs from the standard EA in that the second variation derivative $\delta^2 S/\delta g^m \delta g^n$ is substituted by the second covariant derivative with the Γ^i_{mn} connection.

Let us find $\Gamma^{(1)}$ (10.56) for the $d = 5$ Einstein gravity on the background $R^0_4 \times S_1$ as the first example. Using the requirement defining the metric γ_{mn} we can verify that is has the same form as in 4-dimensional gravity (see also the discussion in Chapter 6)

$$\gamma_{mn} = \sqrt{g}C^{AB,MN} = \frac{\sqrt{g}}{4}(g^{MA}g^{NB} + g^{MB}g^{NA} - g^{AB}g^{MN}). \quad (10.57)$$

It is now easy to construct the connection

$$
\begin{aligned}
T^i_{mn} \equiv T^{PQ,AB}_{MN} &= \tfrac{1}{2}\Big[\tfrac{1}{2}g^{PQ}\delta^{AB}_{MN} + \tfrac{1}{2}g^{AB}\delta^{PQ}_{MN} \\
&\quad - \tfrac{1}{2}(\delta^{AQ}_{MN}g^{PB} + \delta^{AP}_{MN}g^{QB} + \delta^{PB}_{MN}g^{AQ} + \delta^{BQ}_{MN}g^{AP}) \\
&\quad + \tfrac{1}{3}(\delta^{PQ,AB} - \tfrac{1}{2}g^{PQ}g^{AB})g_{MN}\Big]. \qquad (10.58)
\end{aligned}
$$

Consequently, $\delta S/\delta g^m \neq 0$, $T^i_{mn} \neq 0$, $D_m \neq \delta/\delta g^m$, and the unique EA does not coincide with the standard one. Let us choose the gauge condition as

$$\chi^A = g^{AB}\nabla^C h_{BC} - \tfrac{1}{2}\nabla^A h \qquad H^{AB} = g^{AB}. \quad (10.59)$$

The operators N_{AB} and N^{AB} have the form resulting from the above relations

$$N_{AB} = -g_{AB}\Box \qquad N^{AB} = -\frac{g^{AB}}{\Box}. \quad (10.60)$$

Using (10.56)–(10.60), as a result of direct calculation we can obtain

$$\Gamma^{i}_{mn} g_i = -\tfrac{1}{3} C^{AB,PQ} - \tfrac{5}{3} \frac{C^{AB,MN}}{\Box} \times \left[\delta^P_M \nabla_N \nabla^Q + \delta^Q_M \nabla_N \nabla^P - g^{PQ} \nabla_M \nabla_N\right]. \quad (10.61)$$

Then we obtain for F_{mn}

$$-F_{mn} = C^{AB,MN} \Big(\delta^{PQ}_{MN} (\Box + \tfrac{5}{3}\Lambda) - \frac{5\Lambda}{3\Box} \times \left[\delta^P_M \nabla_N \nabla^Q + \delta^Q_M \nabla_N \nabla^P - g^{PQ} \nabla_M \nabla_N\right] \Big)$$

$$\nabla^m_A \frac{\delta \chi^B}{\delta g^m} = g^{AB} \Box. \quad (10.62)$$

Making use of (10.62) in (10.56) we can show that the one-loop correction to the unique EA is

$$\Gamma^{(1)} = \tfrac{1}{2} \{10 \, \mathrm{Sp} \ln(\Box + \tfrac{5}{3}\Lambda) - 5 \, \mathrm{Sp} \ln \Box\}. \quad (10.63)$$

Finally, for the one-loop Vilkovisky–De Witt EA (or unique EA) in $d = 5$ gravity we obtain (see, also [203, 205])

$$\Gamma = \int d^4x \left(\frac{4\pi\rho\Lambda}{\kappa^2} - \frac{1}{32\pi^2\rho^4} \left[10 d_2(-\tfrac{5}{3}\lambda) - 5 d_2(0)\right] \right) \quad (10.64)$$

where $\lambda = \Lambda\rho^2$ and $d_2(y)$ is given below equation (10.23).

It is obvious that the unique EA (10.64) differs considerably from standard EA (10.24) in the same theory. The Vilkovisky–De Witt EA was first obtained using the above-described method in [203] and using another method in [204].

We will now write the one-loop unique EA for the $d = 5$ Einstein gravity on the background $R_4 \times S_1$ as the expansion on four-dimensional curvature and its derivatives (with account of linear curvature terms)

$$\Gamma = \int d^4x \sqrt{g} \left\{ \frac{4\pi\rho\Lambda}{\kappa^2} - \frac{2\pi\rho R}{\kappa^2} - \frac{A(\lambda)}{\rho^4} - \frac{B(\lambda)}{\rho^2} R + \dots \right\}. \quad (10.65)$$

Here $A(\lambda)$ is given by (10.64)

$$A(\lambda) = \frac{5}{16\pi^2} d_2(-\tfrac{5}{3}\lambda) + \frac{5}{32\pi^2} d_2(0). \quad (10.66)$$

The function $B(\lambda)$ can be found in a similar way. However, its calculation is a little more complicated (for detail, see [204, 206])

$$B(\lambda) = -\frac{1}{192\pi^2}\Big[25d_1(-\tfrac{5}{3}\lambda) + 11d_1(0) - \frac{15}{(-\frac{5}{3}\lambda)}\Big(d_2(-\tfrac{5}{3}\lambda) - d_2(0)\Big)\Big]$$

$$d_1(y) = -d_2'(y) = \tfrac{4}{3}\pi y^{3/2} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi^2 n^3} + \frac{2y^{1/2}}{\pi n^2}\right)\exp[-2\pi n y^{1/2}]. \tag{10.67}$$

Now we can say that the gauge and parametrization invariant EA (10.65) is found in one-loop approximation. Let us rewrite conditions of spontaneous compactification (10.28) for EA (10.65) in the form

$$\frac{4\pi\rho\Lambda}{\kappa^2} = \frac{A(\lambda)}{\rho^2} \tag{10.68}$$

$$5A(\lambda) = 2\lambda\frac{\partial A(\lambda)}{\partial\lambda}. \tag{10.69}$$

Equation (10.69) can be obtained from second condition (10.28) after finding $4\pi\rho\Lambda/\kappa^2$ with the help of (10.68).

Using (10.68), (10.69) and (10.65) we obtain the following expression for four-dimensional gravitational constant κ_{ind}^2

$$\frac{\rho^2}{\kappa_{\text{ind}}^2} = B(\lambda) + \frac{A(\lambda)}{2\lambda}. \tag{10.70}$$

Numerical solution of (10.68) and (10.69) gives [204]: $\lambda \approx -0.1176$. Then $\rho^2/\kappa_{\text{ind}}^2 \approx -0.0035/16\pi$. Thus, the conditions of spontaneous compactification are fulfilled when $\rho^2 \approx -0.1176/\Lambda$ $(\Lambda < 0)$. However, a four-dimensional gravitational constant is induced with the wrong sign (antigravity). As a consequence self-consistent dimensional reduction is not physically acceptable (see, also [203–206].

Though $d = 5$ gravity is nothing but an interesting model example the question arises: is it possible to find the way to change the situation with an induced gravitational constant? Let us describe now some possible ways of doing this. Let us consider that the gravitational field is twisted (with antiperiodic boundary conditions on sphere S_1). Therefore the spectrum of $\Box$ on S_1 has the form: $-(l+1/2)^2/\rho^2$. Then the unique EA has the same structure as in (10.65), however, functions $d_1(y)$ and $d_2(y)$ change (they can be found by the method described in section 10.3)

$$d_2(y) = -\tfrac{8}{15}\pi y^{5/2} + \sum_{n=1}^{\infty} \exp(-4\pi n y^{1/2}) \times \left[\frac{3}{32\pi^4 n^5} + \frac{3y^{1/2}}{8\pi^3 n^4} + \frac{y}{2\pi^2 n^3}\right] - \sum_{n=1}^{\infty} \exp(-2\pi n y^{1/2}) \left[\frac{3}{2\pi^4 n^5} + \frac{3y^{1/2}}{\pi^3 n^4} + \frac{2y}{\pi^2 n^3}\right]$$

$$d_1(y) = -d_2'(y) = \tfrac{4}{3}\pi y^{3/2} + \sum_{n=1}^{\infty} \exp(-4\pi n y^{1/2})\left(\frac{1}{4\pi^2 n^3} + \frac{y^{1/2}}{\pi n^2}\right) - \sum_{n=1}^{\infty} \exp(-2\pi n y^{1/2})\left(\frac{1}{\pi^2 n^3} + \frac{2y^{1/2}}{\pi n^2}\right). \tag{10.71}$$

Numerical solution of equations (10.68) and (10.69) with functions (10.71) gives [370] : $\lambda \approx -0.1278$, $\rho^2/\kappa_{\text{ind}}^2 \approx -0.1328/16\pi$. Thus, self-consistent dimensional reduction is again physically unacceptable.

Another possibility lies in changing of initial action. Consider, for example, string-induced multidimensional quantum gravity

$$S = \int \mathrm{d}^d x \sqrt{g}\Big(- \frac{1}{\kappa^2}(R - 2\Lambda) + aG\Big)$$

where a is a dimensional parameter, G is the Gauss–Bonnet combination (in $d = 4$, G is topologically invariant). The unique EA in the theory with this action on the background $R_4^0 \times S_1$ coincides with EA (10.65). The induced gravitational constant in this theory is not calculated yet, it must depend not only upon Λ but, also upon a. The presence of an additional parameter indicates the possibility of the induced positive gravitational constant for some values of parameter a.

Now let us consider $d = 5$ gravity on $R_4^0 \times S_1$ at non-zero temperature. Then

$$\Gamma_1 = -\beta \int \mathrm{d}^4 x \sqrt{g}\left[\frac{A}{\rho^4} + \frac{B}{\rho^2} R + \ldots\right] \tag{10.72}$$

where $\beta = 1/kT$ and T is temperature. We can show that A and B are given by the same expressions as above. However, functions d_1 and d_2 are somewhat more complicated since they also depend on the temperature. Using the technique [319] we can show that choosing periodical boundary conditions for gravitational field we obtain

$$d_1(y,\beta) = \frac{1}{6\pi^2}\left((2\pi\sqrt{y})^3 + 3\sqrt{\frac{2}{\pi}}(2\pi\sqrt{y})^3 \sum_{n,m=-\infty}^{\infty}{}' \frac{K_{3/2}(z)}{z^{3/2}}\right)$$
$$d_2(y,\beta) = -\frac{1}{60\pi^4}\left((2\pi\sqrt{y})^5 - 15\sqrt{\frac{2}{\pi}}(2\pi\sqrt{y})^5 \sum_{n,m=-\infty}^{\infty}{}' \frac{K_{5/2}(z)}{z^{5/2}}\right) \tag{10.73}$$

where $z = 2\pi\sqrt{y}(n^2+(\beta m/2\pi\rho)^2)^{1/2}$, the prime means that the term with $m = n = 0$ is omitted, $K_a(z)$ is a modified Bessel function, d_1 and d_2 for antiperiodic boundary conditions are given in [370] (see also [200]).

We have analysed the conditions (10.68) and (10.69) numerically. Then these conditions can be solved at any temperature. However, in all cases a four-dimensional gravitational constant is induced with the wrong sign. This makes self-consistent dimensional reduction physically unacceptable.

Let us now find the unique EA in Einstein gravity on $R_4^0 \times T_{d-4}$. We will describe another method of calculation of the one-loop unique EA for this purpose. This method is due to Fradkin and Tseytlin [198] (see conditions of application of this method in [198]). According to this technique we have to do the following.

Firstly, it is necessary to find $\delta^2 S$. Secondly, it is necessary to add the Vilkovisky correction to $\delta^2 S$

$$S_V = -T^i_{mn}\frac{\delta S}{\delta g^i}h^m h^n. \tag{10.74}$$

For the theory under consideration

$$S_V = \frac{1}{4\kappa^2}\int \mathrm{d}^d x\sqrt{g}\frac{(d-4)}{(d-2)}\Lambda(h^2_{AB} - \tfrac{1}{2}h^2).$$

Thirdly, we use the α-parameter gauge for calculations. For the theory under consideration this is (10.9) with $\gamma = -1/2$. Finally, one must calculate the standard EA with the substitution $\delta^2 S \to \delta^2 S + S_V$ and take the limit $\alpha \to 0$. The reader can check that as a result for d-dimensional Einstein gravity on $R_4^0 \times T_{d-4}$ we obtain [205, 366]

$$V = \frac{2\Lambda}{\kappa^2}\prod_{i=1}^{d-4} 2\pi\rho_i + \frac{d}{2}\left(\frac{(d-1)}{2}F_{d-4}\Big(-\frac{d\Lambda}{d-2}\Big) - F_{d-4}(0)\right) \tag{10.75}$$

where $F_{d-4}(X)$ is given by (10.25) and (10.26). For $d = 5$, we have the same result (10.64).

Numerical analysis for V (10.75) as well as for Einstein gravity on $R_3^0 \times S_1$ [210] or $R_4^0 \times T_{d-4}$ at non-zero temperature [370] show that spontaneous compactification conditions can be fulfilled. However, the maximum of the effective potential is realized on solutions of spontaneous compactification conditions. It means that spontaneous compactification is not stable. Also, an analysis of the induced gravitational constant, which has not been calculated here, must be given.

Let us consider now the one-loop unique effective action in multidimensional R^2-gravity with action (10.29) on $R_4^0 \times T_{d-4}$. Making use of the same method as above we first find the metric γ_{mn} [205, 209]

$$\gamma_{mn} = \sqrt{g}(\delta^{AB,CD} - a g^{AB} g^{CD}) \tag{10.76}$$

where $a = (4\alpha_1 + \alpha_2)/4(\alpha_1 - \alpha_3)$, $a \neq 1/d$, $a \neq 1$ because of the conditions $\alpha_2 \neq -4\alpha_3$, $\alpha_1 \neq \alpha_3$. Using γ_{mn} from (10.76) we obtain

$$\begin{aligned} T^i_{mn} = \tfrac{1}{2}\Big[& \tfrac{1}{2} g^{AB}\delta^{CD}_{MN} + \tfrac{1}{2} g^{CD}\delta^{AB}_{MN} - \left(\delta^{A(C}_{MN} g^{D)B} + \delta^{B(C}_{MN} g^{D)A}\right) \\ & + \frac{1}{2(da-1)}\left(\delta^{AB,CD} g_{MN} - a g_{MN} g^{AB} g_{CD}\right)\Big]. \end{aligned} \tag{10.77}$$

Substituting (10.77) into (10.74) we obtain

$$\begin{aligned} S_V = \int \mathrm{d}^d x \sqrt{g}\frac{\Lambda}{K}\Big[& \tfrac{1}{4}\left(1 - \frac{da}{2(da-1)}\right) h^2 \\ & - \tfrac{1}{2}\left(1 - \frac{d}{4(da-1)}\right) h^2_{AB}\Big]. \end{aligned} \tag{10.78}$$

The Landau–De Witt gauge in the theory under consideration has the form

$$\begin{aligned} S_{\mathrm{GF}} = \beta_1 \int & \mathrm{d}^d x \sqrt{g}(\nabla_C h^C_A - a\nabla_A h)(g^{AB}\Box + \beta_2 \nabla^A \nabla^B) \\ & \times (\nabla_D h^D_B - a\nabla_B h) \end{aligned} \tag{10.79}$$

where β_1, β_2 are arbitrary and at the end of calculations $\beta_1 \to \infty$.

The one-loop convenient effective action taking S_V into account is (cf. with (10.33))

$$\begin{aligned} \Gamma = S + \tfrac{1}{2}\Big\{ & \mathrm{Sp}\ln\left[\frac{\delta^2}{\delta h_{AB} h_{CD}}(S + S_{\mathrm{GF}} + S_V)\right] \\ & - \mathrm{Sp}\ln\left(g^{AB}\Box + \beta_2 \nabla^A\nabla^B\right) \\ & - 2\,\mathrm{Sp}\ln\left(g^{AB}\Box + (1-2a)\nabla^A\nabla^B\right)\Big\}. \end{aligned} \tag{10.80}$$

The EA (10.80) is the gauge and parametrization independent effective action after taking the limit $\beta_1 \to \infty$.

The operators in (10.80) are non-minimal operators. To find these operators we use the method developed in section 10.3. The reader can repeat this exercise and get ($\beta_1 \to \infty$)

$$\Gamma = -\frac{\Lambda}{K}\prod_{i=1}^{d-4} 2\pi\rho_i \int \mathrm{d}^4x + \tfrac{1}{2}\Big\{\frac{d^2-d-2}{2}$$
$$\times \operatorname{Sp}\ln\left(\Box^2 + \frac{\Box}{K(\alpha_2+4\alpha_3)} + \frac{\Lambda d}{4K(da-1)(\alpha_2+4\alpha_3)}\right)$$
$$+ \operatorname{Sp}\ln\left(\Box^2 + \frac{(d-2)(1-a)\Box}{K(da-1)(\alpha_2+4\alpha_3)} + \frac{\Lambda d}{4K(da-1)(\alpha_2+4\alpha_3)}\right)$$
$$- d\operatorname{Sp}\ln\Box\Big\}. \tag{10.81}$$

It is interesting to note that the dependence on the gauge parameter β_2, which is arbitrary, disappears. This explicitly demonstrates the gauge independence of the unique EA. Note also that the unique EA (10.81) has been found in [209], and for $d = 5$ in [205, 207].

The unique EA has an especially simple form if the Einstein term in action (10.29) is absent. Then

$$\Gamma = -\frac{\Lambda}{K}\prod_{i=1}^{d-4} 2\pi\rho_i \int \mathrm{d}^4x + \tfrac{1}{2}\Big\{\frac{d^2-d}{2}$$
$$\times \operatorname{Sp}\ln\left(\Box^2 + \frac{\Lambda d}{4K(da-1)(\alpha_2+4\alpha_3)}\right) - d\operatorname{Sp}\ln\Box\Big\}. \tag{10.82}$$

It is interesting that for $\alpha_1, \alpha_2, \alpha_3 \to 0$ and $a = 1/2$ (this a corresponds to the metric γ_{mn} of Einstein gravity) this expression leads to the unique EA of Einstein gravity.

The final result for the unique effective potential is

$$V = -\frac{\Lambda}{K}\prod_{i=1}^{d-4} 2\pi\rho_i$$
$$+ \frac{d^2-d-2}{2}\left[F_{d-4}\left(\frac{-1-\sqrt{1-\Lambda K d(\alpha_2+4\alpha_3)/(da-1)}}{2K(\alpha_2+4\alpha_3)}\right)\right.$$
$$\left. + F_{d-4}\left(\frac{-1-\sqrt{1-\Lambda K d(\alpha_2+4\alpha_3)/(da-1)}}{2K(\alpha_2+4\alpha_3)}\right)\right] - dF_{d-4}(0)$$
$$+ F_{d-4}\left[-\frac{(d-2)(1-a)}{2K(da-1)(\alpha_2+4\alpha_3)}\right.$$

$$\times \left(1+\sqrt{1-\frac{\Lambda K d(da-1)(\alpha_2+4\alpha_3)}{(d-2)^2(1-a)^2}}\right)\Bigg]$$
$$+ F_{d-4}\Bigg[-\frac{(d-2)(1-a)}{2K(da-1)(\alpha_2+4\alpha_3)}$$
$$\times \left(1-\sqrt{1-\frac{\Lambda K d(da-1)(\alpha_2+4\alpha_3)}{(d-2)^2(1-a)^2}}\right)\Bigg] \tag{10.83}$$

The numerical analysis for $d = 5$ leads to the same conclusions as for Einstein gravity—instability of spontaneous compactification. (Note that the stability of the $R_4 \times S_1$ space in $d = 5$ quantum gravity has been discussed (using other methods) in [379, 380].)

Thus, we have discussed the gauge and parametrization independent effective action in multidimensional quantum gravity on a torus compactified background. The background under consideration is not a good ground state for the theory. That is why it is interesting to consider more realistic backgrounds (time-dependent spaces, Calabi–Yau manifolds, etc.) Unfortunately, attempts to calculate the unique EA on a curved background have not been successful because of major technical problems. In [210], for example, we tried to calculate the unique EA in Einstein gravity on $R_4^0 \times S_{d-4}$. All the principal questions could be resolved, but great technical difficulties appeared. Perhaps, the use of computers will make it possible to find solutions to these problems.

Another possibility is the application of gauge and parametrization independent effective action in (multidimensional) supergravity. Recently some attempts in this direction have been made (see [198, 213, 366]).

10.6 Two-loop effective action in Einstein gravity

In the previous sections we considered the one-loop effective action in multidimensional quantum gravity. The natural question is connected with the possibility of extending our consideration beyond the one-loop approximation. It is well-known that the investigation of the effective action in quantum gravity beyond the one-loop approximation is quite complicated. For example, the calculation of leading (double-pole) divergences of two-loop effective action in Einstein gravity has been made manually [381, 382]. However, all divergences of two-loop EA on the mass shell have been found in [383] with the help of a computer.

In this section we briefly describe the calculation of the two-loop effective action in multidimensional Einstein gravity on a torus compactified background. More details are given in [384, 385]. We note that we use the same notations in this section as in section 10.2, because we follow the notations of papers [384, 385]. Thus, the background under consideration is $M_N \times T_K$ where M_N is N-dimensional Minkowski space and the action is

$$S = -\frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{|g|}(R - 2\Lambda).$$

We shall write the functional integral for generating functional $W[J, g]$

$$\begin{aligned}\exp \mathrm{i}W[J,g] = \int & \mathrm{D}h_{AB}\, \mathrm{D}\bar{c}^D\, \mathrm{D}c^E \\ & \times \exp\left[\mathrm{i}\{S[g+\kappa h] + S_{\mathrm{GF}} + S_{\mathrm{gh}}\right. \\ & + \int \mathrm{d}^{N+K}x\, J^{AB}(g_{AB} + \kappa h_{AB})\}\end{aligned} \tag{10.84}$$

where

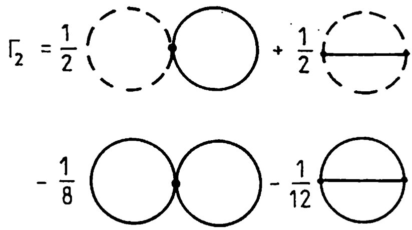
$$\begin{aligned}S_{\mathrm{GF}} &= \tfrac{1}{2}\int \mathrm{d}^{N+K}x\sqrt{|g|}\chi_A[g,h]g^{AB}\chi_B[g,h] \\ \chi_A[g,h] &= G_A^{BC}[g]h_{BC} \qquad G_A^{BC}[g] = \delta_A^{(B}\nabla^{C)} - \tfrac{1}{2}g^{BC}\nabla_A \\ S_{\mathrm{gh}} &= \int \mathrm{d}^{N+K}x\sqrt{|g|}\bar{c}^A G_A^{CE}[g]R_{CE}^B[g+\kappa h]c_B \\ R_{CE}^B[g] &= 2\delta_{(C}^B\nabla_{E)}.\end{aligned}$$

The effective action is defined as the Legendre transform of $W[J, g]$ (see section 2.3)

$$\Gamma = W[J,g] - \int \mathrm{d}^{N+K}x\, J^{AB}g_{AB}. \tag{10.85}$$

The two-loop correction Γ_2 to the EA is defined by terms proportional to κ^4 in the loop expansion of equations (10.84) and (10.85).

The corresponding diagrams are [2] (see section 2.10):



(10.86)

where full lines represent the graviton propagator $G_{(A_1B_1)(A_2B_2)}(x_1, x_2)$ and dashed lines represent the ghost propagator $G_A^B(x, x')$; the vertices are given in [385]. The graviton propagator is

$$G_{(A_1B_1)(A_2B_2)}(x_1, x_2) = \frac{1}{(2\pi)^{N+K}\rho_1 \times \ldots \times \rho_k} \times \sum_{n_1,\ldots,n_k=-\infty}^{\infty} \int \mathrm{d}^N p \exp[-\mathrm{i}p_A(x_1^A - x_2^B)] G_{(A_1B_1)(A_2B_2)}(p) \quad (10.87)$$

where

$$G_{(A_1B_1)(A_2B_2)}(p) = -\frac{4}{p^2 - 2\Lambda}\left(g_{A_1(A_2}g_{B_2)B_1} - \frac{1}{d-2} g_{A_1B_1}g_{A_2B_2}\right)$$
$$p_A = \{p_\alpha, n_a/\rho_a\} \qquad p^2 = p^A p_A \qquad d = N + K.$$

For the ghost propagator, $G_A^B(K) = \delta_A^B/k^2$.

The explicit calculation of diagrams (10.86) is very complicated, so we present here only the final result [385] — the two-loop correction to the effective action

$$\frac{\Gamma^{(2)}}{\int \mathrm{d}^N x} = \frac{\kappa^2}{(2\pi)^{2N+K}\rho_1 \times .. \times \rho_k} \times \sum_{\substack{n_1, \ldots, n_k \\ m_1, \ldots, m_k}=-\infty}^{\infty} \int \mathrm{d}^N p \mathrm{d}^N k \left\{ \frac{H(d)p^2 + 2\Lambda G(d)}{(p^2 - 2\Lambda)(k^2 - 2\Lambda)} \right.$$

$$+\frac{A_1(d)(k_Ap^A)^2+B_1(d)k^2p^2+[A_1(d)+B_1(d)]p^2(k^Ap_A)}{(p^2-2\Lambda)k^2(p+k)^2}$$
$$+\frac{A_2(d)(p_Ak^A)^2+B_2(d)p^2k^2+2[A_2(d)+B_2(d)]p^2(p_Ak^A)}{(p^2-2\Lambda)(k^2-2\Lambda)((p+k)^2-2\Lambda)}$$
$$\left.+\frac{2\Lambda D(d)(p_Ak^A)+4\Lambda^2E(d)}{(p^2-2\Lambda)(k^2-2\Lambda)((p+k)^2-2\Lambda)}+\frac{F(d)(p_Ak^A)}{(p^2-2\Lambda)k^2}\right\}. \tag{10.88}$$

Here $\lambda = 2/(d-2)$, $A_1(d) = \lambda(-\mathrm{d}^2/4 - d - 3) + \mathrm{d}^2 + 3d/2 - 4$, $B_1(d) = \lambda(-d+5) - 2d + 4$

$$\begin{aligned}
A_2(d) = -\tfrac{1}{27}\Big[&\lambda^3\Big(-\tfrac{3}{32}d^6+\tfrac{27}{16}d^5-\tfrac{123}{32}d^4+\tfrac{57}{2}d^3-\tfrac{231}{4}d^2+45d-\tfrac{27}{2}\Big)\\
&+\lambda^2\Big(\tfrac{9}{16}d^5-\tfrac{73}{8}d^4+\tfrac{993}{16}d^3-\tfrac{1345}{8}d^2+142d-45\Big)\\
&+\lambda\Big(-\tfrac{43}{16}d^4+\tfrac{261}{8}d^3-\tfrac{325}{4}d^2+213d-190\Big)\\
&+\tfrac{45}{8}d^3-\tfrac{59}{4}d^2+\tfrac{181}{2}d-99\Big]\\
B_2(d) = -\tfrac{1}{27}\Big[&\lambda^2\Big(\tfrac{1}{2}d^4-5d^3+\tfrac{229}{8}d^2-\tfrac{93}{2}d+\tfrac{81}{2}\Big)\\
&+\lambda\Big(-\tfrac{57}{16}d^3+\tfrac{273}{8}d^2-\tfrac{423}{4}d-\tfrac{207}{2}\Big)+\tfrac{77}{8}d^2-\tfrac{187}{4}d+55\Big]
\end{aligned}$$

$$\begin{aligned}
D(d) &= -\tfrac{1}{96}\lambda^3 d(d^5-15d^4+74d^3-244d^2+232d-96)\\
&\quad+\frac{\lambda^2}{16}(d^5-15d^4+74d^3-244d^2+232d-96)\\
&\quad-\frac{\lambda}{4}(3d^4-34d^3+56d^2-70d-24)\\
&\quad+\tfrac{1}{4}(5d^3-49d^2+76d-40)\\
E(d) &= \Big(-\frac{\lambda^3\mathrm{d}^2}{288}+\frac{\lambda^2 d}{64}\Big)(d^4-12d^3+52d^2-96d+64)\\
&\quad-\frac{\lambda d}{32}(7d^3-54d^2+24d+32)+\frac{d}{48}(21d^2-76d+64)\\
F(d) &= \lambda(-3d+2)+2d^2-2d-2\\
H(d) &= \frac{\lambda^2}{3}\Big(\tfrac{1}{16}d^4-\tfrac{9}{16}d^3+\tfrac{9}{4}d^2-\tfrac{15}{4}d+2\Big)\\
&\quad+\frac{\lambda}{3}\Big(\tfrac{3}{16}d^4-\tfrac{23}{16}d^3+5d^2-\tfrac{35}{4}d+5\Big)\\
&\quad+\tfrac{1}{3}\Big(\tfrac{1}{8}d^4-\tfrac{7}{8}d^3+\tfrac{13}{4}d^2-\tfrac{9}{2}d+2\Big)
\end{aligned}$$

$$G(d) = -\tfrac{1}{64}\Big[\lambda^2 d^2(d^2 - 12d - 4) + \lambda d(d^3 + 6d^2 + 88d - 40) + (d^4 - 8d^3 - 22d^2 + 4d)\Big]. \tag{10.89}$$

We see that the result has a very complicated form. Moreover, we have to calculate momentum integrals and infinite sums in (10.88). Bearing this in mind, we adopt the prescription $N \to N - 2\varepsilon$ (dimensional regularization). Then the integrals as well as the sums become regular.

First of all, we will consider the effective action (10.88) when $\Lambda = 0$. Then, we can change the order of integration and summation. We obtain the amazing result: $\Gamma^{(2)} = 0$ [384, 385]. As was mentioned in section 10.3 the one-loop effective action in Einstein gravity with zero cosmological constant is not zero! It is not obvious why the two-loop correction is *zero* when the one-loop correction is not *zero*.

When $\Lambda \neq 0$ we have calculated the momentum integrals [384, 385] and have analysed the structure of divergences of these integrals as well as the sums. For example, if d is even then there are double poles and if d is odd then the poles are simple. We have analysed the two-loop correction to the effective action in detail when $d = 5$ [385]. We have found that after renormalization of the five-dimensional cosmological constant it follows that (non-minimal substruction)

$$\Lambda = \Lambda_{\rm ren}\left[1 - \frac{4\kappa^4\Lambda_{\rm ren}}{(4\pi)^4}\Big(\tilde{D}(5) - \tfrac{1}{3}\tilde{C}(5)\Big)\Big(\frac{1}{2\varepsilon} + C\Big)\right] \tag{10.90}$$

where $\tilde{D}(5) = -\frac{3}{4}A_2(5) - \frac{1}{2}D(5) + E(5)$, $\tilde{C}(5) = \frac{3}{4}A_1(5) + \frac{1}{2}B_1(5)$, C is some constant [385] and the two-loop correction to effective action is finite (this correction is written in terms of $\Lambda_{\rm ren}$). We will not write the explicit expression for this correction here because it is very complicated [385]. We will just mention that spontaneous compactification on a two-loop level is unstable (see [385]) as well as on a one-loop level (see, section 10.3).

Thus, in this section we have shown the possibility of explicit calculation of two-loop effective action in multidimensional Einstein gravity on a torus compactified background. This calculation is also a necessary step in the investigation of the two-loop Vilkovisky–De Witt effective action in the same theory.

10.7 Vacuum energy in torus compactified strings

There are many common features between torus compactified strings and torus compactified gravity. Of course, there are also some differences. To clarify these points we will discuss in this section vacuum energy in torus compactified strings. We will find vacuum energy for bosonic orientable closed strings on the background $M_{25} \times T_1$, following [386]. We will not describe the situation in detail because there are now many good books on string theory (for example [120]) and this is not the subject of this book.

Let us consider bosonic closed string on background $M_{25} \times T_1$ with the corresponding boundary conditions

$$X(\tau, \sigma = \pi) = X(\tau, \sigma = 0) \bmod 2\pi\rho \tag{10.91}$$

where τ and σ are string coordinates. This boundary condition admits winding numbers $L = l\rho$, where l is an integer. Hence, the normal mode expansion is

$$X(\tau, \sigma) = x + 2\alpha' p\tau + 2L\sigma + \sqrt{2\alpha'}\frac{\mathrm{i}}{2}\sum_{n\neq 0}^{\infty}\frac{1}{n}\Big(\alpha_n \exp[-2\mathrm{i}n(\tau-\sigma)] + \tilde{\alpha}_n \exp[-2\mathrm{i}n(\tau+\sigma)]\Big). \tag{10.92}$$

Here $p = m/\rho$, m is an integer and p is the 26th component of momentum. This 26th component (as in torus compactified gravity) is discrete.

The appearence of winding numbers is a new phenomenon in comparison with Kaluza–Klein theories. We note that after initial papers [389, 390] (super)string vacuum energy and its properties at non-zero temperature were then the subject of investigation (see for example [391–406]). From papers [391–406] we know that a string at non-zero temperature is the same as a string compactified on a one-torus. However, winding numbers are absent in the temperature formulation of strings!

The orthonormal gauge conditions in the light-cone gauge can be integrated over to give the mass operator and constraint [386]

$$\begin{aligned} M^2 = \frac{\alpha'(\text{mass})^2}{2} &= N + \tilde{N} + 2\alpha'\Big(\frac{p}{2}\Big)^2 + \frac{1}{2\alpha'}L^2 \\ &= N + \tilde{N} - 2 + \frac{\alpha'}{2\rho^2}m^2 + \frac{\rho^2 l^2}{2\alpha'} \end{aligned} \tag{10.93}$$

$$N - \tilde{N} = pL = ml \tag{10.94}$$

where the oscillator sum $N, \tilde{N}$ contains both compactified and uncompactified dimensions

$$N = \sum_{\mu=1}^{24}\sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_n^{\mu} \qquad \tilde{N} = \sum_{\mu=1}^{24}\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_n^{\mu}. \tag{10.95}$$

To calculate vacuum energy we start with a sum of the vacuum energies of an infinitely great number of string modes

$$\begin{aligned} V &= -\frac{\mathrm{i}}{2}\,\mathrm{Sp}\ln\left[\frac{\alpha'}{2}P^2+M^2\right] \\ &= \frac{\mathrm{i}}{2}\,\mathrm{Sp}\left[\int_0^1 \mathrm{d}x\, x^{\frac{\alpha'}{2}P^2+M^2-1}\left(\ln\frac{1}{x}\right)^{-1}\right]. \end{aligned} \tag{10.96}$$

Here Sp represents the integral over 25-dimensional momentum P and the discrete sum over modes including discrete momenta and winding numbers. The constraint (10.94) can be implemented by

$$\delta_{N,\tilde{N}+ml} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\sigma'\exp[\mathrm{i}\sigma'(N-\tilde{N}-ml)]. \tag{10.97}$$

We can sum over oscillator modes N and $\tilde{N}$ with the help of

$$f(z) = \prod_{n=1}^{\infty}(1-z^n) = \left[\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}(z^n)^k\right]^{-1} \tag{10.98}$$

Integrating over 25-dimensional momentum, using (10.97) and (10.98) we obtain

$$\begin{aligned} V = &-\frac{1}{4\pi(2\pi\alpha')^{25/2}}\int_0^1 \mathrm{d}x\int_{-\pi}^{\pi}\mathrm{d}\sigma'\left(\ln\frac{1}{x}\right)^{-14} \\ &\times x^{-3}\left|f(x\mathrm{e}^{\mathrm{i}\sigma'})\right|\cdot F_1 \end{aligned} \tag{10.99}$$

where

$$F_1 = \left(\ln\frac{1}{x}\right)^{1/2}\sum_{m,l=-\infty}^{\infty}\mathrm{e}^{-i\sigma' ml}x^{(a^2m^2/2)+(l^2/2a^2)}$$

and $a = \frac{\sqrt{\alpha'}}{\rho}$. It is more convenient to examine the modular invariance if we change x and σ' to a complex variable τ

$$x\mathrm{e}^{\mathrm{i}\sigma'} = \mathrm{e}^{2\pi\mathrm{i}\tau} \qquad \tau = \tau_1 + \mathrm{i}\tau_2$$

$$V = -\frac{\pi}{(2\pi\alpha')^{25/2}}\int_{-1/2}^{1/2}\mathrm{d}\tau_1\int_0^{\infty}\mathrm{d}\tau_2(2\pi\tau_2)^{-14}\mathrm{e}^{4\pi\tau_2}\left|f(\mathrm{e}^{2\pi\mathrm{i}\tau})\right|^{-48}F_1 \tag{10.100}$$

where

$$F_1 = \sqrt{2\pi\tau_2} \sum_{m,l=-\infty}^{\infty} \exp\left\{\frac{i\pi\tau}{2}\left(ma - \frac{l}{a}\right)^2 - \frac{i\pi\tau^*}{2}\left(ma + \frac{l}{a}\right)^2\right\}$$
$$= \sqrt{2\pi\tau_2} \sum_{m=-\infty}^{\infty} \exp[-\pi\tau_2 a^2 m^2]\theta_3\left(m\tau_1\middle|\frac{i\tau_2}{a^2}\right)$$

and θ_3 is the Jacobi theta function.

Modular transformations $SL(2,Z)$ are given as

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1 \tag{10.101}$$

where a, b, c and d are integers. These transformations are generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The integrand in (10.100) is trivially invariant under $\tau \to \tau + 1$. The invariance under $\tau \to -1/\tau$ can be shown by the method of Fourier transform or Jacobi's imaginary transformation (see [386]). Thus, the vacuum energy (10.100) is invariant under modular transformations $SL(2,Z)$. The original integration region $\{-1/2 \leqslant \tau_1 \leqslant 1/2,\ \tau_2 > 0\}$ is divided into infinitely many equivalent regions which can be mapped onto each other by the modular transformations (10.101). To avoid multiple counting of the same string configurations we must try to restrict the integral to one of these equivalent regions, usually to the fundamental region $F = \{-1/2 \leqslant \tau_1 \leqslant 1/2,\ \tau_2 > \sqrt{1-\tau^2}\}$. The modular invariance of vacuum energy (see, for example, [386–388, 391–395]) is another characteristic of string theory which is absent in the Kaluza–Klein theory (see, however, [407]). Of course, we can present the string vacuum energy in the form of one integral which is not modular invariant (see, for example, [397–404]). However, modular invariance is a fundamental symmetry in string theory.

To find something stringy we set (as in [401]) $m = \pm 1$, $l = 0$ in (10.93) in the state with $N = \tilde{N} = 0$

$$M^2 = -2 + \frac{\alpha'}{2\rho^2}.$$

We see that tachyonic states are absent if $\rho^2 > 4\alpha'$. If $\rho^2 > 4\alpha'$ the integrand in (10.100) does not contain divergences. Otherwise, divergences appear.

Thus, the appearance of a critical value of radius $\rho_c = 2\sqrt{\alpha'}$ (the Hagedorn temperature in temperature formulation [389–406]) is one more characteristic of a torus compactified string in comparison with torus compactified multidimensional gravity. In temperature formulation the Hagedorn temperature is probably the critical temperature of first-order phase transitions [401].

In [386] a simple way of substracting the vacuum energy divergences (by hand) has been adopted

$$\left|f(\mathrm{e}^{2\pi i\tau})\right|^{-48} \to \left|f(\mathrm{e}^{2\pi i\tau})\right|^{-48} - 1.$$

It is easy to show that $\rho = \sqrt{\alpha'}$ is a stationary point (absolute minimum) of vacuum energy where the affine Kac–Moody algebra (for a review see [300]) for SU(2)×SU(2) is realized.

In the same way as above we can find the vacuum energy in superstring theory on a torus compactified background [387, 388, 393–395, 398, 401–406]. However, the vacuum energy for superstrings on a flat compactified background vanishes because of supersymmetry if periodic boundary conditions for all the fields are chosen. That is why we need antiperiodic boundary conditions for part of string states to have non-vanishing vacuum energy.

As an example we will consider a closed superstring on the background $M_4 \times T_6$. We will choose periodic boundary conditions for boson states and antiperiodic boundary conditions for fermion states along 5, 6th compactified dimension (along 7, ..., 10th dimensions the fermionic boundary conditions are periodic). As a result we obtain [365, 387, 395]:

$$V = -\frac{1}{128(\pi^2\alpha')^2}\int_F \frac{\mathrm{d}^2\tau}{\tau_2^5}\prod_{i=7}^{10}\frac{F_2(a_i,\tau)}{a_i}E(\tau). \tag{10.102}$$

Here

$$F_2(a_i,\tau) = (a_i^2\tau_2)^{1/2}\sum_{m,n=-\infty}^{\infty}\exp\left[-2\pi mn\tau_1 - \pi\tau_2\left(a_i^2m^2 + \frac{n^2}{a_i^2}\right)\right]$$

$$a_i^2 = \alpha'/\rho_i^2$$

$$
\begin{aligned}
E(\tau) = \frac{128}{a_5 a_6 \tau_2} \sum_{l=0,1/2} \Big\{ & [(-1)^{2l} \varepsilon_l(a_5,\tau)\varepsilon_l(a_6,\tau) \\
& + (-1)^{2l} O_l(a_5,\tau) O_l(a_6,\tau)] \left| \frac{\theta_2(0|\tau)}{\theta_1'(0|\tau)} \right|^8 \\
& + [\varepsilon_l(a_5,\tau) O_l(a_6,\tau) + \varepsilon_l(a_6,\tau) O_l(a_5,\tau)] \left| \frac{\theta_4(0|\tau)}{\theta_1'(0|\tau)} \right|^8 \\
& + [(-1)^{2l} \varepsilon_l(a_5,\tau) O_l(a_6,\tau) \\
& + (-1)^{2l} \varepsilon_l(a_6,\tau) O_l(a_5,\tau)] \left| \frac{\theta_3(0|\tau)}{\theta_1'(0|\tau)} \right|^8 \Big\}.
\end{aligned}
$$

The functions $\varepsilon_{0,1/2}(a,\tau)$ and $O_{0,1/2}(a,\tau)$ are defined by the same relation as $F_2(a_i,\tau)$, where for $\varepsilon(0)$ the sum is taken over the even (odd) winding quantum numbers n and the index $0(1/2)$ means that m is an integer (half-integer). The vacuum energy (10.102) is written in the modular invariant form. This is free of divergences if $\rho_i^2 > 2\alpha'$, where ρ_i is the smallest of ρ_5 and ρ_6. Thus, the critical value of radius also appears in superstring theory.

The problem which demands further investigation, in our opinion, is the study of the quantum properties of superstrings on a curved compactified background.

11 Quantum Properties of Torus Compactified Membranes

11.1 Introduction

The theory of p-branes, i.e. of p-dimensional extended objects, has attracted a lot of attention over the last few years. (The reader is referred to the review articles [411–413]). Such theories provide a natural generalization of the physics of string theories ($p = 1$). Among p-brane theories the supermembrane ($p = 2$) in eleven dimensions [414] has attracted the most interest since, like the superstring in ten dimensions, it is potentially consistent at the quantum level. There are a finite number of values of p and D for which it is classically possible to construct a space–time supersymmetric action which generalizes the Green–Schwarz superstring action to describe a D-dimensional extended object moving in a p-dimensional space–time [415, 458]. However, for values other than ($p = 1$, $D = 10$) and ($p = 2$, $D = 11$) such theories have incurable quantum anomalies [416–418].

The status of supermembranes with regard to the problem of the unification of all fundamental forces is still very unclear. Of fundamental importance is the question of the existence of massless states in the effective low-energy theory. Recent results [419–425] indicate that a spectrum of states with stable zero modes and a mass gap does occur for $D = 11$ supermembranes if the physical space–time is anti-de Sitter [425] but not if it is Minkowski space. Although such results may appear discouraging it is nonetheless clear that a study of the theories of extended objects will greatly add to our understanding of the possible field theories offered by nature.

The theory of relativistic membranes introduced by Dirac [426] has been developed on a classical level in [427–433] where some alternative forms of classical membrane action [427–433] as well as supersymmetric extensions [414, 434, 437] have been developed. (It is interesting to note that dimensional reduction of membrane theory [438, 439] leads to strings.)

In [411–413, 416–425, 438–458] the quantum aspects of (super)membranes have been investigated. In this chapter we are mostly interested in torus compactified quantum membranes, i.e., quantum membranes on a flat torus compactified background [411, 440, 441, 445, 447, 454, 455, 457, 458] (for a discussion of the membrane on a curved background, see [460–462]). Owing to the intrinsically non-linear structure of membranes, we adopt a semiclassical approach in section 11.2 following [440].

11.2 Semiclassical quantization of torus compactified supermembranes

Let us start with the action of a closed supermembrane [414]

$$S = -\frac{1}{4\pi^2}\int \mathrm{d}^3\xi \left[\tfrac{1}{2}\sqrt{-g}g^{ij}\Pi_i^\mu\Pi_{j\mu} - \tfrac{1}{2}\sqrt{-g} + \varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA}\right] \tag{11.1}$$

where $\Pi_i^A = (\Pi_i^\mu,\ \Pi_i^\alpha)$, $\Pi_i^\mu = \partial_i X^\mu - \mathrm{i}\bar{\psi}\Gamma^\mu\partial_i\psi$, $\Pi_i^\alpha = \partial_i\psi^\alpha$, ψ^α is a 32-component Majorana spinor (for a modern discussion of spinors, see book [463]), $\mu = 0,1,\ldots,10$, $\xi^i = (\tau,\sigma,\rho)$ $(i = 0,1,2)$ are the coordinates of world-volume with metric g_{ij}, B_{CBA} is the super-3-form with $dB = H$ (the only non-vanishing component of H is $H_{\mu\nu\alpha\beta} = -\frac{1}{3}((C\Gamma_{\mu\nu})_{\alpha\beta})$. We follow the conventions of Duff *et al* [440]: $\varepsilon^{012} = -1$, $\{\Gamma^\mu,\Gamma^\nu\} = -2\eta^{\mu\nu}$, $\eta^{\mu\nu} = \mathrm{diag}(-1,1,\ldots,1)$ and $\bar{\psi}_\alpha = -\psi^\beta C_{\beta\alpha}$ with antisymmetric charge conjugation matrix $C_{\alpha\beta}$. For $\psi^\alpha = 0$, $B_{CBA} = 0$ we have the action of the bosonic membrane.

The action (11.1) is invariant under world-volume general coordinate transformations, and under Poincaré transformations in the d-dimensional Minkowski space–time. Besides, the action is invariant under space–time supersymmetry transformations

$$\delta\psi = \varepsilon \qquad \delta X^\mu = \mathrm{i}\bar{\varepsilon}\Gamma^\mu\psi \tag{11.2}$$

and local Siegel transformations [414, 464]

$$\delta\psi = (1+\Gamma)k \qquad \delta X^\mu = \mathrm{i}\bar{\psi}\Gamma^\mu\delta\psi \tag{11.3}$$

where $\Gamma = (\mathrm{i}/6\sqrt{-g})\varepsilon^{ijk}\Pi_i^\mu\Pi_j^\nu\Pi_k^\rho\Gamma_{\mu\nu\rho}$, k is a world-volume scalar, space–time Majorana spinor function of τ, σ and ρ.

Working in 1.5 order formalism [438] and using the algebraic equation of motion for g_{ij}

$$g_{ij} = \Pi_i^\mu\Pi_{j\mu} \tag{11.4}$$

we can obtain (with the help of the solution for B_{CBA} in (11.1)) the classical equations of motion:

$$\partial_i(\sqrt{-g}g^{ij}\Pi^\mu_j) + \varepsilon^{ijk}\Pi^\nu_i\partial_j\bar{\psi}\Gamma_\nu{}^\mu\partial_k\psi = 0 \tag{11.5}$$

$$(1-\Gamma)g^{ij}\Pi^\mu_i\Gamma_\mu\partial_j\psi = 0. \tag{11.6}$$

Making use of (11.4) leads to the condition $\Gamma^2 = 1$, and here $\frac{1}{2}(1\pm\Gamma)$ is the projection operator. Therefore, equation (11.6) is the equation of motion for 16 of the 32 components of ψ (because of Siegel invariance of the action). To fix 16 unphysical components of ψ we can use the gauge [440]

$$\Gamma\psi = -\psi. \tag{11.7}$$

Now let us begin the semiclassical quantization of supermembranes on the background $M_9 \times T_2$ [440] (for a discussion of quantization on the $M_4 \times T_7$ background see [411]). The classical solution corresponding to this background can be written in the form

$$X^1 = l_1\rho_1\sigma \qquad X^2 = l_2\rho_2\rho \qquad X^I = 0 \qquad I = 3,\ldots,9 \tag{11.8}$$

$$\psi = 0. \tag{11.9}$$

Here $0 \leqslant \sigma \leqslant 2\pi$, $0 \leqslant \rho \leqslant 2\pi$, ρ_1 and ρ_2 are the radii of torus, l_1 and l_2 are winding numbers (compare with section 10.7). From papers [440, 441] where the quantization of bosonic membrane has been discussed we know that the three general coordinate invariances of membrane can be fixed by a string-inspired light-cone gauge

$$X^+ = p^+\tau \tag{11.10}$$

$$g_{0a} = 0 \qquad (a = 1,2) \tag{11.11}$$

where $X^\pm = (X^0 \pm X^{10})/\sqrt{2}$. In the case of the string, unlike the membrane, this gauge choice turns the non-linear system of equations into a linear system. Of course, the non-linear structure of a membrane is much more complicated.

After fixing the gauge freedoms that are functions of (τ, σ, ρ) coordinates with the help of (11.10) and (11.11) there remains a residual τ-independent reparametrization invariance of (11.10) and (11.11)

$$\delta\xi^0 = 0 \qquad \delta\xi^a = f^a(\sigma,\rho). \tag{11.12}$$

We can use one of the gauge freedoms (11.12) to impose the gauge condition [443]

$$g_{00} = -\gamma \qquad \gamma = \det\gamma_{ab} \qquad \gamma_{ab} = g_{ab} \tag{11.13}$$

at a fixed time τ^*. Really, this equation is valid for all τ [440]. However, we have one more reparametrization freedom in (11.12) that leaves gauge condition (11.13) invariant [443]

$$\delta\xi^0 = 0 \qquad \delta\xi^a = \varepsilon^{ab}\partial_b f(\sigma, \rho) \tag{11.14}$$

where $\varepsilon^{12} = -\varepsilon^{21} = 1$. This gauge freedom will be fixed later.

In the gauge (11.10)–(11.13) the world-volume metric and X^- are

$$g_{ij} = \mathrm{diag}\Big(-(l_1 l_2 \rho_1 \rho_2)^2, (l_1\rho_1)^2, (l_2\rho_2)^2\Big) \tag{11.15}$$

$$X^- = \frac{1}{2p^+}(l_1 l_2 \rho_1 \rho_2)^2 \tau \tag{11.16}$$

and the classical (mass)^2 of the solution, which is given by $-P^\mu P_\mu$, where $P^\mu = \int \mathrm{d}\sigma \mathrm{d}\rho K^\mu$ and $K^\mu = \partial L/\partial \dot{X}^\mu$ is the canonical momentum, is

$$(\mathrm{mass})^2 = (l_1 l_2 \rho_1 \rho_2)^2. \tag{11.17}$$

As in paper [440] we shall set $l_1 = l_2 = \rho_1 = \rho_2 = 1$ for the rest of the calculations. We will restore them in the final results.

Let us consider now the fluctuations around the classical solution. We write $\underline{X} = \underline{X}_{\mathrm{class}} + \underline{Z}$

$$X^1 = \sigma + Z^1 \qquad X^2 = \rho + Z^2 \qquad X^I = Z^I \tag{11.18}$$

where $\underline{Z}$ are the fluctuations, and fermions are pure fluctuations ψ. Substituting these expressions into (11.5) and (11.6), using the gauges (11.7), (11.10), (11.11) and (11.13) and keeping the terms of linear order in fluctuations, we obtain [440]

$$\ddot{Z}^1 = \partial_\sigma\partial_\sigma Z^1 + \partial_\sigma\partial_\rho Z^2 \tag{11.19}$$

$$\ddot{Z}^2 = \partial_\rho\partial_\rho Z^2 + \partial_\sigma\partial_\rho Z^1 \tag{11.20}$$

$$\ddot{Z}^I = \partial_\sigma\partial_\sigma Z^I + \partial_\rho\partial_\rho Z^I \tag{11.21}$$

$$\dot{\psi} = \mathrm{i}\Gamma_2\partial_\sigma\psi - \mathrm{i}\Gamma_1\partial_\rho\psi. \tag{11.22}$$

Now it is useful to fix the remaining gauge invariance (11.14). The gauge $g_{0a} = 0$ gives an equation which can be solved for $\partial_a X^-$. After linearization and integration this equation becomes [440]

$$\partial_\rho Z^1 = \partial_\sigma Z^2 + h(\sigma, \rho) \tag{11.23}$$

with arbitrary $h(\sigma, \rho)$. For fixing gauge symmetry (11.14) one can set $h(\sigma, \rho)$ to zero for all τ [440]. Then, we can rewrite (11.19) and (11.20) as

$$\ddot{Z}^1 = \partial_\sigma\partial_\sigma Z^1 + \partial_\rho\partial_\rho Z^1 \tag{11.24}$$

$$\ddot{Z}^2 = \partial_\sigma \partial_\sigma Z^2 + \partial_\rho \partial_\rho Z^2. \tag{11.25}$$

Using the representation of 11-dimensional Γ-matrices given in [440] we can express ψ in the gauge (11.7) as [440]

$$\psi = (16\sqrt{2}p^+)^{-1/2} \begin{pmatrix} \chi \\ -\mathrm{i}\chi^* \\ -\sqrt{2}p^+\chi \\ -\mathrm{i}\sqrt{2}p^+\chi^* \end{pmatrix} \tag{11.26}$$

where χ is a complex 8-component spinor of SO(7)×U(1). We can rewrite (11.22) with the help of (11.26) as

$$\dot{\chi} = \partial_\sigma \chi^* - \mathrm{i}\partial_\rho \chi^*. \tag{11.27}$$

Substituting (11.18) and (11.26) into (11.11) and (11.13) we can find the first derivatives of X^- on τ, ρ, and σ to quadratic order in the fluctuations (see [440] for explicit expressions).

The general solutions of (11.21), (11.24), (11.25) and (11.27) can be found as [440]:

$$\begin{aligned} \underline{Z} &= \underline{z}_0 + \underline{p}\tau + \frac{1}{\sqrt{2}} \sum_{m^2+n^2\neq 0} \frac{1}{\omega_{mn}} \mathrm{e}^{\mathrm{i}(m\sigma+n\rho)} \\ &\quad \times \left[\underline{\alpha}^+_{mn} \mathrm{e}^{\mathrm{i}\omega_{mn}\tau} + \underline{\alpha}_{-m-n} \mathrm{e}^{-\mathrm{i}\omega_{mn}\tau}\right] \end{aligned} \tag{11.28}$$

$$\begin{aligned} \chi &= \sqrt{2}S_{00} + \sum_{m^2+n^2\neq 0} \left[\frac{m-\mathrm{i}n}{\omega_{mn}} S^+_{mn} \mathrm{e}^{\mathrm{i}\omega_{mn}\tau}\right. \\ &\quad \left. + S_{-m-n} \mathrm{e}^{-\mathrm{i}\omega_{mn}\tau}\right] \end{aligned} \tag{11.29}$$

with $\omega_{mn} = (m^2+n^2)^{1/2}$ (compare with string case (10.92)). It is necessary to distinguish between the unconstrained variables Z^I, $I = 3, \ldots, 9$ for which the commutators are

$$[\dot{Z}^I, Z^J] = -(2\pi)^2 \mathrm{i}\delta^{IJ} \delta(\sigma-\sigma')\delta(\rho-\rho') \tag{11.30}$$

and the constrained variables Z^1 and Z^2. For constrained variables Dirac formalism gives the commutators as (see [440])

$$\begin{aligned} [\dot{Z}^1(\sigma,\rho), Z^1(\sigma',\rho') &= -(2\pi)^2 \mathrm{i}\Big(1 - \frac{\partial^2_\rho}{\nabla^2}\Big)\delta(\sigma-\sigma')\delta(\rho-\rho') \\ [\dot{Z}^2(\sigma,\rho), Z^2(\sigma',\rho')] &= -(2\pi)^2 \mathrm{i}\Big(1 - \frac{\partial^2_\sigma}{\nabla^2}\Big)\delta(\sigma-\sigma')\delta(\rho-\rho') \end{aligned} \tag{11.31}$$

where $\nabla^2 = \partial^2_\sigma + \partial^2_\rho$.

For the unconstrained spinor χ the canonical quantization gives

$$\{\chi^{*A}, \chi^B\} = 2(2\pi)^2 \delta^{AB} \delta(\sigma - \sigma')\delta(\rho - \rho'). \tag{11.32}$$

Here $A, B = 1, \ldots, 8$ are SO(7) spinor indices.

Using (11.30), (11.31) and (11.32) and mode expansion (11.28), (11.29) we can obtain the commutation relations for α and S oscillators [440]

$$\begin{aligned}
[\alpha^1_{mn}, \alpha^{1+}_{m'n'}] &= \frac{m^2}{\omega_{mn}} \delta_{mm'} \delta_{nn'} \\
[\alpha^2_{mn}, \alpha^{2+}_{m'n'}] &= \frac{n^2}{\omega_{mn}} \delta_{mm'} \delta_{nn'} \\
[\alpha^I_{mn}, \alpha^{J+}_{m'n'}] &= \omega_{mn} \delta^{IJ} \delta_{mm'} \delta_{nn'} \\
\{S^A_{mn}, S^{B+}_{m'n'}\} &= \delta^{AB} \delta_{mm'} \delta_{nn'}
\end{aligned} \tag{11.33}$$

$$\begin{aligned}
[p^1, Z^1_0] = [p^2, Z^2_0] &= -\mathrm{i} \qquad [p^I, Z^J_0] = -\mathrm{i}\delta^{IJ} \\
\{S^A_{00}, S^{B+}_{00}\} &= \delta^{AB}.
\end{aligned} \tag{11.34}$$

All other independent commutators vanish. Thus, α^+_{mn}, S^+_{mn} are creation operators and α_{mn}, S_{mn} are annihilation operators.

Now we can get the eleven-dimensional mass formula in the form [440]

$$M^2 = (l_1 l_2 \rho_1 \rho_2)^2 + 2 \sum_{m^2+n^2 \neq 0} \left(\underline{\alpha}^+_{mn} \cdot \underline{\alpha}_{mn} + \omega_{mn} S^{A+}_{mn} S^A_{mn} \right). \tag{11.35}$$

Here the dependence on l_1, l_2, ρ_1, ρ_2 is restored, now $\omega_{mn} = [(ml_2\rho_2)^2 + (nl_1\rho_1)^2]^{1/2}$. As was mentioned in [440] this mass formula is correct without the Casimir energy term. (The bosonic contribution is cancelled by an equal but opposite fermionic contribution.)

Substituting (11.28) and (11.29) into expressions for $\partial_\sigma X^-$ and $\partial_\rho X^-$ [440] we can obtain two constraints

$$l_1 \rho_1 p^1 + N^{(1)}_B + N^{(1)}_F = 0 \tag{11.36}$$

$$l_2 \rho_2 p^2 + N^{(2)}_B + N^{(2)}_F = 0 \tag{11.37}$$

where

$$\begin{aligned}
N^{(1)}_B &= \sum_{m^2+n^2 \neq 0} \frac{m}{\omega_{mn}} \underline{\alpha}^+_{mn} \cdot \underline{\alpha}_{mn} \\
N^{(2)}_B &= \sum_{m^2+n^2 \neq 0} \frac{n}{\omega_{mn}} \underline{\alpha}^+_{mn} \cdot \underline{\alpha}_{-mn} \\
N^{(1)}_F &= \sum_{m^2+n^2 \neq 0} m S^{A+}_{mn} S^A_{mn} \\
N^{(2)}_F &= \sum_{m^2+n^2 \neq 0} n S^{A+}_{mn} S^A_{mn}
\end{aligned} \tag{11.38}$$

p^1 and p^2 are discrete. Thus, we have obtained the mass operator and two constraints for the supermembrane. We see that in this case the number of constraints is one more than in the string case (see section 10.7). Now we can proceed to investigate the vacuum energy of (super)membranes and its comparison with string vacuum energy.

11.3 Vacuum energy of torus compactified membrane and modular invariance

Let us investigate the vacuum energy in a torus compactified membrane and compare the results with the results obtained for strings. For simplicity, we start from a closed bosonic membrane on the background $M_{D-2} \times T_2$. In this case we can obtain the mass operator and two constraints with the help of the technique which led to (11.35), (11.36) and (11.37).

We will present these expressions in the following form (see [441, 454, 455, 457])

$$M^2 = (l_1 l_2 \rho_1 \rho_2)^2 + \sum_{j=1}^{D-3} \sum_{n_1,n_2=-\infty}^{\infty}{}' \omega_{n_1 n_2} (N_{n_1 n_2 j} + \tilde{N}_{n_1 n_2 j} + 1) \tag{11.39}$$

$$l_1 k_1 + \sum_{j=1}^{D-3} \sum_{n_1,n_2=-\infty}^{\infty}{}' n_1 (N_{n_1 n_2 j} - \tilde{N}_{n_1 n_2 j}) = 0 \tag{11.40}$$

$$l_2 k_2 + \sum_{j=1}^{D-3} \sum_{n_1,n_2=-\infty}^{\infty}{}' n_2 (N_{n_1 n_2 j} - \tilde{N}_{n_1 n_2 j}) = 0. \tag{11.41}$$

Here

$$\omega_{n_1 n_2} = [(n_1 l_1 \rho_1)^2 + (n_2 l_2 \rho_2)^2]^{1/2}$$

and we introduced the particle operators $N_{n_1 n_2 j}$, $\tilde{N}_{n_1 n_2 j}$ as in the string case (see [440, 457]).

Let us calculate vacuum energy (effective potential) in the same way as for a string (section 10.7)

$$V = -\frac{\mathrm{i}}{2} \operatorname{Sp} \ln \left(\frac{P^2}{2} + M^2 \right) = \frac{\mathrm{i}}{2} \operatorname{Sp} \int_0^1 \mathrm{d}x \, x^{(P^2/2)+M^2-1} \left(\ln \frac{1}{x} \right)^{-1}. \tag{11.42}$$

Here Sp represents the integration over D-dimensional momentum and P^2 is the square of the D-dimensional momentum.

After integration on $D-2$-dimensional momentum we obtain

$$V = -\frac{1}{2(2\pi)^{(D-2)/2}4\pi^2\rho_1\rho_2}\,\mathrm{Sp}\int_0^1 \mathrm{d}x$$
$$\times\, x^{\left[\frac{1}{2}\left(\frac{k_1^2}{\rho_1^2}+\frac{k_2^2}{\rho_2^2}\right)-1+(l_1l_2\rho_1\rho_2)^2+\sum_{j=1}^{D-3}\sum_{n_1,n_2=-\infty}^{\infty\,\prime}\omega_{n_1n_2}(N_{n_1n_2j}+\tilde{N}_{n_1n_2j}+1)\right]}$$
$$\times\left(\ln\frac{1}{x}\right)^{-1-(D-2)/2}. \qquad (11.43)$$

The prime means that the term with $n_1 = n_2 = 0$ is absent in the sum. Making use of the indentities

$$1 = \int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\exp\left\{\mathrm{i}\sigma\right.$$
$$\left.\times\left[l_1k_1 + \sum_{j=1}^{D-3}\sum_{n_1,n_2=-\infty}^{\infty}{}' n_1\left(N_{n_1n_2j}-\tilde{N}_{n_1n_2j}\right)\right]\right\} \qquad (11.44)$$

$$1 = \int_{-\pi}^{\pi}\frac{\mathrm{d}\tau_1}{2\pi}\exp\left\{\mathrm{i}\tau_1\right.$$
$$\left.\times\left[l_2k_2 + \sum_{j=1}^{D-3}\sum_{n_1,n_2=-\infty}^{\infty}{}' n_2\left(N_{n_1n_2j}-\tilde{N}_{n_1n_2j}\right)\right]\right\} \qquad (11.45)$$

in (11.43) we obtain

$$V = -\frac{1}{2(2\pi)^{(D-2)/2}4\pi^2\rho_1\rho_2}\sum_{k_1,k_2,l_1,l_2=-\infty}^{\infty}\mathrm{Sp}\int_0^1\mathrm{d}x\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma\mathrm{d}\tau_1}{(2\pi)^2}$$
$$\times\left(\ln\frac{1}{x}\right)^{-D/2} x^{\left[\frac{1}{2}\left(\frac{k_1^2}{\rho_1^2}+\frac{k_2^2}{\rho_2^2}\right)-1+(l_1l_2\rho_1\rho_2)^2+(D-3)\sum_{n_1,n_2=-\infty}^{\infty\,\prime}\omega_{n_1n_2}\right]}$$
$$\exp\left\{\mathrm{i}\sigma l_1k_1+\mathrm{i}\tau_1l_2k_2\right.$$
$$+\sum_{j=1}^{D-3}\sum_{n_1,n_2=-\infty}^{\infty}{}'\left[N_{n_1n_2j}(\omega_{n_1n_2}\ln x+\mathrm{i}\sigma n_1+\mathrm{i}\tau_1n_2)\right.$$
$$\left.\left.+\tilde{N}_{n_1n_2j}(\omega_{n_1n_2}\ln x-\mathrm{i}\sigma n_1-\mathrm{i}\tau_1n_2)\right]\right\}. \qquad (11.46)$$

Calculating Sp in (11.46) we obtain

$$V = -\frac{1}{2(2\pi)^{(D-2)/2}4\pi^2\rho_1\rho_2}\sum_{k_1,k_2,l_1,l_2=-\infty}^{\infty}\int_0^1\mathrm{d}x\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma\mathrm{d}\tau_1}{(2\pi)^2}$$

$$\times \left(\ln \frac{1}{x}\right)^{-D/2} x^{\left[\frac{1}{2}\left(\frac{k_1^2}{\rho_1^2}+\frac{k_2^2}{\rho_2^2}\right)-1+(l_1 l_2 \rho_1 \rho_2)^2+(D-3)\sum_{n_1,n_2=-\infty}^{\infty\prime} \omega_{n_1 n_2}\right]}$$
$$\exp\left\{i\sigma l_1 k_1 + i\tau_1 l_2 k_2\right\}$$
$$\times \prod_{n_1,n_2=-\infty}^{\infty}{}' \left|1 - x^{\omega_{n_1 n_2}} e^{i(\sigma n_1 + \tau_1 n_2)}\right|^{-2(D-3)}. \tag{11.47}$$

Let us change the variables $xe^{i\sigma} = e^{2\pi i t} \equiv e^{2\pi i(t_1 + it_2)}$ and $\tau = 2\pi s$. Then we can get [457]

$$V = -\frac{1}{2(2\pi)^D \rho_1 \rho_2} \sum_{l_1,l_2=-\infty}^{\infty} \int_E d^2 t \int_{-1/2}^{1/2} ds (\operatorname{Im} t)^{-D/2}$$
$$\times A \exp\left(2\pi \operatorname{Im} t[(D-3)E_1 + (l_1 l_2 \rho_1 \rho_2)^2]\right) \theta_3\left(l_1 \operatorname{Re} t \middle| \frac{i \operatorname{Im} t}{\rho_1^2}\right)$$
$$\times \theta_3\left(s l_2 \middle| \frac{i \operatorname{Im} t}{\rho_2^2}\right) \tag{11.48}$$

Here E: $\left\{0 \leqslant t_2 < \infty,\ -\frac{1}{2} \leqslant t_1 \leqslant \frac{1}{2}\right\}$,

$$E_1 = \sum_{n_1,n_2}{}' \omega_{n_1 n_2} = Z_2 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \left(\tfrac{1}{2}\right)_\varphi = Z_2 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \left(\tfrac{3}{2}\right)_\varphi.$$
$$A = \prod_{n_1,n_2=-\infty}^{\infty} \left|1 - \exp[-2\pi t_2 \omega_{n_1 n_2} + 2\pi i(n_1 \operatorname{Re} t + s n_2)]\right|^{-2(D-3)}.$$

It is easy to study the asymptotic behaviour of vacuum energy (11.48). When $t_2 \to \infty$ the behaviour of integrand (11.42) is

$$t_2^{-D/2} \exp\{-2\pi t_2 (D-3) E_1 - 2\pi t_2 (l_1 l_2 \rho_1 \rho_2)^2\}.$$

If we change $\sum_{l_1 l_2} \int \to \int \sum_{l_1 l_2}$ then integrand (11.48) is regular when $t_2 \to \infty$. In the ultraviolet limit ($t_2 \to 0$) the integrand behaves as

$$t_2^{-D/2} \sum_{l_1,l_2=-\infty}^{\infty} \exp\left[2(l_1 l_2 \rho_1 \rho_2)^2 \left(-\pi t_2 + \frac{(D-3)\zeta(3)}{\pi t_2^2}\right)\right.$$
$$\left. - \frac{\pi}{t_2}\left(t_1^2 \rho_1^2 l_1^2 + s^2 \rho_2^2 l_2^2\right)\right]. \tag{11.49}$$

Thus, the integrand diverges when $t_2 \to 0$. As in [454, 455] one can introduce the cut-off parameter $\varepsilon(0 < \varepsilon < \infty)$ for regularization. Then the Hagedorn temperature for membranes depends on this parameter [454, 455] (cf. section 10.7).

Now we will discuss the properties of the vacuum energy (11.48). It is evident that the integrand in (11.48) is invariant under $t \to t+1$ (group of discrete translations U consisting of the matrices of type

$$U \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where $b \in Z$). The group U is a congruent (Borel) subgroup of modular group SL$(2, Z)$. Then we can use the theorem proved in the appendix of [393] and rewrite the vacuum energy in modular invariant form [457]. Using the appendix of [393] we easily get the explicit SL$(2, Z)$ invariant vacuum energy

$$\begin{aligned} V = &-\frac{1}{2(2\pi)^D \rho_1 \rho_2} \sum_{l_1, l_2 = -\infty}^{\infty} \sum_{(c,d)=1} \int_F d^2\tau \int_{-1/2}^{1/2} ds (\operatorname{Im} \gamma_{cd}\tau)^{-D/2} \\ &\times \exp\left\{ -2\pi \operatorname{Im} \gamma_{cd}\tau \left[(D-3)E_1 + (l_1 l_2 \rho_1 \rho_2)^2 \right] \right\} \\ &\times \theta_3\left(l_1 \operatorname{Re} \gamma_{cd}\tau \middle| \frac{\mathrm{i}}{\rho_1^2} \operatorname{Im} \gamma_{cd}\tau \right) \theta_3\left(s l_2 \middle| \frac{\mathrm{i}}{\rho_2^2} \operatorname{Im} \gamma_{cd}\tau \right) \\ &\times \prod_{n_1, n = -\infty}^{\infty}{}' \Big| 1 - \exp[-2\pi \operatorname{Im} \gamma_{cd}\tau \omega_{n_1 n_2} \\ &+ 2\pi \mathrm{i}(n_1 \operatorname{Re} \gamma_{cd}\tau + s n_2)] \Big|^{-2(D-3)} . \end{aligned} \tag{11.50}$$

Here F is a fundamental region, $\gamma_{cd} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, the * symbol meaning that any transformation in SL$(2, Z)$ with the bottom row (c, d) can be used as representative of a coset. In the same way we can show that the vacuum energy of bosonic p-branes $(i = 0, \dots, p-1)$ on the background $M_{D-2} \times T_2$ can be presented in the form invariant under $G = [\mathrm{SL}(2, Z)]^{n+1}$ if $p = 2n + 1$ or $G = [\mathrm{SL}(2, Z)]^{n+1} \otimes U$ if $p = 2n + 2$. It is evident that the same results can be obtained for a membrane (or p-brane) on the background M_D. For example, the vacuum energy for a closed bosonic membrane on M_D can be presented in SL$(2, Z)$ invariant form.

These results can be generalized for a $D = 11$ supermembrane. The vacuum energy on the background $M_K \times T_{D-K}$ is equal to zero because of supersymmetry. That is why one has to consider the twisted fields. As an example we can find the vacuum energy for a $D = 11$ supermembrane on the background $S_1(\text{temperature}) \times R_8^0 \times T_2$ (see [455]). In this case the mass operator and constraints are the same (11.35), (11.36), (11.37) as on the background $M_9 \times T_2$. We can do the calculation as above and find the modular non-invariant vacuum energy ([455]). Then it is easy to write this expression in a modular invariant way [457, 458]

$$
\begin{aligned}
V = &- \int_F \frac{\mathrm{d}^2\tau}{\pi^2 \rho_1 \rho_2} \sum_{(c,d)=1} (2\,\mathrm{Im}\,\gamma_{cd}\tau)^{-11/2} \Big[\theta_3\Big(0\Big|\frac{\mathrm{i}\beta^2}{2\,\mathrm{Im}\,\gamma_{cd}\tau}\Big) \\
&- \theta_4\Big(0\Big|\frac{\mathrm{i}\beta^2}{2\,\mathrm{Im}\,\gamma_{cd}\tau}\Big)\Big] \int_{-1/2}^{1/2} \mathrm{d}y \sum_{n_1,n_2,k_1,k_2=-\infty}^{\infty} \exp\Big\{ - \frac{(\rho_1\rho_2 n_1 n_2)^2}{2\pi} \\
&\times \mathrm{Im}\,\gamma_{cd}\tau - \frac{\mathrm{Im}\,\gamma_{cd}\tau}{2\pi}\Big(\frac{n_1^2}{\rho_1^2} + \frac{n_2^2}{\rho_2^2}\Big) + 2\pi\mathrm{i}(n_1 k_1\,\mathrm{Re}\,\gamma_{cd}\tau + n_2 k_2 y)\Big\} \\
&\times \left[{\prod_{m_1,m_2=-\infty}^{\infty}}' \frac{1 - \mathrm{e}^{\left\{-\frac{\omega_{m_1 m_2}}{\pi}\mathrm{Im}\,\gamma_{cd}\tau + 2\pi\mathrm{i} m_1\,\mathrm{Re}\,\gamma_{cd}\tau + 2\pi\mathrm{i} y m_2\right\}}}{1 + \mathrm{e}^{\left\{-\frac{\omega_{m_1 m_2}}{\pi}\mathrm{Im}\,\gamma_{cd}\tau + 2\pi\mathrm{i} m_1\,\mathrm{Re}\,\gamma_{cd}\tau + 2\pi\mathrm{i} y m_2\right\}}} \right]^{-8}
\end{aligned}
\tag{11.51}
$$

Here β is the inverse temperature. The asymptotics of this vacuum energy have been investigated in [455]. It has been shown in [455] that in modular non-invariant formulation (if a cut-off parameter ε is introduced) the Hagedorn temperature is $\beta_c^2 = 112\pi^2\zeta(3)/(\varepsilon\rho_1\rho_2)$. Unlike superstrings the Hagedorn temperature depends on ε. Probably this fact is caused by the use of the semiclassical approximation.

We have shown that vacuum energy in a torus compactified (super)membrane can be presented in a modular invariant form. It would be of great interest to generalize these results. Maybe other larger groups (including $\mathrm{SL}(2, Z)$ as a subgroup) of invariance of membrane theory exist?

11.4 Static potential for bosonic p-branes

Some physical properties of p-branes (and membranes) in the quantum regime may be gleaned from a study of the effective action for various static p-brane configurations. In pacticular, it is reasonably straightforward to study the one-loop approximation and the large d approximation. In the second case, following the calculations developed in [470] for the string case (with applications to quantum chromodynamics) we can systematically develop an expansion for the effective action in powers of $1/d$. The static potential may then be obtained by studying the saddle point equations for the leading order term. Such calculations have been performed in [465–469] for different cases.

In this section we will follow [469] and will consider the Dirac action for bosonic p-branes

$$S_D = k \int \mathrm{d}^{p+1}\xi \sqrt{\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})} \tag{11.52}$$

where k is the p-brane tension, ξ^i, $i = 0, \ldots, p$ denote world-volume coordinates with world-volume metric γ_{ij} and X^μ, $\mu = 0, \ldots, d-1$ denote space–time coordinates with a metric $g_{\mu\nu}$. For $p = 2$ we have a bosonic membrane (the supersymmetric extension of bosonic membrane in Howe–Tucker form is discussed in section 11.2). We use the Euclidean signature in this section.

The equations of motion derived from the action (11.52) have the following form

$$\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j X^\nu g_{\mu\nu}) = \tfrac{1}{2}\sqrt{\gamma}\gamma^{ij}\partial_i X^\nu \partial_j X^\lambda \frac{\partial g_{\nu\lambda}}{\partial X^\mu} \tag{11.53}$$

where γ_{ij} is defined implicitly to be the induced metric

$$\gamma_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}. \tag{11.54}$$

We will consider open, toroidal and spherical p-brane configurations and will take the metric $g_{\mu\nu}$ to be flat in the case of the open p-brane. We will embed the toroidal and spherical p-branes in space–times of topology $S_1 \times \ldots \times S_1 \times R^0_{d-p}$ and $S_p \times R^0_{d-p}$ respectively in order to keep them static. In the spherical case the only non-zero elements of the metric $g_{\mu\nu}$ are given by

$$\begin{aligned} g_{00} &= 1 \qquad g_{d-1,d-1} = R^2 \\ g_{d-j,d-j} &= R^2 \prod_{k=1}^{j-1} \sin^2\Theta_k \qquad j = 2, \ldots, p \\ g_{ii} &= 1 \qquad i = 1, \ldots, d-p-1 \end{aligned} \tag{11.55}$$

where R is the fixed radius of the p-sphere.

In the case of the open or toroidal p-branes we will consider the following classical solution of equation (11.53) and (11.54)

$$\begin{aligned} X^0_{\text{cl}} &= \xi_0 \qquad X^\perp_{\text{cl}} = 0 \qquad X^{d-1}_{\text{cl}} = \xi_1, \ldots, X^{d-p}_{\text{cl}} = \xi_p \\ (\gamma_{\text{cl}})_{ij} &= \eta_{ij} \end{aligned} \tag{11.56}$$

where $X_{\rm cl}^{\perp} = (X^1, \ldots, X^{d-p-1})$ and $(\xi_1, \ldots, \xi_p) \in \mathcal{R} = [0, A_1] \times \ldots \times [0, A_p]$. This represents an open p-brane with fixed boundaries, or a once-wrapped torus. It generalizes the corresponding solution for bosonic membranes [465]. In the case of spherical p-branes we will generalize the result of [466] by taking the once-wrapped sphere on S^p as the classical solution of equations (11.53) and (11.54)

$$\begin{aligned} X_{\rm cl}^0 &\equiv \tau = \xi_0 \qquad X_{\rm cl}^{\perp} = 0 \\ X_{\rm cl}^{d-1} &\equiv \Theta_1 = \xi_1, \ldots, X_{\rm cl}^{d-p} \equiv \Theta_{d-p} = \xi_p \\ (\gamma_{\rm cl})_{ij} \mathrm{d}\xi^i \mathrm{d}\xi^j &= \mathrm{d}\tau^2 + R^2 \mathrm{d}\Omega_p^2. \end{aligned} \tag{11.57}$$

In both cases we can consider the semi-classical quantization of the theory using the background gauge of [465]

$$X^0 = X_{\rm cl}^0 \qquad X^{d-1} = X_{\rm cl}^{d-1}, \ldots, X^{d-p} = X_{\rm cl}^{d-p} \tag{11.58}$$

in which there are no Faddeev–Popov ghosts. In all cases the fields are taken to be periodic in imaginary time, with period T

$$X^{\perp}(0, \xi_1, \ldots, \xi_p) = X^{\perp}(T, \xi_1, \ldots, \xi_p). \tag{11.59}$$

For the open p-brane we take the remaining boundary conditions to be of Dirichelet type

$$\begin{aligned} X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) &= X^{\perp}(\xi_0, \xi_1, 0, \ldots, \xi_p) = \ldots \\ &= X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0) = 0 \\ X^{\perp}(\xi_0, A_1, \xi_2, \ldots, \xi_p) &= X^{\perp}(\xi_0, \xi_1, A_2, \ldots, \xi_p) = \ldots \\ &= X^{\perp}(\xi_0, \xi_1, \ldots, \xi_p, A_p) = 0 \end{aligned} \tag{11.60}$$

while for the toroidal p-brane equations (11.60) will be replaced by periodic boundary conditions

$$\begin{aligned} X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) &= X^{\perp}(\xi_0, A_1, \xi_2, \ldots, \xi_p). \\ X^{\perp}(\xi_0, \xi_1, 0, \ldots, \xi_p) &= X^{\perp}(\xi_0, \xi_1, A_2, \ldots, \xi_p). \\ \ldots \qquad & \qquad \ldots \\ \ldots \qquad & \qquad \ldots \\ \ldots \qquad & \qquad \ldots \\ X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0) &= X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, A_p). \end{aligned} \tag{11.61}$$

In the case of the spherical p-brane, (11.60) is replaced by the appropriate boundary conditions for a p-sphere.

The effective potential is given by

$$V = -\lim_{T\to\infty} \frac{1}{T} \ln \int DX^{\perp} \exp(-S_D) \tag{11.62}$$

where

$$S_D = k \int_0^T d\xi_0 \int_{\mathcal{R}} d^p\xi \sqrt{\gamma} \left[\det(\delta_{ij} + \partial_i X^{\perp} \cdot \partial_j X^{\perp})\right]^{1/2}.$$

In the one-loop approximation (that is, keeping only the quadratic quantum fluctuations) the effective potential is easily found to be [469]

$$V_0 = kA_0 + \tfrac{1}{2}(d-p-1) \sum_{n_1,\ldots,n_p=1}^{\infty} \left(\frac{\pi^2 n_1^2}{A_1^2} + \ldots + \frac{\pi^2 n_p^2}{A_p^2}\right)^{1/2} \tag{11.63}$$

$$V_T = kA_T + \tfrac{1}{2}(d-p-1) \sum_{n_1,\ldots,n_p=-\infty}^{\infty}{}' \left(\frac{4\pi^2 n_1^2}{A_1^2} + \ldots + \frac{4\pi^2 n_p^2}{A_p^2}\right)^{1/2} \tag{11.64}$$

$$V_S = kA_S + \frac{(d-p-1)}{2R} \times \sum_{n=1}^{\infty} \frac{(n+p-2)!}{n!(p-1)!}(2n+p-1)(\sqrt{n(n+p-1})) \tag{11.65}$$

for the open, toroidal and spherical p-branes respectively, where

$$A_0 = A_T = \prod_{i=1}^{p} A_i$$

$$A_S = \left[2\pi^{(p+1)/2} / \Gamma\left(\frac{p+1}{2}\right)\right] R^p.$$

(It is interesting to note that the result is also the same for p-brane actions of the Howe–Tucker form or in the Weyl-invariant form.) The sums in these expressions arise from the evaluation of the determinant of the fluctuation operator with the appropriate boundary conditions. A finite part for the sums may be extracted by zeta-function regularization. Such calculations have been considered in many other contexts in the literature — for example, in Kaluza–Klein theories (see Chapter 10).

Of particular interest is the sign of the one-loop correction term. If it is positive then the Casimir force is repulsive and acts in opposition to the attractive classical force, giving rise to the possibility of a non-point-like groundstate configuration. On the other hand, if it is negative, then the classical and Casimir forces are both attractive and the p-brane will tend to collapse if the boundaries are unclamped.

The open and toroidal sums may be expressed [465–468] in terms of functions $f_0(\rho_1, \ldots, \rho_{p-1})$ and $f_T(\rho_1, \ldots, \rho_{p-1})$ respectively, where

$$\rho_1 = A_2/A_1 \qquad \rho_2 = A_3/A_1 \qquad \ldots, \qquad \rho_{p-1} = A_p/A_1.$$

An analysis shows that for toroidal p-branes f_T is always negative, whereas for open p-branes f_0 can change sign as the ratio of the edge-lengths changes. For membrane, for example, we have (from [466])

$$V = kA_{0(T)} + \frac{(d-3)f_{0(T)}}{2\sqrt{A_{0(T)}}} \tag{11.66}$$

where

$$f_0(\rho) = \frac{\pi}{24}(\rho^{1/2} + \rho^{-1/2}) - \frac{\zeta(3)}{8\pi}(\rho^{3/2} + \rho^{-3/2}) - \frac{1}{4\pi}\sum_{m,n=1}^{\infty}[m^2\rho + n^2\rho^{-1}]^{-3/2}$$

and

$$f_T(\rho) = -\frac{2\zeta(3)}{\pi}(\rho^{3/2} + \rho^{-3/2}) - 4\sum_{m,n=1}^{\infty}[m^2\rho + n^2\rho^{-1}]^{-3/2}.$$

Since $f_{0(T)}(1/\rho) = f_{0(T)}(\rho)$, we can restrict ρ to $0 < \rho \leqslant 1$. As $\rho \to 0$ for constant area A we obtain the limit of string-like configurations. One finds that the dominant term in this limit is the $\rho^{-3/2}$ term. Consequantly, if the area of the membrane is held fixed the Casimir energy decreases to more and more negative values as we approach the string-like configurations, so these configurations are energetically favoured.

Numerical values of the functions $f_{0(T)}$ can be readily calculated in the case of equal edge-lengths [315, 410]

$$f_0(1, \ldots, 1) = \pi \sum_{n_1, \ldots, n_p = 1}^{\infty} [n_1^2 + \ldots + n_p^2]^{1/2} \tag{11.67}$$

$$f_T(1, \ldots, 1) = 2\pi \sum_{n_1, \ldots, n_p = -\infty}^{\infty}{}' [n_1^2 + \ldots + n_p^2]^{1/2}. \tag{11.68}$$

Table 11.1 Numerical values for the sums (11.67), (11.68) as given in [410]. The difference in accuracy to which values are given reflects the fact that the $p = 5, 7$ values quoted in [410] (for torus case) were obtained by interpolation.

p	$f_O(1, \dots, 1)$	$f_T(1, \dots, 1)$
2	0.082	−1.438
3	0.165	−1.676
4	0.151	−1.864
5	0.113	−2.04
6	0.781	−2.344
7	0.052	−2.56
8	0.033	−2.90

These values are listed for $p \leqslant 8$ in table 11.1.

In the case of spherical p-branes numerical values of the sum

$$f_S = \sum_{n=1}^{\infty} \frac{(n+p-2)!}{n!(p-1)!}(2n+p-1)\sqrt{n(n+p-1)} \tag{11.69}$$

can also be obtained for even p. For odd p zeta-function techniques give an expression which diverges. We can of course renormalize the result, however, the sign of f_S will depend on the renormalization scale, μ, which enters via a $\ln \mu$ term. Unless the renormalization scale can be fixed in an unambiguous way no meaning can be given to f_S for odd p. In table 11.2 we quote the numerical values for f_S given in [410].

The results for open and toroidal p-branes of fixed edge-length R and for spherical p-branes may be conveniently summarized as

$$V_{\text{one-loop}} = kA_{0(T)(S)} + \tfrac{1}{2}(d-p-1)\frac{f_{0(T)(S)}}{R} \tag{11.70}$$

where $A_0 = A_T = R^p$, f_0, f_T, f_S are given by (11.67)–(11.69). In each case the classical potential is corrected by a simple $1/R$ term. Since f_T and f_S are negative for all values the p-brane will tend to collapse if released. Since toroidal and spherical p-branes have a non-zero winding number it is of course impossible that these configurations collapse completely to a point—for small R other terms in the effective potential must become important. The open p-brane case is different, however, since f_0 is positive, at least for the values

Table 11.2 Numerical values for the sums (11.69) as given in [410]

p	fs
2	− 0.266
4	− 0.432
6	− 0.552
8	− 0.652
10	− 0.736
20	− 1.046
10^3	− 6.10
10^6	− 145.2
10^{20}	− 8.16×10^8

listed in table 11.1. In this case the potential has a minimum at a finite distance

$$R_0 = \left[\frac{(d-p-1)f_0(1,\ldots,1)}{2kp}\right]^{1/(p+1)} \tag{11.71}$$

corresponding to a non-pointlike groundstate. It is also interesting to note that in the case of a p-brane with higher-derivative terms (a stiff p-brane) the sums in (11.63)–(11.65) are quite different [471].

The calculation of the effective potential in the limit of large space–time dimensionality closely follows that given by Alvarez [470] in string theory. Following the standard way we introduce composite fields σ_{ij} for $\partial_i X^\perp \cdot \partial_j X^\perp$ (in an orthonormal frame) and constraint $\sigma_{ij} = \partial_i X^\perp \cdot \partial_j X^\perp$ by introducing Lagrange multipliers λ^{ij}. We obtain

$$\begin{aligned} Z = \int \mathrm{D}\, X^\perp D\sigma\, \mathrm{D}\,\lambda \exp\Bigg[&-k\int \mathrm{d}^{p+1}\xi\sqrt{h}[\det(\delta_{ij}+\sigma_{ij})]^{1/2} \\ &-\frac{k}{2}\int \mathrm{d}^{p+1}\xi\sqrt{h}\left((\lambda\gamma_{\mathrm{cl}}^{-1})^{ij}\partial_i X^\perp\cdot\partial_j X^\perp - \lambda^{ij}\sigma_{ij}\right)\Bigg] \end{aligned} \tag{11.72}$$

where $h \equiv \det[(\gamma_{\mathrm{cl}})_{ij}]$: for the open and toroidal cases $h = 1$, while for the spherical p-brane $h = R^{2p}\prod_{i=1}^{p-1}(\sin\Theta_i)^{2(p-i)}$. Integrating over $X^\perp$ we find

$$Z = \int \mathrm{D}\,\sigma\,\mathrm{D}\,\lambda \exp(-S_{\mathrm{eff}})$$

where

$$\begin{aligned} S_{\mathrm{eff}} = &\tfrac{1}{2}(d-p-1)\,\mathrm{Tr}\ln\left(-h^{-1/2}\partial_i(\lambda\gamma_{\mathrm{cl}}^{-1})^{ij}h^{1/2}\partial_j\right) \\ &+ kTA_{0(T)(S)}\left[(\det(1+\sigma))^{1/2} - \tfrac{1}{2}\mathrm{Tr}(\lambda\cdot\sigma)\right] \end{aligned} \tag{11.73}$$

and for simplicity we have chosen $A_1 = \ldots = A_p = R$ in the open and toroidal cases. The $1/d$ expansion is systematically generated by expanding (11.73) about its stationary points. The calculation is greatly simplified by the fact that because of symmetry properties of the classical solution together with time translation invariance at large times the ground state configurations of σ and λ can be chosen to be diagonal constant matrices of the form

$$\begin{aligned} \sigma_{ij} &= \mathrm{diag}(\sigma_0, \sigma_1, \ldots, \sigma_1) + \mathrm{O}(1/d) \\ \lambda^{ij} &= \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_1) + \mathrm{O}(1/d). \end{aligned} \tag{11.74}$$

The operator inside the Tr ln term becomes

$$\Delta_{0(T)} = -\lambda_0 \partial_\tau^2 - \lambda_1(\partial_{\xi_1}^2 + \ldots + \partial_{\xi_p}^2) \tag{11.75}$$

in the open and toroidal cases and

$$\Delta_S = -\lambda_0 \partial_\tau^2 - \frac{\lambda_1}{R^2}\Delta_\Omega \tag{11.76}$$

for spherical cases, where Δ_Ω is the Laplacian operator on a p-sphere. Evaluating the Tr ln terms we obtain [469]

$$\begin{aligned} S_{\text{eff}} = kTA_{0(T)(S)}[&(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} \\ &- \tfrac{1}{2}(\sigma_0\lambda_0 + p\sigma_1\lambda_1) + \alpha_{0(T)(S)}(\lambda_1/\lambda_0)^{1/2}] \end{aligned} \tag{11.77}$$

where

$$\begin{aligned} \alpha_{0(T)(S)} &= \frac{(d-p-1)}{2kA_\Omega R^{p+1}} f_{0(T)(S)} \\ A_\Omega &= 2\pi^{(p+1)/2} \Big/ \Gamma\left(\frac{p+1}{2}\right) \end{aligned}$$

for the spherical case and $A_\Omega = 1$ otherwise. For convenience we will drop the subscripts 0, T and S in what follows.

The four saddle-point equations obtained by varying (11.76) are given by

$$\lambda_0 = (1+\sigma_1)^{p/2}(1+\sigma_0)^{-1/2} \tag{11.78}$$

$$\lambda_1 = (1+\sigma_1)^{(p-2)/2}(1+\sigma_0)^{1/2} \tag{11.79}$$

$$\sigma_0 = -\alpha(\lambda_1/\lambda_0^3)^{1/2} \tag{11.80}$$

$$\sigma_1 = \alpha/[p(\lambda_0\lambda_1)^{1/2}]. \tag{11.81}$$

After a little algebraic manipulation we obtain a polynomial equation for $(1+\sigma_1)^{1/2}$ (for even p), or for $(1+\sigma_1)$ (for odd p)

$$(1+\sigma_1)^{(p+1)/2} - (1+\sigma_1)^{(p-1)/2} - \alpha/p = 0. \tag{11.82}$$

We may now eliminate the other functions σ_0, λ_0 and λ_1 to arrive at the following expression for the static potential

$$V = kA(1+\sigma_1)^{(p-1)/4}\left[(1+\sigma_1)^{(p-1)/2} + \tfrac{\alpha}{p}(p+1)\right]^{1/2} \tag{11.83}$$

where σ_1 is given by (11.82). Equation (11.82) is soluble in closed form only for $p = 1, 2, 3, 5, 7$. However, an analysis of the function $P(1+\sigma_1)$ defined by the left-hand side of (11.82) shows that for p even or $p = 5 \bmod 4$ P has at most three roots, and for $p = 3 \bmod 4$ P has at most two roots. The structure of roots of (11.82) (for $p \geqslant 2$) is thus always the same as in the $p = 2$ and $p = 3$ cases, which we will now consider in further detail.

If $p = 2$ then (11.82) is a cubic equation for $(1+\sigma_1)^{1/2}$. It has one real root if $x > 1$, where

$$x \equiv \frac{3\sqrt{3\alpha}}{4} = \frac{3\sqrt{3}(d-3)f}{8kA_\Omega R^3} \tag{11.84}$$

and three real roots if $x \leqslant 1$. The choice of which root to take in the second regime is determined by requiring that the solution matches that of the first regime [465]. We therefore have

$$(1+\sigma_1)^{1/2} = \begin{cases} \frac{2}{\sqrt{3}}\cosh(\frac{1}{3}\operatorname{arccosh} x) & \text{for } x \geqslant 1 \\ \frac{2}{\sqrt{3}}\cos(\frac{1}{3}\arccos x) & \text{for } x \leqslant 1 \end{cases} \tag{11.85}$$

If we write $\Theta_+ \equiv \cosh(\frac{1}{3}\operatorname{arccosh} x)$ and $\Theta_- \equiv \cos(\frac{1}{3}\arccos x)$ then

$$V = \begin{cases} \frac{2}{\sqrt{3}}V_{\rm cl}\left[\sqrt{\Theta_+(\Theta_+ + x)}\right] & \text{for } x \geqslant 1 \\ \frac{2}{\sqrt{3}}V_{\rm cl}\left[\sqrt{\Theta_-(\Theta_- + x)}\right] & \text{for } x \leqslant 1 \end{cases} \tag{11.86}$$

where $V_{\rm cl} = kA$ is the classical potential.

If $p = 3$ then (11.82) is a quadratic equation for $(1+\sigma_1)^{1/2}$. The appropriate solution to take is decided by taking the branch which gives a real potential (in some cases), namely (see [468])

$$\sigma_1 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{4}{3}\alpha}. \tag{11.87}$$

The properties of the potential may be easily deduced in the general case. As $R \to \infty$ we find

$$V \underset{R\to\infty}{\propto} V_{\rm cl} = kA \tag{11.88}$$

i.e., we obtain the classical potential as expected. For toroidal and spherical cases $\alpha < 0$ and consequently the potential becomes complex at a critical distance

$$R_c = \sqrt{\frac{p+1}{p}} \left[\frac{(d-p-1)f_{T(S)}}{2kA_\Omega} \right]^{1/(p+1)} . \qquad (11.89)$$

Thus for these cases we have a tachyonic instability which signals the breakdown of the static approximation. This is perhaps not surprising since, as we observed earlier, the toroidal and spherical p-branes have non-zero winding number and cannot contract all the way down to $R = 0$. Whereas the possibility of such an unbounded contraction posed a dilemma in the one-loop case, it is now definitely ruled out. In the open case the potential is real for all R and we can take the limit $R \to 0$. We find

$$(1+\sigma_1)^{1/2} \underset{R\to 0}{\propto} \left(\frac{\alpha}{p} \right)^{1/(p+1)} \qquad (11.90)$$

and consequently

$$V \underset{R\to 0}{\propto} \sqrt{p+1} \left[k \left(\frac{(d-p-1)f_0}{2p} \right)^p \right]^{1/(p+1)}$$
$$\times \left\{ 1 + \frac{1}{2(p+1)} \left[\frac{2kp}{f_0(d-p-1)} \right]^{2/(p+1)} R^2 + \dots \right] \qquad (11.91)$$

For example, for the open membrane

$$V \underset{R\to 0}{\propto} 0.13[k(d-3)]^{1/3} \left[1 + 2.33 \left(\frac{k}{d-3} \right)^{2/3} R^2 + \dots \right] . \qquad (11.92)$$

So in the open case V is flat at $R = 0$ but with a shifted value with respect to V_{cl}. This is the only minimum of the potential since

$$\frac{dV}{dR} = \frac{kpR^{p-1}(1+\sigma_1)^{3(p-1)/4}}{\left[(1+\sigma_1)^{(p-1)/2} + \frac{\alpha}{p}(p+1) \right]^{1/2}} \qquad (11.93)$$

which vanishes only for $R_0 = 0$. Consequently the effect of the $1/d$ summation is to shift the groundstate of the one-loop approximation to a point-like configuration.

It is interesting to note that for Howe–Tucker or Weyl-invariant actions of bosonic p-branes the static potential has the same form [469].

Thus, in this chapter we have considered some quantum properties of membrane theory in the semiclassical quantization approximation. We have discussed the vacuum energy and the static potential. Of course, there are some other approaches which address the existence of massless states [419–425] and the non-renormalizability of membrane theory using standard arguments (see for example [444]). The results of these papers were very discouraging and, as a consequence, there was a reduction in the number of research papers on membrane theories after 1989. However, in our opinion many problems in membrane theory demand further careful research. To start with there is the problem of the quantization of membranes on a curved background (and the whole related programme of research). Also, account should be taken of all 3-topologies having a consistent quantum theory and this would demand much work, connected with the classification of three-dimensional geometries (all of these should give the contribution to amplitudes, etc).

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Index

Different quantum gravity models have been devised to study the quantum aspects of gravitational interaction. The investigation of these models requires the computation of the effective action — hence the significance of this book.

Effective Action in Quantum Gravity is divided into three parts. The first part is pedagogical in nature and contains an introduction to the field theoretical models. The second part explains the quantum theory of the interacting fields in curved space including renormalization groups and the asymptotic properties of grand unification theories at high curvature. In the third part the authors discuss the problems of quantized gravitational field theory, in particular the quantum theory of higher-derivative gravity, the quantum Kaluza–Klein theories and the quantum theory of strings and membranes.

This book is intended for postgraduate students and researchers in high-energy physics and gravitational theory. Although a knowledge of quantum field theory and gravity is assumed, *Effective Action in Quantum Gravity* can be read without reference to other books or papers.

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