Probabilistic Model & Maximum Likelihood Estimation

This document proposes a probabilistic model that gives an estimate of the probability of spoilage of mushrooms at any storage temperature and explains the MLE approach for fitting the model parameters. The logistic regression structure with binomial observations is considered in the development of the model.

1.1 Probabilistic Model

We observe data for $N = 4$ temperature levels. For each level i, we have:

- **Storage temperature** x_i (in °C)
- **Number of mushrooms** n_i
- **Number of spoiled mushrooms** y_i

Level ID	Storage Temperature x (°C)	Total Mushrooms n	Spoiled Mushrooms y
1	2	30	2
2	8	25	4
3	15	20	5
4	25	30	20

Latent Spoilage Probability

Each mushroom in group i spoils independently with probability p_i. This probability is modeled using the **sigmoid (or logistic) function**:

$$p_i = \sigma(\alpha + \beta x_i), \quad \text{where} \quad \sigma(z) = \frac{1}{1 + e^{-z}}.$$

Here, $\theta = (\alpha, \beta)$ are the unknown **parameters** to be estimated.

Likelihood for Each Group

Given the parameters θ, the observed number of spoiled mushrooms y_i follows a **Binomial distribution**:

$$y_i \mid \theta \sim \text{Binomial}(n_i, p_i)$$

The probability mass function for a single group is:

$$P(y_i \mid \theta) = \binom{n_i}{y_i} p_i^{y_i} (1 - p_i)^{n_i - y_i}.$$

Independence Across Groups

Assuming **independence** across all N groups (given θ), the **full-data likelihood** is the product of the individual likelihoods:

$$P(y \mid \theta) = \prod_{i=1}^{N} \binom{n_i}{y_i} p_i^{y_i} (1 - p_i)^{n_i - y_i}, \quad \text{where} \quad p_i = \sigma(\alpha + \beta x_i).$$

Prior on Parameters

In a Bayesian setting, we place **Gaussian priors** on the parameters α and β:

$$\alpha \sim \mathcal{N}(0, 4), \qquad \beta \sim \mathcal{N}(0, 1).$$

Assuming independence of α and β, the joint prior probability is:

$$P(\theta) = \mathcal{N}(\alpha; 0, 4)\ \mathcal{N}(\beta; 0, 1).$$

Full Generative Model

Putting the likelihood and prior together defines the **full generative model**:

$$P(y, \theta) = P(\theta) \prod_{i=1}^{N} \binom{n_i}{y_i} [\sigma(\alpha + \beta x_i)]^{y_i} [1 - \sigma(\alpha + \beta x_i)]^{n_i - y_i}.$$