$1-\ y'' - 4y' + 4y = 0\ ;\ y_1 = e^{2x}$

$\frac{y_2}{e^{2x}} = u(x)$ $\quad y_2 = ue^{2x}$ $\quad y_2' = u'e^{2x} + u2e^{2x}$

$y_2'' = u''e^{2x} + u'2e^{2x} + u'2e^{2x} + u4e^{2x}$

$y_2'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$

$y'' - 4y' + 4y = 0$

$u''e^{2x} + \cancel{4u'e^{2x}} + \cancel{4ue^{2x}} - 4(\cancel{u'e^{2x}} + \cancel{2ue^{2x}}) + \cancel{4ue^{2x}}$

$u''e^{2x} = 0$ $\quad w = u'$ $\quad w' = u''$

$w'e^{2x} = 0$

$w' = \frac{1}{e^{2x}}$ $\quad \int \frac{dw}{dx} dx = \int \frac{1}{e^{2x}} dx$ $\quad u = -2x$, $du = -2dx$

$w = -\frac{1}{2}\int e^{u}\, du$

$w = -\frac{e^{-2x}}{2} dx + c$

$w = u' = -\frac{e^{-2x}}{2} dx$

$\int \frac{du}{dx} dx = \int -\frac{e^{-2x}}{2} dx$

$u = \frac{e^{-2x}}{4} + c$

$y_2 = ue^{2x}$, $\quad y_2 = \frac{e^{-2x}}{4} e^{2x} + ce^{2x}$

2.- $(1-x^2)y'' + 2xy' = 0; \quad y_1 = 1$

$\frac{y_2}{1} = u(x) \quad y_2 = u \qquad y_2' = u'$

$y_2'' = u''$

$(1-x^2)u'' + 2xu' = 0 \qquad u'' = w' \quad u' = w$

$\frac{1}{u'} u'' = \frac{-2x}{(1-x^2)}$

$\frac{1}{w} \frac{dw}{dx} = \frac{-2x}{1-x^2}$

$\int \frac{1}{w} = \int \frac{-2x}{1-x^2} \qquad v = 1-x^2$

$dv = -2x\, dx$

$\ln(w) = \int \frac{1}{v} dv = \ln|v| = \ln|1-x^2| + c$

$w = 1-x^2+c$

$w = u' = 1-x^2$

$u = \int 1-x^2 = x - \frac{x^3}{3}$

$y_2 = x - \frac{x^3}{3} + c$

3- $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$

$\frac{y_2}{x^4} = u(x)$ $y_2 = ux^4$ $y_2' = u'x^4 + u4x^3$

$y_2'' = u''x^4 + u'4x^3 + u'4x^3 + u12x^2$

$x^2(u''x^4 + 8u'x^3 + 12ux^2) - 7x(u'x^4 + 4ux^3) + 16ux^4$

$u''x^6 + 8u'x^5 + \cancel{12ux^4} - 7u'x^5 - \cancel{28ux^3} + \cancel{16ux^4}$

$u''x^6 + u'x^5 = 0$

$\frac{1}{u'}u'' = \frac{-x^5}{x^6}$

$\frac{1}{u'}u'' = \frac{-1}{x}$ $w = u'$ $w' = u''$

$\frac{1}{w}w' = \frac{-1}{x}$ $\int \frac{1}{w}\,dw = \int -\frac{1}{x}$

$\ln(w) = -\ln(x)$

$w = \frac{1}{x}$

$w = u' = \frac{1}{x}$

$u = \int \frac{1}{x} = \ln(x) + C$

$y_2 = ux^4$ $y_2 = \underline{\ln(x)x^4 + Cx^4}$

4- $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$; $y_1 = x+1$

$y_2 = u(x+1)$

$$y_2' = u'(x+1) + u$$

$$y_2'' = u''(x+1) + u' + u'$$

$$y_2'' = u''(x+1) + 2u'$$

$$(1-2x-x^2)(u''(x+1)+2u') + 2(1+x)(u'(x+1)+u) + 2(u(x+1))$$

$$(1-2x-x^2)(u''(x+1)) + (1+2x+x^2)(+2u') + 2u'(x^2+2x+1) +$$

$$2(1+x)u - 2(x+1)u = 0$$

$$u''(x+1)(1-2x-x^2) = -2u'(1+x^2)-x^2,$$

$$\frac{1}{w}w' = \frac{-2(1-x^2)}{(x+1)(1-2x-x^2)}$$

$$\ln\int\frac{1}{w} = -2\frac{(1-x^2)}{(x+1)(1-2x-x^2)}$$

$$\ln w = -2\int\frac{(1-x^2)}{(x+1)(1-2x-x^2)}$$

~~$5- x^3y'' + 2xy' + 2(1+x)y' - 2y = 0; \quad y_1 = x+1$~~

$5- x^2y'' + 2xy' - 6y = 0 ; \quad y_1 = x^2$

$y_2 = u(x) \qquad y_2 = ux^2 \qquad y' = u'x^2 + u2x$

$y'' = u''x^2 + u'2x + u'2x + 2u$

$y'' = u''x^2 + 4u'x + 2u$

$x^2(u''x^2 + 4u'x + 2u) + 2x(u'x^2 + 2ux) + 6ux^2$

$u''x^4 + 4u'x^3 + \cancel{2ux^2} + 2u'x^3 + \cancel{4ux^2} - \cancel{6ux^2}$

$u''x^4 + 6u'x^3 = 0$

$u''x^4 = -6u'x^3$

$\frac{1}{u'}u'' = \frac{-6}{x} \qquad w = u' \quad w' = u''$

$\int \frac{1}{w} = \int \frac{-6}{x}$

$\ln w = -6\ln|x|$

$w = \frac{1}{x^6}$

$w = u' = \frac{1}{x^6}$

$u = \int \frac{1}{x^6}$

$u = \frac{1}{5x^5} + C$

$y_2 = ux^2 = \left(\frac{1}{5x^5} + C\right)(x^2) = \frac{1}{5x^3} + Cx^2$

6- $xy'' + y' = 0; \quad y_1 = \ln x$

$y'' + \frac{y'}{x} = 0$

$u = c\int \frac{e^{-\int \frac{1}{x} dx}}{(\ln(x))^2} = c\int \frac{1}{x(\ln(x))^2}$

$v = \ln(x)$
$dv = \frac{1}{x} dx$

$u = c\int \frac{1}{(v)^2} dv = \frac{-1}{\ln(x)} + c$

$y_2 = u y_1 = \ln x\left(\frac{-1}{\ln(x)} + c\right) = -1 + c_1 \ln(x)$

$= c + c_1 \ln(x)$

7- $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$

$u = c\int \frac{e^{-\int 2 dx}}{(xe^{-x})^2} = c\int \frac{e^{-2x}}{(xe^{-x})^2} dx = c\int \frac{1}{x^2} dx = \frac{-1}{x} + c$

$u = \frac{-1c}{x} + c$

$y_2 = u y_1 = xe^{-x}\left(\frac{-1c}{x} + c\right) c_1 e^{-x} + cxe^{-x}$

8.- $4x^2y'' + y = 0$; $y_1 = x^{1/2}\ln x$ $P(x) = 0$

$y'' + \frac{1}{4x^2}y = 0$; $u = c\int \frac{1}{(x^{1/2})^2} = \int \frac{1}{x} = \ln(x) + C$

$u = c\ln(x) + C$

$y_2 = uy_1$; $y_2 = x^{1/2}(c\ln(x) + C)$

$y_2 = cx^{1/2}\ln(x) + Cx^{1/2}$

9.- $x^2y'' - xy' + 2y = 0$; $y_1 = x\,\mathrm{sen}(\ln x)$

$y'' - \frac{1}{x}y' + \frac{2y}{x^2}$ $u = c\int \frac{e^{-\int -\frac{1}{x}dx}}{(x\,\mathrm{sen}(\ln x))^2} = c\int \frac{x}{(x\,\mathrm{sen}(\ln x))^2}$

$= c\int \frac{1}{x\sin^2(\ln(x))}$ $v = \ln(x)$, $dv = \frac{1}{x}dx$ $= c\int \frac{1}{\sin^2(v)} = c\int \csc^2(v)dv$

$= -\cot(v) = -\cot(\ln(x)) + C = -c\cot(\ln(x)) + C$

$y_2 = uy_1 = x\,\mathrm{sen}(\ln x)\left(-c\frac{\cos(\ln x)}{\sin(\ln x)} + C\right)$

$y_2 = xc\cos(\ln x) + C_1x\sin(\ln x)$

10- $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2\cos(\ln x)$

$y'' - \frac{3}{x}y' + \frac{5y}{x^2} = 0 \quad v = c\int \frac{e^{-\int -\frac{3}{x}}}{(x^2\cos(\ln x))^2} = c\int \frac{(x)^3}{(x^2\cos(\ln x))^2}$

$= c\int \frac{1}{x\cos^2(\ln x)} \quad \begin{matrix} u = \ln x \\ du = \frac{1}{x}dx \end{matrix} \quad = c\int \frac{1}{\cos^2(u)}du$

$= c\int \sec^2(u)\,du = c\tan(u) = c\tan(\ln x) + c$

$y_2 = vy_1 = x^2\cos(\ln x)\left(c\frac{\sin(\ln x)}{\cos(\ln x)} + c\right)$

$y_2 = cx^2\sin(\ln x) + c_1\cos(\ln x)$