Problem 1 Consider $D \subseteq C \subseteq \mathcal{H}$, where C is closed convex and we assume that $\mathrm{P}_C(0) \in D$.
(i) Show that D has the minimum norm property and that $\mathrm{P}_C(0) = \mathrm{P}_D(0)$.
From the definition of the Projection we deduce

$$\|\mathrm{P}_C(0)\| \leq \|x\|, \quad \forall x \in C \qquad \{?\}$$

thus, as $\overline{\mathrm{conv}}\,(D) \subseteq C$,

$$\|\mathrm{P}_C(0)\| \leq \|x\|, \quad \forall x \in \overline{\mathrm{conv}}\,(D) \qquad (1) \; \boxed{\texttt{eq:clconv}}$$

and therefore

$$\|\mathrm{P}_C(0)\| \leq \|x\|, \quad \forall x \in D. \qquad (2) \; \boxed{\texttt{eq:projond}}$$

By assumption $\mathrm{P}_C(0) \in D$ and therefore $\mathrm{P}_C(0) \in \overline{\mathrm{conv}}\,(D)$. Combining this with (1) gives that $\mathrm{P}_C(0) = \mathrm{P}_{\overline{\mathrm{conv}}\,(D)}(0)$ and proofs the minimum norm property of D. Similarly, we deduce from (2) that $\mathrm{P}_C(0) \in D$ is already the projection of 0 onto D, i.e. $\mathrm{P}_C(0) = \mathrm{P}_D(0)$.
(ii) From the projection theorem we deduce that

$$\langle x_k - \mathrm{P}_C(0), 0 - \mathrm{P}_C(0)\rangle \leq 0$$

?eq:lala?

and therefore

$$\langle x_k, \mathrm{P}_C(0)\rangle \geq \|\mathrm{P}_C(0)\|^2. \qquad \{?\}$$

Furthermore, by Cauchy-Schwarz also have that

$$\|\mathrm{P}_C(0)\|\|x_k\| \geq \langle x_k, \mathrm{P}_C(0)\rangle \geq \|\mathrm{P}_C(0)\|^2. \qquad \{?\}$$

Thus $\langle x_k, \mathrm{P}_C(0)\rangle$ converges to $\|\mathrm{P}_C(0)\|^2$. Now consider

$$\begin{aligned}\|x_k - \mathrm{P}_C(0)\| &= \langle x_k - \mathrm{P}_C(0), x_k - \mathrm{P}_C(0)\rangle \\ &= \langle x_k - \mathrm{P}_C(0), 0 - \mathrm{P}_C(0)\rangle + \langle x_k - \mathrm{P}_C(0), x_k\rangle \qquad \{?\} \\ &\leq \|x_n\|^2 - \langle \mathrm{P}_C(0), x_k\rangle\end{aligned}$$

which proves the strong convergence.
Remark (see Problem 2 and 3) Note that the projection onto a closed convex set is a firmly nonexpansive map and identity minus firmly nonexpansive map is again firmly nonexpansive. Thus, if we look for a projection P in place of $\mathrm{Id} - T$ we can ensure that $T = \mathrm{Id} - P = \mathrm{Id} - (\mathrm{Id} - T)$ is in fact nonexpansive.
Problem 2 Construct an example where $\overline{\mathrm{ran}}\,(\mathrm{Id} - T)$ is not convex, for a nonexpansive operator $T : C \to C$.
Consider $C = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ the uppe half space and let $\mathrm{P} := \mathrm{Id} - T$ be the projection onto the unit ball with center $(0, -1)$. First it is easy to see that that the image of C under the projection is nonconvex as is given by the upper half of surface of the ball (not including the points $(1, -1)$ and $(-1, -1)$). Now we need to check that $\mathrm{ran}\,(T) = \mathrm{ran}\,(\mathrm{Id} - \mathrm{P})$ is contained in the upper half space (since we require that T maps back to its domain). This is however clear as the domain is above the x-axis and all

images of P are below it. Thus, all vecors given by $x - \mathrm{P}x$ point up (and are therefore contained in C).

Problem 3 Construct an example where $\overline{\mathrm{ran}}\,(\mathrm{Id} - T)$ does not posses the minimum norm propert for a nonexpansive operator $T : C \to X$.
Let $\mathrm{Id} - T$ be the projection onto the unit ball, in $\mathbb{R}^2$ with domain $\{(x, 1) \in \mathbb{R}^2 : -1 \le x \le 1\}$. The range of $\mathrm{Id} - T$ is then clearly a subset of the upper half of the unit circle and does not have the minimum norm property.

Problem 4 Pazy's Trichotomy.
i)$T = \mathrm{Id}$.
Clearly $0 \in \overline{\mathrm{ran}}\,(\mathrm{Id} - T) = \{0\}$ and $(T^n x) = x$ for all n is therefore bounded.
ii) Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) := \begin{cases} x + 1, & \mathrm{if} x \le 1; \\ x + \frac{1}{x}, & \mathrm{if} x > 1. \end{cases} \qquad \{?\}$$

Clearly, $\mathrm{ran}\,(\mathrm{Id} - T) = [-1, 0)$. This means that case ii) of Pazy's Trichotomy is occurring. Now we want to check that $\limsup_n T^n x = +\infty$ for some/all x.
Clearly, for any x we have that $T^n x$ is greater than 1 after finitely many steps, so we only consider this case. For $x > 1$ we can see that T is monotone. Let $x > y > 1$, then

$$Tx \ge Ty \Leftrightarrow x + \frac{1}{x} \ge y + \frac{1}{y} \Leftrightarrow x - y \ge \frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy} \qquad \{?\}$$

where the right hand side $x - y \ge \frac{x-y}{xy}$ is true as $x, y > 1$. We will proof the fact that $T^n x$ is unbounded by contradiction. Assume therefore that $T^n x$ is bounded. As the T is monotone $T^n x$ must converge to a limit wich we call θ. Since $T^n x$ converges to θ there exists an $n_0 \in \mathbb{N}$ such that $T^{n_0} x \ge \theta - \frac{1}{\theta}$. By the monotonicity we have that

$$T^{n_0+1} x \ge T\left(\theta - \frac{1}{\theta}\right) = \theta - \frac{1}{\theta} + \frac{1}{\theta - \frac{1}{\theta}} > \theta - \frac{1}{\theta} + \frac{1}{\theta} = \theta, \qquad \{?\}$$

which contradicts the convergence of $T^n x$.
Next, we check that $\lim_n \frac{1}{n} T^n x = 0$. Note that

$$T^2 x = Tx + \frac{1}{Tx} = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}} \le x + \frac{2}{x}. \qquad \{?\}$$

By the same token, we have that

$$T^n x \le x + \frac{n}{x}. \qquad \{?\}$$

Thus

$$\limsup_n \frac{T^n x}{n} \le \limsup_n \left(\frac{x}{n} + \frac{1}{x}\right) \le \frac{1}{x}. \qquad \{?\}$$

We will use this bound now to deduce convergence by considering for arbitrary $k \in \mathbb{N}$

$$\limsup_n \frac{T^n x}{n} = \limsup_n \frac{T^{n+k} x}{n+k} = \limsup_n \frac{T^n(T^k x)}{n+k} \leq \limsup_n \frac{T^n(T^k x)}{n} \leq \frac{1}{T^k x}. \quad \{?\}$$

However, we he have seen earlier that $\lim_k T^k x = +\infty$. This shows that $\lim_n \frac{1}{n} T^n x = 0$.

(iii) $Tx := x - z$, for some fixed $z \neq 0$.
Clearly, $\operatorname{ran}(\mathrm{Id} - T) = \{z\}$. Therefore, $0 \notin \overline{\operatorname{ran}}(\mathrm{Id} - T) = \{z\}$. Furthermore

$$T^n x = x - nz. \quad \{?\}$$

Therefore

$$\lim_n T^n x = -\operatorname{sgn}(z)\,\infty \quad \{?\}$$

and

$$\lim_n \frac{1}{n} \|T^n x\| = \|z\|. \quad \{?\}$$

Problem 5 The Edelstein operator. It is given by

$$T\left((x_k)_{k\in\mathbb{N}}\right) := \left(1 + (x_k - 1)\exp\left(\frac{2\pi i}{k!}\right)\right)_{k\in\mathbb{N}}. \quad \{?\}$$

Check that $(T^n 0)_k = 1 - \exp\left(\frac{2\pi i n}{k!}\right)$.

For $n = 1$, this follows immediately from the definition. Assume it is true for $n - 1$. Then,

$$\begin{aligned}(T^n 0) = (T \circ T^{n-1} 0) = T\left(\left(1 - \exp\left(\frac{2\pi i n}{k!}\right)\right)_{k\in\mathbb{N}}\right) \\ = \left(1 - \exp\left(\frac{2\pi i(n-1)}{k!}\right)\exp\left(\frac{2\pi i}{k!}\right)\right)_{k\in\mathbb{N}} \\ = \left(1 - \exp\left(\frac{2\pi i n}{k!}\right)\right)_{k\in\mathbb{N}}.\end{aligned} \quad \{?\}$$

Next, for $x \in \ell^2$ we have that $\|T^k x - T^{k+1} x\| = \|Tx - x\|$ since T is an isometry. Thus, $\lim_k \|T^k x - T^{k+1} x\| = \|Tx - x\|$, which is strictly greater zero as T has no fixed points.

Problem 6 Let $T : \mathcal{H} \to \mathcal{H}$ be linear, and nonexpansive. Show that $\operatorname{Fix} T = \operatorname{Fix} T^*$.

$$x \in \operatorname{Fix} T \Leftrightarrow Tx = x \implies \langle Tx, x\rangle = \|x\|^2 = \langle T^* x, x\rangle. \quad \{?\}$$

This shows that

$$\left\langle T^* \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle = 1 \quad \{?\}$$

which means that the angle between the two vectors is zero and since $\|T\frac{x}{\|x\|}\| \leq 1$ it must already be one and the two vectors have to be equal. Thus x is also a fixed point of T^*.

Problem 7 Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of real numbers converging to $\lambda \in \mathbb{R}$. We show that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \alpha_k = \lambda. \qquad \{?\}$$

Let $\epsilon > 0$ be arbitrary and choose K so large that $|\alpha_i - \lambda| < \epsilon$ for all $i \geq K$. Then,

$$\left|\frac{1}{k}\sum_{i=1}^{k} \alpha_k - \lambda\right| = \frac{1}{k}\left|\left(\sum_{i=1}^{K} \alpha_i - \lambda\right) + \left(\sum_{i=K}^{k} \alpha_i - \lambda\right)\right| \leq \frac{const.}{k} + \epsilon. \qquad \{?\}$$

The only thing left to do is to choose k large enough such that $\frac{const.}{k} \leq \epsilon$.

Problem 8 Show that $A^{-ⓥ}$ is good notation.

$$\left(A^{ⓥ}\right)^{-1} = (-\mathrm{Id})^{-1} \circ A^{-1} \circ (-\mathrm{Id})^{-1} = (-\mathrm{Id}) \circ A^{-1} \circ (-\mathrm{Id}) = \left(A^{-1}\right)^{ⓥ} \qquad (3)\ \boxed{\texttt{eq:ov}}$$

Problem 9 Show that $(A,B)^{**} := ((A,B)^*)^* = (A,B)$. Clearly $\left(A^{-1}\right)^{-1} = A$. So the only thing to check is that

$$\left(B^{-ⓥ}\right)^{-ⓥ} = B. \qquad \{?\}$$

By (3)

$$\begin{aligned}
\left(B^{-ⓥ}\right)^{-ⓥ} &= \left(\left(\left(B^{ⓥ}\right)^{-1}\right)^{-1}\right)^{ⓥ} = \left(B^{ⓥ}\right)^{ⓥ} \\
&= (-\mathrm{Id}) \circ B^{ⓥ} \circ (-\mathrm{Id}) \\
&= (-\mathrm{Id}) \circ (-\mathrm{Id}) \circ B \circ (-\mathrm{Id}) \circ (-\mathrm{Id}) \\
&= B.
\end{aligned} \qquad \{?\}$$

Problem 10 (Attouch-Thera duality) For maximally monotone A, B we define $K_z := (Az) \cap (-Bz)$ and $Z_k := A^{-1}k \cap B^{-1}(-k)$. Clearly, Z_k is a closed, convex set for any k as the image of of maximally monotone operator is always closed and convex and so is their intersection. Furthermore we have that

$$\begin{aligned}
k \in K_z &\Leftrightarrow k \in (Az) \cap (-Bz) \\
&\Leftrightarrow k \in Az \wedge k \in -Bz \\
&\Leftrightarrow z \in A^{-1}(k) \wedge z \in B^{-1}(-k) \\
&\Leftrightarrow z \in (A^{-1}k) \cap (B^{-1}(-k)) \\
&\Leftrightarrow z \in Z_k.
\end{aligned} \qquad \{?\}$$

Also, all the solution to $0 \in Ax + Bx$ which we call $Z := (A+B)^{-1}(0)$ are given by $\bigcup_{k\in X} Z_k$. To see this, let $z \in Z$ be a solution. Then K_z is not empty, i.e. it contains an element k. Thus, $z \in Z_k$ and therefore $z \in \bigcup_{k\in X} Z_k$.
Conversely, let z be an element of $\bigcup_{k\in X} Z_k$. Then, there exists a k, such that $z \in Z_k$. Thus, $k \in K_z$ which means that K_z is in particular not empty and z is therefore a solution.

Problem 11 (Passty's convexity result) Show that

$$(\operatorname{gra} A) \cap ((x,0) - \operatorname{gra}(-B)) = \{(y,w) \in \operatorname{gra} A \,|\, (x-y,w) \in \operatorname{gra} B\}. \qquad \{?\}$$

This follows directly from the definition of intersection

$$\begin{aligned}(\operatorname{gra} A) \cap ((x,0) - \operatorname{gra}(-B)) &= \{(y,w) \,|\, (y,w) \in \operatorname{gra} A \wedge (x,0) - (y,-w) \in \operatorname{gra} B\} \\ &= \{(y,w) \,|\, (y,w) \in \operatorname{gra} A \wedge (x-y,w) \in \operatorname{gra} B\} \qquad \{?\} \\ &= \{(y,w) \in \operatorname{gra} A \,|\, (x-y,w) \in \operatorname{gra} B\}.\end{aligned}$$

Problem 12 (parallel sum) Show that

$$(A \mathbin{\square} B)z = \bigcup_{y\in X} (Ay) \cap (B(x-y)). \qquad \{?\}$$

We start of by proofing that

$$\bigcup_{y\in X} (Ay) \cap (B(x-y)) \subseteq (A \mathbin{\square} B)z. \qquad \{?\}$$

Let $z \in \bigcup_{y\in X} (Ay) \cap (B(x-y))$. Then there exists a y' such that

$$z \in (Ay') \cap (B(x-y')). \qquad \{?\}$$

Thus

$$z \in Ay' \quad \wedge \quad z \in B(x-y'). \qquad \{?\}$$

Therefore, by inverting the operators

$$y' \in A^{-1}z \quad \wedge \quad x - y' \in B^{-1}z. \qquad \{?\}$$

Summing up the above inclusions gives

$$x \in (A^{-1} + B^{-1})z. \qquad \{?\}$$

Inverting again gives the desired inclusion. The opposite inclusion follows analogously.

Problem 13 (paramonotone operators) Show that $A+\lambda B$ is paramonotone for paramonotone operators A, B. Let $(x, x^*) \in \operatorname{gra}(A+\lambda B)$ and $(y, y^*) \in \operatorname{gra}(A+\lambda B)$ where $x^* = x_A^* + \lambda x_B^*$ for $x_A^* \in Ax$ and $x_B^* \in Bx$ - analogously for y^*. Assume now that

$$\langle x-y, x_A^* + \lambda x_B^* - y_A^* - \lambda y_B^* \rangle = 0. \qquad \{?\}$$

Then, by linearity of the inner product we get

$$\langle x-y, x_A^* - y_A^* \rangle + \lambda \langle x-y, x_B^* - y_B^* \rangle = 0. \qquad \{?\}$$

Since A and B are monotone both of these summands are nonnegative so they both must be zero. Due to the paramonotonicity we get that

$$\{(x, y_A^*), (y, x_A^*)\} \subseteq \operatorname{gra} A \qquad \{?\}$$

and

$$\{(x, y_B^*), (y, x_B^*)\} \subseteq \lambda \operatorname{gra} B. \qquad \{?\}$$

Thus,

$$\{(x, y_A^* + y_B^*), (y, x_A^* + x_B^*)\} \subseteq \operatorname{gra}(A+\lambda B) \qquad \{?\}$$

which proofs the paramonotonicity of $A+\lambda B$.

Problem 14 (orthogonal sum) Let C, D be nonempty closed convex sets. i) $C+D$ is convex.

$$C+D = \{c+d : c \in C, d \in D\} \qquad \{?\}$$

Let $x, y \in C+D$, then for any $\alpha \in (0,1)$

$$\begin{aligned} \alpha x + (1-\alpha) y &= \alpha(c_x + d_x) + (1-\alpha)(c_y + c_d) \\ &= \alpha c_x + (1-\alpha) c_y + \alpha c_y (1-\alpha) d_y \end{aligned} \qquad \{?\}$$

which is again an element of $C+D$ due to the convexity of C and D. ii) If $C \perp D$, then $C+D$ is closed.
Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $C+D$ with limit x. Due to the orthogonality of C and D we can decompose $x_n = c_n + d_n$ in a unique way, for $c_n \in C$ and $d_n \in D$. In particular, $c_n = \mathrm{P}_C x_n$ and analogously $d_n = \mathrm{P}_D x_n$. Due to the nonexpansiveness of the projection

$$\|x_n - x\| \geq \|\mathrm{P}_C x_n - \mathrm{P}_C x\| \qquad \{?\}$$

c_n and d_n must also converge.
iii) Example where $C+D$ is not closed.
Consider $C = \{(x,y) : e^{-x} \leq y\}$ and $D = \{(-x, 0) : x \geq 0\}$. Then, $(0,y) \in C+D$ for all $y > 0$ as we can decompose it through

$$(0,y) = (-x, 0) + (x, y) \qquad \{?\}$$

for an arbitrary x such that $e^{-x} \leq y$. However, the origin is clearly not contained, making the sum not closed.

Problem 15 (projector onto primal solutions) Let A, B be paramonotone and $Z - Z \perp K$. Show that $J_A \circ \mathrm{P}_{Z+K} = \mathrm{P}_Z$.
Let $z_0 \in Z$. Then $Z - z_0 \perp K$. Thus, $Z + K - z_0$ is closed and convex. Therefore,

$$\begin{aligned}\mathrm{P}_{Z+K}(x) &= \mathrm{P}_{z_0+Z+K}(x) = z_0 + \mathrm{P}_{Z+K-z_0}(x - z_0)\\ &= z_0 + \mathrm{P}_{Z-z_0}(x - z_0) + \mathrm{P}_K(x - z_0) = \mathrm{P}_Z(x) + \mathrm{P}_K(x - z_0).\end{aligned} \quad \{?\}$$

Set now $z := \mathrm{P}_Z(x)$. Then,

$$\mathrm{P}_{Z+K}(x) - z = \mathrm{P}_{Z+K}(x) - \mathrm{P}_Z(x) = P_K(x) \in K = K_z \subseteq Az. \quad \{?\}$$

Hence,

$$z = J_A \circ \mathrm{P}_{Z+K}(x) \quad \{?\}$$

i.e.

$$\mathrm{P}_Z(x) = J_A \circ \mathrm{P}_{Z+K}(x). \quad \{?\}$$

Problem 16 (reflected resolvent calculus) Check the following calculus.
i) $R_{C^{-1}} = -R_C$
Let $y = R_{C^{-1}}(x)$, then

$$\begin{aligned}&y = 2\big(\mathrm{Id} + C^{-1}\big)^{-1}(x) - x\\ \Leftrightarrow &\frac{y+x}{2} = \big(\mathrm{Id} + C^{-1}\big)^{-1}(x)\\ \Leftrightarrow &\frac{y+x}{2} + C^{-1}\left(\frac{y+x}{2}\right) \ni x\\ \Leftrightarrow &C^{-1}\left(\frac{y+x}{2}\right) \ni \frac{x-y}{2}\\ \Leftrightarrow &\frac{y+x}{2} \in C\left(\frac{x-y}{2}\right)\\ \Leftrightarrow &x \in \frac{x-y}{2} + C\left(\frac{x-y}{2}\right)\\ \Leftrightarrow &(\mathrm{Id} + C)^{-1}(x) = \frac{x-y}{2}\\ \Leftrightarrow &x - 2(\mathrm{Id} + C)^{-1}(x) = y\\ \Leftrightarrow &- R_C(x) = y.\end{aligned} \quad \{?\}$$

ii) $J_{C^{\circledvee}} = (J_C)^{\circledvee}$
Let $y = J_{C^{\circledvee}}$, then

$$\begin{aligned}&y = (\mathrm{Id} + (-\mathrm{Id}) \circ C \circ (-\mathrm{Id}))^{-1}(x)\\ \Leftrightarrow &x \in y - C(-y)\\ \Leftrightarrow &x \in -(-y + C(-y))\\ \Leftrightarrow &x \in -(\mathrm{Id} + C)(-y)\\ \Leftrightarrow &- (\mathrm{Id} + C)^{-1}(-x) = y\\ \Leftrightarrow &(J_C)^{\circledvee}(x) = y.\end{aligned} \quad \{?\}$$

iii) Show that $R_{C^{-\textcircled{v}}} = \mathrm{Id} - 2(J_C)^{\textcircled{v}}$.
Utilizing i) and ii) we deduce that

$$R_{C^{-\textcircled{v}}} = R_{(C^{\textcircled{v}})^{-1}} \overset{i)}{=} -R_{C^{\textcircled{v}}} = \mathrm{Id} - 2J_{C^{\textcircled{v}}} \overset{ii)}{=} \mathrm{Id} - 2(J_C)^{\textcircled{v}}. \tag*{\{?\}}$$

Problem 17 (backward-backward operator) Give an example where backward-backward is not self-dual. Let B be the normal cone to $\mathbb{R}_+$ and A the identity on $\mathbb{R}$. Then,

$$J_B \circ J_A(-1) = J_B(-1) = 0 \tag*{\{?\}}$$

and

$$J_{B^{-\textcircled{v}}} \circ J_{A^{-1}}(-1) = J_{B^{-\textcircled{v}}}(-1) = (-\mathrm{Id}) \circ B^{-1}(1) = (-\mathrm{Id})(1) = -1. \tag*{\{?\}}$$

Problem 18 Let A, B be paramonotone and $k \in K$. Assume $J_A(z+k) = \mathrm{P}_Z(z = k)$ for all $z \in Z$. Show that $k \in (Z - Z)^{\perp}$.
As a first small interlude, we will proof that

$$J_A(z+k) = z. \tag*{\{?\}}$$

We probably proofed this at some point but I couldn't find it (or it can very easily deduced from our work about Douglas-Rachford). Either way, since A and B are paramonotone $K = K_z$ for all $z \in Z$. This means that every $k \in K$ fullfills

$$k \in Az \cap (-Bz) \tag*{\{?\}}$$

for all $z \in Z$, and in particular

$$k \in Az. \tag*{\{?\}}$$

Thus $k + z \in (\mathrm{Id} + A)(z)$ and

$$J_A(z+k) = z. \tag*{\{?\}}$$

Now we can use the assumption of this problem that $J_A(z+k) = \mathrm{P}_Z(z+k)$ for all $z \in Z$ to deduce that

$$z = P_Z(z+k) \quad , \forall z \in Z. \tag*{\{?\}}$$

From this we deduce via the projection theorem that

$$\left\langle z - z', z + k - z \right\rangle \leq 0, \quad \forall z, z' \in Z. \tag*{\{?\}}$$

Thus

$$\left\langle z - z', k \right\rangle \leq 0, \quad \forall z, z' \in Z. \tag*{\{?\}}$$

And by reversing the roles of z and z' also

$$\left\langle z' - z, k \right\rangle \leq 0, \quad \forall z, z' \in Z. \tag*{\{?\}}$$

Therefore
$$k \perp (Z - Z). \tag{?}$$

Problem 19 Let U be a closed linear subspace of X and $b \in U^{\perp}$
$\{0\}$, $A := N_U$ and $B = \mathrm{Id} + N_{-b+U}$.
From the paper we already know that
$$J_B = -b + \frac{1}{2}\mathrm{P}_U \tag{?}$$
and the Douglas-Rachford operator
$$T = T_{(A,B)} = \mathrm{P}_{U^\perp} + J_B R_U. \tag{?}$$
By induction we check that
$$T^n = P_{U^\perp} + \frac{1}{2^n}\mathrm{P}_U - nb. \tag{?}$$
In order to see this we consider
$$T^{n+1} = T \circ T^n = (\mathrm{P}_{U^\perp} + (-b + \frac{1}{2}\mathrm{P}_U)R_U) \circ \left(P_{U^\perp} + \left(\frac{1}{2}\right)^n \mathrm{P}_U - nb\right). \tag{?}$$
Expanding the above expressions gives
$$\begin{aligned}
&\left(\mathrm{P}_{U^\perp} + (-b + \frac{1}{2}\mathrm{P}_U)R_U\right) \circ \left(P_{U^\perp} + \frac{1}{2^n}\mathrm{P}_U - nb\right) = \\
&= \mathrm{P}_{U^\perp} + 0 - nb + \left((-b + \frac{1}{2}\mathrm{P}_U)R_U\right) \circ \left(P_{U^\perp} + \frac{1}{2^n}\mathrm{P}_U - nb\right) \\
&= \mathrm{P}_{U^\perp} - nb + -b + \frac{1}{2}\mathrm{P}_U R_U P_{U^\perp} + \frac{1}{2}\mathrm{P}_U R_U \frac{1}{2^n}\mathrm{P}_U - \frac{1}{2}\mathrm{P}_U R_U(nb) \\
&= \mathrm{P}_{U^\perp} - (n+1)b + 0 + \frac{1}{2}\mathrm{P}_U \frac{1}{2^n}\mathrm{P}_U - \frac{n}{2}\mathrm{P}_U(b) \\
&= \mathrm{P}_{U^\perp} - (n+1)b + \frac{1}{2^{n+1}}P_U,
\end{aligned} \tag{?}$$
where used the fact that $R_U P_U = P_U R_U = P_U$ multiple times. This finishes the proof.

Coding Problem Suppose $U = \mathbb{R}^2_+$ and V a line in $\mathbb{R}^2$, not necessarely through the origin. We set $A = N_U$ and $B = N_V$. Clearly J_A is then given by the the projection onto the nonnegative orthant which simply set negative coordinates of a point to zero. Similarly J_B is a projection onto the line which can be realized by subtracting a reference point, projection onto the corresponding 1-dimensional subspace and then readding the reference point, where the the projection of a point x onto the subspace genereated by a vector v is given by the formula
$$\frac{\langle x, v\rangle}{\|v\|^2}v. \tag{?}$$

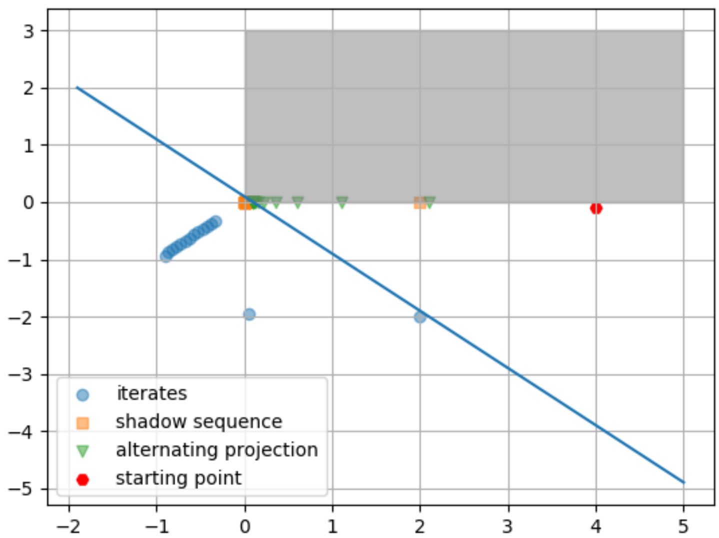

Figure 1: The line has a nonempty intersection with the interior of the nonnegative orthant. Douglas-Rachford converges in finitely many steps to a solution and does so in much less iterations than the alternating projection method.

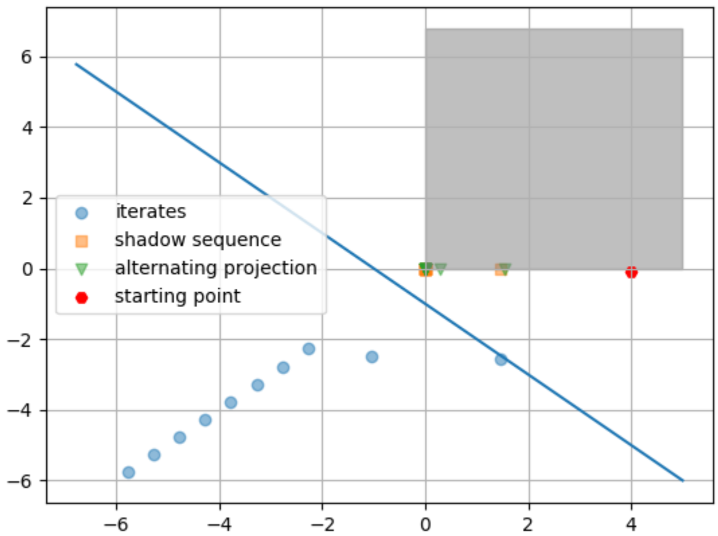

Figure 2: The intersection between line and nonnegative orthant is empty. This means that the problem is infeasible. The iterates thus converge to $+\infty$. The shadow sequence however converges to a normal solution. So do the iterates of the alternating projection method after slightly more iterations.