

Problem 1 Consider $D \subseteq C \subseteq \mathcal{H}$, where C is closed convex and we assume that $P_C(0) \in D$.

(i) Show that D has the minimum norm property and that $P_C(0) = P_D(0)$.
From the definition of the Projection we deduce

$$\|P_C(0)\| \leq \|x\|, \quad \forall x \in C \quad \{?\}$$

thus, as $\overline{\text{conv}}(D) \subseteq C$,

$$\|P_C(0)\| \leq \|x\|, \quad \forall x \in \overline{\text{conv}}(D) \quad (1) \quad \boxed{\text{eq:clconv}}$$

and therefore

$$\|P_C(0)\| \leq \|x\|, \quad \forall x \in D. \quad (2) \quad \boxed{\text{eq:projond}}$$

By assumption $P_C(0) \in D$ and therefore $P_C(0) \in \overline{\text{conv}}(D)$. Combining this with (1) gives that $P_C(0) = P_{\overline{\text{conv}}(D)}(0)$ and proofs the minimum norm property of D . Similarly, we deduce from (2) that $P_C(0) \in D$ is already the projection of 0 onto D , i.e. $P_C(0) = P_D(0)$.

(ii) From the projection theorem we deduce that

$$\langle x_k - P_C(0), 0 - P_C(0) \rangle \leq 0 \quad \text{?eq:lala?}$$

and therefore

$$\langle x_k, P_C(0) \rangle \geq \|P_C(0)\|^2. \quad \{?\}$$

Furthermore, by Cauchy-Schwarz also have that

$$\|P_C(0)\| \|x_k\| \geq \langle x_k, P_C(0) \rangle \geq \|P_C(0)\|^2. \quad \{?\}$$

Thus $\langle x_k, P_C(0) \rangle$ converges to $\|P_C(0)\|^2$. Now consider

$$\begin{aligned} \|x_k - P_C(0)\|^2 &= \langle x_k - P_C(0), x_k - P_C(0) \rangle \\ &= \langle x_k - P_C(0), 0 - P_C(0) \rangle + \langle x_k - P_C(0), x_k \rangle \quad \{?\} \\ &\leq \|x_n\|^2 - \langle P_C(0), x_k \rangle \end{aligned}$$

which proves the strong convergence.

Remark (see Problem 2 and 3) Note that the projection onto a closed convex set is a firmly nonexpansive map and identity minus firmly nonexpansive map is again firmly nonexpansive. Thus, if we look for a projection P in place of $\text{Id} - T$ we can ensure that $T = \text{Id} - P = \text{Id} - (\text{Id} - T)$ is in fact nonexpansive.

Problem 2 Construct an example where $\overline{\text{ran}}(\text{Id} - T)$ is not convex, for a nonexpansive operator $T : C \rightarrow C$.

Consider $C = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ the upper half space and let $P := \text{Id} - T$ be the projection onto the unit ball with center $(0, -1)$. First it is easy to see that the image of C under the projection is nonconvex as is given by the upper half of surface of the ball (not including the points $(1, -1)$ and $(-1, -1)$). Now we need to check that $\text{ran}(T) = \text{ran}(\text{Id} - P)$ is contained in the upper half space (since we require that T maps back to its domain). This is however clear as the domain is above the x -axis and all

images of P are below it. Thus, all vectors given by $x - Px$ point up (and are therefore contained in C).

Problem 3 Construct an example where $\overline{\text{ran}}(\text{Id} - T)$ does not possess the minimum norm property for a nonexpansive operator $T : C \rightarrow X$.

Let $\text{Id} - T$ be the projection onto the unit ball, in \mathbb{R}^2 with domain $\{(x, 1) \in \mathbb{R}^2 : -1 \leq x \leq 1\}$. The range of $\text{Id} - T$ is then clearly a subset of the upper half of the unit circle and does not have the minimum norm property.

Problem 4 Pazy's Trichotomy.

i) $T = \text{Id}$.

Clearly $0 \in \overline{\text{ran}}(\text{Id} - T) = \{0\}$ and $(T^n x) = x$ for all n is therefore bounded.

ii) Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$T(x) := \begin{cases} x + 1, & \text{if } x \leq 1; \\ x + \frac{1}{x}, & \text{if } x > 1. \end{cases} \quad \{?\}$$

Clearly, $\text{ran}(\text{Id} - T) = [-1, 0)$. This means that case ii) of Pazy's Trichotomy is occurring.

Now we want to check that $\limsup_n T^n x = +\infty$ for some/all x .

Clearly, for any x we have that $T^n x$ is greater than 1 after finitely many steps, so we only consider this case. For $x > 1$ we can see that T is monotone. Let $x > y > 1$, then

$$Tx \geq Ty \Leftrightarrow x + \frac{1}{x} \geq y + \frac{1}{y} \Leftrightarrow x - y \geq \frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy} \quad \{?\}$$

where the right hand side $x - y \geq \frac{x - y}{xy}$ is true as $x, y > 1$. We will prove the fact that $T^n x$ is unbounded by contradiction. Assume therefore that $T^n x$ is bounded. As the T is monotone $T^n x$ must converge to a limit which we call θ . Since $T^n x$ converges to θ there exists an $n_0 \in \mathbb{N}$ such that $T^{n_0} x \geq \theta - \frac{1}{\theta}$. By the monotonicity we have that

$$T^{n_0+1} x \geq T\left(\theta - \frac{1}{\theta}\right) = \theta - \frac{1}{\theta} + \frac{1}{\theta - \frac{1}{\theta}} > \theta - \frac{1}{\theta} + \frac{1}{\theta} = \theta, \quad \{?\}$$

which contradicts the convergence of $T^n x$.

Next, we check that $\lim_n \frac{1}{n} T^n x = 0$. Note that

$$T^2 x = Tx + \frac{1}{Tx} = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}} \leq x + \frac{2}{x}. \quad \{?\}$$

By the same token, we have that

$$T^n x \leq x + \frac{n}{x}. \quad \{?\}$$

Thus

$$\limsup_n \frac{T^n x}{n} \leq \limsup_n \left(\frac{x}{n} + \frac{1}{x} \right) \leq \frac{1}{x}. \quad \{?\}$$

We will use this bound now to deduce convergence by considering for arbitrary $k \in \mathbb{N}$

$$\limsup_n \frac{T^n x}{n} = \limsup_n \frac{T^{n+k} x}{n+k} = \limsup_n \frac{T^n(T^k x)}{n+k} \leq \limsup_n \frac{T^n(T^k x)}{n} \leq \frac{1}{T^k x}. \quad \{?\}$$

However, we have seen earlier that $\lim_k T^k x = +\infty$. This shows that $\lim_n \frac{1}{n} T^n x = 0$.

(iii) $Tx := x - z$, for some fixed $z \neq 0$.

Clearly, $\text{ran}(\text{Id} - T) = \{z\}$. Therefore, $0 \notin \overline{\text{ran}}(\text{Id} - T) = \{z\}$. Furthermore

$$T^n x = x - nz. \quad \{?\}$$

Therefore

$$\lim_n T^n x = -\text{sgn}(z) \infty \quad \{?\}$$

and

$$\lim_n \frac{1}{n} \|T^n x\| = \|z\|. \quad \{?\}$$

Problem 5 The Edelstein operator. It is given by

$$T((x_k)_{k \in \mathbb{N}}) := \left(1 + (x_k - 1) \exp\left(\frac{2\pi i}{k!}\right) \right)_{k \in \mathbb{N}}. \quad \{?\}$$

Check that $(T^n 0)_k = 1 - \exp\left(\frac{2\pi i n}{k!}\right)$.

For $n = 1$, this follows immediately from the definition. Assume it is true for $n - 1$. Then,

$$\begin{aligned} (T^n 0) &= (T \circ T^{n-1} 0) = T\left(\left(1 - \exp\left(\frac{2\pi i n}{k!}\right)\right)_{k \in \mathbb{N}}\right) \\ &= \left(1 - \exp\left(\frac{2\pi i(n-1)}{k!}\right) \exp\left(\frac{2\pi i}{k!}\right)\right)_{k \in \mathbb{N}} \\ &= \left(1 - \exp\left(\frac{2\pi i n}{k!}\right)\right)_{k \in \mathbb{N}}. \end{aligned} \quad \{?\}$$

Next, for $x \in \ell^2$ we have that $\|T^k x - T^{k+1} x\| = \|Tx - x\|$ since T is an isometry. Thus, $\lim_k \|T^k x - T^{k+1} x\| = \|Tx - x\|$, which is strictly greater zero as T has no fixed points.

Problem 6 Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be linear, and nonexpansive. Show that $\text{Fix } T = \text{Fix } T^*$.

$$x \in \text{Fix } T \Leftrightarrow Tx = x \implies \langle Tx, x \rangle = \|x\|^2 = \langle T^* x, x \rangle. \quad \{?\}$$

This shows that

$$\left\langle T^* \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle = 1 \quad \{?\}$$

which means that the angle between the two vectors is zero and since $\|T \frac{x}{\|x\|}\| \leq 1$ it must already be one and the two vectors have to be equal. Thus x is also a fixed point of T^* .

Problem 7 Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence of real numbers converging to $\lambda \in \mathbb{R}$. We show that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \alpha_k = \lambda. \quad \{?\}$$

Let $\epsilon > 0$ be arbitrary and choose K so large that $|\alpha_i - \lambda| < \epsilon$ for all $i \geq K$. Then,

$$\left| \frac{1}{k} \sum_{i=1}^k \alpha_k - \lambda \right| = \frac{1}{k} \left| \left(\sum_{i=1}^K \alpha_i - \lambda \right) + \left(\sum_{i=K+1}^k \alpha_i - \lambda \right) \right| \leq \frac{\text{const.}}{k} + \epsilon. \quad \{?\}$$

The only thing left to do is to choose k large enough such that $\frac{\text{const.}}{k} \leq \epsilon$.

Problem 8 Show that $A^{-\textcircled{\vee}}$ is good notation.

$$(A^{\textcircled{\vee}})^{-1} = (-\text{Id})^{-1} \circ A^{-1} \circ (-\text{Id})^{-1} = (-\text{Id}) \circ A^{-1} \circ (-\text{Id}) = (A^{-1})^{\textcircled{\vee}} \quad (3) \quad \boxed{\text{eq:ov}}$$

Problem 9 Show that $(A, B)^{**} := ((A, B)^*)^* = (A, B)$. Clearly $(A^{-1})^{-1} = A$. So the only thing to check is that

$$(B^{-\textcircled{\vee}})^{-\textcircled{\vee}} = B. \quad \{?\}$$

By (3)

$$\begin{aligned} (B^{-\textcircled{\vee}})^{-\textcircled{\vee}} &= \left(\left((B^{\textcircled{\vee}})^{-1} \right)^{-1} \right)^{\textcircled{\vee}} = (B^{\textcircled{\vee}})^{\textcircled{\vee}} \\ &= (-\text{Id}) \circ B^{\textcircled{\vee}} \circ (-\text{Id}) \\ &= (-\text{Id}) \circ (-\text{Id}) \circ B \circ (-\text{Id}) \circ (-\text{Id}) \\ &= B. \end{aligned} \quad \{?\}$$

Problem 10 (Attouch-Thera duality) For maximally monotone A, B we define $K_z := (Az) \cap (-Bz)$ and $Z_k := A^{-1}k \cap B^{-1}(-k)$. Clearly, Z_k is a closed, convex set for any k as the image of of maximally monotone operator is always closed and convex and so is their intersection. Furthermore we have that

$$\begin{aligned} k \in K_z &\Leftrightarrow k \in (Az) \cap (-Bz) \\ &\Leftrightarrow k \in Az \wedge k \in -Bz \\ &\Leftrightarrow z \in A^{-1}(k) \wedge z \in B^{-1}(-k) \\ &\Leftrightarrow z \in (A^{-1}k) \cap (B^{-1}(-k)) \\ &\Leftrightarrow z \in Z_k. \end{aligned} \quad \{?\}$$

Also, all the solution to $0 \in Ax + Bx$ which we call $Z := (A + B)^{-1}(0)$ are given by $\bigcup_{k \in X} Z_k$. To see this, let $z \in Z$ be a solution. Then K_z is not empty, i.e. it contains an element k . Thus, $z \in Z_k$ and therefore $z \in \bigcup_{k \in X} Z_k$. Conversely, let z be an element of $\bigcup_{k \in X} Z_k$. Then, there exists a k , such that $z \in Z_k$. Thus, $k \in K_z$ which means that K_z is in particular not empty and z is therefore a solution.

Problem 11 (Passty's convexity result) Show that

$$(\text{gra } A) \cap ((x, 0) - \text{gra } (-B)) = \{(y, w) \in \text{gra } A \mid (x - y, w) \in \text{gra } B\}. \quad \{?\}$$

This follows directly from the definition of intersection

$$\begin{aligned} (\text{gra } A) \cap ((x, 0) - \text{gra } (-B)) &= \{(y, w) \mid (y, w) \in \text{gra } A \wedge (x, 0) - (y, -w) \in \text{gra } B\} \\ &= \{(y, w) \mid (y, w) \in \text{gra } A \wedge (x - y, w) \in \text{gra } B\} \\ &= \{(y, w) \in \text{gra } A \mid (x - y, w) \in \text{gra } B\}. \end{aligned} \quad \{?\}$$

Problem 12 (parallel sum) Show that

$$(A \square B)z = \bigcup_{y \in X} (Ay) \cap (B(x - y)). \quad \{?\}$$

We start of by proofing that

$$\bigcup_{y \in X} (Ay) \cap (B(x - y)) \subseteq (A \square B)z. \quad \{?\}$$

Let $z \in \bigcup_{y \in X} (Ay) \cap (B(x - y))$. Then there exists a y' such that

$$z \in (Ay') \cap (B(x - y')). \quad \{?\}$$

Thus

$$z \in Ay' \quad \wedge \quad z \in B(x - y'). \quad \{?\}$$

Therefore, by inverting the operators

$$y' \in A^{-1}z \quad \wedge \quad x - y' \in B^{-1}z. \quad \{?\}$$

Summing up the above inclusions gives

$$x \in (A^{-1} + B^{-1})z. \quad \{?\}$$

Inverting again gives the desired inclusion. The opposite inclusion follows analogously.

Problem 13 (paramonotone operators) Show that $A + \lambda B$ is paramonotone for paramonotone operators A, B . Let $(x, x^*) \in \text{gra}(A + \lambda B)$ and $(y, y^*) \in \text{gra}(A + \lambda B)$ where $x^* = x_A^* + \lambda x_B^*$ for $x_A^* \in Ax$ and $x_B^* \in Bx$ - analogously for y^* . Assume now that

$$\langle x - y, x_A^* + \lambda x_B^* - y_A^* - \lambda y_B^* \rangle = 0. \quad \{?\}$$

Then, by linearity of the inner product we get

$$\langle x - y, x_A^* - y_A^* \rangle + \lambda \langle x - y, x_B^* - y_B^* \rangle = 0. \quad \{?\}$$

Since A and B are monotone both of these summands are nonnegative so they both must be zero. Due to the paramonotonicity we get that

$$\{(x, y_A^*), (y, x_A^*)\} \subseteq \text{gra } A \quad \{?\}$$

and

$$\{(x, y_B^*), (y, x_B^*)\} \subseteq \text{gra } B. \quad \{?\}$$

Thus,

$$\{(x, y_A^* + y_B^*), (y, x_A^* + x_B^*)\} \subseteq \text{gra}(A + \lambda B) \quad \{?\}$$

which proves the paramonotonicity of $A + \lambda B$.

Problem 14 (orthogonal sum) Let C, D be nonempty closed convex sets. i) $C + D$ is convex.

$$C + D = \{c + d : c \in C, d \in D\} \quad \{?\}$$

Let $x, y \in C + D$, then for any $\alpha \in (0, 1)$

$$\begin{aligned} \alpha x + (1 - \alpha)y &= \alpha(c_x + d_x) + (1 - \alpha)(c_y + d_d) \\ &= \alpha c_x + (1 - \alpha)c_y + \alpha d_x + (1 - \alpha)d_d \end{aligned} \quad \{?\}$$

which is again an element of $C + D$ due to the convexity of C and D . ii) If $C \perp D$, then $C + D$ is closed.

Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in $C + D$ with limit x . Due to the orthogonality of C and D we can decompose $x_n = c_n + d_n$ in a unique way, for $c_n \in C$ and $d_n \in D$. In particular, $c_n = P_C x_n$ and analogously $d_n = P_D x_n$. Due to the nonexpansiveness of the projection

$$\|x_n - x\| \geq \|P_C x_n - P_C x\| \quad \{?\}$$

c_n and d_n must also converge.

iii) Example where $C + D$ is not closed.

Consider $C = \{(x, y) : e^{-x} \leq y\}$ and $D = \{(-x, 0) : x \geq 0\}$. Then, $(0, y) \in C + D$ for all $y > 0$ as we can decompose it through

$$(0, y) = (-x, 0) + (x, y) \quad \{?\}$$

for an arbitrary x such that $e^{-x} \leq y$. However, the origin is clearly not contained, making the sum not closed.

Problem 15 (projector onto primal solutions) Let A, B be paramonotone and $Z - Z \perp K$. Show that $J_A \circ P_{Z+K} = P_Z$.

Let $z_0 \in Z$. Then $Z - z_0 \perp K$. Thus, $Z + K - z_0$ is closed and convex. Therefore,

$$\begin{aligned} P_{Z+K}(x) &= P_{z_0+Z+K}(x) = z_0 + P_{Z+K-z_0}(x - z_0) \\ &= z_0 + P_{Z-z_0}(x - z_0) + P_K(x - z_0) = P_Z(x) + P_K(x - z_0). \end{aligned} \quad \{?\}$$

Set now $z := P_Z(x)$. Then,

$$P_{Z+K}(x) - z = P_{Z+K}(x) - P_Z(x) = P_K(x) \in K = K_z \subseteq Az. \quad \{?\}$$

Hence,

$$z = J_A \circ P_{Z+K}(x) \quad \{?\}$$

i.e.

$$P_Z(x) = J_A \circ P_{Z+K}(x). \quad \{?\}$$

Problem 16 (reflected resolvent calculus) Check the following calculus.

i) $R_{C^{-1}} = -R_C$

Let $y = R_{C^{-1}}(x)$, then

$$\begin{aligned} y &= 2(\text{Id} + C^{-1})^{-1}(x) - x \\ \Leftrightarrow \frac{y+x}{2} &= (\text{Id} + C^{-1})^{-1}(x) \\ \Leftrightarrow \frac{y+x}{2} + C^{-1}\left(\frac{y+x}{2}\right) &\ni x \\ \Leftrightarrow C^{-1}\left(\frac{y+x}{2}\right) &\ni \frac{x-y}{2} \\ \Leftrightarrow \frac{y+x}{2} &\in C\left(\frac{x-y}{2}\right) \\ \Leftrightarrow x &\in \frac{x-y}{2} + C\left(\frac{x-y}{2}\right) \\ \Leftrightarrow (\text{Id} + C)^{-1}(x) &= \frac{x-y}{2} \\ \Leftrightarrow x - 2(\text{Id} + C)^{-1}(x) &= y \\ \Leftrightarrow -R_C(x) &= y. \end{aligned} \quad \{?\}$$

ii) $J_C^{\textcircled{\vee}} = (J_C)^{\textcircled{\vee}}$

Let $y = J_C^{\textcircled{\vee}}(x)$, then

$$\begin{aligned} y &= (\text{Id} + (-\text{Id}) \circ C \circ (-\text{Id}))^{-1}(x) \\ \Leftrightarrow x &\in y - C(-y) \\ \Leftrightarrow x &\in -(-y + C(-y)) \\ \Leftrightarrow x &\in -(\text{Id} + C)(-y) \\ \Leftrightarrow -(\text{Id} + C)^{-1}(-x) &= y \\ \Leftrightarrow (J_C)^{\textcircled{\vee}}(x) &= y. \end{aligned} \quad \{?\}$$

iii) Show that $R_{C^-\mathbb{V}} = \text{Id} - 2(J_C)^{\mathbb{V}}$.
Utilizing i) and ii) we deduce that

$$R_{C^-\mathbb{V}} = R_{(C^{\mathbb{V}})^{-1}} \stackrel{i)}{=} -R_{C^{\mathbb{V}}} = \text{Id} - 2J_{C^{\mathbb{V}}} \stackrel{ii)}{=} \text{Id} - 2(J_C)^{\mathbb{V}}. \quad \{?\}$$

Problem 17 (backward-backward operator) Give an example where backward-backward is not self-dual. Let B be the normal cone to \mathbb{R}_+ and A the identity on \mathbb{R} . Then,

$$J_B \circ J_A(-1) = J_B(-1) = 0 \quad \{?\}$$

and

$$J_{B^-\mathbb{V}} \circ J_{A^{-1}}(-1) = J_{B^-\mathbb{V}}(-1) = (-\text{Id}) \circ B^{-1}(1) = (-\text{Id})(1) = -1. \quad \{?\}$$

Problem 18 Let A, B be paramonotone and $k \in K$. Assume $J_A(z + k) = P_Z(z + k)$ for all $z \in Z$. Show that $k \in (Z - Z)^\perp$.

As a first small interlude, we will proof that

$$J_A(z + k) = z. \quad \{?\}$$

We probably proofed this at some point but I couldn't find it (or it can very easily deduced from our work about Douglas-Rachford). Either way, since A and B are paramonotone $K = K_z$ for all $z \in Z$. This means that every $k \in K$ fullfills

$$k \in Az \cap (-Bz) \quad \{?\}$$

for all $z \in Z$, and in particular

$$k \in Az. \quad \{?\}$$

Thus $k + z \in (\text{Id} + A)(z)$ and

$$J_A(z + k) = z. \quad \{?\}$$

Now we can use the assumption of this problem that $J_A(z + k) = P_Z(z + k)$ for all $z \in Z$ to deduce that

$$z = P_Z(z + k), \quad \forall z \in Z. \quad \{?\}$$

From this we deduce via the projection theorem that

$$\langle z - z', z + k - z \rangle \leq 0, \quad \forall z, z' \in Z. \quad \{?\}$$

Thus

$$\langle z - z', k \rangle \leq 0, \quad \forall z, z' \in Z. \quad \{?\}$$

And by reversing the roles of z and z' also

$$\langle z' - z, k \rangle \leq 0, \quad \forall z, z' \in Z. \quad \{?\}$$

Therefore

$$k \perp (Z - Z). \quad \{?\}$$

Problem 19 Let U be a closed linear subspace of X and $b \in U^\perp \setminus \{0\}$, $A := N_U$ and $B = \text{Id} + N_{-b+U}$.

From the paper we already know that

$$J_B = -b + \frac{1}{2}P_U \quad \{?\}$$

and the Douglas-Rachford operator

$$T = T_{(A,B)} = P_{U^\perp} + J_B R_U. \quad \{?\}$$

By induction we check that

$$T^n = P_{U^\perp} + \frac{1}{2^n}P_U - nb. \quad \{?\}$$

In order to see this we consider

$$T^{n+1} = T \circ T^n = (P_{U^\perp} + (-b + \frac{1}{2}P_U)R_U) \circ \left(P_{U^\perp} + \left(\frac{1}{2}\right)^n P_U - nb \right). \quad \{?\}$$

Expanding the above expressions gives

$$\begin{aligned} & \left(P_{U^\perp} + (-b + \frac{1}{2}P_U)R_U \right) \circ \left(P_{U^\perp} + \frac{1}{2^n}P_U - nb \right) = \\ &= P_{U^\perp} + 0 - nb + \left((-b + \frac{1}{2}P_U)R_U \right) \circ \left(P_{U^\perp} + \frac{1}{2^n}P_U - nb \right) \\ &= P_{U^\perp} - nb + -b + \frac{1}{2}P_U R_U P_{U^\perp} + \frac{1}{2}P_U R_U \frac{1}{2^n}P_U - \frac{1}{2}P_U R_U (nb) \\ &= P_{U^\perp} - (n+1)b + 0 + \frac{1}{2}P_U \frac{1}{2^n}P_U - \frac{n}{2}P_U(b) \\ &= P_{U^\perp} - (n+1)b + \frac{1}{2^{n+1}}P_U, \end{aligned} \quad \{?\}$$

where used the fact that $R_U P_U = P_U R_U = P_U$ multiple times. This finishes the proof.

Coding Problem Suppose $U = \mathbb{R}_+^2$ and V a line in \mathbb{R}^2 , not necessarily through the origin. We set $A = N_U$ and $B = N_V$. Clearly J_A is then given by the the projection onto the nonnegative orthant which simply set negative coordinates of a point to zero. Similarly J_B is a projection onto the line which can be realized by subtracting a reference point, projection onto the corresponding 1-dimensional subspace and then readding the reference point, where the the projection of a point x onto the subspace generated by a vector v is given by the formula

$$\frac{\langle x, v \rangle}{\|v\|^2} v. \quad \{?\}$$

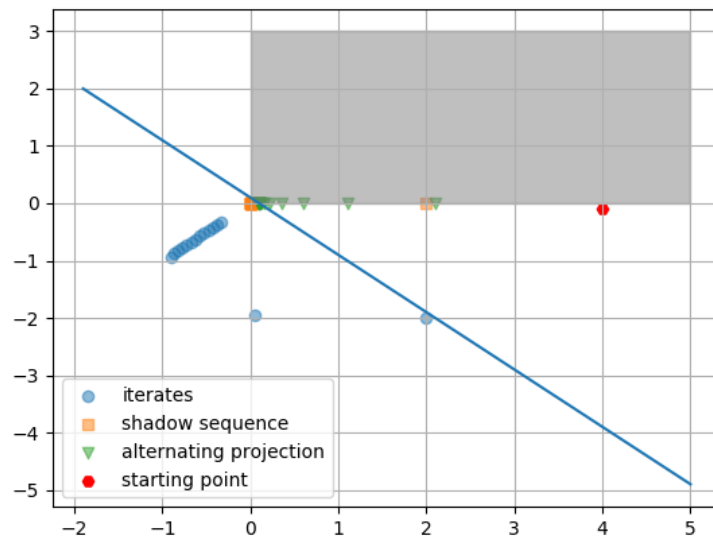


Figure 1: The line has a nonempty intersection with the interior of the nonnegative orthant. Douglas-Rachford converges in finitely many steps to a solution and does so in much less iterations than the alternating projection method.

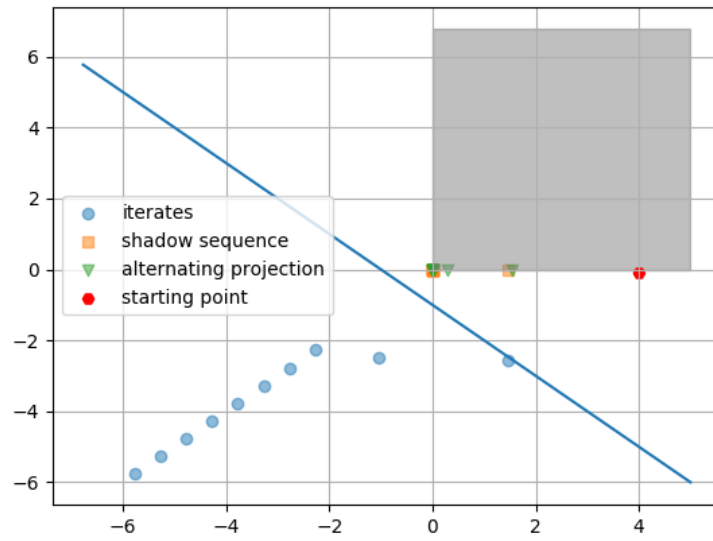


Figure 2: The intersection between line and nonnegative orthant is empty. This means that the problem is infeasible. The iterates thus converge to $+\infty$. The shadow sequence however converges to a normal solution. So do the iterates of the alternating projection method after slightly more iterations.