Matrix games

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Introduction

Given

- Player I (rows, Alice)
- Player II (columns, Bob)
- a *payoff* matrix $A \in \mathbb{R}^{m \times n}$

Every round

1. Alice picks (row) strategy $i \in [m] := \{1, \dots, m\}$ Bob picks (col) strategy $j \in [n]$
2. Bob pays Alice the amount $a_{i,j}$

zero-sum game

Example: penalty game

		kicker	
		L	R
goalkeeper	L	1	-1
	R	-1	1

Figure: penalty game

Example: prisoners dilemma

	Confess A	Stay quiet A
Confess B	6 6	10 0
Stay quiet B	0 10	2 2

Figure

Worst case

- Alice gets $\min_{j\in[n]} a_{i,j}$

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We claim:

$$\max_i \min_j a_{i,j} \leq \min_j \max_i a_{i,j}$$

"Tallest dwarf is not as tall as the smallest giant."

But equality does not hold in general!

Proof of the min-max theorem

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Proof of the min-max theorem

$$
\begin{aligned}
a_{ij} &\leq a_{ij} && \forall i, j \\
a_{ij} &\leq \max_i a_{ij} && \forall i, j \\
\min_j a_{ij} &\leq \min_j \max_i a_{ij} && \forall i
\end{aligned}
$$

Proof of the min-max theorem

$$\begin{aligned} a_{ij} &\leq a_{ij} && \forall i, j \\ a_{ij} &\leq \max_i a_{ij} && \forall i, j \\ \min_j a_{ij} &\leq \min_j \max_i a_{ij} && \forall i \end{aligned}$$

Definition

We call (i^*, j^*) a saddle point (or *Nash equilibrium*) if

$$\max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j}.$$

These are called *pure strategies*.

Rock paper scissors

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Mixed Strategies

With pure strategies we do not always have a saddle point.

von Neumann (1928) — Mixed strategies

- Alice picks strategies $1, \ldots, m$ *with probabilities* $x \in \Delta_m$
- Bob picks strategies $1, \ldots, n$ *with probabilities* $x \in \Delta_n$

Expected gain of Alice is

$$\langle x, Ay \rangle = \sum_{i,j} a_{ij} x_i y_j$$

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Theorem

Saddle point exists Expected gain of Alice = expected loss of Bob

$$\max_{x \in \Delta} \min_{y \in \Delta} \langle x, Ay \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle x, Ay \rangle.$$

Stopping criteria

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle =: v$$

$$\max_{y\in\Delta}\min_{x\in\Delta}\langle x, Ay\rangle = v$$

stopping criterion

$$f_p(x) - f_p(x^*) = f_p(x) - v$$
$$f_d(y^*) - f_d(y) = v - f_d(y)$$

Before we never had the optimal value!

Solution will always be on the boundary:

$$f_p(x) = \max_{y\in\Delta}\langle A^T x, y\rangle = \max_{j}\langle A^T x, e_j\rangle$$

Stopping criteria

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle =: v$$

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$$f_p(x) - f_p(x^*) = f_p(x) - v \leq \epsilon/2$$

$$f_d(y^*) - f_d(y) = v - f_d(y) \leq \epsilon/2$$

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$$\Rightarrow f_p(x) - f_d(y) \leq \epsilon$$

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Consider

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle$$

as a minimization problem

$$\min_{x\in\Delta}\langle x, Ay^*\rangle$$

Then

$$x^* \in \arg\min_{y\in\Delta} f_p(x) \Leftrightarrow \langle \nabla f_p(x^*), x - x^*\rangle \geq 0 \quad \forall x \in D$$

Thus

$$\langle A^T y^*, x - x^*\rangle \geq 0 \quad \forall x \in D$$
$$\langle -Ax^*, y - y^*\rangle \geq 0 \quad \forall y \in D$$

Concatenate the two conditions to get

$$\left\langle \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix} \right\rangle$$

By rewriting $z = (x, y)$ and $F(z) = [A^T y; Ax]$, then

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in \Delta_n \times \Delta_m := C \tag{VI}$$

Variational inequality

If $F = \nabla \varphi$ then (VI) would be equivalent to

$$\min_C \varphi$$