Matrix games

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1 Introduction

Introduction

Given

- Player I (rows, Alice)
- Player II (columns, Bob)
- a *payoff* matrix $A \in \mathbb{R}^{m \times n}$

Every round

1. Alice picks (row) strategy $i \in [m] := \{1, \ldots, m\}$ Bob picks (col) strategy $j \in [n]$
2. Bob pays Alice the amount $a_{i,j}$

zero-sum game

Example: penalty game

		kicker	
		L	R
goalkeeper	L	1	-1
	R	-1	1

Figure: penalty game

Example: prisoners dilemma

	Confess A	Stay quiet A
Confess B	6 6	10 0
Stay quiet B	0 10	2 2

Worst case

- Alice gets $\min_{j \in [n]} a_{i,j}$

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We claim:

$$\max_i \min_j a_{i,j} \leq \min_j \max_i a_{i,j}$$

"Tallest dwarf is not as tall as the smallest giant."

But equality does not hold in general!

Proof of the min-max theorem

$$a_{ij} \leq a_{ij} \qquad \forall i, j$$

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$$
\begin{aligned}
a_{ij} &\le a_{ij} && \forall i, j \\
a_{ij} &\le \max_i a_{ij} && \forall i, j \\
\min_j a_{ij} &\le \min_j \max_i a_{ij} && \forall i
\end{aligned}
$$