Matrix scaling

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1. Introduction

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given: a matrix $A \in \mathbb{R}_{+}^{m\times n}$, vectors $r \in \mathbb{R}_{++}^{m}$ and $c \in \mathbb{R}_{++}^{n}$
find: diagonal matrices X and Y such that for $B = XAY$ it holds:

$$B\mathbb{1}_n = r \quad \text{and} \quad B^T\mathbb{1}_m = c$$

where $\mathbb{1}_n = (1, \ldots, 1)$ exactly n-times. Equivalently

$$\|B_{i,:}\|_1 = r_i \quad \text{and} \|B_{:,j}\| = c_j.$$

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If $\|r\|_1 \neq \|c\|_2$ this is not possible.

Visualization of diagonal scaling

$$
\begin{aligned}
B &= \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix} A \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{bmatrix} \\
&= \begin{bmatrix} a_{1,1}x_1y_1 & a_{1,2}x_1y_2 & \cdots & a_{1,n}x_1y_m \\ \vdots & \ddots & & \\ a_{m,1}x_my_1 & & \cdots & a_{m,n}x_my_m \end{bmatrix}
\end{aligned}
$$

Applications

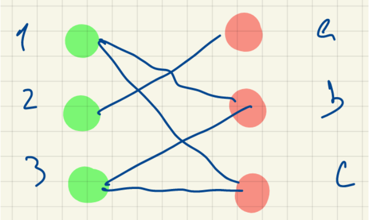
Ill conditioned linear system $Az = b$.

Can multiply both sides by X and substitute $z = Yv$ to get instead

$$XAz = X$$

$(0-1)$ matrices | bipartite graphs

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$



Figure

Definition