Matrix games

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Introduction

Given

- Player I (rows, Alice)
- Player II (columns, Bob)
- a *payoff* matrix $A \in \mathbb{R}^{m \times n}$

Every round

(i) Alice picks (row) strategy $i \in [m] := \{1, \dots, m\}$ Bob picks (col) strategy $j \in [n]$

(ii) Bob pays Alice the amount $a_{i,j}$

zero-sum game

Example: penalty game

		kicker L	kicker R
goalkeeper	L	1	-1
	R	-1	1

Figure: penalty game

Example: prisoners dilemma

	Confess A	Stay quiet A
Confess B	6 6	10 0
Stay quiet B	0 10	2 2

Figure: prisoners dilemma

Worst case

- if Alice chooses strategy i she gets (at least): $\min_{j\in[n]} a_{i,j}$
- Alice can ensure payoff $\max_{i\in[m]} \min_{j\in[n]} a_{i,j}$
- Bob pays (at most) $\min_{j\in[n]} \max_{i\in[m]} a_{i,j}$

We claim:

$$\max_i \min_j a_{i,j} \leq \min_j \max_i a_{i,j}$$

"Tallest dwarf is not as tall as the smallest giant."

But: **No equality in general!**

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Proof of the min-max theorem

$$a_{ij} \leq a_{ij} \qquad \forall i, j$$

$$a_{ij} \leq \max_i a_{ij} \qquad \forall i, j$$

$$\min_j a_{ij} \leq \min_j \max_i a_{ij} \qquad \forall i$$

Definition

We call (i^*, j^*) a saddle point (or *Nash equilibrium*) if

$$\max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j}.$$

These are called *pure strategies*.

Proof of the min-max theorem

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\begin{aligned}
a_{ij} &\le a_{ij} && \forall i, j \\
a_{ij} &\le \max_i a_{ij} && \forall i, j \\
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Rock paper scissors

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Mixed Strategies

With pure strategies we do not always have a saddle point.

von Neumann (1928) — Mixed strategies

- Alice picks strategies $1, \ldots, m$ *with probabilities* $x \in \Delta_m$
- Bob picks strategies $1, \ldots, n$ *with probabilities* $x \in \Delta_n$

Expected gain of Alice is

$$\langle x, Ay \rangle = \sum_{i,j} a_{ij} x_i y_j$$

Theorem

Saddle point exists Expected gain of Alice = expected loss of Bob

$$\max_{x \in \Delta} \min_{y \in \Delta} \langle x, Ay \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle x, Ay \rangle.$$

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Stopping criteria

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle =: v$$

$$\max_{y\in\Delta}\min_{x\in\Delta}\langle x, Ay\rangle = v$$

stopping criterion

$$f_p(x) - f_p(x^*) = f_p(x) - v \leq \epsilon/2$$
$$f_d(y^*) - f_d(y) = v - f_d(y) \leq \epsilon/2$$
$$\Rightarrow f_p(x) - f_d(y) \leq \epsilon$$

Before we never had the optimal value!

Solution will always be on the boundary:

$$f_p(x) = \max_{y\in\Delta}\langle A^T x, y\rangle = \max_j \langle A^T x, e_j\rangle$$

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Consider

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle$$

as a minimization problem

$$\min_{x\in\Delta} f_p(x) = \langle x, Ay^*\rangle.$$

Then, by the **first-order optimality condition**

$$x^* \in \arg\min_{x\in\Delta} f_p(x) \Leftrightarrow \langle \nabla f_p(x^*), x - x^*\rangle \geq 0 \quad \forall x \in \Delta$$

Thus

$$\begin{aligned} \langle A^T y^*, x - x^*\rangle &\geq 0 \quad \forall x \in \Delta \\ \langle -Ax^*, y - y^*\rangle &\geq 0 \quad \forall y \in \Delta \end{aligned}$$

Concatenate the two conditions to get

$$\left\langle \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix} \right\rangle \geq 0.$$

Games as Variational Inequalities

We had:

$$\left\langle \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix} \right\rangle \geq 0.$$

By rewriting $z = (x, y)$ and $F(z) = [A^T y; Ax]$, then

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in \Delta_n \times \Delta_m =: C \tag{VI}$$

Variational inequality

If $F = \nabla \varphi$ then (VI) would be equivalent to

$$\min_{z \in C} \varphi(z)$$

Potential - integrability

Question: Does there exist a potential φ for F, such that $F = \nabla\varphi$

Integrability condition (from calculus)

Is the case if

$$\frac{\partial\varphi}{\partial x\partial y} = \frac{\partial\varphi}{\partial y\partial x}$$

But

$$\frac{\partial F_1}{\partial y} = A^T \neq -A = \frac{\partial F_2}{\partial x}$$

VI as Fixed point equation

$$\begin{aligned} &\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in C \\ \Leftrightarrow\ & z^* = P_C(z^* - F(z^*)) \end{aligned} \tag{FP}$$

Proof.

Applying the property of the projection

$$\langle P_C(x) - x, x' - P_C(x) \rangle \geq 0 \quad \forall x' \in C$$

with (FP), gives

$$\langle z^* - (z^* - F(z^*)), z - z^* \rangle \geq 0 \quad \forall z \in C. \quad \square$$

- should remind us of (projected) gradient descent
- when you see a fixed point equation: iterate!

But is it any good?

$$z_{k+1} = z_k - \alpha F(z_k)$$

Then

$$\begin{aligned} \|z_{k+1}\|^2 &= \|z_k\|^2 - \underbrace{2\alpha\langle F(z_k), z_k\rangle}_{=0} + \alpha^2\|F(z_k)\|^2 \\ &= \|z_k\|^2 + \alpha^2\|F(z_k)\|^2 \end{aligned}$$

Resulting in $\|z_{k+1}\| \geq \|z_k\|$.

$\Rightarrow$ No bueno!

Theorem

We can still show a complexity of $\mathcal{O}(1/\sqrt{k})$ with the same analysis as for subgradient descent for averaged iterates in terms of $f_p(x) - f_d(y)$.

Sketch of the proof

With the notation $g_k = F(z_k)$ we get

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leq \|x_k - \alpha_k g_k - x^*\|^2 \\ &= \|x_k - x^*\|^2 + 2\alpha_k \langle g_k, x^* - x_k\rangle + \alpha^2 \|g_k\|^2.\end{aligned}$$

But now $\langle g_k, x^* - x_k\rangle = [f_d(y_k) - f_p(x_k)]$. Rest of the proof is left as an exercise.