Projected Gradient Descent

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1. Introduction

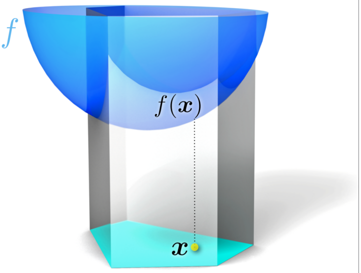
Constrained Optimization

Constrained optimization problem

$$\text{minimize } f(x)$$
$$\text{subject to } x \in C$$

How to solve them

- Project onto C
- transform to *unconstrained problem*



Constrained Optimization

Constrained optimization problem

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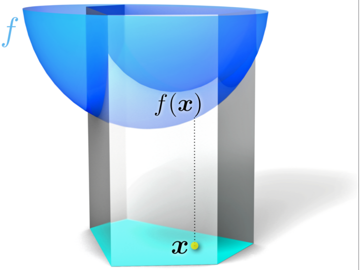
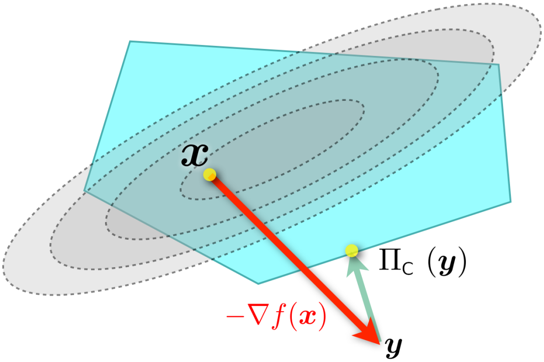
We will focus on:

- **Projected Gradient Descent**



Projected Gradient Descent

Idea: After every step project back onto the set:
$\Pi_C(x) := \arg\min_{y \in C} \|y - x\|$.



Projected subgradient method

$$\text{(constrained setting)} \quad \min_{x \in C} f(x)$$

Algorithm Projected subgradient method

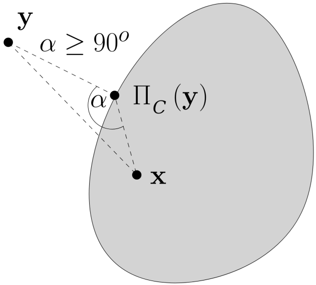
1: **for** $k = 0, 1, \ldots$ **do**
2: Pick $g_k \in \partial f(x_k)$
3: $y_{k+1} = x_k - \alpha g_k$
4: $x_{k+1} = \Pi_C(y_{k+1})$

Properties of the Projection

Fact

Let $C \subseteq \mathbb{R}^d$ be closed and convex, $x \in C$ and $y \in \mathbb{R}^d$.Then

- $\langle x - \Pi_C(y), y - \Pi_C(y) \rangle \leq 0$
- $\|x - \Pi_C(y)\|^2 + \|y - \Pi_c(y)\|^2 \leq \|y - x\|^2$



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Proof.

Since $\Pi_C(x)$ is the minimizer of a differentiable convex function $d_x(y) = \frac{1}{2}\|y - x\|^2$ over C, by the **first-order optimality condition**

$$\begin{aligned} 0 &\leq \langle \nabla d_x(\Pi_C(x)), y - \Pi_C(x) \rangle \\ &= \langle \Pi_C(x) - x, y - \Pi_C(x) \rangle \end{aligned}$$

$\square$

Results for projected GD

Projected subgradient method II

Proof.

We can deduce the exact same inequality as before

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|\Pi_C(x_k - \alpha g_k) - \Pi_C(x^*)\|^2 \\
&\leq \|x_k - \alpha g_k - x^*\|^2 \\
&= \|x_k - x^*\|^2 + 2\alpha\langle g_k, x^* - x_k\rangle + \alpha^2\|g_k\|^2 \\
&\leq \|x_k - x^*\|^2 + 2\alpha(f^* - f(x_k)) + \alpha^2\|g_k\|^2.
\end{aligned}$$

$\square$