

Newton's and Quasi-Newton Methods

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1-dimensional case: Newton-Raphson method

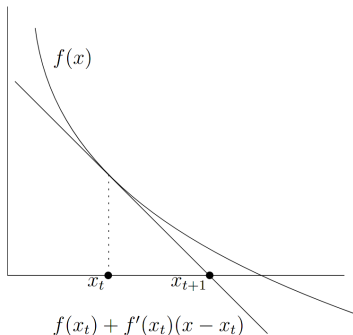
Objective: Find zero of differentiable $f : \mathbb{R} \rightarrow \mathbb{R}$.

Strategy: Solve

$$f(x_k) + f'(x_k)(x - x_k) = 0.$$

Method: Gives

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



The Babylonian method

- ◇ compute square root of $R \in \mathbb{R}_+$
- ◇ find zero of $f(x) = x^2 - R$
- ◇ use Newton-Raphson:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - R}{2x_k} = \frac{1}{2} \left(x_k + \frac{R}{x_k} \right)$$

- ◇ Starting from $x_0 > 0$ we have

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{R}{x_k} \right) \geq \frac{x_k}{2}.$$

- ◇ Starting from $x_0 = R \geq 1$, it takes $\mathcal{O}(\log R)$ steps to get to $x_k - \sqrt{R} < \frac{1}{2}$.

The Babylonian method - Takeoff

Note that

$$x_{k+1} - \sqrt{R} = \frac{1}{2} \left(x_k + \frac{R}{x_k} \right) - \sqrt{R} = \frac{x_k}{2} + \frac{R}{2x_k} - \sqrt{R} = \frac{1}{2x_k} (x_k - \sqrt{R})^2$$

For simplicity $R \geq 1/4$, then $x_k \geq \sqrt{R} \geq 1/2$. Hence

$$x_{k+1} - \sqrt{R} = \frac{1}{2x_k} (x_k - \sqrt{R})^2 \leq (x_k - \sqrt{R})^2$$

If $x_0 - \sqrt{R} < \frac{1}{2}$ (ensured after $\mathcal{O}(\log R)$ steps).

$$x_k - \sqrt{R} \leq (x_0 - \sqrt{R})^{2^k} \leq \left(\frac{1}{2} \right)^{2^k}$$

To achieve $x_k - \sqrt{R} < \epsilon$ we only need $k = \log \log(\epsilon^{-1})$ steps!

The Babylonian method - Example

$R = 1000$, in double arithmetic

- ◇ 7 steps to get to $x_7 - \sqrt{1000} < 1/2$
- ◇ 3 steps to get to $\sqrt{1000}$ up to *machine precision*
- ◇ First phase: \approx **one more correct digit** per iteration
- ◇ Second phase: \approx **double the number of correct digits** per iteration

In practice: $\log \log x \leq 5$.

Newton's method for optimization

Goal: Find global minimum x^* of convex, differentiable function f .

Strategy: Search for zero of derivative.

1-dimensional case: Apply Newton-Raphson method to f' :

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} = x_k - f''(x_k)^{-1} f'(x_k)$$

(requires **twice** differentiable and $f'' > 0$)

d -dimensional case: Newton's methods for minimizing convex $f : \mathbb{R}^d \rightarrow \mathbb{R}$:

$$x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k)$$

Newton's method as adaptive gradient descent

General update scheme:

$$x_{k+1} = x_k - H(x_k)\nabla f(x_k)$$

for some matrix $H(x) \in \mathbb{R}^{d \times d}$.

- ◇ **Newton's method:** $H = \nabla^2 f(x_k)^{-1}$.
- ◇ **Gradient descent:** $H = \alpha \text{Id}$

Newton's methods **adapts** to the local geometry of f at x_k

→ *no need for choosing a stepsize.*

Convergence in one step on quadratic functions

A **quadratic** function

$$f(x) = \frac{1}{2}x^T Mx + q^T x + c$$

is called *nondegenerate* if $M \in \mathbb{R}^{d \times d}$ is invertible.

- ◇ $x^* := M^{-1}q$ is the unique solution of $\nabla f(x) = 0$
- ◇ x^* is the unique global minimum if f is convex

Lemma

On nondegenerate quadratic functions with arbitrary starting point x_0 , Newton's method yields $x_1 = x^$*

Proof.

We have $\nabla f(x) = Mx - q$ and $\nabla^2 f(x) = M$. Therefore

$$x_1 = x_0 - \nabla^2 f(x_0)^{-1} \nabla f(x_0) = x_0 - M^{-1}(Mx_0 - q) = M^{-1}q = x^*. \quad \square$$

Affine Invariance

Newton's method is **affine invariant** (invariant under any invertible affine transformation): Denote the Newton step for h by

$$N_h(x) := x - \nabla^2 h^{-1} \nabla h(x).$$

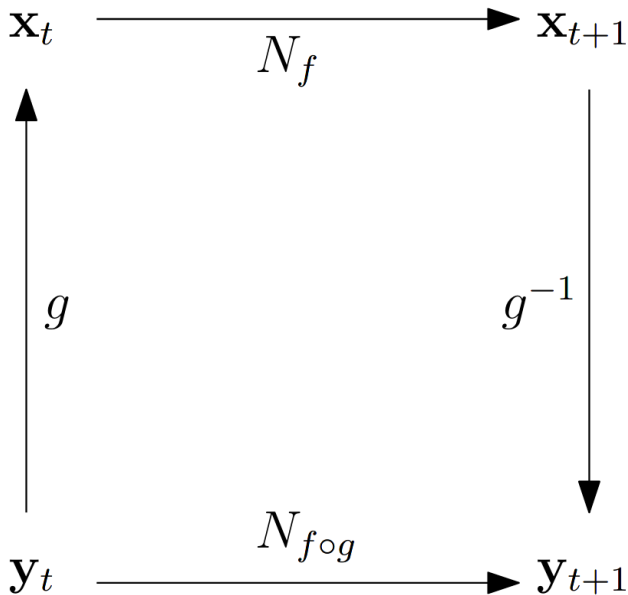
Lemma

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be twice differentiable, $A \in \mathbb{R}^{d \times d}$ an invertible matrix and $b \in \mathbb{R}^d$.

$$g(x) = Ax + b.$$

Then

$$N_{f \circ g} = g^{-1} \circ N_f \circ g$$



Minimizing the second-order Taylor approximation

Alternative interpretation of Newton's method: Minimize (local) **quadratic model** of f .

Lemma

*Let f be conve, twice differentiable and $\nabla^2 f(x) \succ 0$. Then x_{k+1} resulting from **Newton's step** satisfies*

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^d} f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle x - x_k, \nabla^2 f(x_k)(x - x_k) \rangle$$

Local Convergence

We will prove:

Under suitable conditions on f and **close to the minimum** Newton's method approximates solution up to an error ϵ in **$\log \log(1/\epsilon)$** iterations.

- ◇ much faster than anything so far..
- ◇ only locally

We call this a **local convergence** result.

Global convergence statements are more difficult to obtain (some only recently).

Theorem + Technical conditions

Theorem

Let f be convex with unique global minimum x^* , and X a ball around x^* s.t.

(i) *Bounded inverse Hessians:* There exists $\mu > 0$

$$\|\nabla^2 f(x)^{-1}\| \leq \frac{1}{\mu}, \quad \forall x \in X$$

(ii) *Lipschitz continuous Hessians:* There exists $B > 0$

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq B\|x - y\|, \quad \forall x, y \in X$$

Then, for $x_{k+1} = N_f(x_k)$ we have

$$\|x_{k+1} - x^*\| \leq \frac{B}{2\mu} \|x_k - x^*\|^2$$

Super-exponential speed

Corollary

In the setting of previous theorem, if

$$\|x_k - x^*\| \leq \frac{\mu}{B},$$

then

$$\|x_k - x^*\| \leq \frac{\mu}{B} \left(\frac{1}{2}\right)^{2^k - 1}$$

Close to the global minimum, we will reach distance to the minimum less than ϵ in at most $\log \log(1/\epsilon)$ steps.

As for the last phase of Babylonian method.

Super-exponential speed - intuition

- ◇ Almost constant Hessians close to optimality...
- ◇ so f behaves almost like a quadratic
- ◇ on which Newton's converge in one step

Lemma

If

$$\|x_0 - x^*\| \leq \frac{\mu}{B}$$

the Hessians in Newton's method satisfy the *relative error bound*

$$\frac{\|\nabla^2 f(x_k) - \nabla^2 f(x^*)\|}{\|\nabla^2 f(x^*)\|} \leq \left(\frac{1}{2}\right)^{2^k - 1}.$$

Proof of convergence theorem

We abbreviate $H = \nabla^2 f(x_k)$, $x = x_k$, $x^+ = x_{k+1}$

$$\begin{aligned}x^+ - x^* &= x - x^* - H^{-1} \nabla f(x) \\&= x - x^* + H^{-1} (\nabla f(x^*) - \nabla f(x)) \\&= x - x^* + H^{-1} \int_0^1 H(x + t(x^* - x)) (x^* - x) dt,\end{aligned}$$

where we used the fundamental theorem of calculus

$$\int_a^b h'(t) dt$$

with

$$\begin{aligned}h(t) &= \nabla f(x + t(x^* - x)) \\h'(t) &= \nabla^2 f(x + t(x^* - x)) (x^* - x).\end{aligned}$$

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Proof of convergence theorem II

So far

$$x^+ - x^* = x - x^* + H^{-1} \int_0^1 H(x + t(x^* - x))(x^* - x) dt$$

With

$$x - x^* = H(x)^{-1} \int_0^1 -H(x)(x^* - x)$$

we get

$$x^+ - x^* = H^{-1} \int_0^1 (H(x + t(x^* - x)) - H(x))(x^* - x) dt.$$

Using norms

$$\|x^+ - x^*\| \leq \|H^{-1}\| \left\| \int_0^1 H(x + t(x^* - x)) - H(x)(x^* - x) dt \right\|$$

Proof of convergence theorem III

$$\begin{aligned}\|x^+ - x^*\| &= \|H^{-1}\| \left\| \int_0^1 (H(x + t(x^* - x)) - H(x))(x^* - x) dt \right\| \\ &\leq \|H^{-1}\| \|x^* - x\| \int_0^1 \|H(x + t(x^* - x)) - H(x)\| dt\end{aligned}$$

Use **bounded inverse Hessians** and **Lipschitz continuity** of the Hessian to conclude

$$\begin{aligned}\|x^+ - x^*\| &\leq \frac{1}{\mu} \|x^* - x\| \int_0^1 B \|t(x^* - x)\| dt \\ &= \frac{B}{\mu} \|x^* - x\|^2 \int_0^1 t dt = \frac{B}{2\mu} \|x - x^*\|^2.\end{aligned}$$

Downside of Newton's method

Computational bottleneck in every step:

- ◇ compute Hessian
- ◇ invert Hessian or solve $\nabla^2 f(x_k) \Delta x = -\nabla f(x_k)$

Matrix has size $d \times d$, taking $\mathcal{O}(d^3)$ to invert.

In many applications the dimension d is large (too large to even store Hessian).

When training a ML model d is the number of parameters of our ML model (number of features for linear model).

The secant method

Another iterative method for finding zeros in 1-d. Recall Newton-Raphson:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

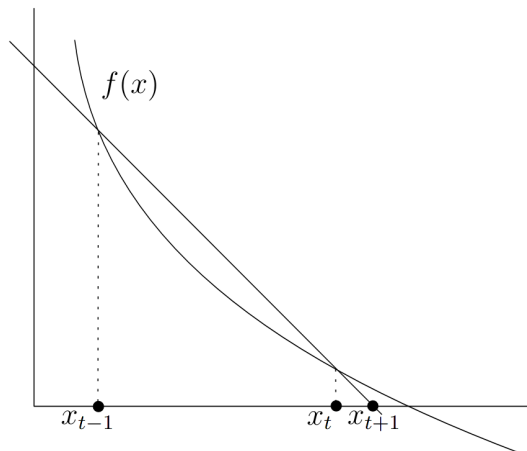
Use **finite difference approximation** of $f'(x_k)$:

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

We obtain the **secant method**:

$$x_{k+1} = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}.$$

The secant method II



Constructs the line through $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$

The secant method III

- ◇ is a *derivative-free* version of the Newton-Raphson method.
- ◇ **For optimization**: Apply secant method to f' to optimize f :

$$x_{k+1} = x_k - f'(x_k) \frac{x_k - x_{k-1}}{f'(x_k) - f'(x_{k-1})}.$$

- ◇ yields a **second-derivative free** version of Newton's method.

What about higher dimensions? Can't divide vectors..

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The secant condition

In 1-d:

$$H_k := \frac{f'(x_k) - f'(x_{k-1})}{x_k - x_{k-1}} \approx f''(x_k)$$
$$\Leftrightarrow f'(x_k) - f'(x_{k-1}) = H_k(x_k - x_{k-1}),$$

the **secant condition**.

- ◇ Newton's method: $x_{k+1} = x_k - f''(x_k)^{-1}f'(x_k)$
- ◇ Secant method: $x_{k+1} = x_k - H_k^{-1}f'(x_k)$

Quasi-Newton methods

$$\nabla f(x_k) - \nabla f(x_{k-1}) = H_k(x_k - x_{k-1}) \approx \nabla^2 f(x_k)(x_k - x_{k-1})$$

We therefore hope that $H_k \approx \nabla^2 f(x_k)$.

$$\text{Secant: } x_{k+1} = x_k - H_k^{-1} \nabla f(x_k)$$

- ◇ $d = 1$: unique number H_k satisfying the secant condition
- ◇ $d > 0$: secant condition $\nabla f(x_k) - \nabla f(x_{k-1}) = H_k(x_k - x_{k-1})$ has infinitely many symmetric solutions

*Any scheme of choosing in each step of the secant method a symmetric H_k that satisfies the secant condition defines a **Quasi-Newton method**.*

Quasi-Newton methods II

- ◇ Newton's method is a Quasi-Newton method if and only if f is a nondegenerate **quadratic** function.
- ◇ Hence, Quasi-Newton methods do not generalize Newton's method but form a family of *related* algorithms.
- ◇ First Quasi-Newton method by William Davidon in 1956
- ◇ But the paper got rejected for lacking a convergence analysis,
- ◇ was finally officially published in 1991
- ◇ **methods of choice** in a number of relevant machine learning applications

Developing a Quasi-Newton method

- ◇ We want to avoid matrix inversion \Rightarrow directly deal with the inverse H_k^{-1}
- ◇ **Given:** iterates x_{k-1}, x_k and matrix H_{k-1}^{-1}
- ◇ **Seeking:** next matrix H_k^{-1} needed in next Quasi-Newton step

$$x_{k+1} = x_k + H_k^{-1} \nabla f(x_k)$$

- ◇ How to choose H_k^{-1} ?
- ◇ Newton's method: $\nabla^2 f(x_k)$ fluctuates only very little in the region of very fast convergence.
- ◇ Makes sense to have $H_k \approx H_{k-1}$ or $H_k^{-1} \approx H_{k-1}^{-1}$

Greenstadt's family of Quasi-Newton methods

Greenstadt [Gre70]: Update

$$H_k^{-1} = H_{k-1}^{-1} + E_k,$$

with E_k an error matrix.

- ◇ Try to **minimize error** subject to H_k satisfying the **secant condition**!

Simple error measure: Frobenius norm

$$\|E\|_F^2 := \sum_{i=1}^d \sum_{j=1}^d E_{ij}^2$$

BFGS method

- ◇ special version of Greenstadt
- ◇ is named after Broyden, Fletcher, Goldfarb and Shanno who all came up with it independently around 1970. Greenstadt's name is mostly forgotten.
- ◇ Newton's method needs to compute and invert Hessians \rightarrow cost of $\mathcal{O}(d^3)$ per iteration
- ◇ any method in Greenstadt's family avoids computation of Hessian. Only gradients are needed.
- ◇ In the BFGS method, the cost per iteration drops to $\mathcal{O}(d^2)$.
- ◇ even this can be prohibitive \rightarrow limited memory BFGS
- ◇ uses observation that we do not need H_k^{-1} , only $H_k^{-1} \nabla f(x_k)$