# Matrix games

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## Introduction

Given

⋄ Player I (rows, Alice)

⋄ Player II (columns, Bob)

⋄ a *payoff* matrix $A \in \mathbb{R}^{m \times n}$

Every round

(i) Alice picks (row) strategy $i \in [m] := \{1, \ldots, m\}$ Bob picks (col) strategy $j \in [n]$

(ii) Bob pays Alice the amount $a_{i,j}$

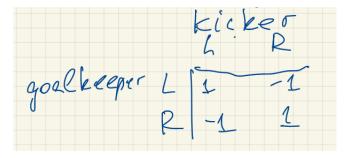**zero-sum game**

## Example: penalty game



Figure: penalty game

## Example: prisoners dilemma



|            | Confess A | Stay quiet A |
|------------|-----------|--------------|
| Confess B  |      6    |       10     |
|            |  6        |   0          |
| Stay quiet B |    0    |        2     |
|            | 10        | 2            |

Figure: prisoners dilemma

## Worst case

⋄ if Alice chooses strategy $i$ she gets (at least): $\min_{j\in[n]} a_{i,j}$

⋄ Alice can ensure payoff $\max_{i\in[m]} \min_{j\in[n]} a_{i,j}$

⋄ Bob pays (at most) $\min_{j\in[n]} \max_{i\in[m]} a_{i,j}$

We claim:

$$\max_i \min_j a_{i,j} \le \min_j \max_i a_{i,j}$$

"Tallest dwarf is not as tall as the smallest giant."

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## Proof of the min-max theorem

$$a_{ij} \leq a_{ij} \qquad \forall i, j$$
$$a_{ij} \leq \max_i a_{ij} \qquad \forall i, j$$
$$\min_j a_{ij} \leq \min_j \max_i a_{ij} \qquad \forall i$$

### Definition

We call $(i^*, j^*)$ a saddle point (or *Nash equilibrium*) if

$$\max_i a_{ij^*} = a_{i^* j^*} = \min_j a_{i^* j}.$$

These are called *pure strategies*.

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## Rock paper scissors

|   | $R$ | $P$ | $S$ |
|---|-----|-----|-----|
| $R$ | 0 | $-1$ | 1 |
| $P$ | 1 | 0 | $-1$ |
| $S$ | $-1$ | 1 | 0 |

## Mixed Strategies

*With pure strategies we do not always have a saddle point.*

### von Neumann (1928) — Mixed strategies

◇ Alice picks strategies $1, \ldots, m$ *with probabilities* $x \in \Delta_m$

◇ Bob picks strategies $1, \ldots, n$ *with probabilities* $x \in \Delta_n$

Expected gain of Alice is

$$\langle x, Ay \rangle = \sum_{i,j} a_{ij} x_i y_j$$

### Theorem

*Saddle point exists Expected gain of Alice = expected loss of Bob*

$$\max_{x \in \Delta} \min_{y \in \Delta} \langle x, Ay \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle x, Ay \rangle.$$

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## Stopping criteria

$$\min_{x \in \Delta} \max_{y \in \Delta} \langle x, Ay \rangle =: v$$

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### stopping criterion

$$f_p(x) - f_p(x^*) = f_p(x) - v \le \epsilon/2$$

$$f_d(y^*) - f_d(y) = v - f_d(y) \le \epsilon/2$$

$$\Rightarrow f_p(x) - f_d(y) \le \epsilon$$

Before we never had the optimal value!

Solution will always be on the boundary:

$$f_p(x) = \max_{y \in \Delta} \langle A^T x, y \rangle = \max_j \langle A^T x, e_j \rangle$$

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Solution will always be on the boundary:

$$f_p(x) = \max_{y \in \Delta} \langle A^T x, y \rangle = \max_j \langle A^T x, e_j \rangle$$

Consider

$$\min_{x\in\Delta}\max_{y\in\Delta}\langle x, Ay\rangle$$

as a minimization problem

$$\min_{x\in\Delta} f_p(x) = \langle x, Ay^*\rangle.$$

Then, by the **first-order optimality condition**

$$x^* \in \arg\min_{x\in\Delta} f_p(x) \Leftrightarrow \langle \nabla f_p(x^*), x - x^*\rangle \geq 0 \quad \forall x \in \Delta$$

Thus

$$\langle A^T y^*, x - x^*\rangle \geq 0 \quad \forall x \in \Delta$$
$$\langle -Ax^*, y - y^*\rangle \geq 0 \quad \forall y \in \Delta$$

Concatenate the two conditions to get

$$\left\langle \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \left( \begin{array}{c} x^* \\ y^* \end{array} \right), \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x^* \\ y^* \end{array} \right) \right\rangle \geq 0.$$

## Games as Variational Inequalities

We had:

$$\left\langle \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ y^* \end{pmatrix} \right\rangle \geq 0.$$

By rewriting $z = (x, y)$ and $F(z) = [A^T y; Ax]$, then

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in \Delta_n \times \Delta_m =: C \qquad \text{(VI)}$$

**Variational inequality**

If $F = \nabla \varphi$ then (VI) would be equivalent to

$$\min_{z \in C} \varphi(z)$$

## Potential - integrability

**Question:** Does there exist a potential $\varphi$ for $F$, such that $F = \nabla\varphi$

### Integrability condition (from calculus)

Is the case if

$$\frac{\partial\varphi}{\partial x\partial y} = \frac{\partial\varphi}{\partial y\partial x}$$

But

$$\frac{\partial F_1}{\partial y} = A^T \neq -A = \frac{\partial F_2}{\partial x}$$

## VI as Fixed point equation

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in C$$
$$\Leftrightarrow z^* = P_C(z^* - F(z^*)) \tag{FP}$$

### Proof.

Applying the property of the projection

$$\langle P_C(x) - x, x' - P_C(x) \rangle \geq 0 \quad \forall x' \in C$$

with (FP), gives

$$\langle z^* - (z^* - F(z^*)), z - z^* \rangle \geq 0 \quad \forall z \in C. \quad \square$$

⋄ should remind us of (projected) gradient descent
⋄ when you see a fixed point equation: iterate!

## But is it any good?

$$z_{k+1} = z_k - \alpha F(z_k)$$

Then

$$\|z_{k+1}\|^2 = \|z_k\|^2 - \underbrace{2\alpha\langle F(z_k), z_k \rangle}_{=0} + \alpha^2 \|F(z_k)\|^2$$

$$= \|z_k\|^2 + \alpha^2 \|F(z_k)\|^2$$

Resulting in $\|z_{k+1}\| \geq \|z_k\|$.

$$\Rightarrow \textbf{No bueno!}$$

We can still show

### Theorem

*complexity of $\mathcal{O}(1/\sqrt{k})$ for **averaged iterates** in terms of $f_p(x) - f_d(y)$.*

$\diamond$ with the same analysis as for subgradient descent

## Sketch of the proof

With the notation $g_k = F(z_k)$ we get

$$
\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leq \|x_k - \alpha_k g_k - x^*\|^2 \\
&= \|x_k - x^*\|^2 + 2\alpha_k \langle g_k, x^* - x_k \rangle + \alpha^2 \|g_k\|^2.
\end{aligned}
$$

But now $\langle g_k, x^* - x_k \rangle = [f_d(y_k) - f_p(x_k)]$. Rest of the proof is left as an exercise.