Projected Gradient Descent

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1. Introduction

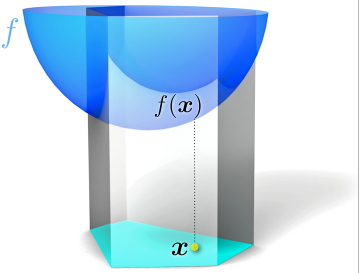
Constrained Optimization

Constrained optimization problem

$$\text{minimize} f(x)$$
$$\text{subject to } x \in C$$

How to solve them

- Project onto C
- transform to *unconstrained problem*



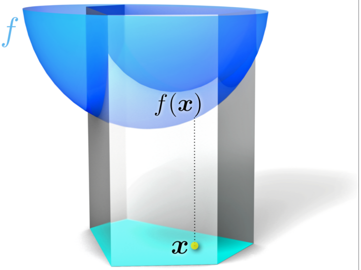
Constrained Optimization

Constrained optimization problem

$$\text{minimize} f(x)$$
$$\text{subject to } x \in C$$

We will focus on:

- **Projected Gradient Descent**



Projected Gradient Descent

Projected subgradient method

$$\text{(constrained setting)} \qquad \min_{x \in C} f(x)$$

Algorithm Projected subgradient method

1: **for** $k = 0, 1, \ldots$ **do**
2: Pick $g_k \in \partial f(x_k)$
3: $x_{k+1} = P_C(x_k - \alpha g_k)$

By using the fact that the projection is a contraction

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|$$

Projected subgradient method II

Proof.

We can deduce the exact same inequality as before

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|P_C(x_k - \alpha g_k) - x^*\|^2 \\ &\leq \|x_k - \alpha g_k - x^*\|^2 \\ &= \|x_k - x^*\|^2 + 2\alpha\langle g_k, x^* - x_k\rangle + \alpha^2\|g_k\|^2 \\ &\leq \|x_k - x^*\|^2 + 2\alpha(f^* - f(x_k)) + \alpha^2\|g_k\|^2.\end{aligned}$$

□