Magnetic helicity density and its flux in weakly inhomogeneous turbulence

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A gauge invariant and hence physically meaningful definition of magnetic helicity density for random fields is proposed, using the Gauss linking formula, as the density of correlated field line linkages. This definition is applied to the random small scale field in weakly inhomogeneous turbulence, whose correlation length is small compared with the scale on which the turbulence varies. For inhomogeneous systems, with or without boundaries, our technique then allows one to discuss the local magnetic helicity density evolution, in a gauge independent fashion, which was not possible earlier. This evolution equation is governed by local sources (owing to the mean field) and by the divergence of a magnetic helicity flux density. The role of magnetic helicity fluxes in alleviating catastrophic quenching of mean field dynamos is discussed.

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In fluid dynamics, magnetohydrodynamics, and in plasma physics, helicities of various kinds are used, e.g. kinetic, magnetic, current, and cross helicities [1]. These helicities are defined as volume integrals over corresponding helicity densities. Helicity densities provide invaluable insight into origin and redistribution of 'swirl' in the system. The intuitive understanding is helped further by the fact that the integrals over some of the helicity densities are conserved by the nonlinear interactions, e.g. magnetic and cross helicities in magnetohydrodynamics and kinetic helicity in fluid dynamics. This is particularly useful in the case of magnetic helicity which plays an important role in tokamaks [2], the sun [3], and now also in dynamo theory [4]. Magnetic helicity is special because it is not only a quadratic invariant of ideal magnetohydrodynamics (MHD), but can also be better conserved than energy, even in resistive MHD [5]. It is this feature which allows strong constraints to be placed on the nonlinear behavior of all the above systems using helicity conservation. The magnetic helicity is usually defined as $H_{\rm M} = \int \boldsymbol{A}\cdot\boldsymbol{B}\,{\rm d}V$, where $\boldsymbol{A}$ is the vector potential such that the magnetic field is given by $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Under a gauge transformation $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$, which leaves $\boldsymbol{B}$ invariant, one has $H' = H + \int \Lambda\boldsymbol{B}\cdot{\rm d}\boldsymbol{S}$. So H is only gauge invariant if the $\boldsymbol{B}$ field vanishes sufficiently rapidly at the boundary of the integration volume.

In most practical contexts, like the sun or galaxies, however, the field does not vanish on the boundaries. A possible remedy to the gauge problem might be to consider instead the gauge-invariant *relative* magnetic helicity, defined by subtracting the helicity of a reference vacuum field [6]. But the flux of relative helicity is cumbersome to work with for arbitrarily shaped boundaries. Also the concept of a density of relative helicity is not meaningful, since it is defined only as a volume integral.

For some applications [7, 8] it has been useful to consider instead the evolution of the current helicity density, $H_C = \boldsymbol{J}\cdot\boldsymbol{B}$, where $\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}/\mu_0$ is the current density and μ_0 is the vacuum permeability. Note that H_C, as well as its flux, are locally well defined, explicitly gauge invariant, observationally measurable, and the small scale contribution to H_C is also the quantity that enters the closure expression for the nonlinear α effect [9]. The major disadvantage is that one loses the conceptually simple form of the magnetic helicity conservation law. We propose here instead an alternate means to define magnetic helicity density for the random small scale field, using the more basic Gauss linking formula for helicity, which can be directly applied to discuss magnetic helicity density and its flux even in systems with boundaries.

In the following we work with random small scale quantities, denoted by lower case characters and defined as the departure from the corresponding mean field quantity, e.g. $\boldsymbol{b} = \boldsymbol{B} - \overline{\boldsymbol{B}}$ for the magnetic field, $\boldsymbol{j} = \boldsymbol{J} - \overline{\boldsymbol{J}}$ for the current density, and $\boldsymbol{u} = \boldsymbol{U} - \overline{\boldsymbol{U}}$ for the velocity. Throughout this paper we adopt ensemble averages which, in practice, are commonly approximated as spatial averages over one or two coordinate directions [4]. However, the approach developed below applies also to the case without a mean magnetic field. Therefore, specific applications to the mean field dynamo (MFD) will be postponed until the end of this paper.

Magnetic helicity density. Given the random small scale magnetic field $\boldsymbol{b}(\boldsymbol{x},t)$ one can also define the magnetic helicity directly in terms of the field, as the linkage of its flux, using the Gauss's linking formula [6, 10]

$$h_G = \frac{1}{4\pi}\int\int \boldsymbol{b}(\boldsymbol{x})\cdot\left[\boldsymbol{b}(\boldsymbol{y})\times\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right]{\rm d}^3x\,{\rm d}^3y, \quad (1)$$

where both integrations extend over the full volume. Suppose we define an auxiliary field

$$\boldsymbol{a}_C(\boldsymbol{x}) = \frac{1}{4\pi}\int \boldsymbol{b}(\boldsymbol{y})\times\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,{\rm d}^3y, \quad (2)$$

then this field satisfies $\boldsymbol{\nabla}\times\boldsymbol{a}_C = \boldsymbol{b}$, and $\boldsymbol{\nabla}\cdot\boldsymbol{a}_C = 0$, and one can write $h_G = \int \boldsymbol{a}_C\cdot\boldsymbol{b}\,{\rm d}^3x$. This is the origin of the textbook definition of magnetic helicity in what is known as the Coulomb gauge for the vector potential. Provided the field is closed over the integration volume, this definition can be applied in any other gauge. Note however that for an open system with boundaries, it is *not* useful to go to the definition involving the vector potential, which is now of course gauge dependent. We therefore take the point of view here that the magnetic helicity h_G defined by Eq. (1) is the more basic definition of the topological property determining the links associated with magnetic fields, and not the definition in terms of the vector potential. We will see below that this then also allows us to define naturally a gauge invariant magnetic helicity density for random fields, as long as the correlation scale of the field is much smaller than the size of the system, as the density of correlated links of the field.

For this consider $\boldsymbol{b}$ to be a random field with a correlation function $\overline{b_i(\boldsymbol{x},t)b_j(\boldsymbol{y},t)} = M_{ij}(\boldsymbol{r},\boldsymbol{R})$. Here we have defined the difference $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ and the mean $\boldsymbol{R} = (\boldsymbol{x}+\boldsymbol{y})/2$, keeping in mind that, for weakly inhomogeneous turbulence, the two-point correlation $M_{ij}(\boldsymbol{r},\boldsymbol{R})$ and in fact all two-point correlations below, vary rapidly with $\boldsymbol{r}$ but vary slowly with $\boldsymbol{R}$. Then taking the ensemble average of h_G, we have

$$\overline{h}_G = \frac{1}{4\pi}\int \mathrm{d}^3R \int \mathrm{d}^3r\ \epsilon_{ijk} M_{ij}(\boldsymbol{r},\boldsymbol{R})\,\frac{r_k}{r^3}. \qquad (3)$$

Now suppose the correlation scale l of the random small scale field $\boldsymbol{b}$ is much smaller than the system scale R_{S}. That is suppose there exists an intermediate scale L such that $l \ll L \ll R_{\mathrm{S}}$, and $M_{ij}(\boldsymbol{r},\boldsymbol{R}) \to 0$ as $|\boldsymbol{r}| \to L \gg l$. Then one can do the $\boldsymbol{r}$ integral even by restricting to the intermediate scale L, and still capture all the dominant contributions to the integral. This then motivates us to define the magnetic helicity density h of the random small scale field as $\overline{h}_G = \int \mathrm{d}^3R\, h(\boldsymbol{R})$. Here,

$$h(\boldsymbol{R}) = \frac{1}{4\pi}\int_{L^3} \mathrm{d}^3r\ \epsilon_{ijk} M_{ij}(\boldsymbol{r},\boldsymbol{R})\,\frac{r_k}{r^3}, \qquad (4)$$

where we can formally let $L \to \infty$. The above expression for $h(\boldsymbol{R})$ in Eq. (4) is our proposal for the helicity density of the random small scale field $\boldsymbol{b}$. Evidently, $h(\boldsymbol{R})$ is explicitly gauge invariant. A qualitative description would be to say that the magnetic helicity density of a random small scale field is the density of correlated links of the field. We can now derive the evolution equation for $h(\boldsymbol{R})$ and also meaningfully (in a gauge invariant manner) talk about its flux. Note that this has not been possible before, although many papers [11, 12, 13] have appealed to the notion of a magnetic helicity flux density in some qualitative fashion. We will see that the magnetic helicity evolution equation we derive, first reproduces the known evolution equation for homogeneous turbulence and generalizes it for the inhomogeneous case by introducing possible fluxes of helicity.

Magnetic helicity density evolution. It is much simpler to work out the evolution equation for $h(\boldsymbol{R})$ by first going to Fourier space, using the two-scale approach of Ref. [14], where all two-point correlations are assumed to vary rapidly with $\boldsymbol{r}$ and slowly with $\boldsymbol{R}$. Consider the equal time, ensemble average of the product $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)}$. The common dependence of f and g on t is assumed and will not explicitly be stated. Let $\hat{f}(\boldsymbol{k}_1)$ and $\hat{g}(\boldsymbol{k}_2)$ be the Fourier transforms of f and g, respectively. We can express this correlation as $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)} = \int \Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3k$, with

$$\Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R}) = \int \overline{\hat{f}(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{g}(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{R}}\,\mathrm{d}^3K. \qquad (5)$$

Here $\boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)$ and $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$. In what follows we require the correlation tensors,

$$v_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_i,\hat{u}_j,\boldsymbol{k},\boldsymbol{R}), \qquad (6)$$

$$m_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{b}_i,\hat{b}_j,\boldsymbol{k},\boldsymbol{R}), \qquad (7)$$

$$\chi_{jk}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_j,\hat{b}_k,\boldsymbol{k},\boldsymbol{R}), \qquad (8)$$

of the $\boldsymbol{u}$ and $\boldsymbol{b}$ fields, as well as the cross correlation between these two fields, in Fourier space. In MFD theory, the turbulent emf is given by $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u}\times\boldsymbol{b}}$, whose components are $\overline{\mathcal{E}}_i(\boldsymbol{R}) = \epsilon_{ijk}\int \chi_{jk}(\boldsymbol{k},\boldsymbol{R})\,\mathrm{d}^3k$. Further, in Fourier space we have for the magnetic helicity density,

$$h(\boldsymbol{R}) = \int\int \epsilon_{ijk}\,\overline{\hat{b}_i(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_j(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})} \times \left(\frac{\mathrm{i}k_k}{k^2}\right)\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{R}}\,\mathrm{d}^3k\,\mathrm{d}^3K. \qquad (9)$$

We should remark, that for an inhomogeneous system, the Coulomb gauge magnetic helicity density, say $h_C = \overline{\boldsymbol{a}_C\cdot\boldsymbol{b}}$, would have $(k_k + K_k/2)/(\boldsymbol{k}+\boldsymbol{K}/2)^2$ replacing k_k/k^2 in the Fourier space expression of Eq. (9). The two expressions are identical for the homogeneous case, and even in the weakly inhomogeneous case up to first order terms in K/k, but not in general. So $h(\boldsymbol{R}) \neq h_C(\boldsymbol{R})$ in general.

In order to compute the magnetic helicity evolution, we use the induction equation for $\boldsymbol{b}$ in Fourier space, $\partial\hat{b}_i(\boldsymbol{k})/\partial t = -\epsilon_{ipq}\mathrm{i}k_p\hat{e}_q$. Here $\hat{e}_q$ is the Fourier transform of the small scale electric field e_q, which is given by [4, 15]

$$\boldsymbol{e} = -\boldsymbol{u}\times\overline{\boldsymbol{B}} - \overline{\boldsymbol{U}}\times\boldsymbol{b} - \boldsymbol{u}\times\boldsymbol{b} + \overline{\boldsymbol{u}\times\boldsymbol{b}} + \eta\boldsymbol{j}. \qquad (10)$$

Substituting this in the time derivative of Eq. (9), we get after some straightforward algebra,

$$\begin{aligned}\frac{\partial h(\boldsymbol{R})}{\partial t} = &\int\int\Big\{-2\int\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_q(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\\ &+ 2\frac{K_jk_q}{k^2}\,\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_j(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\\ &- \frac{K_sk_s}{k^2}\,\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_q(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\Big\}\\ &\times\ \mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{R}}\,\mathrm{d}^3K\,\mathrm{d}^3k. \end{aligned} \qquad (11)$$

We denote the integrals over the 3 terms in curly brackets above as A_1, A_2, and A_3, respectively. From the above definition of Φ, the first term is simply $A_1 = -2\overline{\boldsymbol{e}\cdot\boldsymbol{b}}$, or

$$A_1 = 2\overline{\boldsymbol{b}\cdot(\boldsymbol{u}\times\overline{\boldsymbol{B}})} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}} = -2\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}, \qquad (12)$$

where $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u}\times\boldsymbol{b}}$ is the turbulent emf. Note that in the limit of homogeneous turbulence only the term A_1 survives. This is because the other 2 terms which involve a large K_i in the integrand, will introduce a large scale R_i derivative on evaluating the integral over $\boldsymbol{K}$, and this vanishes in the homogeneous case. So, for the homogeneous case we recover a local generalization (without volume integration) of the magnetic helicity conservation equation [4]

$$\partial h/\partial t = -2\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}} \quad \text{(from } A_1 \text{ term only).} \tag{13}$$

In the inhomogeneous case, since the terms A_2 and A_3 are scalars which depend on K_i in the integrand, and hence a large scale R_i derivative, they will contribute purely to the flux of helicity. The only term which involves volume generation of helicity density is the A_1 term, which we see involves correlations no higher than the two-point one. This is in contrast to the current helicity evolution, which involved undetermined triple correlations in their volume generation [7].

Let us now evaluate the helicity fluxes given by A_2 and A_3. This involves straightforward but tedious algebra. We also work out the flux to the lowest order in the R derivative. There are again 3 main types of contributions due to different parts of the electric field $\boldsymbol{e}$. First there is a contribution proportional to $\overline{\boldsymbol{B}}$ due to that part $\boldsymbol{e} = -\boldsymbol{u}\times\overline{\boldsymbol{B}} + \ldots$. In Fourier space this gives $\hat{e}_q(\boldsymbol{k}) = \epsilon_{qlm}\int \hat{u}_m(\boldsymbol{k}-\boldsymbol{k}')\hat{\overline{B}}_l(\boldsymbol{k}')\mathrm{d}^3k'$. We substitute this into the expression for A_2 and A_3, change variables to $\boldsymbol{K}' = \boldsymbol{K} - \boldsymbol{k}'$, use the definition for χ_{ij} and evaluate the integrations over $\boldsymbol{K}'$ and $\boldsymbol{k}'$ retaining only terms to lowest order in the R derivatives. We then get $A_2 = -\nabla_j\overline{\mathcal{F}}_j^{\mathrm{VC}}$, and $A_3 = -\nabla_j\overline{\mathcal{F}}_j^{\mathrm{A}}$, where the mean field dependent fluxes $\overline{\mathcal{F}}_j^{\mathrm{VC}}$ and $\overline{\mathcal{F}}_j^{\mathrm{A}}$ are given by [16]

$$\begin{aligned}\overline{\mathcal{F}}_i^{\mathrm{VC}} &= 2\epsilon_{qlm}\overline{B}_l(\boldsymbol{R})\int \frac{\mathrm{i}k_q}{k^2}\,\chi_{mi}\,\mathrm{d}^3k,\\ \overline{\mathcal{F}}_i^{\mathrm{A}} &= -\epsilon_{qlm}\overline{B}_l(\boldsymbol{R})\int \frac{\mathrm{i}k_i}{k^2}\,\chi_{mq}\,\mathrm{d}^3k.\end{aligned} \tag{14}$$

Note that $\overline{\mathcal{F}}_i^{\mathrm{A}}$ only depends on the antisymmetric part of the cross correlation χ_{mq}, whereas $\overline{\mathcal{F}}_i^{\mathrm{VC}}$ is sensitive to the symmetric part as well. Now consider the contribution proportional to the mean velocity from the part of the electric field $\boldsymbol{e} = -\overline{\boldsymbol{U}}\times\boldsymbol{b} + \ldots$. The evaluation of this follows the same steps as in evaluating (14) except that one can map $u_m \to b_l$ and $\overline{B}_l \to \overline{U}_m$. This gives $A_2 + A_3 = -\nabla_j\overline{\mathcal{F}}_j^{\mathrm{bulk}}$, where the flux due to bulk motions is given by

$$\overline{\mathcal{F}}_i^{\mathrm{bulk}} = \epsilon_{qlm}\overline{U}_m(\boldsymbol{R})\int\left(2\frac{\mathrm{i}k_q}{k^2}\,m_{li} - \frac{\mathrm{i}k_i}{k^2}\,m_{lq}\right)\mathrm{d}^3k. \tag{15}$$

Indeed if the magnetic correlations were isotropic then it is easy to simplify this further and one gets $\overline{\mathcal{F}}_i^{\mathrm{bulk}} = h\overline{U}_i$, exactly as one should for an advective flux!

The contribution to the fluxes from $\boldsymbol{e} = -\boldsymbol{u}\times\boldsymbol{b}$, see Eq. (10), introduces triple correlations in the flux, which then need a closure theory to evaluate. We denote this flux term as $\overline{\mathcal{F}}_i^{\mathrm{triple}}$. However, since this triple correlation comes only in the flux, and not the volume generation term, it is likely that its value can be constrained by a conservation law. This will be examined in more detail in the future. Note also that the contribution to the helicity flux from $\boldsymbol{e} = \overline{\boldsymbol{u}\times\boldsymbol{b}} + ..$ is zero and that the resistive contribution from $\boldsymbol{e} = \eta\boldsymbol{j} + \ldots$ is likely to be negligible compared to the terms that we retain.

Further simplification of the helicity fluxes for use in say MFD models, requires the evaluation of the turbulent emf tensor χ_{ij}, which can be done only under a closure scheme. Of course one needs to evaluate χ_{ij} only to the lowest order in R_i derivatives. In the minimal tau approximation closure, for example, triple correlations, T_{lk}, which arise now in the evolution equation for $\partial\chi_{lk}/\partial t$, are assumed to provide relaxation of the turbulent emf or χ_{lk} and one takes $T_{lk} = -\chi_{lk}/\tau$, where τ is the relaxation time [4, 17, 18]. There is some justification for this closure from numerical simulations [19]. We concentrate below on nonrotating but helical turbulence. For such turbulence we have from [4, 18], $\chi_{lk} = -\tau\mathrm{i}\boldsymbol{k}\cdot\overline{\boldsymbol{B}}(v_{lk} - m_{lk})$. Using this expression to evaluate the fluxes in Eq. (14) one gets

$$\overline{\mathcal{F}}_i^{\mathrm{VC}} = 2\tau\epsilon_{qlm}\overline{B}_s\overline{B}_l\int\frac{k_sk_q}{k^2}\,(v_{mi} - m_{mi})\,\mathrm{d}^3k, \tag{16}$$

$$\overline{\mathcal{F}}_i^{\mathrm{A}} = -\tau\epsilon_{qlm}\overline{B}_s\overline{B}_l\int\frac{k_sk_i}{k^2}\,(v_{mq} - m_{mq})\,\mathrm{d}^3k. \tag{17}$$

Here $\overline{\mathcal{F}}_i^{\mathrm{VC}}$ is a nonlinear generalization of the magnetic helicity flux obtained by Vishniac and Cho (VC-flux) [13]. The fluxes used by Kleeorin et al. [12] can arise from both $\overline{\mathcal{F}}_i^{\mathrm{A}}$ and the contributions of the antisymmetric parts of the correlations to $\overline{\mathcal{F}}_i^{\mathrm{VC}}$. We also point out that corresponding current helicity fluxes derived in [7], although qualitatively similar to the magnetic helicity fluxes derived above, differ in quantitative details.

Putting all our results together we can write for the evolution of the magnetic helicity density

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot\overline{\boldsymbol{\mathcal{F}}} = -2\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}, \tag{18}$$

where the flux is $\overline{\mathcal{F}}_i = \overline{\mathcal{F}}_i^{\mathrm{VC}} + \overline{\mathcal{F}}_i^{\mathrm{A}} + \overline{\mathcal{F}}_i^{\mathrm{bulk}} + \overline{\mathcal{F}}_i^{\mathrm{triple}}$. We should emphasize that Eq. (18) is a *local* magnetic helicity conservation law. If one could not define a gauge-invariant magnetic helicity density, one can only have an integral (global) conservation law, as in previous studies.

We now draw attention to a number of features of the magnetic helicity fluxes. (i) There are no fluxes if the turbulence is homogeneous. (ii) $\overline{\mathcal{F}}_i^{\mathrm{VC}} = 0$ and $\overline{\mathcal{F}}_i^{\mathrm{A}} = 0$ if the turbulence is isotropic but inhomogeneous. The bulk and possibly the triple correlation fluxes can survive. (iii) The VC and A-fluxes are proportional to two-point correlations (χ_{ij}) and the mean field $\overline{\boldsymbol{B}}$, see Eq. (14); this is unlike the 2-D case where corresponding fluxes of the squared vector potential arise purely in triple correlations [20]. (iv) If there is equipartition, i.e. $v_{ij} = m_{ij}$, the VC and A-fluxes vanish. However, shear can generate anisotropic pieces to v_{ij} and m_{ij} in an asymmetric manner. This is because shear acting on an underlying isotropic MHD turbulence can generate anisotropic pieces, through the terms $\partial\boldsymbol{u}/\partial t \sim -\boldsymbol{u}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{U}} + ...$ and $\partial\boldsymbol{b}/\partial t = +\boldsymbol{b}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{U}} + ...$. Since these generation terms have different signs, shear can break the symmetry between v_{ij} and m_{ij} and lead to net fluxes. (v) From a calculation akin to that in [22] for nonhelical turbulence, the velocity contribution to the VC-flux is given by $\overline{\mathcal{F}}_i^{\rm VC} = \phi_{ijk}\overline{B}_j\overline{B}_k$, where

$$\phi_{ijk} = \bar{C}_{\rm VC}\, \epsilon_{ijl}\, \overline{\mathsf{S}}_{lk}. \tag{19}$$

Here $\overline{\mathsf{S}}_{lk} = \frac{1}{2}(\overline{U}_{l,k} + \overline{U}_{k,l})$ is the mean rate of strain tensor and $\bar{C}_{\rm VC} \sim \frac{4}{5}(u\tau)^2$ is a dimensionless coefficient of order unity.

Since the small scale magnetic helicity opposes the α effect [9], its loss through corresponding magnetic helicity fluxes can alleviate this quenching effect [11, 12, 13, 21, 22]. Note that the α effect quantifies the contribution of $\overline{\boldsymbol{\mathcal{E}}}$ that is aligned with the mean field, i.e. $\overline{\boldsymbol{\mathcal{E}}} = \alpha\overline{\boldsymbol{B}} + ...$ for the simplest case of a scalar α effect. For a closed system, Eq. (13) applies, and in the stationary limit, this predicts $\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}} = -\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$, which tends to zero as $\eta \to 0$, for any reasonable spectrum of current helicity. This leads to a catastrophic quenching of the turbulent emf parallel to $\overline{\boldsymbol{B}}$. In the presence of helicity fluxes, however, we have $\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}} = -\frac{1}{2}\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\mathcal{F}}} - \eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$, in the stationary limit, and $\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}}$ need not be catastrophically quenched. So the turbulent magnetic helicity fluxes worked out here are crucial for the efficient working of the mean field dynamo. Numerical work in determining the α effect did show a 30-fold increase [8] in simulations which allowed helicity fluxes to develop. However, a more convincing demonstration of the importance of helicity fluxes comes from a dynamo simulation in the presence of shear showing that only with open boundaries a significant large scale field of equipartition field strength develops [23].

In summary we have proposed here a gauge invariant definition of magnetic helicity density for random fields in weakly inhomogeneous systems, which can also have boundaries. This is particularly useful in the context of MFDs since one can then meaningfully discuss magnetic helicity fluxes. We have then been able to derive an evolution equation for the local magnetic helicity density and show that they naturally involve helicity fluxes, which may alleviate the problems associated with MFDs [22]. Our work therefore lays the conceptual foundation for the many discussions of the effects of helicity fluxes already existing in the literature and future explorations.

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