

Dynamic Multiscaling in Two-Dimensional Fluid Turbulence

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We obtain, by extensive direct numerical simulations, time-dependent and equal-time structure functions for the vorticity, in both quasi-Lagrangian and Eulerian frames, for the direct-cascade regime in two-dimensional fluid turbulence with air-drag-induced friction. We show that different ways of extracting time scales from these time-dependent structure functions lead to different dynamic-multiscaling exponents, which are related to equal-time multiscaling exponents by different classes of bridge relations; for a representative value of the friction we verify that, given our error bars, these bridge relations hold.

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The scaling properties of both equal-time and time-dependent correlation functions close to a critical point, say, in a spin system, have been understood well for nearly four decades [1,2]. By contrast, the development of a similar understanding of the multiscaling properties of equal-time and time-dependent structure functions in the inertial range in fluid turbulence still remains a major challenge because it requires interdisciplinary studies that must use ideas from both nonequilibrium statistical mechanics and turbulence [3–12]. We develop, therefore, a complete characterization of the rich multiscaling properties of time-dependent vorticity structure functions for the direct-cascade regime of 2D turbulence in fluid films with friction, which we study via a direct numerical simulation (DNS). Such a characterization has not been possible hitherto because it requires very long temporal averaging to obtain good statistics for *quasi-Lagrangian* structure functions [13], which are considerably more complicated than their conventional, Eulerian counterparts as we show below. Our DNS study yields a variety of interesting results that we summarize informally before providing technical details and precise definitions: (a) We calculate equal-time and time-dependent vorticity structure functions in Eulerian and quasi-Lagrangian frames [13]; (b) we then show how to extract an infinite number of different time scales from such time-dependent structure functions; (c) next we present generalizations of the dynamic-scaling *Ansatz*, first used in the context of critical phenomena [2] to relate a diverging relaxation time τ to a diverging correlation length ξ via $\tau \sim \xi^z$, where z is the dynamic-scaling exponent. These generalizations yield, in turn, an infinity of dynamic-multiscaling exponents [5,6,8–12]. (d) A suitable extension of the multifractal formalism [4], which provides a rationalization of the multiscaling of equal-time structure functions in turbulence, yields linear bridge

relations between dynamic-multiscaling exponents and their equal-time counterparts [5,6,8–12]; our study provides numerical evidence in support of such bridge relations.

The statistical properties of fully developed, homogeneous, isotropic turbulence are characterized, among other things, by the equal-time, order- p , longitudinal-velocity structure function $S_p(r) \equiv \langle [\delta u_{\parallel}(r, t)]^p \rangle$, where $\delta u_{\parallel}(r, t) \equiv [\mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{r}/r]$, $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity at point \mathbf{x} and time t , and $r \equiv |\mathbf{r}|$. In the inertial range $\eta_d \ll r \ll L$, $S_p(r) \sim r^{\zeta_p}$, where ζ_p , η_d , and L , are, respectively, the equal-time exponent, the dissipation scale, and the forcing scale. The pioneering work [3] of Kolmogorov (*K41*) predicts simple scaling with $\zeta_p^{K41} = p/3$ for 3D homogeneous, isotropic fluid turbulence. However, experiments and numerical simulations show marked deviations from *K41* scaling, especially for $p \geq 4$, with ζ_p a nonlinear, convex function of p ; thus, we have multiscaling of equal-time velocity structure functions. To examine dynamic multiscaling, we must obtain the order- p , time-dependent structure functions $\mathcal{F}_p(r, t)$, which we define precisely below, extract from these the time scales $\tau_p(r)$, and then the dynamic-multiscaling exponents z_p via dynamic-multiscaling *Ansätze* like $\tau_p(r) \sim r^{z_p}$. This task is considerably more complicated than its analog for the determination of the equal-time multiscaling exponents ζ_p [5–12] for the following two reasons. (I) In the conventional Eulerian description, the sweeping effect, whereby large eddies drive all smaller ones directly, relates spatial separations r and temporal separations t linearly via the mean-flow velocity, whence we get trivial dynamic scaling with $z_p = 1$, for all p . A quasi-Lagrangian description [5,13] eliminates sweeping effects so we calculate time-dependent, quasi-Lagrangian

vorticity structure functions from our DNS. (II) Such time-dependent structure functions, even for a fixed order p , do not collapse onto a scaling function, with a unique, order- p , dynamic exponent. Hence, even for a fixed order p , there is an infinity of dynamic-multiscaling exponents [5,6,8–12]; roughly speaking, to specify the dynamics of an eddy of a given length scale, we require this infinity of exponents.

Statistically steady fluid turbulence is very different in 3D and 2D; the former exhibits a direct cascade of energy whereas the latter shows an inverse cascade of kinetic energy from the energy-injection scale to larger length scales and a direct cascade in which the enstrophy goes towards small length scales [14]. In many physical realizations of 2D turbulence, there is an air-drag-induced friction. In this direct-cascade regime, velocity structure functions show simple scaling but their vorticity counterparts exhibit multiscaling [15–17], with exponents that depend on the friction. Time-dependent structure functions have not been studied in 2D fluid turbulence; the elucidation of the dynamic multiscaling of these structure functions, which we present here, is an important step in the systematization of such multiscaling in turbulence.

We numerically solve the forced, incompressible, 2D Navier-Stokes equation with air-drag-induced friction, in the vorticity (ω)–stream-function (ψ) representation with periodic boundary conditions:

$$\partial_t \omega - J(\psi, \omega) = \nu \nabla^2 \omega - \mu \omega + f, \quad (1)$$

where $\nabla^2 \psi = \omega$, $J(\psi, \omega) \equiv (\partial_x \psi)(\partial_y \omega) - (\partial_x \omega)(\partial_y \psi)$, and the velocity $\mathbf{u} \equiv (-\partial_y \psi, \partial_x \psi)$. The coefficient of friction is μ and f is the external force. We work with both Eulerian and quasi-Lagrangian fields. The latter are defined with respect to a Lagrangian particle, which was at the point ξ_0 at time t_0 , and is at the position $\xi(t|\xi_0, t_0)$ at time t , such that $d\xi(t|\xi_0, t_0)/dt = \mathbf{u}[\xi(t|\xi_0, t_0), t]$, where \mathbf{u} is the Eulerian velocity. The quasi-Lagrangian velocity field \mathbf{u}^{QL} is defined [13] as follows:

$$\mathbf{u}^{\text{QL}}(\mathbf{x}, t|\xi_0, t_0) \equiv \mathbf{u}[\mathbf{x} + \xi(t|\xi_0, t_0), t]; \quad (2)$$

likewise, we can define the quasi-Lagrangian vorticity field ω^{QL} in terms of the Eulerian ω . To obtain this quasi-Lagrangian field we use an algorithm developed in Ref. [18], described briefly in the Supplemental Material [19].

To integrate the Navier-Stokes equations we use a pseudospectral method with the 2/3 rule for the removal of aliasing errors [17] and a second-order Runge-Kutta scheme for time marching with a time step $\delta t = 10^{-3}$. We force the fluid deterministically on the second shell in Fourier space. And we use $\mu = 0.1$, $\nu = 10^{-5}$, and $N = 2048^2$ collocation points [20]. We obtain a turbulent but statistically steady state with a Taylor microscale $\lambda \simeq 0.2$, Taylor-microscale Reynolds number $\text{Re}_\lambda \simeq 1400$, and a box-size eddy-turnover time $\tau_{\text{eddy}} \simeq 8$. We remove

the effects of transients by discarding data up to time $\simeq 80\tau_{\text{eddy}}$. We then obtain data for averages of time-dependent structure functions for a duration of time $\simeq 100\tau_{\text{eddy}}$. The energy spectrum averaged over the same time interval is shown in Fig. 1(a).

The equal-time, order- p , vorticity structure functions we consider are $S_p^\phi(r) \equiv \langle [\delta\omega^\phi(r, t)]^p \rangle \sim r^{z_p^\phi}$, for $\eta_d \ll r \ll L$, where $\delta\omega^\phi(r, t) = [\omega^\phi(\mathbf{x} + \mathbf{r}, t) - \omega^\phi(\mathbf{x}, t)]$, the angular brackets denote an average over the nonequilibrium statistically steady state of the turbulent fluid, and the superscript ϕ is either E , in the Eulerian case, or QL , in the quasi-Lagrangian case; for notational convenience we do not include a subscript ω on S_p^ϕ and the multiscaling exponent z_p^ϕ . We assume isotropy here, but show below how to extract the isotropic parts of S_p^ϕ in a DNS. We also use the time-dependent, order- p vorticity structure functions

$$\mathcal{F}_p^\phi(r, \{t_1, \dots, t_p\}) \equiv \langle [\delta\omega^\phi(r, t_1) \dots \delta\omega^\phi(r, t_p)] \rangle; \quad (3)$$

here t_1, \dots, t_p are p different times; clearly, $\mathcal{F}_p^\phi(r, \{t_1 = \dots = t_p = 0\}) = S_p^\phi(r)$. We concentrate on the case $t_1 = t_2 = \dots = t_l \equiv t$ and $t_{l+1} = t_{l+2} = \dots = t_p = 0$, with $l < p$, and, for simplicity, denote the resulting time-dependent structure function as $\mathcal{F}_p^\phi(r, t)$; shell-model studies [8,9] have shown that the index l does not affect dynamic-multiscaling exponents, so we suppress it henceforth. Given $\mathcal{F}_p^\phi(r, t)$, it is possible to extract a characteristic time scale $\tau_p(r)$ in many different ways. These time scales can, in turn, be used to extract the order- p dynamic-multiscaling exponents z_p via the dynamic-multiscaling Ansatz $\tau_p(r) \sim r^{z_p}$. If we obtain the order- p , degree- M , integral time scale

$$\mathcal{T}_{p,M}^{I,\phi}(r) \equiv \left[\frac{1}{S_p^\phi(r)} \int_0^\infty \mathcal{F}_p^\phi(r, t) t^{(M-1)} dt \right]^{(1/M)}, \quad (4)$$

we can use it to extract the *integral* dynamic-multiscaling exponent $z_{p,M}^{I,\phi}$ from the relation $\mathcal{T}_{p,M}^{I,\phi} \sim r^{z_{p,M}^{I,\phi}}$. Similarly, from the order- p , degree- M , *derivative* time scale

$$\mathcal{T}_{p,M}^{D,\phi} \equiv \left[\frac{1}{S_p^\phi(r)} \frac{\partial^M}{\partial t^M} \mathcal{F}_p^\phi(r, t) \Big|_{t=0} \right]^{(-1/M)}, \quad (5)$$

we obtain the derivative dynamic-multiscaling exponent $z_{p,M}^{D,\phi}$ via the relation $\mathcal{T}_{p,M}^{D,\phi} \sim r^{z_{p,M}^{D,\phi}}$.

Equal-time vorticity structure functions in 2D fluid turbulence with friction exhibit multiscaling in the direct-cascade range [15–17]. For the case of 3D homogeneous, isotropic fluid turbulence, a generalization of the multifractal model [4], which includes time-dependent velocity structure functions [5,9,11,12], yields linear bridge relations between the dynamic-multiscaling exponents and their equal-time counterparts. For the direct-cascade regime in our study, we replace velocity structure functions

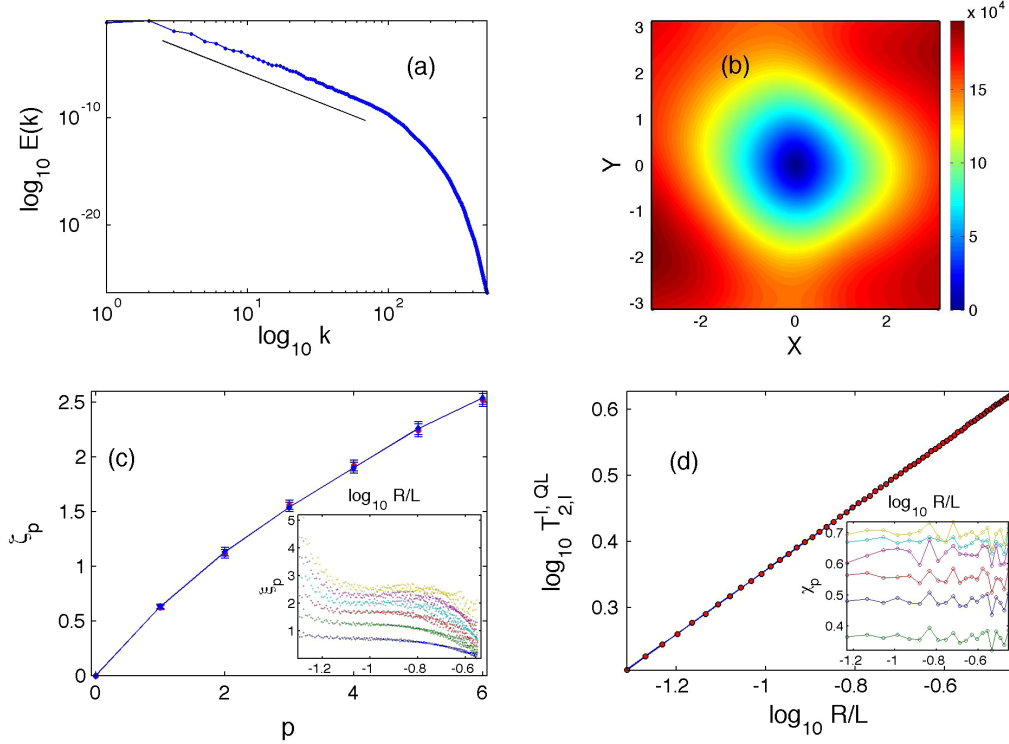


FIG. 1 (color online). (a) Log-log (base 10) plot of the energy spectrum $E(k)$ versus k . The black line with slope -4.1 is shown for reference. (b) Pseudocolor plot of the equal-time, quasi-Lagrangian, second-order vorticity structure function $S_2^{\text{QL}}(\mathbf{R})$ averaged over the origin \mathbf{r}_c [we use $\mathbf{r}_c = (i, j)$, $2 \leq i, j \leq 5$]. (c) Plots of the equal-time, vorticity, multiscaling exponents ξ_p^ϕ versus p for Eulerian (red circles) and quasi-Lagrangian (blue diamonds) fields (error bars are comparable to the size of the symbols); the inset shows the local slopes ξ_p , obtained as defined in the text, versus the separation, from $p = 1$ (bottom) to $p = 6$ (top). (d) Log-log (base 10) plot of the order-2, degree-1, integral time scale $T_{2,1}^{I,QL}(R)$ versus the separation R showing our data points (open red circles) and the best-fit line (full black) in the scaling range; the inset shows the local slopes χ_p , obtained as defined in the text, versus the separation, from $p = 1$ (bottom) to $p = 6$ (top).

by vorticity structure functions and thus obtain the following bridge relations for time-dependent vorticity structure functions in 2D fluid turbulence with friction:

$$z_{p,M}^{I,\phi} = 1 + [\zeta_{p-M}^\phi - \zeta_p^\phi]/M, \quad (6)$$

$$z_{p,M}^{D,\phi} = 1 + [\zeta_p^\phi - \zeta_{p+M}^\phi]/M. \quad (7)$$

The vorticity field $\omega^\phi = \langle \omega^\phi \rangle + \omega'^\phi$ can be decomposed into the time-averaged mean flow $\langle \omega^\phi \rangle$ and the fluctuations ω'^ϕ about it. To obtain good statistics for vorticity structure functions it is important to eliminate any anisotropy in the flow by subtracting out the mean flow from the field. Therefore, we redefine the order- p , equal-time structure function to be $S_p^\phi(\mathbf{r}_c, \mathbf{R}) \equiv \langle |\omega'^\phi(\mathbf{r}_c + \mathbf{R}) - \omega'^\phi(\mathbf{r}_c)|^p \rangle$, where \mathbf{R} has magnitude R and \mathbf{r}_c is an origin. We next use $S_p^\phi(\mathbf{R}) \equiv \langle S_p^\phi(\mathbf{r}_c, \mathbf{R}) \rangle_{\mathbf{r}_c}$, where the subscript \mathbf{r}_c denotes an average over the origin [we use $\mathbf{r}_c = (i, j)$, $2 \leq i, j \leq 5$]. These averaged structure functions are isotropic, to a good approximation for small R , as can be seen from the illustrative pseudocolor plot of $S_2^{\text{QL}}(\mathbf{R})$ in Fig. 1(b). The purely isotropic parts of such

structure functions can be obtained [17,21] via an integration over the angle θ that \mathbf{R} makes with the x axis; i.e., we calculate $S_p^\phi(R) \equiv \int_0^{2\pi} S_p^\phi(\mathbf{R}) d\theta$ and then the equal-time multiscaling exponent ζ_p^ϕ , the slopes of the scaling ranges of log-log plots of $S_p^\phi(R)$ versus R . The mean of the local slopes $\xi_p \equiv d(\log S_p^\phi)/d(\log R)$ in the scaling range yields the equal-time exponents, and their standard deviations give the error bars. The equal-time vorticity multiscaling exponents, with $1 \leq p \leq 6$, are given for Eulerian and quasi-Lagrangian cases in columns 2 and 3, respectively, of Table I; they are equal, within error bars, as can be seen most easily from their plots versus p in Fig. 1(c).

We obtain the isotropic part of $\mathcal{F}_p^\phi(R, t)$ in a similar manner. Equations (4) and (5) now yield the order- p , degree- M integral and derivative time scales (see the Supplemental Material [19]). Slopes of linear scaling ranges of log-log plots of $\mathcal{T}_{p,M}^{I,\phi}(R)$ versus R yield the dynamic-multiscaling exponent $z_{p,M}^{I,\phi}$. A representative plot for the quasi-Lagrangian case, $p = 2$ and $M = 1$, is given in Fig. 1(d); we fit over the range $-1.2 < \log_{10}(r/L) < -0.55$ and obtain the local slopes χ_p with

TABLE I. Order- p ; equal-time, Eulerian exponents ζ_p^E ; equal-time, quasi-Lagrangian exponents ζ_p^{QL} ; integral-scale, dynamic-multiscaling exponent $z_{p,1}^{I,QL}$ from the bridge relation and the values of ζ_p^{QL} in column 3; $z_{p,1}^{I,QL}$ from our calculations of time-dependent structure functions; the derivative-time exponents $z_{p,2}^{D,QL}$ from the bridge relation and the values of ζ_p^{QL} in column 3; $z_{p,2}^{D,QL}$ from our calculations of the time-dependent structure function. The error estimates are obtained as described in the text.

Order (p)	ζ_p^E	ζ_p^{QL}	$z_{p,1}^{I,QL}$ [Eq. (6)]	$z_{p,1}^{I,QL}$	$z_{p,2}^{D,QL}$ [Eq. (7)]	$z_{p,2}^{D,QL}$
1	0.62 ± 0.009	0.63 ± 0.008	0.366 ± 0.008	0.37 ± 0.02	0.55 ± 0.02	0.53 ± 0.02
2	1.13 ± 0.009	1.13 ± 0.008	0.50 ± 0.02	0.48 ± 0.01	0.62 ± 0.02	0.62 ± 0.2
3	1.561 ± 0.009	1.54 ± 0.01	0.59 ± 0.02	0.57 ± 0.01	0.64 ± 0.02	0.65 ± 0.01
4	1.92 ± 0.01	1.90 ± 0.01	0.64 ± 0.02	0.63 ± 0.01	0.68 ± 0.03	0.68 ± 0.01
5	2.24 ± 0.01	2.26 ± 0.01	0.64 ± 0.02	0.65 ± 0.02	0.70 ± 0.03	0.70 ± 0.02
6	2.52 ± 0.02	2.54 ± 0.02	0.72 ± 0.03	0.67 ± 0.02	0.71 ± 0.04	0.71 ± 0.03

successive, nonoverlapping sets of 3 points each. The mean values of these slopes yield our dynamic-multiscaling exponents (column 5 in Table I), and their standard deviations yield the error bars. We calculate the degree- M , order- p derivative-time exponents by using a sixth-order, finite-difference scheme to obtain $\mathcal{T}_{p,M}^{D,\phi}$ and then the dynamic-multiscaling exponents $z_{p,M}^{D,\phi}$. Our results for the quasi-Lagrangian case with $M = 2$ are given in column 7 of Table I. We find, furthermore, that both the integral and derivative bridge relations (6) and (7) hold within our error bars, as shown for the representative values of p and M considered in Table I (compare columns 4 and 5 for the integral relation and columns 6 and 7 for the derivative relation). Note also that the values of the integral and the derivative dynamic-multiscaling exponents are markedly different from each other (compare columns 5 and 7 of Table I).

The Eulerian structure functions $\mathcal{F}_p^E(R, t)$ also lead to nontrivial dynamic-multiscaling exponents, which are equal to their quasi-Lagrangian counterparts (see Supplemental Material [19]). The reason for this initially surprising result is that, in 2D turbulence, the friction controls the size of the largest vortices, provides an infrared cutoff at large length scales, and thus suppresses the sweeping effect. We have demonstrated this in the Supplemental Material [19]. Had the sweeping effect not been suppressed, we would have obtained trivial dynamic scaling for the Eulerian case.

The calculation of dynamic-multiscaling exponents has been limited so far to shell models for 3D, homogeneous, isotropic fluid [6,8,9,11,12] and passive-scalar turbulence [10]. We have presented the first study of such dynamic multiscaling in the direct-cascade regime of 2D fluid turbulence with friction by calculating both quasi-Lagrangian and Eulerian structure functions. Our work brings out clearly the need for an infinity of time scales and associated exponents to characterize such multiscaling, and it verifies, within the accuracy of our numerical calculations, the linear bridge relations (6) and (7) for a representative value of μ . We find that friction also suppresses sweeping effects so, with such friction, even Eulerian vorticity

structure functions exhibit dynamic multiscaling with exponents that are consistent with their quasi-Lagrangian counterparts.

Experimental studies of Lagrangian quantities in turbulence have been increasing steadily over the past decade [22]. We hope, therefore, that our work will encourage studies of dynamic multiscaling in turbulence. Furthermore, it will be interesting to check whether the time scales considered here can be related to the persistence time scales for 2D turbulence [23].

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Note added.—Just as we were preparing this study for publication we became aware of a recent paper [24] on a related study for 3D fluid turbulence.

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- [19] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.107.184503> for a description of (i) our algorithm for obtaining quasi-Lagrangian fields in a pseudospectral simulation, (ii) the numerical determination of integral time scales from time-dependent structure functions, and (iii) the dynamic multi-scaling of Eulerian structure functions in the presence of friction.
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