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MASTER'S THESIS

Sequential Cognition Processes: A Framework For Reasoning with Non-Monotonic Logics

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Declaration of Authorship

I, Axel IND, declare that this thesis titled, "Sequential Cognition Processes: A Framework For Reasoning with Non-Monotonic Logics" and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

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Signed:

_____

Date:

_____

"Everything should be made as simple as possible, but no simpler."

Albert Einstein

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Abstract

Faculty of Engineering
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by Axel IND

Approaches to cognitive modelling with non-monotonic logics have thus far been largely *ad hoc* and poorly standardised, making inter-model comparisons difficult. As an attempt to systematically represent non-monotonic logics in a framework that standardises cognitive modelling under these logics without sacrificing their expressiveness, we introduce the Sequential Cognition Process (SCP). Under the assumption that human reasoning can be represented as a sequence of distinct cognitive operations on an initial knowledge base, SCPs provide a consistent framework for generating and evaluating models of human cognition. Using an adapted interpretation of the Weak Completion Semantics (WCS) and Reiter's Default Logic, SCPs are able to accurately model several classical experiments in cognitive modelling. We use the SCP framework to model both general case reasoners – which arrive at the most frequently observed conclusions – and poorly-studied individual case reasoners – which do not. We illustrate the use of the SCP framework through examination of the Wason Selection Task and Suppression Task.

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List of Abbreviations

SCP	**S**equential **C**ognitive **P**rocesses
nSCP	**n**on-**S**equential **C**ognitive **P**rocesses
WCS	**W**eak **C**ompletion **S**emantics
WST	**W**ason **S**election **T**ask
CTM	**C**ognitive **T**ransition **M**odel
pCTM	**p**artial **C**ognitive **T**ransition **M**odel

Chapter 1

Introduction

1.1   Overview

The human mind is complex. So complex that thousands of approaches from dozens of fields have failed to capture its complexity. The sheer size of the brain – containing over 5000 times as many neurons as the largest practical neural networks (Mocanu et al., 2018) – and our limited understanding of the fundamental learning processes it employs mean that researchers can neither completely describe nor predict human actions nor model human thought processes. Instead, much of the current state of the art in cognitive modelling relies on one of two general approaches: creating systems that structurally approximate the human brain, and creating systems which approximate a more abstract intuition of cognition. The first type of system encompasses many algorithms related to machine learning and deep learning; the second type, with which this paper is concerned, has existed in some form for far longer than humans have been thinking creatures (Smirnova et al., 2015). Every shark stalking its prey, every tiny proto-mammal hiding from a hungry dinosaur, and every man driving to work in the morning has applied this type of reasoning when trying to make predictions about the actions of other agents in their world. By applying case-specific reasoning to a known world state we are able to make imperfect, but quick, predictions about the mental state of other agents in our world.

Due in part to the difficulty of cloning dinosaurs to hunt participants in cognitive research experiments, and partly due to concerns about the ease with which they could answer questionnaires on the experience later, most cognitive tasks used by researchers tend to be more dull than the examples above.

Non-monotonic logics have proven able to adequately model a large number of standard cognitive reasoning tasks such as The Wason Selection Task (Wason, 1968) and Suppression Task (Byrne, 1989). These approaches, though effective and well-founded in isolation, are often unable to integrate or be compared to data models of other tasks, even when they rely on the same underlying logic. Although the non-monotonic logics themselves are generally carefully described, procedures ranging from best practise in deciding appropriate knowledge bases to the mechanisms by which inferences should be interpreted tend to be re-imagined on a case-by-case basis.

Further, cognitive frameworks using non-monotonic logics are almost always designed to describe the most common (general) conclusion drawn by participants in the experiment. Modelling other individual reasoners or classes of reasoners who differ from the norm is often a non-trivial process. It has been shown that the Weak Completion Semantics (Hölldobler, 2015) is able to model the four most general cases of the Wason Selection Task under the assumption that reasoners who differ from the general case (*deviant reasoners*) follow a sequence of mental processes that is still highly similar to that of the general reasoner (Breu et al., 2019).

This thesis introduces the Sequential Cognition Process (SCP) which generalises the assumption of sequential cognitive operations, each of which uses a collection of epistemic states as input and produces a collection of epistemic states as output.

1.2 Current State of the Art

The current state of the art in cognitive modelling using non-monotonic logics has begun to show a shift towards a computational approach to problem solving (Dietz, Holldobler, and Ragni, 2012). Stenning and Van Lambalgen, 2012 argued, in a general sense, that modelling human reasoning should be done primarily towards an appropriate representation of the underlying cognitive processes, and secondarily, it should be evaluated empirically with respect to this representation. This philosophy is in stark contrast to that of many other areas of artificial intelligence research, in which the underlying structure of the agent does not need to mimic existing cognitive processes, and instead the primary evaluation criterion is generally the effectiveness of the agent at making predictions on unseen data sets.

This thesis makes extensive reference to the work of Dietz, Holldobler, and Ragni, 2012, Dietz, Hölldobler, and Wernhard, 2014, and Ragni, Kola, and Johnson-Laird, 2017, and their contributions to the emerging field of cognitive modelling.

The author of this thesis has previously shown the suitability of the Weak Completion Semantics for modelling individual reasoners in the Wason Selection Task (Breu et al., 2019) and the SCP framework that this thesis introduces is an extension of the intuition of discrete, sequential processes underlying cognition. The question of how to model individual reasoners is a pernicious problem in cognitive modelling and work by Breu et al., 2019, as well modelling tools like `CCOBRA` (Nicolas, 2018) have been able to use traditional modelling techniques, with minor variations, to replicate more than simply the most common reasoner responses.

1.3 Contributions of the SCP Framework

Using SCPs and a set of well-founded cognitive operations, it is possible to apply traditional search techniques to problems in cognitive modelling with non-monotonic logics that have previously required expert-made models. SCPs introduce a number of desirable properties: they introduce a partially standardised (though extensible) set of allowable cognitive operations, they standardise the structure of what constitutes an epistemic state, they are easily modified to accommodate deviant reasoners when a well-founded general model already exists, and their sequential structure makes them well-suited to scoring algorithms that allow intra- and inter-experimental modelling and comparisons.

Logics which are able to interpret conditionals of the form "If ϕ, then ψ" are varied in both their implementation and underlying principles. Work by Heit, Hahn, and Feeney, 2005 has shown evidence that not all information is derived, stored, or retrieved symmetrically and thus, it is unlikely that a single logical system will be able to fully model tasks which utilise information whose interpretation is dependant on multiple structurally and interpretationally diverse sources of information. The SCP framework enables researchers to use robust search techniques which make no assumptions about conditional interpretation at formulation time, and rather, branches to incorporate multiple possible epistemic knowledge structures at run-time. This technique enables multiple non-monotonic logics to be used across a single cognition task.

This thesis also introduces a new method for modelling reasoners in the WST which makes use of the interpretation of a conditional and its contraposition to provide a more intuitive method of applying the WCS to the task. Section 6.2 shows that Dietz, Hölldobler,

and Wernhard, 2014's method of adding a subset of allowable abducables to a knowledge base to model the general case of the WST can also be used to model an individual reasoner case of the Suppresion Task in which suppression is not observed.

1.4 Thesis Layout

This thesis begins with a discussion of the mathematical preliminaries related to logic programming and non-monotonic logics - particularly the WCS and Reiter's default logic (Chapter 2). Chapter 3 introduces several classical experiments in the field of cognitive modelling and discusses an existing non-monotonic interpretation of each. Chapter 4 introduces the SCP framework and the concept of SCP Tasks, SCPs, and Realised SCPs. Chapter 5 introduced the concept of the cognitive toolbox which contains well-founded tools to allow the SCP framework to model a wide variety of cognitive tasks. Chapter 6 shows how SCPs can be used to model the classical experiments introduced in the Chapter 3 in terms of both individual and generalised reasoners. Chapter 7 discusses how SCPs might be compared to one another to determine which is most plausible or to identify common motifs that are present across experiments. Chapter 8 briefly discusses the implementation of the SCP framework in Python 3 and the core philosophies of its design. Finally, Chapter 9 summarises the contributions of this thesis and discusses potential extensions to the SCP framework.

Chapter 2

Mathematical Preliminaries

2.1 Propositional Logic

Propositional logic is one of the simplest classical logics and concerns itself with binary truth values. $\top$ denotes a tautology, and $\bot$ denotes a contradiction. An alphabet $\Sigma = \{v_1, ..., v_n\}$ defines the set of atoms available to the language. Every atom $v_i \in \Sigma$ has the domain $\{v_i, \bar{v}_i\}$. An (atom, domain) pair which associates an atom with a value in its domain is called a literal.

A formula $\theta \in \zeta$ from the set of all formuli ζ is defined recursively over all conjunction $\wedge$, disjunction $\vee$, and negation $\neg$ operations as follows:

$$\theta = \{v_i \in \Sigma | \neg\theta | \theta \vee \theta | \theta \wedge \theta\}$$

The material implication $(\psi \leftarrow \phi)$ precisely substitutes for the formula $(\neg\psi \vee \phi)$ and is often used, classically to denote an "if ϕ then ψ" relationship between variables. However, in Section 2.3 other interpretations of the "if ϕ then ψ" relationship are examined in a non-monotonic setting.

An interpretation of Σ, $I_w(\Sigma)$ assigns each $v_i \in \Sigma$ to one value in its domain as defined by possible world w. The interpretation of a formula θ, $I_w(\theta)$ is recursively calculated using the associated truth tables of the operators involved, as defined in Table 2.1.

2.2 Possible Worlds

A possible world $w \in W_\sigma$ from the set of all possible worlds W over a language Σ is $(v_0, w_0) \wedge ... \wedge I(v_n, w_n)$ and is the conjunction of each atom and an associated value from its domain. Possible worlds represent one setting of the variables in the alphabet that could hypothetically hold. A model of a propositional formula $\theta \in \zeta$ is a possible world w in which θ is evaluated to $\top$ (also called satisfying) and is written $w \models \theta$.

A model $Mod(\theta)$ denotes the set of possible worlds in which θ holds – that is, $Mod(\theta) = \{w | w \models \theta\}$. The operator $\models$ is overloaded so that $\phi \models \psi$ if and only if $Mod(\phi) \subseteq Mod(\psi)$ for $\phi, \psi \in \zeta$.

$\wedge$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

$\vee$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\neg$	
$\top$	$\bot$
$\bot$	$\top$

$\rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

$\leftarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

TABLE 2.1: Truth tables for standard operators in propositional logic.

$(\psi\|\phi)$	$\phi \models \top$	$\phi \models u$	$\phi \models \bot$
$\psi \models \top$	verification	non-applicability	non-applicability
$\psi \models u$	verification	non-applicability	non-applicability
$\psi \models \bot$	falsification	non-applicability	non-applicability

TABLE 2.2: Evaluation of conditional rules using a 3-valued *de Finetti* truth table.

The *least model* of a theory θ is model M of θ such that there is exists no other model M' of θ such that $M \subset M'$.

A set of propositional clauses W is called *deductively closed*, written th(W) if it contains every formula ϕ that can be logically deduced from W.

2.3 Conditionals

Defeasible knowledge is information which is assumed, but not guaranteed, to hold. Many non-monotonic logics treat this notion as a form of typicality. A condition $(\psi|\phi)$ is used to represent the defeasible rule "if ϕ then ψ". Following Wason, 1968 we do not use the implication $\leftarrow$ to represent a conditional, and instead opt for a three-valued interpretation of what Baratgin, Over, and Politzer, 2014 called a *de Finetti truth table* (Table 2.2). This truth table for conditionals differs from the standard implication in that it treats rules of the form $(\psi|\phi)$ as inapplicable when $\phi \models \bot$ or $\phi \models u$ and more closely follows human intuition about how rules are evidenced in the absence of their antecedent.

2.3.1 Interpretation of conditionals

The precise interpretation of conditionals is a topic for philosophical and logical debate. Approaches, both probabilistic and logical, can be used to define precisely what a conditional means in the context of a logical system. In this thesis we adopt adopt Stenning, Van Lambalgen, et al., 2008's interpretation of conditionals as licenses for implication.

Thus, the conditional $(\psi|\phi)$ precisely means $\psi \leftarrow \phi \wedge \neg ab$ where ab is called an abnormality predicate. The abnormality predicate captures an element of uncertainty in the interpretation of the conditional and means that the conditional now reflects the statement "If ϕ and nothing abnormal is known, then ψ".

The choice of appropriate value assignments to the abnormality predicate is non-trivial and often requires expert knowledge to design. However, over the course of this thesis we will adopt an algorithmic approach to abnormality creation (Algorithm 1) that will serve to mimic the hand-designed abnormalities used by Dietz, Holldobler, and Ragni, 2012, and

Breu et al., 2019.

Algorithm 1: `Conditional to License for Implication`

Function *Create Abnormalities(KB)* **is**

  for $(\psi|\phi) \in KB[\Delta]$ **do**

    $k :=$ the lowest natural number for which $\text{ab}_k \notin \text{KB}$;

    all dependencies $:= [A|(\psi|A) \in \text{KB}'[\Delta]]$;

    for $A \in$ *all dependencies* **do**

      head $:= \psi$;

      current dependencies $:=$ all dependencies $- A$;

      if *current dependencies* $= \{\}$ **then**

        current dependencies $:= \bot$;

      end

      body $:= (\text{current dependencies}_1 \vee ... \vee \text{current dependencies}_n)$;

      `/* Add the conditional as a license for implication to the set`
      `    of rules.                                               */`

      $\text{KB}'[S] := \text{KB} \cup (\text{head} \leftarrow A \wedge \neg\text{ab}_k)$;

      $\text{KB}'[S] := \text{KB}'[S] \cup \text{ab}_k \leftarrow \neg body$;

    end

  end

  `/* Remove all conditionals now that they have been interpreted as`
  `    licences for implication.                                  */`

  $\Delta := \{\}$;

  return KB'

end

2.4 Logic Programming

2.4.1 Knowledge Base

In propositional logic, a knowledge base $S = \{\phi_1, ..., \phi_n\}$ for the language Σ is a set of propositional rules from the set of formula ζ_Σ. A propositional knowledge base is used to encode certain knowledge and relationships about the world. S is called *consistent* only if there exists some possible world $w \in W$ such that every rule in $I_w(\text{KB}_{prop})$ evaluates to $\top$.

    More generally, a non-monotonic knowledge base $\text{KB} = (S, \Delta)$ contains both a set of propositional rules (S) and a set of defeasible/conditional rules ($\Delta = \{\Delta_1, ..., \Delta_m\}$). A non-monotonic knowledge base tolerates a conditional $(\psi|\phi) \in \Delta$ if and only if there exists some possible world $w \in W$ that verifies $(\psi|\phi)$ and does not falsify any conditional in Δ. KB is said to be *consistent* if and only if there exists of partition of Δ, $(\Delta_0, ..., \Delta_n)$ such that Δ_i is tolerated by Δ_{i+k}, where k is any natural number not greater than m.

    When $\text{KB} = (S = \{\text{rule}_1, ..., \text{rule}_n\}, \Delta = \{\})$, we will sometimes ignore Δ and refer only to the value of S whilst still using the term 'KB'.

2.5 Non-Monotonic Logics

In the field of non-monotonic logics, reasoning is represented as a collection of defeasible inferences. Unlike in classical logic, conclusions need not hold in perpetuity, or even in the same model and revision is always possible. Monotonic logics are not capable of describing human reasoning in experiments like the Suppression Task (Dietz, Holldobler, and Ragni, 2012) because they lack this revisionist characteristic. Ragni et al., 2016 showed that two-valued logics were not sufficient to model human cognition.

$\rightarrow$	$\top$	u	$\bot$
$\top$	$\top$	u	$\bot$
u	$\top$	$\top$	u
$\bot$	$\top$	$\top$	$\top$

TABLE 2.3: A table showing the implication operator in 3-valued Łukasiewicz logic.

A large number of non-monotonic frameworks exist in the literature (McDermott and Doyle, 1980), each applicable to a different subset of cognitive problem space, and each modelling their problem space with various degrees of success. In the simplest formulation, a non-monotonic logic is an extension to a classical logic which introduces a preference relation $\leftarrow_p$. This preference relation states that, given some number of derivable facts in a knowledge base, the fact derived using the most preferential rule is to be derived first and cannot be overwritten by a less preferable assignment.

Although the non-monotonic logics discussed in this chapter have simple extensions to first-order logic, we instead restrict ourselves to a propositional format throughout this thesis.

2.6 The Weak Completion Semantics

The Weak Completion Semantics is a non-monotonic logic which procedurally encodes several well-known cognitive phenomena. The WCS makes use of 3-valued Łukasiewicz logic (Table 2.3), rather than Kripke-Kleene logic which has been shown inadequate to model several aspects of cognition, including Suppression (Dietz, Holldobler, and Ragni, 2012). It adds abnormalities to non-ground inferences, and replaces the classical inference ($\leftarrow$), with a bijective ($\leftrightarrow$).

The Weak Completion of a program P is defined as follows:

1. Replace all clauses of the form $A \leftarrow body_1, ..., A \leftarrow body_n$ with $A \leftarrow body_1 \vee ... \vee body_n$.

2. Replace all occurrences of $\leftarrow$ with $\leftrightarrow$.

Applying this procedure to P results in wc P which is the *weak completion* of P.

The next requirement to apply the WCS framework is the introduction of a semantic operator ϕ_{SvL} (Stenning, Van Lambalgen, et al., 2008). Let J be the result of applying the semantics operator to an interpretation I and logic program P. Then J is defined as follows:

$$J^\top = \{A| \text{ there exists a clause } A \leftarrow Body \in P$$
$$\text{with } I(Body) = \top\}$$

$$J^\bot = \{A| \text{ there exists a clause } A \leftarrow Body \in P$$
$$\text{and for all clauses } A \leftarrow Body \in P$$
$$\text{we find } I(Body) = \bot\}$$

Using $I = <\emptyset, \emptyset>$, the least model of P ($lm_Ł wcP$) can be calculated by iterating $\phi_{SvL,P}$. The WCS guarantees a unique least model for a propositional knowledge base P when interpreted with Łukasiewicz logic.

2.7   Reiter's Default Logic

Reteir's Default Logic (Reiter, 1980) is a non-monotonic framework which allows us to divide the inferential capabilities of a system into those facts and inference rules which are always true (as in classical logic) and those which are are usually true. The second type of inference is the eponymous default rule.

A Reiter Default Theory (W, D) consists of a background theory W, and a set of default rules D. For our purposes we will restrict both to a propositional, rather than a first-order logic domain.

The set of default rules D consists of a set of tritonic clauses of the form $\delta = \frac{\text{pre}(\delta):\text{just}(\delta)}{\text{cons}(\delta)}$. Where $\text{pre}(\delta)$ is a propositional clause called the precondition, $\text{just}(\delta)$ is a set of propositional clauses and is called the justification, and $\text{cons}(\delta)$ is a propositional clause called the consequence. To maintain consistency, we restrict our background theory to propositional clauses of the form (head $\leftarrow$ body) and restrict D so that, for any $\delta \in D$, $\text{cons}(\delta)$ is of the form (head $\leftarrow$ body).

A default δ is *applicable* to a deductively closed set $\text{th}(W)$ if and only if $\text{pre}(\delta) \in \text{th}(W)$ and there exists no $\text{just}_i(\delta) \in \text{just}(\delta)$ such that $\neg\text{just}(\delta) \in th(W)$.

A default process $\Pi = (\delta_1, ..., \delta_n)$ consists of a sequence of default rules, $\delta_i \in D$. Following Ragni et al., 2017, we define the sets $\text{In}(\Pi) = \{\text{th}(W \cup \text{cons}(\delta) | \delta \in \Pi)\}$ and $\text{Out}(\Pi) = \{\neg A | A \in \text{just}(\delta), \delta \in \Pi\}$, where $\text{In}(\Pi)$ is called the *inset*, and $\text{Out}(\Pi)$ is called the *outset*. A default process is called *successful* if and only if $\text{In}(\Pi) \cap \text{Out}(\Pi) = \emptyset$. A default process if called *closed* if and only if there exists no $\delta \in D$, where δ is applicable to $\text{In}(\Pi)$ and $\delta \notin \Pi$. The set E is called an extension of default process D if and only if $E = \text{In}(\Pi)$, for some successful closed process Π.

We write $W \models_D^{\text{Reiter}} \psi$ if and only if $\psi \in \bigcap \epsilon$, where ϵ is the set of all valid extensions of the default theory (W, D).

It should be immediately apparent from this formulation that, for many possible sets of rules in default theories, the number of extensions is non-monotonic as it is dependent on the number of orders in which default rules are applied successfully. A very significant restriction on Reiter's default logic is the complexity of computing deductively closed sets.

The closed world assumption $\frac{:\neg\perp}{\perp}$ states that all information which is not known to be true is assumed to be false. The closed world assumption can be used to ensure that, for every variable $v_i \in \Sigma$, $I_w(v_i) \models \top$ or $I_w(v_i) \models \perp$.

Chapter 3

Experiments

3.1 Suppression Task

3.1.1 Intuitive Overview

The Suppression Task refers to an experiment conducted by Byrne, 1989 and is a classical example of the inadequacy of monotonic logics for modelling human reasoning. In classical logic, if our knowledge base S is such that $I(S) \models \phi$, then it must be the case that $I(S \cup \theta) \models \phi$, for some new rule θ. However, in the Suppression Task, participants no longer draw classically valid inferences when new information is added. The task is described with three statements:

1. If she has an essay to write (e), she will study late in the library (l).

2. If the library is open (o), she will study late in the library (l).

3. She has an essay to write (e).

For the purposes of this thesis we will interpret the first and second statements as the conditional rules ($l|e$), and ($l|o$). The final rule ($e \leftarrow \top$) is regarded as a certain fact not subject to revision.

Given only the rule ($e \leftarrow \top$) and conditional ($l|e$), the participants consistently concluded that she would study late in the library, seemingly interpreting the conditional ($l|e$) as the implicative fact ($l \leftarrow e$) and drawing the classical logic inference $\frac{l \leftarrow e, e}{l}$ with *modus ponens*. But when given the additional conditional ($l|o$), participants no longer believe that they have enough information to judge whether she will study late in the library, and a significant portion of them no longer draw the classical conclusion. This effect, called Suppression, demonstrates the need for something more than classical logic for modelling human reasoning.

3.1.2 Modelling the Suppression Task: WCS

Work by Dietz, Hölldobler, and Wernhard, 2014 has shown that the WCS is an adequate non-monotonic logic for modelling the Suppression Task. Under the WCS and Łukasiewicz 3-valued logic, the task can be modelled using an epistemic knowledge base KB_{noSup}, with:

$$\text{KB}_{\text{noSup}} = (S = \{e \leftarrow \top\}, \Delta = \{(l|e)\})$$

Which represents the task without the suppressing conditional ($l|o$). Interpreting the conditional as a license for implication, as in Section 2.3.1, the knowledge base can be rewritten as a propositional knowledge base:

$$\text{KB}_{\text{noSup}} = \{e \leftarrow \top, l \leftarrow e \wedge \neg \text{ab}_1, \text{ab}_1 \leftarrow \bot\})$$

Weakly completing KB_{noSup} results in:

$$\text{wc } KB_{noSup} = \{e \leftrightarrow \top, l \leftrightarrow e \wedge \neg ab_1, ab_1 \leftrightarrow \bot\}$$

Finally application of the semantic operator results in $\top = \{e\}, \bot = \{ab_1\}$ after the first execution and converges to the least model $\top = \{e, l\}, \bot = \{ab_1\}$.

After application of the semantic operator l is true in the least model, and so participants conclude that she will study late in the library (as when KB_{noSup} is evaluated classically). However, in the case where Suppression is observed, the same process yields a different result because of the presence of the extra conditional $(l|o)$.

Now the initial knowledge base KB_{sup} includes the conditional $(l|o)$:

$$KB_{sup} = (S = \{e \leftarrow \top\}, \Delta = \{(l|e), (l|o)\})$$

And, when rewritten as a propositional knowledge base:

$$KB_{sup} = \{e \leftarrow \top, l \leftarrow e \wedge \neg ab_1, l \leftarrow o \wedge \neg ab_2, ab_1 \leftarrow \neg o, ab_2 \leftarrow \neg e\}$$

Where the abnormalities added to the interpretation of conditionals $(\psi|body)$ which share ψ as a head are considered dependent on one another. Weakly completing this knowledge base results in:

$$\text{wc } KB_{sup} = \{e \leftrightarrow \top, l \leftrightarrow (e \wedge \neg ab_1) \vee (o \wedge \neg ab_2), ab_1 \leftrightarrow \neg o, ab_2 \leftrightarrow \neg e\}$$

Applying the semantic operator to wc KB_{sup} converges to the least model $\top = \{e\}, \bot = \{ab_2\}$ after the first application. The variable l remains unknown in the least model. Suppression has now been demonstrated because $l \notin \text{lm wc } KB_{sup}$, even thought $l \in \text{lm wc } KB_{noSup}$.

3.1.3 Modelling the Suppression Task: Reiter's Default Logic

Ragni et al., 2017 showed that it was not possible to model the suppression task using Reiter's Default Logic. They created a default theory (W, D'_P) as an illustration of this point.

A naive Default Theory for the suppression task (W, D_{naive}) can be used to model the classical logic response to the suppression task (i.e. that she will study late in the library), by treating the conditional as an implication without an associated abnormality predicate.

$$W = \{e \leftarrow \top\}$$

$$D_{naive} = \{\delta_1 : \frac{e \leftarrow \top : l \leftarrow \top}{l \leftarrow \top}, \}$$

$$D'_{naive} = \{\delta_1 : \frac{e \leftarrow \top : l \leftarrow \top}{l \leftarrow \top}, \delta_2 : \frac{o \leftarrow \top : l \leftarrow \bot}{l \leftarrow \top}\}$$

The only closed and successful default process that can generated for this task is $\Pi_{naive} = (\delta_1)$, which results in the inset $In(\Pi_{naive}) = th(th(\{e \leftarrow \top\}) \cup \{l \leftarrow \top\})$. When the set of default rules is extended to create D'_{naive}, Π_{naive} remains the only extension and has the inset $In(\Pi'_{naive}) = th(th(\{e \leftarrow \top\}) \cup \{l \leftarrow \top\})$. Thus, in every extension e of (W, D_{naive}), we find that $(l \leftarrow \top) \in In(e)$, and in every extension e' of (W, D'_{naive}) we find that $(l \leftarrow \top) \in In(e')$. Thus, the extension of the set of default rules in (W, D) to create (W, D') has failed to demonstrate a suppressing effect on the variable l.

Even generalising our WCS approach to the Suppression Task and defining the sets of

	{D}	{D,3}	{D,3,7}	{D,7}
Abstract	36	39	5	19

TABLE 3.1: The canonical results of the Wason Selection Task for the Abstract case.

	p	pq	$pq\bar{q}$	$p\bar{q}$
Abstract	36	39	5	19
Everyday	23	37	11	29
Deontic	13	19	4	64

TABLE 3.2: The canonical results of the Wason Selection Task over the Abstract, Everyday, and Deontic Cases

defaults D_P, and D'_P which capture intuition about abnormalities, fails to demonstrate suppression.

$$D_P = \{\delta_1 : \frac{e \leftarrow \top : \mathsf{ab}_1 \leftarrow \bot}{l \leftarrow \top}, \delta_2 : \frac{\top : \mathsf{ab}_1 \leftarrow \top}{\mathsf{ab}_1 \leftarrow \top}\}$$

$$D'_P = \left\{ \begin{array}{ll} \delta_1 : \frac{e \leftarrow \top : \mathsf{ab}_1 \leftarrow \bot}{l \leftarrow \top}, & \delta_2 : \frac{o \leftarrow \top : \mathsf{ab}_2 \leftarrow \bot}{l \leftarrow \top}, \\ \delta_3 : \frac{o \leftarrow \bot : \mathsf{ab}_1 \leftarrow \top}{\mathsf{ab}_1 \leftarrow \top}, & \delta_4 : \frac{e \leftarrow \bot : \mathsf{ab}_2 \leftarrow \top}{\mathsf{ab}_2 \leftarrow \top} \end{array} \right\}$$

For the theory (W, D_P), the only closed and successful default processes which produce extensions are $\Pi_P^1 = (\delta_1, \delta_2)$ and $\Pi_P^2 = (\delta_2, \delta_1)$, with $(l \leftarrow \top) \in \text{In}(\Pi_P^1)$ and $(l \leftarrow \top) \in \text{In}(\Pi_P^2)$. For the theory (W, D'_P) the only closed and successful default process is $\Pi'_P = (\delta_1)$ and $(l \leftarrow \top) \in \text{In}(\Pi'_P)$, thus neither theory demonstrates suppression.

3.2 Wason Selection Task

Another widely studied task in the psychological literature is the Wason Selection Task (WST), which asks participants to draw conclusions about which variables are able to falsify a given rule – a process captured in classical logic by the *Modus Tolens*[1] deductive argument. The WST has many formulations (including the Abstract, Everyday, and Deontic cases), and this thesis will examine one of the most mostly widely researched cases: the Abstract case.

The Abstract case of the WST presents the subject with four cards laid out on a table as shown in Figure 3.1. Their face-up sides read $D, K, 3,$ and 7. Participants are told that each

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[1]If $a \rightarrow b$, then $\neg b \rightarrow \neg a$
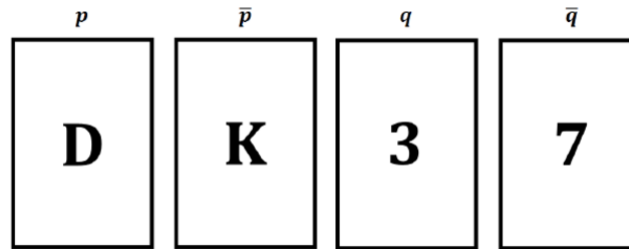
FIGURE 3.1: The Abstract case of the Wason Selection Task

D	3	$(3 \leftarrow D)$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

TABLE 3.3: Propositional evaluation of the implication $3 \leftarrow D$ for each possible truth value of 3 and D.

D	3	$(3\|D)$
$\top$	$\top$	verification
$\top$	u	non-applicability
$\top$	$\bot$	falsification
$\bot$	$\cdot$	non-applicability
u	$\cdot$	non-applicability

TABLE 3.4: De-finetti evaluation of the conditional $(3|D)$ for each possible truth value of 3 and D.

card has a number on one side and a letter on the other. They are then asked which cards must be turned over to test the rule:

> "If there is a D on one side of the card, then the other side shows 3."

This task requires the participants to find possible defeaters for the conditional rule. If we assume a classical interpretation of conditionals of the form "If ψ then ϕ", Table 3.3 shows that the cards pairs $(D, 3)$ and $(D, 7)$ are both possible defeaters to the rule $(3 \leftarrow D)$ because it can only be falsified when the interpretation maps $(D, \top)$. Any observed card which allows a possible world with both of those assignments is a possible defeater. If a participant were to turn the D card and find a number that was not 3, or were to a turn the card 7 and find the letter D on the other side, the classical interpretation of conditional would not hold. Thus these two cards must be turned to check the rule. However, human subjects very seldom choose this classical answer set and instead overwhelmingly prefer to turn the cards D and 3. Table 3.1 shows the most common card selection sets for participants.

A great many cognitive theories have been applied to these results, with varying degrees of success (Ragni et al., 2017). This paper will focus on the WCS interpretation of the task provided by Ragni, Kola, and Johnson-Laird, 2017. This approach assumes an initial knowledge base KB containing only the conditional $(3|D)$ and an interpretation of the conditional as a license for implication.
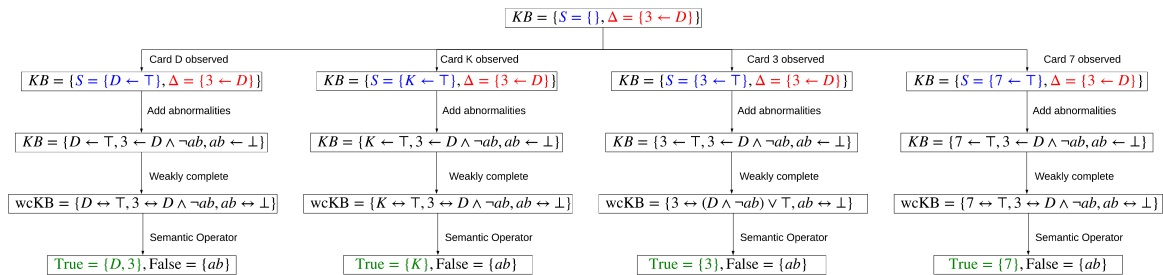
FIGURE 3.2: Application of the WCS to the abstract case of the WST for the basic weak completion approach.

Observation O	Explanation ϵ	lm wc $(KB \cup \epsilon)$	Turn Function
D	$\{D \leftarrow \top\}$	$\{D,3\}\{ab_1\}$	turn
K	$\{K \leftarrow \top\}$	$\{K\}\{ab_1\}$	do not turn
3	$\{D \leftarrow \top\}$	$\{D,3\}\{ab_1\}$	turn
7	$\{7 \leftarrow \top\}$	$\{7\}\{ab_1\}$	do not turn

TABLE 3.5: Dietz' computational logic approach to the abstract case of the WST.

As a general strategy for deciding whether to turn a card C, given a possible world w, we will turn the card C if $I_w((3|D)) \neq$ non-applicability under the *de Finetti* truth table from Section 2.3. Table 3.4 shows how the conditional $(3|D)$ is evaluated under this interpretation. This interpretation intentionally leaves the relationship between C and w vague and does not define what constitutes a valid possible world.

Dietz, Holldobler, and Ragni, 2012 showed that the abstract case of the WST could be modelled using the WCS and 3-valued Łukasiewicz logic. Using the conditional as the initial logic program, we write KB $= (S = \{\}, \Delta = (3|D))$. Treating conditionals as licenses for implication, KB can be rewritten as KB $= \{3 \leftarrow D \wedge \neg ab_1\}$ where ab_1 is an abnormality predicate. Now lm wc(KB) $= \{\top = \{\}, \bot = \{ab\}\}$. Using the least model lm wc(KB) as possible world $I_{\text{lm wc(KB)}}((3|D)) =$ non-applicability and we do not turn the card. In order to adapt the WCS to the WST they introduced the concept of observation support.

The set of abducibles A corresponds to positive and negative facts which are not assigned in KB. For the WST as formulated here $A = \{D \leftarrow \top, D \leftarrow \bot, 3 \leftarrow \top, 3 \leftarrow \bot, K \leftarrow \top, K \leftarrow \bot, 7 \leftarrow \top, 7 \leftarrow \bot\}$. An observation O is *explained* by a minimal explanation $\epsilon \in A$ if and only if O holds in lm wc$(KB \cup \epsilon)$. Table 3.5 shows the least models that can be drawn for the observations D, 3, K, and 7. If a least model obtained in this way contains atoms and atomic assignments that could verify or falsify the conditional (i.e. make $I_{\text{lm wc}(KB \cup \epsilon)}(3|D) \neq$ non-applicability when using the *de Finetti* truth table), then the observed card should be turned. Figure 3.2 shows the derivation of lm wc$(KB \cup \epsilon)$ for $\epsilon = D, \epsilon = 3, \epsilon = K$, and $\epsilon = 7$.[2]

This simple interpretation of the WST which applies abduction to the initial knowledge base to derive the observation is sufficient to model the general reasoner. It does not, however, explain any of the other deviant cases which are observed in Table 3.1. Why, for example, should it be considered accurate if it gives no information about the classically accurate choice of the D and 7 cards, chosen by 19 participants, a significant proportion? Instead, consider how these aberrant reasoners could be modelled. Perhaps they use extra computational steps or omit steps? This author has previously shown that the WCS is able to model these individual reasoners (Breu et al., 2019)[3] by including three simple processes: *basic weak completion*, *abduction* (as we have already discussed in this section), and *contraposition*. By including these three processes, and stochastically controlling when they are activated or silenced, it has been shown that not only is this extended framework able to model the major cases of the WCS, but very close approximations to empirical frequency results can also be drawn for all three cases of the WST (Table 3.2).

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[2]This approach differs in description from Dietz' original one, in which an observed card o was turned if the least model assigned non-u values to every variable in the conditional and explained o, but this formulation is identical in practice.

[3]My contribution to this paper was the initial proposal of the extensions, their integration, the silencing mechanism used for modelling individual reasoners, and the broad idea of the stochastic evaluation technique itself.

Iteration	$\top$	$\bot$
0		
1	7	ab_1, ab_2
2	7	$3, ab_1, ab_2$
3	$7, D'$	$3, ab_1, ab_2$
4	$7, D'$	$3, ab_1, ab_2, D$

TABLE 3.6: Applying the Weak Completion Semantics and Contraposition to the 7 card of the WST.

Basic Weak Completion

Basic weak completion ignores the existence of other least models which may validate or invalidate the inference $(3|D)$ and, instead handles only those cases where $\epsilon \leftrightarrow (O \leftarrow \top)$. That is, only information about the currently observed card is considered in the abductive framework. Figure 3.2 illustrates this process for each of the four observations: $D, 3, K$, and 7. Now only wc $\mathrm{lm}(KB \cup (D \leftarrow \top))$ provides assignments which for which the de Finetti interpretation of $(3|D) \neq$ non-applicability. Observing the card 3 no longer leads to the decision to turn the card because only the case $\epsilon = (3 \leftarrow \top)$ is considered and the case $\epsilon = (D \leftarrow \top)$ is not considered.

Contraposition

Contraposition implicitly makes use of *modus tolens*, usually assumed to be silenced in human cognition, to derive certain canonical cases in the WST. Using contraposition it becomes possible to model individual participants who choose to turn the cards D and 7 (the classical response).

Contraposition is applied as follows for the initial knowledge base KB:

1. For each conditional $(\psi|\phi) \in KB'$:

   - Add the conditional $(\phi'|X)$ to Δ, where X is the other possible value for that card face[4].

   - Add the fact $(\phi \leftarrow \neg\phi')$ to S.

2. Interpret conditionals as licenses for implication.

3. Apply the Abductive case of the WCS as normal.

4. Turn the card if and only if some or $I_{\mathrm{lm\,WC\,}(KB\cup\epsilon)}(\psi|\phi) \neq$ non-applicability or some $I_{\mathrm{lm\,WC\,}(KB\cup\epsilon)}(\phi'|X) \neq$ non-applicability, and lm WC $(KB \cup \epsilon)$ explains O.

Using this technique, and applying basic weak completion – i.e. only considering the case $\epsilon \leftrightarrow (O \leftarrow \top)$ – it is possible to model the classical case of the WST. The resolution for the 7 card is as follows:

$$KB = (S = \{\}, \Delta = \{(3|D)\})$$

$$KB' = (S = \{D \leftarrow \neg D'\}, \Delta = \{(3|D), (D'|7)\})$$

$$KB' = \{3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, D \leftarrow \neg D', ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$$

$$(KB' \cup (7 \leftarrow \top)) = \{3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, D \leftarrow \neg D', ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 7 \leftarrow \top\}$$

---

[4] $(D, K), (3, 7)$
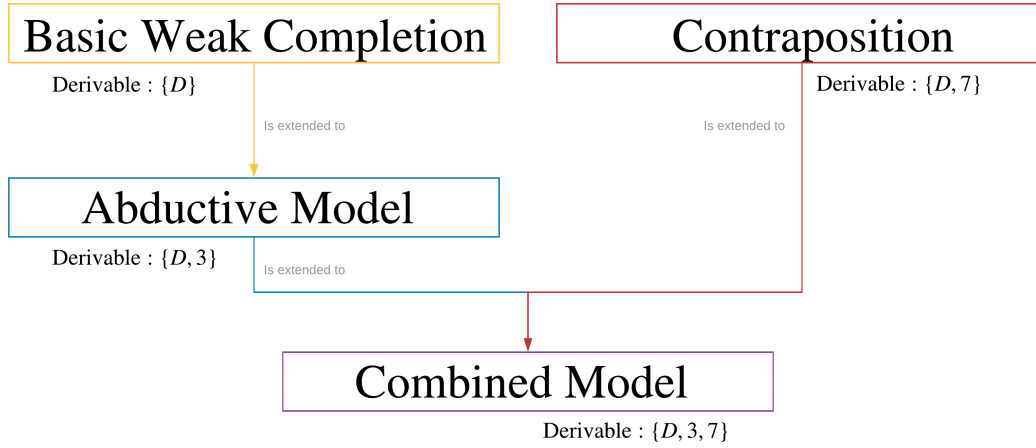
FIGURE 3.3: Using the abduction and contraposition extensions to derive all four canonical cases. Arrows indicate that the origin model produces a subset of the turn instructions of the endpoint.

$$\text{wc}\left(\text{KB}' \cup (7 \leftarrow \top)\right) = \left\{ \begin{array}{ccc} 3 \leftrightarrow D \wedge \neg\text{ab}_1, & D' \leftrightarrow 7 \wedge \neg\text{ab}_2, & D \leftrightarrow \neg D', \\ \text{ab}_1 \leftarrow \bot, & \text{ab}_2 \leftarrow \bot, & 7 \leftrightarrow \top \end{array} \right\} \tag{3.1}$$

Table 3.6 shows the computation of lm $\text{wc}(KB' \cup (7 \leftarrow \top))$. Because adding explanation $(7 \leftarrow \top)$ leads to a least model which can falsify the conditional using our *de Finetti* truth table, it is possible to deduce that the card 7 must be turned. Figure 3.3 illustrates the use of these extensions to derive all four of the canonical cases of the WST.

Combined Model

By combining our contraposition and abductive models for inferences, it is possible to model the case where the cards D, 3, and 7 are turned. Combining the models $(A_i, ..., A_n)$ is done by turning a card c if and only if c would be turned in some model A_i.

This work has shown that all canonical cases of the weak completion semantics can be modelled by activating or silencing these three processes on the initial knowledge base. The intuition that a discrete and systematic succession of well-founded processes could be used to model multiple different results for the same experiment was the inspiration for the SCP framework which this thesis introduces.

Chapter 4

Sequential Cognition Processes

4.1 SCPs: An Intuitionist Description

Although there is evidence that the brain can perform several simultaneous operations when considering a task (such as when considering an image (Sigman and Dehaene, 2008), the SCP framework assumes that at some points in reasoning about a given task, the mental processes of the agent converge to a set of epistemic states, called a *state point*. Whatever happens between these points of convergence can contain any number of parallel processes. The collection of processes that occur between any two state points in a reasoning task is called a *cognitive operation*. It follows that a basic validity requirement for a cognitive operation is that it takes a set of epistemic states as input and produces a set of epistemic states as output. Figure 4.1a outlines an SCP-like structure that is powerful enough to model any cognitive task that involves an epistemic state transition. However, it does not provide any useful information; the nature of the processes followed is completely undescribed. Suppose, instead, that some cognitive task is being modelled, and that researchers have reason to believe that The Weak Completion Semantics should play a part in their model. Under this new restriction, and assuming a sufficiently expressive epistemic state, Figure 4.1a is still an accurate model of the process, but now so is Figure 4.1b. By sacrificing some of the ambiguity – and, thus, expressiveness – of the model, the information content of the model description has increased. This trade-off is a feature of the SCP framework and finding the right depth of complexity to model the task accurately and still provide meaningful information is more art than science at present.

4.2 SCPs: Mathematical Formulation

An SCP Task $\Pi = (s_i, M, f(), \gamma)$ consists of an initial epistemic state point s_i, a known set of cognitive operations M, a desired final output γ, and an external function $f()$ which generates those final outputs by translating the final epistemic state into an empirically grounded set of possible responses.

(A) Unrestricted SCP. (B) WCS guaranteed to occur.
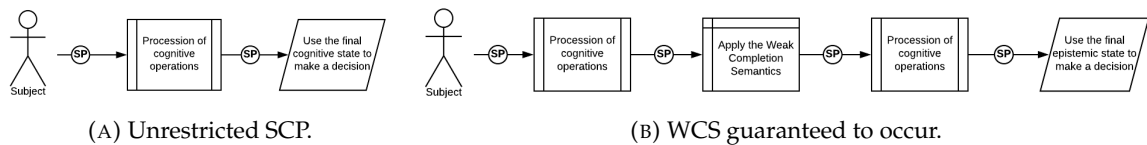
FIGURE 4.1: The most general description on an SCP with and without guaranteeing the WCS is applied at least once. An agent transitions from one epistemic state to another and then uses it to make a decision. *SP* nodes indicate state points.

An *epistemic state*, also called a *base point*, describes all information which the agent *believes* to be true at a given moment in its cognition. An epistemic state is an *n*-tuple of relevent structural variables, each of which describes some aspect of the belief of the agent at that moment (for example, a non-monotic KB as described in Section 2.4.1, is an epistemic state $s_i = (S, \Delta)$ containing structural variables S for a set of propositional rules, and Δ for a set of conditionals beliefs.)

A state point p is defined recursively by $p = \{\bar{p} \oplus Q\}$ where $\bar{p}$ is a base point, and Q is a set of state points, and $\oplus$ represents the exclusive-or operation[1]. State point containment $\in_s$ for state points p and q is defined recursively as follows:

$$p \in_s q = \begin{pmatrix} p \in q & \text{True} \\ \exists_{r \in p} p \in_s r = \text{True} & \text{True} \\ \text{Otherwise} & \text{False} \end{pmatrix}$$

It is never the case that $p \in_s p$.

A cognitive operation $m = (\chi, e), m \in M$ consists of a precondition χ and a process e, such that for an input state point p, every base point $\bar{p} \in_s p$ is either accepted as input ($\bar{p} \models m[\chi]$ under whatever definition of $\models$ is used for the complex operation m), or else rejected. Every base point is evaluated by the complex operation in isolation (no other base point $\bar{q}$ can affect the output of m on base point $\bar{p}$).

$J[\bar{p}, m] = p'$ describes the *application* of a base point $\bar{p}$ to a cognitive operation $m \in M$ to produce an output state point p', where p' is a set of base points. p' is *defined* if and only if $\bar{p}$ is a valid input for m (Section 4.5.1). In this way, a single single base point may produce multiple base points as output, reflecting potentially non-monotonic or non-deterministic properties of cognitive operations.

We overload the functionality of J so that $J[p, m] = p'$, where p is a state point (rather than just a base point), and define $J[p, m]$ as follows: $J[p, m] = [\texttt{replace}(\bar{p} \in_s p, J[\bar{p}, m])]$. $J[p, m] = p'$ is *defined* if and only if some $J[\bar{p} \in p, m]$ is defined.

It follows that the depth of a state point is directly related to the number of complex operations which have been performed in the SCP prior to its occurrence. If a base point does not meet some predefined precondition required by that cognitive operation (discussed in detail in Section 4.5.1), it is either ignored completely and not processed (*cruel application*), or passed exactly as is to the next complex operation (*lenient application*).

A cognitive operation is called monotonic if it always yields a base point as an output given a base point input ($J[\bar{p}, m] = \bar{p}'$).

This paper will focus on cases where the type of cognitive state remains constant, but it is possible for $J[\bar{p}, m] = p'$ where $\bar{p}$ is of structure (α) and $\bar{p}' \in p'$ to be of structure (β). Future models of human cognition may well rely on background knowledge which draws inferences from multiple types of non-monotonic logics.

A *cognitive transition model* (CTM) describes an ordered procession of cognitive operations following an initial epistemic state as follows: CTM $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$ where s_i is a state point, and m_k is a cognitive operation. A *partial cognitive transition model* (pCTM) $\pi = (m_1 \longmapsto ... \longmapsto m_n)$ describes a procession of cognitive operations without an an initial state point.

The CTM $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$, for SCP task $\Pi = (s_i, M, f(), \gamma)$ is *defined* if and only if $J[p_k, m_k]$ is defined for $p_k = J[p_{k-1}, m_k]$, $k > 2$, and $J[s_i, m_1]$ is defined. The set of possible CTMs for SCP task Π is given by $L[\Pi] = \{\pi\}$, where π is a defined CTM for Π.

An SCP $\mu = (\pi, f())$ pairs a CTM π with an external activation function $f()$ (Section 4.6). An SCP $\mu = (\pi, f())$ is defined for SCP task $\Pi = (s_i, M, f(), \gamma)$ if and only if $\pi \in L[\Pi]$.

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[1] $(X \oplus Y) = ((X \cup Y) - (X \cap Y))$ for sets X and Y.

Finally, we define a *realised SCP* $r = (k, f())$ where $k \in K[\pi]$, and $K[\pi] = \{s_i \longmapsto (m_1, \bar{p}_1) \longmapsto ... \longmapsto (m_n, \bar{p}_n)\}$ where, for every pair $(m_i, \bar{p}_i)$, $\bar{p}_i \in_s J[\bar{p}_{i-1}, m_i]$. Realised SCPs describe a single agent's interpretation of an SCP and associate only a single epistemic state, rather than a state point to the output each cognitive operation in an SCP.

$r = (k, f())$ is a realised SCP for SCP $\mu = (\pi, f())$, with $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$ and k is given by $k = (s_i \longmapsto (m_i, \bar{p}_1 \in J[\bar{p}_0, m]) \longmapsto ... \longmapsto (m_n, \bar{p}_n \in J[\bar{p}_{n-1}, m]))$.

A cognitive operation is called *monotonic* if it is guaranteed to produce only one base point as output for each base point passed as input. When every cognitive operation in M is monotonic, it is trivially easy to transform an SCP into a realised SCP and vice versa (Lemma 1). In cases where SCPs are monotonic, we will use the SCP and its realised SCP interchangeably. The number of realised SCPs for an SCP is the same as the number of base points in the final state point of that SCP.

4.3 SCP Tasks vs. SCPs vs. Realised SCPs

SCP tasks describe a problem that needs to be solved and the limitations which the solution must to adhere to; SCPs describe a single 'recipe' that can be used to describe a sequence of well-founded, non-monotonic operations which an agent might use to model a problem; and realised SCPs describe how adherence to specific SCP can result in some agent coming to a conclusion that matches empirical data.

For a given SCP task there can be many possible SCPs (the question of how to choose the most probable of these candidate SCPs is discussed in Section 7), or no possible SCPs at all.

Figure 4.2 explores a situation in which students are asked if they will pass their next exam. Though all subsequent examples in this paper will consider SCPs in a more formal context which is grounded in specific nonmonotonic logical interpretations, this conceptual approach captures the intuition of the different levels of granularity which SCP Tasks, SCPs, and realised SCPs.

4.4 Epistemic States

The choice of epistemic state is dependent on the properties that are known or suspected to be true for the cognitive task as a whole. For example, a researcher working on drawing inferences using boolean Propositional Logic can be certain that any SCP they create should be expressive enough to pass a classical, propositional knowledge base and possible world to a cognitive operation. Thus, it might suffice to simply define $s_i = (S)$ where S is a set of propositional rules. By contrast, a researcher using the WCS requires a system capable of both communicating a set of rules to the next complex operation, and of describing the results of repeated applications of the semantic operator. To model the Suppression Task or WCS it might seem intuitive simply to append a set of conditional rules Δ to the state used for the propositional case and to use $lm \, wc(s_i)$ as an external function to evaluate the SCP. And, though this approach works for many examples, it makes it impossible to perform further processing in the SCP *after* applying the semantic operator. What if the conclusion drawn was meant to form part of the background knowledge of another process? In practice, we will see that using $s_k = (S, \Delta, V, R)$, where V is a set of (variable name, value pairs) and R is a categorization variable, seems able to model all aspects of the WCS, including information related to the least model.

As yet, there are no definite rules for creating an epistemic state, but Albert Einstein's famous advice from 1950 still rings true: "Everything should be made as simple as possible,

My "Task"

I have an exam upcoming. ← Task Context

I have not studied for my exam. ← Initial World State

I can either study for the exam, or not study. ← Cognitive Operations

SCP Task

If I do not study, I will certainly fail. ←

If I do study, I may fail or I may pass. ← Operation effects

The outcome that matters to me is whether I pass or fail. ← Evaluation Function

I desire an outcome in which I pass. ← Target Evaluation

My "Strategy"

SCPs

Work Understood = False ... Work Understood = False

Don't Study ... Read Chapter

Write Exam ... Write Exam

Realised SCPs

Work Understood = False ... Work Understood = False ... Work Understood = False

Don't Study ... Read Chapter ... Read Chapter

Work Understood = False ... Work Understood = False ... Work Understood = True

Write Exam ... Write Exam ... Write Exam

Exam Passed = False ... Exam Passed = False ... Exam Passed = True

The "Outcome" of my strategy

FIGURE 4.2: An SCP-like examination of a mental task, showing the relative levels of granularity of SCP Tasks, SCPs, and Realised SCPs.

but no simpler." The ideal epistemic state is one that enables every reasonable cognitive operation in M that might help model the problem, without adding superfluous functionality that might render searching the SCP space infeasible.

4.4.1 Common State Point Structural Variables

Though state points are largely unbounded in the information that they can contain and communicate, this thesis will restrict itself to epistemic state structures which carry intuition both from the underlying logics which they utilise and from properties of search in the field of AI Planning (Korf, 1987).

The possible world: V

The possible world variable V encodes information about the current world state the agent believes itself to be in. As in Section 2.2, the possible world variable consists of a conjunction of variable pairs of the form $(\text{atom}, \text{value} \in \text{domain}_{\text{atom}})$, where atom occurs only once. Unlike in a true possible world, there exist atoms $\in \Sigma$ which do not occur in V.

$V[\phi]$ returns the pair $(\phi, \text{value}) \in V$. An interpretation of formula ϕ using V, $I_v[\phi]$ occurs as with any possible world except that, when an atomic name a which occurs in ϕ when $V[a]$ is undefined is given the default value of that language during computation (u in Łukasiewicz logic, and $\bot$ in propositional logic.). This is a necessity born of the fact that infinite space is required to explicitly store every possible variable. The reason that V is used throughout this thesis is threefold: it ensures that the possible world of the agent at any state of its cognition is always known; it allows inferences drawn using one logic to be utilised as background knowledge in another logic; and in practical implementation it can prevent recalculation of complex rules that have been previously interpreted.

Propositional Rule Set: S

The propositional rule set S is defined precisely as the knowledge base was defined in Section 2.4.1.

The set of conditionals: Δ

The set of conditionals Δ is defined precisely as in the conditional knowledge base in Section 2.4.1.

4.4.2 The Categorization Variable

The final property of a cognitive operation that needs to be discussed is how it is able to interact with the categorization variable R. Imagine a case drawn from Saldanha, Hölldobler, and Rocha, 2017 where the difference between creating abnormalities for obligate and factual conditionals is discussed. The intuition behind the authors' work can be summarised by saying that there are two different types of conditional statement, those that *have to be* true, and those that are *usually* true. If it rains r, I will normally take my umbrella u (in the absence of something abnormal happening, like a plague of umbrella-stealing gnomes). This a statement that is usually true, but some things are definitely true. For example, when it rains, water has to fall out of the sky s. This is an essential property of rain, not subject to abnormalities. Thus one useful set of categorizations for researchers seeking cosmic knowledge of weather patterns may be: $R = \{obligate : \{(u|r)\}, factual : \{(s|r)\}\}$. With these labels we might then expect that the process followed by the operation $m \in M$ which creates abnormalities would treat the two conditionals in the epistemic state differently, because of

their assignments in *R*. *R* then is a way of expressing meta information to the subsequent cognitive operations, and it is completely possible that some operation m_k might change *R* is such a way that future operation m_{k+1} produces different output. This is a technique we will exploit in several examples in this paper.

4.5 Cognitive Operations

The set of possible complex operations *M* determines many attributes of the achievable final state point p_n. If every $m \in M$ is monotonic and the initial state s_i is a base point, then p_n will be a base point. If some cognitive operation is computationally complex or produces a very large number of output state points, then search using that base point becomes less efficient. If some complex operation m' (such as weakly completing) is known or believed to occur in the SCP, then a restriction on the cognitive states exists such that either the initial state is of a format suitable as input for m', or there exists another cognitive operation which is able to output a state point which contains base points of a suitable format.

More abstractly, the set of cognitive operations should be well-founded in the literature. The set of possible complex operations is infinite and an SCP only meaningfully describes human cognition when it contains cognitive operations that have been justified empirically (*modus ponens-modus tolens* asymmetry, suppression, denial of the antecedent, etc.).

4.5.1 Pre-conditions and Effects

The precondition χ of a cognitive operation *m* refers to those conditions which the input state point must satisfy in order for that operation to be considered valid. An SCP is valid if and only if every cognitive operation it contains is valid. For example, one might have a cognitive process describing Julie's plans for a night out on the town. Let us imagine that the SCP task describing her night out includes the operation `goHomeByCar`. Semantically, this operation should take her *isHome* variable and set it to True.

The situations in which `goHomeByCar` can reasonably occur in an SCP are at the researchers' discretion. Researcher 1 might feel that it can be allowed at any point in the SCP and will simply have no effect for those input epistemic states in which she is already at home. Researcher 2 might argue that `goHomeByCar` is applicable only when *isHome*=False. Researcher 3 knows that only the cognitive operation `goToClub` changes the *isHome* variable, and so argues that *isHome* is only applicable after `goToClub` occurs. There are merits to the arguments of each researcher.

Researcher 1 argues for a property called *trivial validity*, that is that SCPs should always be considered valid, without any need for evaluation. This approach, however has one significant drawback: it cannot handle changes in the structure of epistemic state inputs. Imagine an epistemic operation called `dontDriveDrunk` which corresponds to our party animal realising that she shouldn't drive home if she's been drinking. Imagine further that this operation takes base points which are either of propositional or default structure (Section 2.7) and outputs an epistemic state of default structure which contains the new rule. If `goHomeByCar` took only a propositional state as input, then any sequence where `dontDriveDrunk` occurs as the previous operation will cause the SCP to fail because of the input state is not of an allowed type. Trivial validity then, is not sufficient for modelling SCPs in which state structure changes.

Researcher 2 has opted for a *variable validity* approach. She has reasoned, correctly, that the `goHomeByCar` only results in an epistemic state change when when *isHome* is not True. She therefore, feels that it only makes sense to reduce redundancy and search complexity in creating the SCP by only allowing the action to occur when it can be said to have an effect.

Validity	Full SCP evaluation	Uniform Epistemic Structure	Operator Silencing Knowledge
Trivial	X	✓	X
Variable	✓	X	X
Operator	X	X	✓
Hybrid	X	X	X

TABLE 4.1: SCP property requirements for precondition types in cognitive operations.

This argument has some merit from an intuitive perspective, but presents an unpleasant question: what if there are state points in which only some base points actually meet the precondition? To compound the troubles with this variable state approach to preconditions is the fact that it is not possible to determine SCP validity without evaluating the SCP at runtime, which could be slow for large SCPs.

Researcher 3 has taken an *operator validity* approach, instead has focused on the structure of the SCP. This approach allows SCP validity to be determined without explicitly evaluating an SCP, one need simply search through the cognitive operations in the SCP to make sure that no operation has a precondition operation which has not yet occurred. This approach also presents drawbacks, it requires explicit knowledge of operation interactions, and adding a new cognitive operation to M in the SCP Task might force several other operations change their interactions. Further, there might be other cognitive operations which mean that the precondition operation is no longer in effect. Imagine a third operation `goHomeByTrain` which also sets *isHome* to True. Now the sequence of operations $x \longmapsto$ `goToClub` $\longmapsto$ `goHomeByTrain` $\longmapsto$ `goHomeByCar` would still be valid with Researcher 2's original requirement that `goToClub` occurred previously in SCP. However, it is obvious that `goHomeByTrain` has negated the effect of `goToClub`. Evidently this approach is not appropriate in any case in which other operations can silence the effect of those operations mentioned in preconditions.

Every approach shows strengths and weaknesses. A final approach to consider is the *hybrid validity* approach. With this approach, all cognitive operations are assumed to be valid, provided that the output base points of the previous operation are of a suitable input structure for the current operation. Hybrid validity is an appropriate approach for all SCPs in which every cognitive operation has a known output structure. Though hybrid validity does not have the best-case search properties of the other approaches, its universal appropriateness means that it should generally be the starting point for generating SCPs. The hybrid approach is followed implicitly throughout the rest of this thesis and preconditions are omitted.

4.5.2 Validity, Satisfaction, and Optimality

Validity

As discussed in Section 4.5.1 validity can be defined for a cognitive operation and its input. Realised SCPs describe both the operation and the input for an SCP and allow us to define SCP validity as follows:

A realised SCP $r = (k \in K[\pi], f())$, $k = (s_i \longmapsto (m_1, s_1) \longmapsto ... \longmapsto (m_n, s_n))$ with evaluation function $f()$ is valid if and only if s_i is a state point, every $m \in \pi$ is valid (according the validity requirements defined by the researcher), and $f(k)$ is defined.

An SCP $(\pi, f())$ is valid if and only if there exists some realised SCP $r = (k \in K[\pi], f())$ which is valid.

An SCP Task $\Pi = (s_i, M, \gamma, f())$ is valid if and only if there exists some SCP $((\pi, f())$, with $(\pi = (s_i \longmapsto m_1 \in M \longmapsto ... \longmapsto m_n \in M)$ which is valid.

Validity does not require that $f(\pi) = \gamma$, only that external function $f()$ is able to make some prediction or set of predictions based on π.

Satisfaction

A realised SCP $r = (k \in K[\pi], f())$, $k = (s_i \longmapsto (m_1, s_1) \longmapsto ... \longmapsto (m_n, s_n))$ with evaluation function $f()$ satisfies goal condition γ, written $r \models \gamma$ if and only if r is valid, and $f(k) \models \gamma$.

An SCP $(\pi, f())$ satisfies γ, written $(\pi, f()) \models \gamma$, if and only if there exists some realised SCP $r = (k \in K[\pi], f())$ for which $f(k) \models \gamma$.

An SCP Task $\Pi = (s + i, M, \gamma, f())$ satisfies γ, written $\Pi \models \gamma$, if and only if there exists some SCP $((\pi, f())$, with $(\pi = (s_i \longmapsto m_1 \in M \longmapsto ... \longmapsto m_n \in M)$ which satisfies γ.

Heuristic Searches and machine learning techniques are can be to find satisfying solutions for situations in which it is possible for an answer to be good enough for practical purposes, but search time or space is infeasible.

Optimality

Optimality refers to finding the best possible SCP to describe a problem according to whatever criteria are used to evaluate the SCP (Section 5.3.1). As with all exhaustive search techniques, optimality can be guaranteed for searches of restricted depth even when SCP space contains infinite loops.

A realised SCP $r = (k \in K[\pi], f())$, $k = (s_i \longmapsto (m_1, s_1) \longmapsto ... \longmapsto (m_n, s_n))$ generated from SCP $(\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$ with evaluation function $f()$ is optimal for goal condition γ and heuristic function $g()$, if and only if $r \models \gamma$, and $\forall r' = (k' \in [K[\pi]], f()), g(r) \geq g(r')$ for all case in which $f(r') \models \gamma$.

An SCP $(\pi, f())$ generated from $\Pi = (s + i, M, \gamma, f())$ is optimal for goal condition γ and heuristic function $g()$, if and only if $(\pi, f()) \models \gamma$ and there exists no other γ - satisfying SCP $(\pi, f())$ which can be generated from Π for which $g(\pi') > g(\pi)$.

Optimality is not defined for SCP Tasks.

Formally, an SCP $\pi = (\pi_0 \longmapsto ... \longmapsto \pi_n)$ generated from SCP Task $\Pi = (s_i, M, \gamma, f())$ with evaluation function $h()$ is optimal if and only if $f(\pi)$ is satisfying and there exists no SCP π' such that $g(\pi) < g(\pi')$.

In an SCP context a solution may be optimal for a given task or set of empirical data, but only globally valid or satisfying. A significant part of the appeal of the SCP framework is the potential to use high-scoring local solutions to several tasks, or from several reasoners, to predict which are most likely by searching for evidence of common structural motifs in the disparate solutions.

4.5.3 Credulous and Skeptical Validity

A planning task may permit an SCP in which all or only some of the associated realised SCPs are valid, satisfying, or optimal. An SCP is called *credulously valid* if $f(\pi)$ is valid for *at least one* epistemic state in the final state p_n. An SCP is called *sceptically valid* if $f(\pi)$ is valid for *every* epistemic state in the final state. In cases where all operations are monotonic, sceptical validity is the same as credulous validity. Similar logic to applies the concepts of sceptical satisfaction, credulous satisfaction, sceptical optimality, and credulous optimality.

4.6 External Evaluation Functions

The external evaluation function $f()$ is responsible for evaluating an SCP or reaslised SCP in order to map it to the observed or predicted empirical data observed in respondents. The $f()$ function in general is an unbounded tool for making a decision with a CTM.

The process of defining $f()$ forces researchers to specify what information in the epistemic state corresponds to empirical data. The exact procedure by which $f()$ makes this decision should correspond to the researcher's chosen criteria for decision making.

The question of how to select the most appropriate external evaluation function is not a focus of this thesis, but, where necessary, all used evaluation functions will be justified with respect to either common-sense reasoning, or evidence will be given that such a function is already used in a more informal way in the existing literature for that experiment. External evaluation functions will be provided using pseudocode.

A goal $\gamma = \{P_1 : \{\alpha_1, ... \alpha_m\}, ..., P_n : \{\beta_1, ... \beta_p\}\}$ is a labelled set of properties, each of which correspond to some empirically observed response to the task at hand. For example, in the Suppression Task (Section 3.1) we might define:

$$\gamma_{\text{sup}} = \{'\text{conclusion}' : '\text{we are unsure if she will study late in the library}'\}$$

The precise mechanism by which the conclusions in γ_{sup} are reached is assumed to be unknown, and it is the purpose of the SCP framework to find a well-founded cognitive process, and corresponding external evaluation function, which matches this empirical answer set.

We say that a realised SCP, $r = (k, f())$, *satisfies* a goal $\gamma = \{P_1 : \{\alpha_1, ... \alpha_m\}, ..., P_n : \{\beta_1, ... \beta_p\}\}$, written $f(k) \models \gamma$, if and only if for every property label $P_i \in f(k)$, we also find that $P_i \in \gamma$, and $f(k)[P_i] \in \gamma[P_i]$.

We say that an SCP, $\mu = (\pi, f())$, *strictly satisfies* a goal γ, written $f(\pi) \models_{\text{strict}} \gamma$, if and only if, *for every* realised SCP $r = (k = K[\pi], f())$ we find that $f(k) \models \gamma$.

We say that an SCP, $\mu = (\pi, f())$, *weakly satisfies* a goal γ, written $f(\pi) \models_{\text{weak}} \gamma$, if and only if, *for some* realised SCP $r = (k = K[\pi], f())$ we find that $f(k) \models \gamma$.

These two definitions of satisfaction for SCPs will be used later to motivate an alternate method for modelling individual reasoners in the Wason Selection Task using SCPs (Section 6.3). When an SCP is monotonic, strict and weak satisfaction are identical and we simply write $f(\pi) \models_{\text{weak}} \gamma$.

An external evaluation function $f()$ is said to be *final state dependent* if the only decision criteria $f(\pi)$, with $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$, uses are those of the final epistemic state p_n, where $p_i = J[p_{i-1}, m_i]$, $i \geq 1$ and $p_0 = s_i$. In final state dependent external evaluation function, the structural properties of the SCP or rSCP being evaluated cannot affect the output of f. In Chapter 6 we will only concern ourselves with final state dependent external evaluation functions, but it is important to note that other kinds may have practical usefulness.

Finally, we define the *preferred response* $p()$ for the output of an external evaluation function $f()$ and CTM π as follows:

Algorithm 2: $\mathrm{p}(f(\pi, \mathrm{pref}))$: a method which removes unpreferred responses from a list and replaces them with the preferred response (if it was in the original list).

Function $p(f(\pi), pref)$ **is**
    responses:=$f(\pi)$;
    finalResponses:=$\{\}$ **for** *category* $\in$ *responses* **do**
        if *pref[category]* $\in$ *responses[category]* **then**
          | finalResponses[category]=pref[category]
        else
          | finalResponses[category]=responses[category]
        end
        return finalResponses
    end
end

A preferred response allows us to specify a hierachy of achievable outputs from an SCP or realised SCP and a practical application will be shown for modelling individual reasoners in the WST (Section 6.3).

4.7 Search in SCP-Space

As with any process in which one input can produce one or multiple outputs, it is possible to search through SCP space in order to find those SCPs which meet certain criteria. Those criteria might be *validity*, *satisfaction* (conditions in γ are satisfied), or *optimality* (there exists no better solution to this problem) as defined in Section 4.5.2.

SCPs lend themselves particularly well to forward search techniques (Korf, 1996) but also have some potential using backwards or biderectional search (De Champeaux, 1983).

Searching through solutions to an SCP task takes one of two forms: *De Novo search*, and *Insertion search*. De Novo search generates a set of solution SCPs that meet the optimality, satisfaction, or validity requirements of the researcher from scratch, using only the information contained in the planning task. Insertion Search changes an existing SCP which models a particular response in order to model a reasoner with differing responses. Section 4.7.1 and Section 4.7.2 discuss the philosophy, applications and mechanical considerations of these two search approaches.

4.7.1 De Novo SCP Searching

De Novo (from new) search is a search technique in which a final desirable world state is achieved by generating a sequence of cognitive operations and using a known initial state as the input prior to the first cognitive function. An SCP Task contains all the information required to conduct a de Novo search through the space of allowable SCPs for a given cognitive task. The exact search techniques used can be easily varied, but we will consider de Novo search in terms of a breadth-first traversal Zhou and Hansen, 2006.

Figure 4.3 illustrates the process by which breadth first search over an SCP task $\Pi = (x, M, \gamma, f())$ can be conducted. The search terminates if a structural inconsistency between two states in $\pi = (x \longmapsto A_0 \longmapsto ... \longmapsto A_n)$ occurs (e.g. output state point structure of A_k does not match expected input structure of A_{k+1}). Operator sequences are added to the list of solutions if and only if they meet the validity requirements of the search being used.
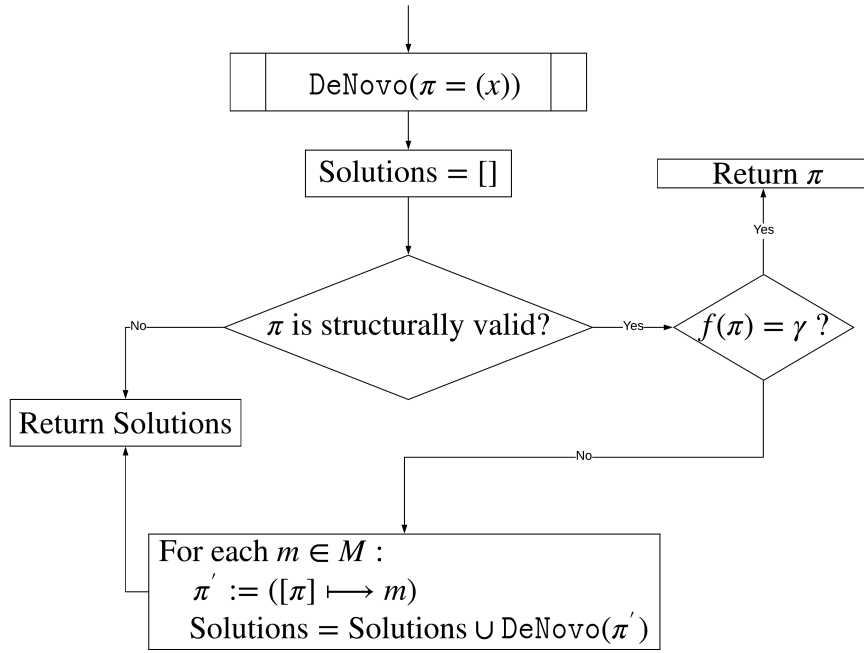
FIGURE 4.3: A breadth-first search algorithm for De Novo SCP search with SCP Task $\Pi = (x, M, \gamma, f())$.

In practice it becomes necessary to limit the search depth of the algorithm to search tractable in most cases, because, in some cases, search space and solution space may both be infinitely large, as suggested by Lemma 5.

4.7.2 Insertion Search

Changing the structure of an existing SCP allows researchers to model deviations from standard reasoning for a task. For example, in Section 6.2 we show that slight modifications to an SCP that models the standard case of the suppression task can lead to an SCP which is able to model reasoners who achieve the classical valid conclusion that she will study late in the library.

The insertion search modifies an existing SCP:

$$\mu = (\pi = (s_i \longmapsto A_0 \longmapsto ... \longmapsto A_n), f())$$

to produce a new SCP:

$$\mu' = (\pi' = \{(s_i \longmapsto B_0^* \longmapsto A_0 \longmapsto B_1^* \longmapsto ... \longmapsto B_n^* \longmapsto A_n \longmapsto B_{n+1}^*), f()\})$$

Where $B_i^* = (B_0 \longmapsto ... \longmapsto B_n)$, $B \in M$ or is the trivial operation T which returns the input state point as output. When describing the SCP, T operations are generally omitted.

In general, Insertion Search is a more difficult search type than De-Novo Search (which is equivalent to an Insertion Search in which the initial SCP is empty). The reason for this is that it is possible for a single insertion to result in an invalid SCP, but adding additional cognitive states may revalidate the SCP (Lemma 3).

This consideration means that it is potentially necessary to insert an unbounded number of new cognitive operations at each point in π to find those insertions which are valid. When

the total number of new cognitive operations to insert is bounded by N, however, it becomes possible to generate every possible subsequence of operation insertions of length $\geq N - k$, where k is the number of insertions already used in the search, for each position B_i^* and determine for which of the resulting SCP cognitive operations sequences $f(\pi')$ is valid.

Chapter 5

Cognitive Operations: The Cognitive Toolbox

5.1 Overview

As discussed in Section 4.1, and illustrated in Figure 4.1a the simplest imaginable description of sequential human cognition consists of a single step transforming the initial task information (along with background information) into an epistemic state that can be externally interpreted to choose a response. That is, $\mu = (\pi = (S_i \longmapsto m), f())$ can model any cognitive task with a complex enough definition of m.

To understand why this is a poor choice of SCP, consider a general limitation of the SCP framework: we can only change the output of an SCP by changing the initial epistemic state, or by inserting/deleting a cognitive operation at an existing state point. Consider a case where the initial epistemic state is immutable. Thus, for $f(x \longmapsto A)$, the only possible changes are: $f(x \longmapsto \bar{B} \longmapsto A \longmapsto \bar{B})$, $f(x \longmapsto A \longmapsto \bar{B})$, $f(x \longmapsto \bar{B} \longmapsto A)$, $f(x \longmapsto \bar{B})$, and $f(x)$. Where $\bar{B}$ denotes the action of drawing some sequence of random, but valid cognitive state transitions. There is no way to change the operations that occur in A. Thus, even though Figure 4.1a may be an appropriate model for the general reasoner of a given task, it imposes the implicit assumption that all other test subjects who achieved differing results, achieved them only by either completely ignoring the black-box process the general reasoners used, or else by inserting new cognitive operations only at the very start or end of the processing task. In Section 6.2 and Section 6.3 we will explore examples of cases where assumptions like that are simply not reflected by empirical data.

Instead of using a single black-box operation to describe all cognition, it follows logically to use multiple smaller black box operations, each taking a state point as input and resulting in a state point as output. Using this interpretation, cognitive operations for the general reasoner should be as extensive as possible, provided that they do not force implausible changes to describe deviant users or when used in other cognitive tasks. Figure 4.1a and Figure 4.1b in the previous chapter, captured this intuition as it applies to the WCS. Indeed, as a non-monotonic logic that is already explicitly divided into discrete steps in the literature (Section 2.6), its individual steps are obvious candidate cognitive operations.

It is worth noting that this increased ability to model deviant reasoners comes at the cost of forcing more state points into the SCP being used.

The remainder of this section is devoted to expressing the exact properties, limitations, and concrete descriptions of cognitive operations, as well as defining several well-founded cognitive operations algorithmically with respect to their preconditions, effects, and output properties in Section 5.4.

M	Finite Length Constraint	Finite Solution Space	Guaranteed Solution
Ω^*	None	X	✓
Ω^*	Yes	X	✓
Ω	None	X	✓
Ω	Yes	X	✓
finite	None	not guaranteed	X
finite	Yes	✓	X

TABLE 5.1: Solution space considerations of SCPs tasks as determined by length constraints and choice of cognitive operations.

5.2 Cognitive Operation Space

In principle, the number of cognitive operations available for use in an SCP is unlimited. We denote the space of all cognitive operations Ω, and the space of all pCTMs ($m_0 \in \Omega \longmapsto ... \longmapsto m_n \in \Omega$) with Ω^*. Similarly, given the SCP Task $\Pi = (s_i, M, f()\gamma)$, M denotes the set of allowable cognitive operations for that task and $M^* = \{(m_1 \longmapsto ... \longmapsto m_n | m_i \in M)\}$ is the space of all pCTMs for that task. There cannot and never will be an exhaustive search procedure to evaluate every possible way of manipulating an epistemic state (Lemmas 4 and 5 illustrate the existence of valid SCPs for infinitely many different cognitive operations and valid SCPs of any length, respectively). However, there are ways of limiting the number of allowable cognitive operations. Definition 2 defines those cases in which there is redundancy in the set of all possible cognitive operations, and Definition 3 focuses this concept on specific cognitive tasks which are to be modelled using SCPs for a given SCP Task Π. Redundant cognitive operations can be removed from the set of allowable cognitive operations M without changing the domain of $\{f(\pi)|\pi$ where is a valid SCP of $\Pi\}$.

5.3 A Limited Set of Allowable Operations

In practice it is impossible compute the set of all candidate SCPs when the set of allowable cognitive operations M in a planning task is Ω^* or even Ω. Without a length constraint Lemma 5 shows that sometimes it is not even possible to compute all possible SCPs for many SCP tasks with a finite set of allowable operations. Table 5.1 summarises the properties of the solution space under different SCP task conditions.

5.3.1 A Limited Set of Cognitive Operations

Readers should recognize that using the space of all possible cognitive operations for M results in a computationally impossible task. No matter the other conditions imposed there will be infinitely many solutions to the given SCP task which satisfy $f()$.

But, beyond being just computationally infeasible, there are empirical and philosophical objections to this approach. The first is that the human brain itself does not have infinite processing or space capabilities and so any model which allows for unrestricted data sizes in the set of allowable cognitive operations M or in the number of resultant state points M is not biologically well-founded[1].

---

[1]Because Ω is the set of all possible cognitive operations there must be at least one $m \in M$ such that, for any given input s_i, $J[s_i, m]$ results in a state point of information content greater than the capacity of any storage system to hold.

The second consideration is that there is considerable evidence in the literature that many problems can be modelled using logical frameworks which describe cognitive behaviour across a variety of tasks – for example, the Diversity Principle(Heit, Hahn, and Feeney, 2005). These frameworks reflect a more general philosophy in cognitive psychology which is the idea of consistent cognitive motifs in reasoning. Using an infinite M shows no discrimination at the time of task-formulation between those operations for which there is significant evidence and those which are fundamentally improbable.

5.3.2 A Limited SCP Length

As discussed in Section 5.3.1, the human brain is constrained by a set of physical, computational, and storage limitations. In all cases except those which are trivial or repetitive it would be computationally infeasible for the human mind to mimic an arbitrarily long SCP. For this reason, SCPs of infinite length, though possible in a mathematical formulation, tend to violate the spirit of the SCP Framework which is to *accurately* model human cognition processes, not simply their conclusions.

It is worth noting that SCPs containing infinite loops can exist and can be evaluated at some time points n, they simply cannot be evaluated in their entirety. For example, the SCP $\mu = ((s_i \longmapsto A), f())$, if the definition of cognitive operation A includes a call to A every time it is evaluated.

5.4 Purpose-Selected Cognitive Operations

5.4.1 Overview

This section discusses the intuition and precise definition of several cognitive operations which will be used throughout the remainder of this paper. These operations allow the SCP Framework to mimic non-monotonic logics for the experiments discussed in Chapter 6.

Accessing a specific element α_k in the structural variables of base point $\bar{p} = (\alpha_1, .., \alpha_n)$ is short-handed to $\bar{p}[\alpha_k]$. $\bar{p}[\alpha_k]$ returns α_k if and only if $\alpha_k \in \bar{p}$ and returns the empty set otherwise. This indexing notation is also used to append a structural variable to $\bar{p}$ and $\bar{p}[\alpha_j] := t$ removes $\bar{p}[\alpha_j]$ from $\bar{p}$ if it already exists and creates a new structural variable $\alpha_j = t$ in the resulting state point p'.

A cognitive operation m is always evaluated in terms of each base point $\bar{p} \in p$ of the input state point, and each base point can be evaluated independently of the others. For the definitions that follow, a base point $\bar{p}$ is assumed as input and a state point is returned as output.

Section 5.4.2 discusses the concept of preconditions in cognitive operations, generally, and details an approach to determining cognitive operation validity called Hybrid Validity. Subsequent subsection define and justify several well-founded cognitive operations which will be made use of when modelling cognitive tasks with SCPs.

5.4.2 Preconditions

When using Hybrid Validity, function $\texttt{functionName}_\chi(\bar{p})$ determines the validity of the input base points. The input alphabet $\Sigma_\chi = \{\alpha_1, ..., \alpha_k\}$ of a cognitive function m describes the set of structural variables which must be present in the epistemic states $\bar{p} \in_S p$ of the input

state point p. The alphabet of structural variables in p is given by Σ_p. Hybrid validity, as mentioned in Section 4.5.2, can be algorithmically defined as follows:

Algorithm 3: $\texttt{functionName}_\chi(\bar{p}))$: Hybrid validity requirements for $J[p, m]$ to be defined.

Function $\chi\ (\bar{p})$ **is**

    /* Where each structural variable α_i must be present in p to validate the precondition.   */

    $\Sigma_\chi = \{\alpha_1, ..., \alpha_n\}$;

    if $\forall v \in \Sigma_\chi, v \in \Sigma_p$ **then**

      | **return** "Defined"

    else

      | **return** "Undefined"

    end

end

All cognitive operations which we define in this section will be assumed to have a known Σ_χ and will only proceed to their effect e if $\texttt{functionName}_\chi(\bar{p}) =$ "Defined". The output state point of a defined $J[\bar{p}, m]$ is the returned value of $m[e](\bar{p})$.

5.4.3 Propositional Logic

An SCP for evaluating propositional logic programs, as defined in Section 2.1, must be able to take a set of rules and facts and draw the set of all valid inferences from that set using the standard propositional rules discussed in Section 2.1. To that purpose, we introduce a cognitive operation called $\texttt{th}$ which draws all classically valid inferences from a set of logical rules[2].

$\texttt{th}$ is by its nature a contrived cognitive operation and there is no expectation that it truly reflects human cognition. Instead, it serves as a benchmark operation with which to compare the results derived from knowledge base-driven logical approaches. Even a simpler cognitive operation called $\texttt{SAT}$ which determines if the given knowledge base is satisfiable still represents an *NP-complete* problem and is infeasible for any large knowledge base (Schaefer, 1978).

However, variations on the intuition of these operations have real applications in cognitive modelling with SCPs and will be discussed throughout this section.

Algorithm 4: $\texttt{th}(\bar{p})$: generates the potentially infinite set of possible classical inferences from $\bar{p}['S']$.

Function $e(\bar{p})$ **is**

    $\bar{p}['S'] := \bar{p}['S'] \cup [(\text{atom name} \leftarrow \text{value}) \mid (\text{atom name}, \text{value}) \in \bar{p}['V']$ and value $\neq u$;

    $\bar{p}['S'] := \bar{p}['S'] \cup [A \mid \text{clause } A \text{ would not make } \bar{p}['S'] \text{ inconsistent}]$;

    $\bar{p}['V'] := [B \mid B \leftarrow body \in \bar{p}['S'] \text{ and } I(body) = \top \text{ or } I(body) = \bot]$;

    return $\bar{p}$

end

A realised SCP $r = (k = K[\pi], f())$ is said to *model the classical inference*, if $f(k) = f(s_i \longmapsto \texttt{th})$, where s_i is the initial state point of the CTM π. An SCP π is said to model the classical inference if some realised SCP generated from π models the classical inference.

---

[2]The reason that we do not introduce the functionality of the $\texttt{th}$ cognition operation in the external evaluation function $f()$ is that doing so would make it impossible to directly use the derived theorems as part of another SCP.

5.4.4 Variable Insertion

Variable insertion is a single cognitive operation with a highly intuitive interpretation and justification. By its nature, the set of variables $\bar{p}[text'V']$ in an epistemic state $\bar{p}$ is not a true possible world (as discussed in Section 4.4.1), and may omit variables present in the real possible world. It should be noted that omitted variables may become apparent later. For example, when interpreting a conditional as a license for implication, or when storing the result of a calculated mathematical problem.

It seems reasonable then that a cognitive operation `addV` which appends a variable to base point V would be required to expand the knowledge base to incorporate new facts.

Algorithm 5: `addV`$(\bar{p})$: adds a variable v, defined *a priori*

```
/* The variable to be added, defined before SCP execution.           */
```
$v := \alpha = (\text{atom name}, \text{value});$
Function $e(\bar{p})$ **is**
> ```
> /* Remove all possible assignments of the variable from the list of
>    variables                                                        */
> ```
> $\bar{p}['V'] := \bar{p}['V'] - [\text{atom name}, x \in \text{Domain}_{\text{atom name}}];$
> ```
> /* Add the new variable and its assignment to the list of variables
>    */
> ```
> $\bar{p}['V'] := \bar{p}['V'] \cup \alpha;$
> **return** $\bar{p}$

end

In order to ensure that the SCP still acts as a pipeline of information, it is necessary that `Add`α for variable $\alpha = (\text{atom name}, \text{value})$ is defined as part of the allowable operations of the SCP Task Π which generates the SCPs and realised SCPs that will make use of it. The core principle of this pipeline framework is that each usable component (cognitive operation) of that framework which is not the input value should not be modified.

5.4.5 Variable Deletion

Following on from the `addV` operation, the `remove` operations captures human intuition of forgetting. Consider the sequence of numbers: 1, 44, 27, 8, 0 , -4, 6, 7, 346, 7, 74, 7, 234, -55, 2.4, 18. Now without looking back at the numbers, ask yourself some questions: how many numbers were there? Were any of them prime? How many numbers were repeated? In all probability you are not entirely sure. This simple thought experiment provides support for our first extension, the idea that variables can be "forgotten", that is, that information that existed in the knowledge base at one point in time might no longer exist at a later timepoint.

This is not the only imaginable case where a variable might be removed from the knowledge base of the person being modelled. The size of the knowledge base used for cognitive modelling is always implicitly restricted to relevant variables. Only those variables whose values might reasonably be expected to affect the final conclusions drawn with regard to the research question should be considered. Finding which variables and rules are relevant is, however, non-trivial. For another real-life example, imagine a mystery novel: Three hundred pages of plot descriptions, character actions, and dialogues. In a good murder mystery novel every piece of information that reveals the killer's identity is hidden in the story itself, yet we do not hold every fact and interaction in the book in our epistemic model of the book, so discerning the identity of the killer remains a mystery until the last page. But when the mystery is solved, many details that we internalised while reading (and recalled in retrospect) suddenly make the conclusion seem obvious. We have not forgotten this information,

we had merely incorrectly deemed it irrelevant at the time and ignored it in our cognitive processing.

Forgetting a variable in an SCP is not-trivial task, just as removing a variable in an AI planning task is non-trivial. Multiple intuitions for how it may best be done occur, but we will keep this definition as simple as possible for the examples that will be discussed in Chapter 6. We introduce three cognitive operations $\texttt{remove}_\top$, $\texttt{remove}_\bot$, and $\texttt{remove}_u$. $\texttt{remove}_\top$ assumes that conditionals and rules which depend on the removed variable are validated by its removal. $\texttt{remove}_\bot$ assumes that rules and conditionals are invalidated by the removal. $\texttt{remove}_u$ assumes that no conclusion can be drawn without this information. These three approaches can be differentiated by these examples:

- "If aliens invade, then I will hide in my bunker."

- "If I do not fail my exam, then I will graduate."

- "Unless I am a boy, I am a girl."

In the first example, the presumed default position is that I will not hide in my bunker. In the second, the default position is that I will graduate. And in the third example there is no apparent default position. In the absence of the conditions in each each of these conditionals we would probably conclude that, I am probably not hiding in my bunker, I probably graduated, and that my gender is uncertain. This observation implies that a mechanism by which to determine the plausibility of default positions is needed to translate such default theories into a non-monotonic framework. In the absence of such a mechanism we will generally assume $\texttt{remove}_\bot$, as with the closed-world assumption.

The variables to be deleted are specified in categorization variable at $\bar{p}['R']['delete']$ for a given base point $\bar{p}$. If a variable does not exist, no change occurs.

Algorithm 6: $\texttt{remove}_\perp(\bar{p})$: removes a variable name v, defined *a priori*

Function *e($\bar{p}$)* **is**

    $\bar{p}' = \texttt{clone}(\bar{p})$;

    toDel $:= \bar{p}'['R']['delete']$;

    for *varName $\in$ toDel* **do**

        Remove *varName* from the possible world variable $\bar{p}'['V']$.;

        $\bar{p}' := \texttt{rem}(\bar{p}', varName)$;

    end

    return $\bar{p}'$

end

Function $\texttt{rem}(\bar{p}, v)$ **is**

    for *rule $=$ (head $\leftarrow$ body) $\in \bar{p}['S']$* **do**

        if *head $= v$* **then**

           $\bar{p}['S'] := \bar{p}['S'] -$ rule;

        end

        if *$v \in$ body* **then**

           replace all subclauses $(v \vee \phi), (v \wedge \phi) \in \bar{p}['S']$ with (ϕ);

        end

    end

    for *rule $=$ (head$|$body) $\in \bar{p}['\Delta']$* **do**

        if *head $= v$* **then**

           $\bar{p}['\Delta'] := \bar{p}['\Delta'] -$ rule;

        end

        if *$v \in$ body* **then**

           replace all subclauses $(v \vee \phi), (v \wedge \phi) \in \bar{p}['\Delta']$ with (ϕ);

        end

    end

    `/* This is where remove`$_\perp$` differs from remove`$_\top$`, and remove`$_u$` */`

    replace all rules (head $\leftarrow v) \in \bar{p}['S']$ with $\perp$;

    replace all rules (head $\leftarrow \neg v) \in \bar{p}['S']$ with $\neg\perp$;

    replace all rules (head$|v$) with $\perp$;

    replace all rules (head$|\neg v$) with $\neg\perp$;

    return $\bar{p}$

end

5.4.6 Variable Fixation

The next case of a potential complex operation to add to the search space of our SCPs is the idea of Variable Fixing. The idea that some conclusions can be fixed *a priori*. This operation may prove contentious to those who are experienced with mathematical logic, but holds some merit when justified from a psychological perspective.

Consider a person who strongly doubts the effectiveness of vaccines, we will call her Karen. Karen started her day convinced that giving her child the MMR vaccine is more dangerous than the disease itself. Later that day Karen spoke to her doctor who strongly advised that she vaccinate her child. He offered her a variety of peer-reviewed papers and studies that showed the relative safety of the vaccination. Karen listened carefully to the trained medical professional, and then went home. After some thought Karen decided that he was wrong, and her opinion on vaccines didn't change.

In this example Karen shows a very powerful type of cognitive bias, the unwillingness to change her opinions, despite powerful evidence to the contrary. This phenomenon has been observed across a great many fields of study, from medical psychology(Brown et al., 2010)(Wroe et al., 2005) to political sciences(Tappin, Leer, and McKay, 2017). In the context of cognitive modelling with logics, it indicates that some mental rules or variables are immutable, regardless of new evidence or valid beliefs that would logically contradict them. Non-monotonic logics, as a class, are already capable of dealing with bias effects, as non-monotonic logics are built on the basis of a preference operation.

In order to implement this operation, we will make use of the categorization variable R first discussed in Section 4.4.1.

Algorithm 7: `FixV`$(\bar{p})$: fixes a variable name v, defined *a priori*

```
/* The variable to be fixed, defined before SCP execution.         */
```
$v := (\alpha = (\text{atom name}, \text{value}));$
Function *e($\bar{p}$)* **is**
$\quad \bar{p}['\text{V}'] := \bar{p}['\text{V}'] - (\text{atom name}, \text{value} \in \text{domain}) \in \bar{p}['\text{V}']) \cup \alpha;$
$\quad \bar{p}['\text{R}']['\text{fixed}'] := \bar{p}['\text{R}']['\text{fixed}'] \cup v;$
$\quad$ **return** $\bar{p}$
end

This algorithm removes 'atom name' from the possible world V and re-adds it with the value to which it is to be fixed. The atom is then added to the list of fixed variables kept by $\bar{p}['\text{R}']$. This list of fixed variables is used by other operations which evaluate state points later in the SCP or realised SCP, and, as per their encoding, they will not modify the assignment of this variable.

5.4.7   Adding Abnormalities

As discussed in Section 2.3.1, one way to interpret conditionals in reasoning tasks is as licenses for implication. One possible definition for a cognitive operation that performs this same task by transforming defeasible conditional facts to propositional logic rules is as given

by Algorithm 8.

Algorithm 8: `addAB(`$\bar{p}$`)`

Function *e(*$\bar{p}$*)* **is**

    `/* Variables for which an abnormality has already been created    */`

    for $(\psi|\phi) \in \bar{p}['\Delta]'$ **do**

        $k :=$ the lowest natural number for which $\text{ab}_k \notin \bar{p}['S']$;

        all dependencies $:= [A|(\psi|A) \in \bar{p}['\Delta']]$;

        for $A \in$ *all dependencies* **do**

            head $:= \psi$;

            `/* All other variables which can make `ψ` true are treated as`
            `    though they could also falsify `ψ`.                          */`

            current dependencies $:=$ all dependencies $- A$;

            if *current dependencies* $= \{\}$ **then**

                current dependencies $:= \bot$;

            end

            body $:= ($current dependencies$_1 \lor ... \lor$ current dependencies$_n)$;

            `/* Add the conditional as a license for implication to the set`
            `    of rules.                                              */`

            $\bar{p}['S'] := \bar{p}['S'] \cup ($head $\leftarrow A \land \neg\text{ab}_k)$;

            $\bar{p}['S'] := \text{ab}_k \leftarrow \neg body$;

        end

        `/* Add the new abnormality to the possible world `V`.        */`

        $\bar{p}['V'] := \bar{p}['V'] \cup (\text{ab}_k, u)$;

    end

    `/* Remove all conditionals now that they have been interpreted as`
    `    licences for implication.                                   */`

    $\Delta := \{\}$;

    return $\bar{p}$

end

5.4.8 Weakly Completing

Weak completion is an essential part of the WCS. Under the assumption that the WCS is a valid representation of human cognition in at least one scenario, any comprehensive set of cognitive operations *M* must be able to mimic the Weak Completion of a knowledge base. Algorithm 9 shows one way in which this can be achieved.

Algorithm 9: `wc(`$\bar{p}$`)`

Function *e(*$\bar{p}$*)* **is**

    Replace all rules $\in \bar{p}['S']$ of the form $A \leftarrow$ body$_1, ..., A \leftarrow$ body$_n$ with
     $A \leftarrow$ body$_1 \lor ... \lor$ body$_n$;

    Replace all occurrences of $\leftarrow \in \bar{p}['S']$ with $\leftrightarrow$;

    return $\bar{p}$

end

5.4.9 Semantic Operator

Another essential step in the WCS, the semantic operator is used to assign all variables to either True or False (explictly), or to Unknown (implicitly). Algorithm 10 illustrates one

way in which this cognitive operation can be implemented for any epistemic state of the form $s = (KB, V, ...)$.

Algorithm 10: `semantic`$(\bar{p})$

Function *e($\bar{p}$)* **is**

    /* All least models that can achieved by applying the semantic
       operator to the current state point.   */

    $p' = \{\phi_{SvL}(\bar{p})\};$

    return p'

end

Function $\phi_{SvL}(\bar{p})$ **is**

    if *there exists a clause $A \leftarrow body \in \bar{p}['S']$ with $I_V(body) = \top$* **then**

        if $A \notin \bar{p}[fixed]$ **then**

            | $\bar{p}['V'][A] := \top$

        end

    end

    if *there exists a clause $A \leftarrow body \in \bar{p}['S']$ and for all clauses $A \leftarrow body \in \bar{p}['S']$ we find $I_V(body) = \bot$* **then**

        if $A \notin \bar{p}['fixed']$ **then**

            | $\bar{p}['V'][A] := \bot$

        end

    end

    return $\bar{p}$

end

5.4.10   Adding Abducibles

Next, we define a cognitive operation which uses the set of possible abducibles, given by $\bar{p}['R']['abducibles']$. We have already used the intuition of this operation to handle the WST and derive the general answer set. Later we will see that it can also be used to model individual reasoners in the Suppression Task. `addExp` adds possible explanations ϵ to the set of rules S in the epistemic state and is defined as follows:

Algorithm 11: `addExp`

Function *e($\bar{p}$)* **is**

    abducibles:=$\bar{p}['abducibles'];$

    $p' := [];$

    for *every unique subset $A \in abducibles$* **do**

        newP $:= \bar{p};$

        newP$['S'] =$ newP$['S'] \cup \{a \in A \leftarrow \top\};$

        newP$['\epsilon'] = A;$

        $p' := p' \cup$ newP;

    end

    return p'

end

5.4.11   Default Inferencing

If we assume that Reiter's default logic is a valid model of cognition for at least one task, it follows that an comprehensive formulation of M must encode a cognitive operation for drawing inferences from a set of default rules. Algorithm 12 describes the procedure for

drawing such inferences from an epistemic state state of the form $s = (W, D)$ where W is a deductively closed set of inference rules, and D is a set of default rules of the form $\frac{\text{pre}(\delta):\text{just}(\delta)}{\text{cons}(\delta)}$.

Algorithm 12: `default`$(\bar{p})$: encodes the inference of valid default processes.

> **Function** $e(\bar{p})$ **is**
> > Let theory $= (W, D)$, where $W = \bar{p}['W']$, and $D = \bar{p}['D']$;
> > Determine Π, the set of all closed, successful default processes for default theory (W, D);
> > **return** `createP`$(\bar{p}, \Pi)$
>
> **end**
> **Function** $createP(\bar{p}, \Pi)$ **is**
> > $p' = []$ **for** $\pi \in \Pi$ **do**
> > > $\bar{p}' := $ `clone`$(\bar{p})$;
> > > $\bar{p}'['W'] := \bar{p}'[W] \cup \text{In}(\pi)$;
> > > /* If no default processes were previously applied, simply add the default process.   */
> > > $\bar{p}'['R'][\'\delta'] := \bar{p}'['R'][\'\delta'] \cup \delta$;
> > > Add any inferences of the form head $\leftarrow$ body, where head is an atomic name, and body is a truth value, as the pair (head, body) to $\bar{p}'['V']$;
> > > $p' := p' \cup \bar{p}'$;
> > > **return** p'
> >
> > **end**
>
> **end**

As with the `th` cognitive operation, `default` is only a practically useful function if some approximation of the set of deductively closed inferences can be generated or evaluated.

5.5 Cognitive Operations as Aggregates

Any valid pCTM can be combined into a single cognitive operation (Lemma 6). This leads to an obvious and interesting idea. That of finding well-founded, uninterrupted epistemic operation sequences which are effective in modelling a variety of tasks and representing them as a single epistemic operation. We call this approach *aggregating*, and it draws inspiration from recent advances in the field of reinforcement learning(Drummond, 2002) in which simple sequences of actions which solve recurring subtasks across multiple solutions in the learning domain are treated a new, combined, and callable action when solving future problems in that domain.

This approach introduces the desirable property of shortening the total length of the SCP for a given set of cognitive tasks. However, it is evident from Lemma 7 that any SCP task in which an aggregated subsequence of operations replaces those individual operations in M may be less expressive than the same formulation in which the aggregated operations are still present.

For example, given SCP Task $\Pi = (s_i, M, f(), \gamma)$, with `wc` $\in M$ and `semantic` $\in M$, the cognitive processes `wc` and `semantic` could be combined together to form the pCTM `wcs` $= ($`wc` $\longmapsto$ `semantic`$)$. If we restrict solution SCPs for Π to those which contain `wcs`, then we guarantee that first the weak completion operation, and then the semantic operation must be applied in that order and with no other cognitive operations occurring between them. This approach may lead to fewer satisfying SCPs, but the found solution may generally be more plausible than those which are discounted.

The use of aggregates then, is a trade-off between sacrificing cross-reasoner accuracy and optimising a set of desirable heuristic properties such as length minimisation or limiting the number of state points in the final state if using a final state dependent SCP. It is worth noting that any valid pCTM can be treated as a single valid cognitive operation when using a final state dependent evaluation function without affecting the domain of the SCP Task.

In many ways this compromise follows the philosophy of cognitive modelling in general. A neuron-by-neuron approach to predicting human behaviour (if one were ever possible) would give us extreme accuracy in modelling any human reasoner, but is too complex to be practical, and even if it were, would provide no abstracted information with which to find common motifs and inferences among reasoners. On the other extreme, modelling reasoners as (*input*, *output*) pairs perfectly captures the average predictions of a population, but proves very inaccurate for unseen individuals. State-of-the-art approaches to cognitive modelling like neural networks and non-monotonic logical frameworks seek generalizations which accurately capture the responses of most reasoners across as many tasks as possible by approximating human motifs in human reasoning and applying across multiple tasks and inputs.

A researcher could decide that it is unreasonable to expect any cognitive operations to seperate `wc` and `semantic`, as they both form part of the same nonmonotonic logic. They might then create the new aggregate complex operation `wcs` which applies the WCS, and, thus, guarantees a unique model, in its entirety as follows:

Algorithm 13: `wcs`$(\bar{p})$

Function *e($\bar{p}$)* **is**
$\quad \bar{p}' := \texttt{wc}(\bar{p});$
$\quad p' := \texttt{semantic}(\bar{p}');$
$\quad$ **return** p'
end

It is worth noting that, when `wcs` $\in M$, and `wc` $\notin M$, `semantic` $\notin M$, some satisfying SCPs no longer exist. For example, it is possible to model the general case of the Suppression by only applying the semantic operator, and not weakly completing (this solution does not guarantee a unique least model, but does correctly model the task). Aggregate cognitive operations reduce the SCP space that needs to be explored to find satisfying SCPs and can enforce properties like the unique least model, but often do so at the cost of obscuring other solutions.

Researchers could even go a step further and define the cognitive operation `abwcs` which first interprets the conditionals in $p['\Delta']$ and then applies the WCS to the result. This algorithm is as follows:

Algorithm 14: `abwcs`$(\bar{p})$

Function *e($\bar{p}$)* **is**
$\quad \bar{p}' := \texttt{addAB}(\bar{p});$
$\quad \bar{p}' := \texttt{wc}(\bar{p});$
$\quad p' := \texttt{semantic}(\bar{p}');$
$\quad$ **return** p'
end

The reader will note that `abwcs` now provides precisely the same functionality as the examples of the WCS we examined in Chapter 3, and Chapter 6 will show that these three operations in sequence can indeed be modelled in the SCP framework and come to the same conclusions as when the WCS is applied in the original experiments.

The question of how to generate appropriate complex operation aggregates is a complex problem and is not discussed further in this thesis. It is, however a topic in which the author

has significant interest.

5.6 Epistemic State Structure Changes with Cognitive Operations

The concept and mathematics of cognitive operations whose output states points which differ in structure from their input state point structure are fairly straightforward. Some cognitive operation $A \in \Omega^*$ with input base point structure $\text{struct}(s_k)$ and output base point structure $\text{struct}(s_{k+1})$ is called a *structural transformation operation* if and only if $\text{struct}(s_k) \neq \text{struct}(s_{k+1})$.

The intuition behind structural state changes goes back to Albert Einstein's quote that opened this thesis "Everything should be made as simple as possible, but no simpler.". In theory there is no restriction on SCP structure that would prevent every epistemic state passing an arbitrarily large number of variables and arguments. An SCP meant to represent a cognitive task where participant responses are consistent with propositional logic, for example, could be accurately represented with an initial state $s_x = (S)$ where S is simply the knowledge base of facts and rules given; but a researcher might equally use an epistemic state $s_y = (S, V)$ where a variable V is intended to store the values of each variable in S; or even $s_z = (S, V, D)$ where D is a set of default rules. All three of these approaches would give an SCP the information required to evaluate the propositional task, but, if we know the task structure does not deal with uncertainty or show variation in reasoner responses, it becomes apparent that the set of default rules D is unnecessary.

All of the possibilities given hold for the propositional task because s_y and s_z are both supersets of s_x and s_x is sufficient to model the task. Thus, provided the researchers do not believe that the task will make use of cognitive operations which require more complex state structures, s_x is the simplest solution which meets the requirements of the researchers.

An obvious extension to the discussion above is the fact that a transformation from one state point structure to another when the input structure is a subset of the output structure is as easy as appending more structural variables to the input state. In the example above a propositional logic compliant state point s_x could be transformed into a WCS compliant state point s_y or a default rule compliant state point s_z.

Unfortunately, structural changes caused by appending variables are not the only type of structural changes. Obvious cases of structural transformation where they do not apply spring to mind. Such as the case where a more expressive state point must be changed to a less expressive state point structure. Imagine a case where a cognitive operation B transforms a case point of $\text{struct}(s_z)$ to one of $\text{struct}(s_y)$. An intuitive example of this might be a hypothetical mental operation which handles biases in thinking. A student might begin at an epistemic state which believes "usually, if I study for tests, I fail" and, through bias confirmation, come to believe "If i study for tests, I fail", transitioning from a default rule, to immutable rule.

In cases like this, it is up to the researcher to make several decisions: do I believe that the now empty set of variables D no longer provides meaningful information? And how do I represent the change in the variables still present in the output state? These questions are very often task or function specific and cannot be answered in a general sense.

5.7 Two Approaches to Creating Cognitive Operations

5.7.1 Theoretical Approximation

It seems reasonable that, given our desire to accurately model reasoning in participants, researchers creating cognitive functions for SCPs would choose to limit the set of allowable

cognitive operations to those for which there is a sound theoretical basis.

The first approach to creating cognitive operations we will consider is the use of cognitive operations for which there is already evidence in our existing models of human cognition. This evidence can take a wide variety of forms: it may follow from our knowledge of physics (for example, the impossibility of storing a set of unrelated variables above a certain physical threshold); evolutionary biology (there cannot exist a class of organisms without some kind of reproductive drive); anatomy (there must exist cognitive processes that facilitate voluntary muscle activation); sociology (some neuronal connections seem to restrict the number of close friends a human can maintain (Gonçalves, Perra, and Vespignani, 2011)); or any other field of science. Using this information can allow researchers to predict what cognitive functions may play a role in observed empirical data.

For the most part, theoretical approximation is a very powerful and well-justified mechanism for determining which cognitive operation could plausibly considered in the cognitive toolbox for a given task. Restricting the space of other cognitive operations which may play a part is also enabled by all the scientific fields mentioned above, complexity theory tells us that humans are unlikely to apply exhaustive classical logic reasoning to a task when the set of variables and rules to be considered is large because of physical limitations related to solving *NP-complete* or harder tasks efficiently.

At present, the foundation of non-monotonic reasoning in cognitive modelling is this theoretical mindset, a well-founded explanation is found by borrowing from a relevent field (often psychology) and then tested against empirical evidence and across multiple cognitive tasks to show evidence that is a reasonable approximation of a mechanism in human cognition.

This approach was used throughout this chapter and is analogous to the argument made by Stenning and Van Lambalgen, 2012 that modelling reasoning should be done primarily towards an appropriate theoretical representation, and then secondarily towards approximating empirical reasons

5.7.2 Empirical Appoximation

Contrasting theoretical approximation is the field of constrained empirical approximation. Techniques ranging from neural networks and Markov models to genetically inspired evolutionary algorithms consistently outperform humans in a huge variety of fields (Saad, 2004). Drawing on enormous databases of empirical data, modern artificial intelligence researchers have come to view machine learning as the most plausible way to create the first real thinking machine.

pseudo-cognitive operations are a class of cognitive operation which meet the mathematical definition of cognitive operations, but do not have either theoretical or empirical support for its existence in human cognition. Arguably, almost every cognitive operation in Ω and aggregated operation in Ω^* meets this definition. The operation `th` which outputs an epistemic state containing the deductive closure of its input states is one example of a pseudo-cognitive operation.

By utilizing a large set of simplistic pseudo-cognitive operations and using reinforcement learning techniques such as subtask learning (Drummond, 2002), it may well be possible to generate complex aggregate operations from these simple building blocks. And aggregate operations which are shown to be viable across different cognitive tasks will have strong empirical, but not theoretical, support for their existence as motifs in human reasoning.

Chapter 6

Modelling Experiments with SCPs

6.1 Overview

As with any cognitive framework, the most important metric for judging the success of SCPs comes from testing how well SCPs can approximate empirical data across tasks. This chapter will show the suitability of SCPs with a common set of allowable cognitive operations for modelling several well-studied experiments in cognitive modelling.

In particular, we will show that the Wason Selection Task, and Suppression Task can be modelled at both the general and individual reasoner level with SCPs in a formulation that is both intuitive and consistent with the WCS approach already described in Chapter 3. For the Suppression Task we also show that an SCP interpretation using Reiter's Default Logic remains unable to show Suppression in the default theory from Section 3.1.3.

6.2 Suppression Task

6.2.1 Suppression Task: WCS

Whilst modelling the Suprression Task in this section, we will continue to use the notation and variables defined in Section 3.1. Below is a brief restatement of the phrasing of the problem:

1. If she has an essay to write (e), she will study late in the library (l).

2. If the library is open (o), she will study late in the library (l).

3. She has an essay to write (e).

To model the Suppression Task in the SCP framework, we must first define the SCP Task $\Pi_{sup} = (s_i, M, f_{\text{sup}}(), \gamma)$ which outlines the restrictions and goals of the model. The structure of s_i must be robust enough to capture information regarding a set of rules, variable assignments in the possible world, and conditionals. To this end we define:

$$s_i = \{s_i^{\text{el}}, s_i^{\text{elo}}\}$$

$$s_i^{\text{el}} = \{S, \Delta_{\text{noSup}}, V, \text{name} = \text{'el'}\}$$

$$s_i^{\text{elo}} = \{S, \Delta_{\text{Sup}}, V, \text{name} = \text{'elo'}\}$$

For the the only unconditional rule, that she has an essay to write, we define $S = \{(e \leftarrow \top)\}$. The set of conditional rules $\Delta_{\text{noSup}} = \{l|e\}$ captures the probable assumption that she will study late in the library if she has an essay to write, $\Delta_{\text{sup}} = \{(l|e), (l|o)\}$ captures the addition of the conditional that if the library is open she will study late in the library. And the possible world $V = \{(e, u), (l, u), (o, u)\}$ sets all variable names occurring in S and δ to

unknown. The structural variable *name* discriminates the two epistemic states and, because there is no cognitive operation in *M* which can add, remove or change or change *name*, it is guaranteed that every base point in the final state will have the name 'el' or 'elo' indicating from which initial epistemic base point it was generated.

The second consideration *M*, denotes the set of possible cognitive operations which may occur in the resulting SCPs and realised SCPs of Π_{sup}. Following the intuition we used in Section 3.1, we choose cognitive operations that enable application of the WCS and interpretation of conditionals. To that end we define:

$$M = \{\texttt{addAB}, \texttt{semantic}, \texttt{wc}\}$$

The external evaluation function should translate the SCP into a decision that can be compared with empirical data. In the Suppression Task we will treat $f_{\text{sup}}()$ as a function that makes a single choice from the set {*'she will study late in the library'*, *she will not study late in the library*, *we are uncertain if she will study late in the library*}. Under the assumption only the final state point p_n determines whether she will study late, we might define the final state dependant external evaluation function $f_{\text{sup}}()$ as follows:

Algorithm 15: f_{sup}: determine if Suppression has been observed between the two formulation 'el' and 'elo'.

Function $f_{sup}(\pi)$ **is**

    /* This is a final state evaluation SCP.   */

    Let $R = \{(k \in K[\pi], f_{\text{sup}}())\}$ be the set of realised SCPs of the form $r_i = (k, f_{\text{sup}}())$ which can be generated from π.;

    Let $P = \{\bar{p}_1, ..., \bar{p}_n\}$, where $\bar{p}_i$ is the final state of r_i;

    /* Get all final epistemic state which can be generated from each of the two initial epistemic states   */

    $P_{\text{el}} := \{\bar{p} \in P | \bar{p}['name'] = 'el'\}$;

    $P_{\text{elo}} := \{\bar{p} \in P | \bar{p}['name'] = 'elo'\}$;

    if *there is a response* $\in responses(P_{el})$ *which is not* $\in responses(P_{elo})$ **then**

        | **return** 'suppression observed'

    else

        | **return** 'no suppression'

    end

end

Function $responses(p)$ **is**

    | **return** $[\texttt{response}(\bar{p}) | \bar{p} \in p]$

end

Function $response(\bar{p})$ **is**

    if $I_{\bar{p}[V]}(l) \models \top$ **then**

        | **return** 'she will study late in the library'

    else if $I_{\bar{p}[V]}(l) \models \bot$ **then**

        | **return** 'she will not study late in the library'

    else

        | **return** 'we are uncertain if she will study late'

    end

end

It is worth noting that one could also define f_{sup} in such a way that it iterates over every variable assignment for P_{el} and P_{elo} in the final state point and checks if any inference is drawn in P_{el} that is not drawn in P_{elo}. However, the purpose of the external evaluation function is to transform an SCP into an output from the set of *empirically observed* outputs,
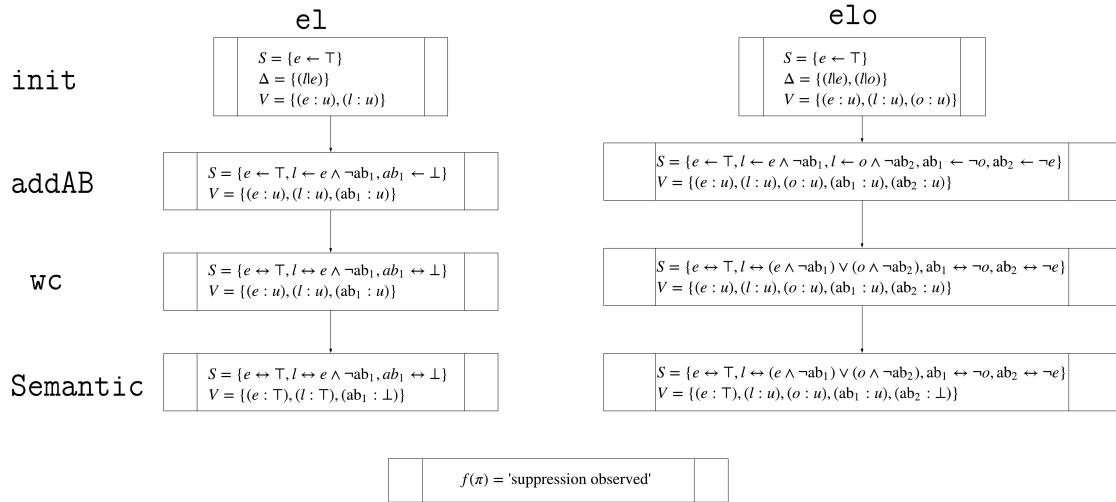
FIGURE 6.1: Suppression for the SCP $\mu_{\text{sup}} = (\pi = (s_i \longmapsto \texttt{addAB} \longmapsto \texttt{wc} \longmapsto \texttt{semantic}), f())$

.

and in this case the empirical knowledge is a response to the question "will she study late in the library?".

Next, we define the empirical result we wish to emulate γ. In this case we will define γ_{noSup} to model cases where no suppression is observed, and γ_{sup} to model cases where suppression is observed.

$$\gamma_{\text{noSup}} = \text{no suppression}$$

$$\gamma_{\text{sup}} = \text{suppression observed}$$

Armed with these definitions, it is possible to model the task Π as follows:

$$\Pi = (s_i, M, f(), \gamma_{\text{sup}})$$

Applying De Novo search to Π returns the satisfying SCP μ_{sup}:

$$\mu_{\text{sup}} = (\pi = (s_i \longmapsto \texttt{addAB} \longmapsto \texttt{wc} \longmapsto \texttt{semantic}), f())$$

Figure 6.1 illustrates the realised SCPs for μ_{sup} and illustrates that $\mu_{\text{sup}} \models \gamma$. This example has served to show that suppression can modelled in the SCP framework.

To illustrate the ability of the framework to model individual reasoners, we examine those reasoners who, despite knowing both that if she has an essay to write she will study late in the library and that if the library is open she will study late in the library, still do not demonstrate suppression, and instead draw the classical conclusion that she will study late in the library. To this end we define a second SCP Task Π_{deviant}.

$$\Pi_{\text{deviant}} = (s_{\text{dev}}, M', f_{\text{sup}}(), \gamma_{\text{noSup}})$$

$$s_{\text{dev}} = \{s_{\text{dev}}^{\text{el}}, s_{\text{dev}}^{\text{elo}}\}$$

$$s_{\text{dev}}^{\text{el}} = \{S, \Delta_{\text{noSup}}, V, \text{name} = \text{'el'}, R\}$$

$$s_{\text{dev}}^{\text{elo}} = \{S, \Delta_{\text{Sup}}, V, \text{name} = \text{'elo'}, R\}$$

$$R = \{\text{delete} : \{o\}, \text{explanations} : \{o \leftarrow \top, o \leftarrow \bot\}\}$$

Where s_{dev} is identical to s_i but includes the categorization structural variable R with a sub-variables called *'delete'*, and *'explanations'*.

$$M' = \{\texttt{addAB}, \texttt{semantic}, \texttt{wc}, \texttt{remove}_\perp, \texttt{addExp}\}$$

M' contains $\texttt{remove}_\perp$ which is able to delete any set of variables mentioned in $R['delete']$ of the epistemic state. $\texttt{addExp}$ allows us to add abducible facts to an epistemic state, following the intuition we first used to model the WST in Section 3.2. To see the explanations and justifications for these two cognitive operations, refer to Section 5.4.5, and Section 5.4.10.

Equipped with these two cognitive operations, there are several SCPs which can model Π_{deviant}.

Figure 6.2 shows the realised SCPs for one satisfying SCP for Π_{deviant}. This example shows that deleting the variable o from the epistemic state, prevents Suppression from occurring because the abnormality $(\text{ab}_1 \leftarrow \neg o)$ is the cause of Suppression in Π, and in Π' the abnormality is now given by $(\text{ab}_1 \leftarrow \neg \top)$.

In the Appendix, Figure 1 and Figure 2 show that the non-monotonic cognitive operation $\texttt{addExp}$ can prevent suppression by adding abducibles related to the free variable o.[1] This is, as far as the author of this thesis is aware, the first use of Hölldobler, 2015's abducible approach to the WST to model the Suppression Task. By giving o an assignment, the abnormality $(\text{ab}_1 \leftrightarrow \neg o)$ no longer evaluates to u, and across all final epistemic states. Together, the combined 'elo' epistemic states now return every $\texttt{response}()$ that the combined 'el' epistemic states do.

Interestingly, using $f_{\sup()}$, the case where the abducible $(o \leftarrow \perp)$ is added, results in the previous unseen response 'she will not study late in the library' to be returned in $\texttt{responses}(P_{\text{elo}})$ but not in $\texttt{responses}(P_{\text{el}})$. This is not an instance of suppression because the new response is only generated from the initial epistemic state $s_{\text{dev}}^{\text{elo}}$ whose information is a superset of the information in $s_{\text{dev}}^{\text{el}}$.

Both of these examples illustrate the derivation of the classical logical response to the suppression task as a modification of an SCP which did illustrate suppression. This evidences one of the core claims of the SCP Framework which is that it is a reasonable tool for modelling individual reasoners in a cognitive task.

The value of these examples lies in that they show that classical logic response to a cognitive task is not necessarily derived through the use of classical logic.

6.2.2 Suppression Task: Reiter's Default Logic

As discussed in Section 3.1.3, Reiter's Default Logic is not able to model the Suppression Task. This remains the case when using an SCP approximation of this logic. This brief example is provided as evidence that SCPs are able to approximate the Reiter's Default Logic interpretation of the task, rather than to propose any solutions which would make suppression possible.

In order to create this model, we introduce the new cognitive operation $\texttt{I}_W$. $\texttt{I}_W$ interprets the variable $\bar{p}['W']$ of input base point $\bar{p}$, and sets the variables $\bar{p}['V']$ in a base point $\bar{p}' \in p'$

---

[1]The output is too large to graph attractively, and should be viewed in its entirety to confirm that this approach is effective.
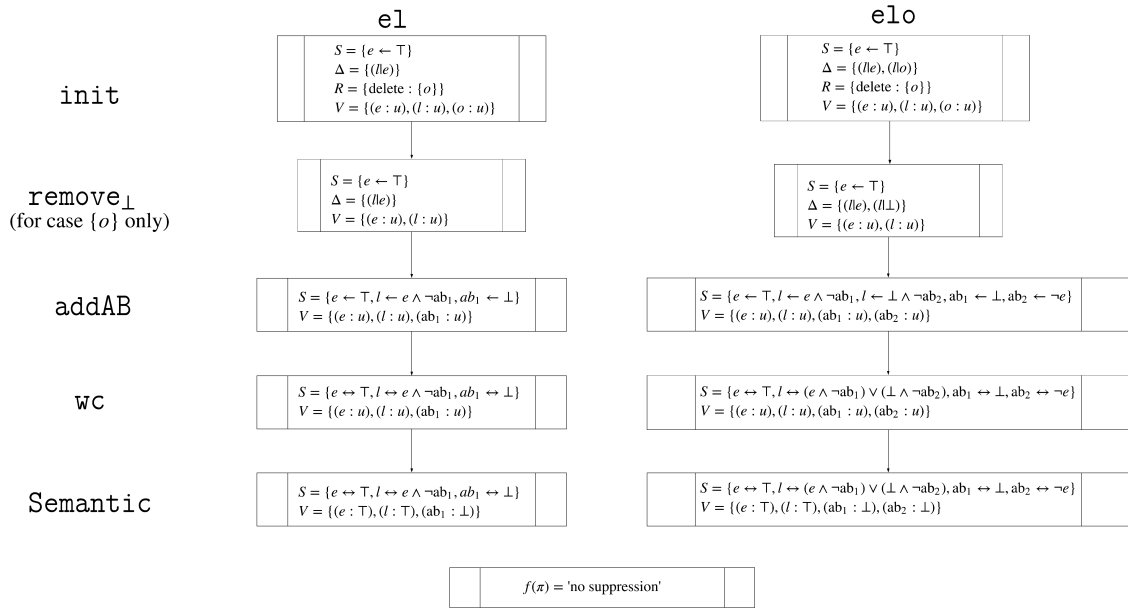
FIGURE 6.2: Inhibition of Suppression for the SCP $\mu_{\text{del}} = (\pi, f_{\text{sup}}()), \pi = (s_i \longmapsto \texttt{remove}_\perp \longmapsto \texttt{addAB} \longmapsto \texttt{wc} \longmapsto \texttt{semantic}).$

.

of the output state point p' to a valid possible world configuration for those variables.

Algorithm 16: $\texttt{I}_w$ generates possible world assignments for a propositional knowledge base W.

Function $e(\bar{p})$ **is**
    possibleAss:=$\{w | w$ is a valid possible setting of the variables in $\bar{p}['V']$ for
      propositional knowledge base $\bar{p}['W']\}$;
    $p' = \{\}$;
    for *unique* $w \in possibleAss$ **do**
        $\bar{p}' := \texttt{clone}(\bar{p})$;
        $\bar{p}'['V'] := w$;
        $p' := p' \cup \bar{p}'$;
    end
    return p'
end

Equipped with this new cognitive operation we might define the SCP Task Π_{default} as follows:

$$\Pi_{\text{default}} = (s_d, M_d, f_{\text{sup}}(), \gamma_{\text{sup}})$$

$$s_d = \{s_d^{el}, s_d^{elo}\}$$

$$s_d^{el} = (W, D_{el}, V_{el})$$

$$s_d^{elo} = (W, D_{elo}, V_{elo})$$

Where the initial state s_d consists of epistemic states containing the initial information that she will study late in the library, as well as extra conditional information interpreted as a set of defaults, as in Section 3.1.3, and a possible world variable. Each epistemic state in the initial state point encodes the information required to form a default theory of the form

(W, D).

$$W = \{e \leftarrow \top\}$$

$$D_{el} = \{\delta_1 : \frac{e \leftarrow \top : \mathbf{ab}_1 \leftarrow \bot}{l \leftarrow \top}, \delta_2 : \frac{\top : \mathbf{ab}_1 \leftarrow \top}{\mathbf{ab}_1 \leftarrow \top}\}$$

$$D_{elo} = \{\delta_1 : \frac{e \leftarrow \top : \mathbf{ab}_1 \leftarrow \bot}{l \leftarrow \top}, \delta_2 : \frac{o \leftarrow \top : \mathbf{ab}_2 \leftarrow \bot}{l \leftarrow \top}, \delta_3 : \frac{o \leftarrow \bot : \mathbf{ab}_1 \leftarrow \top}{\mathbf{ab}_1 \leftarrow \top}, \delta_4 : \frac{e \leftarrow \bot : \mathbf{ab}_2 \leftarrow \top}{\mathbf{ab}_2 \leftarrow \top}\}$$

$$V_{el} = \{(e, u), (l, u)\}$$

$$V_{elo} = \{(e, u), (l, u), (o, u)\}$$

To show the use of SCPs in the default logic interpretation of the suppression task, we limit the set of allowable cognitive operations M_d as follows:

$$M_d = \{\texttt{default}, \texttt{I}_w\}$$

De Novo search reveals no satisfying SCPs for Π_{default}, as we would expect from a faithful adaptation of Reiter's Default Logic which is unable to demonstrate suppression. The modified SCP Task $\Pi'_{\text{default}} = (s_d, M_d, f_{\text{sup}}(), \gamma_{\text{noSup}})$, does have a satisfying SCP.

$$\mu'_d = (\pi'_d, f_{\text{noSup}})$$
$$\pi'_d = (s_d \longmapsto \texttt{default} \longmapsto \texttt{I}_W)$$

We find that suppression is not observed. We define the final state points of the epistemic states s_d^{el} and s_d^{elo} which we will call P^{el} and P^{elo}, respectively. We find that $(l, \top) \in P_i^{el}$, and $(l, \top) \in P_j^{elo}$ for every base point $P_i^{el} \in_S P^{el}$ and $P_i^{elo} \in_S P^{elo}$. For this reason, $f_{\text{sup}}(\pi'_d) \models \gamma$.

These results mimic the intuition and data transformations steps used in Section 3.1.3, but do so in a more structured way. When considered in conjunction with the SCP generated for the WST interpretation of the Suppression Task, the simplicity of the SCP framework for modelling cognitive tasks across different logics is evident.

6.3 Wason Selection Task

In order to model the abstract case of the WST, we define the SCP Task $\Pi = (s_i, M, f_{\text{WST}}(), \gamma)$. Π is intended to model the most most common card choice made by reasoners: turning both D and 3. The choice of external evaluation function for this task (the turn function), if we are to follow the intuition from Section 3.1.2, has two requirements: it must be able to capture whether an observation O can be explained by the least model of the weakly completed program, and it must ensure that variable assignments which could falsify or verify the conditional rule $(3|D)$ using our *de Finetti* truth table (Table 2.2), are present in the least model.

To this end, we add a set of abducibles to the categorization variable R, and combine this with the variables for the set of propositional rules S, conditional rules Δ, and possible world V to create the initial state point s_i.

Next, we require an intuition for the contents of s_i. In this case, abduction can be simulated by the SCP in one of three ways. The first is to create a unique SCP for each possible explanation for each observation as we did for the classical WCS interpretation of the task in Section 3.1.2. But the fact that SCPs may contain an initial state point, rather than just a single epistemic state, enables a more elegant solution that captures the full scope of possible (*observation*, *explanation*) pairings. The other two options are to start with a state point containing each abductive case, or to create cognitive operations which add the abducible cases at computation time. Opting for the latter, we define:

$$s_i = (S, \Delta, R)$$

$$S = \{\}$$

$$\Delta = \{(3|D)\}$$

$$R = \{\text{abducibles} : \{D \leftarrow \top, D \leftarrow \bot, K \leftarrow \top, K \leftarrow \bot, 3 \leftarrow \top, 3 \leftarrow \bot, 7 \leftarrow \top, 7 \leftarrow \bot\}\}$$

Because SCPs aim to capture as much cognitive information as possible within the CTM as possible, we will prefer this third case and make use of the categorization variable R to specify both the set of abducibles, and, later, the chosen explanations (ϵ). In Section 11 we defined the complex operation `addExp` which creates a new epistemic state as output for each possible combination of the abducibles that could be added to $\bar{p}['S']$ of input base point $\bar{p}$.

For the purposes of readability, we will restrict examination of the abducibles to cases that are of length 1, and contain only positive facts[2].

---

[2]A reader wishing to see this process with respect to the full set of abducibles should examine the `WST.py` file in the SCP implementation.

Next, we define the final state dependent external evaluation function $f_{WST}()$ as follows:

Algorithm 17: f_{WST}

Function $f_{WST}(\pi)$ **is**

    `/* This is a final state evaluation SCP.                    */`

    Let $R = [(k \in K[\pi], f())]$ be the set of realised SCPs $r_i = (k, f())$ which can be generated from π;

    Let $P = [\bar{p}_1, ..., \bar{p}_n]$, where $\bar{p}_i$ is the final epistemic state of r_i;

    Let $\epsilon = [\epsilon_1, ..., \epsilon_n]$, where $\epsilon_i = \bar{p}_i[R]['\epsilon']$;

    Let $O = \{(D, \top), (K, \top), (3, \top), (7, \top)\}$ be the set of observations;

    `/* We include the contrapositive case (D'|7) because it will be shown`
        `to be useful in modelling individual reasoners later.        */`

    Let $C = \{(3|D), (D'|7)\}$ be the set of conditionals which must be falsified or verified;

    Output={};

    for $\bar{p}_i \in P$ **do**

        run isExplanation($\bar{p}_i, O$);

    end

    for *each minimal explanation e for observation o* **do**

        if $I_{e['V']}(C)$ *either falsifies or verifies some conditional in C* **then**

            Output[o]:=Output[o]$\cup$ 'Turn Card';

        else

            Output[o]:=Output[o]$\cup$ 'Do not turn Card';

        end

    end

    return Output

end

Function $isExplanation(\bar{p}_i, O)$ **is**

    for *each* $o \in O$ **do**

        if $I_{\bar{p}_i['V']}(\bar{p}_i['S']) \models o$ **then**

            `/* Then `$\bar{p}_i$` explains `o`.                          */`

            Add $\bar{p}_i$ to the explanations for o.

        end

    end

end

Next we create our goals. We will make one goal for each of the canonical cases of the WST. The structure of the goal associates a turn function with every possible observation that needs to be explained.

$$\gamma_{D,3} = \{(D : \text{turn card}), (3 : \text{turn card}), (K : \text{do not turn}), (7 : \text{do not turn})\}$$

$$\gamma_D = \{(D : \text{turn card}), (3 : \text{do not turn}), (K : \text{do not turn}), (7 : \text{do not turn})\}$$

$$\gamma_{D,7} = \{(D : \text{turn card}), (3 : \text{do not turn}), (K : \text{do not turn}), (7 : \text{turn card})\}$$

$$\gamma_{D,3,7} = \{(D : \text{turn card}), (3 : \text{turn card}), (K : \text{do not turn}), (7 : \text{turn card})\}$$

Next, we define the set of allowable cognitive operations M. In this case, because our previous modelling of the general case of the WCS required the addition of explanations from the set of abducibles, we will include the operation addExp (as seen in the SCP Modelling of individual cases of the Suppression Task).

$$M = \{\text{addAB}, \text{semantic}, \text{wc}, \text{AddExp}\}$$

init

$S = \{\}$
$\Delta = \{(3|D)\}$
$V = \{D : u, K : u, 3 : u, 7 : u\}$
$R = \{\text{abducibles}:\{D,K,3,7\}\}$

addAB

$S = \{3 \leftarrow D \wedge \neg ab,\}$
$\Delta = \{\}$
$V = \{D : u, K : u, 3 : u, 7 : u\}$
$R = \{\text{abducibles}:\{D,K,3,7\}\}$

addExp

$\epsilon = \{D\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftarrow \bot, D \leftarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{D\}\}$

$\epsilon = \{K\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftarrow \bot, K \leftarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{K\}\}$

$\epsilon = \{3\}$

$S = \{3 \leftarrow D \wedge \neg ab, ab \leftarrow \bot, 3 \leftarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{3\}\}$

$\epsilon = \{7\}$

$S = \{3 \leftarrow D \wedge \neg ab, ab \leftarrow \bot, 7 \leftarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{7\}\}$

WC

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, D \leftrightarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{D\}\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, K \leftrightarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{K\}\}$

$S = \{3 \leftrightarrow (D \wedge \neg ab) \vee \top, ab \leftrightarrow \bot\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{3\}\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, 7 \leftrightarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : u, ab : u\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{7\}\}$

semantic

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, D \leftrightarrow \top\}$
$V = \{D : \top, K : u, 3 : \top, 7 : u, ab : \bot\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{D\}\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, K \leftrightarrow \top\}$
$V = \{D : u, K : \top, 3 : u, 7 : u, ab : \bot\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{K\}\}$

$S = \{3 \leftrightarrow (D \wedge \neg ab) \vee \top, ab \leftrightarrow \bot\}$
$V = \{D : u, K : u, 3 : \top, 7 : u, ab : \bot\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{3\}\}$

$S = \{3 \leftrightarrow D \wedge \neg ab, ab \leftrightarrow \bot, 7 \leftrightarrow \top\}$
$V = \{D : u, K : u, 3 : u, 7 : \top, ab : \bot\}$
$R = \{\text{abducibles}:\langle D,K,3,7\rangle, \epsilon : \{7\}\}$

Partial Turn Function

$f(\pi_{D=D}) = \{(D : \text{'turn card'})\}$
$f(\pi_{D=K}) = \{\}$
$f(\pi_{D=3}) = \{(3 : \text{'turn card'})\}$
$f(\pi_{D=7}) = \{\}$

$f(\pi_{D=D}) = \{\}$
$f(\pi_{D=K}) = \{(K : \text{'do not turn'})\}$
$f(\pi_{D=3}) = \{\}$
$f(\pi_{D=7}) = \{\}$

$f(\pi_{D=D}) = \{D : \text{'turn card'}\}$
$f(\pi_{D=K}) = \{\}$
$f(\pi_{D=3}) = \{3 : \text{'do not turn'}\}$
$f(\pi_{D=7}) = \{\}$

$f(\pi_{D=D}) = \{\}$
$f(\pi_{D=K}) = \{\}$
$f(\pi_{D=3}) = \{\}$
$f(\pi_{D=7}) = \{7 : \text{'do not turn'}\}$

$f_{\text{WST}}(\pi) = \{(D : \text{'turn card'}), (K : \text{'do not turn'}), (3 : \text{'turn card'}), (K : \text{'do not turn'}), (3 : \text{'turn card','do not turn'}), (7 : \text{'do not turn'})\}$

$f_{\text{WST}}^{\text{pref}}(\pi) = \{(D : \text{'turn card'}), (K : \text{'do not turn'}), (3 : \text{'do not turn'}), (7 : \text{'do not turn'})\}$

FIGURE 6.3: Realised SCPs for the SCP interpretation of the WST where $\mu_{D,3} = (\pi, f_{\text{WST}})$, and $\mu_D = (\pi, f_{\text{WST}}^{\text{pref}})$, with $\pi = (s_i \longmapsto$ addAB $\longmapsto$ addExp $\longmapsto$ WC $\longmapsto$ semantic). Results for $\mu_{\{}D,3\}$ mimic those observed in Hölldobler, 2015's WST interpretation which concludes the canonical case $D,3$. Results for μ_D conclude the canonical case D.

The overall planning task, Π, can now be formulated for the general case of the WST.

$$\Pi = (s_i, M, f_{\text{WST}}(), \gamma_{D,3})$$

De Novo search for the SCP Plan yields several SCP (differentiated by where the `addExp` cognitive operation occurs), including this one:

$$\mu_{D,3} = (\pi, f_{\text{WST}}())\}$$

$$\pi = (s_i \longmapsto \texttt{addAB} \longmapsto \texttt{addExp} \longmapsto \texttt{wc} \longmapsto \texttt{semantic})$$

$$f_{\text{WST}}(\pi) = \left\{ \begin{array}{ll} (\text{D} : \{\text{turn card}, \text{do not turn}\}), & (3 : \{\text{turn card}, \text{do not turn}\}), \\ (\text{K} : \{\text{do not turn}\}), & (7 : \{\text{do not turn}\}) \end{array} \right\}$$

Figure 6.3 demonstrates the calculation of $f(\pi)$ for each positive explanation ϵ of length 1. When this task is run for minimal explanations of unrestricted length, the set of conclusions for each card returned by $f_{\text{WST}}()$ do not change. We find that $\mu_{D,3} \models_{\text{weak}} \gamma_{D,3}$ and that we should turn the cards D, and 3, precisely as in Dietz, Hölldobler, and Wernhard, 2014's model. In the figure the partial turn function $f_{\pi_{0=x}}$ means: 'if x were the only observation in O, what would be the result of $f_{\text{WST}}(\pi)$?'. It is provided only for illustrative purposes.

Thus, it has been shown that the SCP Framework is capable of modelling the general response to the abstract case of the WST.

In order to model the goals γ_D, $\gamma_{D,7}$, and $\gamma_{D,3,7}$, we must again change our SCP Task. These extensions are easily introduced and, the author hopes, will serve as motivation of the practical and intuitive usefulness of the SCP Framework.

The simplest of these case to model is γ_D. This could done by creating the new SCP Task Π_D.

$$\Pi_D = (s_i, M, \gamma_D, f_{\text{WST, simple}})$$

And defining $f_{\text{WST, simple}}$ precisely as f_{WST} was defined, except that the `isExplanation` function in Algorithm 18 is changed to that of Algorithm 19.

Algorithm 18: `isExplanation` function as it appears in f_{WST}.

Function $isExplanation(\bar{p}_i, O)$ **is**

    for *each* $o \in O$ **do**

        if $I_{\bar{p}_i['V']}(\bar{p}_i['S']) \models o$ **then**

            /* Then $\bar{p}_i$ explains o.   */

            Add $\bar{p}_i$ to the explanations for o.

        end

    end

end

Algorithm 19: `isExplanation` function as it appears in $f_{\text{WST, simple}}$.

Function $isExplanation(\bar{p}_i, O)$ **is**

    for *each* $o \in O$ **do**

        /* :   */

        We require that the observation being explained is an element of the proposed explanations. **if** $I_{\bar{p}_i['V']}(\bar{p}_i['S']) \models o$ *AND* $o \in \bar{p}_i['R']['\epsilon']$ **then**

            /* Then $\bar{p}_i$ explains o.   */

            Add $\bar{p}_i$ to the explanations for o.

        end

    end

end

This approach mimics the logic we used to create the Simple WCS Extension in Section 3.1. However, there is an alternative approach which the author prefers.

One should note that, while $\mu_{D,3} \models_{\text{weak}} \gamma_{D,3}$, it is also the case that that 'do not turn' occurs in $f_{\text{WST}}(\pi)[K]$, $f_{\text{WST}}(\pi)[3]$, and $f_{\text{WST}}(\pi)[7]$, but is not in $f_{\text{WST}}(\pi)[D]$. If we treat $f_{\text{WST}}(\pi)$ sceptically and assume that we only turn a card if there is *no reason not to turn it*, we arrive at the conclusion that only D should be turned.

We define a set of preferred responses which prefers the 'do not turn' response as follows (using preferred responses as defined in Section 4.6):

$$\text{pref} = \{(D : \text{do not turn}), (3 : \text{do not turn}), (K : \text{do not turn}), (7 : \text{do not turn})\}$$

$$p(f_{\text{WST}}(\pi), \text{pref}) = \{(D : \text{turn card}), (3 : \text{do not turn}), (K : \text{do not turn}), (7 : \text{do not turn})\}$$

Now it is clear that $p(f_{\text{WST}}(\pi), \text{pref}) = \gamma_D$. To return this to an acceptable SCP format, we may simply define $f_{\text{WST}}^{\text{pref}} = p(f_{\text{WST}}(\pi))$ and observe that $\mu_D \models \gamma_D$ for SCP $\mu_D = (\pi, f_{\text{WST}}^{\text{pref}}())$. Figure 6.3 illustrates the process by which the canonical case D is derived.

Implementing contraposition in the SCP framework we have defined is surprisingly simple. We simply need to define a new SCP Task $\Pi_{D,3,7}$ as follows:

$$\Pi_{D,3,7} = (s_i', M, \gamma_{D,3,7}, f_{\text{WST}}())$$

$$s_i' = (S', \Delta', R)$$

$$S' = \{(D \leftarrow \neg D')\}$$

$$\Delta' = \{(3|D), (D'|7)\}$$

Simply by adding the contrapositive conditional and its variable initialization, search returns an SCP $\mu_{D,3,7} = (\pi', f_{\text{WST}}())$, $\pi' = (s_i' \longmapsto \texttt{addAB} \longmapsto \texttt{addExp} \longmapsto \texttt{wc} \longmapsto \texttt{semantic})$,

init

$S = \{D \leftarrow \neg D'\}$
$\Delta = \{(3|D),(D'|7)\}$
$V = \{D : u, K : u, 3 : u, 7 : u, D : u\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}\}$

addAB

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$
$\Delta = \{\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}\}$

$\epsilon = \{3\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 3 \leftarrow \top\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{3\}\}$

$\epsilon = \{7\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 7 \leftarrow \top\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{7\}\}$

addExp

$\epsilon = \{D\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, D \leftarrow \top\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{D\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 3 \leftarrow \top\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{3\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 7 \leftarrow \top\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{7\}\}$

WC

$S = \{D \leftarrow \neg D' \vee \top, 3 \leftarrow D \wedge \neg ab_1, ab_1 \leftarrow \bot, D' \leftarrow 7 \wedge \neg ab_2, ab_2 \leftarrow \bot\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{D\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow (D \wedge \neg ab_1) \vee \top, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{3\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$
$V = \{(D : u),(K : u),(3 : u),(7 : u),(D' : u),(ab_1 : u),(ab_2 : u)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{7\}\}$

semantic

$S = \{D \leftarrow \neg D' \vee \top, 3 \leftarrow D \wedge \neg ab_1, ab_1 \leftarrow \bot, D' \leftarrow 7 \wedge \neg ab_2, ab_2 \leftarrow \bot\}$
$V = \{(D : \top),(K : u),(3 : \top),(7 : u),(D' : u),(ab_1 : \bot),(ab_2 : \bot)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{D\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow (D \wedge \neg ab_1) \vee \top, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$
$V = \{(D : u),(K : u),(3 : \top),(7 : u),(D' : u),(ab_1 : \bot),(ab_2 : \bot)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{3\}\}$

$S = \{D \leftarrow \neg D', 3 \leftarrow D \wedge \neg ab_1, D' \leftarrow 7 \wedge \neg ab_2, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot, 7 \leftarrow \top\}$
$V = \{(D : \bot),(K : u),(3 : \bot),(7 : \top),(D' : \top),(ab_1 : \bot),(ab_2 : \bot)\}$
$R = \{\text{abducibles:} \{D, K, 3, 7\}, \epsilon : \{7\}\}$

Partial Turn Function

$f(\pi_{o=D}) = \{(D : \text{'turn card'})\}$
$f(\pi_{o=K}) = \{\}$
$f(\pi_{o=3}) = \{(3 : \text{'turn card'})\}$
$f(\pi_{o=7}) = \{\}$

$f(\pi_{o=D}) = \{D : \text{'turn card'}\}$
$f(\pi_{o=K}) = \{\}$
$f(\pi_{o=3}) = \{3 : \text{'do not turn'}\}$
$f(\pi_{o=7}) = \{\}$

$f(\pi_{o=D}) = \{D : \text{'turn card'}\}$
$f(\pi_{o=K}) = \{\}$
$f(\pi_{o=3}) = \{3 : \text{'turn card'}\}$
$f(\pi_{o=7}) = \{7 : \text{'turn card'}\}$

$f_{\text{WST}}(\pi') = \{(D : \text{'turn card'}), (K : \text{'do not turn'}), (3 : \{\text{'turn card', 'do not turn'}\}), (7 : \text{'turn card'})\}$

$f_{\text{WST}}^{\text{pref}}(\pi') = \{(D : \text{'turn card'}), (K : \text{'do not turn'}), (3 : \text{'do not turn'}), (7 : \text{'turn card'})\}$
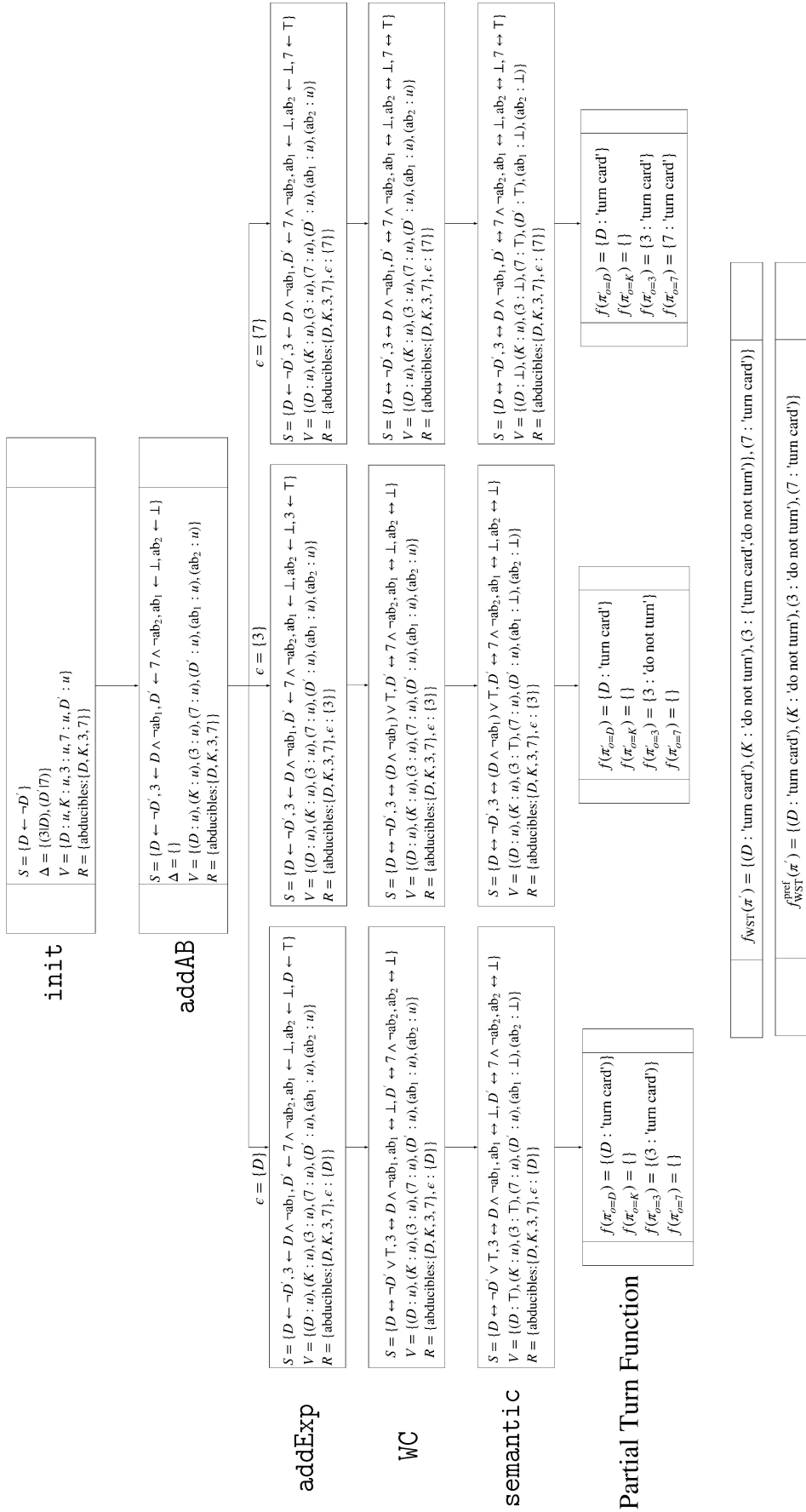
FIGURE 6.4: Realised SCPs for the SCP interpretation of the WST where $\mu_{D,3,7} = (\pi', f_{\text{WST}})$, and $\mu_{D,7} = (\pi', f_{\text{WST}}^{\text{pref}})$, with $\pi = (s_i' \longmapsto$ addAB $\longmapsto$ addExp $\longmapsto$ wc $\longmapsto$ semantic). Results for $\mu_{D,3,7}$, $\mu_{D,7}$ conclude the canonical cases $\{D, 3, 7\}$ and $\{D, 7\}$, respectively. The case for $\epsilon = \{K\}$ is omitted, for reasons of readability.

with $f_{\text{WST}}(\pi') \models_{\text{weak}} \gamma_{D,3,7}$. It is worth noting that π and π' differ only in the contents of their respective initial state points.

Finally, the case where the D and 7 cards are turned (the classical inference), can be modelled by combining aspects of the SCPs above. We define the SCP Task $\Pi_{D,7}$:

$$\Pi_{D,7} = (s'_i, M, f_{\text{WST}}^{\text{pref}}(), \gamma_{D,7})$$

And running De Novo search over this Task reveals that the SCP $\mu_{D,7} = (\pi', f_{\text{WST}}^{\text{pref}}())$ is a satisfying SCP for the goal $\gamma_{D,7}$.

Figure 6.4 illustrates $\mu_{D,3,7}$ and $\mu_{D,7}$, and demonstrates that only the external evaluation function differs between them. The interested reader is also pointed to `example_wcs_wst.py`, where they can see this process for the full set of abducibles and should note some of the interesting effects of negative explanations on the turn functions.

Thus, it has been shown that two CTMs π, and π', in conjunction with the external activation functions $f_{\text{WST}}()$ and $f_{\text{WST}}^{\text{pref}}()$ are able to model individual reasoners in the Wason Selection Task for all four of the canonical cases.

Chapter 7

Comparing SCPs

7.1 Why do we need to compare SCPs?

The ability to compare the feasibility of different solutions is an essential step in any computational process in which multiple approaches produce the desired output. Consider a toy example of an SCP task which describes the mental process needed to bake a cake:

$$\Pi = (s_0, M, \gamma, f())$$

$$s_0 = (V = (cakeBaked : \bot))$$
$$M = \{\texttt{mixIngredients}, \texttt{bakeIngredients}, \texttt{doTheLaundry}\}$$

$$f(x) = \begin{cases} cakeBaked \models \top & \text{True} \\ cakeBaked \models \bot & \text{False} \end{cases}$$

$$\gamma = True$$

Without specifying the precise details of the complex operations in M, and simply using our intuition of the effects of these actions, we can draw some candidate SCPs. Candidate SCPs such as $\mu = (\pi = (s_0 \longmapsto \texttt{mixIngredients}), f())$ do not result in the cake being baked and so are discounted immediately. However, consider a case where the modelling algorithm being used has come up with two possible SCPs to explain the processes chosen by the participant:

$$\mu_1 = (\pi_1 = (s_0 \longmapsto \texttt{mixIngredients} \longmapsto \texttt{bakeIngredients}), f()) \tag{7.1}$$

$$\mu_2 = (\pi_2 = (s_0 \longmapsto \texttt{mixIngredients} \longmapsto \texttt{doTheLaundry} \longmapsto \texttt{bakeIngredients}), f()) \tag{7.2}$$

Intuitively, both of these operational sequences would result in a cake being baked and both $f(\pi_1) \models \gamma$ and $f(\pi_1) \models \gamma$. Thus, both are candidate solutions to the SCP task given. However, both of these solutions may not be equally *plausible*. Why would someone need to do their laundry to make a cake? We can concoct wild scenarios in which the participant's house is so full of dirty laundry that access to the oven it restricted, but this seems implausible. Most readers would agree that Equation 7.1 is more plausible that Equation 7.2.

This toy example is evidence that, in at least some cases, one can confidently prefer one SCP to another. The question now arises: how do we precisely, and consistently prefer one SCP over another? This question is not easy to answer, and this chapter is devoted to proposing candidate solutions which may be able to quantitatively score and select preferable SCPs.

Section 7.2 discusses the question of how to compare different SCPs found using search for a single task. Section 7.3 discuses the more general problem of comparing SCPs even when the SCPs Tasks from which they were generated differ. Section 7.3.1 introduces the Needleman-Wunsch Algorithm for string matching whose underlying principles have allowed us to quantify questions of homology and evolutionary relationships in biology.

7.2 Comparing Generated SCPs

7.2.1 Scoring

Cognitive modelling is a science that exists, in part, to replicate the empirical results of human reasoning. It suffers from a painful truth: just because a solution is simple, elegant and seemingly well-justified, it does not follow that that solution is correct. Indeed, that solution might completely fail to explain experimental data from a cognitive task, and must then be discounted. However in the fields of string-matching, etymology, and homological evolution (Sweetser, 1990) (Needleman and Wunsch, 1970) , mathematically consistent approaches to scoring are still generally a good starting point. And so we carry that assumption into the field of cognitive modelling and assume that, in the absence of directly contradictory empirical data, certain properties related to finding the optimal sequence of cognitive operations are desirable, whilst others are not.

In general the easiest way to compare two distinct objects is to quantify some subset of their properties and use these properties to rank the objects. Continuing with the toy example in Section 7.1 we will attempt to create a common-sense scoring mechanism to determine whether μ_1 or μ_2 is a more cognitively plausible solution to baking a cake. A great many possible criteria exist for scoring these two SCPs, but we will focus on just two of them: the length of the SCP, and the plausibility of each cognitive operation that occurs in either SCP.

SCP Length

Perhaps the simplest and most intuitive way to decide which of two SCPs is best suited to solving a specific problem is to prefer the shortest one. In the field of Bioinformatics, one of the earliest approaches to determine which two organisms from a set were more closely related was to directly estimate how many genetic mutations (insertions, deletions, value changes) would be necessary to turn each of these genetic sequences into each other sequence. The same logic can be applied to SCPs for those SCPs generated using either *De Novo* or *Insertion Search* (Section 4.7).

If we define the length $|\mu|$ of an SCP $\mu = \{\pi, f()\}$ to be number of cognitive operations present in π. In the case of *De Novo* search, we assume assume that the optimal length of a solution to Π is an SCP $\mu^* = (\pi^*, f())$ with length $|\mu^*| = 0$. This optimal solution obviously does not exist in this case $f(s_0) \not\models (cakeBaked = \top)$, but it serves as a way of implicitly preferring shorter SCPs, as those will require fewer insertion operations to satisfy $f(x)$.

Using this simple test criteria: $|\mu_1| < |\mu_2|$. For this reason μ_1 is preferred because it requires only 2 operations to transform the ideal SCP $f(s_0)$ into $f(s_0 \longmapsto \texttt{mixIngredients} \longmapsto \texttt{bakeIngredients})$, rather than the 3 required for μ_2.

Though simple, this scoring procedure provides the foundations upon which more complex scoring algorithms will be built for the remainder of this section.

Limitations

Unfortunately, the SCP Length approach fails to capture anything about the *relationship* between the SCPs being scored. Let us examine the following three SCPs, all generated whilst modelling the Wason Selection Task (Chapter 6.2):

$$\pi_{\text{WST}} = (s_{WST} \longmapsto \texttt{addAB} \longmapsto \texttt{addExp} \longmapsto \texttt{wc} \longmapsto \texttt{semantic})$$

$$\pi_{\text{prop}} = (s_{WST} \longmapsto \texttt{th})$$

$$\mu_{D,3} = (\pi_{\text{WST}}, f_{\text{WST}}())$$

$$\mu_{D,7} = (\pi_{\text{WST}}, f_{\text{WST}}^{\text{pref}}())$$

$$\mu'_{D,7} = (\pi_{\text{prop}}, f_{\text{WST}}^{\text{pref}}())$$

Using SCP Length, $|\mu_{D,7}| = 4$ and $|\mu'_{D,7}| = 1$, we would would conclude that $\mu'_{D,7}$ is the more probable SCP, because its CTM contains a smaller number of cognitive operations.

In isolation, given only $|\mu_{D,7}|$ and $\mu'_{D,7}$, this conclusion makes some sense. $\mu'_{D,7}$ is indeed significantly shorter, and can be achieved with a single cognitive operation. But our information about these SCPs comes with more context than that. Assume that we are certain that $\mu_{D,3}$ is an accurate model for some other related cognitive task. Now it seems plausible that the solution to a related cognitive task may be an offshoot of an underlying core process. From this perspective, if we belief that $\mu_{D,3}$ has an underlying model in common (save for some 'mutations' which add, remove, or change cognitive operations) with the solution to the task to turn the D and 3 cards, $\mu_{D,7}$ seems significantly more likely than $\mu'_{D,7}$.

7.3   Comparing Related SCPs

One of the core principles of biological scoring algorithms is the intuition that the genetic structures being analysed share a common ancestor at some point in the past. With this assumption it becomes possible to model precisely what genetic mutations one sequence might have undergone to become the other.

This intuition carries surprisingly well into the field of cognitive modelling where researchers very often search for common core mechanisms shared between different reasoners, or across different cognitive tasks. Concepts like the Diversity Principle (Heit, Hahn, and Feeney, 2005), and denial of the antecedent are often shown be powerful predictors of participant responses.

To this end, I propose that the concept of a shared underlying cognitive process in cognitive modelling can be considered analogous to the concept of shared ancestry in genetic sequences. If we assume that the SCP Framework is a valid mechanism for human cognition, it becomes possible to use these biological principle to compare and score SCPs.

7.3.1   The Needleman-Wunsch Algorithm

In the field of Bioinformatics, one of the earliest approaches to determine which two organisms from a set were more closely related was to directly estimate how many genetic mutations (insertions, deletions, and mutations) would be necessary to transform the genetic sequence of one of those animals into the other. Needleman and Wunsch, 1970 proposed a simple scoring algorithm to determine how similar two genetic sequences were to one another by computing how many insertions, deletions, and mutations the first sequence would have to undergo to become the second sequence, and assigning a quantitative penalty or reward to each of these events.

Event	Process	Score
$x = y$	Match	1
$x \neq y, x \neq \text{'-'}, y \neq \text{'-'}$	Mismatch	-1
$x = \text{'-'},$	Insert	-1
$y = \text{'-'}$	Delete	-1

TABLE 7.1: A very simple table for scoring SCPs with the Needleman-Wunsch algorithm for inputs x and y.

		A	T	T	A	C	A
	0	-1	-2	-3	-4	-5	-6
A	-1	1	0	-1	-2	-3	-4
T	-2	0	2	1	0	-1	-2
G	-3	-1	1	1	0	-1	-2
C	-4	-2	0	0	0	1	0
T	-5	-3	-1	1	0	0	0

TABLE 7.2: Scoring matrix for the sequences $a = ATTACA$ and $b = ATGCT$ using the scoring properties defined in Table 7.1.

Needleman's recursion makes use of a dynamic programming approach and *score matrix* D to score two sequences efficiently ($O(nm)$ time and $O(nm)$ space complexity for sequences of length n and m, respectively). The recursion for his algorithm for sequences a and b is as follows:

$$D_{0,j} = \text{insertion cost} \times j$$

$$D_{i,0} = \text{deletion cost} \times i$$

$$D_{i,j} = \min \begin{pmatrix} D_{i-1,j-1} & + & s(a_i, b_j) \\ D_{i-1,j} & + & s(a_i, -) \\ D_{i,j-1} & + & s(-, b_j) \end{pmatrix}$$

Where an example of the costs $s(x, y)$ is given by Table 7.1. Matches are usually positively scored (desirable), and mismatches, insertions, and deletions, are usually negatively scored (undesirable).

Needleman also defined a traceback algorithm to show what this optimal global alignment was. The traceback begins at the bottom right corner of the D matrix and for, position $D_{i,j}$, calculates which neighbouring value $D_{i-1,j-1}$, $D_{i-1,j}$, $D_{i,j-1}$ would result in $D_{i,j}$ when added to the cost of match/mismatch, insertion and deletion respectively. We will not detail this aspect of the algorithm beyond this point because our interest lies in the scoring itself, rather than the traceback.

Table 7.2 shows the scoring matrix for the sequences $a = ATTACA$ and $b = ATGCT$, and the optimal alignment is given by:

$$\begin{pmatrix} A & T & T & A & C & A \\ A & - & T & G & C & T \end{pmatrix}$$

Where '$-$' denotes an insertion or deletion.

	s_{WST}	addAB	addExp	wc	semantic
	0	-1	-2	-3	-4	-5
s_{WST}	-1	1	0	-1	-2	-3
addAB	-2	0	2	1	0	-1
addExp	-3	-1	1	3	2	1
wc	-4	-2	0	2	4	3
semantic| -5	-3	-1	1	3	5

TABLE 7.3: Needleman-Wunsch comparison of $\mu_{D,3}$ to $\mu_{D,7}$

	s_{WST}	addAB	addExp	wc	semantic
	0	-1	-2	-3	-4	-5
s_{WST}	-1	1	0	-1	-2	-3
th	-2	0	-1	-2	-3	-4

TABLE 7.4: Needleman-Wunsch comparison of $\mu_{D,3}$ to $\mu'_{D,7}$

Needleman-Wunsch for SCPs

The initial extension of the Needleman-Wunsch algorithm to the SCP framework is surprisingly simple. Because known SCPs are guaranteed to consist of a linear, finite sequence of cognitive operations, and comparison between cognitive operations is possible (two cognitive operations are equal if they have the same name), the only thing the SCP framework lacks is a way to represent insertions and deletions. To this end we define the `insert` cognitive operation.

Algorithm 20: `insert`$(\bar{p})$: an empty cognitive operation used as a place-holder for scoring.

    Function $e(\bar{p})$ **is**
    │   **return** $\bar{p}$
    end

Like the `th` operation, `insert` has no function in an SCP Task, and is not assumed to be a function with an analogue in human reasoning. Instead it is a placeholder function that represents some unknown function in the shared cognitive process from which both SCPs are assumed to originate.

Finally, we must decide how the initial epistemic state is to be scored. For now we will simply treat s_i as if it were a cognitive operation for scoring purposes.

Now we can compare our candidate solution SCPs $\mu_{D,7}$ and $\mu'_{D,7}$ using the score system from Table 7.1 and the Needleman-Wunsch Algorithm. Table 7.4 shows global alignment score of -4 for ($\mu_{D,3}, \mu'_{D,7}$), whereas Table 7.3 has a global alignment score of 5 for ($\mu_{D,3}, \mu_{D,7}$) and so $\mu_{D,7}$ is believed to be more similar to $\mu_{D,3}$.

$$\text{align}(\mu_{D,3}, \mu'_{D,7}) = \begin{pmatrix} s_{\text{WST}} & \text{addAB} & \text{addExp} & \text{wc} & \text{semantic} \\ s_{\text{WST}} & \text{addAB} & \text{addExp} & \text{wc} & \text{semantic} \end{pmatrix}$$

$$\text{align}(\mu_{D,3}, \mu_{D,7}) = \begin{pmatrix} s_{\text{WST}} & \text{addAB} & \text{addExp} & \text{wc} & \text{semantic} \\ s_{\text{WST}} & \text{insert} & \text{insert} & \text{insert} & \text{th} \end{pmatrix}$$

Cognitive Op	Match	Mismatch	Insertion
s_i	3	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-1
`addAB`	1	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-1
`addExp`	1	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-1
`semantic`	1	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-1
`th`	5	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-5
`wc`	1	$\frac{s_{\text{ext}}(x,\texttt{insert})+s_{\text{ext}}(y,\texttt{insert})}{2}$	-1

TABLE 7.5: A very simple table for scoring SCPs with the Needleman-Wunsch algorithm for inputs x and y.

Extending Needleman-Wunsch for SCPs

We have now seen that SCP similarity can be computed using the Needleman-Wunsch algorithm. `SCP_scoring_improved` in the implementation of the SCP framework takes this approach one step further and assigns unique scores to matches, mismatches, and insertion operations for every cognitive operation. This allows the researcher to specify intuition about the relative complexity of each cognitive operation.

The most important change in this formulation is the creation of a more expressive scoring function. Whilst, $s(x,y)$ in Table 7.1 was only able to compare x and y on the basis of their similarity to one another or if they were an insertion operation, the extended score function such as the one in s_{ext} in Table 7.5 is also able to differentiate the type of cognitive operations passed as input. The cost of a mismatch is set to some function of the two inputs, in this case their average insertion costs.[1] This change necessitates a slightly different recursion because $D_{0,j}$ and $D_{i,0}$ are now determined by the cost of inserting a specific cognitive operation at a given point, rather than a single fixed insertion cost.

$$D_{0,0} = 0$$

$$D_{0,j} = D_{0,j-1} + s_{\text{ext}}(a_{j-1},\texttt{insert})$$

$$D_{i,0} = D_{i-1,0} + s_{\text{ext}}(b_{i-1},\texttt{insert})$$

$$D_{i,j} = \min \begin{pmatrix} D_{i-1,j-1} & + & s_{\text{ext}}(a_i,b_j) \\ D_{i-1,j} & + & s_{\text{ext}}(a_i,\texttt{insert}) \\ D_{i,j-1} & + & s_{\text{ext}}(\texttt{insert},b_j) \end{pmatrix}$$

This approach allows us to penalise the insertion of certain cognitive operations which are considered improbable to occur in human reasoning (such as the `th` operation), it allows researchers to be specific about the relative likelihood of certain operations being inserted, or can be used to highly favour certain cognitive motifs (for example, by rewarding matches of `wc` and `semantic` operations, the recursion now implicitly prefers solutions which make use of the WCS framework).

The scoring matrices for $\text{align}_{\text{ext}}(\mu_{D,3}, \mu_{D,7})$ and $\text{align}_{\text{ext}}(\mu_{D,3}, \mu'_{D,7})$ are provided in the Appendix (Table 1 and 2) for the interested reader.

---

[1]The alternative to this approach is to have a second table which describes the mismatch cost of every possible pairing of cognitive operation inputs.

It is my belief that this technique of SCP scoring and comparison offers a great deal of insight into which cognitive operations are to be preferred, and, over a sufficiently large number of well-founded SCPs for different cognitive tasks, may help to shape our future theories about what general processing motifs really inform certain aspects of human cognition.

7.3.2 Limitations of the Needleman-Wunsch Algorithm

The Needleman-Wunsch algorithm is generally considered a good introductory algorithm to those learning bioinformatics, but it lacks many of the useful features of more advanced algorithms in the field. In a similar way, it was used here to show the suitability of similarity scoring algorithms to SCPs, logical extensions to more advanced string-matching algorithms are easily imagined.

A significant limitation of the Needleman-Wunsch algorithm, and by extensions our algorithm, is the inability to deal with crossovers. That is, the algorithm cannot handle cases where operators or groups of operators are shifted into a new position (imagine cutting the middle out of a piece of string and retying it to the end of the string). This limitation is imposed by many such algorithms to limit the computational complexity of the computation, without this limitation string matching becomes NP-hard.

The Smith-Waterman algorithm (Smith, Waterman, et al., 1981) is a simple variation on the Needleman-Wunsch algorithm and is used to find optimal local alignments. It could be used to find subsequences of complex actions that are preserved between different SCPs. These subsequences (for example the pCTM (`addAB` $\longmapsto$ `wc`) may be shown to be present in the models of several different cognitive tasks, and may show support for common motifs in general human cognition.

Chapter 8

Program Design

8.1 Implementing the SCP Framework

The SCP framework has been implemented in `Python 3` as a series of modules which follow strong object-oriented programming principles and which support simple extensions and revisions. This implementation is used to demonstrate the practical approaches to generating and evaluating SCPs; and to model a series of cognitive tasks for which empirical results already exist. The library containing this implementation and its associated documentation is available at github.com/AxelInd/SCP_Implementation.

At present, concrete implementations for epistemic state structures and cognitive operation formulations related to propositional logic, the WCS, and Reiter's default logic are provided.

8.2 A Modular Programming Approach

The principle of modular programming is followed in this implementation to the greatest extent possible, thus maximising the ease of modification and upkeep for the program. Figure 8.1 provides a simplified class diagram to explain how the major components of the system relate to one another.

8.2.1 Most Important Modules

- `SCPFramework/basicLogic`: an implementation of basic two and three-valued logic.

- `SCPFramework/SCP_Task`: implements both the SCP Task (Π) object and a search function for generating SCPs which meet the given goals.

- `SCPFramework/CTM`: implements the CTM π and defines the $J[p, m]$, the application of a cognitive operation to an input state point

- `SCPFramework/CognitiveOperations`: defines the set of cognitive operations as well as the their input and output parameters. This module can be extended to define other well-founded cognitive operations.

- `SCPFramework/StatePointOperations`: defines $f(\pi) \models_{\text{strict}} \gamma$, and $f(\pi) \models_{\text{weak}} \gamma$ determining if a given SCP meets a given set of output criteria.

- `SCPFramework/truthTables`: a set of truth tables for each operator defined in `basicLogic`.

- `ScoringAlgorithms/SCP_scoring`: an implementation of the Needleman-Wunsch Algorithm for SCPs described in Chapter 7

- `ScoringAlgorithms/SCP_scoring_improved`: an implementation of the extended Needleman Wunsch Algorithm for SCPs described in Chapter 7
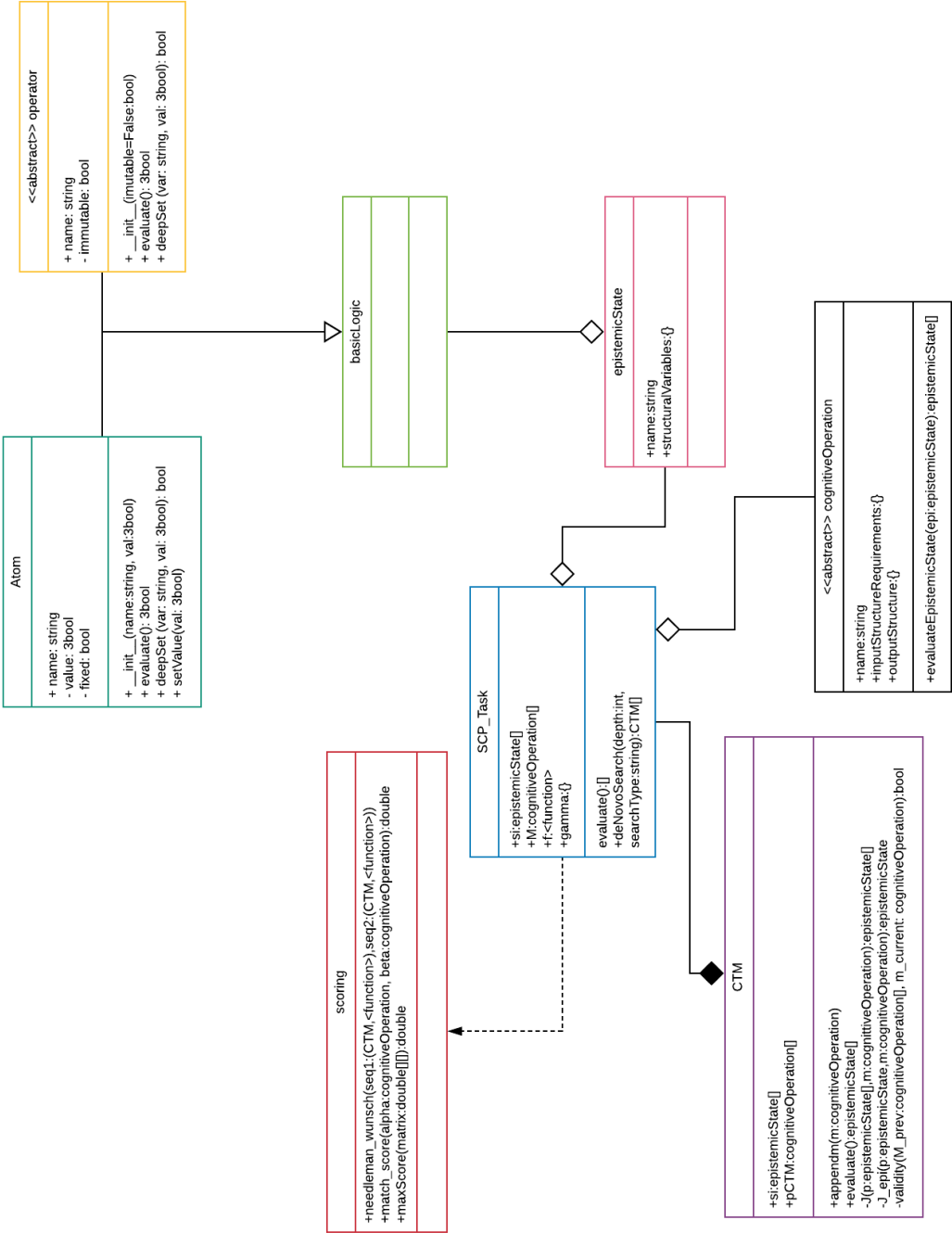
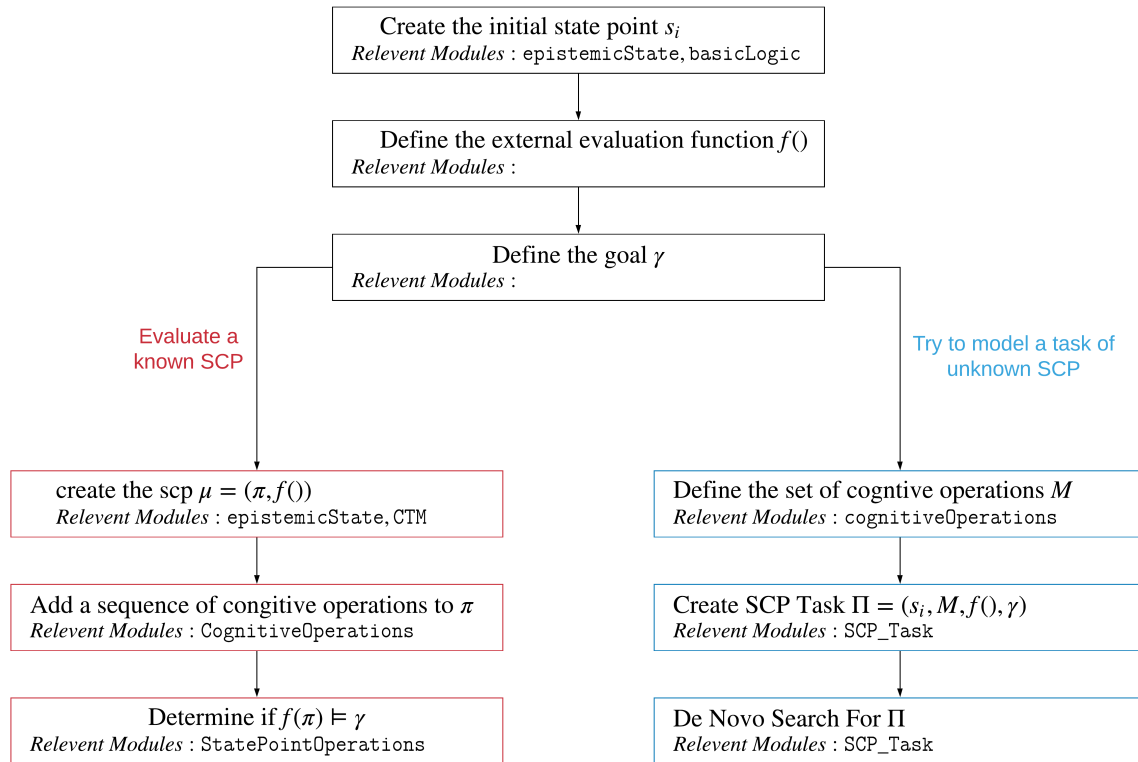FIGURE 8.1: Class diagram for the implementation of the SCP Framework.

FIGURE 8.2: Two use cases for the implementation of the SCP framework

8.2.2 Program Flow

Diagram 8.2 illustrates the basic programmatic flow used to model either a known SCP or known empirical result set using the SCP Framework. In the appendix, Figure 3 provides a code snippet to show exactly how the general case of the Suppression Task is modelled in the SCP framework.

8.2.3 Modelled Experiments

- `example_wcs_suppressionTask.py`: models the Suppression Task for both general and individual cases as discussed in Chapter 6.

- `example_wcs_wst.py`: models the Wason Selection Task for all four canonical cases of the abstract formulation as discussed in Chapter 6.

- `example_default_various.py`: models of the Suppression Task, Penguin Case, and Nixon Diamond implemented with a default-logic capable SCP.

8.3 External Framework Integration

At present work is ongoing to integrate the SCP framework with the `CCOBRA` (Nicolas, 2018) cognitive modelling environment.

Chapter 9

Conclusions and Future Work

9.1 Conclusion

This thesis has introduced, and implemented, a new computational framework into the field of cognitive modelling which is able to capture much of the nuance of other non-monotonic frameworks, whilst still maintaining a consistent technique for task formulation and evaluation.

Over the course of this thesis we have: introduced three levels of granularity into the description of cognitive tasks by formalising the mathematical structure of SCP Plans, SCPs, and Realised SCPs; established a viable search procedure for generating new SCPs from SCP Tasks; defined a set of cognitive operations that can be used for modelling both general and individual case of the Wason Selection Task and Suppression Task using the Weak Completion Semantics; shown that the intuition and functionality of both the WCS and Reiter's Default Logic can be encoded using the SCP framework; modelled the WST and Suppression Task using entirely novel approaches for both the general and individual reasoner; established both theoretical and empirical evidence for SCPs as models of human cognition; and implemented a robust scoring system for comparing SCPs under the assumption that they both represent deviations from an underlying cognitive process.

SCPs represent a novel and powerful framework for modelling non-monotonic logics. They have been shown capable of modelling the Suppression Task and Wason Selection under the WCS for both general and individual cases. SCPs provide a dynamic framework incorporating cognitive operations which are applicable across different logics, provided that those logics share structural features that are compatible with some subset of the known set of cognitive operations. Because the input and output epistemic states need not be structurally similar, SCPs may represent the first approach to modelling human cognition that is able to integrate multiple non-monotonic logic frameworks at run-time and at search-time.

SCPs represent a new frontier for Cognitive Modelling in non-monotonic logics and research into their capabilities and limitations may help create more robust and mathematically consistent explanations and predictions for human behaviour across an extensive array of cognitive tasks.

9.2 Future Work

One of the most obvious extensions of the SCP framework, is the creation of cognitive operations which are able to model first-order logical systems, rather than the propositional systems to which this thesis has been restricted. The introduction of techniques from machine learning, artificial intelligence planning and reinforcement learning would allow for several desirable extensions of the SCP Framework, including but not limited to: more empirically well-founded score tables when comparing SCPs, the generation of heuristic techniques to search through SCPs and SCP Plans with large branching factors, and the development of

techniques to generate novel cognitive processes which are evidenced in SCPs across different cognitive tasks. From a more theoretical outlook, formal properties of SCPs which combine techniques from several cognitive models should be identified and described. From a practical perspective, integration of the implemented SCP framework with existing modelling tools like CCOBRA will make the framework far more accessible to researchers who wish to generate models for observed cognitive phenomena.

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Appendix

Definitions and Proofs

Definition 1. *The expansion operation **exp** operation precisely substitutes a pCTM for its constituent elements as follows:*

1. *Let F be a pCTM such that: $F = (A \longmapsto B \longmapsto C)$, with $A \in \Omega^*$, $B \in \Omega^*$, $C \in \Omega^*$, and $B = (B_1 \longmapsto ... \longmapsto B_n)$, $B_i \in \Omega^*$.*

2. *Then $F = (A \longmapsto exp(B) \longmapsto C) = (A \longmapsto B_1 \longmapsto ... \longmapsto B_n \longmapsto C)/$*

Definition 2. *Given a cognitive operation sequence $A \in \Omega^*$, a final state dependant external evaluation function $f()$, and goal γ, A is **redundant** iff one of the following holds:*

- *$f(s_i \longmapsto exp(B) \longmapsto exp(C)) \models x'$ and $f(s_i \longmapsto exp(A)) \models x'$ for every viable epistemic state x, $B \in \Omega^*$, $C \in \Omega^*$, where neither B nor C is the empty set or the trivial operation T (See Definition 1, for a definition of the exp operation).*

- *$f(s_i \longmapsto exp(A)) = f(s_i)$ for every viable state point s_i.*

- *$f(s_i \longmapsto exp(B)) = c$ for all $B \in \Omega^*$, where c is constant.*

Definition 3. *Given a limited set of cognitive operations $M = \{A_0, ..., A_n\}$, $A_i \in \Omega^*$, a final state dependant external evaluation function $f()$, and goal γ, $A \in M^*$ is **task redundant** iff one of the following holds:*

- *$f(s_i \longmapsto exp(B) \longmapsto exp(C)) \models x'$ and $f(s_i \longmapsto exp(A)) \models x'$ for every viable epistemic state x, $B \in M^*$, $C \in M^*$, where neither B nor C is the empty set or the trivial operation T (See Definition 1, for a definition of the exp operation). .*

- *$f(s_i \longmapsto exp(A)) = f(s_i)$ for every viable state point s_i.*

- *$f(s_i \longmapsto exp(B)) = c$ for all $B \in M^*$, where c is constant.*

- *There exists no sequences $B \in M^*$, $C \in M^*$, and state point s_i such that $\mu = (\pi, f())$ is a valid SCP, with $\pi = \{s_i \longmapsto B \longmapsto A \longmapsto C)\}$.*

Lemma 1. *Let $\mu = (\pi, f())$ be a monotonic SCP, with $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$, $m_i \in \Omega^*$. $r = (k \in K[\pi], f())$ is the only realised SCP of μ, with $K[\pi] = \{s_i \longmapsto (m_1, \bar{p}_1) \longmapsto ... \longmapsto (m_n, \bar{p}_n)\}$, and for every pair $(m_i, \bar{p}_i)$, $\bar{p}_i \in_s J[\bar{p}_{i-1}, m_i]$. Then it is trivially easy to derive μ from r, and vice versa.*

Proof.

- Determining r from μ:

  – Let $R' = \{k \in K[\pi], f()\}$ be the set of all realised SCPs of μ.

  – μ is monotonic and, by definition, R' is either the empty set $\varnothing$ or contains only single realised SCP $r' = (k, f())$ with known structure for π.

– Because every realised SCP for μ must be in R', and r is realised SCP for μ, it follows that $r \in R'$.

– Now, $r \in R'$, $r' \in R'$, and R' contains only a single realised SCP.

– Thus, $r = r'$. And because the structure of r' is known, the structure of r is known.

• Determining μ from r:

– The realised SCP r, by definition, contains both the details and relative location of each $m_i \in \pi$ which occurs in the SCP which created it.

– if $(m_i, \cdot) \prec (m_j, \cdot)$ in k, then $m_i \prec m_j$ in π.

– The initial state point s_i is given by the first element in k.

– Thus, all information needed to generate π is present in r.

– The external activation, $f()$, of r is, by definition, the same as that of the SCP which generated it.

– Thus $f()$ and π are known for the SCP μ.

– $\therefore \mu = (\pi, f())$ has been determined from r.

• Determining μ from r:

– Lemma 2 states that μ can be determined from any realised SCP that can be generated from π, this remains true when there is only one realised SCP.

$\square$

Lemma 2. *Let* $\mu = (\pi, f())$ *be an SCP, with* $\pi = (s_i \longmapsto m_1 \longmapsto ... \longmapsto m_n)$, $m_i \in \Omega^*$. $R = \{k \in K[\pi], f()\}$ *is the set of realised SCP of* μ, *with* $K[\pi] = \{s_i \longmapsto (m_1, \bar{p}_1) \longmapsto ... \longmapsto (m_n, \bar{p}_n)\}$ *and for every pair* $(m_i, \bar{p}_i)$, $\bar{p}_i \in_s J[\bar{p}_{i-1}, m_i]$. *Then it is trivially easy to derive* μ *from R.*

Proof.

• Let some $r \in R$ be a realised SCP of π.

• The realised SCP r, by definition, contains both the details and relative location of each $m_i \in \pi$ which occurs in the SCP which created it.

• if $(m_i, \cdot) \prec (m_j, \cdot)$ in k, then $m_i \prec m_j$ in π.

• The initial state point s_i is given by the first element in k.

• Thus, all information needed to generate π is present in r.

• The external activation, $f()$, of r is, by definition, the same as that of the SCP which generated it.

• Thus $f()$ and π are known for the SCP μ.

• $\therefore \mu = (\pi, f())$ has been determined from r.

$\square$

Lemma 3. *Given a set of cognitive operations M, an external activation function* $f()$, *an initially valid CTM* $\pi = \{s_i \longmapsto m_0 \in M \longmapsto ... \longmapsto m_n \in M\}$, *and a valid SCP* $\mu = (\pi, f)$, *inserting multiple operations into* π *to create the CTM* π'' *may result in a valid SCP* $\mu'' = (\pi'', f())$, *when inserting only one operation to create* $\mu' = (\pi', f())$ *would result in an invalid SCP.*

Proof.

1. Assume hybrid validity (Section 4.5.1).

2. Let $\pi = (s_i)$.

3. Let s_i be of structure (S)

4. Let $f(\pi) = \top$.

5. Let $\gamma = \top$.

6. Let $\mu = (\pi, f())$ be a valid SCP.

7. Let $m_1 \in M$ be a cognitive operation with output structure (S). And precondition structure $\chi = \{S, V\}$.

8. Let $m_2 \in M$ be a cognitive operation with output structure (S, V). And precondition structure $\chi = \{\}$.

9. Then $\mu' = (\pi', f()), \pi' = (s_i \longmapsto m_1)$ is not a valid SCP.

10. However, $\mu'' = (\pi'', f()), \pi'' = (s_i \longmapsto m_2 \longmapsto m_1)$ is a valid SCP.

$\square$

Lemma 4. *There are an infinite number of possible, valid SCPs, that can meet any goal condition γ from input s_i provided that at least one SCP exists that can reach a final state dependent goal condition γ from input s_i.*

Proof.

1. There exists an infinite set $M^{\text{add}} \in \Omega$ of unique cognitive operations which add a new variable from the set of all possible variable names $p \in \Sigma$.

2. There exists an infinite set $M^{\text{rem}} \in \Omega$ of unique cognitive operations which remove a variable from the set of all possible variable names $p \in \Sigma$.

3. It follows that there exist infinite set of pairs M^{addRem}, with $(m_i^{\text{add}} \in M^{\text{add}}, m_i^{\text{rem}} \in M^{\text{rem}}) \in W$ where m_i^{add} adds a variable $p \in \Sigma$ to the resulting state point, and m_i^{rem} removes that variable from the resulting statepoint.

4. Then $f(x \longmapsto A) = f(s_i \longmapsto m_i^{\text{add}} \longmapsto m_i^{\text{rem}} \longmapsto A)$ for all $(m_i^{\text{add}}, m_i^{\text{rem}}) \in M^{\text{addRem}}$.

$\square$

Lemma 5. *If there exists an SCP which meets a goal condition γ of length n, using a final state dependant external evaluation function $f()$ then there exists an SCP that meets γ of length $n + k$ where k is any natural number.*

Proof.

- Proof 9.2 states that if $f(s_i \longmapsto A) \models \gamma$ then $f(s_i \longmapsto m_i^{\text{add}} \longmapsto m_i^{\text{rem}} \longmapsto A) \models \gamma$ for all $(m_i^{\text{add}}, m_i^{\text{rem}}) \in M^{\text{addRem}}$.

- Let X be a pCTM of the form $X = (m_0^{\text{add}} \longmapsto m_0^{\text{rem}} \longmapsto \ldots \longmapsto m_N^{\text{add}} \longmapsto m_N^{\text{rem}})$ where $(m_i^{\text{add}}, m_i^{\text{rem}}) \in M^{\text{addRem}}$.

- For odd lengths:

  1. It follows that if $f(s_i \longmapsto A) \models \gamma$ then $f(x \longmapsto A \longmapsto \exp(X)) \models \gamma$ (Definition 1).

  2. Because X always contains an even number of cognitive operations, the length of the resulting SCP must be $n' = 1 + v$ where v is any even number.

- For even lengths:

  1. Let $T \in \Omega$ be the trivial operation, that is, it returns the input state point as output.

  2. Notice that $f(x \longmapsto A \longmapsto T) \models \gamma$ iff $f(x \longmapsto A) \models \gamma$.

  3. Thus, $f(x \longmapsto A \longmapsto \exp(X) \longmapsto T) \models \gamma$ and is of length $n' = 2 + v$, where v is any even number.

$\square$

Lemma 6. *Given an SCP* $\mu = (\pi, f())$, $\pi = (s_i \longmapsto A_0 \longmapsto \ldots \longmapsto A_n)$, *where* $f()$ *is a final state dependant external evaluation function,* s_i *is a state point, and* $A_i \in \Omega^*$, *any subsequence* $A' = (A_k \longmapsto \ldots \longmapsto A_{k+l})$, $(k+l) < n$ *of the cognitive operations in* π *is a valid pCTM and, thus, a valid aggregate cognitive operation.*

Proof.

1. Let $p_i = J[p_{i-1}, A_{i-1}]$, when $i \geq 1$.

2. Let $p_0 = s_i$.

3. Given that μ is a valid SCP, every $J[p_i, A_i]$ must take a valid state point as input p_i and result in a valid state point as output p_{i+1}.

4. It follows that A' takes a valid state point as input, because A_k took a valid state point as input.

5. It follows that A' produces a valid state point as output, because A_{k+l} produced a valid state point as output.

6. Therefore, A' is a valid cognitive operation.

$\square$

Lemma 7. *Given an SCP Task* $\Pi = (s_i, f(), M, \gamma)$ *in which* $M = \{A_0, \ldots, A_n\}$, $A \in \Omega^*$ *and a second SCP Task* $\Pi' = (s_i, M', f(), \gamma)$ *in which* $M' = (M \setminus \{A_k, \ldots, A_m\} \cup A')$, *where* $A' = (A_k \longmapsto \ldots \longmapsto A_m)$ *is a pCTM, then some ordering of the other operations in* $(\Pi \setminus \Pi')$, Π' *is as expressive or less expressive than* Π.

Proof.

No more expressive:

1. Let $\mu' = (\pi', f())$, with $\pi' = (s_i \longmapsto A_p \longmapsto \ldots \longmapsto A' \longmapsto \ldots \longmapsto A_q)$, be a solution to Π'.

2. Let $\mu'' = (\pi'', f())$, with $\pi'' = (s_i \longmapsto A_p \longmapsto \ldots \longmapsto \exp(A')] \longmapsto \ldots \longmapsto A_q)$.

3. Then μ'' is a solution to Π' (Definition 1).

4. μ'' is a valid solution to Π because π'' uses only cognitive operations which occur in Π, s_i is the initial epistemic state, and $f(\pi'') \models \gamma$.

Less Expressive: Proof by counterexample

1. If we assume that Π' is strictly as expressive or more expressive than Π, then there exist no cases in which Π' is less expressive.

2. Let $\gamma = \top$, and $f(s_i) = \top$ for all valid input π in which no cognitive operation occurs more than once.

3. Let $s_i = \{\}$, $M = \{A_0, A_1\}$, $M' = \{A'\}$, $A' = \{A_0 \longmapsto A_1\}$.

4. Then satisfying SCPs of Π are $((s_i), f())$, $((s_i \longmapsto A_0), f())$, $((s_i \longmapsto A_1), f())$, $((s_i \longmapsto A_0 \longmapsto A_1), f())$ and $((s_i \longmapsto A_1 \longmapsto A_0), f())$.

5. Satisfying SCPs of Π' are $((s_i), f())$, and $((s_i \longmapsto A'), f())$.

6. Thus, Π' has fewer solutions that Π. A contradiction.

$\square$

```
=================ABDUCIBLE SUPPRESSION=====================
Epistemic state:

===>el<===
S:: [(e ↔ T), (l ↔ (e ∧ (¬ ab_1))), (ab_1 ↔ ⊥)]
Delta:: []
V:: [(e:True), (l:True), (ab_1:False)]
R:: {'abducibles': []}

response(p_bar): She will study late in the library
------------------------------
Epistemic state:

===>el<===
S:: [(e ↔ T), (l ↔ (e ∧ (¬ ab_1))), (ab_1 ↔ ⊥), (o ↔ T)]
Delta:: []
V:: [(e:True), (l:True), (ab_1:False)]
R:: {'abducibles': [(o ← T)]}

response(p_bar): She will study late in the library
------------------------------
Epistemic state:

===>el<===
S:: [(e ↔ T), (l ↔ (e ∧ (¬ ab_1))), (ab_1 ↔ ⊥), (o ↔ ⊥)]
Delta:: []
V:: [(e:True), (l:True), (ab_1:False)]
R:: {'abducibles': [(o ← ⊥)]}

response(p_bar): She will study late in the library
------------------------------
Epistemic state:

===>el<===
S:: [(e ↔ T), (l ↔ (e ∧ (¬ ab_1))), (ab_1 ↔ ⊥), (o ↔ (T ∨ ⊥))]
Delta:: []
V:: [(e:True), (l:True), (ab_1:False)]
R:: {'abducibles': [(o ← T), (o ← ⊥)]}

response(p_bar): She will study late in the library
------------------------------
Epistemic state:

===>elo<===
S:: [(e ↔ T), (l ↔ ((e ∧ (¬ ab_1)) ∨ (o ∧ (¬ ab_2)))), (ab_1 ↔ (¬ o)), (ab_2 ↔ (¬ e))]
Delta:: []
V:: [(e:True), (l:None), (o:None), (ab_1:None), (ab_2:False)]
R:: {'abducibles': []}

response(p_bar): We are uncertain if she will study late in the library
------------------------------
Epistemic state:
```

FIGURE 1: Epistemic states demonstrating that `addExp` can be used to model the individual case of the Suppression Task where Suppression is prevented. Shown for SCP $\mu = (\pi, f_{\text{sup}}())$, $\pi = (s_{\text{dev}} \longmapsto \texttt{addAB} \longmapsto \texttt{addExp} \longmapsto \texttt{wc} \longmapsto \texttt{semantic})$. Part1.

```
===>elo<===
S:: [(e ↔ T), (l ↔ ((e ∧ (¬ ab_1)) ∨ (o ∧ (¬ ab_2)))), (ab_1 ↔ (¬ o)), (ab_2 ↔ (¬ e)), (o ↔ T)]
Delta:: []
V:: [(e:True), (l:True), (o:True), (ab_1:False), (ab_2:False)]
R:: {'abducibles': [(o ← T)]}

response(p_bar): She will study late in the library
------------------------------
Epistemic state:

===>elo<===
S:: [(e ↔ T), (l ↔ ((e ∧ (¬ ab_1)) ∨ (o ∧ (¬ ab_2)))), (ab_1 ↔ (¬ o)), (ab_2 ↔ (¬ e)), (o ↔ ⊥)]
Delta:: []
V:: [(e:True), (l:False), (o:False), (ab_1:True), (ab_2:False)]
R:: {'abducibles': [(o ← ⊥)]}

response(p_bar): She will not study late in the library
------------------------------
Epistemic state:

===>elo<===
S:: [(e ↔ T), (l ↔ ((e ∧ (¬ ab_1)) ∨ (o ∧ (¬ ab_2)))), (ab_1 ↔ (¬ o)), (ab_2 ↔ (¬ e)), (o ↔ (T ∨ ⊥))]
Delta:: []
V:: [(e:True), (l:True), (o:True), (ab_1:False), (ab_2:False)]
R:: {'abducibles': [(o ← T), (o ← ⊥)]}

response(p_bar): She will study late in the library
------------------------------
predictions:  {'el': ['She will study late in the library', 'She will study late in the library', 'She will study late in the
library', 'She will study late in the library'], 'elo': ['We are uncertain if she will study late in the library', 'She will
study late in the library', 'She will not study late in the library', 'She will study late in the library']}

f(pi) models gamma_noSup? :  True
```

FIGURE 2: Epistemic states demonstrating that `addExp` can be used to model the individual case of the Suppression Task where Suppression is prevented. Shown for SCP $\mu = (\pi, f_{\sup}())$, $\pi = (s_{\mathrm{dev}} \longmapsto \texttt{addAB} \longmapsto \texttt{addExp} \longmapsto \texttt{wc} \longmapsto \texttt{semantic})$. Part2.

```python
def standardSuppression():
    print ("==============================================================")
    print ("================STANDARD SUPPRESSION======================")
    print ("==============================================================")
    #The epistemic state for the initial state of a reasoner who only knows
    #conditional ( l | e )
    basePoint1=epistemicState.epistemicState('el')
    delta1=["( l | e )"]
    S1 = ["( e <- T )"]
    delta1AsLogic = scpNotationParser.stringListToBasicLogic(delta1)
    S1AsLogic = scpNotationParser.stringListToBasicLogic(S1)
    basePoint1['S']=S1AsLogic
    basePoint1['Delta']=delta1AsLogic
    basePoint1['V']=[e,l]

    #The elo case expressed as the addition of information to the el case
    basePoint2=copy.deepcopy(basePoint1)
    basePoint2.setName('elo')
    #The possible starting states for the SCP
    extraConditional=["( l | o )"]
    extraConditionalAsLogic = scpNotationParser.stringListToBasicLogic(extraConditional)
    basePoint2['Delta']=basePoint2['Delta']+extraConditionalAsLogic
    basePoint2['V']=basePoint2['V']+[o]

    #Create the first state point
    statePoints=[basePoint1,basePoint2]
    s_i=statePoints

    f=f_suppression_studyLate
    #The desired output of the external evaluation function
    gamma={'el':'She will study late in the library',
           'elo':'We are uncertain if she will study late in the library'}

    #test ctm
    c = CTM.CTM()
    c.setSi(s_i)
    c.appendm(ADDAB)
    c.appendm(WC)
    c.appendm(SEMANTIC)

    predictions = f(c)
    print ('predictions: ', predictions)
    print ("Lenient Interp")
    print (StatePointOperations.predictionsModelsGamma_lenient(predictions,gamma))
    print("Strict Interp")
    print (StatePointOperations.predictionsModelsGamma_strict(predictions,gamma))
    print ("f(pi) models gamma_Sup? : ",
           (StatePointOperations.predictionsModelsGamma_lenient(predictions,gamma)))
```

FIGURE 3: Code snippet showing how the Suppression Task is modelled in the SCP Framework.

Additional Tables

	s_{WST}	addAB	AddExp	wc	semantic
	0	-1	-2	-3	-4	-5
s_{WST}	-1	3	2	1	0	-1
th	-6	-2	0.0	-1.0	-2.0	-3.0

TABLE 1: Extended Needleman-Wunsch comparison of $\mu_{D,3}$ to $\mu'_{D,7}$.

	s_{WST}	addAB	AddExp	wc	semantic
	0	-1	-2	-3	-4	-5
s_{WST}	-1	3	2	1	0	-1
addAB	-2	2	4	3	2	1
addExp	-3	1	3	5	4	3
wc	-4	0	2	4	6	5
semantic	-5	-1	1	3	5	7

TABLE 2: Extended Needleman-Wunsch comparison of $\mu_{D,3}$ to $\mu_{D,7}$.