Approximation Theory for Distant Bang Calculus

Kostia Chardonnet, Jules Chouquet, Axel Kerinec

$$(\Lambda) \qquad M, N ::= x \mid \lambda x.M \mid (MN)$$

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$$(\lambda x.M)N \mapsto_\beta M\{N/x\}$$

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Reaching normal forms?

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$$\mathsf{I} = \lambda x.x \qquad \mathsf{I}x \to_\beta x$$
$$\Omega = (\lambda x.xx)(\lambda x.xx) \qquad \Omega \to_\beta \Omega \to_\beta \cdots \to_\beta \Omega \to_\beta \cdots$$

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Reaching normal forms?

$$
\begin{aligned}
\mathsf{I} &= \lambda x.x & \mathsf{I}x &\to_\beta x \\
\Omega &= (\lambda x.xx)(\lambda x.xx) & \Omega &\to_\beta \Omega \to_\beta \cdots \to_\beta \Omega \to_\beta \cdots \\
\mathsf{Y} &= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) & \mathsf{Y} &\to_\beta \lambda f.(f((\lambda x.f(xx))(\lambda x.f(xx)))) \\
& & &\to_\beta \lambda f.(f(f((\lambda x.f(xx))(\lambda x.f(xx))))) \\
& & &\twoheadrightarrow_\beta \lambda f.(f(f(f(\cdots))))
\end{aligned}
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Y &= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) & Y &\to_\beta \lambda f.(f((\lambda x.f(xx))(\lambda x.f(xx)))) \\
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& & &\twoheadrightarrow_\beta \lambda f.(f(f(f(\cdots))))
\end{aligned}
$$

Definition (Solvability)

M is solvable: $\exists x_1, \ldots x_n, M_1, \ldots, M_k$ s.t. $(\lambda x_1 \ldots x_n.M)M_1 \cdots M_k \twoheadrightarrow_\beta I$

Definition (Böhm tree of M)

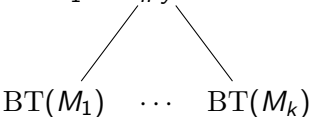- If $M \twoheadrightarrow_\beta \lambda x_1 \ldots x_n.y\, M_1 \cdots M_k$ then

$$\mathrm{BT}(M) = \lambda x_1 \ldots x_n.y$$

$$\mathrm{BT}(M_1) \quad \cdots \quad \mathrm{BT}(M_k)$$

- Otherwise

$$\mathrm{BT}(M) = \bot$$

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$$
\begin{aligned}
Y &= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \\
&\twoheadrightarrow_\beta \lambda f.(f(f(\cdots)))
\end{aligned}
$$

$$\mathrm{BT}(Y)$$

$$\|$$

$$\lambda f$$

$$|$$

$$f$$

$$|$$

$$f$$

$$\vdots$$

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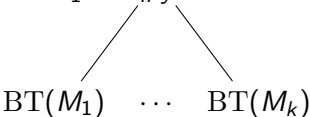
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$$

$$\mathrm{BT}(Y)$$

$$\shortparallel$$

$$\lambda f$$

$$|$$

$$f$$

$$|$$

$$f$$

$$\vdots$$

Theorem (Solvability and Böhm Tree)

M *insolvable iff* $\mathrm{BT}(M) = \bot$.

Function	
$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$	

Function	λ-calculus
$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$	$(\lambda x.M)N = \sum_{n=0}^{\infty} \frac{1}{n!}(\lambda x.M)[\underbrace{N, \ldots, N}_{n}]$

A differential λ-calculus:

- resource-sensitive: in N can only replace one occurrence of x
- strongly normalising: each resource term has a normal form

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Theorem (Commutation [Ehrhard and Regnier 08])

$$\mathrm{nf}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$$

Call-by-Name

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$(\lambda x.xx)((\lambda y.M)N)$

Evaluation Strategies

Call-by-Name

$$(\lambda x.xx)((\lambda y.M)N) \quad \rightarrow_\beta \quad ((\lambda y.M)N)((\lambda y.M)N)$$

Call-by-Name

$$(\lambda x.xx)((\lambda y.M)N) \quad \begin{aligned} &\to_\beta \quad ((\lambda y.M)N)((\lambda y.M)N) \\ &\to_\beta \quad M\{N/y\}((\lambda y.M)N) \end{aligned}$$

Call-by-Name

$$
\begin{aligned}
(\lambda x.xx)((\lambda y.M)N) \quad &\rightarrow_\beta \quad ((\lambda y.M)N)((\lambda y.M)N) \\
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Call-by-Value

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$$V, U ::= x \mid \lambda x.M$$
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Solvability $\Rightarrow$ PROBLEM

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Solvability $\Rightarrow$ PROBLEM

Definition (Srcutability)

M is scrutable:

$$\exists x_1, \ldots x_n, M_1, \ldots, M_k, V \; s.t. \; (\lambda x_1 \ldots x_n.M)M_1 \cdots M_k \twoheadrightarrow_{\beta_v} V$$

- Call-by-Push-Value (CbPV) [Levy 99]

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- untyped version = Bang-calculus [Ehrhard and Guerrieri 16].

Definition (Bang)

Terms $\quad M, N := x \mid MN \mid \lambda x.M \mid !M \mid \mathbf{der}(M)$

Values $\quad V, W := x \mid !M$

Reduction rules:
$$
\begin{array}{lll}
(\lambda x.M)(V) & \rightarrow_b & M\{V/x\} \\
\mathbf{der}(!M) & \rightarrow_b & M
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$\Rightarrow$ PROBLEM
Stuck redexes in CbV and Bang

λ_v^σ-calculus

Permutation Rules

$$(\lambda x.M)NN' \quad \mapsto_{\sigma_1} \quad (\lambda x.MN')N$$
$$V((\lambda x.M)N) \quad \mapsto_{\sigma_3} \quad (\lambda x.VM)N$$

λ_v^σ-calculus

Permutation Rules		[Carraro and Guerrieri 14]

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$$\mathcal{T}(\mathrm{BT}_V(M)) = NF(\mathcal{T}(M))$$

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Theorem (Scrutability and BT)

M unscrutable iff $\mathrm{BT}(M) = \emptyset$.

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Theorem (Scrutability and BT)

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$$\Rightarrow \text{PROBLEM}$$
Bang extension difficult

Definition (Distant Mechanism)

List contexts: $L := < \cdot > \mid L[M/x]$

Distant Versions

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Reductions: $L\langle\lambda x.M\rangle N \to L\langle M[N/x]\rangle \qquad M[L\langle V\rangle/x] \to L\langle M\{V/x\}\rangle$

Without capture of free variables.

Distant Versions

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Without capture of free variables.

Distant CbV [Accatoli and Paolini 12],
Distant CbN/Distant Bang [Bucciarelli, Kesner, Ríos and Viso 23]

Link between them

Translation into dBang [Bucciarelli, Kesner, Ríos and Viso 23]:

CbN	CbV
$x^n = x$	$x^v = \,!x$
$(\lambda x.M)^n = \lambda x M^n$	$(\lambda x.M)^v = \,!(\lambda x.M^v)$
$(M\ N)^n = M^n!N^n$	$(M\ N)^v = \begin{cases} L\langle P \rangle N^v & \text{if } M^v = L\langle !P \rangle \\ \mathbf{der}(M^v)\ N^v & \text{otherwise} \end{cases}$
$(M[N/x])^n = M^n[!N^n/x]$	$(M[N/x])^v = M^v[N^v/x]$

Link between them

Translation into dBang [Bucciarelli, Kesner, Ríos and Viso 23]:

CbN	CbV
$\begin{aligned} x^n &= x \\ (\lambda x.M)^n &= \lambda x M^n \\ (M\ N)^n &= M^n ! N^n \\ (M[N/x])^n &= M^n[! N^n/x] \end{aligned}$	$\begin{aligned} x^v &= !x \\ (\lambda x.M)^v &= !(\lambda x.M^v) \\ (M\ N)^v &= \begin{cases} L\langle P\rangle N^v & \text{if } M^v = L\langle !P\rangle \\ \mathbf{der}(M^v)\ N^v & \text{otherwise} \end{cases} \\ (M[N/x])^v &= M^v[N^v/x] \end{aligned}$

Theorem (Translation of meaningfulness [Kesner, Arrial and Guerrieri 24])

- M is dCbN-meaningful iff M^n is meaningful.
- M is dCbV-meaningful iff M^v is meaningful.

Definition (Distant Bang)

Terms $M, N := x \mid MN \mid \lambda x.M \mid !M \mid \textbf{der}(M) \mid M[N/x]$
(Lists Contexts $L :=< \cdot > \mid L[M/x]$

Reduction rules:

$$
\begin{aligned}
L\langle \lambda x.M \rangle N &\mapsto_{db} L\langle M[N/x] \rangle \\
M[L\langle !N \rangle /x] &\mapsto_{db} L\langle M\{N/x\} \rangle \\
\textbf{der}(L\langle !M \rangle) &\mapsto_{db} L\langle M \rangle
\end{aligned}
$$

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$$M[L\langle !N\rangle/x] \quad \mapsto_{db} \quad L\langle M\{N/x\}\rangle$$
$$\textbf{der}(L\langle !M\rangle) \quad \mapsto_{db} \quad L\langle M\rangle$$

Exemple : Let $\textbf{der}(!(\lambda y.\lambda x.x))(\Omega)(!I) \rightarrow_{db} (\lambda y.\lambda x.x)(\Omega)(!I) \rightarrow_{db}$
$(\lambda x.x)[\Omega/y](!I) \rightarrow_{db} x[!I/x][\Omega/y] \rightarrow_{db} I[\Omega/y]$

Ressource calculus for dBang

Definition (Ressource calculus $\delta\mathtt{Bang}$)

(terms) $m, n := x \mid mn \mid \lambda x.m \mid \mathbf{der}(m) \mid m[n/x] \mid [m_1, \ldots, m_k]$

The reduction relation:

- $l\langle \lambda x.m \rangle n \Rightarrow_\delta \{l\langle m[n/x]\rangle\}$

- $m[l\langle [n_1, \ldots, n_k]\rangle/x] \Rightarrow_\delta$
  $\begin{cases} \displaystyle\bigcup_{\sigma \in P_k} l\langle m\{n_{\sigma(1)}/x_1, \ldots, n_{\sigma(k)}/x_k\}\rangle & \text{if } k = d_x(m) \\ \emptyset & \text{otherwise.} \end{cases}$

- $\mathbf{der}(l\langle [m_1, \ldots, m_k]\rangle) \Rightarrow_\delta \{l\langle m_1\rangle\}$ if $k = 1$ and $\emptyset$ otherwise.

Taylor Expansion for dBang

Being an approximant of a term:

$$\frac{}{x \lhd_! x} \qquad \frac{m \lhd_! M}{\lambda x\, m \lhd_! \lambda x\, M} \qquad \frac{m \lhd_! M}{\mathbf{der}(m) \lhd_! \mathbf{der}(M)} \qquad \frac{m \lhd_! M \qquad n \lhd_! N}{mn \lhd_! MN}$$

$$\frac{m \lhd_! M \qquad n \lhd_! N}{m[n/x] \lhd_! M[N/x]} \qquad \frac{m_1 \lhd_! M \quad \cdots \quad m_k \lhd_! M}{[m_1, \ldots, m_k] \lhd_! \, !M} \; (k \in \mathbb{N})$$

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$$\frac{m \lhd_! M \qquad n \lhd_! N}{m[n/x] \lhd_! M[N/x]} \qquad \frac{m_1 \lhd_! M \quad \cdots \quad m_k \lhd_! M}{[m_1, \ldots, m_k] \lhd_! \,!M} \,(k \in \mathbb{N})$$

Definition (Taylor expansion and Taylor normal form))

$$\mathcal{T}(M) = \{ m \in \delta\mathrm{Bang} \mid m \lhd_! M \}$$

Taylor Expansion for dBang

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$$\frac{m \lhd_! M \qquad n \lhd_! N}{m[n/x] \lhd_! M[N/x]} \qquad \frac{m_1 \lhd_! M \quad \cdots \quad m_k \lhd_! M}{[m_1, \ldots, m_k] \lhd_! \,!M} \ (k \in \mathbb{N})$$

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$$\mathcal{T}(M) = \{ m \in \delta\mathtt{Bang} \mid m \lhd_! M \} \qquad NF(\mathcal{T}(M)) = \bigcup_{m \lhd_! M} \mathbf{nf}(m)$$

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$$\frac{m \lhd_! M \qquad n \lhd_! N}{m[n/x] \lhd_! M[N/x]} \qquad \frac{m_1 \lhd_! M \quad \cdots \quad m_k \lhd_! M}{[m_1, \ldots, m_k] \lhd_! \,!M}\ (k \in \mathbb{N})$$

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Being an approximant of a term:

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$$\frac{m \triangleleft_! M \qquad n \triangleleft_! N}{m[n/x] \triangleleft_! M[N/x]} \qquad \frac{m_1 \triangleleft_! M \quad \cdots \quad m_k \triangleleft_! M}{[m_1, \ldots, m_k] \triangleleft_! \,!M} \;(k \in \mathbb{N})$$

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$$\mathcal{T}(M) = \{ m \in \delta\mathrm{Bang} \mid m \triangleleft_! M \} \qquad NF(\mathcal{T}(M)) = \bigcup_{m \triangleleft_! M} \mathbf{nf}(m)$$

$$
\begin{array}{ccc}
M & \rightarrow & N \\
\triangledown & & \triangledown \\
m & \rightarrow & n
\end{array}
$$

Taylor Expansion for dBang

Being an approximant of a term:

$$\frac{}{x \lhd_! x} \qquad \frac{m \lhd_! M}{\lambda x\, m \lhd_! \lambda x\, M} \qquad \frac{m \lhd_! M}{\mathbf{der}(m) \lhd_! \mathbf{der}(M)} \qquad \frac{m \lhd_! M \qquad n \lhd_! N}{mn \lhd_! MN}$$

$$\frac{m \lhd_! M \qquad n \lhd_! N}{m[n/x] \lhd_! M[N/x]} \qquad \frac{m_1 \lhd_! M \quad \cdots \quad m_k \lhd_! M}{[m_1, \ldots, m_k] \lhd_! \,!M} \; (k \in \mathbb{N})$$

Definition (Taylor expansion and Taylor normal form))

$$\mathcal{T}(M) = \{ m \in \delta\mathrm{Bang} \mid m \lhd_! M \} \qquad NF(\mathcal{T}(M)) = \bigcup_{m \lhd_! M} \mathbf{nf}(m)$$

$$\begin{array}{ccc} M & \rightarrow & N \\ \triangledown & & \triangledown \\ m & \rightarrow & n \end{array}$$

Lemmas: Given $m \in NF(\mathcal{T}(M))$ then $\exists M'$ s.t. $M \rightarrow^* M'$ and $m \lhd_! M'$.
Given $M \rightarrow^* N$ then $NF(\mathcal{T}(M)) = NF(\mathcal{T}(N))$.

dBang $+ \perp$

$$\boxed{\text{dBang} + \perp}$$

Approximants:

$$
\begin{array}{rcl}
A & := & B \mid \lambda x A \mid !A \mid A[A_!/x] \mid \perp \\
B & := & x \mid A_\lambda A \mid \textbf{der}(A_!) \\
A_! & := & B \mid \lambda x.A \mid A_![A_!/x] \\
A_\lambda & := & B \mid !A \mid A_\lambda[A_!/x]
\end{array}
$$

$$\text{dBang} + \perp$$

Approximants:

$$
\begin{array}{rcl}
A & := & B \mid \lambda x A \mid !A \mid A[A_!/x] \mid \perp \\
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A_! & := & B \mid \lambda x.A \mid A_![A_!/x] \\
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\end{array}
$$

$\sqsubseteq$ least contextual closed preorder:
$$\forall M \in \text{dBang}_\perp, F[\perp] \subseteq F[M]$$

$$\text{dBang} + \bot$$

Approximants:

$$
\begin{aligned}
A &:= B \mid \lambda x A \mid {!}A \mid A[A_!/x] \mid \bot \\
B &:= x \mid A_\lambda A \mid \textbf{der}(A_!) \\
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Definition (Approximants of a term)

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Böhm tree for dBang

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Definition (Taylor of Böhm)

$$\mathcal{T}(\bot) = \emptyset$$

Commutation Theorem (dBang)

Definition (Taylor of Böhm)

$$\mathcal{T}(\bot) = \emptyset \qquad \mathcal{T}(\texttt{BT(M)}) = \cup_{\texttt{a} \in \mathcal{A}(\texttt{M})} \mathcal{T}(\texttt{a})$$

Definition (Taylor of Böhm)

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Lemmas: (1) Let $A \sqsubseteq M$, then $\mathcal{T}(A) \subseteq \mathcal{T}(M)$.
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(2) Let $A \sqsubseteq M$, then $\mathcal{T}(A) \subseteq NF(\mathcal{T}(M))$.
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Theorem (Commutation)

$$\mathcal{T}(\mathtt{BT(M)}) = \mathtt{NF}(\mathcal{T}(\mathtt{M}))$$

$$M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x]$$
$$L\langle \lambda x.M \rangle\ N \rightarrow L\langle M[N/x] \rangle \qquad\qquad M[N/x] \rightarrow M\{N/x\}$$

Distant Call-by-Name

$$M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x]$$
$$L\langle \lambda x.M \rangle \; N \rightarrow L\langle M[N/x] \rangle \qquad\qquad M[N/x] \rightarrow M\{N/x\}$$

Ressource approximants subset of δBang

- $x \lhd_n x$
- $\lambda x.m \lhd_n \lambda x.M$ if $m \lhd_n M$.
- $m[n_1, \ldots, n_k] \lhd_n MN$ if $m \lhd_n M$ and $n_i \lhd_n N$ for any $i \leq k$
- $m[[n_1, \ldots, n_k]/x] \lhd_n M[N/x]$ if $m \lhd_n M$ and $n_i \lhd_n N$ for any $i \leq k$

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$$M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x]$$
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- $m[[n_1, \ldots, n_k]/x] \triangleleft_n M[N/x]$ if $m \triangleleft_n M$ and $n_i \triangleleft_n N$ for any $i \leq k$

Definition (Taylor dCbN)

$\mathcal{T}^n(M) = \{ m \in \delta\text{Bang} \mid m \triangleleft_n M \}$

Distant Call-by-Name

$$M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x]$$
$$L\langle \lambda x.M \rangle\ N \to L\langle M[N/x] \rangle \qquad\qquad M[N/x] \to M\{N/x\}$$

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$$L\langle \lambda x.M \rangle \, N \to L\langle M[N/x] \rangle \qquad\qquad M[N/x] \to M\{N/x\}$$

Definition (Approximants)

$$A \quad ::= \quad N_\lambda \mid \lambda x.A$$
$$N_\lambda \quad ::= \quad x \mid \bot \mid N_\lambda A$$

Distant Call-by-Name

$$M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x]$$
$$L\langle\lambda x.M\rangle\ N \to L\langle M[N/x]\rangle \qquad M[N/x] \to M\{N/x\}$$

Definition (Approximants)

$$
\begin{aligned}
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$$

Definition (Böhm Tree)

- $\mathcal{A}(M) = \{A \mid M \to^* N, A \sqsubseteq N\}$
- $\mathtt{BT(M)} = \cup \mathcal{A}(\mathtt{M})$

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid M[N/x] \qquad\qquad V ::= x \mid \lambda x.M$$
$$L\langle \lambda x.M \rangle\ N \to L\langle M[N/x] \rangle \qquad\qquad M[V/x] \to M\{V/x\}$$

Distant Call-by-Value

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid M[N/x] \qquad V ::= x \mid \lambda x.M$$
$$L\langle \lambda x.M \rangle\ N \to L\langle M[N/x] \rangle \qquad M[V/x] \to M\{V/x\}$$

Ressource approximants subset of δBang

- $[x, \ldots, x]_k \lhd_v x$ for any $k \in \mathbb{N}$.
- $[\lambda x.m_1, \ldots, \lambda x.m_k] \lhd_v \lambda x.M$ if $m_i \lhd_v M$ for any $i \leq k$.
- $\mathbf{der}(m)n \lhd_v MN$ if $m \lhd_v M$, $n \lhd_v N$ and $M \notin V$
- $mn \lhd_v VN$ if $[m] \lhd_v V$ and $n \lhd_v N$
- $m[n/x] \lhd_v M[N/x]$ if $m \lhd_v M$ and $n \lhd_v N$.

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Ressource approximants subset of $\delta \mathtt{Bang}$

- $[x, \ldots, x]_k \triangleleft_v x$ for any $k \in \mathbb{N}$.
- $[\lambda x.m_1, \ldots, \lambda x.m_k] \triangleleft_v \lambda x.M$ if $m_i \triangleleft_v M$ for any $i \leq k$.
- $\mathbf{der}(m)n \triangleleft_v MN$ if $m \triangleleft_v M$, $n \triangleleft_v N$ and $M \notin V$
- $mn \triangleleft_v VN$ if $[m] \triangleleft_v V$ and $n \triangleleft_v N$
- $m[n/x] \triangleleft_v M[N/x]$ if $m \triangleleft_v M$ and $n \triangleleft_v N$.

Definition (Taylor dCbV)

$$\mathcal{T}^v(M) = \{m \in \delta \mathtt{Bang} \mid m \triangleleft_v M\}$$

Distant Call-by-Value

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid M[N/x] \qquad V ::= x \mid \lambda x.M$$
$$L\langle \lambda x.M \rangle\ N \to L\langle M[N/x] \rangle \qquad M[V/x] \to M\{V/x\}$$

Distant Call-by-Value

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid M[N/x] \qquad V ::= x \mid \lambda x.M$$
$$L\langle \lambda x.M \rangle\ N \to L\langle M[N/x] \rangle \qquad M[V/x] \to M\{V/x\}$$

Definition (Approximants)

$$A\ ::= N_\lambda \mid \lambda x.A \mid A[N_V/x]$$
$$N_\lambda ::= x \mid \bot \mid N_\lambda A \mid A_\lambda[N_V/x]$$
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Distant Call-by-Value

$$M, N ::= x \mid \lambda x.M \mid M \; N \mid M[N/x] \qquad V ::= x \mid \lambda x.M$$
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Link with dBang

Theorem (Translation of Taylor)

$$\mathcal{T}^n(M) = \mathcal{T}(M^n) \qquad \mathcal{T}^v(M) = \mathcal{T}(M^v).$$

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$$\mathcal{T}^n(M) = \mathcal{T}(M^n) \qquad \mathcal{T}^v(M) = \mathcal{T}(M^v).$$

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$$\Rightarrow \text{PROBLEM}$$
Need reverse simulation

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Arrial, Guerrieri, and Kesner. The benefits of diligence. 24

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Reverse simulation up to a few steps is enough

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Commutation Theorem for everyone!

Meaningfulness

$$
\begin{array}{llll}
\textit{Surface} & S & ::= & <\,\cdot\,>\,|\ SM\ |\ MS\ |\ \lambda x.S\ |\ \mathbf{der}(S)\ |\ S[M/x]\ |\ M[S/x] \\
\textit{Tests} & T & ::= & <\,\cdot\,>\,|\ TM\ |\ (\lambda x.T)M
\end{array}
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$$
\begin{array}{llll}
Surface\ CbN & S_n & ::= & < \cdot > \mid S_n\ M \mid \lambda x.S_n \mid S_n[N/x] \\
Surface\ CbV & S_v & ::= & < \cdot > \mid S_v\ M \mid M\ S_v \mid S_v[M/x] \mid M[S_v/x]
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Definition (Meaningfulness)

$M \in \texttt{dBang}$ meaningful: $\exists T$ and N s.t. $T\langle M \rangle \rightarrow^{*}_{dbs}\ !N$.

$\Rightarrow$ similarly dCbN/dCbV meaningfulness is defined under surface contexts.

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\textit{Surface} & S & ::= & <\cdot> \mid SM \mid MS \mid \lambda x.S \mid \mathbf{der}(S) \mid S[M/x] \mid M[S/x] \\
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Taylor and Böhm for full contexts

Taylor and Meaningfulness

Theorem (Meaningfulness and Taylor expansion)

- $M \in \mathtt{dBang}$, if M is meaningful, then $NF(\mathcal{T}(M)) \neq \emptyset$.

Taylor and Meaningfulness

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- $M \in \mathtt{dBang}$, if M is meaningful, then $NF(\mathcal{T}(M)) \neq \emptyset$.
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Taylor and Meaningfulness

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- Restriction to a clash-free fragment: $(x!x)(x!x)$ is meaningless and cannot be given an empty Taylor normal form.
- $\Rightarrow$ Restriction to terms reached by a translation from dCbN and dCbV.

A study of Taylor, Böhm and meaningfulness
for dBang, dCbV and dCbN

A study of Taylor, Böhm and meaningfulness
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A comparable study: Böhm and Taylor for All! Dufour and Mazza, 2024

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Want to study an extensional version:
Extensional taylor expansion, Blondeau-Patissier, Clairambault, and
Auclair, 2024