

# Approximation Theory for Distant Bang Calculus

Kostia Chardonnet, Jules Chouquet, Axel Kerinec  
(a Work in Progress)

$$(\Lambda) \quad M, N ::= x \mid \lambda x. M \mid (MN)$$

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$$Y = \lambda f. (\lambda x. f(xx))(\lambda x. f(xx)) \quad Y \rightarrow_{\beta} \lambda f. (f((\lambda x. f(xx))(\lambda x. f(xx)))) \\ \rightarrow_{\beta} \lambda f. (f(f((\lambda x. f(xx))(\lambda x. f(xx))))) \\ \rightarrow_{\beta} \lambda f. (f(f(f(\cdots))))$$

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### Definition (Solvability)

$M$  is solvable:  $\exists x_1, \dots, x_n, M_1, \dots, M_k$  s.t.  $(\lambda x_1 \dots x_n. M)M_1 \dots M_k \twoheadrightarrow_{\beta} I$

Definition (Böhm tree of  $M$ )

- If  $M \rightarrow_{\beta} \lambda x_1 \dots x_n.y M_1 \dots M_k$  then

$$\text{BT}(M) = \lambda x_1 \dots x_n.y$$

```
graph TD; BT_M["BT(M) = λx₁...xₙ.y"] --> BT_M1["BT(M₁)"]; BT_M --> BT_M2["..."]; BT_M --> BT_Mk["BT(Mₖ)"]
```

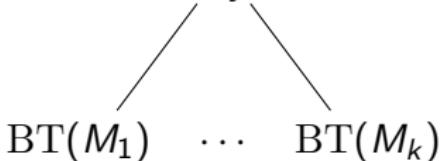
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$$\begin{array}{lcl} Y & = & \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \\ \rightarrow_{\beta} & & \lambda f.(f(f(\dots))) \end{array}$$

$$\text{BT}(Y)$$

||

$\lambda f$

|

$f$

|

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graph TD
    BT_M[BT(M)] --> BT_M1[BT(M1)]
    BT_M --> BT_Mk[BT(Mk)]
  
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- Otherwise

$$\text{BT}(M) = \perp$$

$$\begin{array}{ccl} Y & = & \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \\ & \rightarrow_{\beta} & \lambda f.(f(f(\dots))) \end{array}$$

$$\begin{array}{c} \text{BT}(Y) \\ || \\ \lambda f \\ | \\ f \\ | \\ f \\ \vdots \end{array}$$

## Theorem (Solvability and Böhm Tree)

$M$  insolvable iff  $\text{BT}(M) = \perp$ .

<i>Function</i>	
$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x - a)^n$	

<i>Function</i>	$\lambda$ -calculus
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Theorem (Commutation [Ehrhard and Regnier 08])

$$\text{nf}(\mathcal{T}(M)) = \mathcal{T}(\text{BT}(M))$$

# Evaluation Strategies

*Call-by-Name*

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$$\begin{aligned} (\lambda x. xx)((\lambda y. M)N) &\rightarrow_{\beta} ((\lambda y. M)N)((\lambda y. M)N) \\ &\rightarrow_{\beta} M\{N/y\}((\lambda y. M)N) \end{aligned}$$

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Definition (Scrutability)

$M$  is scrutable:

$$\exists x_1, \dots x_n, M_1, \dots, M_k, V \text{ s.t. } (\lambda x_1 \dots x_n. M)M_1 \dots M_k \rightarrow_{\beta_v} V$$

- Call-by-Push-Value (CbPV) [Levy 99]

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### Definition (Bang)

Terms  $M, N := x \mid MN \mid \lambda x. M \mid !M \mid \mathbf{der}(M)$

Values  $V, W := x \mid !M$

Reduction rules:

$$\begin{array}{lll} (\lambda x. M)(V) & \rightarrow_b & M\{V/x\} \\ \mathbf{der}(!M) & \rightarrow_b & M \end{array}$$

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$\Rightarrow$  PROBLEM

Stuck redexes in CbV and Bang

## Permutation Rules

[Carraro and Guerrieri 14]

$$\begin{array}{lll} (\lambda x.M)NN' & \mapsto_{\sigma_1} & (\lambda x.MN')N \\ V((\lambda x.M)N) & \mapsto_{\sigma_3} & (\lambda x.VM)N \end{array}$$

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## Theorem (Scrutability and BT)

$$M \text{ unscrutable iff } \text{BT}(M) = \emptyset.$$

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Bang extension difficult

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## Definition (Distant Mechanism)

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Without capture of free variables.

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Without capture of free variables.

Distant CbV [Accatoli and Paolini 12],

Distant CbN/Distant Bang [Bucciarelli, Kesner, Ríos and Viso 23]

## Link between them

Translation into dBang [Bucciarelli, Kesner, Ríos and Viso 23]:

$CbN$	$CbV$
$x^n = x$	$x^\nu = !x$
$(\lambda x.M)^n = \lambda x M^n$	$(\lambda x.M)^\nu = !(\lambda x.M^\nu)$
$(M \ N)^n = M^n!N^n$	$(M \ N)^\nu = \begin{cases} L(P)N^\nu & \text{if } M^\nu = L(!P) \\ \mathbf{der}(M^\nu) \ N^\nu & \text{otherwise} \end{cases}$
$(M[N/x])^n = M^n[!N^n/x]$	$(M[N/x])^\nu = M^\nu[N^\nu/x]$

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Theorem (Translation of meaningfulness [Kesner, Arrial and Guerrieri 24])

- $M$  is  $dCbN$ -meaningful iff  $M^n$  is meaningful.
- $M$  is  $dCbV$ -meaningful iff  $M^\vee$  is meaningful.

## Definition (Distant Bang)

Terms  $M, N := x \mid MN \mid \lambda x. M \mid !M \mid \mathbf{der}(M) \mid M[N/x]$   
(Lists Contexts  $L := <\cdot> \mid L[M/x]$ )

Reduction rules:

$$\begin{array}{lll} L\langle \lambda x. M \rangle N & \mapsto_{db} & L\langle M[N/x] \rangle \\ M[L\langle !N \rangle/x] & \mapsto_{db} & L\langle M\{N/x\} \rangle \\ \mathbf{der}(L\langle !M \rangle) & \mapsto_{db} & L\langle M \rangle \end{array}$$

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Exemple : Let  $\mathbf{der}(!(\lambda y. \lambda x. x))(\Omega)(!I) \rightarrow_{db} (\lambda y. \lambda x. x)(\Omega)(!I) \rightarrow_{db} (\lambda x. x)[\Omega/y](!I) \rightarrow_{db} x[!I/x][\Omega/y] \rightarrow_{db} I[\Omega/y]$

# Ressource calculus for dBang

## Definition (Ressource calculus $\delta$ Bang)

(terms)  $m, n := x \mid mn \mid \lambda x. m \mid \mathbf{der}(m) \mid m[n/x] \mid [m_1, \dots, m_k]$

The reduction relation:

- $I\langle \lambda x. m \rangle n \Rightarrow_{\delta} \{I\langle m[n/x] \rangle\}$
- $m[I\langle [n_1, \dots, n_k] \rangle / x] \Rightarrow_{\delta} \begin{cases} \bigcup_{\sigma \in P_k} I\langle m\{n_{\sigma(1)}/x_1, \dots, n_{\sigma(k)}/x_k\} \rangle & \text{if } k = d_x(m) \\ \emptyset & \text{otherwise.} \end{cases}$
- $\mathbf{der}(I\langle [m_1, \dots, m_k] \rangle) \Rightarrow_{\delta} \{I\langle m_1 \rangle\}$  if  $k = 1$  and  $\emptyset$  otherwise.

# Taylor Expansion for dBang

Being an approximant of a term:

$$\frac{m \triangleleft! M}{x \triangleleft! x} \quad \frac{m \triangleleft! M}{\mathbf{der}(m) \triangleleft! \mathbf{der}(M)} \quad \frac{m \triangleleft! M \quad n \triangleleft! N}{mn \triangleleft! MN}$$
$$\frac{m \triangleleft! M \quad n \triangleleft! N}{m[n/x] \triangleleft! M[N/x]} \quad \frac{m_1 \triangleleft! M \quad \cdots \quad m_k \triangleleft! M}{[m_1, \dots, m_k] \triangleleft! !M} \quad (k \in \mathbb{N})$$

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Definition (Taylor expansion and Taylor normal form))

$$\mathcal{T}(M) = \{m \in \delta\text{Bang} \mid m \triangleleft! M\}$$

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Lemmas: Given  $m \in NF(\mathcal{T}(M))$  then  $\exists M' \text{ s.t. } M \rightarrow^* M'$  and  $m \triangleleft! M'$ .  
Given  $M \rightarrow^* N$  then  $NF(\mathcal{T}(M)) = NF(\mathcal{T}(N))$ .

## Böhm tree for dBang

dBang + ⊥

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dBang +  $\perp$

Approximants:

$$A := B \mid \lambda x A \mid !A \mid A[A! / x] \mid \perp$$

$$B := x \mid A_\lambda A \mid \mathbf{der}(A!)$$

$$A! := B \mid \lambda x. A \mid A![A! / x]$$

$$A_\lambda := B \mid !A \mid A_\lambda[A! / x]$$

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$\text{dBang} + \perp$

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$\sqsubseteq$  least contextual closed preorder:

$$\forall M \in \text{dBang}_\perp, F[\perp] \subseteq F[M]$$

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$\text{dBang} + \perp$

Approximants:

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## Theorem (Commutation)

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# Distant Call-by-Name

$$\begin{array}{c} M, N ::= x \mid \lambda x. M \mid MN \mid M[N/x] \\ L\langle \lambda x. M \rangle \ N \rightarrow L\langle M[N/x] \rangle \qquad \qquad \qquad M[N/x] \rightarrow M\{N/x\} \end{array}$$

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Ressource approximants subset of  $\delta$ Bang

- $x \triangleleft_n x$
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- $m[n_1, \dots, n_k] \triangleleft_n MN$  if  $m \triangleleft_n M$  and  $n_i \triangleleft_n N$  for any  $i \leq k$
- $m[[n_1, \dots, n_k]/x] \triangleleft_n M[N/x]$  if  $m \triangleleft_n M$  and  $n_i \triangleleft_n N$  for any  $i \leq k$

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Definition (Taylor dCbN)

$$\mathcal{T}^n(M) = \{m \in \delta\text{Bang} \mid m \triangleleft_n M\}$$

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$$\begin{array}{lcl} A & ::= & N_\lambda \mid \lambda x. A \\ N_\lambda & ::= & x \mid \perp \mid N_\lambda A \end{array}$$

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- **der**( $m$ ) $n \triangleleft_v MN$  if  $m \triangleleft_v M$ ,  $n \triangleleft_v N$  and  $M \notin V$
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Need reverse simulation

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Commutation Theorem for everyone!

# Meaningfulness

*Surface Tests*       $S ::= <\cdot> \mid SM \mid MS \mid \lambda x. S \mid \mathbf{der}(S) \mid S[M/x] \mid M[S/x]$

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$M \in \text{dBang}$  meaningful:  $\exists T$  and  $N$  s.t.  $T\langle M \rangle \rightarrow_{dbs}^* !N$ .

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Taylor and Böhm for full contexts

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## Theorem (Meaningfulness and Taylor expansion)

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⇒ Restriction to terms reached by a translation from dCbN and dCbV.

# Conclusion

A study of Taylor, Böhm and meaningfulness  
for dBang, dCbV and dCbN

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A comparable study: Böhm and Taylor for All! Dufour and Mazza, 2024

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Want to study an extensional version:

Extensional taylor expansion, Blondeau-Patissier, Clairambault, and Auclair, 2024