

Approximation Theory for Distant Bang Calculus

Kostia Chardonnet, Jules Chouquet, Axel Kerinec
(a Work in Progress)

$$(\Lambda) \quad M, N ::= x \mid \lambda x.M \mid (MN)$$

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$$(\lambda x.M)N \mapsto_{\beta} M\{N/x\}$$

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Reaching normal forms?

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Reaching normal forms?

$$\begin{array}{ll} I = \lambda x.x & Ix \rightarrow_{\beta} x \\ \Omega = (\lambda x.xx)(\lambda x.xx) & \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \cdots \rightarrow_{\beta} \Omega \rightarrow_{\beta} \cdots \end{array}$$

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Definition (Solvability)

M is solvable: $\exists x_1, \dots, x_n, M_1, \dots, M_k$ s.t. $(\lambda x_1 \dots x_n.M)M_1 \dots M_k \twoheadrightarrow_{\beta} I$

Definition (Böhm tree of M)

- If $M \twoheadrightarrow_{\beta} \lambda x_1 \dots x_n. y M_1 \dots M_k$ then

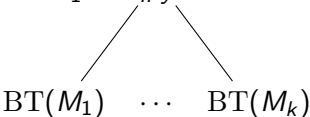
$$\text{BT}(M) = \lambda x_1 \dots x_n. y$$
$$\text{BT}(M_1) \quad \dots \quad \text{BT}(M_k)$$

- Otherwise

$$\text{BT}(M) = \perp$$

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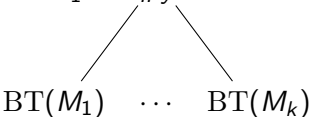
$$\begin{aligned} Y &= \lambda f. (\lambda x. f(xx)) (\lambda x. f(xx)) \\ &\twoheadrightarrow_{\beta} \lambda f. (f(f(\dots))) \end{aligned}$$

$$\text{BT}(Y)$$

$$\begin{array}{c} \parallel \\ \lambda f \\ | \\ f \\ | \\ f \\ \vdots \end{array}$$

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Theorem (Solvability and Böhm Tree)

$$M \text{ insolvable iff } \text{BT}(M) = \perp.$$

<i>Function</i>	
$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x - a)^n$	

<i>Function</i>	<i>λ-calculus</i>
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A differential λ -calculus:

- resource-sensitive: in N can only replace one occurrence of x
- strongly normalising: each resource term has a normal form

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Theorem (Commutation [Ehrhard and Regnier 08])

$$\text{nf}(\mathcal{T}(M)) = \mathcal{T}(\text{BT}(M))$$

Call-by-Name

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$(\lambda x.xx)((\lambda y.M)N)$

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$$(\lambda x.xx)((\lambda y.M)N) \rightarrow_{\beta} ((\lambda y.M)N)((\lambda y.M)N)$$

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$$\begin{aligned} (\lambda x.xx)((\lambda y.M)N) &\rightarrow_{\beta} ((\lambda y.M)N)((\lambda y.M)N) \\ &\rightarrow_{\beta} M\{N/y\}((\lambda y.M)N) \end{aligned}$$

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Definition (Srcutability)

M is scrutable:

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- Call-by-Push-Value (CbPV) [Levy 99]

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- untyped version = Bang-calculus [Ehrhard and Guerrieri 16].

Definition (Bang)

Terms $M, N := x \mid MN \mid \lambda x.M \mid !M \mid \mathbf{der}(M)$

Values $V, W := x \mid !M$

Reduction rules:

$$\begin{array}{ll} (\lambda x.M)(V) & \rightarrow_b M\{V/x\} \\ \mathbf{der}(!M) & \rightarrow_b M \end{array}$$

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\Rightarrow PROBLEM

Stuck redexes in CbV and Bang

Permutation Rules

[Carraro and Guerrieri 14]

$$(\lambda x.M)NN' \mapsto_{\sigma_1} (\lambda x.MN')N$$

$$V((\lambda x.M)N) \mapsto_{\sigma_3} (\lambda x.VM)N$$

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$$\mathcal{T}(\text{BT}_V(M)) = NF(\mathcal{T}(M))$$

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Theorem (Scrutability and BT)

M unscrutable iff $\text{BT}(M) = \emptyset$.

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\Rightarrow PROBLEM

Bang extension difficult

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List contexts: $L := < \cdot > \mid L[M/x]$

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Reductions: $L\langle \lambda x.M \rangle N \rightarrow L\langle M[N/x] \rangle \quad M[L\langle V \rangle/x] \rightarrow L\langle M\{V/x\} \rangle$

Without capture of free variables.

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Distant CbV [Accatoli and Paolini 12],

Distant CbN/Distant Bang [Bucciarelli, Kesner, Ríos and Viso 23]

Link between them

Translation into dBang [Bucciarelli, Kesner, Ríos and Viso 23]:

<i>CbN</i>	<i>CbV</i>
$x^n = x$	$x^\nu = !x$
$(\lambda x.M)^n = \lambda x.M^n$	$(\lambda x.M)^\nu = !(\lambda x.M^\nu)$
$(M\ N)^n = M^n!N^n$	$(M\ N)^\nu = \begin{cases} L\langle P \rangle N^\nu & \text{if } M^\nu = L\langle !P \rangle \\ \mathbf{der}(M^\nu) N^\nu & \text{otherwise} \end{cases}$
$(M[N/x])^n = M^n[!N^n/x]$	$(M[N/x])^\nu = M^\nu[N^\nu/x]$

Link between them

Translation into dBang [Bucciarelli, Kesner, Ríos and Viso 23]:

<i>CbN</i>	<i>CbV</i>
$x^n = x$	$x^\vee = !x$
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$(M\ N)^n = M^n !N^n$	$(M\ N)^\vee = \begin{cases} L\langle P \rangle N^\vee & \text{if } M^\vee = L\langle !P \rangle \\ \mathbf{der}(M^\vee) N^\vee & \text{otherwise} \end{cases}$
$(M[N/x])^n = M^n[!N^n/x]$	$(M[N/x])^\vee = M^\vee[N^\vee/x]$

Theorem (Translation of meaningfulness [Kesner, Arrial and Guerrieri 24])

- M is dCbN-meaningful iff M^n is meaningful.
- M is dCbV-meaningful iff M^\vee is meaningful.

Definition (Distant Bang)

Terms $M, N := x \mid MN \mid \lambda x.M \mid !M \mid \mathbf{der}(M) \mid M[N/x]$
 (Lists Contexts $L := \langle \cdot \rangle \mid L[M/x]$

Reduction rules:

$$\begin{aligned}
 L\langle \lambda x.M \rangle N &\mapsto_{db} L\langle M[N/x] \rangle \\
 M[L\langle !N \rangle / x] &\mapsto_{db} L\langle M\{N/x\} \rangle \\
 \mathbf{der}(L\langle !M \rangle) &\mapsto_{db} L\langle M \rangle
 \end{aligned}$$

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Exemple : Let $\mathbf{der}(!(\lambda y.\lambda x.x))(\Omega)(!I) \rightarrow_{db} (\lambda y.\lambda x.x)(\Omega)(!I) \rightarrow_{db}$
 $(\lambda x.x)[\Omega/y](!I) \rightarrow_{db} x[!I/x][\Omega/y] \rightarrow_{db} I[\Omega/y]$

Definition (Ressource calculus δ Bang)

(terms) $m, n := x \mid mn \mid \lambda x.m \mid \mathbf{der}(m) \mid m[n/x] \mid [m_1, \dots, m_k]$

The reduction relation:

- $I\langle \lambda x.m \rangle n \Rightarrow_{\delta} \{I\langle m[n/x] \rangle\}$
- $m[I\langle [n_1, \dots, n_k] \rangle / x] \Rightarrow_{\delta} \begin{cases} \bigcup_{\sigma \in P_k} I\langle m\{n_{\sigma(1)}/x_1, \dots, n_{\sigma(k)}/x_k\} \rangle & \text{if } k = d_x(m) \\ \emptyset & \text{otherwise.} \end{cases}$
- $\mathbf{der}(I\langle [m_1, \dots, m_k] \rangle) \Rightarrow_{\delta} \{I\langle m_1 \rangle\}$ if $k = 1$ and \emptyset otherwise.

Taylor Expansion for dBang

Being an approximant of a term:

$$\begin{array}{c} \frac{}{x \triangleleft_! x} \quad \frac{m \triangleleft_! M}{\lambda x \, m \triangleleft_! \lambda x \, M} \quad \frac{m \triangleleft_! M}{\mathbf{der}(m) \triangleleft_! \mathbf{der}(M)} \quad \frac{m \triangleleft_! M \quad n \triangleleft_! N}{mn \triangleleft_! MN} \\[1em] \frac{m \triangleleft_! M \quad n \triangleleft_! N}{m[n/x] \triangleleft_! M[N/x]} \quad \frac{m_1 \triangleleft_! M \quad \dots \quad m_k \triangleleft_! M}{[m_1, \dots, m_k] \triangleleft_! !M} \quad (k \in \mathbb{N}) \end{array}$$

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Definition (Taylor expansion and Taylor normal form))

$$\mathcal{T}(M) = \{m \in \delta\mathbf{Bang} \mid m \triangleleft_! M\}$$

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$$\begin{array}{ccc} M & \rightarrow & N \\ \nabla & & \nabla \\ m & \rightarrow & n \end{array}$$

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 \frac{m \triangleleft_! M \quad n \triangleleft_! N}{m[n/x] \triangleleft_! M[N/x]} \quad \frac{m_1 \triangleleft_! M \quad \dots \quad m_k \triangleleft_! M}{[m_1, \dots, m_k] \triangleleft_! !M} \quad (k \in \mathbb{N})
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$$\begin{array}{ccc}
 M & \rightarrow & N \\
 \nabla & & \nabla \\
 m & \rightarrow & n
 \end{array}$$

Lemmas: Given $m \in NF(\mathcal{T}(M))$ then $\exists M'$ s.t. $M \rightarrow^* M'$ and $m \triangleleft_! M'$.
 Given $M \rightarrow^* N$ then $NF(\mathcal{T}(M)) = NF(\mathcal{T}(N))$.

$\text{dBang} + \perp$

dBang + \perp

Approximants:

$$A \quad := \quad B \mid \lambda x A \mid !A \mid A[A_! / x] \mid \perp$$
$$B \quad := \quad x \mid A_\lambda A \mid \mathbf{der}(A_!)$$
$$A_! \quad := \quad B \mid \lambda x. A \mid A_![A_! / x]$$
$$A_\lambda \quad := \quad B \mid !A \mid A_\lambda[A_! / x]$$

dBang + \perp

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$$A_\lambda := B \mid !A \mid A_\lambda[A_! / x]$$

\sqsubseteq least contextual closed preorder:

$$\forall M \in \text{dBang}_\perp, F[\perp] \subseteq F[M]$$

dBang + \perp

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\sqsubseteq least contextual closed preorder:

$$\forall M \in \mathbf{dBang}_\perp, F[\perp] \subseteq F[M]$$

Definition (Approximants of a term)

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dBang + \perp

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Theorem (Commutation)

$$\mathcal{T}(\text{BT}(M)) = \text{NF}(\mathcal{T}(M))$$

$$\begin{array}{l} M, N ::= x \mid \lambda x.M \mid MN \mid M[N/x] \\ L\langle \lambda x.M \rangle N \rightarrow L\langle M[N/x] \rangle \qquad M[N/x] \rightarrow M\{N/x\} \end{array}$$

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Ressource approximants subset of δ Bang

- $x \triangleleft_n x$
- $\lambda x.m \triangleleft_n \lambda x.M$ if $m \triangleleft_n M$.
- $m[n_1, \dots, n_k] \triangleleft_n MN$ if $m \triangleleft_n M$ and $n_i \triangleleft_n N$ for any $i \leq k$
- $m[[n_1, \dots, n_k]/x] \triangleleft_n M[N/x]$ if $m \triangleleft_n M$ and $n_i \triangleleft_n N$ for any $i \leq k$

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$$\mathcal{T}^n(M) = \{m \in \delta\text{Bang} \mid m \triangleleft_n M\}$$

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$$\begin{array}{l} A ::= N_\lambda \mid \lambda x.A \\ N_\lambda ::= x \mid \perp \mid N_\lambda A \end{array}$$

Distant Call-by-Name

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$$\begin{array}{ll} M, N ::= x \mid \lambda x.M \mid M N \mid M[N/x] & V ::= x \mid \lambda x.M \\ L\langle \lambda x.M \rangle N \rightarrow L\langle M[N/x] \rangle & M[V/x] \rightarrow M\{V/x\} \end{array}$$

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Commutation Theorem for everyone!

Surface

$S ::= \langle \cdot \rangle \mid SM \mid MS \mid \lambda x.S \mid \mathbf{der}(S) \mid S[M/x] \mid M[S/x]$

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Surface CbN $S_n ::= \langle \cdot \rangle \mid S_n M \mid \lambda x.S_n \mid S_n[N/x]$

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Meaningfulness

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$M \in \mathbf{dBang}$ meaningful: $\exists T$ and N s.t. $T \langle M \rangle \rightarrow_{dbs}^* !N$.

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Taylor and Böhm for full contexts

Theorem (Meaningfulness and Taylor expansion)

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- \Rightarrow Restriction to terms reached by a translation from dCbN and dCbV.

A study of Taylor, Böhm and meaningfulness
for dBang, dCbV and dCbN

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A comparable study: Böhm and Taylor for All! Dufour and Mazza, 2024

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Want to study an extensional version:

Extensional taylor expansion, Blondeau-Patissier, Clairambault, and
Auclair, 2024