AuW Recap

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Cannot be used during the exam, but it's a nice short recap of everything done in the second semester.

1 Last semester

Go read again about MST algorithms and so on.

2 Graphentheorie

2.1 Zusammenhang

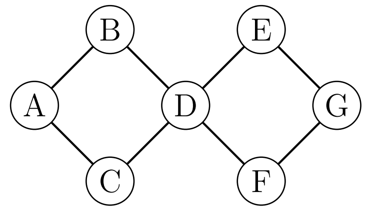
Definition 1. A graph $G = (V, E)$ is *k-zusammenhängend* if $|V| \geq k + 1$ and for all $X \subseteq V,\ |X| < k$ the following is true:

$$\text{The graph } G[V \setminus X] \text{ is zusammengehängend}$$

Definition 2. A graph $G = (V, E)$ is *k-kanten-zusammenhängend* if for all $X \subseteq E,\ |X| < k$ the following is true:

$$\text{The graph } G(V, E \setminus X) \text{ is zusammengehängend}$$

Note 1. A graph can be both 2-kanten-zusammengehängend and be only 1-zusammengehängend at the same time. Example:



Definition 3. In a zusammengehängend Graph, *Artikulationsknoten* disconnect the graph when removed. Only 1-zusammengehängend graphs can have Artikulationsknoten

Theorem 1. *In zusammenhängende Graphs it's possible to find Artikulationsknoten in $\mathcal{O}(|E|)$ if an adjacency list is used.*

Definition 4. A zusammenhängend Graph may contain *Brücke*. In this case, it's **not 2-kanten-zusammenhängend**.

An edge is a bridge if it disconnects the graph when removed.

Theorem 2. *Brücke can also be computed in $\mathcal{O}(|E|)$ using an adjacency list.*

Definition 5. $G = (V, E)$ is zusammenhängend. For $e, f \in E$ we define the relation

$$e \sim f \Leftrightarrow e = f \text{ or there is a Kreis containing both edges}$$

This is an equivalence relation. Each equivalence class is called a Block (plural Blöcke).

2.2 Kreise

Definition 6. An *Eulertour* in a graph G is a closed path (*Zyklus*) that contains every edge $\in E$ exactly once.

A graph containing an Eulertour is called *eulersch*.

Definition 7.

Theorem 3.

$$\textit{A graph is eulersch} \Leftrightarrow deg(v) \textit{ is even for all vertices}$$

Theorem 4. *In a connected and eulersch Graph it's possible to find an Eulertour in $\mathcal{O}(|E|)$.*

Definition 8. A *Hamintonkreis* in G is a cycle that goes through every vertex exaclty once. A graph containing an Hamintonkreis is called *hamintonsch.*

Theorem 5. *The algorithm seen in class can find an Hamintonkreis in time* $\mathcal{O}(n^2 \cdot 2^n)$ *and memory* $\mathcal{O}(n \cdot 2^n)$*, where* $n = |V|$*.*

Theorem 6. *A bipartite graph* $G = (A \uplus B, E)$ *cannot contain an Hamintonkreis.*

Theorem 7 (Dirac)**.** *A graph* G *with* $V \geq 3$ *in which every vertex has at least* $|V|/2$ *neighbors is hamintonsch.*

Definition 9. In a complete graph K_n (all vertices are connected together), the metric Traveling Salesman Problem consists in finding an Hamintonkreis C with minimal cost (distance).

Definition 10. An α-Approximationsalgorithmus of this problem finds an H.kreis C so that

$$\sum_{e \in C} l(e) \leq \alpha \cdot opt(K_n, l)$$

Meaning that it finds a solution worse by the optimal solution by a factor α.

Theorem 8. *If there's an* α*-Approximationsalgorithmus with* $\alpha > 1$ *for the TSP with running time* $\mathcal{O}(f(n))$ *then there's also an algorithm that decides whether a graph with* n *vertices is hamintonsch in* $\mathcal{O}(f(n))$*.*

Theorem 9. *For the metric TSP there's a 2-Approximationsalgorithmus with running time* $\mathcal{O}(n^2)$*. It find the MST in* $\mathcal{O}(n^2)$ *and then uses it to find the H.k.*

2.3 Matching

Definition 11. A set of edges $M \subseteq E$ is called *Matching* in a graph G if

$$\forall e, f \in M \ (e \cap f = \emptyset)$$

- A vertex v is said to be *überdeckt* by a matching M if the matching contains an edge containing v.
- A matching M is called *perfektes Matching* if every vertex is überdeckt (equivalent: $|M| = |V|/2$).

Definition 12. A matching M is said to be:

- *inklusionsmaximal* if $M \cup \{e\}$ is not a matching for all $e \in E \setminus M$.
- *kardinalitätsmaximal* if $|M| \geq |M'|$ for all matchings M' in G.

Note that *kardinalitätsmaximal* $\Rightarrow$ *inklusionsmaximal* (the opposite might not be true).

Algorithm 1: Greedy-Matching

Result: The inklusionsmaximales matching M
while $E \neq \emptyset$ **do**
    choose an edge $e \in E$;
    $M \leftarrow M \cup \{e\}$;
    remove e and all incident edges from G;

Theorem 10. *The greedy-matching algorithm finds an inklusionsmaximal Matching in time $\mathcal{O}(|E|)$ for which the following holds: $|M_{Greedy}| \geq \frac{1}{2}|M_{max}|$, where M_{max} is a kardinalitätsmaximales Matching.*

Theorem 11 (Berge)**.** *Let M be a matching in G that is not k.maximal, then there is an augmenting path to M.*

Algos at page 65 and 67

Theorem 12. *Is n even K_n a complete graph, then it's possible to find a minimal perkeftes Matching in time $\mathcal{O}(n^3)$*

Theorem 13. *There's a 3/2-Approximationsalgorithmus for the TSP that runs in $\mathcal{O}(n^3)$.*

Definition 13. *Nachbarnschaft einer Knotenmenge $X \subseteq V$:*

$$N(X) := \bigcup_{v \in X} N(v)$$

Theorem 14 (Hall, Heiratssatz)**.** *For a bipartite graph $G = (A \uplus B, E)$ theres a matching M with $|M| = |A|$ if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.*

Definition 14. A bipartite graph is called *k-regulär* if every vertex has degree k.

Theorem 15. *Let G be a k-regular bipartite graph. Then there's $M_1, ..., M_k$ so that $E = M_1 \uplus M_2 \uplus ... \uplus M_k$ and all $M_i, 1 \leq i \leq k$ are perfect matchings.*

Theorem 16. *Is $G = (V, E)$ a 2^k-regular bipartite Graph, then it's possible to find a perfect matching in $\mathcal{O}(|E|)$.*

2.4 Färbungen

Definition 15. A *(Knoten)-Färbung* (vertex coloring) of a graph G with k colors is $c : V \to [k]$, so that

$$c(u) \neq c(v) \ \forall \{u, v\} \in E$$

The *chromatische Zahl* (chromatic number) $X(G)$ of a graph is the minimal amount of colors that can be used to color G.

Theorem 17. *A graph is bipartite if and only if doesn't contain any Kreis of odd length.*

Theorem 18 (Vierfarbensatz)**.** *Every map can be colored with 4 colors.*

Algorithm 2: Greedy-Färbung

Data: G
Result: array c mapping each vertex to a color
$c(v_1) \leftarrow 1$;
for $i = 2, \ldots, n$ **do**
  $c(v_i) \leftarrow \min \{k \in \mathbb{N} \mid k \neq c(u) \text{ for all } u \in N(v_i) \cap \{v_1, ..., v_{i-1}\}\}$

Theorem 19. *Let G be a connected graph and $C(G)$ the amount of colors used by the Greedy-Färbung algorithm. Then*

$$\mathcal{X}(G) \leq C(G) \leq \Delta(G) + 1$$

Where $\Delta(G) := \max_{v \in V} deg(v)$ is the max degree in the graph.

The running time is $\mathcal{O}(|E|)$ if an adjacency list is used.

Theorem 20 (Brooks)**.** *Let G be a connected graph that is neither complete nor an odd Kreis ($G \neq K_n$ and $G \neq C_{2n+1}$). Then*

$$\mathcal{X}(G) \leq \Delta(G)$$

And there's an algorithm that can color the vertices of G in time $\mathcal{O}(|E|)$ and with $\Delta(G)$ colors.

Theorem 21 (Mycielski-Konstruktion)**.** *For all $k \geq 2$ there's a triangle-free graph G_k with $\mathcal{X}(G) \geq k$.*

Theorem 22. *Every 3-färbbaren graph can be colored in time $\mathcal{O}(|E|)$ with $\mathcal{O}(\sqrt{|V|})$ colors.*

3 Randomized algorithms