AuW Recap

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Cannot be used during the exam, but it's a nice short recap of everything done in the second semester.

1 Last semester

Go read again about MST algorithms and so on.

2 Zusammenhang

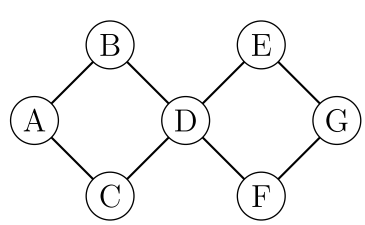
Definition 1. A graph $G = (V, E)$ is *k-zusammenhängend* if $|V| \geq k + 1$ and for all $X \subseteq V,\ |X| < k$ the following is true:

The graph $G[V \setminus X]$ is zusammengehängend

Definition 2. A graph $G = (V, E)$ is *k-kanten-zusammenhängend* if for all $X \subseteq E,\ |X| < k$ the following is true:

The graph $G(V, E \setminus X)$ is zusammengehängend

Note 1. A graph can be both 2-kanten-zusammengehängend and be only 1-zusammengehängend at the same time. Example:



Definition 3. In a zusammengehängend Graph, *Artikulationsknoten* disconnect the graph when removed. Only 1-zusammengehängend graphs can have Artikulationsknoten

Theorem 1. *In zusammenhängende Graphs it's possible to find Artikulationsknoten in $\mathcal{O}(|E|)$ if an adjacency list is used.*

Definition 4. A zusammenhängend Graph may contain *Brücke*. In this case, it's **not 2-kanten-zusammenhängend**.

An edge is a bridge if it disconnects the graph when removed.

Theorem 2. *Brücke can also be computed in $\mathcal{O}(|E|)$ using an adjacency list.*

Definition 5. $G = (V, E)$ is zusammenhängend. For $e, f \in E$ we define the relation

$$e \sim f \Leftrightarrow e = f \text{ or there is a Kreis containing both edges}$$

This is an equivalence relation. Each equivalence class is called a Block (plural Blöcke).

3 Kreise

Definition 6. An *Eulertour* in a graph G is a closed path (*Zyklus*) that contains every edge $\in E$ exactly once.

A graph containing an Eulertour is called *eulersch*.

Definition 7.

Theorem 3.

$$\textit{A graph is eulersch} \Leftrightarrow deg(v) \textit{ is even for all vertices}$$

Theorem 4. *In a connected and eulersch Graph it's possible to find an Eulertour in $\mathcal{O}(|E|)$.*

Definition 8. A *Hamintonkreis* in G is a cycle that goes through every vertex exaclty once. A graph containing an Hamintonkreis is called *hamintonsch*.

Theorem 5. *The algorithm seen in class can find an Hamintonkreis in time $\mathcal{O}(n^2 \cdot 2^n)$ and memory $\mathcal{O}(n \cdot 2^n)$, where $n = |V|$.*

Theorem 6. *A bipartite graph $G = (A \uplus B, E)$ cannot contain an Hamintonkreis.*

Theorem 7 (Dirac)**.** *A graph G with $V \geq 3$ in which every vertex has at least $|V|/2$ neighbors is hamintonsch.*

Definition 9. In a complete graph K_n (all vertices are connected together), the metric Traveling Salesman Problem consists in finding an Hamintonkreis C with minimal cost (distance).

Definition 10. An α-Approximationsalgorithmus of this problem finds an H.kreis C so that

$$\sum_{e \in C} l(e) \leq \alpha \cdot opt(K_n, l)$$

Meaning that it finds a solution worse by the optimal solution by a factor α.

Theorem 8. *If there's an α-Approximationsalgorithmus with $\alpha > 1$ for the TSP with running time $\mathcal{O}(f(n))$ then there's also an algorithm that decides whether a graph with n vertices is hamintonsch in $\mathcal{O}(f(n))$.*

Theorem 9. *For the metric TSP there's a 2-Approximationsalgorithmus with running time $\mathcal{O}(n^2)$. It find the MST in $\mathcal{O}(n^2)$ and then uses it to find the H.k.*

4 Matching

Definition 11. A set of edges $M \subseteq E$ is called *Matching* in a graph G if

$$\forall e, f \in M \ (e \cap f = \emptyset)$$

- A vertex v is said to be *überdeckt* by a matching M if the matching contains an edge containing v.
- A matching M is called *perfektes Matching* if every vertex is überdeckt (equivalent: $|M| = |V|/2$).

Definition 12. A matching M is said to be:

- *inklusionsmaximal* if $M \cup \{e\}$ is not a matching for all $e \in E \setminus M$.

- *kardinalitätsmaximal* if $|M| \geq |M'|$ for all matchings M' in G.

Note that *kardinalitätsmaximal* $\Rightarrow$ *inklusionsmaximal* (the opposite might not be true).

Theorem 10. *The greedy-matching algorithm finds an inklusionsmaximal Matching in time $\mathcal{O}(|E|)$ for which the following holds: $|M_{Greedy}| \geq \frac{1}{2}|M_{max}|$, where M_{max} is a kardinalitätsmaximales Matching.*

Theorem 11 (Berge)**.** *Let M be a matching in G that is not kardinalitätsmaximal, then there is an augmented path to M.*