

Analysis 2 Recap

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1 Linear differential equations

Definition L. *linear Differential equation. Homogeneous if $b = 0$, inhomogeneous otherwise.*

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Theorem 2.2.3. *... y is k -times differentiable ...*

For the homogeneous equation, given a choice of x_0 and (y_0, \dots, y_{k-1}) there's a unique solution $f \in S$ such that

$$f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1} \quad (1)$$

For the inhomogeneous equation with a b continuous on the interval, the set of solutions S_b is the set of functions $f + f_0$ where $f \in S$. Again, for any x_0 and (y_0, \dots, y_{k-1}) there's a unique solution such that ((1)).

If $b \neq 0$ then S_b is not a vector space.

Proposition 2.3.1. *Any solution of $y' + ay = 0$ is in the form $f(x) = ze^{-A(x)}$, where A is a primitive of a and $z \in \mathbb{C}$. Unique solution is $f(x) = y_0 e^{A(x_0) - A(x)}$*

To solve the inhomogeneous equation $y' + ay = b$, the prev solution is used. Using Variation of the constant we replace z with $z(x)$ and then $y' + ay = b \Leftrightarrow z'(x) = b(x) e^{A(x)}$ and $f_0(x) = C(x)e^{-A(x)}$, where $C(x)$ is a primitive of $z'(x)$.

1.1 Constant coefficients

Definition L. *et $a_0, \dots, a_{k-1} \in \mathbb{C}$. Linear differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$. Homogeneous solution is in the form $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{C}$. We have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geq 0$ and x .*

Conclusion: $f(x)$ is a solution iff $P(\alpha) = 0$, where $P(X) = X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0$.

This polynomial of degree k has k roots (counted with multiplicity). There exist complex numbers $\alpha_1, \dots, \alpha_k$ such that $P(X) = (X - \alpha_1) \dots (X - \alpha_k)$. This is the companion or characteristic polynomial of the homogeneous diff. equation.

- No multiple roots When $\alpha_i \neq \alpha_j$ for all i, j .

Solution of the homogeneous equation ($b = 0$): form $f(x) = z_1 e^{\alpha_1 x} + \dots + z_k e^{\alpha_k x}$. Unique solution with $f(x_0) = y_0, \dots, f^{(k-1)}(x_0) = y_{k-1}$ can be obtained by viewing z_i as unknowns. Substitute $x = x_0$ in the formula for f and solve for z_1, \dots, z_k (linear system).

- Multiple roots Assume α is a multiple root of order j of the polynomial P , with $2 \leq j \leq k$. Then

$f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = x e^{\alpha x}$, ..., $f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$ are linearly independent and are solutions of the h.l.d.e.

Example S. *suppose $P(X) = X(X-4)^3(X-(1+i))(X-(1-i))$, then the solutions are $f_0(x) = 1$ (sol. for $X = 0$), $f_1(x) = e^{4x}$, $f_2(x) = x e^{4x}$, $f_3(x) = x^2 e^{4x}$, $f_4(x) = e^{(1+i)x}$, $f_5(x) = e^{(1-i)x}$*

Now the inhomogeneous equation ($b \neq 0$):

Should avoid variation of the constants. Can use special cases:

1. $b(x) = x^d e^{\beta x}$ for some integer $d \geq 0$ and an item β which is NOT a root of P , then the solution is of the form $f(x) = Q(x) e^{\beta x}$, where Q is a polynomial of degree d .
2. $b(x) = x^d \cos(\beta x)$ or $b(x) = x^d \sin(\beta x)$ for some integer $d \geq 0$ and β is NOT a root of P , then one can transform it to a combination of complex exponentials or look for a solution of the form $f(x) = Q_1(x) \cos(\beta x) + Q_2(x) \sin(\beta x)$, Q_1, Q_2 have degree d .
3. $b(x)$ is in the form of the previous two but IS a root of multiplicity j , then one looks for $f(x) = Q(x) e^{\beta x}$, with Q of degree $q + j$.
4. Special case $\beta = 0$ of the previous 3 (b polynomial of degree $d \geq 0$): if 0 is NOT a root, look for a solution f (polynomial) of deg d , or degree $d + j$ if 0 IS a root, where j is the multiplicity of 0.

1.2 Variation of the constants for degree ≥ 2

Does not require the coefficients to be constants, but it makes it easier.

Inhomogeneous equation

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Solutions f_1, \dots, f_k for the homogeneous equations must be found first.

We then search for a solution of the form $f(x) = z_1(x)f_1(x) + \dots + z_k(x)f_k(x)$, such that we have (for all x):

$$\begin{cases} z_1'(x)f_1(x) + \dots + z_k'(x)f_k(x) = 0 \\ z_1'(x)f_1'(x) + \dots + z_k'(x)f_k'(x) = 0 \\ \dots \\ z_1'(x)f_1^{(k-2)}(x) + \dots + z_k'(x)f_k^{(k-2)}(x) = 0 \end{cases}$$

Example C. *ase $k = 2$: Write again $f = z_1 f_1 + z_2 f_2$ and the constraint $z_1' f_1 + z_2' f_2 = 0$.*

2 Differential in \mathbb{R}^n

Definition 3.3.5. $f : X \mapsto \mathbb{R}$ has a partial derivative with respect to the i -th variable if the function

$$g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$$

is differentiable for all $x_0 \in X$ on the set $I = \{t \in \mathbb{R} \mid (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$.

Its derivative $g'(x_{0,i})$ is denoted

$$\frac{\partial f}{\partial x_i}(x_0), \partial_{x_i}(x_0), \partial_i(x_0)$$

Proposition 3.3.7. $x \in \mathbb{R}^n$ open, f, g functions from X to \mathbb{R}^m . Let $1 \leq i \leq n$.

1. if f, g have partial derivatives of i -th coordinate on X , then $f + g$ also does. $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$
2. if the previous is true and $m = 1$, then fg also does and $\partial_{x_i}(fg) = \partial_{x_i}(f) \cdot g + f \cdot \partial_{x_i}(g)$
3. If the previous is true and $g(x) \neq 0$ for all $x \in X$, then f/g has a partial derivative $\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2$

Definition 3.3.9. $f : X \mapsto \mathbb{R}^m$ has partial derivatives on X . Write $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$.

For any $x \in X$, the **Jacobi Matrix** (m rows, n columns) of f at x is defined as

$$J_f(x) = (\partial_{x_i} f_j(x))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$$

Definition 3.3.11. 1. If all partial derivatives of $f : X \mapsto \mathbb{R}$ exist at $x_0 \in X$, then the column vector

$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \dots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

is called the **gradient** of f at x_0 .

2. Let $f = (f_1, f_2, \dots, f_n) : X \mapsto \mathbb{R}^n$ and all partial derivatives of all coordinates f_i of f exist at $x_0 \in X$. Then

$$\text{Tr}(J_f(x_0)) = \sum_{i=1}^n \partial_{x_i} f_i(x_0)$$

is the trace of the Jacobi Matrix and is called the **divergence** of f at x_0 , also $\text{div}(f)(x_0)$

Definition 3.4.2. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function. Let u be a linear map $\mathbb{R}^n \mapsto \mathbb{R}^m$ and $x_0 \in X$. We say that f is differentiable at x_0 with differential u if

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{1}{\|x - x_0\|} (f(x) - f(x_0) - u(x - x_0)) = 0$$

We then denote $df(x_0) = u$. If it is differentiable at every $x_0 \in X$, then it is differentiable on X .

Then, close to x_0 , we can approximate $f(x)$ by $g(x) = f(x_0) + u(x - x_0)$

Proposition 3.4.4. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function differentiable on X . Then

1. f is **continuous** on X .
2. f admits partial derivatives on X with respect to each variable.
3. Assume that $m = 1$. Let $x_0 \in X$ and $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ be the differential of f at x_0 . We then have $\partial_{x_i} f(x_0) = a_i$ for $1 \leq i \leq n$

Proposition 3.4.6. $X \subset \mathbb{R}^n$ open, $f : x \mapsto \mathbb{R}^m$, $g : X \mapsto \mathbb{R}^m$ differentiable functions on X .

1. $f + g$ is differentiable on X with differential $d(f + g) = df + dg$.

2. If $m = 1$, then fg is differentiable. If we also have $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable.

Proposition 3.4.7. If f has all partial derivatives on X and they are all continuous on X , then f is differentiable on X . The matrix of the differential $df(x_0)$ is the Jacobi Matrix of f at x_0 .

Proposition 3.4.9 (Chain Rule). Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open. Let $f : X \mapsto Y$ and $g : Y \mapsto \mathbb{R}^p$ be differentiable functions. Then $g \circ f : X \mapsto \mathbb{R}^p$ is differentiable on X , and for any $x \in X$, its differential is given by

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

The Jacobi Matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0))J_f(x_0) \text{ (matrix product)}$$

Proposition (Chain Rule case). For the form $f(g_1(x), \dots, g_k(x))$, the chain rule is

$$\frac{d}{dx}f(g_1(x), \dots, g_k(x)) = \sum_{i=1}^k \left(\frac{d}{dx}g_i(x) \right) D_i f(g_1(x), \dots, g_k(x))$$

Where $D_i f$ is the derivative of f with respect to the i -th argument and $D_i f(z)$ is the value of this derivative at z .

Definition L. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ differentiable. Let $x_0 \in X$ and $u = df(x_0)$ be the differential of f at x_0 . The graph of the affine linear approximation

$$g(x) = f(x_0) + u(x - x_0)$$

from \mathbb{R}^n to \mathbb{R}^m , or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

is called the **tangent space** at x_0 to the graph of f .

Definition 3.4.13 (Directional Derivative). Let $X \subset \mathbb{R}^n$ be open, $f : X \mapsto \mathbb{R}^m$ a function.

Let $v \in \mathbb{R}^n$ be a non-zero vector and $x_0 \in X$.

We say that f has **directional derivative** $w \in \mathbb{R}^m$ in the direction v if the function g defined on the set I has a derivative at $t = 0$ and this is equal to w .

$$g(t) = f(x_0 + tv), \quad I = \{t \in \mathbb{R} \mid x_0 + tv \in R\}$$

Other words: limit is equal to w

$$\lim_{\substack{t \rightarrow 0 \\ t \neq 0}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Proposition 3.4.15. X open, f differentiable. Then for any $x_0 \in X$ and non-zero v , the function has a directional derivative at x_0 in the direction v , equal to $df(x_0)(v)$

Definition 3.5.1. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$. We say that f is of class:

- C^1 is differentiable on X and all partial derivatives are continuous.
- C^k if differentiable on X and all partial derivatives $\partial_{x_i} f : X \mapsto \mathbb{R}^m$ are of class C^{k-1} .
- C^∞ if $f \in C^k(X; \mathbb{R}^m)$ for all $k \geq 1$

Set of functions of class C^k from X to \mathbb{R}^m is denoted $C^k(X; \mathbb{R}^m)$

Proposition 3.5.4 (Mixed derivatives commute). $X \subset \mathbb{R}^n$ open and $f : X \mapsto \mathbb{R}^m$ of class C^k . Then the partial derivatives of order k are independent of the order in which the partial derivatives are taken: for any variables x_1, x_2, \dots, x_n we have

$$\partial_{x_1, x_2, \dots, x_n} f = \partial_{x_2, x_1, \dots, x_n} f = \dots \text{ (all combinations)}$$

Definition 3.5.9 (Hessian). Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ a C^2 function. For $x \in X$, the **Hessian matrix** of f at x is the **symmetric square matrix**

$$H_f(x) = \text{Hess}_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n}$$

Definition (Second partial derivative test). Using the Hessian matrix and its determinant, one can find critical points of a function and (most of the times) determine whether it's a min/max or a saddle point.

Define $D(a, b) = \det(H_f(a, b))$.

- **Two dimensions:** to find all critical points one looks for solutions of $f_x(a, b) = f_y(a, b) = 0$ and then characterizes them as follows:

1. If $D(a, b) > 0$ and $f_{XX}(a, b) > 0$ then (a, b) is a local minimum of f .
2. If $D(a, b) > 0$ and $f_{XX}(a, b) < 0$ then (a, b) is a local maximum.
3. If $D(a, b) < 0$ then (a, b) is a saddle point.
4. If $D(a, b) = 0$ then the test is inconclusive.

- **Multiple variables:** one must look at the eigenvalues of the Hessian matrix at (a, b) .

1. If all eigenvalues are positive, then it's a local minimum.
2. If all eigenvalues are negative, then it's a maximum.
3. If there's both positive and negative eigenvalues, then it's a saddle point.

4. Otherwise, the test is inconclusive.

Example (Change of variable). Idea: create h , which is f on a different coordinate system.

Open set $U \subset \mathbb{R}^n$ containing the new variables (y_1, \dots, y_n) and a change of variable $g : U \mapsto X$ that expresses (x_1, \dots, x_n) in terms of (y_1, \dots, y_n) .

Consider $x_1 = g_1(y_0, \dots, y_n)$, $x_n = g_n(y_1, \dots, y_n)$

Composite $h = f \circ g : U \mapsto \mathbb{R}$ is the function f expressed in terms of the new variables y .

Polar coordinates: Map $g : U \mapsto \mathbb{R}^2$, $g(r, \theta) = (r \cos \theta, r \sin \theta)$. Replace f by $h : h(r, \theta) = f(r \cos \theta, r \sin \theta)$.

$$J_g(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

2.1 Taylor Polynomials

Definition 3.7.1 (Taylor polynomials). Let $k \geq 1$ be an integer, $f : X \mapsto \mathbb{R}$ a function of class C^k on X , and fix $x_0 \in X$. The k -th Taylor polynomial of f at point x_0 is the poly in n variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \dots + \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}(x_0) y_1^{m_1} \dots y_n^{m_n}$$

where the last sum ranges over the tuples of n positive integers such that the sum is k .

Case $n = 1$ (one variable):

$$T_k f(y; x_0) = f(x_0) + f'(x_0)y + \frac{f''(x_0)}{2}y^2 + \dots + \frac{f^{(k)}(x_0)}{k!}y^k$$

Proposition 3.7.3 (Taylor Approximation). $k \geq 1$, $X \subset \mathbb{R}^n$ open, $f : X \mapsto \mathbb{R}$ a C^k function. For x_0 in X , we define $E_k f(x; x_0)$ by

$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$

then we have

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{e_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

2.2 Critical Points

Proposition 3.8.1. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ be differentiable. If $x_0 \in X$ is a local maximum or a local minimum, then we have (equivalent) for $1 \leq i \leq n$:

$$df(x_0) = 0, \quad \nabla f(x_0) = 0, \quad \frac{\partial f}{\partial x_i}(x_0) = 0$$

Definition 3.8.2 (Critical Point). Let X be open and f be differentiable. A point x_0 is called a **critical point** of f if $\nabla f(x_0) = 0$.

Definition 3.8.6 (Non-degenerate critical point). f of class C^2 . A critical point x_0 is **non-degenerate** if the Hessian matrix has non-zero determinant.

Corollary 3.8.7. X open and $f : X \mapsto \mathbb{R}$ of class C^2 . Let x_0 be a non-degenerate critical point of f . Let p, q be the number of positive and negative eigenvalues of $\text{Hess}_f(x_0)$

1. if $p = n$, equivalently if $q = 0$, the function f has a local minimum at x_0 .
2. if $q = n$, equivalently if $p = 0$, f has a local maximum at x_0 .
3. Otherwise, the function f does not have a local extremum at x_0 , equivalently it has a saddle point at x_0 .

2.3 Lagrange multipliers

Proposition 3.9.2 (Lagrange Multiplier). Let $X \subset \mathbb{R}^n$ be open and $f, g : X \mapsto \mathbb{R}$ be class C^1 . If $x_0 \in X$ is a local extremum of f restricted to the set $Y = \{x \in X \mid g(x) = 0\}$ ($\nabla f(x_0)$ can be non-zero!), then either $\nabla g(x_0) = 0$ or there exist λ such that

$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$

In other words, (x_0, λ) is a critical point of $h(x, \lambda) = f(x) - \lambda g(x)$. Value λ is the Lagrange Multiplier at x_0 .

2.4 The inverse and implicit functions theorems

Definition 3.10.1 (Change of variable).

Theorem 3.10.2 (Inverse function theorem). $X \subset \mathbb{R}^n$ open and $f : X \mapsto \mathbb{R}^n$ differentiable. If the jacobian matrix of f at $x_0 \in X$ is invertible ($\det(J_f(x_0)) \neq 0$) then f is a change of variable around x_0 .

Moreover, $J_g(f(x_0)) = J_f(x_0)^{-1}$.

In addition, if f is of class C^k , then g is also of class C^k .

Theorem 3.10.4 (Implicit function theorem). Let $X \subset \mathbb{R}^{n+1}$ be open, $g : X \mapsto \mathbb{R}$ be of class C^k with $k \geq 1$. Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $g(x_0, y_0) = 0$.

Assume that $\partial_y g(x_0, y_0) \neq 0$.

Then there exists an open set $U \subset \mathbb{R}^n$ containing x_0 , an open interval $I \subset \mathbb{R}$ containing y_0 , and a function $f : u \mapsto \mathbb{R}$ of class C^k such that the system of equations

$$\begin{cases} g(x, y) = 0 \\ x \in U, y \in I \end{cases}$$

is equivalent with $y = f(x)$. In particular, $f(x_0) = y_0$. Moreover, the gradient of f at x_0 is

$$\nabla f(x_0) = -\frac{1}{(\partial_y g)(x_0, y_0)} \nabla_x g(x_0, y_0)$$

where $\nabla_x g = (\partial_{x_1} g, \dots, \partial_{x_n} g)$

3 Integration in \mathbb{R}^n

3.1 Line integrals

Definition 4.1.1. Uses scalar product in \mathbb{R}^n .

1. Let $I = [a, b]$ be a closed and bounded interval in \mathbb{R} . Let $f(t) = (f_1(t), f_2(t), \dots, f_n(t))$ be continuous (f_i is continuous). Then we define

$$\int_a^b f(t) dt = \left(\int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right)$$

2. A **parametrized curve** in \mathbb{R}^n is a continuous map $\gamma : [a, b] \mapsto \mathbb{R}^n$ that is piecewise C^1 , i.e., there's $k \geq 1$ and a partition

$$a = t_0 < t_1 < \dots < t_{k-1} < t_k = b$$

such that the restriction of f to $]t_{j-1}, t_j[$ is C^1 for $1 \leq j \leq k$. Then we say that γ is a parametrized curve, or path, between $\gamma(a)$ and $\gamma(b)$.

3. Let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a parametrized curve. Let $X \subset \mathbb{R}^n$ be a subset containing the image of γ . Let $g : X \mapsto \mathbb{R}^n$ be a continuous function. Then the integral

$$\int_a^b g(\gamma(t)) \gamma'(t) dt \in \mathbb{R}$$

is called the **line integral** of g along γ . Denoted

$$\int_{\gamma} g(s) \cdot ds \quad \text{or} \quad \int_{\gamma} g(s) \cdot d\vec{s}$$

When working with line integrals, we say that $f : X \mapsto \mathbb{R}^n$ is a **vector field**.

Proposition idk. This integral of continuous functions $I \mapsto \mathbb{R}^n$ (one variable) satisfies

$$\int_a^b (f(t) + g(t)) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$$

and

$$\int_a^b f(t) dt = - \int_b^a f(t) dt$$

Definition 4.1.4. Let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a parametrized curve. An **oriented reparametrization** of γ is a parametrized curve $\sigma : [c, d] \mapsto \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$, differentiable on $]a, b[$, strictly increasing and satisfies $\varphi(a) = c, \varphi(b) = d$, where $\varphi : [c, d] \mapsto [a, b]$ is a continuous map.

Proposition 4.1.5. Let γ be a parametrized curve in \mathbb{R}^n , σ an oriented reparametrization of γ . Let X be a set containing the image of γ (or, equivalently, the image of σ), and $f : X \mapsto \mathbb{R}^n$ a continuous function.

Then the line integrals are the same:

$$\int_{\gamma} f(s) \cdot d\vec{s} = \int_{\sigma} f(s) \cdot d\vec{s}$$

Definition 4.1.8. Let $X \subset \mathbb{R}^n$ and $f : X \mapsto \mathbb{R}^n$ a continuous vector field.

If for any $x_1, x_2 \in X$ the line integral is independent of the choice of γ in X from x_1 to x_2 , then we say that the vector field is **conservative**.

Remark 4.1.9. Equivalently, f is conservative iff

$$\int_{\gamma} f(s) \cdot d\vec{s} = 0$$

for any **closed** parametrized curve γ in X . A curve is closed if $\gamma(a) = \gamma(b)$.

Theorem Hidden in the page (gradient vector conservative). If X is open, then any vector field of the form $f = \nabla g$, where g is of class C^1 on X , is conservative.

Theorem 4.1.10. Let X be open and f a conservative vector field. Then there exist a C^1 function g on X such that $f = \nabla g$.

If any two points on X can be joined by a parametrized curve, then g is unique up to addition of a constant: if $\nabla g_1 = f$, then $g - g_1$ is constant on X .

Remark 4.1.11. To say that Any two points of X can be joined by a parametrized curve means that, for all $x, y \in X$, there exist a p.c. $\gamma : [a, b] \mapsto X$ such that $\gamma(a) = x, \gamma(b) = y$. When this is true, we say that X is **path-connected**.

(it is true whenever X is **convex**)

If f is a conservative vector field on X , then a function g such that $\nabla g = f$ is called a **potential** for f . Note that g is not unique and can differ of at least a constant.

Proposition 4.1.13. Let $X \subset \mathbb{R}^n$ be an open set, $f : X \mapsto \mathbb{R}^n$ a vector field of class C^1 . Write

$$f(x) = (f_1(x), \dots, f_n(x))$$

If f is conservative, then we have, for any integers with $1 \leq i \neq j \leq n$:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Example 4.1.14. Consider $f(x, y, z) = (y^2, xz, 1)$. Clearly, $\partial_y(y^2) = 2y \neq z = \partial_x(xz)$ Then f can't be conservative.

Definition 4.1.15. A subset $X \subset \mathbb{R}^n$ is **star shaped** if there exists $x_0 \in X$ so that for all $x \in X$, the line connecting the two is contained in X .

Then X is star-shaped around x_0 .

Theorem 4.1.17. Let $X \subset \mathbb{R}^n$ be star shaped and f be a C^1 vector field such that on X , for any $1 \leq i \neq j \leq n$:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad (2)$$

Then f is **conservative**

Definition 4.1.20. Let $X \subset \mathbb{R}^3$ be an open set and $f : X \mapsto \mathbb{R}^3$ a C^1 vector field. Then the curl of f , $\text{curl}(f)$, is the continuous vector field on X

$$\text{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

Where $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$

If $\text{curl}(f) = 0$, then condition (2) holds!

Remark 4.1.21 (remember the definition with determinant). With (e_1, e_2, e_3) being the canonical basis of \mathbb{R}^3 and expanding with $\partial_x \cdot f_i = f_i \partial_x = \partial_x f_i$:

$$\text{curl}(f) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

3.2 The Riemann integral in \mathbb{R}^n

For any bounded closed $X \subset \mathbb{R}^n$ and continuous function $f : X \mapsto \mathbb{R}$, one can define the integral of f over X , denoted

$$\int_X f(x) dx \in \mathbb{R}$$

The integral satisfies the properties:

1. **(Compatibility)** if $n = 1$ and $X = [a, b]$ is an interval ($a \leq b$), then the integral of f over X is the Riemann integral of f :

$$\int_{[a, b]} f(x) dx = \int_a^b f(x) dx$$

2. **(Linearity)** if f, g are continuous on X and $a, b \in \mathbb{R}$, then

$$\int_X (af_1(x) + bf_2(x)) dx = a \int_X f_1(x) dx + b \int_X f_2(x) dx$$

3. **(Positivity)** if $f \leq g$, then

$$\int_X f(x) dx \leq \int_X g(x) dx$$

and especially, if $f \geq 0$, then

$$\int_X f(x) dx \geq 0$$

Moreover, if $Y \subset X$ is compact and $f \geq 0$, then

$$\int_Y f(x) dx \leq \int_X f(x) dx$$

4. **(Upper bound and triangle inequality)** Since $-|f| \leq f \leq |f|$, we have

$$\left| \int_X f(x) dx \right| \leq \int_X |f(x)| dx$$

and since $|f + g| \leq |f| + |g|$

$$\left| \int_X (f(x) + g(x)) dx \right| \leq \int_X |f(x)| dx + \int_X |g(x)| dx$$

5. **(Volume)** if $f = 1$, then the integral of f is the "volume" in \mathbb{R}^n of the set X . If $f \geq 0$ in general, the integral of f is the volume of the set

$$\{(x, y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x)\} \subset \mathbb{R}^{n+1}$$

In particular, if X is a bounded "rectangle", say

$$X = [a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$$

and $f = 1$, then

$$\int_X dx = (b_n - a_n) \cdot \dots \cdot (b_1 - a_1)$$

We write $\text{Vol}(X)$ or $\text{Vol}_n(X)$ for the volume of X .

6. **(Multiple Integral, or Fubini's Theorem)** If $n_1, n_2 \geq 1$ are integers such that $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^{n_1}$, let

$$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$$

Let X_1 be the set of x_1 such that Y_{x_1} is not empty. Then X_1 is compact in \mathbb{R}^{n_1} and Y_{x_1} is compact in \mathbb{R}^{n_2} for all $x_1 \in X_1$. If the function

$$g(x_1) = \int_{Y_{x_1}} f(x_1, x_2) dx_2$$

on X_1 is continuous, then

$$\int_X f(x_1, x_2) dx = \int_{X_1} g(x_1) dx_1 = \int_{X_1} \left(\int_{Y_{x_1}} f(x_1, x_2) dx_2 \right) dx_1$$

Similarly, exchanging the role of x_1 and x_2 , we have

$$\int_X f(x_1, x_2) dx = \int_{X_2} \left(\int_{Z_{x_2}} f(x_1, x_2) dx_1 \right) dx_2$$

Where $Z_{x_2} = \{x_1 \mid (x_1, x_2) \in X\}$, if the integral over x_1 is a continuous function.

7. **(Domain additivity)** if X_1 and X_2 are compact subsets of \mathbb{R}^n and f is continuous on $X_1 \cup X_2$, then

$$\int_{X_1 \cup X_2} f(x) dx + \int_{X_1 \cap X_2} f(x) dx = \int_{X_1} f(x) dx + \int_{X_2} f(x) dx$$

Notice that $X_1 \cap X_2$ is also compact, so all integrals exist. If $X_1 \cap X_2$ is empty, then the integral over it is equal to 0, shortening the formula to a more convenient form. This is also true if the intersection is **negligible** (Def 4.2.3).

Definition 4.2.3. 1. Let $1 \leq m \leq n$. A **parametrized m-set** in \mathbb{R}^n is a continuous map

$$f : [a_1, b_1] \times \dots \times [a_m, b_m] \mapsto \mathbb{R}^n$$

which is C^1 on

$$[a_1, b_1] \times \dots \times [a_m, b_m]$$

2. A subset $B \subset \mathbb{R}^n$ is **negligible** if there exist an integer $k \geq 0$ and parametrized m_i -sets $f_i : X_i \mapsto \mathbb{R}^n$ with $1 \leq n \leq k$ and $m_i < n$ such that

$$X \subset f_1(X_1) \cup \dots \cup f_k(X_k)$$

Example 4.2.4. Any subset of the real axis $\mathbb{R} \times \{0\}$ is negligible in \mathbb{R}^2 .

More generally, if $H \subset \mathbb{R}^n$ is an affine subspace of dimension $m < n$, then any subset of \mathbb{R}^n that is contained in H is negligible.

Proposition 4.2.5. Let $X \subset \mathbb{R}^n$ be a compact set. Assume that X is negligible. Then for any continuous function on X we have

$$\int_X f(x) dx = 0$$

3.3 Improper integrals

Some basic definitions on \mathbb{R}^2 .

Let $I \subset \mathbb{R}$ be a bounded interval, $J = [a, +\infty[$ for some $a \in \mathbb{R}$. Let f be a continuous function on $X = J \times I$. We say that f is **Riemann-integrable** on X if the following limit exists

$$\lim_{x \rightarrow +\infty} \int_{[a, x] \times I} f(x, y) dx dy = \lim_{x \rightarrow +\infty} \int_I \left(\int_a^x f(x, y) dx \right) dy$$

The equality is a case of Fubini's Theorem. We denote this limit with

$$\int_{J \times I} f(x, y) dx dy$$

Similarly, let f be continuous on \mathbb{R}^2 . Assume that $f \geq 0$. We say that f is Riemann-integrable on \mathbb{R}^2 if the following limit exists

$$\lim_{R \rightarrow +\infty} \int_{[-R, R]^2} f(x, y) dx dy$$

This limit is called the **integral of f over \mathbb{R}^2** and denoted

$$\int_{\mathbb{R}^2} f(x, y) dx dy$$

Fubini formula for this:

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x, y) dy \right) dx$$

Remark 4.3.1. In all these cases we also often say that "the integral converges" to indicate that a function is Riemann-integrable on an unbounded set.

If $|f| \leq g$ and the integral of g is Riemann-integrable on an unbounded set, then f also does.

3.4 The change of variable formula

Analogue of the one for one-variable calculus

$$\left(\int f(g(x)) g'(x) dx = \int f(y) dy \right)$$

Let $\bar{X}, \bar{Y} \subset \mathbb{R}^n$ be compact subsets. Let $\varphi : \bar{X} \mapsto \bar{Y}$ be a continuous map.

We assume that we can write $\bar{X} = X \cup B$ and $\bar{Y} = Y \cup C$ where

- the sets X, Y are open.
- The sets B, C are negligible (Def ??def:4.2.3)

- the restriction of φ to the open set X is a C^1 bijective map from X to Y .

Then $J_\varphi(x)$ is invertible at all $x \in X$. We assume that we can find a continuous function on \bar{X} that restricts to $\det(J_\varphi(x))$ on X (we have a formula for the Jacobian, so this is obvious in most cases). Abuse notation and write it even if $x \in B$.

Remark 4.4.1. There is no assumption concerning the image of B .

Sometimes φ is the restriction of a C^1 map $\mathbb{R}^n \mapsto \mathbb{R}^n$, in which case the last issue doesn't require any argument.

Theorem 4.4.2 (Change of variable formula). In the situation described above, for any continuous function f on \bar{Y} , we have

$$\int_{\bar{X}} f(\varphi(x)) |\det(J_\varphi(x))| dx = \int_{\bar{Y}} f(y) dy$$

To remember, when $y = \varphi(x)$, then $dy = |\det(J_\varphi(x))| dx$.

Special cases:

- When $\varphi(x) = x + x_0$ (translation): φ is affine-linear and $J_\varphi(x) = 1_n$ (identity matrix). The change of variable formula becomes, for any compact subset \bar{X} and any continuous function f on $x_0 + \bar{X}$:

$$\int_{\bar{X}} f(x + x_0) dx = \int_{x_0 + \bar{X}} f(x) dx$$

- When φ is a restriction of a bijective linear map, namely $\varphi(x) = Ax$, where A is an invertible matrix of size n . Then $J_\varphi(x) = A$ for all $x \in \mathbb{R}^n$ with constant determinant $\det(A)$. Let $\bar{X} = X \cup B$ be compact as above and $\bar{Y} = \varphi(\bar{X})$. Then $\varphi(\bar{X}) = \varphi(X) \cup \varphi(B)$. The change of variable formula becomes (for any continuous f on \bar{Y})

$$\int_{\bar{X}} f(\varphi(x)) dx = \frac{1}{|\det(A)|} \int_{\bar{Y}} f(y) dy$$

Example (Standard examples). 1. **Polar coordinates** (r, θ) are useful for integrating over a disc in \mathbb{R}^2 centered at 0, or more generally over a disk sector $\delta = \delta(a, b, R)$ defined

$$0 \leq r \leq R, \quad -\pi < a \leq \theta \leq b < \pi$$

We compute the Jacobian det. and obtain

$$\int_{\delta} f(x, y) dx dy = \int_0^R \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

- Spherical Coordinates** (r, θ, φ) in \mathbb{R}^3 , integrate on balls centered at 0. Jacobian determinant is $-r^2 \sin(\varphi)$. To integrate a function f over a ball of radius R we use

$$\int_B f(x, y, z) dx dy dz = \int_0^R \int_0^{2\pi} \int_0^\pi f(\bar{x}, \bar{y}, \bar{z}) r^2 \sin(\varphi) \, dr \, d\theta \, d\varphi$$

With $\bar{x} = r \cos(\theta) \sin(\varphi)$, $\bar{y} = r \sin(\theta) \sin(\varphi)$, $\bar{z} = r \cos(\varphi)$.

3.5 Geometric applications of integrals

Welp, applications that can actually be useful

- (Center of mass)** Let X be a compact subset of \mathbb{R}^n of positive volume. The *center of mass* (or *barycenter*) of X is the point $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ with

$$\bar{x}_1 = \frac{1}{Vol X} \int_X x_i dx$$

Intuitively, x_i is the average over X of the i -th coordinate and \bar{x} is the point where X is "perfectly balanced" (could be outside of X !).

- (Surface area)** Consider a function $f : [a, b] \times [c, d] \mapsto \mathbb{R}$ which is C^1 on the open interval. Let

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in [a, b] \times [c, d], \, z = f(x, y)\} \subset \mathbb{R}^3$$

be the graph of f . Intuitively, this is a surface and it has area

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x, y))^2 + (\partial_y f(x, y))^2} dx dy$$

Analogue for a function $f : [a, b] \mapsto \mathbb{R}$ (**length**):

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

3.6 The Green formula

Definition 4.6.1. A **simple closed parametrized curve** $\gamma : [a, b] \mapsto \mathbb{R}^2$ is a closed ($\gamma(a) = \gamma(b)$) parametrized curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or they are a, b , and such that $\gamma'(t) \neq 0$ for $a < t < b$ (if γ is only piecewise C^1 , this condition only applies where $\gamma'(t)$ exists).

Theorem 4.6.3 (Green's formula). Let $X \subset \mathbb{R}^2$ be a compact set with a boundary ∂X that is the union of finitely many simple closed parametrized curves $\gamma_1, \dots, \gamma_k$. Assume that

$$\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^2$$

has the property that X lies always "to the left" of the tangent vector $\gamma'(t)$ based at $\gamma_i(t)$.

Let $f = (f_1, f_2)$ be a vector field of class C^1 defined on some open set containing X . Then we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy = \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}$$

If the orientation is not met, the curve can be "reversed", e.g., replaced with $\tilde{\gamma} = \gamma(1 - t)$, which reverses the orientation of the tangent vector.

Corollary 4.6.5. Let $X \subset \mathbb{R}^2$ compact set with boundary ∂X that is the union of finitely many s.c.p.c $\gamma_1, \dots, \gamma_k$. Assume that $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) : [a_i, b_i] \mapsto \mathbb{R}^2$ has the property that X always lies "left" of the tangent vector. Then we have

$$Vol(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \gamma'_{i,2}(t) dt$$

3.7 The Gauss-Ostrogradski formula

Analogue of the Green formula in \mathbb{R}^3 .

Definition 4.7.1. A parametrized surface $\Sigma : [a, b] \times [c, d] \mapsto \mathbb{R}^3$ is a 2-set in \mathbb{R}^3 such that the rank of the J . matrix is 2 at all $(s, t) \in]a, b[\times]c, d[$ Note that 2 is the max rank (there are two variables).

Definition 4.7.3 (vector product). Let x, y be two linearly independent vectors in \mathbb{R}^3 . The vector product (or cross product) $z = x \times y$ is the unique vector such that (x, y, z) is a basis of \mathbb{R}^3 (z perpendicular to the plane generated by x, y , also pairwise lin. indep.) with $\det(x, y, z) \geq 0$ and

$$\|z\| = \|x\| \cdot \|y\| \cdot \sin(\theta)$$

Where $\theta = \angle(x, y)$

If x, y are not lin. indep., then we define $x \times y = 0$, the zero vector.

Ez formula for canonical base only (remember the oral exam back in the day!):

$$x \times y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Also $y \times x = -x \times y$.

Theorem 4.7.6 (Gauss-Ostrogradski formula). Let $X \subset \mathbb{R}^3$ be a compact set with a boundary ∂X that is a parametrized surface $\Sigma : [a, b] \times [c, d] \mapsto \mathbb{R}^3$.

Assume that Σ is injective in the open interval, and that the normal vector of Σ points away from the surface at all points.

Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ be the unit exterior normal vector.

Let $f = (f_1, f_2, f_3)$ be a C^1 vector field defined on some open set containing X . Then we have

$$\int_X \operatorname{div}(f) dx dy dz = \int_{\Sigma} (f \cdot \vec{u}) d\sigma$$

Clarify: $\div(f)$ is the divergence of the vector field f , $\operatorname{div}(f) = \partial_x f + \partial_y f + \partial_z f$.

4 Trigonometry table

Angle (deg)	0	30	45	60	90	180
Angle (rad)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.	0
cot	N.D.	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	N.D.

5 Limit Cheat Sheet

$x \in \mathbb{R}, \quad a, b \in \mathbb{R}^+, \quad n \in \mathbb{N}$	
$\lim_{n \rightarrow \infty} a^n = +\infty$ if $a > 1$	$\lim_{n \rightarrow \infty} a^n = 0$ if $0 < a < 1$
$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$ if $a > 0$
$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$	
$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$
$\lim_{x \rightarrow \infty} \frac{\ln^b(x)}{x^a} = 0$	$\lim_{x \rightarrow 0} x^a \ln^b(x) = 0$
$\lim_{x \rightarrow \infty} \frac{e^{ax}}{x^b} = +\infty$	$\lim_{x \rightarrow \infty} \frac{x^b}{e^{ax}} = 0$

6 Derivative Cheat Sheet

Properties

$$(cf)' = cf'(x) \qquad (f \pm g)' = f'(x) \pm g'(x)$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(g(f(x))) = g'(f(x))f'(x)$$

$f(x)$	$f'(x)$
x	1
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
\sqrt{x}	$\frac{1}{2\sqrt{x}}, \quad x \neq 0$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$ x $	$x > 0 \implies 1, \text{ or } x < 0 \implies -1, x \neq 0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x) = \frac{1}{x \ln(a)}$	
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\coth(x)$	$-\frac{1}{\sinh^2(x)} = 1 - \coth^2(x)$
$\operatorname{asinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{acosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{atanh}(x)$	$\frac{1}{1-x^2}$

7 Integral Cheat Sheet

	$\int f(x)dx = F(x) + C$
Per parti	$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$
Per sostituzione immediata	$\int g(f(x))f'(x)dx = G(f(x)) + C$
Per sostituzione (cambiamento di variabile)	$\int g(x)dx = \int g(f(t))f'(t)dt$ with $x = f(t)$
Integrale logaritmico	$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$

$f(x)$	$F(X)(without + C)$
a	$\frac{ax}{x^{n+1}}$
x^n	$\frac{1}{n+1}$
$\frac{1}{x}$	$\ln x $
\sqrt{x}	$\frac{2}{3}x\sqrt{x}$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b} \ln \left \frac{x-a}{x-b} \right $
$\frac{ax+b}{cx+d}$	$\frac{ax}{c} - \frac{ad-bc}{c^2} \ln cx+d $
$\frac{1}{x^2+a^2}$	$\frac{1}{a} \arctan \left(\frac{x}{a} \right)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $
e^x	e^x
$\ln(x)$	$x(\ln(x) - 1)$
a^x	$\frac{a^x}{\ln(a)}$
$\log_a(x)$	$x(\log_a(x) - \log_a(e))$
xe^{ax}	$\frac{1}{a^2}(ax-1)e^{ax}$
$x \ln(ax)$	$\frac{x^2}{4}(2 \ln(ax) - 1)$
$\sin(x)$	$-\cos(x)$
$\arcsin(x)$	$x \arcsin(x) + \sqrt{1-x^2}$
$\cos(x)$	$\sin(x)$
$\arccos(x)$	$x \arccos(x) - \sqrt{1-x^2}$
$\tan(x)$	$-\ln \cos(x) $
$\arctan(x)$	$x \arctan(x) - \frac{1}{2} \ln(1+x^2)$
$\cot(x)$	$\ln \sin(x) $
$\operatorname{arccot}(x)$	$x \operatorname{arccot}(x) + \frac{1}{2} \ln(1+x^2)$
$\sin^2(x)$	$\frac{1}{2}(x - \sin(x) \cos(x))$
$\cos^2(x)$	$\frac{1}{2}(x + \sin(x) \cos(x))$
$\tan^2(x)$	$\tan(x) - x$
$\sqrt{x^2+a}$	$\frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2} \ln \left x + \sqrt{x^2+a} \right $
$\frac{1}{\sqrt{x^2+a}}$	$\ln \left x + \sqrt{x^2+a} \right $
$\sqrt{r^2-x^2}$	$\frac{1}{2}x\sqrt{r^2-x^2} + \frac{r^2}{2} \arcsin \left(\frac{x}{r} \right)$
$\frac{1}{\sqrt{r^2-x^2}}$	$\arcsin \left(\frac{x}{r} \right)$

More?