Analysis 2 Recap

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1 Linear differential equations

Definition L. *inear Differential equation. Homogeneous if $b = 0$, inhomogeneous otherwise.*
$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1y' + a_0y = b$$

Theorem 2.2.3. *...y is k-times differentiable ...*

For the homogeneous equation, given a choice of x_0 and $(y_0, ..., y_{k-1})$ there's a unique solution $f \in \mathcal{S}$ such that
$$f(x_0) = y_0,\ f'(x_0) = y_1,\ \dots,\ f^{(k-1)}(x_0) = y_{k-1} \qquad (1)$$

For the inhomogeneous equation with a b continous on the interval, the set of solutions $\mathcal{S}_b$ is the set of functions $f + f_0$ where $f \in \mathcal{S}$. Again, for any x_0 and $(y_0, ..., y_{k-1})$ there's a unique solution such that ((1)).

If $b \neq 0$ then $\mathcal{S}_b$ is not a vector space.

Proposition 2.3.1. *Any solution of $y' + ay = 0$ is in the form $f(x) = ze^{-A(x)}$, where A is a primitive of a and $z \in \mathbb{C}$. Unique solution is $f(x) = y_0e^{A(x_0)-A(x)}$*

To solve the inhomogeneous equation $y' + ay = b$, the prev solution is used. Using *Variation of the constant* we replace z with $z(x)$ and then $y' + ay = b \Leftrightarrow z'(x) = b(x)\ e^{A(x)}$ and $f_0(x) = C(x)e^{-A(x)}$, where $C(x)$ is a primitive of $z'(x)$.

1.1 Constant coefficients

Definition L. *et $a_0, ..., a_{k-1} \in \mathbb{C}$. Linear differentian equation $y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1y' + a_0y = b$. Homogeneous solution is in the form $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{C}$. We have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geq 0$ and x.*

Conclusion: $f(x)$ is a solution iff $P(\alpha) = 0$, where $P(X) = X^k + a_{k-1}X^{k-1} + ... + a_1X + a_0$.

This polynomial of degree k has k roots (counted with multiplicity). There exist complex numbers $\alpha_1, ..., \alpha_k$ such that $P(X) = (X-\alpha_1)...(X-\alpha_k)$. This is the companion or characteristic polynomial of the homogeneous diff. equation.

- No multiple roots When $\alpha_i \neq \alpha_j$ for all i, j.

  Solution of the homogeneous equation ($b = 0$): form $f(x) = z_1e^{\alpha_1 x} + ... + z_ke^{\alpha_k x}$. Unique solution with $f(x_0) = y_0$, ..., $f^{(k-1)}(x_0) = y_{k-1}$ can be obtained by viewing z_i as unknowns. Substitute $x = x_0$ in the formula for f and solve for $z_1, ..., z_k$ (linear system).

- Multiple roots Assume α is a multiple root of order j of the polynomial P, with $2 \leq j \leq k$. Then
  $$f_{\alpha,0}(x) = e^{\alpha x},\ f_{\alpha,1}(x) = xe^{\alpha x},\ \dots,\ f_{\alpha,j-1}(x) = x^{j-1}e^{\alpha x}$$
  are linearly independent and are solutions of the h.l.d.e.

  Example S. *uppose $P(X) = X(X-4)^3(X-(1+i))(X-(1-i))$, then the solutions are $f_0(x) = 1$ (sol. for $X = 0$), $f_1(x) = e^{4x}$, $f_2(x) = xe^{4x}$, $f_3(x) = x^2e^{4x}$, $f_4(x) = e^{(1+i)x}$, $f_5(x) = e^{(1-i)x}$*

Now the inhomogeneous equation ($b \neq 0$):
Should avoid variation of the constants. Can use special cases:

1. $b(x) = x^de^{\beta x}$ for some integer $d = ge0$ and an item β which is NOT a root of P, then the solution is of the form $f(x) = Q(x)e^{\beta x}$, where Q is a polynomial of degree d.

2. $b(x) = x^d\cos(\beta x)$ or $b(x) = x^d\sin(\beta x)$ for some integer $d \geq 0$ and β is NOT a root of P, then one can transform it to a combination of complex exponentials or look for a solution of the form $f(x) = Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)$, Q_1, Q_2 have degree d.

3. $b(x)$ is in the form of the previous two but IS a root of multiplicity j, then one looks for $f(x) = Q(x)e^{\beta x}$, with Q of degree $q + j$.

4. Special case $\beta = 0$ of the previous 3 (b polynomial of degree $d \geq 0$): if 0 is NOT a root, look for a solution f (polynomial) of deg d, or degree $d + j$ if 0 IS a root, where j is the multiplicity of 0.

1.2 Variation of the constants for degree ge 2

Does not require the coefficients to be constants, but it makes it easier.

Inhomogeneous equation
$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1y' + a_0y = b$$
Solutions $f_1, ..., f_k$ for the homogeneous equations must be found first.

We then search for a solution of the form $f(x) = z_1(x)f_1(x) + ... + z_k(x)f_k(x)$, such that we have (for all x):
$$\begin{cases} z_1'(x)f_1(x) + ... + z_k'(x)f_k(x) = 0 \\ z_1'(x)f_1'(x) + ... + z_k'(x)f_k'(x) = 0 \\ ... \\ z_1'(x)f_1^{(k-2)}(x) + ... + z_k'(x)f_k^{(k-2)}(x) = 0 \end{cases}$$

Example C. *ase $k = 2$: Write again $f = z_1f_1 + z_2f_2$ and the constraint $z_1'f_1 + z_2'f_2 = 0$.*

2 Differential in $\mathbb{R}^n$

Definition 3.3.5. *$f : X \mapsto \mathbb{R}$ has a partial derivative with respect to the i-th variable if the function*
$$g(t) = f(x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n})$$
is differentiable for all $x_0 \in X$ on the set $I = \{t \in \mathbb{R} \mid (x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n}) \in X\}$.

Its derivative $g'(x_{0,i})$ is denoted
$$\frac{\partial f}{\partial x_i}(x_0),\ \partial_{x_i}(x_0),\ \partial_i(x_0)$$

Proposition 3.3.7. *$x \subset \mathbb{R}^n$ open, f, g functions from X to $\mathbb{R}^m$. Let $1 \leq 1 \leq n$.*

1. *if f, g have partial derivatives of i-th coordinate on X, then $f + g$ also does. $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$*

2. *if the previous is true and $m = 1$, then fg also does and $\partial_{x_i}(fg) = \partial_{x_i}(f) \cdot g + f \cdot \partial_{x_i}(g)$*

3. *If the previous is true and $g(x) \neq 0$ for all $x \in X$, then f/g has a partial derivative $\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2$*

Definition 3.3.9. *$f : X \mapsto \mathbb{R}^m$ has partial derivatives on X. Write $f(x) = (f_1(x), f_2(x), ..., f_m(x))$.*

*For any $x \in X$, the **Jacobi Matrix** (m rows, n columns) of f at x is defined as*
$$J_f(x) = (\partial_{x_i}f_i(x))_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$$

Definition 3.3.11. 1. *If all partial derivatives of $f : X \mapsto \mathbb{R}$ exist at $x_0 \in X$, then the column vector*
$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1}f(x_0) \\ ... \\ \partial_{x_n}f(x_0) \end{pmatrix}$$
*is called the **gradient** of f at x_0.*

2. *Let $f = (f_1, f_2, ..., f_n) : X \mapsto \mathbb{R}^n$ and all partial derivatives of all coordinates f_i of f exist at $x_0 \in X$. Then*
$$Tr(J_f(x_0)) = \sum_{i=1}^{n} \partial_{x_i}f_i(x_0)$$
*is the trace of the Jacobi Matrix and is called the **divergence** of f at x_0, also $div(f)(x_0)$*

Definition 3.4.2. *Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function. Let u be a linear map $\mathbb{R}^n \mapsto \mathbb{R}^m$ and $x_0 \in X$. We say that f is differentiable at x_0 with differential u if*
$$\lim_{\substack{x\to x_0 \\ x\neq 0}} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$$
We then denote $df(x_0) = u$. If it is differentiable at every $x_0 \in X$, then it is differentiable on X.

Then, close to x_0, we can approximate $f(x)$ by $g(x) = f(x_0) + u(x - x_0)$

Proposition 3.4.4. *Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function differentiable on X. Then*

1. *f is **continuous** on X.*

2. *f admits partial derivatives on X with respect to each variable.*

3. *Assume that $m = 1$. Let $x_0 \in X$ and $u(x_1, ..., x_n) = a_1x_1 + ... + a_nx_n$ be the differential of f at x_0. We then have $\partial_{x_i}f(x_0) = a_i$ for $1 \leq i \leq n$*

Proposition 3.4.6. *$X \subset R^n$ open, $f : x \mapsto R^m$, $g : X \mapsto R^m$ differentiable functions on X.*

1. *$f + g$ is differentiable on X with differential $d(f + g) = df + df$.*

2. *If $m = 1$, then fg is differentiable. If we also have $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable.*

Proposition 3.4.7. *If f has all partial derivatives on X and they are all continuous on X, then f is differentiable on X. The matrix of the differential $df(x_0)$ is the Jacobi Matrix of f at x_0.*

Proposition 3.4.9 (Chain Rule). *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open. Let $f : X \mapsto Y$ and $g : Y \mapsto R^p$ be differentiable functions. Then $g \circ f : X \mapsto \mathbb{R}^p$ is differentiable on X, and for any $x \in X$, its differential is given by*

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

The Jacobi Matrix satisfies

$$J_{g\circ f}(x_0) = J_g(f(x_0))J_f(x_0) \text{ (matrix product)}$$

Definition L. *et $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ differentiable. Let $x_0 \in X$ and $u = df(x_0)$ be the differential of f at x_0. The graph of the affine linear approximation*

$$g(x) = f(x_0) + u(x - x_0)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m$, or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

*is called the **tangent space** at x_0 to the graph of f.*

Definition 3.4.13 (Directional Derivative). *Let $X \subset \mathbb{R}^n$ be open, $f : X \mapsto \mathbb{R}^m$ a function.*

Let $v \in \mathbb{R}^n$ be a non-zero vector and $x_0 \in X$.

*We say that f has **directional derivative** $w \in \mathbb{R}^m$ in the direction v if the function g defined on the set I has a derivative at $t = 0$ and this is equal to w.*

$$g(t) = f(x_0 + tv), \quad I = \{t \in \mathbb{R} \mid x_0 + tv \in R\}$$

Other words: limit is equal to w

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Proposition 3.4.15. *X open, f differentiable. Then for any $x_0 \in X$ and non-zero v, the function has a directional derivative at x_0 in the direction v, equal to $df(x_0)(v)$*

Definition 3.5.1. *Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$. We say that f is of class:*

- *C^1 is differentiable on X and all partial derivatives are continuous.*
- *C^k if differentiable on X and all partial derivatives $\partial_{x_i} f : X \mapsto \mathbb{R}^m$ are of class C^{k-1}.*
- *C^∞ if $f \in C^k(X; \mathbb{R}^m)$ for all $k \geq 1$*

Set of functions of class C^k from X to $\mathbb{R}^m$ is denoted $C^k(X; \mathbb{R}^m)$

Proposition 3.5.4 (Mixed derivatives commute). *$X \subset \mathbb{R}^n$ open and $f : X \mapsto \mathbb{R}^m$ of class C^k. Then the partial derivatives of order k are independent of the order in which the partial derivatives are taken: for any variables $x_1, x_2, ..., x_n$ we have*

$$\partial_{x_1,x_2,...,x_n} f = \partial_{x_2,x_1,...,x_n} f = ... \text{ (all combinations)}$$

Definition 3.5.9 (Hessian). *Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ a C^2 function. For $x \in X$, the **Hessian matrix** of f at x is the **symmetrix square** matrix*

$$H_f(x) = \mathrm{Hess}_f(x) = (\partial_{x_i,x_j} f)_{1\leq i,j\leq n}$$

Example (Change of variable). *Idea: create h, which is f on a different coordinate system.*

Open set $U \subset \mathbb{R}^n$ containing the new variables $(y_1, ..., y_n)$ and a change of variable $g : U \mapsto X$ that expresses $(x_1, ..., x_n)$ in terms of $(y_1, ..., y_n)$.

Consider $x_1 = g_1(y_0, ..., y_n), \quad x_n = g_n(y_1, ..., y_n)$

Composite $h = f \circ g : U \mapsto \mathbb{R}$ is the function f expressed in terms of the new variables y.

Polar coordinates: Map $g : U \mapsto \mathbb{R}^2$, $g(r, \theta) = (r \cos\theta, r \sin\theta)$. Replace f by h: $h(r, \theta) = f(r \cos\theta, r \sin\theta)$.

$$J_g(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

2.1 Taylor Polynomials

Definition 3.7.1 (Taylor polynomials). *Let $k \geq 1$ be an integer, $f : X \mapsto \mathbb{R}$ a function of class C^k on X, and fix $x_0 \in X$. The k-th Taylor polynomial of f at point x_0 is the poly in n variables of degree $\leq k$ given by*

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) y_i + ...$$

$$+ \sum_{m_1+...+m_n=k} \frac{1}{m_1!...m_n!} \frac{\partial^k f}{\partial x_1^{m_1} ... \partial x_n^{m_n}}(x_0) y_1^{m_1} ... y_n^{m_n}$$

where the last sum ranges over the tuples of n positive integers such that the sum is k.

Case $n = 1$ (one variable):

$$T_k f(y; x_0) = f(x_0) + f'(x_0)y + \frac{f''(x_0)}{2} y^2 + ... + \frac{f^{(k)}(x_0)}{k!} y^k$$

Proposition 3.7.3 (Taylor Approximation). *$k \geq 1$, $X \subset \mathbb{R}^n$ open, $f : X \mapsto \mathbb{R}$ a C^k function. For x_0 in X, we define $E_k f(x; x_0)$ by*

$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$

then we have

$$\lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{e_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

2.2 Critical Points

Proposition 3.8.1. *Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ be differentiable. If $x_0 \in X$ is a local maximum or a local minimum, then we have (equivalent) for $1 \leq i \leq n$:*

$$df(x_0) = 0, \ \nabla f(x_0) = 0, \ \frac{\partial f}{\partial x_i}(x_0) = 0$$

Definition 3.8.2 (Critical Point). *Let X be open and f be differentiable. A point x_0 is called a **critical point** of f if $\nabla f(x_0) = 0$.*

Definition 3.8.6 (Non-degenerate critical point). *f of class C^2. A critical point x_0 is **non-degenerate** if the Hessian matrix has non-zero determinant.*

Corollary 3.8.7. *X open and $f : X \mapsto \mathbb{R}$ of class C^2. Let x_0 be a non-degenerate critical point of f. Let p, q be the number of positive and negative eigenvalues of $Hess_f(x_0)$*

1. *if $p = n$, equivalently if $q = 0$, the function f has a local minimum at x_0.*
2. *if $q = n$, equivalently if $p = 0$, f has a local maximum at x_0.*
3. *Otherwise, the function f does not have a local extremum at x_0, equivalently it has a saddle point at x_0.*

2.3 Lagrange multipliers

Proposition 3.9.2 (Lagrange Multiplier). *Let $X \subset \mathbb{R}^n$ be open and $f, g : X \mapsto \mathbb{R}$ be class C^1. If $x_0 \in X$ is a local extremum of f restricted to the set $Y = \{x \in X \mid g(x) = 0\}$ ($\nabla f(x_0)$ can be non-zero!), then either $\nabla g(x_0) = 0$ or there exist λ such that*

$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$

In other words, (x_0, λ) is a critical point of $h(x, \lambda) = f(x) - \lambda g(x)$.

Value λ is the Lagrange Multiplier at x_0.

2.4 The inverse and implicit functions theorems

Definition 3.10.1 (Change of variable).

Theorem 3.10.2 (Inverse function theorem). *$X \subset \mathbb{R}^n$ open and $f : X \mapsto \mathbb{R}^n$ differentiable. If the jacobian matrix of f at $x_0 \in X$ is invertible ($det(J_f(x_0)) \neq 0$) then f is a change of variable around x_0.*

Moreover, $J_g(f(x_0)) = J_f(x_0)^{-1}$.

In addition, if f is of class C^k, then g is also of class C^k.

Theorem 3.10.4 (Implicit function theorem). *Let $X \subset \mathbb{R}^{n+1}$ be open, $g : X \mapsto \mathbb{R}$ be of class C^k with $k \geq 1$. Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $g(x_0, y_0) = 0$.*

Assume that $\partial_y g(x_0, y_0) \neq 0$.

Then there exists an open set $U \subset \mathbb{R}^n$ containing x_0, an open interval $I \subset \mathbb{R}$ containing y_0, and a function $f : u \mapsto \mathbb{R}$ of class C^k such that the system of equations

$$\begin{cases} g(x, y) = 0 \\ x \in U, \ y \in I \end{cases}$$

is equivalent with $y = f(x)$. In particular, $f(x_0) = y_0$. Moreover, the gradient of f at x_0 is

$$\nabla f(x_0) = -\frac{1}{(\partial_y g)(x_0, y_0)} \nabla_x g(x_0, y_0)$$

where $\nabla_x g = (\partial_{x_1} g, ..., \partial_{x_n} g)$

3 Integration in $\mathbb{R}^n$

3.1 Line integrals

Definition 4.1.1. *Uses scalar product in $\mathbb{R}^n$.*

1. *Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $f(t) = (f_1(t), f_2(t), ..., f_n(t))$ be continuous (f_i is continuous). Then we define*

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, ..., \int_a^b f_n(t)dt \right)$$

2. *A **parametrized curve** in $\mathbb{R}^n$ is a continuous map $\gamma : [a, b] \mapsto \mathbb{R}^n$ that is piecewise C^1, i.e, there's $k \geq 1$ and a partition*

$$a = t_0 < t_1 < ... < t_{k-1} < t_k = b$$

such that the restriction of f to $]t_{j-1}, t_j[$ is C^1 for $1 \leq j \leq k$. Then we say that γ is a parametrized curve, or pathx, between $\gamma(a)$ and $\gamma(b)$.

3. *Let gamma* : $[a,b] \mapsto \mathbb{R}^n$ *be a parametrized curve. Let* $X \subset \mathbb{R}^n$ *be a subset containing the image of* γ. *Let* $g : X \mapsto \mathbb{R}^n$ *be a continuous function. Then the integral*
$$\int_a^b g(\gamma(t))\gamma'(t)dt \in \mathbb{R}$$
is called the ***line integral*** *of* g *along* γ. *Denoted*
$$\int_\gamma g(s) \cdot ds \quad or \quad \int_\gamma g(s) \cdot d\vec{s}$$

When working with line integrals, we say that $f : X \mapsto \mathbb{R}^n$ is a **vector field**.

Proposition idk. *This integral of continuous functions* $I \mapsto \mathbb{R}^n$ *(one variable) satisfies*
$$\int_a^b (f(t)+g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$$
and
$$\int_a^b f(t)dt = -\int_b^a f(t)dt$$

Definition 4.1.4. *Let gamma* : $[a,b] \mapsto \mathbb{R}^n$ *be a parametrized curve. An* ***oriented reparametrization*** *of* γ *is a parametrized curve* $\sigma : [c,d] \mapsto \mathbb{R}^n$ *such that* $\sigma = \gamma \circ \varphi$, *differentiable on* $]a,b[$, *strictly increasing and satisfies* $\varphi(a) = c, \varphi(b) = d$, *where* $\varphi : [c,d] \mapsto [a,b]$ *is a continuous map.*

Proposition 4.1.5. *Let* γ *be a parametrized curve in* $\mathbb{R}^n$, σ *an oriented reparametrization of* γ. *Let* X *be a set containing the image of* γ *(or, equivalently, the image of* σ*), and* $f : X \mapsto \mathbb{R}^n$ *a continuous function.*

Then the line integrals are the same:
$$\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$$

Definition 4.1.8. *Let* $X \subset \mathbb{R}^n$ *and* $f : X \mapsto \mathbb{R}^n$ *a continuous vector field.*

If for any $x_1, x_2 \in X$ *the line integral is independent of the choice of* γ *in* X *from* x_1 *to* x_2, *then we say that the vector field is* ***conservative***.

Remark 4.1.9. *Equivalently,* f *is conservative iff*
$$\int_\gamma f(s) \cdot d\vec{s} = 0$$
for any ***closed*** *parametrized curve* γ *in* X. *A curve is closed if* $\gamma(a) = \gamma(b)$.

Theorem Hidden in the page (gratient vector conservative). *If* X *is open, then any vector field of the form* $f = \nabla g$, *where* g *is of class* C^1 *on* X, *is conservative.*

Theorem 4.1.10. *Let* X *be open and* f *a conservative vector field. Then there exist a* C^1 *function* g *on* X *such that* $f = \nabla g$.

If any two points on X *can be joined by a parametrized curve, then* g *is unique up to addition of a constant: if* $\nabla g_1 = f$, *then* $g - g_1$ *is constant on* X.

Remark 4.1.11. *To say that Any two points of* X *can be joined by a parametrized curve means that, for all* $x, y \in X$, *there exist a p.c.* $\gamma : [a,b] \mapsto X$ *such that* $\gamma(a) = x, \gamma(b) = y$. *When this is true, we say that* X *is* ***path-connected***.

(it is true whenever X *is* ***convex****)*

If f *is a conservative vector field on* X, *then a function* g *such that* $\nabla g = f$ *is called a* ***potential*** *for* f. *Note that* g *is not unique and can differ of at least a constant.*

Proposition 4.1.13. *Let* $X \subset \mathbb{R}^n$ *be an open set,* $f : X \mapsto \mathbb{R}^n$ *a vector field of class* C^1. *Write*
$$f(x) = (f_1(x), ..., f_n(x))$$
If f *is conservative, then we have, for any integers with* $1 \le i \ne j \le n$:
$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Example 4.1.14. *Consider* $f(x,y,z) = (y^2, xz, 1)$. *Clearly,* $\partial_y(y^2) = 2y \ne z = \partial_x(xz)$ *Then* f *can't be conservative.*

Definition 4.1.15. *A subset* $X \subset \mathbb{R}^n$ *is* ***star shaped*** *if there exists* $x_0 \in X$ *so that for all* $x \in X$, *the line connecting the two is contained in* X.

Then X *is star-shaped around* x_0.

Theorem 4.1.17. *Let* $X \subset \mathbb{R}^n$ *be star shaped and* f *be a* C^1 *vector field such that on* X, *for any* $1 \le i \ne j \le n$:
$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad (2)$$
Then f *is* ***conservative***

Definition 4.1.20. *Let* $X \subset \mathbb{R}^3$ *be an open set and* $f : X \mapsto \mathbb{R}^3$ *a* C^1 *vector field. Then the curl of* f, $curl(f)$, *is the continuous vector field on* X
$$curl(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$
Where $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

If $curl(f) = 0$, ***then condition*** (2) ***holds!***

Remark 4.1.21 (remember the definition with determinant). *With* (e_1, e_2, e_3) *being the canonical basis of* $\mathbb{R}^3$ *and expanding with* $\partial_x \cdot f_i = f_i \partial_x = \partial_x f_i$:
$$curl(f) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

3.2 The Riemann integral in $\mathbb{R}^n$

For any bounded closed $X \subset \mathbb{R}^n$ and continuous function $f : X \mapsto \mathbb{R}$, one can define *the integral of* f *over* X, denoted
$$\int_X f(x)dx \quad \in \mathbb{R}$$
The integral satisfies the properties:

1. **(Compatibility)** if $n = 1$ and $X = [a,b]$ is an interval ($a \le b$), then the integral of f over X is the Riemann integral of f:
$$\int_{[a,b]} f(x)dx = \int_a^b f(x)dx$$

2. **(Linearity)** if f, g are continuous on X and $a, b \in \mathbb{R}$, then
$$\int_X (af_1(x) + bf_2(x))dx = a\int_X f_1(x)dx + b\int_X f_2(x)dx$$

3. **(Positivity)** if $f \le g$, then
$$\int_X f(x)dx \le \int_X g(x)dx$$
and especially, if $f \ge 0$, then
$$\int_X f(x)dx \ge 0$$
Moreover, if $Y \subset X$ is compact and $f \ge 0$, then
$$\int_Y f(x)dx \le \int_X f(x)dx$$

4. **(Upper bound and triangle inequality)** Since $-|f| \le f \le |f|$, we have
$$\left|\int_X f(x)dx\right| \le \int_X \left|f(x)\right|dx$$
and since $|f+g| \le |f| + |g|$
$$\left|\int_X (f(x)+g(x))dx\right| \le \int_X \left|f(x)\right|dx + \int_X \left|g(x)\right|dx$$

5. **(Volume)** if $f = 1$, then the integral of f is the *"volume"* in $\mathbb{R}^n$ of the set X. If $f \ge 0$ in general, the integral of f is the volume of the set
$$\{(x,y) \in X \times \mathbb{R} \mid 0 \le y \le f(x)\} \subset \mathbb{R}^{n+1}$$
In particular, if X is a bounded "rectangle", say
$$X = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$$
and $f = 1$, then
$$\int_X dx = (b_n - a_n) \cdot \dots \cdot (b_1 - a_1)$$
We write $Vol(X)$ or $Vol_n(X)$ for the volume of X.

6. **(Multiple Integral, or Fubini's Theorem)** If $n_1, n_2 \ge 1$ are integers such that $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^{n_1}$, let
$$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$$
Let X_1 be the set of x_1 such that Y_{x_1} is not empty. Then X_1 is compact in $\mathbb{R}^{n_1}$ and Y_{x_1} is compact in $\mathbb{R}^{n^2}$ for all $x_1 \in X_1$. If the function
$$g(x_1) = \int_{Y_{x_1}} f(x_1, x_2)dx_2$$
on X_1 is continuous, then
$$\int_X f(x_1,x_2)dx = \int_{X_1} g(x_1)dx_1 = \int_{X_1}\left(\int_{Y_{x_1}} f(x_1,x_2)dx_2\right)dx_1$$
Similarly, exchanging the role of x_1 and x_2, we have
$$\int_X f(x_1,x_2)dx = \int_{X_2}\left(\int_{Z_{x_2}} f(x_1,x_2)dx_1\right)dx_2$$
Where $Z_{x_2} = \{x_1 \mid (x_1, x_2) \in X\}$, if the integral over x_1 is a continuous function.

7. **(Domain additivity)** if X_1 and X_2 are compact subsets of $\mathbb{R}^n$ and f is continuous on $X_1 \cup X_2$, then
$$\int_{X_1 \cup X_2} f(x)dx + \int_{X_1 \cap X_2} f(x)dx = \int_{X_1} f(x)dx + \int_{X_2} f(x)dx$$
Notice that $X_1 \cap X_2$ is also compact, so all integrals exist. If $X_1 \cap X_2$ is empty, then the integral over it is equal to 0, shortening the formula to a more convenient form. This is also true if the intersection is **negligible** (Def 4.2.3).

Definition 4.2.3. *1. Let* $1 \le m \le n$. *A* ***parametrized m-set*** *in* $\mathbb{R}^n$ *is a contiuous map*
$$f : [a_1, b_1] \times ... \times [a_m, b_m] \mapsto \mathbb{R}^n$$
which is C^1 *on*
$$]a_1, b_1[\times ... \times]a_m, b_m[$$

2. *A subset $B \subset \mathbb{R}^n$ is **negligible** if there exist an integer $k \geq 0$ and parametrized m_i-sets $f_i : X_i \mapsto \mathbb{R}^n$ with $1 \leq n \leq k$ and $m_i < n$ such that*
$$X \subset f_1(X_1) \cup ... \cup f_k(X_k)$$

Example 4.2.4. *Any subset of the real axis $\mathbb{R} \times \{0\}$ is negligible in $\mathbb{R}^2$.*
More generally, if $H \subset \mathbb{R}^n$ is an affine subspace of dimension $m < n$, then any subset of $\mathbb{R}^n$ that is contained in H is negligible.

Proposition 4.2.5. *Let $X \subset \mathbb{R}^n$ be a compact set. Assume that X is negligible. Then for any continuous function on X we have*
$$\int_X f(x)dx = 0$$

3.3 Improper integrals

Some basic definitions on $\mathbb{R}^2$.

Let $I \subset \mathbb{R}$ be a bounded interval, $J = [a, +\infty[$ for some $a \in \mathbb{R}$. Let f be a continuous function on $X = J \times I$. We say that f is **Riemann-integrable** on X if the following limit exists
$$\lim_{x\to+\infty} \int_{[a,x]\times I} f(x,y)dxdy = \lim_{x\to+\infty} \int_I \left(\int_a^x f(x,y)dx \right) dy$$
The equality is a case of Fubini's Theorem. We denote this limit with
$$\int_{J\times I} f(x,y)dxdy$$
Similarly, let f be continuous on $\mathbb{R}^2$. Assume that $f \geq 0$. We say that f is Riemann-integrable on $\mathbb{R}^2$ if the following limit exists
$$\lim_{R\to+\infty} \int_{[-R,R]^2} f(x,y)dxdy$$
This limit is called the **integral of f over $\mathbb{R}^2$** and denoted
$$\int_{\mathbb{R}^2} f(x,y)dxdy$$
Fubini formula for this:
$$\int_{\mathbb{R}^2} f(x,y)dxdy = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dx \right) dy$$

Remark 4.3.1. *In all these cases we also often say that "the integral converges" to indicate that a function is Riemann-integrable on an unbounded set.*
If $|f| \leq g$ and the integral of g is Riemann-integrable on an unbounded set, then f also does.

3.4 The change of variable formula

Analogue of the one for one-variable calculus
$$\left(\int f(g(x))g'(x)dx = \int f(y)dy \right)$$
Let $\bar{X}, \bar{Y} \subset \mathbb{R}^n$ be compact subsets. Let $\varphi : \bar{X} \mapsto \bar{Y}$ be a continuous map.

We assume that we can write $\bar{X} = X \cup B$ and $\bar{Y} = Y \cup C$ where

- the sets X, Y are open.
- The sets B, C are negligible (Def **??**def:4.2.3)
- the restriction of φ to the open set X is a C^1 bijective map from X to Y.

Then $J_\varphi(x)$ is invertible at all $x \in X$. We assume that we can find a continuous function on $\bar{X}$ that restricts to $\det(J_\varphi(x))$ on X (we have a formula for the Jacobian, so this is obvious in most cases). Abuse notation and write it even if $x \in B$.

Remark 4.4.1. *There is no assumption concerning the image of B.*
Sometimes φ is the restriction of a C^1 map $\mathbb{R}^n \mapsto \mathbb{R}^n$, in which case the last issue doesn't require any argument.

Theorem 4.4.2 (Change of variable formula). *In the situation described above, for any continuous function f on $\bar{Y}$, we have*
$$\int_{\bar{X}} f(\varphi(x))|\det(J_\varphi(x))|dx = \int_{\bar{Y}} f(y)dy$$

To remember, when $y = \varphi(x)$, then $dy = |\det(J_\varphi(x))|dx$.
Special cases:

1. When $\varphi(x) = x + x_0$ (translation): φ is affine-linear and $J_\varphi(x) = 1_n$ (identity matrix). The change of variable formula becomes, for any compact subset $\bar{X}$ and any continuous function f on $x_0 + \bar{X}$:
$$\int_{\bar{X}} f(x+x_0)dx = \int_{x_0+\bar{X}} f(x)dx$$
2. When φ is a restriction of a bijective linear map, namely $\varphi(x) = Ax$, where A is an invertible matrix of size n. Then $J_\varphi(x) = A$ for all $x \in \mathbb{R}^n$ with constant determinant $\det(A)$. Let $\bar{X} = X \cup B$ be compact as above and $\bar{Y} = \varphi(\bar{X})$. Then $\varphi(\bar{X}) = \varphi(X) \cup \varphi(B)$. The change of variable formula becomes (for any continuous f on $\bar{Y}$)
$$\int_{\bar{X}} f(\varphi(x))dx = \frac{1}{|\det(A)|} \int_{\bar{Y}} f(y)dy$$

Example (Standard examples). 1. ***Polar coordinates*** *(r, θ) are useful for integrating over a disc in $\mathbb{R}^2$ centered at 0, or more generally over a disk sector $\delta = \delta(a, b, R)$ defined*
$$0 \leq r \leq R, \quad -\pi < a \leq \theta \leq b < \pi$$
We compute the Jacobian det. and obtain
$$\int_\delta f(x,y)dxdy = \int_0^R \int_a^b f(r\cos\theta, r\sin\theta)r\ dr\ d\theta$$

2. ***Spherical Coordinates*** *(r, θ, φ) in $\mathbb{R}^3$, integrate on balls centered at 0. Jacobian determinant is $-r^2\sin(\varphi)$. To integrate a function f over a ball of radius R we use*
$$\int_B f(x,y,z)dxdydz = \int_0^R \int_0^{2\pi} \int_0^\pi f(\bar{x},\bar{y},\bar{z})r^2\sin(\varphi)\ dr\ d\theta\ d\varphi$$
With $\bar{x} = r\cos(\theta)\sin(\varphi)$, $\bar{y} = r\sin(\theta)\sin(\varphi)$, $\bar{z} = r\cos(\varphi)$.

3.5 Geometric applications of integrals

Welp, applications that can actually be useful

1. **(Center of mass)** Let X be a compact subset of $\mathbb{R}^n$ of positive volume. The *center of mass* (or *barycenter*) of X is the point $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ with
$$\bar{x}_1 = \frac{1}{VolX} \int_X x_i dx$$
Intuitively, x_i is the average over X of the i-th coordinate and $\bar{x}$ is the point where X is "perfectly balanced" (could be outside of X!).
2. **(Surface area)** Consider a function $f : [a,b] \times [c,d] \mapsto \mathbb{R}$ which is C^1 on the open interval. Let
$$\Gamma = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in [a,b] \times [c,d],\ z = f(x,y)\} \subset \mathbb{R}^3$$
be the graph of f. Intuitively, this is a surface and it has area
$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x,y))^2 + (\partial_y f(x,y))^2} dxdy$$
Analogue for a function $f : [a,b] \mapsto \mathbb{R}$ (**length**):
$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

3.6 The Green formula

Definition 4.6.1. *A **simple closed parametrized curve** $\gamma : [a,b] \mapsto \mathbb{R}^2$ is a closed ($\gamma(a) = \gamma(b)$) parametrized curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or they are a, b, and such that $\gamma'(t) \neq 0$ for $a < t < b$ (if γ is only piecewise C^1, this condition only applies where $\gamma'(t)$ exists).*

Theorem 4.6.3 (Green's formula). *Let $X \subset \mathbb{R}^2$ be a compact set with a boundary ∂X that is the union of finitely many simple closed parametrized curves $\gamma_1, ..., \gamma_k$. Assume that*
$$\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^2$$
has the property that X lies always "to the left" of the tangent vector $\gamma'(t)$ based at $\gamma_i(t)$.
Let $f = (f_1, f_2)$ be a vector field of class C^1 defined on some open set containing X. Then we have
$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy = \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}$$
If the orientation is not met, the curve can be "reversed", e.g., replaced with $\tilde{\gamma} = \gamma(1-t)$, which reverses the orientation of the tangent vector.

Corollary 4.6.5. *Let $X \subset \mathbb{R}^2$ compact set with boundary ∂X that is the union of finitely many s.c.p.c $\gamma_1, ..., \gamma_k$. Assume that*
$$\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) : [a_i, b_i] \mapsto \mathbb{R}^2$$
has the property that X always lies "left" of the tangent vector. Then we have
$$Vol(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t)\gamma'_{i,2}(t)dt$$

3.7 The Gauss-Ostrogradski formula

Analogue of the Green formula in $\mathbb{R}^3$.

Definition 4.7.1. *A parametrized surface $\Sigma : [a,b] \times [c,d] \mapsto \mathbb{R}^3$ is a 2-set in $\mathbb{R}^3$ such that the rank of the J. matrix is 2 at all $(s,t) \in]a,b[\times]c,d[$*
Note that 2 is the max rank (there are two variables).

Definition 4.7.3 (vector product). *Let x, y be two linearly independent vectors in $\mathbb{R}^3$. The vector product (or cross product) $z = x \times y$ is the unique vector such that (x, y, z) is a basis of $\mathbb{R}^3$ (z perpendicular to the plane generated by x, y, also pairwise lin. indep.) with $\det(x, y, z) \geq 0$ and*
$$\|z\| = \|x\| \cdot \|y\| \cdot \sin(\theta)$$
Where $\theta = \angle(x, y)$
If x, y are not lin. indep., then we define $x \times y = 0$, the zero vector.

Ez formula for *canonical base only (remember the oral exam back in the day!)*:
$$x \times y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$
Also $y \times x = -x \times y$.

Theorem 4.7.6 (Gauss-Ostrogradski formula). *Let $X \subset \mathbb{R}^3$ be a compact set with a boundary ∂X that is a parametrized surface $\Sigma : [a,b] \times [c,d] \mapsto \mathbb{R}^3$.*

Assume that Σ is injective in the open interval, and that the normal vector of Σ points away from the surface at all points.

Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ be the unit exterior normal vector.

Let $f = (f_1, f_2, f_3)$ be a C^1 vector field defined on some open set containing X. Then we have

$$\int_X div(f)dxdydz = \int_\Sigma (f \cdot \vec{u})d\sigma$$

Clarify: $\div(f)$ is the divergence of the vector field f, $div(f) = \partial_x f + \partial_y f + \partial_z f$.

4 Limit Cheat Sheet

$$x \in \mathbb{R}, \quad a, b \in \mathbb{R}^+, \quad n \in \mathbb{N}$$

$\lim_{n\to\infty} a^n = +\infty$ if $a > 1$ $\lim_{n\to\infty} \sqrt[n]{n} = 1$ $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$	$\lim_{n\to\infty} a^n = 0$ if $0 < a < 1$ $\lim_{n\to\infty} \sqrt[n]{a}$ if $a > 0$
$\lim_{x\to 0} \frac{\sin x}{x} = 1$	$\lim_{x\to 0} \frac{\tan x}{x} = 1$
$\lim_{x\to\infty} \frac{\ln^b(x)}{x^a} = 0$ $\lim_{x\to\infty} \frac{e^{ax}}{x^b} = +\infty$	$\lim_{x\to 0} x^a \ln^b(x) = 0$ $\lim_{x\to\infty} \frac{x^b}{e^{ax}} = 0$

5 Derivative Cheat Sheet

Properties

$$(cf)' = cf'(x) \qquad (f \pm g)' = f'(x) \pm g'(x)$$

$$(fg)' = f'g + fg' \qquad (\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(g(f(x))) = g'(f(x))f'(x)$$

$f(x)$	$f'(x)$
x	1
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$, $x \neq 0$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\lvert x\rvert$	$x > 0 \implies 1$, or $x < 0 \implies -1, x \neq 0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x) = \frac{1}{x\ln(a)}$	
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\mathrm{arccot}(x)$	$-\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\coth(x)$	$-\frac{1}{\sin^2(x)} = 1 - \coth^2(x)$
$\mathrm{asinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\mathrm{acosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\mathrm{atanh}(x)$	$\frac{1}{1-x^2}$

6 Integral Cheat Sheet

$$\int f(x)dx = F(x) + C$$

Per parti	$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$
Per sostituzione immediata	$\int g(f(x))f'(x)dx = G(f(x)) + C$
Per sostituzione (cambiamento di variabile)	$\int g(x)dx = \int g(f(t))f'(t)dt$ with $x = f(t)$
Integrale logaritmico	$\int \frac{f'(x)}{f(x)}dx = \ln\lvert f(x)\rvert + C$

$f(x)$	$F(X)(without + C)$
a	ax
x^n	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sqrt{x}$	$\frac{2}{3}x\sqrt{x}$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b}\ln\left\lvert\frac{x-a}{x-b}\right\rvert$
$\frac{ax+b}{cx+d}$	$\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
e^x	e^x
$\ln(x)$	$x(\ln(x) - 1)$
a^x	$\frac{a^x}{\ln(a)}$
$\log_a(x)$	$x(\log_a(x) - \log_a(e))$
xe^{ax}	$\frac{1}{a^2}(ax-1)e^{ax}$
$x\ln(ax)$	$\frac{x^2}{4}(2\ln(ax) - 1)$
$\sin(x)$	$-\cos(x)$
$\arcsin(x)$	$x\arcsin(x) + \sqrt{1-x^2}$
$\cos(x)$	$\sin(x)$
$\arccos(x)$	$x\arccos(x) - \sqrt{1-x^2}$
$\tan(x)$	$-\ln\lvert\cos(x)\rvert$
$\arctan(x)$	$x\arctan(x) - \frac{1}{2}\ln(1+x^2)$
$\cot(x)$	$\ln\lvert\sin(x)\rvert$
$\mathrm{arccot}(x)$	$x\,\mathrm{arccot}(x) + \frac{1}{2}\ln(1+x^2)$
$\sin^2(x)$	$\frac{1}{2}(x - \sin(x)\cos(x))$
$\cos^2(x)$	$\frac{1}{2}(x + \sin(x)\cos(x))$
$\tan^2(x)$	$\tan(x) - x$
$\sqrt{x^2+a}$	$\frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2}\ln\left\lvert x + \sqrt{x^2+a}\right\rvert$
$\frac{1}{\sqrt{x^2+a}}$	$\ln\left\lvert x + \sqrt{x^2+a}\right\rvert$
$\sqrt{r^2-x^2}$	$\frac{1}{2}x\sqrt{r^2-x^2} + \frac{r^2}{2}\arcsin\left(\frac{x}{r}\right)$
$\frac{1}{\sqrt{r^2-x^2}}$	$\arcsin\left(\frac{x}{r}\right)$

More?