

Analysis 2 Recap

Axel Montini
amontini@student.ethz.ch

January 2, 2022

master, 2dbd481f9c65d8e7936ca3eb94790a278456be9a

1 Linear differential equations

Definition L. *linear Differential equation. Homogeneous if $b = 0$, inhomogeneous otherwise.*

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Theorem 2.2.3. *... y is k -times differentiable ...*

For the homogeneous equation, given a choice of x_0 and (y_0, \dots, y_{k-1}) there's a unique solution $f \in S$ such that

$$f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1} \quad (1)$$

For the inhomogeneous equation with a b continuous on the interval, the set of solutions S_b is the set of functions $f + f_0$ where $f \in S$. Again, for any x_0 and (y_0, \dots, y_{k-1}) there's a unique solution such that (1).

If $b \neq 0$ then S_b is not a vector space.

Proposition 2.3.1. *Any solution of $y' + ay = 0$ is in the form $f(x) = ze^{-A(x)}$, where A is a primitive of a and $z \in \mathbb{C}$. Unique solution is $f(x) = y_0 e^{A(x_0) - A(x)}$*

To solve the inhomogeneous equation $y' + ay = b$, the prev solution is used. Using *Variation of the constant* we replace z with $z(x)$ and then $y' + ay = b \Leftrightarrow z'(x) = b(x) e^{A(x)}$ and $f_0(x) = C(x)e^{-A(x)}$, where $C(x)$ is a primitive of $z'(x)$.

1.1 Constant coefficients

Definition L. *et $a_0, \dots, a_{k-1} \in \mathbb{C}$. Linear differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$. Homogeneous solution is in the form $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{C}$. We have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geq 0$ and x .*

Conclusion: $f(x)$ is a solution iff $P(\alpha) = 0$, where $P(X) = X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0$.

This polynomial of degree k has k roots (counted with multiplicity). There exist complex numbers $\alpha_1, \dots, \alpha_k$ such that $P(X) = (X - \alpha_1) \dots (X - \alpha_k)$. This is the companion or characteristic polynomial of the homogeneous diff. equation.

- No multiple roots When $\alpha_i \neq \alpha_j$ for all i, j .

Solution of the homogeneous equation ($b = 0$): form $f(x) = z_1 e^{\alpha_1 x} + \dots + z_k e^{\alpha_k x}$. Unique solution with $f(x_0) = y_0, \dots, f^{(k-1)}(x_0) = y_{k-1}$ can be obtained by viewing z_i as unknowns. Substitute $x = x_0$ in the formula for f and solve for z_1, \dots, z_k (linear system).

- Multiple roots Assume α is a multiple root of order j of the polynomial P , with $2 \leq j \leq k$. Then

$f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = x e^{\alpha x}$, ..., $f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$ are linearly independent and are solutions of the h.l.d.e.

Example S. *suppose $P(X) = X(X-4)^3(X-(1+i))(X-(1-i))$, then the solutions are $f_0(x) = 1$ (sol. for $X = 0$), $f_1(x) = e^{4x}$, $f_2(x) = x e^{4x}$, $f_3(x) = x^2 e^{4x}$, $f_4(x) = e^{(1+i)x}$, $f_5(x) = e^{(1-i)x}$*

Now the inhomogeneous equation ($b \neq 0$):

Should avoid variation of the constants. Can use special cases:

1. $b(x) = x^d e^{\beta x}$ for some integer $d = ge0$ and an item β which is NOT a root of P , then the solution is of the form $f(x) = Q(x)e^{\beta x}$, where Q is a polynomial of degree d .
2. $b(x) = x^d \cos(\beta x)$ or $b(x) = x^d \sin(\beta x)$ for some integer $d \geq 0$ and β is NOT a root of P , then one can transform it to a combination of complex exponentials or look for a solution of the form $f(x) = Q_1(x) \cos(\beta x) + Q_2(x) \sin(\beta x)$, Q_1, Q_2 have degree d .
3. $b(x)$ is in the form of the previous two but IS a root of multiplicity j , then one looks for $f(x) = Q(x)e^{\beta x}$, with Q of degree $q + j$.
4. Special case $\beta = 0$ of the previous 3 (b polynomial of degree $d \geq 0$): if 0 is NOT a root, look for a solution f (polynomial) of deg d , or degree $d + j$ if 0 IS a root, where j is the multiplicity of 0.

1.2 Variation of the constants for degree ge 2

Does not require the coefficients to be constants, but it makes it easier.

Inhomogeneous equation

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Solutions f_1, \dots, f_k for the homogeneous equations must be found first.

We then search for a solution of the form $f(x) = z_1(x)f_1(x) + \dots + z_k(x)f_k(x)$, such that we have (for all x):

$$\begin{cases} z_1'(x)f_1(x) + \dots + z_k'(x)f_k(x) = 0 \\ z_1'(x)f_1'(x) + \dots + z_k'(x)f_k'(x) = 0 \\ \dots \\ z_1'(x)f_1^{(k-2)}(x) + \dots + z_k'(x)f_k^{(k-2)}(x) = 0 \end{cases}$$

Example C. *ase $k = 2$: Write again $f = z_1 f_1 + z_2 f_2$ and the constraint $z_1' f_1 + z_2' f_2 = 0$.*

Definition 3.3.5. $f : X \mapsto \mathbb{R}$ has a partial derivative with respect to the i -th variable if the function

$g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$ is differentiable for all $x_0 \in X$ on the set $I = \{t \in \mathbb{R} \mid (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$.

Its derivative $g'(x_{0,i})$ is denoted

$$\frac{\partial f}{\partial x_i}(x_0), \partial_{x_i}(x_0), \partial_i(x_0)$$

Proposition 3.3.7. $x \subset \mathbb{R}^n$ open, f, g functions from X to \mathbb{R}^m . Let $1 \leq i \leq n$.

1. if f, g have partial derivatives of i -th coordinate on X , then $f + g$ also does. $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$
2. if the previous is true and $m = 1$, then fg also does and $\partial_{x_i}(fg) = \partial_{x_i}(f) \cdot g + f \cdot \partial_{x_i}(g)$
3. If the previous is true and $g(x) \neq 0$ for all $x \in X$, then f/g has a partial derivative $\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2$

Definition 3.3.9. $f : X \mapsto \mathbb{R}^m$ has partial derivatives on X . Write $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$.

For any $x \in X$, the **Jacobi Matrix** (m rows, n columns) of f at x is defined as

$$J_f(x) = (\partial_{x_i} f_j(x))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

Definition 3.3.11. 1. If all partial derivatives of $f : X \mapsto \mathbb{R}$ exist at $x_0 \in X$, then the column vector

$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \dots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

is called the **gradient** of f at x_0 .

2. Let $f = (f_1, f_2, \dots, f_n) : X \mapsto \mathbb{R}^n$ and all partial derivatives of all coordinates f_i of f exist at $x_0 \in X$. Then

$$\text{Tr}(J_f(x_0)) = \sum_{i=1}^n \partial_{x_i} f_i(x_0)$$

is the trace of the Jacobi Matrix and is called the **divergence** of f at x_0 , also $\text{div}(f)(x_0)$

Definition 3.4.2. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function. Let u be a linear map $\mathbb{R}^n \mapsto \mathbb{R}^m$ and $x_0 \in X$. We say that f is differentiable at x_0 with differential u if

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{1}{\|x - x_0\|} (f(x) - f(x_0) - u(x - x_0)) = 0$$

We then denote $df(x_0) = u$. If it is differentiable at every $x_0 \in X$, then it is differentiable on X .

Then, close to x_0 , we can approximate $f(x)$ by $g(x) = f(x_0) + u(x - x_0)$

Proposition 3.4.4. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ be a function differentiable on X . Then

1. f is **continuous** on X .
2. f admits partial derivatives on X with respect to each variable.
3. Assume that $m = 1$. Let $x_0 \in X$ and $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ be the differential of f at x_0 . We then have $\partial_{x_i} f(x_0) = a_i$ for $1 \leq i \leq n$

Proposition 3.4.6. $X \subset \mathbb{R}^n$ open, $f : x \mapsto \mathbb{R}^m$, $g : X \mapsto \mathbb{R}^m$ differentiable functions on X .

1. $f + g$ is differentiable on X with differential $d(f + g) = df + dg$.
2. If $m = 1$, then fg is differentiable. If we also have $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable.

Proposition 3.4.7. If f has all partial derivatives on X and they are all continuous on X , then f is differentiable on X . The matrix of the differential $df(x_0)$ is the Jacobi Matrix of f at x_0 .

Proposition 3.4.9 (Chain Rule). Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open. Let $f : X \mapsto Y$ and $g : Y \mapsto \mathbb{R}^p$ be differentiable functions. Then $g \circ f : X \mapsto \mathbb{R}^p$ is differentiable on X , and for any $x \in X$, its differential is given by

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

The Jacobi Matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0))J_f(x_0) \text{ (matrix product)}$$

Definition L. et $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$ differentiable. Let $x_0 \in X$ and $u = df(x_0)$ be the differential of f at x_0 . The graph of the affine linear approximation

$$g(x) = f(x_0) + u(x - x_0)$$

from \mathbb{R}^n to \mathbb{R}^m , or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

is called the **tangent space** at x_0 to the graph of f .

Definition 3.4.13 (Directional Derivative). Let $X \subset \mathbb{R}^n$ be open, $f : X \mapsto \mathbb{R}^m$ a function.

Let $v \in \mathbb{R}^n$ be a non-zero vector and $x_0 \in X$.

We say that f has **directional derivative** $w \in \mathbb{R}^m$ in the direction v if the function g defined on the set I has a derivative at $t = 0$ and this is equal to w .

$$g(t) = f(x_0 + tv), \quad I = \{t \in \mathbb{R} \mid x_0 + tv \in R\}$$

Other words: limit is equal to w

$$\lim_{\substack{t \rightarrow 0 \\ t \neq 0}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Proposition 3.4.15. X open, f differentiable. Then for any $x_0 \in X$ and non-zero v , the function has a directional derivative at x_0 in the direction v , equal to $df(x_0)(v)$

Definition 3.5.1. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}^m$. We say that f is of class:

- C^1 is differentiable on X and all partial derivatives are continuous.
- C^k if differentiable on X and all partial derivatives $\partial_{x_i} f : X \mapsto \mathbb{R}^m$ are of class C^{k-1} .
- C^∞ if $f \in C^k(X; \mathbb{R}^m)$ for all $k \geq 1$

Set of functions of class C^k from X to \mathbb{R}^m is denoted $C^k(X; \mathbb{R}^m)$

Proposition 3.5.4 (Mixed derivatives commute). $X \subset \mathbb{R}^n$ open and $f : X \mapsto \mathbb{R}^m$ of class C^k . Then the partial derivatives of order k are independent of the order in which the partial derivatives are taken: for any variables x_1, x_2, \dots, x_n we have

$$\partial_{x_1, x_2, \dots, x_n} f = \partial_{x_2, x_1, \dots, x_n} f = \dots \text{ (all combinations)}$$

Definition 3.5.9 (Hessian). Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ a C^2 function. For $x \in X$, the **Hessian matrix** of f at x is the **symmetric square matrix**

$$H_f(x) = \text{Hess}_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n}$$

Example (Change of variable). Idea: create h , which is f on a different coordinate system.

Open set $U \subset \mathbb{R}^n$ containing the new variables (y_1, \dots, y_n) and a change of variable $g : U \mapsto X$ that expresses (x_1, \dots, x_n) in terms of (y_1, \dots, y_n) .

Consider $x_1 = g_1(y_0, \dots, y_n)$, $x_n = g_n(y_1, \dots, y_n)$

Composite $h = f \circ g : U \mapsto \mathbb{R}$ is the function f expressed in terms of the new variables y .

Polar coordinates: Map $g : U \mapsto \mathbb{R}^2$, $g(r, \theta) = (r \cos \theta, r \sin \theta)$. Replace f by h : $h(r, \theta) = f(r \cos \theta, r \sin \theta)$.

$$J_g(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

1.3 Taylor Polynomials

Definition 3.7.1 (Taylor polynomials). Let $k \geq 1$ be an integer, $f : X \mapsto \mathbb{R}$ a function of class C^k on X , and fix $x_0 \in X$. The k -th Taylor polynomial of f at point x_0 is the poly in n variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \dots$$

$$+ \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}(x_0) y_1^{m_1} \dots y_n^{m_n}$$

where the last sum ranges over the tuples of n positive integers such that the sum is k .

Case $n = 1$ (one variable):

$$T_k f(y; x_0) = f(x_0) + f'(x_0)y + \frac{f''(x_0)}{2}y^2 + \dots + \frac{f^{(k)}(x_0)}{k!}y^k$$

Proposition 3.7.3 (Taylor Approximation). $k \geq 1$, $X \subset \mathbb{R}^n$ open, $f : X \mapsto \mathbb{R}$ a C^k function. For x_0 in X , we define $E_k f(x; x_0)$ by

$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$

then we have

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{e_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

1.4 Critical Points

Proposition 3.8.1. Let $X \subset \mathbb{R}^n$ be open and $f : X \mapsto \mathbb{R}$ be differentiable. If $x_0 \in X$ is a local maximum or a local minimum, then we have (equivalent) for $1 \leq i \leq n$:

$$df(x_0) = 0, \quad \nabla f(x_0) = 0, \quad \frac{\partial f}{\partial x_i}(x_0) = 0$$

Definition 3.8.2 (Critical Point). Let X be open and f be differentiable. A point x_0 is called a **critical point** of f if $\nabla f(x_0) = 0$.

Definition 3.8.6 (Non-degenerate critical point). f of class C^2 . A critical point x_0 is **non-degenerate** if the Hessian matrix has non-zero determinant.

Corollary 3.8.7. X open and $f : X \mapsto \mathbb{R}$ of class C^2 . Let x_0 be a non-degenerate critical point of f . Let p, q be the number of positive and negative eigenvalues of $\text{Hess}_f(x_0)$

1. if $p = n$, equivalently if $q = 0$, the function f has a local minimum at x_0 .

2. if $q = n$, equivalently if $p = 0$, f has a local maximum at x_0 .

3. Otherwise, the function f does not have a local extremum at x_0 , equivalently it has a saddle point at x_0 .

2 Limit Cheat Sheet

$x \in \mathbb{R}, \quad a, b \in \mathbb{R}^+, \quad n \in \mathbb{N}$	
$\lim_{n \rightarrow \infty} a^n = +\infty$ if $a > 1$	$\lim_{n \rightarrow \infty} a^n = 0$ if $0 < a < 1$
$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$ if $a > 0$
$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$	
$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$
$\lim_{x \rightarrow \infty} \frac{\ln^b(x)}{x^a} = 0$	$\lim_{x \rightarrow 0} x^a \ln^b(x) = 0$
$\lim_{x \rightarrow \infty} \frac{e^{ax}}{x^b} = +\infty$	$\lim_{x \rightarrow \infty} \frac{x^b}{e^{ax}} = 0$

3 Derivative Cheat Sheet

Properties

$$(cf)' = cf'(x) \qquad (f \pm g)' = f'(x) \pm g'(x)$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(g(f(x))) = g'(f(x))f'(x)$$

$f(x)$	$f'(x)$
x	1
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
\sqrt{x}	$\frac{1}{2\sqrt{x}}, x \neq 0$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$ x $	$x > 0 \implies 1, \text{ or } x < 0 \implies -1, x \neq 0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x) = \frac{1}{x \ln(a)}$	
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\coth(x)$	$-\frac{1}{\sinh^2(x)} = 1 - \coth^2(x)$
$\operatorname{asinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{acosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{atanh}(x)$	$\frac{1}{1-x^2}$

$f(x)$	$F(X)(without + C)$
a	$\frac{ax}{x^{n+1}}$
x^n	$\frac{1}{n+1}$
$\frac{1}{x}$	$\ln x $
\sqrt{x}	$\frac{2}{3}x\sqrt{x}$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b} \ln \left \frac{x-a}{x-b} \right $
$\frac{ax+b}{cx+d}$	$\frac{ax}{c} - \frac{ad-bc}{c^2} \ln cx+d $
$\frac{1}{x^2+a^2}$	$\frac{1}{a} \arctan \left(\frac{x}{a} \right)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $
e^x	$\frac{e^x}{a^x}$
$\ln(x)$	$x(\ln(x)-1)$
a^x	$\frac{\ln(a)}{x}$
$\log_a(x)$	$x(\log_a(x) - \log_a(e))$
xe^{ax}	$\frac{1}{a^2}(ax-1)e^{ax}$
$x \ln(ax)$	$\frac{x^2}{4}(2 \ln(ax) - 1)$
$\sin(x)$	$-\cos(x)$
$\arcsin(x)$	$x \arcsin(x) + \sqrt{1-x^2}$
$\cos(x)$	$\sin(x)$
$\arccos(x)$	$x \arccos(x) - \sqrt{1-x^2}$
$\tan(x)$	$-\ln \cos(x) $
$\arctan(x)$	$x \arctan(x) - \frac{1}{2} \ln(1+x^2)$
$\cot(x)$	$\ln \sin(x) $
$\operatorname{arccot}(x)$	$x \operatorname{arccot}(x) + \frac{1}{2} \ln(1+x^2)$
$\sin^2(x)$	$\frac{1}{2}(x - \sin(x) \cos(x))$
$\cos^2(x)$	$\frac{1}{2}(x + \sin(x) \cos(x))$
$\tan^2(x)$	$\tan(x) - x$
$\sqrt{x^2+a}$	$\frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2} \ln \left x + \sqrt{x^2+a} \right $
$\frac{1}{\sqrt{x^2+a}}$	$\ln \left x + \sqrt{x^2+a} \right $
$\sqrt{r^2-x^2}$	$\frac{1}{2}x\sqrt{r^2-x^2} + \frac{r^2}{2} \arcsin \left(\frac{x}{r} \right)$
$\frac{1}{\sqrt{r^2-x^2}}$	$\arcsin \left(\frac{x}{r} \right)$

More?

4 Integral Cheat Sheet

	$\int f(x)dx = F(x) + C$
Per parti	$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$
Per sostituzione immediata	$\int g(f(x))f'(x)dx = G(f(x)) + C$
Per sostituzione (cambiamento di variabile)	$\int g(x)dx = \int g(f(t))f'(t)dt$ with $x = f(t)$
Integrale logaritmico	$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$