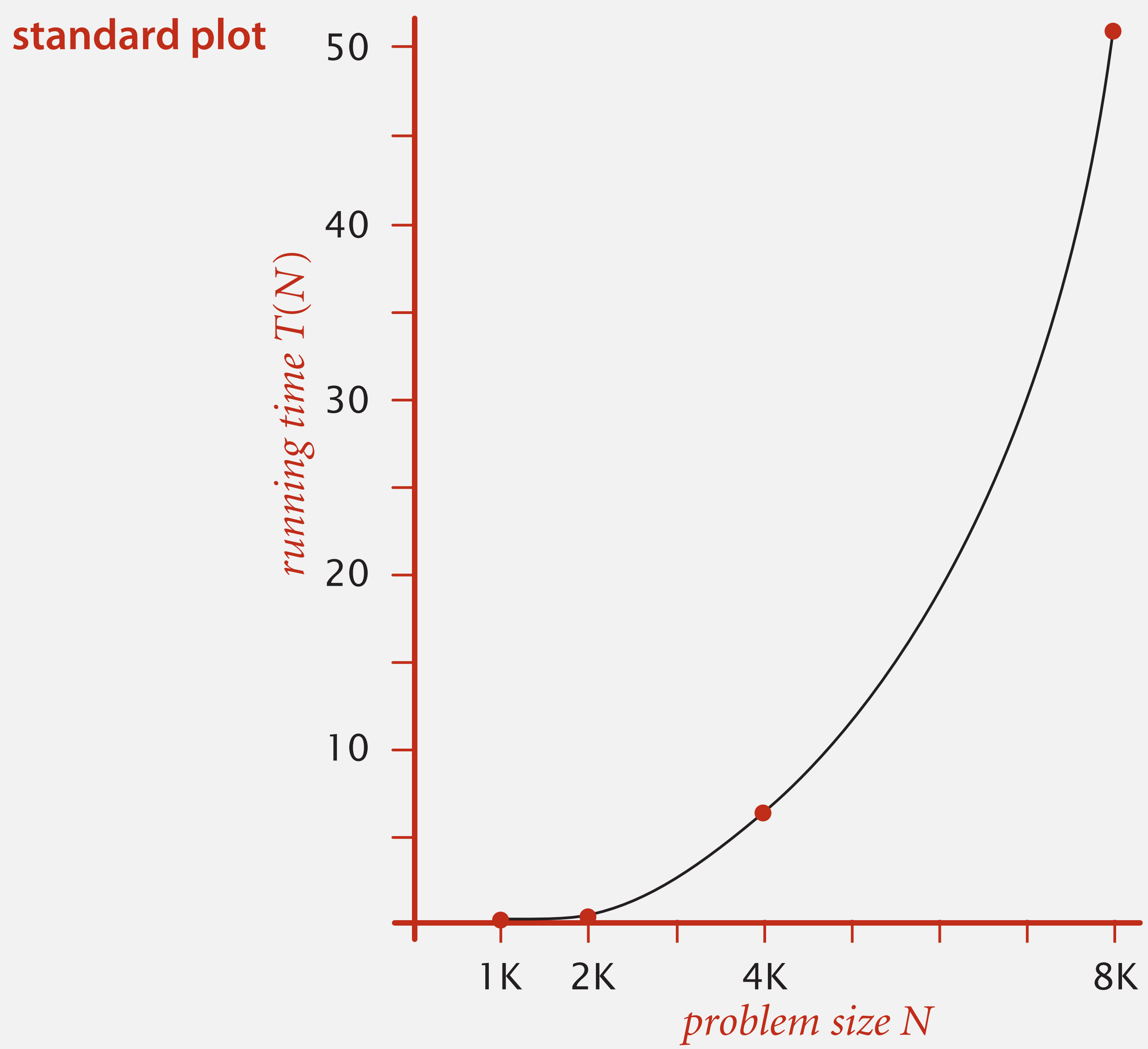
Empirical analysis

Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

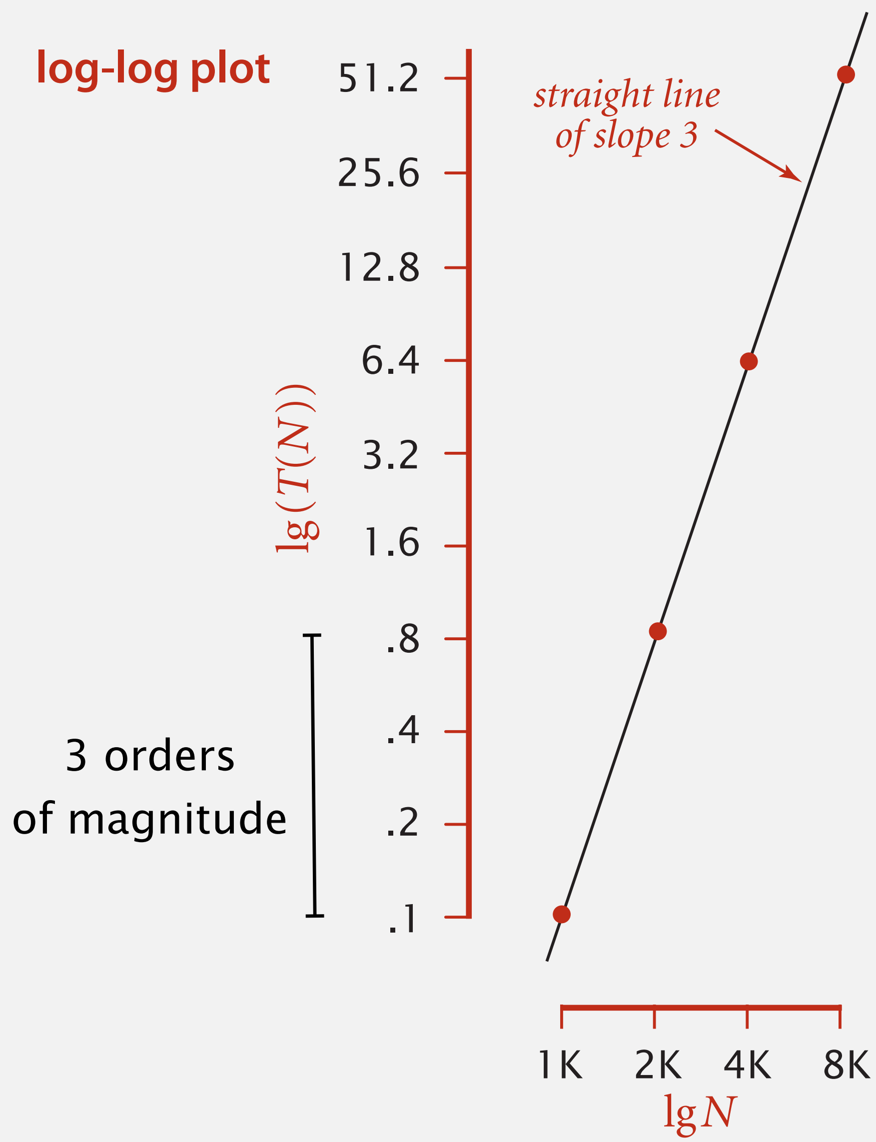
Data analysis

Standard plot. Plot running time $T(N)$ vs. input size N.



Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



$$\lg(T(N)) = b \lg N + c$$
$$b = 2.999$$
$$c = -33.2103$$

$$T(N) = a\, N^{b}, \text{ where } a = 2^{c}$$

power law

Regression. Fit straight line through data points: $a\, N^{b}$.

slope

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Run program, doubling the size of the input.

N	time (seconds) †	ratio	lg ratio
250	0		–
500	0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8	3
8,000	51.1	8	3

lg (6.4 / 0.8) = 3.0

seems to converge to a constant b ≈ 3

$$\frac{T(2N)}{T(N)} = \frac{a(2N)^b}{aN^b} = 2^b$$

Hypothesis. Running time is about $a\,N^{\,b}$ with $b = \lg$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of N) and solve for a.

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1

$51.1 \;=\; a \times 8000^3$

$\Rightarrow \;\; a \;=\; 0.998 \times 10^{-10}$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.

almost identical hypothesis

to one obtained via linear regression

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent in power law

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

determines constant in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

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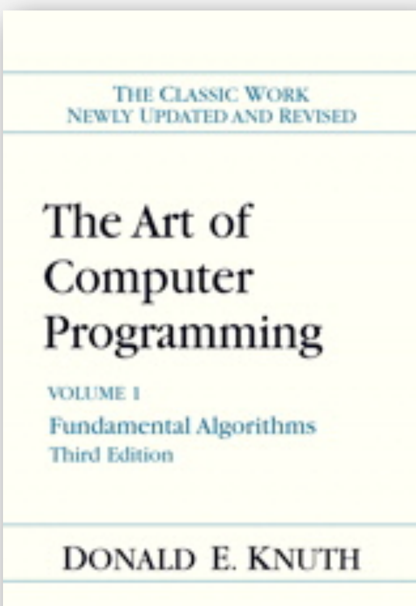
1.4 Analysis of Algorithms

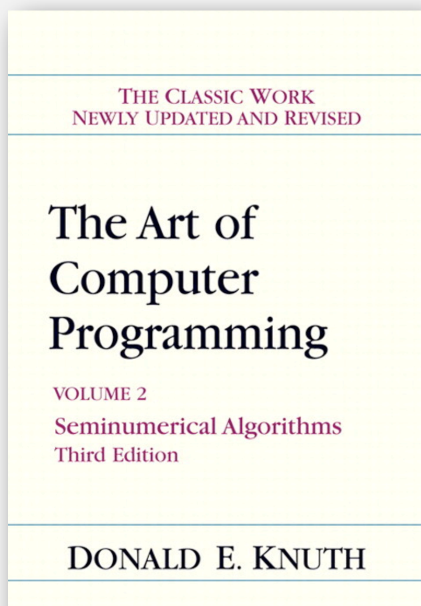
- introduction
- observations
- mathematical models
- order-of-growth classifications
- theory of algorithms
- memory

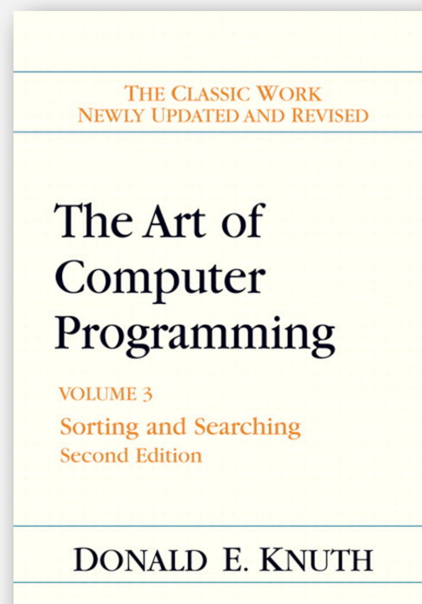
Mathematical models for running time

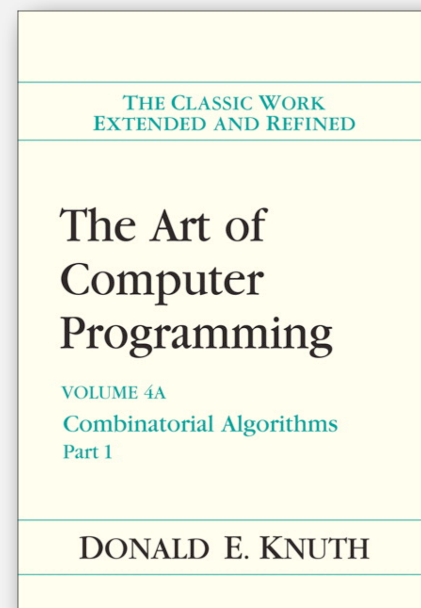
Total running time: sum of cost × frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.









Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.

Cost of basic operations

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	`a + b`	2.1
integer multiply	`a * b`	2.4
integer divide	`a / b`	5.4
floating-point add	`a + b`	4.6
floating-point multiply	`a * b`	4.2
floating-point divide	`a / b`	13.5
sine	`Math.sin(theta)`	91.3
arctangent	`Math.atan2(y, x)`	129
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Cost of basic operations

Observation. Most primitive operations take constant time.

operation	example	nanoseconds †
variable declaration	`int a`	c_1
assignment statement	`a = b`	c_2
integer compare	`a < b`	c_3
array element access	`a[i]`	c_4
array length	`a.length`	c_5
1D array allocation	`new int[N]`	$c_6\ N$
2D array allocation	`new int[N][N]`	$c_7\ N^2$

Caveat. Non-primitive operations often take more than constant time.

novice mistake: abusive string concatenation

Example: 1-SUM

Q. How many instructions as a function of input size N ?

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0)
      count++;
```

N array accesses

operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
increment	N to $2\ N$

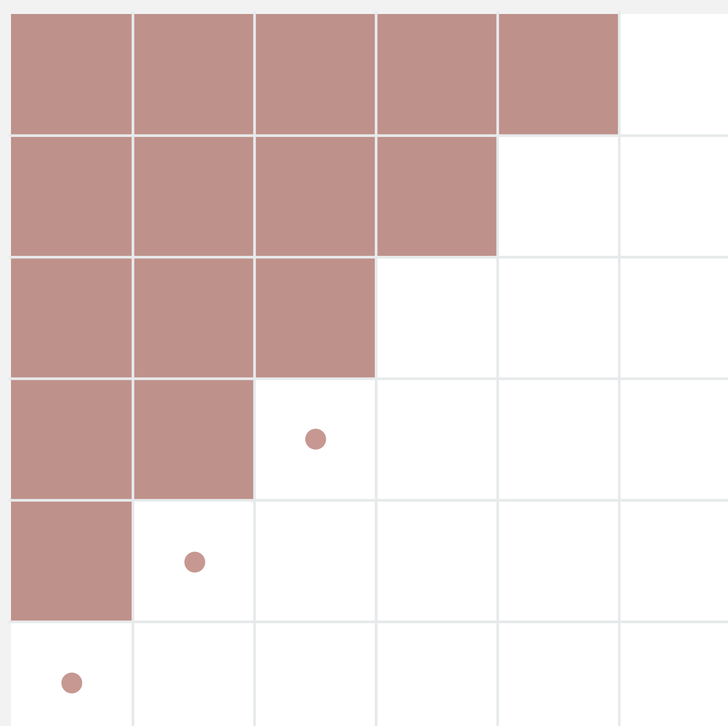
Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) \;=\; \frac{1}{2} N\,(N-1) \;=\; \binom{N}{2}$$

Pf. [n even]



$$0 + 1 + 2 + \ldots + (N-1) \;=\; \underbrace{\frac{1}{2}N^2}_{\text{half of}} \;-\; \underbrace{\frac{1}{2}N}_{\text{half of}}$$

Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1)$$
$$= \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$½ (N + 1) (N + 2)$
equal to compare	$½ N (N - 1)$
array access	$N (N - 1)$
increment	$½ N (N - 1)$ to $N (N - 1)$

tedious to count exactly

Simplifying the calculations

“ It is convenient to have a ***measure of the amount of work involved in a computing process****, even though it be a very* ***crude*** *one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and* ***we shall therefore only attempt to count the number of multiplications and recordings****. ”* — *Alan Turing*

ROUNDING-OFF ERRORS IN MATRIX PROCESSES

By A. M. TURING

(*National Physical Laboratory, Teddington, Middlesex*)

[Received 4 November 1947]

SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known ‘Gauss elimination process’, it is found that the errors are normally quite moderate: no exponential build-up need occur.

Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$½ (N + 1) (N + 2)$
equal to compare	$½ N (N - 1)$
array access	$N (N - 1)$
increment	$½ N (N - 1)$ to $N (N - 1)$

cost model = array accesses

(we assume compiler/JVM do not optimize any array accesses away!)

Simplification 2: tilde notation

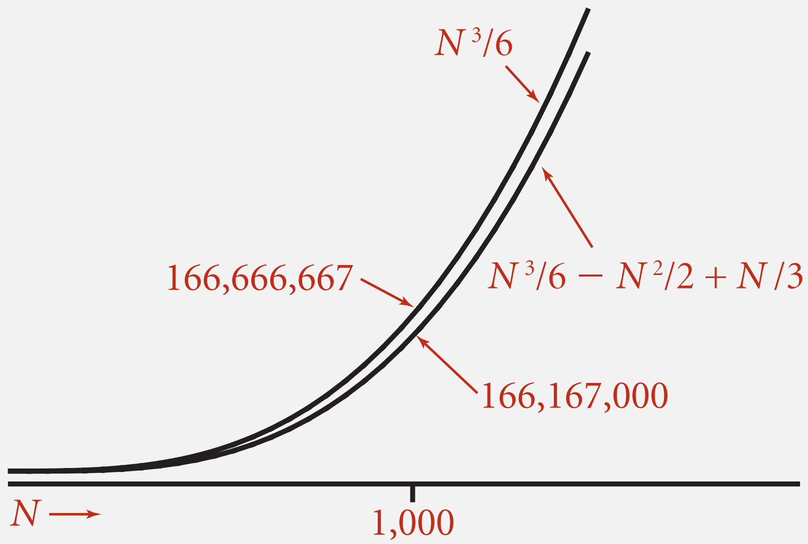
- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

Ex 1. $\frac{1}{6} N^3 + 20 N + 16 \quad \sim \frac{1}{6} N^3$

Ex 2. $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \quad \sim \frac{1}{6} N^3$

Ex 3. $\frac{1}{6} N^3 \underbrace{- \frac{1}{2} N^2 + \frac{1}{3} N}_{\text{discard lower-order terms}} \quad \sim \frac{1}{6} N^3$

(e.g., N = 1000: 166.67 million vs. 166.17 million)



Leading-term approximation

Technical definition. $f(N) \sim g(N)$ means $\displaystyle \lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2}\,(N + 1)\,(N + 2)$	$\sim \frac{1}{2}\,N^2$
equal to compare	$\frac{1}{2}\,N\,(N - 1)$	$\sim \frac{1}{2}\,N^2$
array access	$N\,(N - 1)$	$\sim N^2$
increment	$\frac{1}{2}\,N\,(N - 1)$ to $N\,(N - 1)$	$\sim \frac{1}{2}\,N^2$ to $\sim N^2$

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N(N-1) = \binom{N}{2}$$

A. $\sim N^2$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Diversion: estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \ldots + N$.

$$\sum_{i=1}^{N} i \sim \int_{x=1}^{N} x\,dx \sim \frac{1}{2}N^2$$

Ex 2. $1^k + 2^k + \ldots + N^k$.

$$\sum_{i=1}^{N} i^k \sim \int_{x=1}^{N} x^k dx \sim \frac{1}{k+1}N^{k+1}$$

Ex 3. $1 + 1/2 + 1/3 + \ldots + 1/N$.

$$\sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x} dx = \ln N$$

Ex 4. 3-sum triple loop.

$$\sum_{i=1}^{N}\sum_{j=i}^{N}\sum_{k=j}^{N} 1 \sim \int_{x=1}^{N}\int_{y=x}^{N}\int_{z=y}^{N} dz\,dy\,dx \sim \frac{1}{6}N^3$$

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

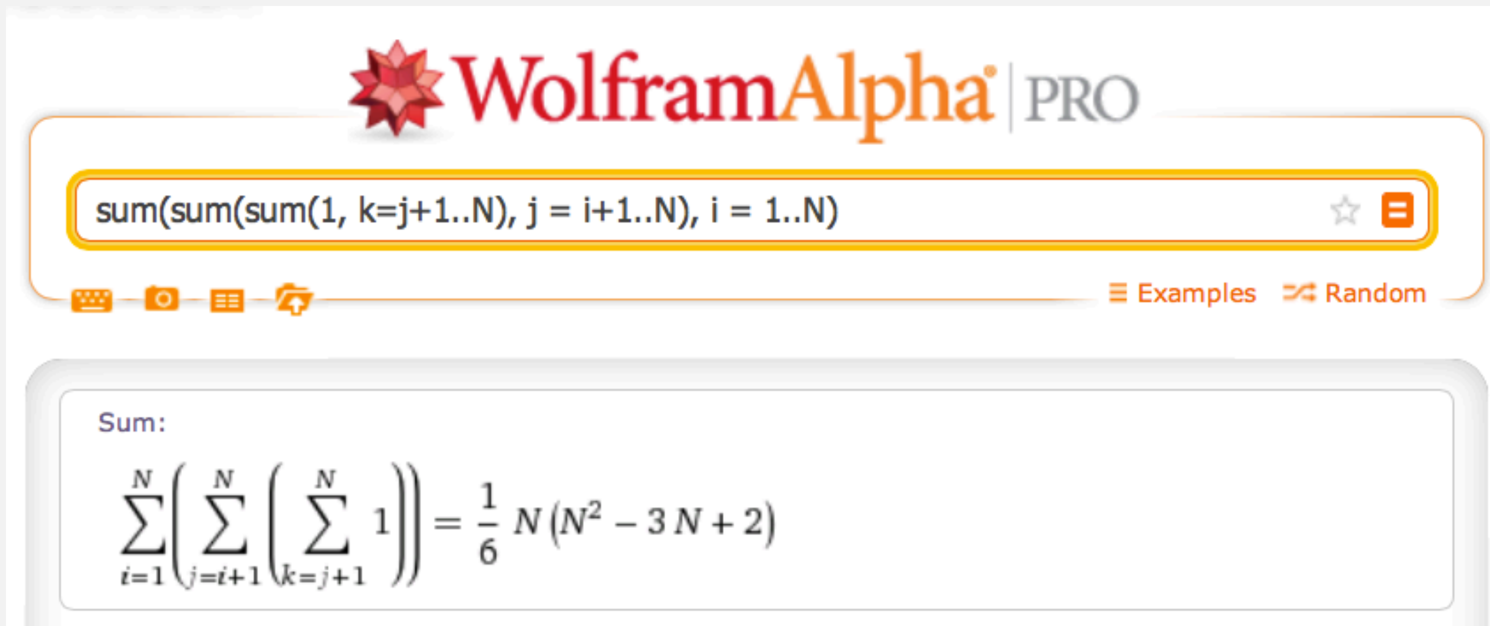
$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

Caveat. Integral trick doesn't always work!

Estimating a discrete sum

Q. How to estimate a discrete sum?

A3. Use Maple or Wolfram Alpha.



wolframalpha.com

```
[wayne:nobel.princeton.edu] > maple15
    |\^/|     Maple 15 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2011
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> factor(sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N));

                              N (N - 1) (N - 2)
                              -----------------
                                      6
```

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

costs (depend on machine, compiler)

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

$A =$ array access
$B =$ integer add
$C =$ integer compare
$D =$ increment
$E =$ variable assignment

frequencies
(depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(N) \sim c N^3$.

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1.4 Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Common order-of-growth classifications

Definition. If $f(N) \sim c\, g(N)$ for some constant $c > 0$, then the order of growth of $f(N)$ is $g(N)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the running time of this code is N^3.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. With running times.

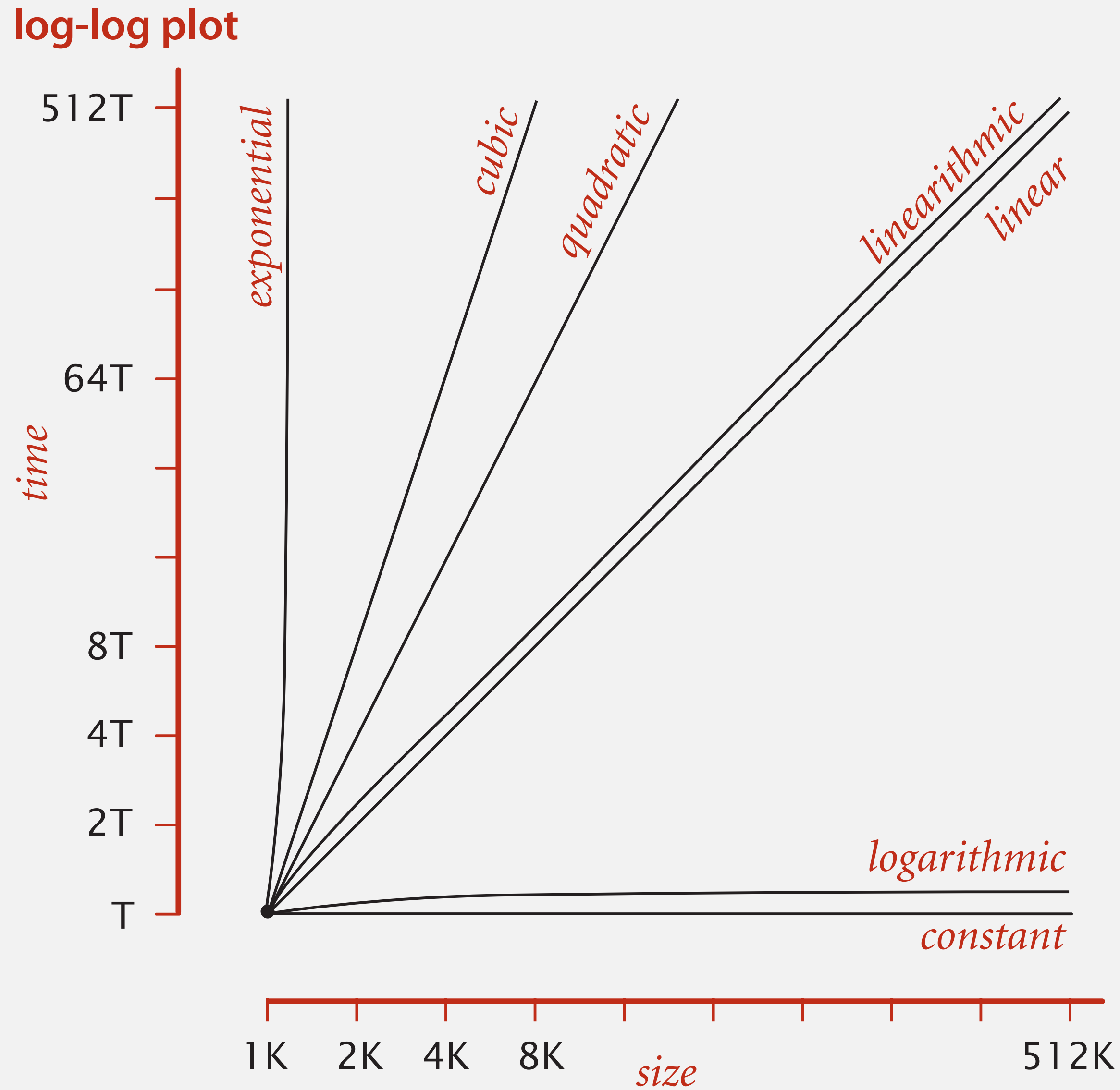
where leading coefficient
depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$$1, \ \log N, \ N, \ N \log N, \ N^2, \ N^3, \text{ and } 2^N$$

suffices to describe the order of growth of most common algorithms.



Typical orders of growth

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2N) / \mathrm{T}(N)$
1	**constant**	`a = b + c;`	statement	add two numbers	1
$\log N$	**logarithmic**	`while (N > 1)` `{   N = N / 2;  ...   }`	divide in half	binary search	~ 1
N	**linear**	`for (int i = 0; i < N; i++)` `{  ...        }`	loop	find the maximum	2
$N \log N$	**linearithmic**	[see mergesort lecture]	divide and conquer	mergesort	~ 2
N^2	**quadratic**	`for (int i = 0; i < N; i++)` `  for (int j = 0; j < N; j++)` `    {  ...        }`	double loop	check all pairs	4
N^3	**cubic**	`for (int i = 0; i < N; i++)` `  for (int j = 0; j < N; j++)` `    for (int k = 0; k < N; k++)` `      {  ...        }`	triple loop	check all triples	8
2^N	**exponential**	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

1.4.5, a,b.g

Finnið ~ nálgun fyrir eftirfarandi stærðir

a. $N + 1$

b. $1 + \frac{1}{N}$

g. $\frac{N^{100}}{2^N}$

1.4.6 c.

```
int sum = 0;
for (int i = 1; i < N; i *= 2)
   for (int j = 0; j < N; j++)
      sum++;
```

1.4.6 c (breytt).

```
int sum = 0;
for (int i = 1; i < N; i *= 2)
   for (int j = 1; j < i; j *= 2)
     sum++;
```

```
int sum = 0;
for (int i = 0; i < N; i++)
   for (int j = 1; j < i; j *= 2)
     sum++;
```

Sum of strings

```
String s = "";
for (int i = 1; i < N; i++)
  s += 'a';
```

Practical implications of order-of-growth

growth rate	problem size solvable in minutes			
	1970s	1980s	1990s	2000s
1	any	any	any	any
log N	any	any	any	any
N	millions	tens of millions	hundreds of millions	billions
N log N	hundreds of thousands	millions	millions	hundreds of millions
N^2	hundreds	thousand	thousands	tens of thousands
N^3	hundred	hundreds	thousand	thousands
2^N	20	20s	20s	30

Bottom line. Need linear or linearithmic alg to keep pace with Moore's law.

Practical implications of order-of-growth

growth rate	problem size solvable in minutes				time to process millions of inputs			
	1970s	1980s	1990s	2000s	1970s	1980s	1990s	2000s
1	any	any	any	any	instant	instant	instant	instant
log N	any	any	any	any	instant	instant	instant	instant
N	millions	tens of millions	hundreds of millions	billions	minutes	seconds	second	instant
N log N	hundreds of thousands	millions	millions	hundreds of millions	hour	minutes	tens of seconds	seconds
N^2	hundreds	thousand	thousands	tens of thousands	decades	years	months	weeks
N^3	hundred	hundreds	thousand	thousands	never	never	never	millennia

Practical implications of order-of-growth

growth rate	name	description	effect on a program that runs for a few seconds	
			time for 100x more data	size for 100x faster computer
1	constant	independent of input size	–	–
$\log N$	logarithmic	nearly independent of input size	–	–
N	linear	optimal for N inputs	a few minutes	100x
$N \log N$	linearithmic	nearly optimal for N inputs	a few minutes	100x
N^2	quadratic	not practical for large problems	several hours	10x
N^3	cubic	not practical for medium problems	several weeks	4–5x
2^N	exponential	useful only for tiny problems	forever	1x

Binary search demo

Goal. Given a sorted array and a key, find index of the key in the array?

Binary search. Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.

successful search for 33

6	13	14	25	33	43	51	53	64	72	84	93	95	96	97
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

lo (index 0) hi (index 14)

Binary search: Java implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

```
public static int binarySearch(int[] a, int key)
{
   int lo = 0, hi = a.length-1;
   while (lo <= hi)
   {
      int mid = lo + (hi - lo) / 2;
      if      (key < a[mid]) hi = mid - 1;
      else if (key > a[mid]) lo = mid + 1;
      else return mid;
  }
  return -1;
}
```

one "3-way compare"

Invariant. If `key` appears in the array `a[]`, then `a[lo]` ≤ `key` ≤ `a[hi]`.

Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \lg N$ key compares to search in a sorted array of size N.

Def. $T(N)$ = # key compares to binary search a sorted subarray of size ≤ N.

Binary search recurrence. $T(N) \le T(N/2) + 1$ for $N > 1$, with $T(1) = 1$.

left or right half (floored division)

possible to implement with one 2-way compare (instead of 3-way)

Pf sketch. [assume N is a power of 2]

$$
\begin{aligned}
T(N) &\le T(N/2) + 1 && \text{[given]} \\
&\le T(N/4) + 1 + 1 && \text{[apply recurrence to first term]} \\
&\le T(N/8) + 1 + 1 + 1 && \text{[apply recurrence to first term]} \\
&\vdots \\
&\le T(N/N) + 1 + 1 + \ldots + 1 && \text{[stop applying, T(1) = 1]} \\
&= 1 + \lg N
\end{aligned}
$$