3.2 Stability exponents for a toy model

a)

In[2]:=
```
(*creating the dynamical system and determine r0*)
rDot[r_, μ_] := μ * r - r^3;
θDot [r_, μ_, ω_, ν_] := ω + ν * r^2;
rSols = Solve[rDot[r, μ] == 0, r];
r0 = r /. rSols[[3]]
```

Out[5]= $\sqrt{\mu}$

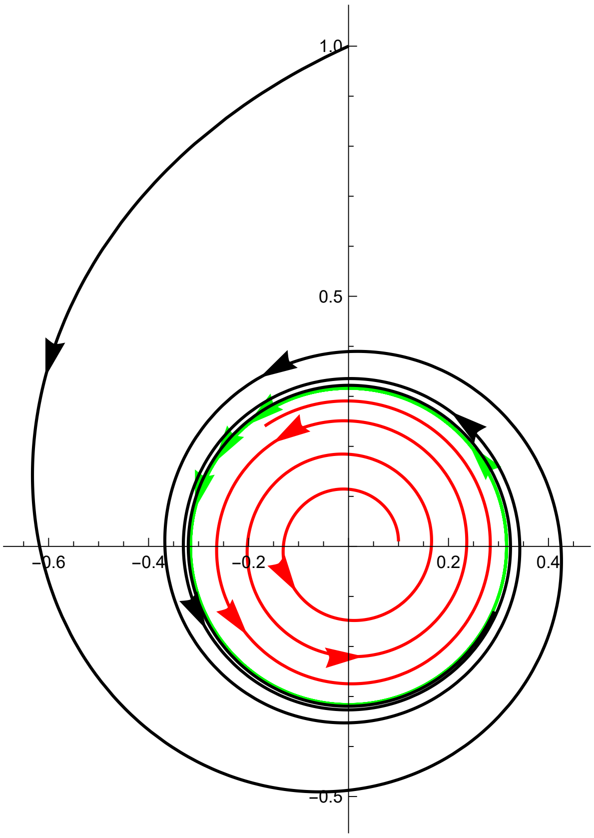
In[6]:=
```
(*calculating the velocity and also the
 circumference of circle to get the period time*)
velocity = r0 * θDot[r0, μ, ω, ν];
distance = 2 * Pi * r0;
periodT = distance / velocity
```

Out[8]= $\frac{2\pi}{\mu\nu+\omega}$

b)

In[9]:=
```
(*Dynamical system 2*)
X1Dot[X1_, X2_] := 1 / 10 * X1 - X2^3 - X1 * X2^2 - X1^2 * X2 - X2 - X1^3;
X2Dot[X1_, X2_] := X1 + 1 / 10 * X2 + X1 * X2^2 + X1^3 - X2^3 - X1^2 * X2;

F[{X1_, X2_}] := {X1Dot[X1, X2], X2Dot[X1, X2]};




limitCycle = NDSolve[{X1'[t] == F[{X1[t], X2[t]}][[1]], X2'[t] == F[{X1[t], X2[t]}][[2]],
    X1[0] == Sqrt[1 / 10], X2[0] == 0.01}, {X1, X2}, {t, 0, 20}];
outside = NDSolve[{X1'[t] == F[{X1[t], X2[t]}][[1]],
    X2'[t] == F[{X1[t], X2[t]}][[2]], X1[0] == 0.1, X2[0] == 0.01}, {X1, X2}, {t, 0, 20}];
inside = NDSolve[{X1'[t] == F[{X1[t], X2[t]}][[1]],
    X2'[t] == F[{X1[t], X2[t]}][[2]], X1[0] == 0, X2[0] == 1}, {X1, X2}, {t, 0, 20}];

p1 = ParametricPlot[Evaluate[{X1[t], X2[t]} /. limitCycle],
    {t, 0, 20}, PlotStyle → Green] /. Line[x_] →
    {Arrowheads[{{0.05, 0.1}, {0.05, 0.4}, {0.05, 0.6}, {0.05, 0.7}}], Arrow[x]};
p2 = ParametricPlot[Evaluate[{X1[t], X2[t]} /. outside],
    {t, 0, 20}, PlotStyle → Red] /. Line[x_] →
    {Arrowheads[{{0.05, 0.1}, {0.05, 0.4}, {0.05, 0.6}, {0.05, 0.7}}], Arrow[x]};
p3 = ParametricPlot[Evaluate[{X1[t], X2[t]} /. inside],
    {t, 0, 20}, PlotStyle → Black] /. Line[x_] →
    {Arrowheads[{{0.05, 0.1}, {0.05, 0.4}, {0.05, 0.6}, {0.05, 0.7}}], Arrow[x]};
Show[ p1, p2, p3, PlotRange → All]
(*The green trajectory is the limit cycle,
the red trajectory is a trajectory from inside going outside,
and the black trajectory coming outside and go into the limit cycle*)
```

Out[18]=



c)

```
(*We get that μ = 1/10,
ω = 1 and ν = 1. I did the solution by pen and paper so see separate
  pdf file in openTa for the solution to the answer*)
```

d)

In[ ]:=
```
(* Write the dynamical system 2 again*)
X1Dot[X1_, X2_] := 1 / 10 * X1 - X2^3 - X1 * X2^2 - X1^2 * X2 - X2 - X1^3;
X2Dot[X1_, X2_] := X1 + 1 / 10 * X2 + X1 * X2^2 + X1^3 - X2^3 - X1^2 * X2;
Dsystem =
  {X1'[t] == 1 / 10 * X1[t] - X2[t]^3 - X1[t] * X2[t]^2 - X1[t]^2 * X2[t] - X2[t] - X1[t]^3,
   X2'[t] == X1[t] + 1 / 10 * X2[t] + X1[t] * X2[t]^2 + X1[t]^3 - X2[t]^3 - X1[t]^2 * X2[t]};
(* Creating the J matrix*)
J = {{D[X1Dot[X1[t], X2[t]], X1[t]], D[X1Dot[X1[t], X2[t]], X2[t]]},
  {D[X2Dot[X1[t], X2[t]], X1[t]], D[X2Dot[X1[t], X2[t]], X2[t]]}}
```

Out[ ]=

$$\left\{\left\{\frac{1}{10} - 3\,X1[t]^2 - 2\,X1[t]\times X2[t] - X2[t]^2,\ -1 - X1[t]^2 - 2\,X1[t]\times X2[t] - 3\,X2[t]^2\right\},\right.$$
$$\left.\left\{1 + 3\,X1[t]^2 - 2\,X1[t]\times X2[t] + X2[t]^2,\ \frac{1}{10} - X1[t]^2 + 2\,X1[t]\times X2[t] - 3\,X2[t]^2\right\}\right\}$$

In[∘]:=

```
ω = 1;
ν = 1;
μ = 1/10;
periodT = 2π/(μ ν + ω);
r0 = Sqrt[μ];
sol = NDSolve[Join[{Dsystem[[1]], Dsystem[[2]],
    M11'[t] == J[[1]][[2]] * M21[t] + J[[1]][[1]] * M11[t],
    M12'[t] == J[[1]][[2]] * M22[t] + J[[1]][[1]] * M12[t],
    M21'[t] == J[[2]][[2]] * M21[t] + J[[2]][[1]] * M11[t],
    M22'[t] == J[[2]][[2]] * M22[t] + J[[2]][[1]] * M12[t],
    X1[0] == r0, X2[0] == 0, M11[0] == M22[0] == 1, M12[0] == M21[0] == 0}],
  {X1, X2, M11, M12, M21, M22}, {t, 0, periodT}]
```

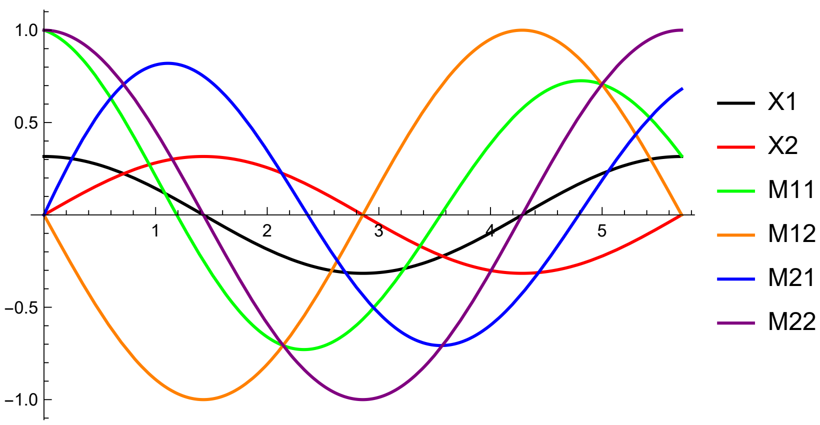
Out[∘]=

```
{{X1 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ],
  X2 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ],
  M11 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ],
  M12 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ],
  M21 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ],
  M22 → InterpolatingFunction[ Domain: {{0., 5.71}} Output: scalar ]}}
```

In[∘]:=

```
Plot[{X1[t] /. sol, X2[t] /. sol, M11[t] /. sol,
  M12[t] /. sol, M21[t] /. sol, M22[t] /. sol}, {t, 0, periodT},
 PlotLegends → {"X1", "X2", "M11", "M12", "M21", "M22"},
 PlotStyle → {Black, Red, Green, Orange, Blue, Purple}]
```

Out[∘]=



e)

In[ ]:=
```
(*Taking out the different
 M:s and then put them into matrix for a better representation/visualizing*)
M11 = M11[periodT] /. sol[[1]];
M12  = M12[periodT] /. sol[[1]];
M21 = M21[periodT] /. sol[[1]];
M22 = M22[periodT] /. sol[[1]];
Mmatrix = {{M11, M12}, {M21, M22}};
Mmatrix // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.319053 & 2.12317\times10^{-8} \\ 0.680947 & 1. \end{pmatrix}$$

f)

In[ ]:=
```
stabilityExpSOfSep = Log[Eigenvalues[Mmatrix]] / periodT
```

Out[ ]= $\{5.78753\times10^{-9}, -0.2\}$

In[ ]:=
```
(*sigma1≤sigma2 therefore sigma1 is the
  second term in the above vector and sigma2 the first term*)
sigma1 = stabilityExpSOfSep[[2]]
sigam2  = stabilityExpSOfSep[[1]]
```

Out[ ]= -0.2

Out[ ]= 5.78753×10^{-9}