2.3 Hopf Bifurcation

a).

we have two systems

$$\begin{cases} \dot{x} = \mu x - 3y - 1x^3 \\ \dot{y} = 3x - \mu y + 2y^3 \end{cases} \quad (1)$$

$$\begin{cases} \dot{x} = \mu x + y - x^2 \\ \dot{y} = -x + \mu y + 2x^2 \end{cases} \quad (2)$$

The system at the bifurcation takes the form:

$$\begin{cases} \dot{x} = -wy + f(x,y) \\ \dot{y} = wx + g(x,y) \end{cases}$$

from this we can se that:

$$\boxed{w_{(1)} = 3 \qquad \text{and} \qquad w_{(2)} = -1}$$

b) Determine f and g for the System ① and ②.

with $W_{(1)} = 3 \Rightarrow$

$$\Rightarrow \begin{cases} \dot{x} = -3y + f(x,y) \\ \dot{y} = 3x + g(x,y) \end{cases}$$

Set these equal to system ① when $\mu = 0$

$\Rightarrow$:

$$\dot{x} = -3y + f(x,y) = -3y - 1x^3$$

$$\dot{y} = 3x + g(x,y) = 3x + 2y^3$$

$$\Rightarrow \begin{cases} f_{(1)}(x,y) = -x^3 \\ g_{(2)}(x,y) = 2y^3 \end{cases}$$

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with $W_{(2)} = -1 \Rightarrow$

$$\Rightarrow \begin{cases} \dot{x} = y + f(x,y) \\ \dot{y} = -x + g(x,y) \end{cases}$$

Set those equal System ② where $\mu = 0$

$$\Rightarrow \begin{cases} \dot{x} = y + f(x,y) = y - x^2 \\ \dot{y} = -x + g(x,y) = -x + 2x^2 \end{cases}$$

this gives us:

$$\begin{cases} f_{(2)}(x,y) = -x^2 \\ g_{(2)}(x,y) = 2x^2 \end{cases}$$

answer b):

$$f_{(1)}(x,y) = -x^3$$

$$g_{(2)}(x,y) = 2y^3$$

$$f_{(2)}(x,y) = -x^2$$

$$g_{(2)}(x,y) = 2x^2$$

c) : $16a = f_{xxx} + f_{xy} + g_{xxy} + g_{yy}$
$$+ \frac{1}{w}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} - g_{yy}) - f_{xx} g_{xx} + f_{yy} g_{yy}\right]$$

consider system (1) :

$$f_{\cancel{x}xx} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}(-x^3)\right) = \frac{\partial^2}{dx^2}\left(-3x^2\right) =$$

$$= \frac{\partial}{\partial x}(-6x) = -6$$

$$f_{xx} = \frac{\partial^2}{\partial x^2}\left(-(x^3)\right) = -6x$$

$$g_{yyy} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial y}(2y^3)\right) = \frac{\partial^2}{\partial y^2}\left(6y^2\right)$$

$$= 12$$

$$g_{yy} = 12y$$

all the other derivatives that are asked for is 0 since $f = -x^3$ only depends on x and $g = 2y^3$ only depends on y

$$\Rightarrow \quad 16a = -6 + 12 = 6$$

$$a = \frac{6}{16} = \frac{3}{8}$$

since $a = \frac{3}{8} > 0$ there is a subcritical bifurcation

<u>consider system (2)</u>

$$f_{xxx} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}(-x^2)\right) = \frac{\partial^2}{\partial x^2}(-2x) = 0$$

$$f_{xx} = \frac{\partial}{\partial x}(-2x) = -2$$

$$g_{xxx} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}(2x^2)\right) = \frac{\partial^2}{\partial x^2}(4x) = 0$$

$$g_{xx} = \frac{\partial}{\partial x}(4x) = 4$$

rest of the asked derivatives are zero

$$\Rightarrow \quad 16a = -\frac{1}{1}\left[-(-2\cdot 4)\right] =$$

$$= -1\cdot 8 = -8$$

$$\Rightarrow \quad a = -\frac{8}{16} = -\frac{1}{2}$$

Since $a = -\frac{1}{2} < 0$, there is a supercritical bifurcation.