Motion of damped pendulum

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{\gamma}{m}\dot{\theta} \qquad (1)$$

We are given that:

$$\dot{\theta} = w$$

$\Rightarrow$ that we can write equation (1) as

$$\dot{w} = -\frac{g}{l}\sin(\theta) - \frac{\gamma}{m}w$$

So now we have a two dimensional dynamical system for w and θ:

$$\begin{cases} \dot{\theta} = w & (2) \\ \dot{w} = -\frac{g}{L}\sin(\theta) - \frac{\gamma}{m}w & (3) \end{cases}$$

Then we can transforme into dimensionless variables (x, y, t') with

$$\theta = \theta_0 x \qquad (4)$$

$$w = w_0 y \qquad (5)$$

$$t = t_0 t' \qquad (6)$$

by substitute these (4, 5, 6) into 2 and 3

we get:

$$\dot{\theta} = \frac{d}{dt}(\theta) = \frac{\theta_0}{t_0}\frac{dx}{dt'} = w_0 y$$

$$\dot{w} = \frac{d}{dt}(w) = \frac{w_0}{t_0}\frac{dy}{dt'} = -\frac{g}{L}\sin(\theta_0 x) - \frac{\gamma}{m} w_0 y$$

t_0 has unit s so we set $t_0 = \sqrt{\frac{L}{g}}$

since $g = \frac{m}{s^2}$ and $L = m$ $\Rightarrow$ $\sqrt{\frac{L}{g}}$.

has the dimension $\sqrt{\frac{m}{\frac{m}{s^2}}} = \sqrt{s^2} = s$

so $\boxed{t_0 = \sqrt{\frac{L}{g}}}$ gives:

$$\frac{dx}{dt'} = \frac{w_0 y \cdot t_0}{\theta_0} = \frac{w_0 y \sqrt{\frac{L}{g}}}{\theta_0} = \sqrt{\frac{L}{g}}\, y \cdot \frac{w_0}{\theta_0}$$

$$\frac{dy}{dt'} = \left(-\frac{g}{L}\sin(\theta_0 x) - \frac{\gamma}{m} w_0 y\right) \cdot \frac{t_0}{w_0}$$

$$\frac{dy}{dt'} = -\frac{g}{L}\sqrt{\frac{L}{g}}\,\frac{\sin(\theta_0 x)}{w_0} - \frac{\gamma}{m}\,y\,\sqrt{\frac{L}{g}}$$

it was given that

$$\begin{cases} \dfrac{dx}{dt'} = y \\[2ex] \dfrac{dy}{dt'} = -\sin(x) - \alpha y \end{cases}$$

compare those

to the ones we calculated

$$\begin{cases} \dfrac{dx}{dt'} = \sqrt{\dfrac{L}{g}}\,y \cdot \dfrac{w_0}{\theta_0} \\[2ex] \dfrac{dy}{dt'} = -\dfrac{g}{L}\sqrt{\dfrac{L}{g}}\,\dfrac{\sin(\theta_0 x)}{w_0} - \dfrac{\gamma}{m}\,y\sqrt{\dfrac{L}{g}} \end{cases}$$

we get :
$$\sqrt{\frac{L}{g}}\ \frac{w_0}{\theta_0} = 1$$

$\theta_0 = 1$ (since we want the term $\sin(x)$ in $\frac{dy}{dt'}$)

$$\boxed{\theta_0 = 1} \Rightarrow \boxed{w_0 = \sqrt{\frac{g}{L}}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dt'} = -\frac{g}{Lw_0}\sqrt{\frac{L}{g}}\sin(x) - \frac{\gamma}{m}y\sqrt{\frac{L}{g}}$$

$$= -\frac{g}{L}\sqrt{\frac{L}{g}}\sqrt{\frac{L}{g}}\sin(x) - \frac{\gamma}{m}y\sqrt{\frac{L}{g}} =$$

$$= -\frac{g}{L}\cdot\frac{L}{g}\sin(x) - \frac{\gamma}{m}y\sqrt{\frac{L}{g}} =$$

$$= -\sin(x) - \frac{\gamma}{m}y\sqrt{\frac{L}{g}} = -\sin(x) - \sigma y$$

$$\Rightarrow \boxed{\sigma = \frac{\gamma}{m}\sqrt{\frac{L}{g}}}$$