2.4 Homoclinic bifurcation

a)

In[ ]:=
```
Solve[{μ * x + y - x^2 == 0, -x + μ * y + 2 * x^2 == 0}, {x, y}]
```

Out[ ]=

$$\left\{\{x \to 0, y \to 0\}, \left\{x \to \frac{1 + \mu^2}{2 + \mu}, y \to \frac{1 - 2\mu + \mu^2 - 2\mu^3}{(2 + \mu)^2}\right\}\right\}$$

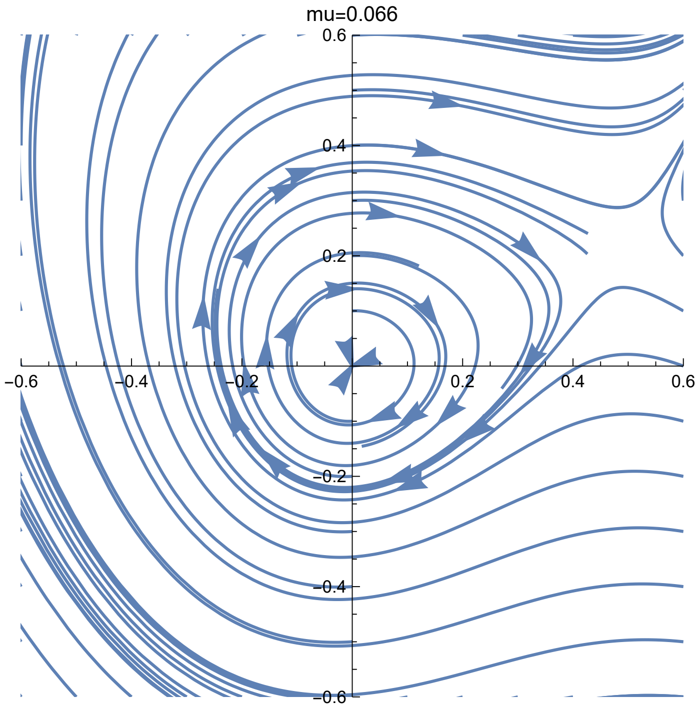
In[109]:=
```
(*Check different values of μ and found that μ=0.066 gives a
  homoclinical bifurcation*)
tEnd = 10;
tStart = 0;
xMin = -0.6;
xMax = 0.6;
yMin = -0.6;
yMax = 0.6;
systemSolver[x0_, y0_] =
  NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2, y'[t] == -x[t] + μ * y[t] + 2 x[t]^2,
      x[0] == x0, y[0] == y0} /. μ → 0.066, {x, y}, {t, tStart, tEnd}];

table1 = Table[{0, y}, {y, yMin, yMax, 0.1}];
table2 = Table[{xMin, y}, {y, yMin, yMax, 0.1}];
table3 = Table[{xMax, y}, {y, yMin, yMax, 0.1}];
table4 = Table[{x, yMin}, {x, xMin, xMax, 0.1}];
table5 = Table[{x, yMax}, {x, xMin, xMax, 0.1}];
initialCondition = Join[table1, table2, table3, table4, table5];

Show[Table[ParametricPlot[Evaluate[
      {x[t], y[t]} /. systemSolver[initialCondition[[i, 1]], initialCondition[[i, 2]]]],
     {t, tStart, tEnd}, PlotRange → {{xMin, xMax}, {yMin, yMax}}],
    {i, Length[initialCondition]}] /.
  Line[x_] :→ {Arrowheads[{0., 0.04, 0.04, 0.04, 0.}], Arrow[x]},
 PlotLabel → "mu=0.066"]
```

NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Out[118]=



In[119]:=

c)

In[ ]:=
```
ODEs = {x'[t] == u * x[t], y'[t] == s * y[t], x[0] == γ, y[0] == 1 };
sol = DSolve[ODEs, {x[t], y[t]}, t];
Solve[(x[t] /. sol[[1]][[1]]) == 1, t]
```

Out[ ]= $\left\{\left\{t \to \boxed{\frac{2\, i\, \pi\, c_1 + \text{Log}\left[\frac{1}{\gamma}\right]}{u} \;\text{ if }\; c_1 \in \mathbb{Z}}\right\}\right\}$

```
(*answer is log(1/γ)/u*)
```

d)

In[ ]:=
```
fixedPoints = Solve[{μ * x + y - x^2 == 0, -x + μ * y + 2 * x^2 == 0}, {x, y}];
xSaddlePoint = fixedPoints[[2]][[1]]
```

Out[ ]= $x \to \frac{1+\mu^2}{2+\mu}$

In[ ]:=
```
(*taking the derivative of xDot with respect to x and then y*)
xDot[x_, y_, μ_] := μ x + y - x^2;
yDot[x_, y_, μ_] := -x + μ y + 2 x^2;
derivativeXDotx = D[xDot[x, y, μ], x]
derivativeXDoty = D[xDot[x, y, μ], y]
```

Out[ ]= $-2\,x + \mu$

Out[ ]= 1

In[ ]:=
```
(*taking the derivative of yDot with respect to x and then y*)
derivativeYDotx = D[yDot[x, y, μ], x]
derivativeYDoty = D[yDot[x, y, μ], y]
```

Out[ ]= $-1 + 4\,x$

Out[ ]= μ

```
(*From the above partial derivatives, we create the jacobian
 matrix, with the saddle x as the x parameter*)
jacobian = {{derivativeXDotx /. xSaddlePoint, derivativeXDoty},
   {derivativeYDotx /. xSaddlePoint, derivativeYDoty}}
```

Out[ ]= $\left\{\left\{\mu - \frac{2\left(1+\mu^2\right)}{2+\mu}, 1\right\}, \left\{-1 + \frac{4\left(1+\mu^2\right)}{2+\mu}, \mu\right\}\right\}$

In[ ]:=
```
eigenVals = Eigenvalues[jacobian]
```

Out[ ]= $\left\{\frac{-1 + 2\,\mu - \sqrt{5 + 9\,\mu^2 + 4\,\mu^3 + \mu^4}}{2+\mu}, \frac{-1 + 2\,\mu + \sqrt{5 + 9\,\mu^2 + 4\,\mu^3 + \mu^4}}{2+\mu}\right\}$

In[ ]:=
```
answer = eigenVals[[2]]
```

Out[ ]=

$$\frac{-1 + 2\mu + \sqrt{5 + 9\mu^2 + 4\mu^3 + \mu^4}}{2 + \mu}$$