4.2 a)

$$D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln(I_q(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} \quad (1)$$

$$I_q(\varepsilon) = \sum_{j=0}^{N_{box}} P_j^q(\varepsilon) \quad (2)$$

$$P_j(\varepsilon) = \binom{n}{k} p^k (1-p)^{n-k} \quad (4)$$

(4) in (2) $\Rightarrow$:

$$I_q(\varepsilon) = \sum_k^n \binom{n}{k} \left[p^k (1-p)^{n-k}\right]^q \quad (5)$$

we can use the following binomial theorem:

$$\sum_k^n \binom{n}{k} r^k = (1+r)^n \quad (6)$$

we rewrite equation (5) to:

$$I_q(\varepsilon) = \sum_k^n \binom{n}{k} \left[p^q (1-p)^{-q}\right]^k (1-p)^{nq}$$

using the above binomial theorem (6) we get:

$$I_q(\varepsilon) = \left(1 + p^q (1-p)^{-q}\right)^n (1-p)^{nq} =$$

$$= \left((1-p)^q + p^q\right)^n \quad (7)$$

② in ① $\Rightarrow$:

$$D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln\left[\left((1-p)^q + p^q\right)^n\right]}{\ln\left[\frac{1}{\varepsilon}\right]} =$$

$$\boxed{\begin{array}{l} \varepsilon = (1/3)^n \\ p = 1/3 \end{array}}$$

$$= \frac{1}{1-q} \frac{n \ln\left[(1-p)^q + p^q\right]}{\ln(3^n)} =$$

$$= \frac{1}{1-q} \frac{\cancel{n} \ln\left[(1-p)^q + p^q\right]}{\cancel{n} \ln(3)} =$$

$$= \frac{1}{1-q} \frac{\ln\left[\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right]}{\ln[3]}$$

So Answer: $$D_q = \frac{1}{1-q} \frac{\ln\left[\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right]}{\ln[3]}$$