

# A numerical study of fixation probabilities for strategies in the Iterated Prisoner's Dilemma

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## Abstract

The Iterated Prisoner's Dilemma is a well established framework for the study of emergent behaviour. In this paper an extensive numerical study of the evolutionary dynamics of this framework are presented.

Fixation probabilities for Moran processes are obtained for 172 different strategies. This is done in both a standard 200 turn interaction and a noisy setting.

To the authors knowledge this is the largest such study. It allows for insights about the behaviour and performance of strategies with regard to their survival in an evolutionary setting.

## 1 Introduction

Main questions are:

1. What strategies are good invaders?
2. What strategies are good at resisting invasion?
3. How do 1 and 2 change as a function of population size?

A key point here is that the relative fitness of a strategy depends on the population distribution. The original Moran process assumes a relative fitness of  $r$  of one strategy over the other, giving a fixation probability for the starting population  $(i, N - i)$  (when  $r = 1$ )

$$\rho = \frac{1 - r^{-i}}{1 - r^{-N}}$$

and  $\rho = 1/N$  if  $r = 1$  (the neutral fixation probability).

This corresponds to a game matrix  $[[1, 1], [r, r]]$  (or  $[[r, r], [1, 1]]$ ), which is of course not what we have – it's a little complicated because our "fitness" is not the payout from the game matrix, rather the sum of the total scores of all the interactions each round. So ALLC and TFT are neutral wrt to each other because they will have the same score each round, giving an effective fitness landscape  $f(i, N - i) = A[i, N - i]^T$  given by the matrix  $A = [[1, 1], [1, 1]]$ . This means that noise and the number of turns per Moran round are significant parameters. I think we should fix the turns at 200; some recent authors run the turns to infinity (to reach stationarity on the sub-"Markov process" on the states (C, C), (C, D), (D, C), (D, D)) but we can't analytically compute the stationary distribution for strategies that use more than one round of memory (and it's not really a Markov process for more than one round of memory anyway). Plus it's unrealistic, and ultimately just amounts to a transform of the game matrix.

To see if one strategy is not neutral with respect to another, we want to empirically measure the fixation probability and compare to the neutral rate. To do this right we need a lot of counts, since we're estimating a binomial probability  $p$  with variance  $p(1 - p)/k$  and  $p$  is close to  $1/N$ . To get the variance small you need something like  $k > 1000$  observations (we can work out the precise requirements).

Note we're not estimating  $r$  for each strategy (pair) since we're in a frequency dependent situation, so we need to look at the population states  $(1, N - 1)$  and  $(N - 1, 1)$  for every pair of strategies, i.e. we can't assume that we're in a  $\rho \leftrightarrow 1 - \rho$  symmetry. More precisely,  $\rho_{(1, N - 1)} = 1 - \rho_{(N - 1, 1)}$  in general. However we can (for fun) compute  $r$  from  $\rho$  with Newton's method (it's not easily invertible for  $N > 3$ ), or take a Bayesian approach on what the distribution of  $\rho$  is and then compute a distribution for  $r$  in the usual way.

A nice addition would be, for an interesting combination of strategies, to measure the fixation value for all  $(i, N - i)$  and compare to the above formula for the value of  $r$  derived from the  $(1, N - 1)$  case. This would show how much we deviate from frequency independence.

Beyond the raw data, we should try to estimate the strategies that are 1) most resistant to invasion 2) the best invaders 3) "most neutral"

as a function of  $N$  across the entire population of strategies. This can really open up if you want to say optimize a parameterized strategy to be most resistant to invasion (a topic of future work, perhaps) – for example Random( $p$ ) for what  $p$  is best?

The existing notebook attempts to get at 1 and 2 by looking at the distributions of fixation probabilities for each strategy – that’s what the box plots for each  $N$  try to visualize for particular  $N$ , and the ”Player Rankings by Median vs. Population Size” for how the cooperative strategies become more successful as  $N$  increases. That plot is the main takeaway IMO, and reinforces the ”evolution of cooperation” narrative that’s so popular. We can tie back to Press and Dyson here – yes, ZD strategies are good Head-to-Head and in small populations, but they aren’t great when the population size gets bigger. How much bigger? Even at  $N=4$  there is a dramatic decline for ZD-extort. Note that this goes against the claims of Stewart and Plotkin (they claimed that ZD strategies basically dominate the Moran process no matter how much memory you allow). This also matches our tournament results – ZD strategies win matches but not tournaments.

It would be great to see how the ensemble strategies (meta strategies) fare, if we don’t mind burning the CPU cycles. I left them out of my initial analysis.

Further Variants (possible additions or future papers): \* Noise \* Spatial structure \* More than two types in the population \* Modified Moran processes (e.g. Fermi selection with the strength of selection coefficient) \* Altered game matrices

Noise is especially interesting because a lot of the cooperative strategies are going to appear neutral to each other (since neither will cast a D unprovoked). A little bit of noise should shuffle the ranks around quite a bit, and show off the abilities of e.g. OmegaTFT. Might be worth including at least one of the ”Player Rankings by Median vs. Population Size” plots for some value of noise (such as 0.05).

More future work: \* Mutation – for mutation we no longer have fixation, rather a stationary distribution. This may require some more programming to compute efficiently (perhaps my stationary library). There’s a lot of interesting work to do here.

Ip think we’d want to include a few of the heatmaps in the final section of the notebook for some interesting cases, like FoolMeOnce, EvolvedLookerUp, etc. Pushing  $N$  higher will make all the plots more interesting. How high we can get  $N$ ? I’d really like to get it to  $N=11$ .

Structure:

- Overview of Moran processes;
- Review of the literature ([1, 3, 2]);
- Short discussion about the Axelrod library.

I’m happy to write this section. We can lift some references from one of my papers on the Moran process.

## 2 Methodology

To carry out this large numerical experiment 172 strategies are used from [4]. These include 169 default strategies in the library at the time (excluding strategies classified as having a long run time) as well as the following 3 finite state machine strategies:

Appendix A shows all the players in question. More information about each player can be obtained in the documentation for [4]. The memory depth of the used strategies is shown in Table 1a.

Memory Depth	0	1	2	3	4	5	6	9	10	11	12	16	20	40	200	$\infty$
Count	3	31	12	8	2	6	1	1	5	1	1	2	2	2	1	94

(a) Memory depth

Stochastic	Count
False	123
True	49

(b) Stochastic versus deterministic

Table 1: Summary of properties of used strategies

All strategies are paired and these pairs are used in 1000 repetitions of a Moran process assuming a starting population of  $(N/2, N/2)$ . This is repeated for even  $N$  between 2 and 14. The fixation probability is then estimated for each value of  $N$ .

Note that due to the high computational cost of these experiments, for any given interaction between two players within the Moran process the outcome is sampled from a pre computed cache of 10000 match outcomes. This is carried out using the approximate Moran process implemented in [4].

As an example, Figure 1 shows the scores between two players that over the 10000 outcomes gives 6817 different scores.

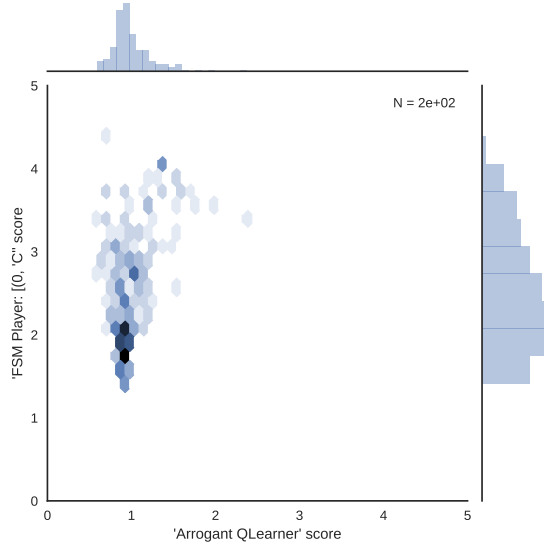


Figure 1: All possible scores for the pair of strategies that have the most different number of match outcomes

### 3 Validation

As described in [3] Consider the payoff matrix:

$$M = \begin{pmatrix} a, b \\ c, d \end{pmatrix} \quad (1)$$

The expected payoffs of  $i$  players of the first type in a population with  $N - i$  players of the second type are given by:

$$F_i = \frac{a(i-1) + b(N-i)}{N-1} \quad (2)$$

$$G_i = \frac{ci + d(N-i-1)}{N-1} \quad (3)$$

With an intensity of selection  $\omega$  the fitness of both strategies is given by:

$$f_i = 1 - \omega + \omega F_i \quad (4)$$

$$g_i = 1 - \omega + \omega G_i \quad (5)$$

The transitions within the birth death process that underpins the Moran process are then given by:

$$p_{i,i+1} = \frac{if_i}{if_i + (N-i)g_i} \frac{N-i}{N} \quad (6)$$

$$p_{i,i-1} = \frac{(N-i)g_i}{if_i + (N-i)g_i} \frac{i}{N} \quad (7)$$

$$p_{ii} = 1 - p_{i,i+1} - p_{i,i-1} \quad (8)$$

Using this it is a known result that the fixation probability of the first strategy in a population of  $i$  individuals of the first type (and  $N - i$  individuals of the second. We have:

$$x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^j \gamma_j}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^j \gamma_j} \quad (9)$$

where:

$$\gamma_j = \frac{p_{j,j-1}}{p_{j,j+1}}$$

Using this comparisons of  $x_{N/2}$  are shown in Figure 2. Note that these are all deterministic strategies and show a perfect match up between the expected value of (9) and the actual Moran process.

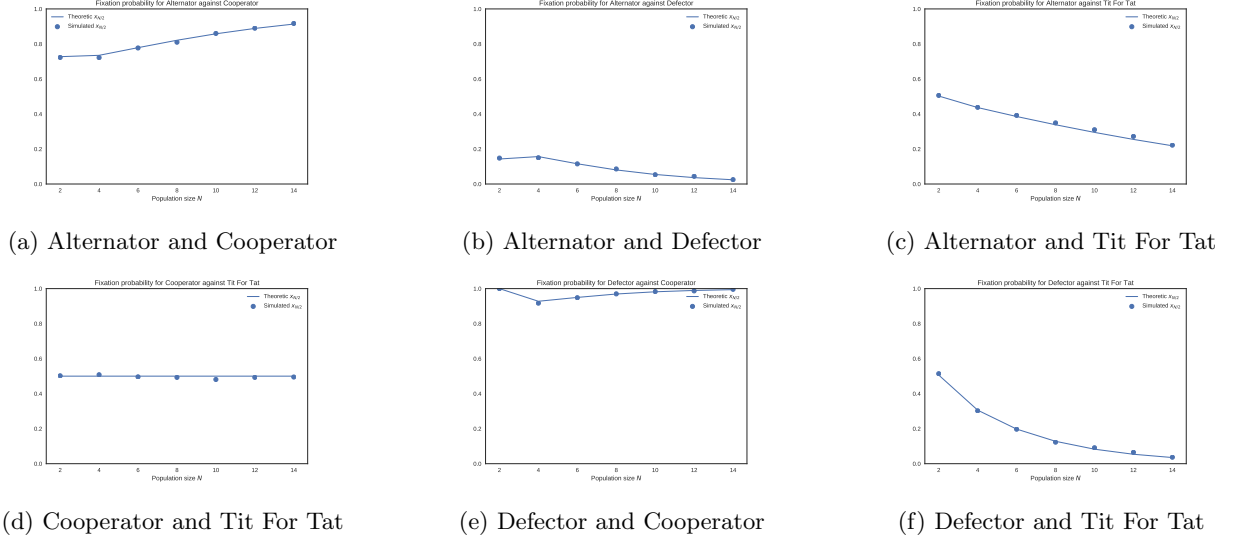


Figure 2: Comparison of theoretic and actual Moran Process fixation probabilities for **deterministic** strategies

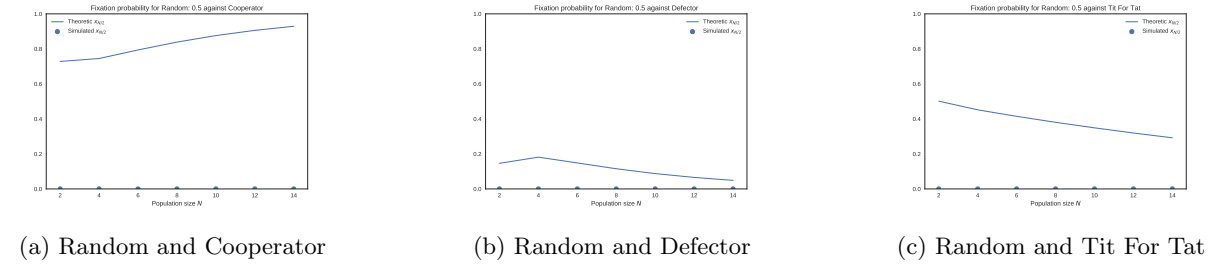


Figure 3: Comparison of theoretic and actual Moran Process fixation probabilities for **stochastic** strategies

Figure ?? shows the fixation probabilities for stochastic strategies. These are no longer a good match which highlights the weakness of the analytical formulae the relies on the average payoffs. A detailed analysis of the 172 strategies considered will be shown in the next Section.

## 4 Numerical results

Structure:

- General overview of the data obtained;
- Inclusion of most of the work in `Moran.ipynb`.

## 5 Conclusion

## References

- [1] Christopher Lee, Marc Harper, and Dashiell Fryer. “The Art of War: Beyond Memory-one Strategies in Population Games”. In: *Plos One* 10.3 (2015), e0120625. ISSN: 1932-6203. DOI: 10.1371/journal.pone.0120625. URL: <http://dx.plos.org/10.1371/journal.pone.0120625>.

- [2] P.A.P. Moran. “Random Processes in Genetics”. In: April (1957), pp. 60–71.
- [3] Martin A Nowak. *Evolutionary Dynamics: Exploring the Equations of Life*. Cambridge: Harvard University Press. ISBN: 0674023382. DOI: 10.1086/523139.
- [4] The Axelrod project developers. *Axelrod: v2.9.0*. Apr. 2016. DOI: 499122. URL: <http://dx.doi.org/10.5281/zenodo.499122>.

## A List of players

- |                               |  |                                   |
|-------------------------------|--|-----------------------------------|
| 1. Adaptive                   | 39. Eventual Cycle Hunter                                  | 76. Hard Go By Majority: 20       |
| 2. Adaptive Tit For Tat: 0.5  | 40. Evolved ANN  | 77. Hard Go By Majority: 40       |
| 3. Aggravater                 | 41. Evolved ANN 5  | 78. Hard Go By Majority: 5        |
| 4. ALLCorALLD                 | 42. Evolved ANN 5 Noise 05                                 | 79. Hard Prober                   |
| 5. Alternator                 | 43. Evolved FSM 4  | 80. Hard Tit For 2 Tats           |
| 6. Alternator Hunter          | 44. Evolved FSM 16   | 81. Hard Tit For Tat              |
| 7. AntiCycler                 | 45. Evolved FSM 16 Noise 05                                | 82. Hesitant QLearner             |
| 8. Anti Tit For Tat           | 46. EvolvedLookerUp1.1.1                                   | 83. Hopeless                      |
| 9. Adaptive Pavlov 2006       | 47. EvolvedLookerUp2.2.2                                   | 84. Inverse                       |
| 10. Adaptive Pavlov 2011      | 48. Evolved HMM 5  | 85. Inverse Punisher              |
| 11. Appeaser                  | 49. Feld: 1.0, 0.5, 200                                    | 86. Joss: 0.9                     |
| 12. Arrogant QLearner         | 50. Firm But Fair  | 87. Knowledgeable Worse and Worse |
| 13. Average Copier            | 51. Fool Me Forever  | 88. Level Punisher                |
| 14. Better and Better         | 52. Fool Me Once   | 89. Limited Retaliate: 0.1, 20    |
| 15. BackStabber: ('D', 'D')   | 53. Forgetful Fool Me Once: 0.05                           | 90. Limited Retaliate 2: 0.08, 15 |
| 16. Bully                     | 54. Forgetful Grudger                                      | 91. Limited Retaliate 3: 0.05, 20 |
| 17. Calculator                | 55. Forgiver   | 92. Math Constant Hunter          |
| 18. Cautious QLearner         | 56. Forgiving Tit For Tat                                  | 93. Naive Prober: 0.1             |
| 19. Champion                  | 57. Fortress3  | 94. MEM2                          |
| 20. CollectiveStrategy        | 58. Fortress4  | 95. Negation                      |
| 21. Contrite Tit For Tat      | 59. GTFT: 0.33   | 96. Nice Average Copier           |
| 22. Cooperator                | 60. General Soft Grudger:<br>n=1,d=4,c=2                   | 97. Nydegger                      |
| 23. Cooperator Hunter         | 61. Soft Go By Majority                                    | 98. Omega TFT: 3, 8               |
| 24. Cycle Hunter              | 62. Soft Go By Majority: 10                                | 99. Once Bitten                   |
| 25. Cycler CCCCD              | 63. Soft Go By Majority: 20                                | 100. Opposite Grudger             |
| 26. Cycler CCCD               | 64. Soft Go By Majority: 40                                | 101. $\pi$                        |
| 27. Cycler CCD                | 65. Soft Go By Majority: 5                                 | 102. Predator                     |
| 28. Cycler DC                 | 66. $\phi$   | 103. Prober                       |
| 29. Cycler DDC                | 67. Gradual  | 104. Prober 2                     |
| 30. Cycler CCCDCD             | 68. Gradual Killer: ('D', 'D', 'D',<br>'D', 'D', 'C', 'C') | 105. Prober 3                     |
| 31. Davis: 10                 | 69. Grofman  | 106. Prober 4                     |
| 32. Defector                  | 70. Grudger  | 107. Pun1                         |
| 33. Defector Hunter           | 71. GrudgerAlternator                                      | 108. PSO Gambler 1.1.1            |
| 34. Desperate                 | 72. Grumpy: Nice, 10, -10                                  | 109. PSO Gambler 2.2.2            |
| 35. DoubleCrosser: ('D', 'D') | 73. Handshake  | 110. PSO Gambler 2.2.2 Noise 05   |
| 36. Doubler                   | 74. Hard Go By Majority                                    | 111. PSO Gambler Mem1             |
| 37. EasyGo                    | 75. Hard Go By Majority: 10                                | 112. Punisher                     |
| 38. Eatherley                 |  | 113. Raider                       |
|                               |  | 114. Random: 0.5                  |
|                               |  | 115. Random Hunter                |

116. Remorseful Prober: 0.1	146. Tricky Defector	'D', 8, 'C'), (3, 'C', 3, 'C'), (3, 'D', 12, 'D'), (4, 'C', 6, 'C'), (4, 'D', 3, 'C'), (5, 'C', 11, 'C'), (5, 'D', 8, 'D'), (6, 'C', 13, 'D'), (6, 'D', 14, 'C'), (7, 'C', 4, 'D'), (7, 'D', 2, 'D'), (8, 'C', 14, 'D'), (8, 'D', 8, 'D'), (9, 'C', 0, 'C'), (9, 'D', 10, 'D'), (10, 'C', 8, 'C'), (10, 'D', 15, 'C'), (11, 'C', 6, 'D'), (11, 'D', 5, 'D'), (12, 'C', 6, 'D'), (12, 'D', 9, 'D'), (13, 'C', 9, 'D'), (13, 'D', 8, 'D'), (14, 'C', 8, 'D'), (14, 'D', 13, 'D'), (15, 'C', 4, 'C'), (15, 'D', 5, 'C')], 1, C
117. Resurrection	147. Tullock: 11	171. FSM Player: [(0, 'C', 13, 'D'), (0, 'D', 12, 'D'), (1, 'C', 3, 'D'), (1, 'D', 4, 'D'), (2, 'C', 14, 'D'), (2, 'D', 9, 'D'), (3, 'C', 0, 'C'), (3, 'D', 1, 'D'), (4, 'C', 1, 'D'), (4, 'D', 2, 'D'), (5, 'C', 12, 'C'), (5, 'D', 6, 'C'), (6, 'C', 1, 'C'), (6, 'D', 14, 'D'), (7, 'C', 12, 'D'), (7, 'D', 2, 'D'), (8, 'C', 7, 'D'), (8, 'D', 9, 'D'), (9, 'C', 8, 'D'), (9, 'D', 0, 'D'), (10, 'C', 2, 'C'), (10, 'D', 15, 'C'), (11, 'C', 7, 'D'), (11, 'D', 13, 'D'), (12, 'C', 3, 'C'), (12, 'D', 8, 'D'), (13, 'C', 7, 'C'), (13, 'D', 10, 'D'), (14, 'C', 10, 'D'), (14, 'D', 7, 'D'), (15, 'C', 15, 'C'), (15, 'D', 11, 'D')], 1, C
118. Retaliate: 0.1	148. Two Tits For Tat	
119. Retaliate 2: 0.08	149. VeryBad	
120. Retaliate 3: 0.05	150. Willing	
121. Revised Downing: True	151. Winner12	
122. Ripoff	152. Winner21	
123. Risky QLearner	153. Win-Shift Lose-Stay: D	
124. SelfSteem	154. Win-Stay Lose-Shift: C	
125. ShortMem	155. Worse and Worse	
126. Shubik	156. Worse and Worse 2	
127. Slow Tit For Two Tats	157. Worse and Worse 3	172. FSM Player: [(0, 'C', 0, 'C'), (0, 'D', 3, 'C'), (1, 'C', 5, 'D'), (1, 'D', 0, 'C'), (2, 'C', 3, 'C'), (2, 'D', 2, 'D'), (3, 'C', 4, 'D'), (3, 'D', 6, 'D'), (4, 'C', 3, 'C'), (4, 'D', 1, 'D'), (5, 'C', 6, 'C'), (5, 'D', 3, 'D'), (6, 'C', 6, 'D'), (6, 'D', 6, 'D'), (7, 'C', 7, 'D'), (7, 'D', 5, 'C')], 1, C
128. Slow Tit For Two Tats 2	158. ZD-Extort-2: 0.1111111111111111, 0.5	
129. Sneaky Tit For Tat	159. ZD-Extort-2 v2: 0.125, 0.5, 1	
130. Soft Grudger	160. ZD-Extort-4: 0.23529411764705882, 0.25, 1	
131. Soft Joss: 0.9	161. ZD-GTFT-2: 0.25, 0.5	
132. SolutionB1	162. ZD-GEN-2: 0.125, 0.5, 3	
133. SolutionB5	163. ZD-SET-2: 0.25, 0.0, 2	
134. Spiteful Tit For Tat	164. $e$	
135. Stalker: D	165. Meta Hunter: 6 players	
136. Stochastic Cooperator	166. Meta Hunter Aggressive: 7 players	
137. Stochastic WSLs: 0.05	167. Meta Majority Memory One: 31 players	172. FSM Player: [(0, 'C', 0, 'C'), (0, 'D', 3, 'C'), (1, 'C', 5, 'D'), (1, 'D', 0, 'C'), (2, 'C', 3, 'C'), (2, 'D', 2, 'D'), (3, 'C', 4, 'D'), (3, 'D', 6, 'D'), (4, 'C', 3, 'C'), (4, 'D', 1, 'D'), (5, 'C', 6, 'C'), (5, 'D', 3, 'D'), (6, 'C', 6, 'D'), (6, 'D', 6, 'D'), (7, 'C', 7, 'D'), (7, 'D', 5, 'C')], 1, C
138. Suspicious Tit For Tat	168. Meta Winner Memory One: 31 players	
139. Tester	169. NMWE Memory One: 31 players	
140. ThueMorse	170. FSM Player: [(0, 'C', 7, 'C'), (0, 'D', 1, 'C'), (1, 'C', 11, 'D'), (1, 'D', 11, 'D'), (2, 'C', 8, 'D'), (2,	
141. ThueMorseInverse		
142. Thumper		
143. Tit For Tat		
144. Tit For 2 Tats		
145. Tricky Cooperator		