Applied Numa 1

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Problem 1

This problem deal with solving numerical experiments to a simplified Landau-Lifshitz equation for $\mathbf{m}$ given the following ODE:

$$\frac{d\mathbf{m}}{dt} = \mathbf{a} \times \mathbf{m} + \alpha \mathbf{a} \times (\mathbf{a} \times \mathbf{m}), \tag{1}$$

$$\mathbf{m}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}, \quad \alpha = 0.1$$

Listed below are plots of the components of the $\mathbf{m}$ vector. They are shown with different time-ranges to symbolise its change with time, t. The Runge-Kutta 3 method is used to gain these results.

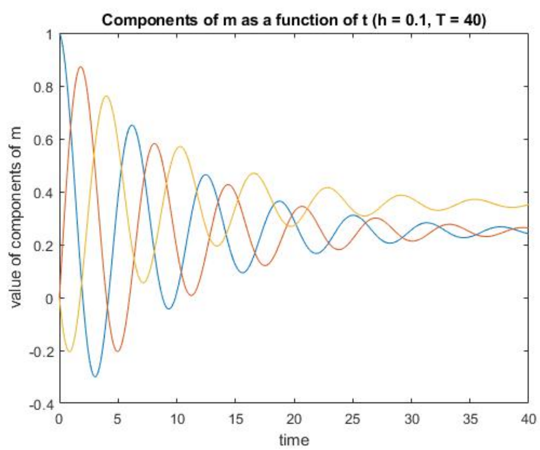

Figure 1: T=40.

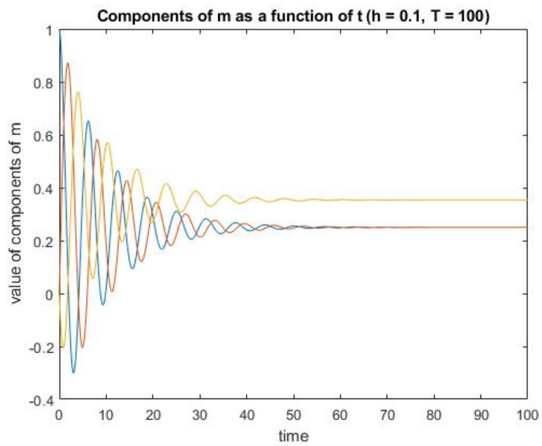

Figure 2: T=100.

Two 3D plots with varying t support that $\frac{\mathbf{m}}{|\mathbf{m}|}$ does become parallel to $\mathbf{a}$. The vectors are merged in the second plot which result in that only the $\mathbf{a}$ (red line) is visible.
The values in the first plot do converge as $T \lim \to \infty$. This makes sense as the derivative (2) goes to 0 if and only if the vectors become parallel, as that is the case when the cross product of two vectors is 0.

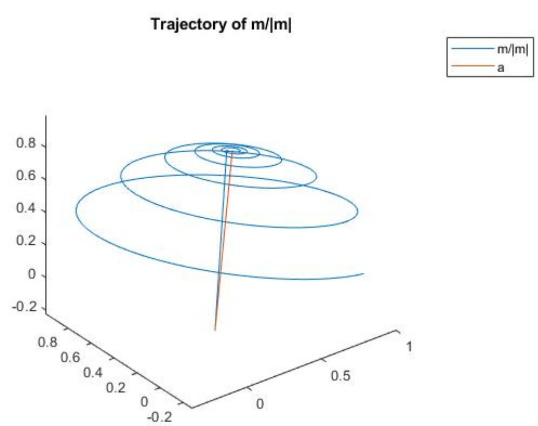

Figure 3: T=40.

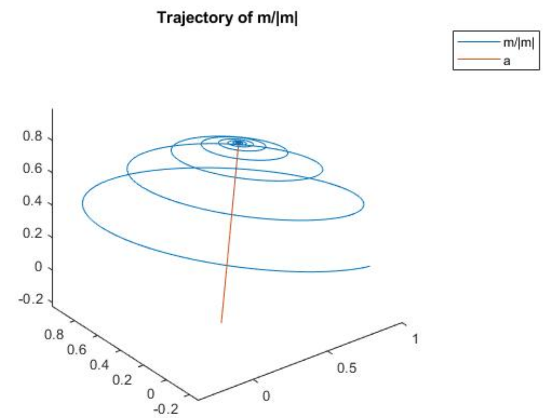

Figure 4: T=100.

Listed below is a log-log plot with the difference $|\hat{\mathbf{m}}_N - \hat{\mathbf{m}}_{2N}|$ and h^3, the latter is arrived at via trial and error. The two lines are shown to be parallel for small stepsizes which implies that the method have an estimated order of accuracy of 3.

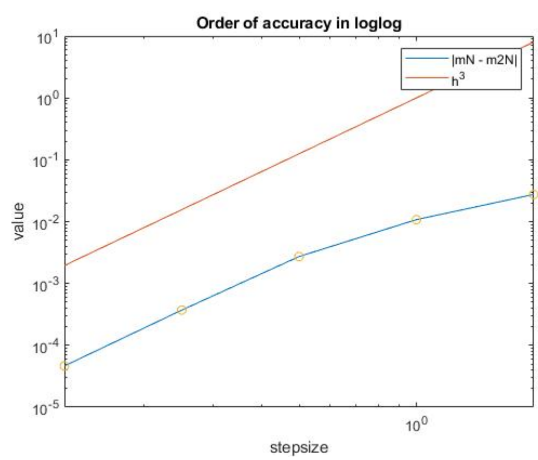

Figure 5: loglog-plot showing order of accuracy.

The matrix A from the standard form

$$\frac{d\mathbf{m}}{dt} = A\mathbf{m}, \tag{2}$$

is computed by hand and the eigenvalues for the aforementioned matrix are thereafter computed with Matlab (exists in the Matlab code) and the eigenvalues are $\lambda_1 = 0, \lambda_2 = -0.1 + i, \lambda_3 = -0.1 - i$. For a method to be asymptotically stable it must have the property of $|1 + h\lambda| < 1$ or $h\lambda \in S$. h here denotes the step-length we take in our numerical method. Seeing as explicit methods contain the bound, the case of the 0 eigenvalue is ignored.
To then identify the h_{stab}, the eigenvector is numerically extended until it is no longer in the region S (the region S here is the one mentioned in the announcements), the selection of λ_2 or λ_3 is irrelevant as they are both symmetrical and so is the region S. $h_{stab} = 2.142$ with a margin of error of 0.0001 are resulted through iteration.

From the two plots below it is seen that the previously made numerical solution diverge for $h > h_{stab}$ and converge for $h < h_{stab}$.

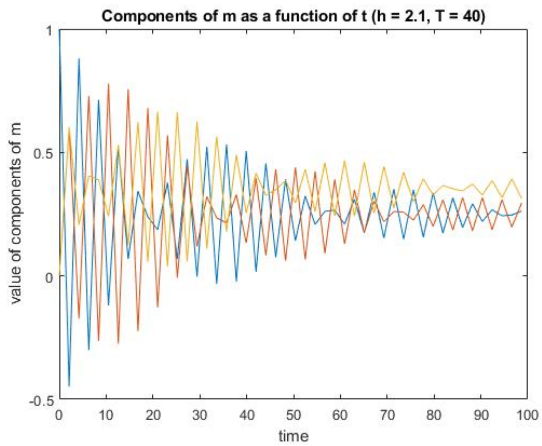

Figure 6: h=2.1.

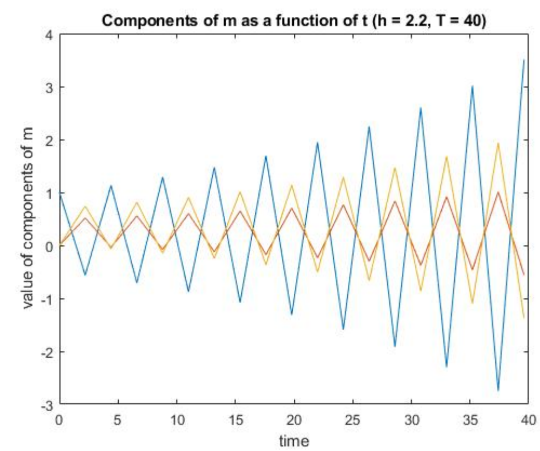

Figure 7: h=2.2.

Problem 2

To be able to instantiate the AB4 method we utilize the RK3 to gain the first 3 values and $r(0), r'(0)$.

A satellites trajectory, $\mathbf{r}$(t) is given by the following ODE:

$$\frac{d^2\mathbf{r}}{dt^2} = -(1-\mu)\frac{\mathbf{r}-\mathbf{r_0}}{|\mathbf{r}-\mathbf{r_0}|^3} - \mu\frac{\mathbf{r}-\mathbf{r_1}}{|\mathbf{r}-\mathbf{r_1}|^3} + 2N\frac{d\mathbf{r}}{dt} + \mathbf{r}$$

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{r}(0) = \begin{pmatrix} 1.2 \\ 0 \end{pmatrix}, \quad \mathbf{r}'(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The length unit is equal to the distance between the earth and the moon. The earth center is at $_0$ and the moon center is at r_1.

$$\mathbf{r_0} = \begin{pmatrix} -\mu \\ 0 \end{pmatrix}, \quad \mathbf{r_1} = \begin{pmatrix} 1-\mu \\ 0 \end{pmatrix}, \quad \mu = \frac{1}{82.45}$$

Listed below are plots containing the trajectory and its velocity. Listed in the Matlab code can the rewriting of the ODE system be found.

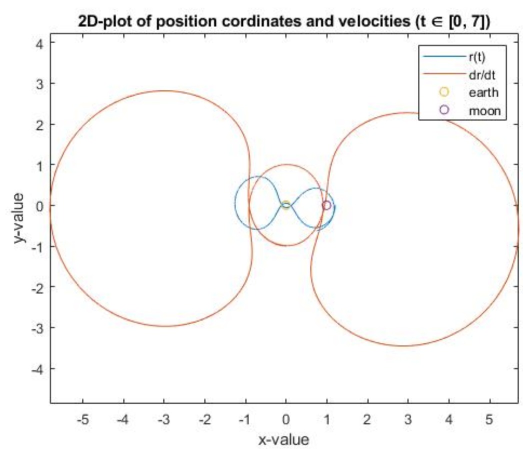

Figure 8: 2D-plot showing position coordinates and velocities

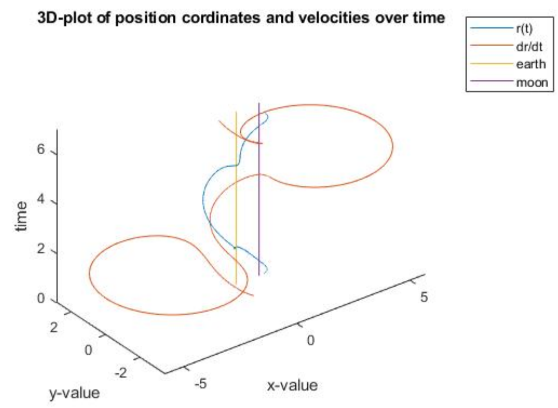

Figure 9: 3D-plot showing position coordinates and velocities with time

When comparing the various r_h and $r_{h/2}$ to establish the T_{acc}, two time increment for $r_{h/2}$ is done for each step for the r_h for them to end up at the same time t. This is done for the three methods: Explicit Euler, Runge Kutta 3 (from part 1) and Adams-Bashforth 4, until the tolerance was no longer fulfilled.

Listed below are the results and plots from the comparison.

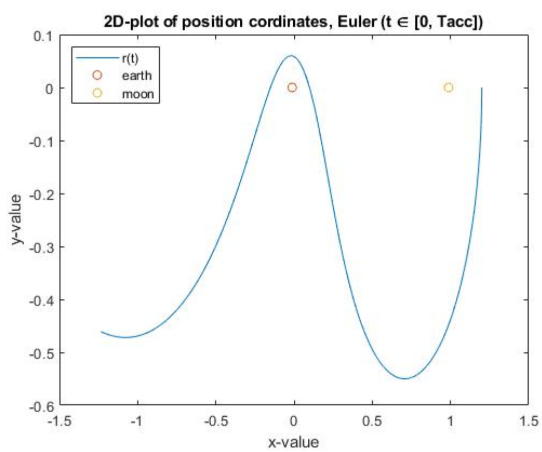

Figure 10: Plot of solution using Explicit Euler, $t \in [0,\ T_{acc}]$.

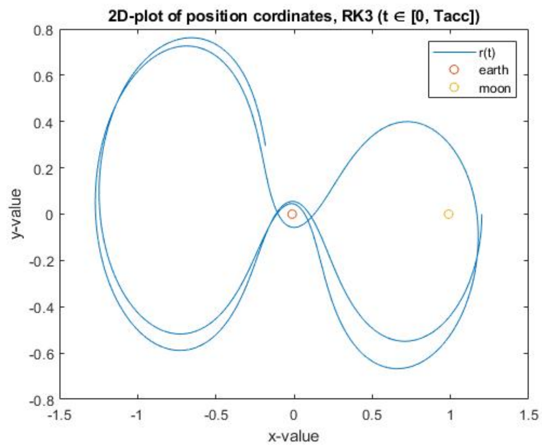

Figure 11: Plot of solution using Runge-Kutta 3, $t \in [0,\ T_{acc}]$.

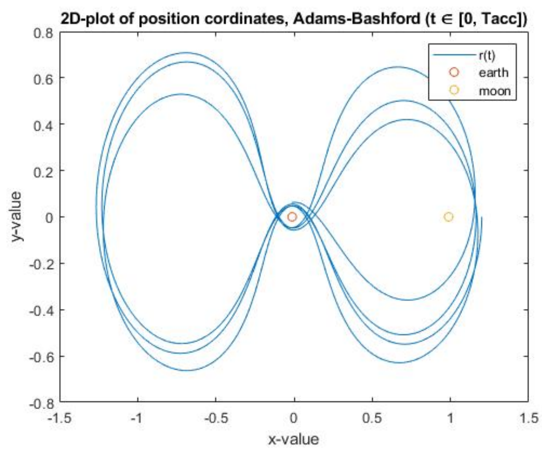

Figure 12: Plot of solution using Adams-Bashforth 4, $t \in [0,\ T_{acc}]$.

The T_{acc} for the given methods show result in suspected order. Considering only looking at the various orders of the methods used they are ordered as such, Euler<RK3<AB4, which is in line with the findings. One should however mention that the RK3 method is used in gaining the initial points for the AB4 method, which should lower the order by one degree. However it is still probable that the AB4 has a larger T_{acc}, seeing as if it had initial points from a fourth order method it would had order four. Additionally one-step methods should be more susceptible to sudden variations and multi-step methods less so seeing as they also consider data from previous time steps. To clarify AB4 here is a

multi-step method. The T_{acc} for the methods is as follows, for Euler $T_{acc} \approx 2.05$, for RK3 $T_{acc} \approx 10.97$ and finally for AD4 $T_{acc} \approx 19.71$

The largest and smallest time-steps is $2.3 \cdot 10^{-4}$ and roughly 0.1 respectively, and it used a total of 904 time-steps. Which is less than our AB4 method, seeing as our $T_{acc} \approx 19.71$ and $h = 0.001$ then $19.71/0.001 = 19710$ steps are taken.

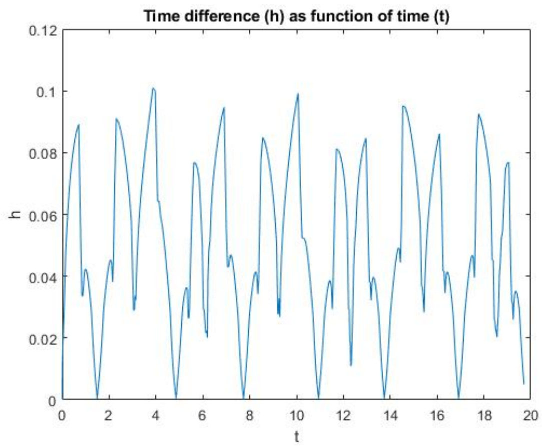

Figure 13: Plot showing time difference as function of time.

We suspect here that when the satellite comes closer to earth the gravitational pull makes it slow down, and in contrast be faster further away from earth, this can also be seen from the comet plots. Thus we have a symmetric plot, symbolizing the satellite approaching and moving further away from earth.

As for the numerical method with adaptive step-size, seeing as near the earth the position and speed vary more considerably (By the gravitational force), thus there is a need for more precise step-size.

Problem 3

The problem studied is the known as Robertson's problem and is a *stiff* system containing the following set of ODEs:

$$\frac{dx_1}{dt} = -r_1 x_1 + r_1 x_2 x_3, \quad x_1(0) = 1$$

$$\frac{dx_2}{dt} = r_1 x_1 - r_2 x_2 x_3 - r_3 x_2^2, \quad x_2(0) = 0$$

$$\frac{dx_3}{dt} = r_2 x_2^2, \quad x_3(0) = 0$$

r_1, r_2 and r_3 are rate constants with the following values: $r_1 = 0.004$, $r_2 = 10^4$ and $r_3 = 3 \times 10^7$.

The RK3 method is used to solve the given non-linear system, in doing so it is detectable empirically that for the cases $N = 125, 250, 500$ the system enviably diverge, and for the other cases $N = 1000, 2000$ the system become stable, as seen from the loglog-plots. This is done for the time interval [0, 1].

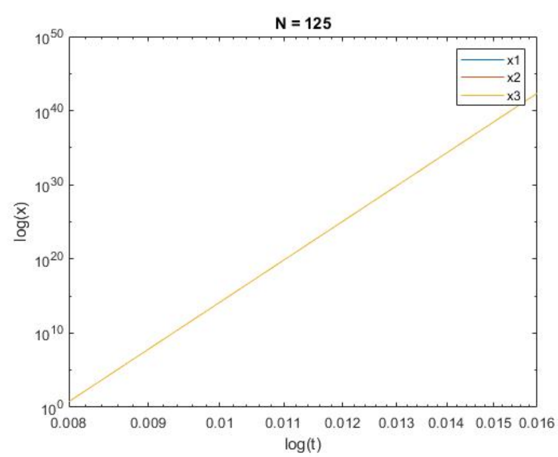

Figure 14: loglog-plot, solved using ODE23 with N = 125.

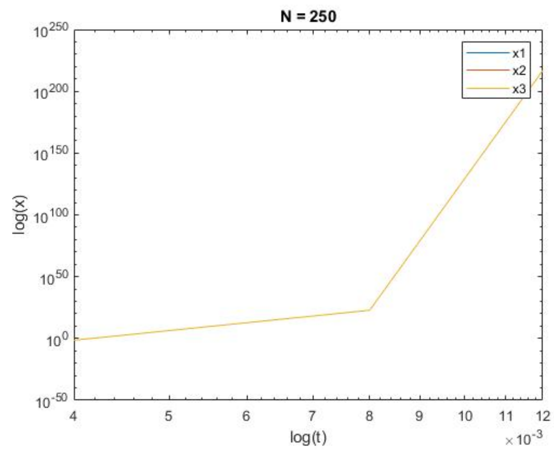

Figure 15: loglog-plot, solved using ODE23 with N = 250.

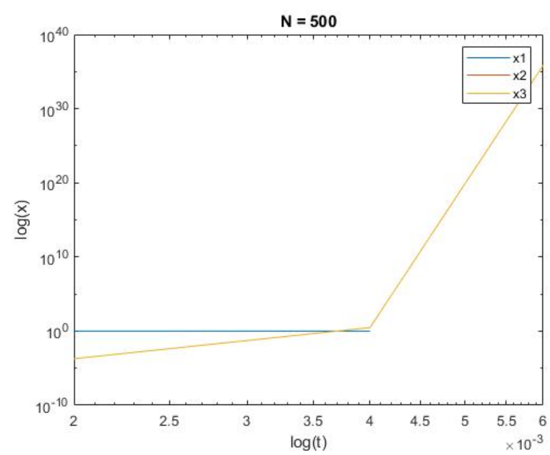

Figure 16: loglog-plot, solved using ODE23 with N = 500.

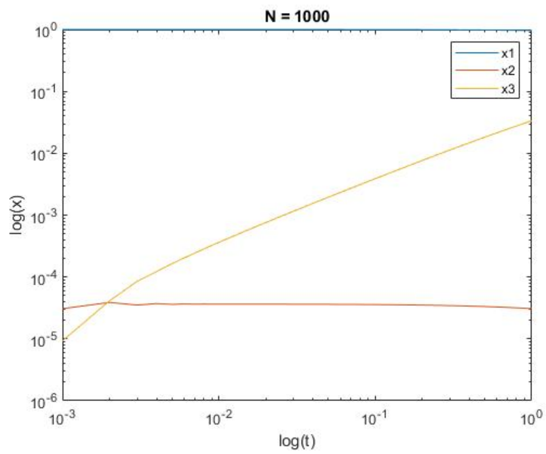

Figure 17: loglog-plot, solved using ODE23 with N = 1000.

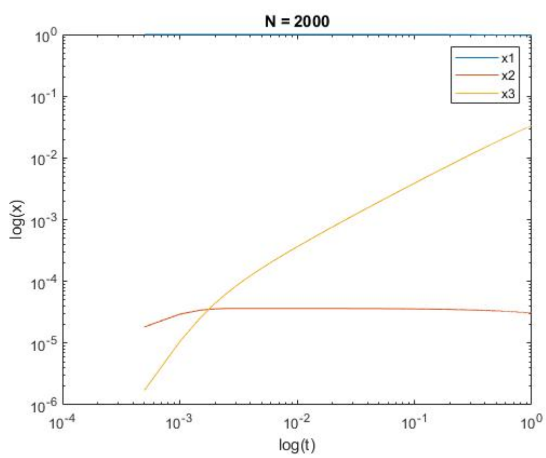

Figure 18: loglog-plot, solved using ODE23 with N = 2000.

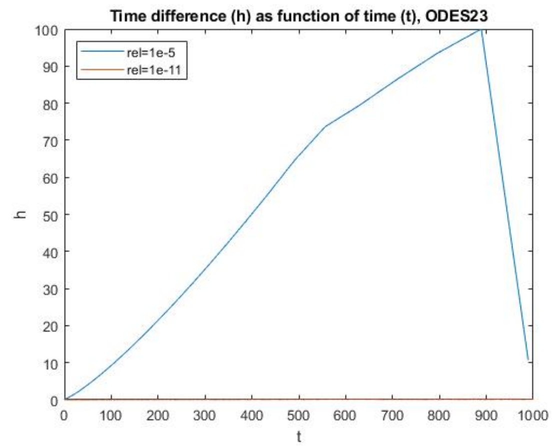

Figure 19: Time-step, h, plotted as a function of time with ODE23S

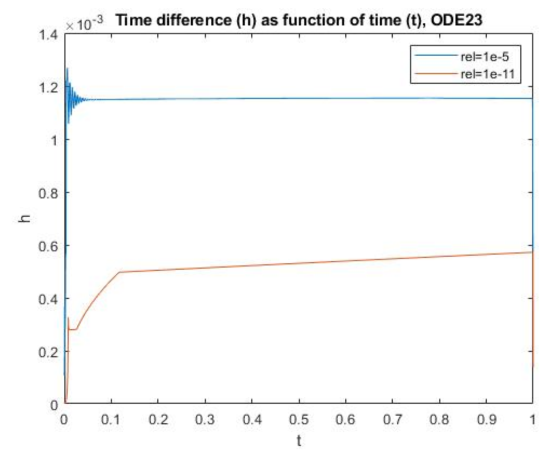

Figure 20: Time-step, h, plotted as a function of time with ODE23

Steps	ODE23	ODE23s
Rel 10^{-5}	879	115
Rel 10^{-11}	3360	58607

The rather obvious statement is that given a lower relative tolerance a smaller step-size is required, which stands to reason seeing as we want our solution to be more precise.

But, why the difference between the explicit and implicit method? Particularly in Stiff ODE problems a characteristic is that explicit method require just much smaller time-steps, h, because of the requirement $h < h_{stab} << h_{acc}$, whereas the implicit method has no demand of a stability limit.

Since ODE23 solves an equation using adaptive time-steps depending on the absolute and relative tolerance, no precise estimation of h_{acc} can be acquired. The maximum time-step might be too large for some areas and the minimum might be unnecessarily small for other areas. This implies that the result from the ODE23 does not inform anything related to the stability-limit. Additionally if one would have to select one it would be the smallest, the reasoning is as follows. The most unstable time steps require the smallest h so from the adaptive method we can find how small our h needs to be for a constant step-size. However specifically for the ODE23 it could just change its h.