Markov process

This computer exercise report submitted in fulfillment of the requirements for the course

in

SF2863 Systems Engineering

by

SF2863 2021 HA1 55

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1. Problem formulation

A big ferry uses two engines which makes it go forward, both the engines can break down making the ferry move forward at a slower pace, if however both break down the ferry stops. The operation of the ferry is assumed to be a Markov process in continuous time.

Engine one breaks down according to a $\exp(\lambda_1)$ distribution, likewise is engine two, $\exp(\lambda_2)$. Moreover, the boat has three repairmen working on it whom can help in repairing a broken engine, this is vital since the repair time of a engine is modelled as $\exp(n_k \mu_k)$, where n_k denotes the amount of repairmen at given engine.

Lastly, let $V(t)$ denote the speed of the ferry, if both engines function properly it is going at a speed of v, if only engine one is working it is moving at v_1 and lastly if only engine 2 is working it is moving at a speed of v_2.

- $v = 18$, $v_1 = 10$, $v_2 = 15$
- $\lambda_1 = 14$, $\lambda_2 = 20$
- $u_1 = 10$, $u_2 = 8$
- $h = 0.001$

2. Analytic solution

1

We wish to determine the states of the Markov process in the assignment. It is readily seen that there are 4 states, both engines working, none working and two additional where only one is properly operating. Below listed is a drawn picture explaining the states.

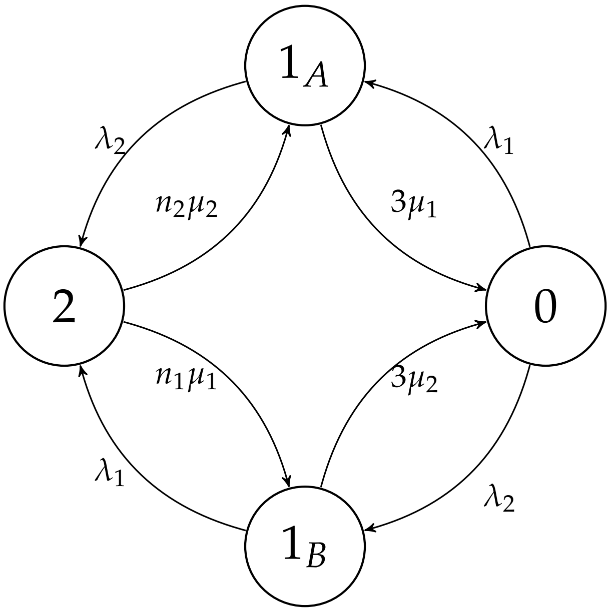


Figure 2.1: State description

First let $X(t) = \#\, of\ engines\ not\ working$, note here too that we have the range of $X(t) = \{0, 1_A, 1_B, 2\}$ where 1_A and 1_B denotes which unique engine is broken down.

Then it is seen that if both are working, the transition rate (seeing as they are exponentially distributed) are with a rate λ_1 to the engine 1_A and at a rate of λ_2 to engine 1_B, this in turn explains the rate at which they move towards $X(t) = 2$ since the other engine is broken it has to be by the other engines rate. In terms of repairing, when only one is broken we assume all three repairmen are trying to restore it, making the various repair rates $3\mu_1$ and $3\mu_2$. However when both are broken we now consider a more nuanced case where n_1 and n_2 can be any divide of the three repairmen to the two broken engines. Making the rates $n_1\mu_1$ and $n_2\mu_2$.

As an additional comment, the case of repairing both at the same time, that is from going to 2 broken engines to none, or the possibility of repairing one while the other breaking at the exact same time is neglected here. Even though there is a possibility for this to happen it is 0 almost surely.

2

The intensity matrix can be determined by identifying the q_{ij} from range in $X(t)$, we should also note that in the intensity matrix it is necessary for the row sum to be zero. This is used in determining the negative diagonal entries.

$$Q = \begin{pmatrix} -(n_2\mu_2 + n_1\mu_1) & n_2\mu_2 & n_1\mu_1 & 0 \\ \lambda_2 & -(\lambda_2 + 3\mu_1) & 0 & 3\mu_1 \\ \lambda_1 & 0 & -(\lambda_1 + 3\mu_2) & 3\mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\lambda_1 + \lambda_2) \end{pmatrix}$$

3

To be able to determine if there is a stationary distribution we refer to the following theorem.

For a finite irreducible Markov process there always exists a unique steady state probabilities π that solve the steady state equations,

$$\boldsymbol{\pi} Q = 0, \quad \sum_j \pi_j = 1.$$

Seeing as our process is finite, as we have a finite range for $X(t)$ and it is indeed a Markov process by the statement of the problem. Then the only thing left to show is that it is irreducible, but as we only have one class, that is, all states interact with one another (which can be seen in the picture or from the transition matrix) we know it is indeed irreducible, and satisfy the assumptions of the theorem. Thus we know it will indeed have a steady state distribution for our problem.

4

In solving the steady state distribution we must solve the following systems of equations,

$$\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\end{pmatrix}\begin{pmatrix}-(n_2\mu_2+n_1\mu_1) & n_2\mu_2 & n_1\mu_1 & 0\\ \lambda_2 & -(\lambda_2+3\mu_1) & 0 & 3\mu_1\\ \lambda_1 & 0 & -(\lambda_1+3\mu_2) & 3\mu_2\\ 0 & \lambda_1 & \lambda_2 & -(\lambda_1+\lambda_2)\end{pmatrix}=0$$

and

$$\pi_1+\pi_2+\pi_3+\pi_4=1.$$

From the problem statement we acquire 5 equations, one of which is redundant seeing as it is a linear combination of each other.

The equation system was solved using Matlab for the 4 cases firstly, $n_1 = 0, n_2 = 3$, secondly, $n_1 = 1, n_2 = 2$, thirdly, $n_1 = 2, n_2 = 1$ and finally $n_1 = 3, n_2 = 0$. Then the following stationary solution was given.

Table 2.1: Different values for $\boldsymbol{\pi}$ with different n_1, n_2

	π_1	π_2	π_3	π_4
$n_1 = 0, n_2 = 3$	0.2869	0.2269	0.1677	0.3186
$n_1 = 1, n_2 = 2$	0.2632	0.1746	0.2392	0.3230
$n_1 = 2, n_2 = 1$	0.2431	0.1304	0.2999	0.3267
$n_1 = 3, n_2 = 0$	0.2258	0.0924	0.3519	0.3299

5

To find the average velocity we simply make the matrix multiplication with Π (Matrix with our solutions in the previous question, as in the table above) with the velocity matrix

$$V^T = \begin{pmatrix}v_0, v_2, v_1, v\end{pmatrix}$$

Then we acquired for the various reparation strategies the following average velocities,

Table 2.2: Different values for $\overline{v}$ with different n_1, n_2

	$n_1 = 0, n_2 = 3$	$n_1 = 1, n_2 = 2$	$n_1 = 2, n_2 = 1$	$n_1 = 3, n_2 = 0$
$\overline{v}$	10.8141	10.8254	10.8349	10.8431

3. Continuous time approach

6a

Firstly, some theoretical background as motivation to the MATLAB code. Let T_i denote the dwell time, and T_{ij} the time that it spends in state i before jumping to state j. We note that from the lecture notes we know that,

$$T_i \in \exp\left(\sum_{j\neq i} q_{ij}\right).$$

Seeing as this is the case we must thus note that the expected dwell time we spend in state i before jumping to state j is described by $E[T_i] = \frac{1}{\sum_{j\neq i} q_{ij}}$ which for our first state, namely when we have no broken engines, is then given by

$$E[T_0] = \frac{1}{\lambda_1 + \lambda_2}.$$

In our simulation we generate two randomised times with the MATLAB command *exprnd* (dependant on the state) with the transition rates to the other states to which we can jump. We then choose to jump to the state to with the smallest transition time. Once at the new state the exact same simulation is done again with new transition rates to the other states.

7a

We wish to run a continuous simulation for the Markov process to simulate the problem statement above. We model it in the manner seen in matlab script HW1Part2. The T values was increased manually till the tolerance was satisfied. With $T = 20000$, averages were obtained for (π_1, π_2, π_3,π_4) for different strategies shown in Table 3.1. The relative error on 1% were calculated running the simulation twice and comparing the two different results by

$$\frac{\Pi - \Pi_{old}}{\Pi_{old}}$$

where Π_{old} is matrix of $\boldsymbol{\pi}$ as row vectors for every strategy for the first run, similarly Π is same matrix but for second run. By taking differences and dividing with every element with corresponding element Π_{old}, relative errors were obtained. And by taking the max value of the absolute value of this matrix, we obtain maximum error which is 0.0085 for the simulation we had. This value will differ every time as transition times are generated randomly with *exprnd*, but the error generally gets smaller as T increases.

Table 3.1: Different values for $\boldsymbol{\pi}$ with different n_1, n_2

	π_1	π_2	π_3	π_4
$n_1 = 0, n_2 = 3$	0.2864	0.2272	0.1678	0.3186
$n_1 = 1, n_2 = 2$	0.2632	0.1751	0.2390	0.3227
$n_1 = 2, n_2 = 1$	0.2434	0.1300	0.3002	0.3264
$n_1 = 3, n_2 = 0$	0.2257	0.0924	0.3522	0.3297

8a

Those values were used to estimate the expected speed of the ferry which is demonstrated in Table 3.2 that were calculated by $(\pi_1, \pi_2, \pi_3, \pi_4) \cdot (v_0, v_2, v_1, v)^T$.

Table 3.2: Different values for $\overline{v}$ with different n_1, n_2

	$n_1 = 0, n_2 = 3$	$n_1 = 1, n_2 = 2$	$n_1 = 2, n_2 = 1$	$n_1 = 3, n_2 = 0$
$\overline{v}$	10.820	10.8246	10.8270	10.8432

4. Discretization approach

6b

From the derivation in the lecture notes, we shall note that we attain the P matrix in the following manner,

$$Q = \lim_{h \to 0^+} \frac{P(h) - I}{h},$$

rewriting the equations gives,

$$P(h) \approx Qh + I,$$

and so for each entry of the P matrix we have, where $i \neq j$

$$p_{ii}(h) \approx 1 + hq_{ii}, \quad p_{ij}(h) \approx hq_{ij}.$$

Since we have a given h in the assignment we can approximate a definite expression for Q. Thus

$$P = Qh + I = \begin{pmatrix} 0.97 & 0 & 0.03 & 0 \\ 0.02 & 0.95 & 0 & 0.03 \\ 0.014 & 0 & 0.962 & 0.024 \\ 0 & 0.014 & 0.02 & 0.966 \end{pmatrix}.$$

Note lastly that the row sums here does indeed equal one.

7b

The MATLAB code for the discrete case is made as follows. Once in a state i, a value via *rand* is generated, that is a value that is $X(t) \in \mathcal{U}(0,1)$ (uniformly distributed). Dependant on which value we get and where it lies in the intervals,$< p_{i,j}$, $p_{i,j} < p_{i,j+1}$, $p_{i,j} + p_{i,j+1} < p_{i,j} + p_{i,j+1} + p_{i,j+2}$ and so forth. If the value falls within that interval the process will move to that state.

For each N, two sequences are made to identify the probability matrix, by looking at the amount of "times" in which they are in a state divided by the combined times. Thus we get the probability of being in a specific state. Then we compare these two sequences by taking the maximum difference and checking if they satisfy the relative error. If however the relative error, 0.01, is not satisfied then we increase the N. Seeing as we have a stochastic process obviously the results will vary.

As a last note the relative error was checked by looking at if the largest difference was larger than 0.01, if it was the case then the iterations would continue.

Below listed is the average of the two matrices which satisfy the tolerance condition in the assignment, MATLAB code HwDiscrete.m., i.e our P matrix.

Table 4.1: Different values for $\boldsymbol{\pi}$ with different n_1, n_2

	π_1	π_2	π_3	π_4
$n_1 = 0, n_2 = 3$	0.2877	0.2261	0.1678	0.3185
$n_1 = 1, n_2 = 2$	0.2634	0.1737	0.2399	0.3229
$n_1 = 2, n_2 = 1$	0.2431	0.1306	0.3000	0.3263
$n_1 = 3, n_2 = 0$	0.2258	0.0916	0.3534	0.3292

For this specific run of the code, a $N = 10000000$ was gained after iterating.

8b

Those values were used to estimate the expected speed of the ferry which is demonstrated in Table 3.2 that were calculated by $(\pi_1, \pi_2, \pi_3, \pi_4) \cdot (v_0, v_2, v_1, v)^T$.

Table 4.2: Different values for $\overline{v}$ with different n_1, n_2

	$n_1 = 0, n_2 = 3$	$n_1 = 1, n_2 = 2$	$n_1 = 2, n_2 = 1$	$n_1 = 3, n_2 = 0$
$\overline{v}$	10.8020	10.8177	10.8330	10.8344

5. Comparison and Reliability

9

For the continuous case we attained a time of 32.69 seconds and for the discrete case 4.62 seconds (for that specific iteration). The results seem plausible and realistic as the discrete case has to do fewer randomised computations. Further we note that, T for the continuous case was, $T = 20000$ and for the discrete case it became $T = N \cdot h = 10000000 \cdot 0.001 = 10000$. Once again these values change dependant on when we ran it multiple times.

10

Seeing as we want to compute the expected time till we reach the state of two broken engines, we remodel our previous Markov chain as before but with the case, two broken engines as an absorption state. That is, it will remain in that state once reached. We use the following formula to find the expected time till absorption,

$$t_i = \frac{1}{q_i} + \sum_{k \in G \setminus \{i\}} \frac{q_{ik}}{q_i} t_k, i \in \{1, 2, 3\}.$$

Thus we have,

$$t_1 = \frac{1}{q_1} + \frac{q_{12}}{q_1} t_2 + \frac{q_{13}}{q_1} t_3$$

$$t_2 = \frac{1}{q_2} + \frac{q_{21}}{q_2} t_1 + \frac{q_{23}}{q_2} t_3$$

$$t_3 = \frac{1}{q_3} + \frac{q_{31}}{q_3} t_1 + \frac{q_{32}}{q_3} t_2$$

The solution can be found in the following MATLAB file HW1Systems.m.

Table 5.1: Time, t_i for absorption in state 0 starting in i

t_1	t_2	t_3
0.1036	0.1143	0.1393

A. Appendix: Task description

Home assignment number 1, 2021, in SF2863 Systems Engineering.

Instructor: Per Enqvist.

The home assignments are a mandatory part of the course. In total, there are two home assignments and you need to collect 4.0 out of 6.0 points in order to pass this part of the course. Home assignment 1 can give up to 2.0 points.

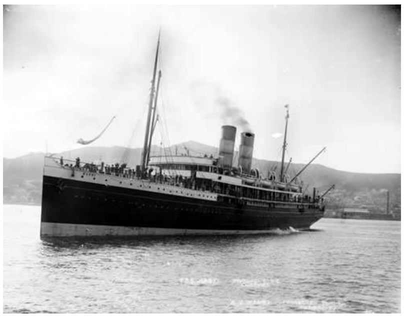
The home assignments can, in addition, give bonus points on the exam. If a home assignment is handed in before the deadline, then the points awarded on the assignment will also count as bonus points on the exam (hence, home assignment 1 can give up to 2.0 bonus points on the exam). In order to get bonus points on the exam the report should be submitted to canvas before 23:59, Tuesday, November 16, 2021.

This home assignment should be done in groups of 2 persons, and one report per group is to be submitted to canvas. See the information on canvas on how to form groups and submit your results. State your name, and email address on the front of the report.

In the report you should describe in your own words how the problem was solved. In particular, carefully describe how you defined your states, modelled the problem and implemented the simulations. Do not explain the code, but rather the mathematical ideas. You should not need more than 4 pages for the report. The answers to the questions in the assignment should be given in the main report and numbered as in this text. In particular, it should not be necessary to look at your code to understand how you define and calculate things. Relevant print-outs and plots should be included in the report. You may use any computer program to facilitate the computations, but you have to include your computer code in the submission. We use automated plagiarism-control tools and will also make random checks that you have not copied parts of another groups report or computer code!

Questions should be posed in the discussion group in canvas.

System and data
We consider a big ferry with two engines that each drives one propeller. If one engine breaks down the ferry can still move, but it runs a bit slower, and if both breaks down the ferry stops.
We assume that the operation of the ferry is well described by a simple Markov process in continuous time.



Assume that engine number 1 breaks down after some time that is modelled as a stochastic variable with $\mathrm{Exp}(\lambda_1)$ distribution. Similarly, engine number 2 breaks down after some time that is $\mathrm{Exp}(\lambda_2)$ distributed. Three repairmen are working on the ferry for repairing the engines. The repair time for engine k is $\mathrm{Exp}(n_k\mu_k)$ distributed, where n_k is the number of repairmen working on engine k.
Let $V(t)$ denote the speed of the ferry at time t. If both the engines are working the ferry is running at a speed which is v, if only engine 1 is working the speed is v_1 and if only engine 2 is working the speed is v_2.

Parameter values

- $v = 18$, $v_1 = 10$, and $v_2 = 15$.
- $\lambda_1 = 14$, $\lambda_2 = 20$.
- $\mu_1 = 10, \mu_2 = 8$
- $h = 0.001$

Now, the system is a Markov process that will depend on how many workers are assigned to repair the engines if they break down. If only one engine is broken it is obvious that it is optimal to let all three repairmen work on it. If both engines are broken it is not so clear how to distribute the workers. To decide this you should for each different repair strategy determine the average speed of the ferry at steady state and this way choose the best strategy.
You should do this both analytically and by simulation.

Analytic solution
First, you should determine the average speed using the theory of Markov processes. For this approach you need to (for all possible repair strategies)

1. define the states of the Markov process,

2. determine the intensity matrix,

3. motivate why there should exist a stationary distribution

4. determine the stationary distributions (for all repair strategies using the intensity matrices defined above.)

5. determine the average speed of the ferry from this stationary distribution.

You may use a computer program to solve the equation systems.

Next you should simulate the system numerically and determine the average speed by ergodic estimates.
This will be done in two different ways; in (a) using a direct simulation of the continuous time process, and in (b) using a time discretization approach.

Continuous time approach
Here we simulate the continuous time process by randomly generating dwell times. Start the process at time 0 with both engines of the ferry functioning. Then you should use that the jump times are exponentially distributed.
Hint: If you use Matlab an exponentially distributed variable can be generated using the command `exprnd`, where you specify the mean of the variable (which is inversely proportional to the intensity).
You can use several exponential distributions, or one exponential distribution and a uniform distribution, to determine how long you remain in a state and which will be the next state.

6a. Describe in detail how you determine the time to the next jump and where you jump.

7a. Run the simulation for a time interval $[0,\ T]$.
By simulating the process we mean that you start the process in some state and then you determine one realization of the process by using the probability description of the process to determine how long you remain in a state and where you transition.

If you run the simulation twice you would get different results. Comparing the two different results can give an indication on if the averages have converged or not. Choose (the magnitude of) T large enough so that the errors are less than 1%.

8a. Take averages to determine estimates of the expected speed of the ferry.

Hint: Since a large number of time steps have to be made to get reasonable ergodic estimates you may note that it is enough to keep track on how long time the Markov process has spent so far in each state to determine the estimates for the stationary probabilities and the average speed.

Discretization approach

Now a discretization of the continuous time process will be made.
Start the process at time 0 with both engines of the ferry functioning. Then make a discretization of the time axis so that we only consider the process at times $t_k = kh$ for $k = 0, 1, 2, \cdots, N$. Then the probability of a jump to another state during the time interval $[t_k, t_{k+1}]$ can be determined (approximatively) using the intensity matrix for the continuous time Markov process. (You could use the matrix exponential function to determine the transition probabilities, but we would like you to use the approximative expressions that hold for small h used in the lectures.)
If the time step h is small enough this approximation is good.

6b. Determine the transition matrix of a discrete time Markov chain that will approximate the continuous time process.
Hint: Check if the transition matrix you obtain has row sums equal to one.

7b. Now use this discretization to simulate the process.
By simulating the process we mean that you start the process in some state and then you determine one realization of the process by using the transition probabilities to determine which transitions occur.

If you run the simulation twice you would get different results. Comparing the two different results can give an indication on if the averages have converged or not. Choose (the magnitude of) N large enough so that the errors are less than 1%.

8b. Take averages to determine estimates of the expected speed of the ferry.

Hint: If you use Matlab, a stochastic variable with uniform probability in [0, 1] can be generated using the command `rand`.
The approach just described has also the purpose of showing the connection between discrete time and continuous time Markov chains.

Comparison

9. How does the computation times between the three different methods compare?
How does the simulation lengths compare, *i.e.*, hN and T?

Hint: In Matlab the computation time can be determined using `tic, toc` or `profile`.

Reliability - average time to full break down

10. Use a method of your choice to determine the average time it takes until the first time that the ferry comes to a halt due to a total engine break down. Assuming that the ferry starts with both engines working.

Good luck!