Project Report

2. Root finding

2.1

Given that $\tan(\theta) = 2\cot(\beta)\frac{M^2 sin^2(\beta)-1}{M^2(\gamma+\cos(2\beta))+2}$, when $\theta = 0, \tan(\theta) = 0$:

$$0 = 2\cot(\beta)\frac{M^2 sin^2(\beta)-1}{M^2(\gamma+\cos(2\beta))+2}$$

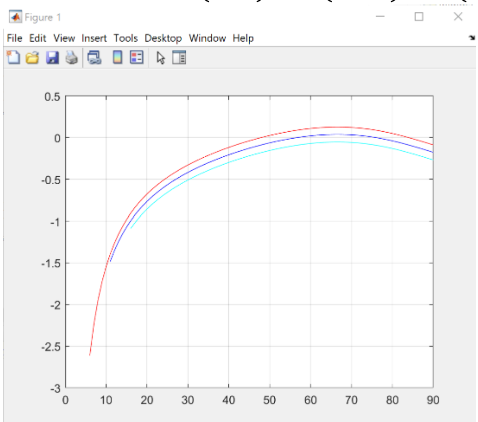
Hence, either $2\cot(\beta) = 0$ or $M^2 sin^2(\beta) - 1 = 0$

1. $2\cot(\beta) = 0, \beta = 90^{\circ} \times Z, Z = integer$. Since in this question, β stands for the shock wave angle, $0 < \beta < 180$, thus $\beta_1 = 90^{\circ}$
2. $M^2 sin^2(\beta) - 1 = 0, M^2 sin^2(\beta) = 1, sin^2(\beta) = \frac{1}{M^2}, \sin(\beta) = \pm\frac{1}{M}, \beta_2 = \arcsin\left(\frac{1}{M}\right)$

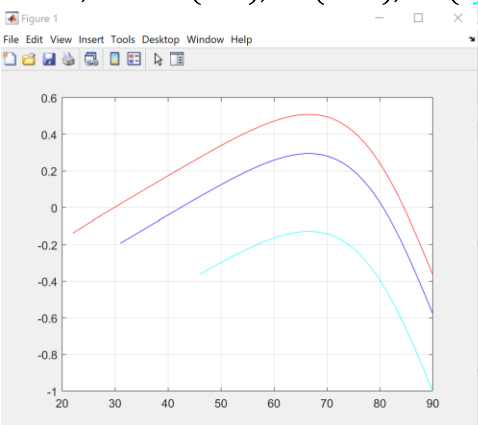
$$hence, \beta_U = 90^{\circ}, \beta_L = \arcsin\left(\frac{1}{M}\right)$$

2.2

a) $M = 1.5, \theta = 5^{\circ}(red), 10^{\circ}(blue), 15^{\circ}(cyan)$



b) $M = 5, \theta = 20^{\circ}(red), 30^{\circ}(blue), 45^{\circ}(cyan)$



Observed from the graph, when **both β_U and β_L exist**, for the **same M**, when θ increases, β_U decreases and β_L increases.

In terms of θ_{max}, for M=1.5, when $\theta = 15^{\circ}$, there's not a root, θ_{max} must be between 10° and 15°.

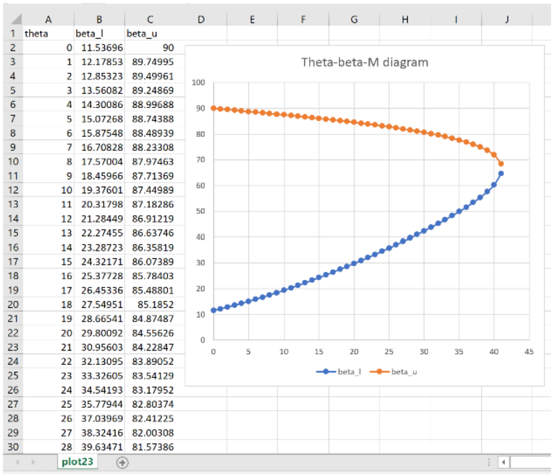
for M=5, when $\theta = 45^{\circ}$, there's no root, θ_{max} must be above 30° below 45°.

2.3

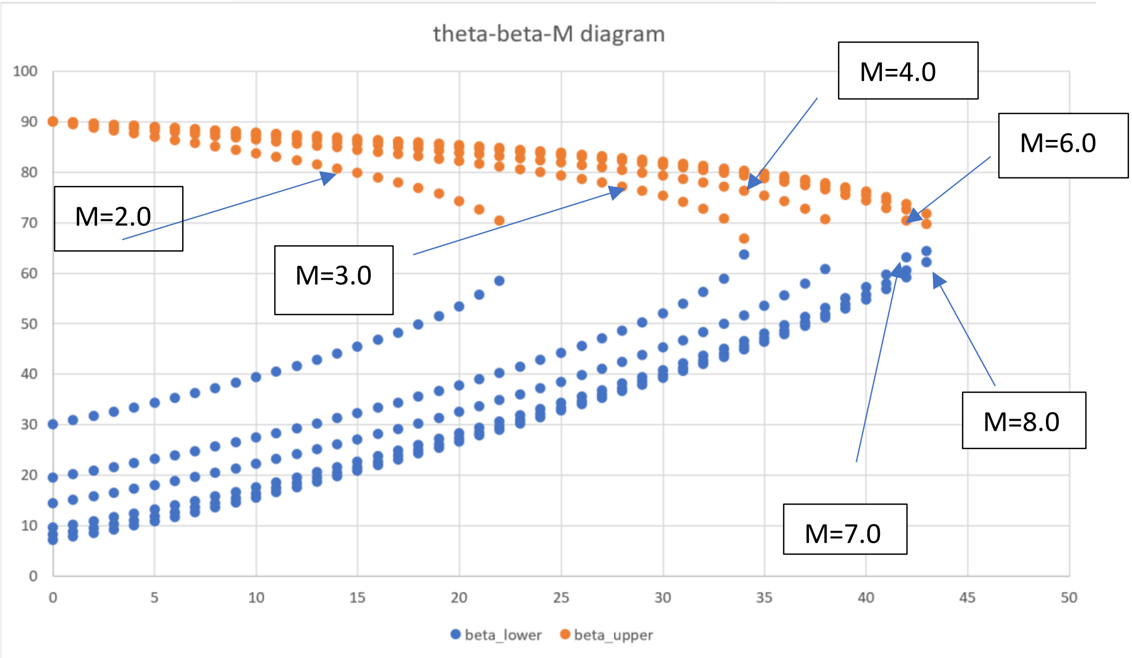
a) For $M = 5.0, \theta = 20^{\circ}$, I chose the two extremes: $\beta_U = 90^{\circ}, \beta_L = \arcsin\left(\frac{1}{M}\right) = 11.54$ as the initial guesses for the roots.

```
/* set the initial guess */
ibeta_l = asin(1.0 / m);
ibeta_u = IBETAU * FACTOR;
```

b)



	A	B	C
1	theta	beta_l	beta_u
2	0	11.53696	90
3	1	12.17853	89.74995
4	2	12.85323	89.49961
5	3	13.56082	89.24869
6	4	14.30086	88.99688
7	5	15.07268	88.74388
8	6	15.87548	88.48939
9	7	16.70828	88.23308
10	8	17.57004	87.97463
11	9	18.45966	87.71369
12	10	19.37601	87.44989
13	11	20.31798	87.18286
14	12	21.28449	86.91219
15	13	22.27455	86.63746
16	14	23.28723	86.35819
17	15	24.32171	86.07389
18	16	25.37728	85.78403
19	17	26.45336	85.48801
20	18	27.54951	85.1852
21	19	28.66541	84.87487
22	20	29.80092	84.55626
23	21	30.95603	84.22847
24	22	32.13095	83.89052
25	23	33.32605	83.54129
26	24	34.54193	83.17952
27	25	35.77944	82.80374
28	26	37.03969	82.41225
29	27	38.32416	82.00308
30	28	39.63471	81.57386



c)

3. Regression

First, focus on the second row of matrix in each side, expand the equation:

$$a \sum_{i=1}^{N} x_i + b \cdot N = \sum_{i=1}^{N} y_i$$

Divide by N

$$a\sum_{i=1}^{N}\frac{x_i}{N}+b=\sum_{i=1}^{N}\frac{y_i}{N}$$

Simplify the equation

$$a\bar{x}+b=\bar{y}$$

1. $b=\bar{y}-a\bar{x}$

Recall from lecture that:

$$S=\sum_{i=1}^{N}(y_i-ax_i-b)^2$$

To minimize S, find the derivative of S:

$$\frac{\partial S}{\partial a}=\sum_{i=1}^{N}2(y_i-ax_i-b)(-x_i)=0$$

Substitute b from equation 1

$$\sum_{i=1}^{N}2(y_i-ax_i-\bar{y}+a\bar{x})(-x_i)=0$$

$$\sum_{i=1}^{N}-x_i\,y_i+ax_i{}^2+x_i\bar{y}-a\bar{x}x_i=0$$

Separate into two sums:

$$\sum_{i=1}^{N}(x_iy_i-x_i\bar{y})=\ a\sum_{i=1}^{N}(x_i{}^2-\bar{x}x_i)$$

$$a=\frac{\sum_{i=1}^{N}(x_iy_i-x_i\bar{y})}{\sum_{i=1}^{N}(x_i{}^2-\bar{x}x_i)}$$

Since

$$\sum_{i=1}^{N}(x_i/N)=\bar{x}$$

So that we can obtain:

$$\textstyle\sum_{i=1}^{N}(\bar{x}^2-\bar{x}x_i)=0\ \text{ and }\ \sum_{i=1}^{N}(\bar{x}\bar{y}-\bar{x}y_i)=0$$

Add these two zero terms to both denominator and numerator:

$$a=\frac{\sum_{i=1}^{N}(x_iy_i-x_i\bar{y})+0}{\sum_{i=1}^{N}(x_i{}^2-\bar{x}x_i)+0}=\frac{\sum_{i=1}^{N}(x_iy_i-x_i\bar{y})+\sum_{i=1}^{N}(\bar{x}\bar{y}-\bar{x}y_i)}{\sum_{i=1}^{N}(x_i{}^2-\bar{x}x_i)+\sum_{i=1}^{N}(\bar{x}^2-\bar{x}x_i)}$$

Refine and combine the terms:

$$a=\frac{\sum_{i=1}^{N}(x_iy_i-2x_i\bar{y}+\bar{x}\bar{y})}{\sum_{i=1}^{N}(x_i{}^2-2\bar{x}x_i+\bar{x}^2)}=\frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{N}(x_i-\bar{x})^2}$$

When $\sum_{i=1}^{N}(x_i - \bar{x})^2 = 0$, this linear regression will fail.

4. Linear Algebraic Systems

Start from the first row, Multiply each term by

$$c_2/a_1$$

the new row becomes

$$[c_1, c_2 b_1/a_1 | b_1 Q_1/a_1]$$

then minus the second row by this new row, the second row now becomes

$$[a_2 - c_2 b_1/a_1, b_2 \mid Q_2 - Q_1 c_2/a_1]$$

from now on each processed row only has **two terms left**.

for i_{th} $(i = 2, 3, \ldots, N)$ row c_i, a_i, b_i continue applying **Gauss elimination**:

1. Multiply the $(i-1)_{th}$ row by $c_i/a_{i-1}{}^*$, the first term in $(i-1)_{th}$ row is c_i, the second term becomes $c_i b_{i-1}/a_{i-1}{}^*$, Q becomes
$$Q_i - Q_{i-1}{}^* c_i/a_{i-1}.$$
2. Minus the i_{th} row by the new row, now the i_{th} row become
$$\left[a_i - \frac{c_i b_{i-1}}{a_{i-1}{}^*}, b_i \middle| Q_i - Q_{i-1}{}^* c_i/a_{i-1}\right]$$
3. Now the matrix becomes the form of the second given matrix.

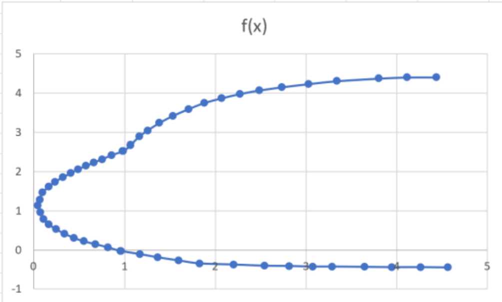
$$\begin{bmatrix} a_1^* & b_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2^* & b_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3^* & b_3 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & a_{N-1}^* & b_{N-1} \\ 0 & \ldots & 0 & 0 & 0 & a_N^* \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \\ \vdots \\ \vdots \\ Q_N^* \end{Bmatrix}$$

4. Hence, start from the last row, going from bottom to top, for $i = N$, since only one term left,
$$a_i^* x_i = Q_i^*$$
$$x_i = Q_i^*/a_i^*$$
5. For other rows, $i = 1, 2, \ldots, N-1$,
$$a_i^* x_i + b_i x_{i+1} = Q_i^*$$
6. Hence
$$x_i = (Q_i^* - b_i x_{i+1})/a_i^*$$

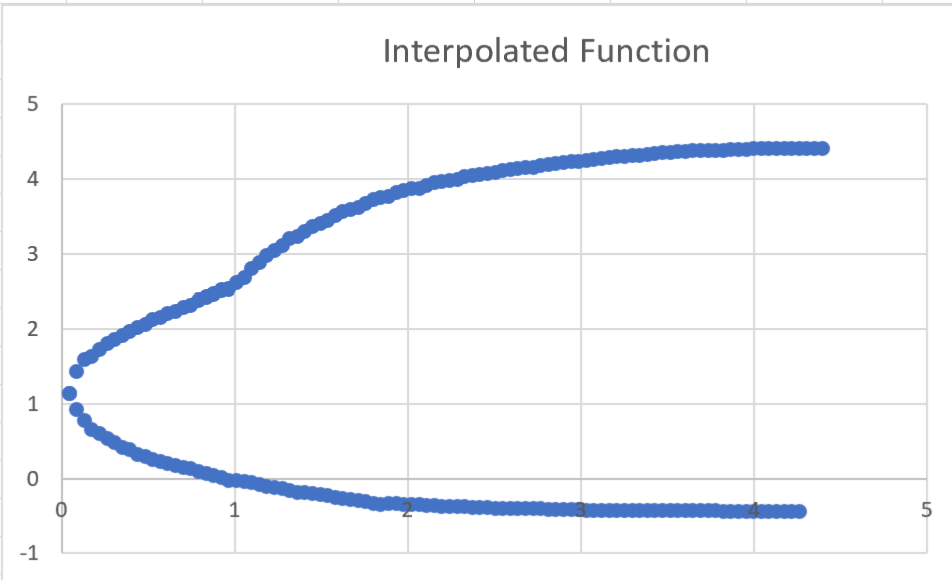
5. Interpolation

Plotted Data in Excel

x	f(x)
4.440706	4.403875
4.116905	4.403875
3.805719	4.374069
3.33894	4.307004
3.027754	4.232489
2.733389	4.150522
2.485282	4.068554
2.275021	3.979136
2.068966	3.867362
1.879731	3.748137
1.707317	3.591654
1.530698	3.412817
1.383516	3.233979
1.253154	3.040238
1.160639	2.883756
1.068124	2.675112
0.98402	2.52608
0.862069	2.421759
0.756939	2.317437
0.668629	2.235469
0.58032	2.146051
0.49201	2.064083
0.412111	1.959762
0.323802	1.85544
0.239697	1.736215
0.172414	1.609538
0.100925	1.467958
0.071489	1.281669
0.046257	1.132638



The interpolated functions plotted:

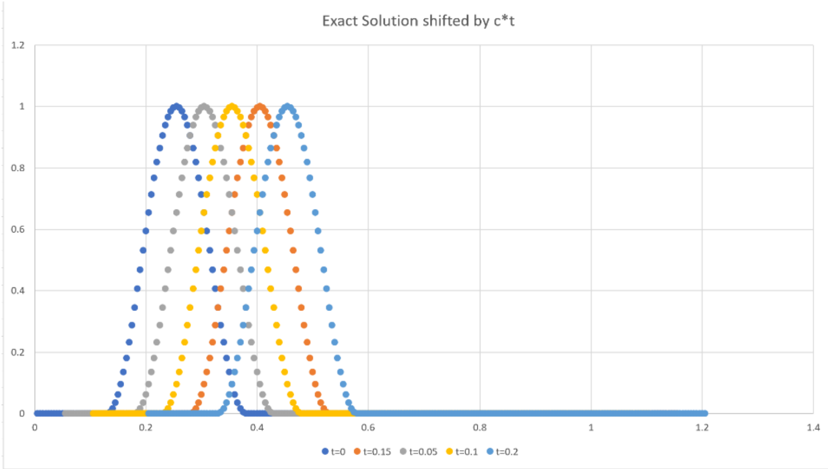


In order to search the corresponded interval by x, two guarding mechanisms were introduced to check if x in within the interval that we desired:

- ```
  target>=interp[i].xo) && (target<=(interp[i].xo+interp[i].h)))
  ```
  one is applied when the order of x variables is ascending in interpolated functions
- ```
  target<=interp[i].xo) && (target>=(interp[i].xo+interp[i].h)
  ```
  The other on is applied when the order of x variable is descending.

6. Differentiation, differential equations

Below is the graph of the **exact solution**:

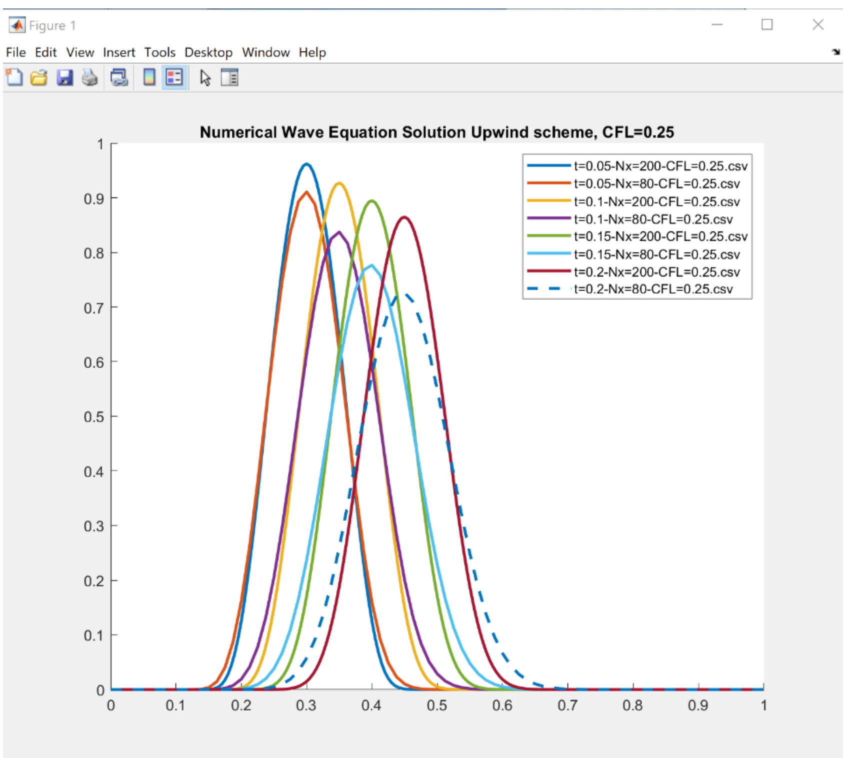


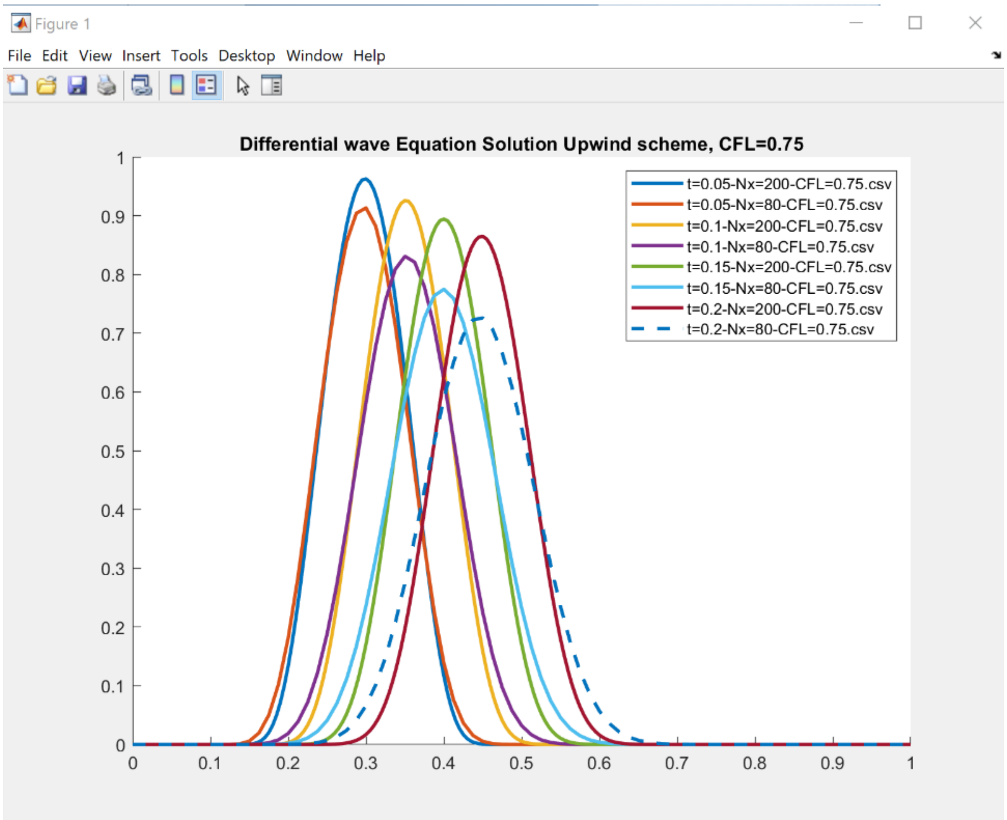
It is easily seen that it shows a wave that shifts from left to right along x axis during a time. Compared with the exact solution, the **central scheme** gives a **more accurate prediction** than **upwind method**.

- **How does the agreement of the numerical prediction with the exact solution depend on Δx?**

  Apply variable controlling method, keep CFL value unchanged, plot the curve with $Nx = 80$ and $Nx = 200$ in the same grid, here are the results:

  - **The Upwind scheme:**



○ 
Figure 1
File Edit View Insert Tools Desktop Window Help
Differential wave Equation Solution Upwind scheme, CFL=0.75
t=0.05-Nx=200-CFL=0.75.csv
t=0.05-Nx=80-CFL=0.75.csv
t=0.1-Nx=200-CFL=0.75.csv
t=0.1-Nx=80-CFL=0.75.csv
t=0.15-Nx=200-CFL=0.75.csv
t=0.15-Nx=80-CFL=0.75.csv
t=0.2-Nx=200-CFL=0.75.csv
t=0.2-Nx=80-CFL=0.75.csv

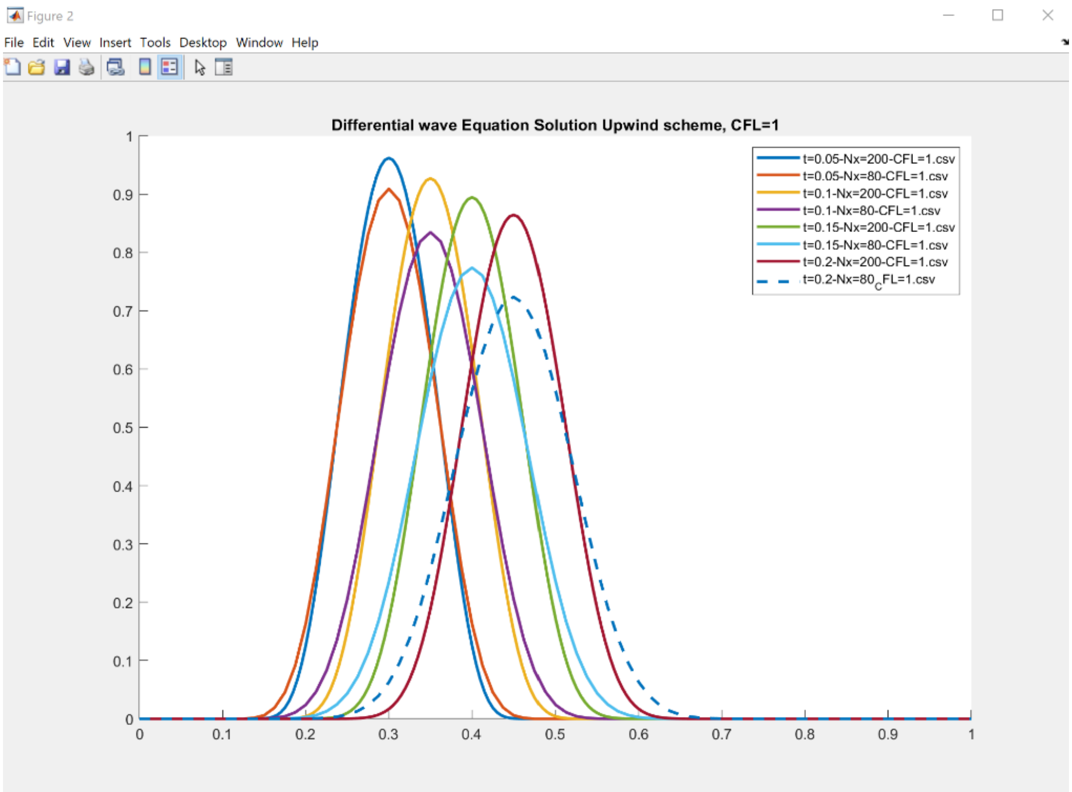


Figure 2
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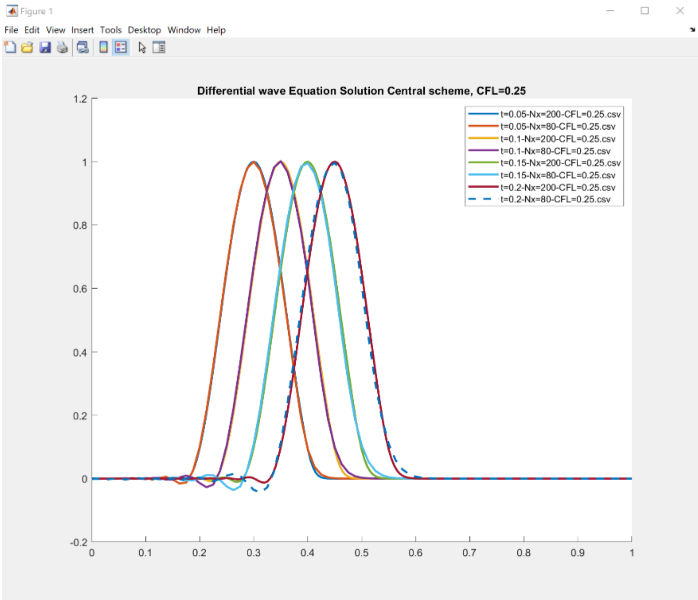
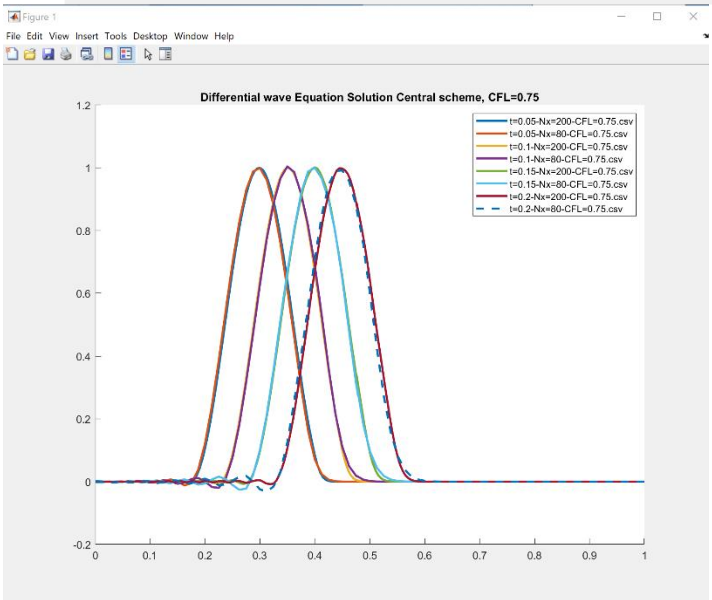
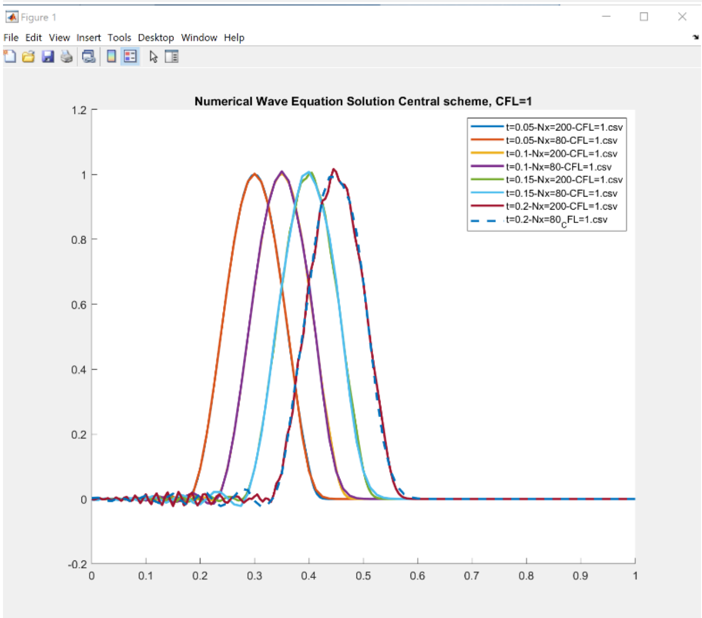
Differential wave Equation Solution Upwind scheme, CFL=1
t=0.05-Nx=200-CFL=1.csv
t=0.05-Nx=80-CFL=1.csv
t=0.1-Nx=200-CFL=1.csv
t=0.1-Nx=80-CFL=1.csv
t=0.15-Nx=200-CFL=1.csv
t=0.15-Nx=80-CFL=1.csv
t=0.2-Nx=200-CFL=1.csv
t=0.2-Nx=80_CFL=1.csv

The central scheme:

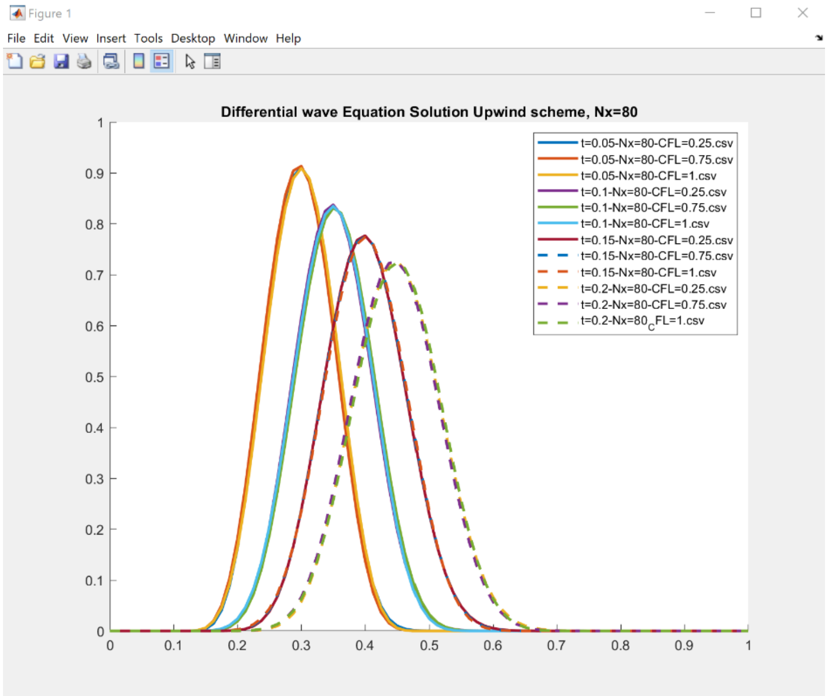


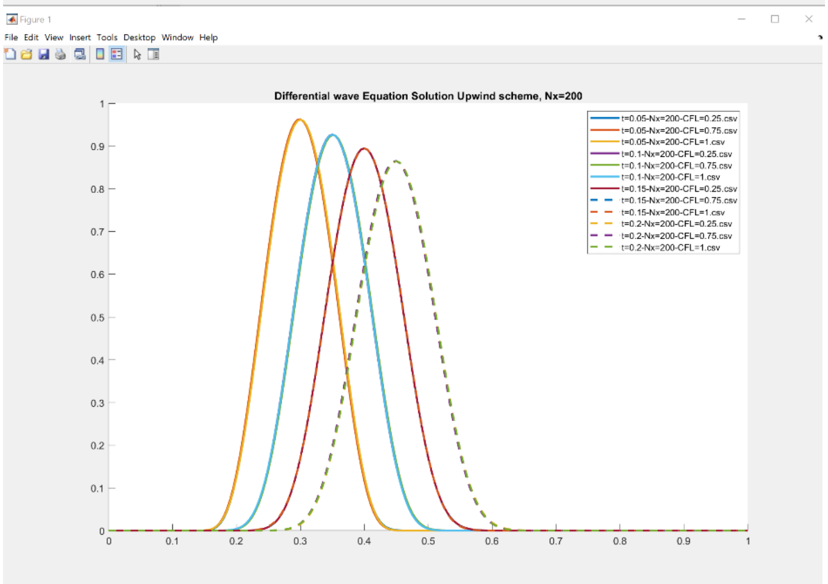




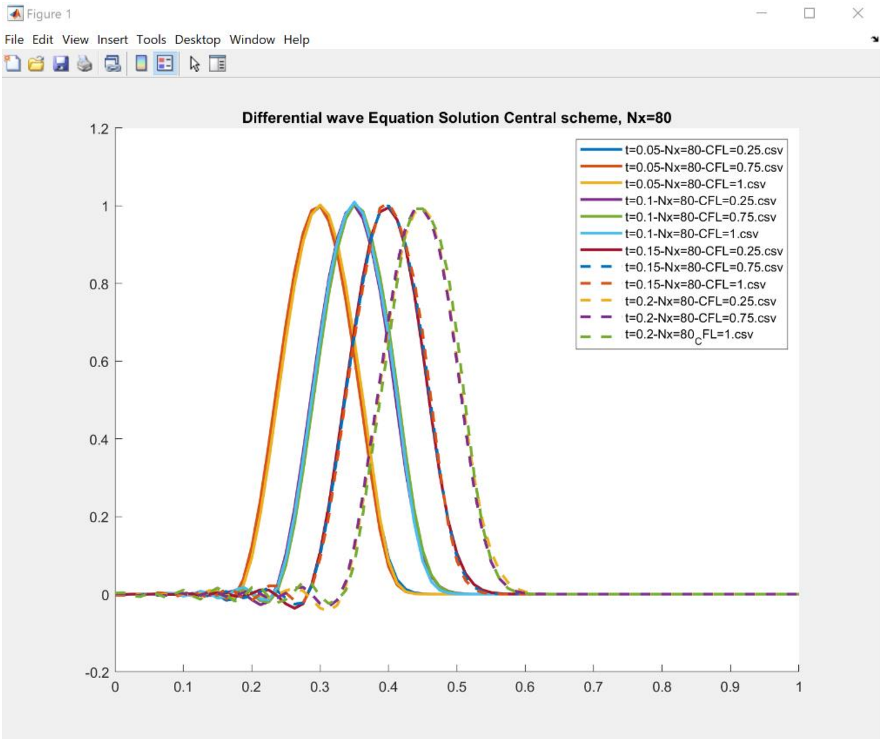
Discussion: The grid resolution determines the accuracy of the numerical predicted solution. For all CFL values, the curves with $Nx = 200$, have closer peak of wave values with the exact solution than those with $Nx = 80$ as well as smoother curves. This effect is more obvious in Upwind scheme. In addition, higher resolution also generates more stable curves in horizonal direction, observed from Central scheme graphs. Therefore, higher resolution provides more accurate prediction than lower resolution.

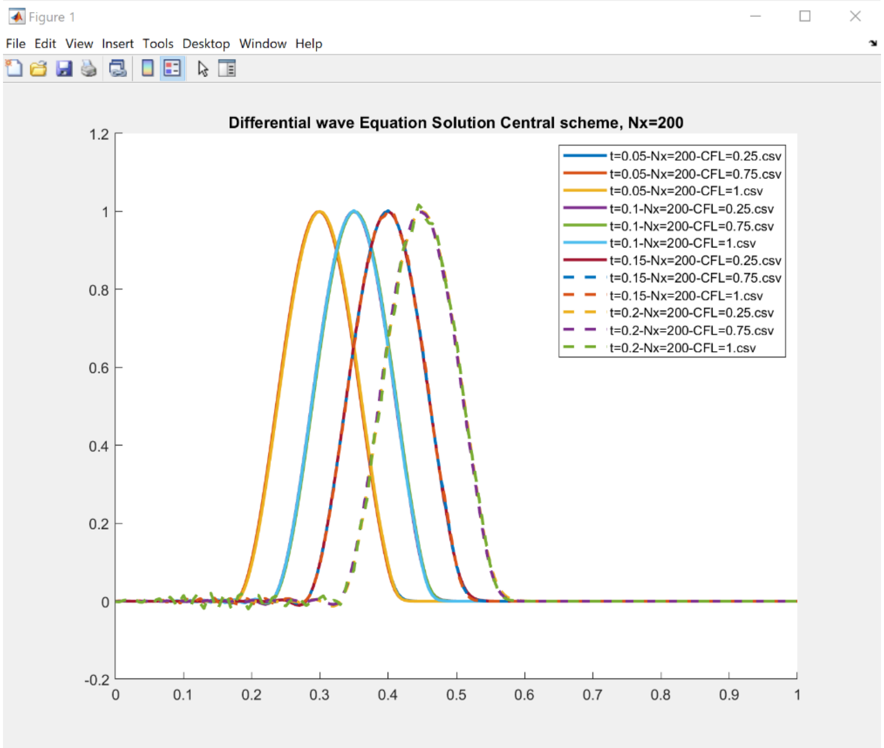
- **How does the agreement of the numerical prediction with the exact solution depend on the CFL number?**
  - **Upwind Scheme**





  - **Central Scheme**



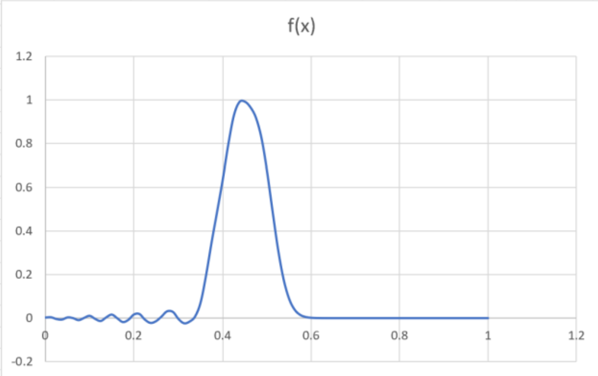


Discussion: CFL value also determines the accuracy, not for Δx but for Δt, with lower CFL value comes higher time resolution. From the graphs, especially in Central scheme, the effect of different CFL values is significant, the curves with lower CFL value possess **smaller, more stable error area** before entering the wave. Hence, CFL number provides more accurate prediction, more agreement with the exact solution.

- **What happens if I chose CFL>1?**

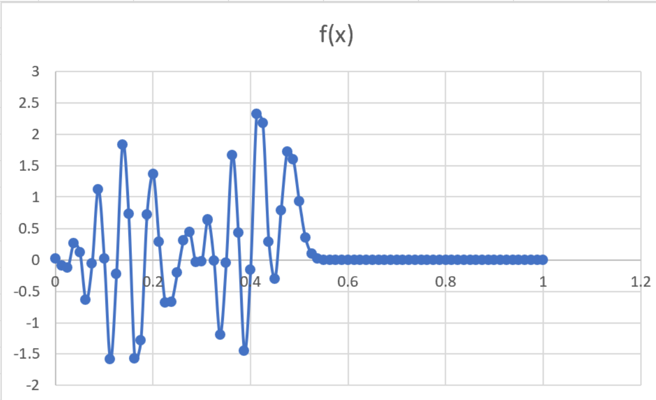
Discussion: An appropriate CFL Number is meant to make the code run stably. Once the CFL number exceeds 1, the code can not generate stable curve predictions anymore. When CFL is higher, so as the influence. Below it the case **when CFL=1.002:**

	A	B
1	x	f(x)
2	0	0.002679
3	0.0125	0.004041
4	0.025	-0.00429
5	0.0375	-0.00646
6	0.05	0.003427
7	0.0625	0.000051
8	0.075	-0.00934
9	0.0875	0.001112
10	0.1	0.01087
11	0.1125	-0.00355
12	0.125	-0.01337
13	0.1375	0.003632
14	0.15	0.016457
15	0.1625	-0.00067
16	0.175	-0.0182
17	0.1875	-0.00795
18	0.2	0.016742
19	0.2125	0.019074
20	0.225	-0.0069
21	0.2375	-0.02205
22	0.25	-0.01421
23	0.2625	0.007186
24	0.275	0.031383
25	0.2875	0.027971
26	0.3	-0.00488
27	0.3125	-0.02384
28	0.325	-0.0156
29	0.3375	0.008442



When CFL = 2:

	A	B
1	x	f(x)
2	0	0.026828
3	0.0125	-0.08979
4	0.025	-0.12587
5	0.0375	0.272345
6	0.05	0.123842
7	0.0625	-0.64114
8	0.075	-0.05621
9	0.0875	1.118683
10	0.1	0.022333
11	0.1125	-1.58915
12	0.125	-0.21764
13	0.1375	1.835915
14	0.15	0.726308
15	0.1625	-1.57493
16	0.175	-1.28803
17	0.1875	0.718396
18	0.2	1.365525
19	0.2125	0.290235
20	0.225	-0.68308
21	0.2375	-0.67413
22	0.25	-0.20191
23	0.2625	0.314167
24	0.275	0.450384
25	0.2875	-0.03487
26	0.3	-0.01924
27	0.3125	0.636448
28	0.325	-0.01019
29	0.3375	-1.19144
30	0.35	-0.04539



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