Math 589: Advanced Probability 2

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Abstract

My course notes for Ad Prob 2

1 Central Limit Theorem

1.1 Motivation

Consider $\{X_n : n \geq 1\}$ i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$ and define $S_n := \sum_{j=1}^n X_j$. From Ad Prob 1 and SLLN we know that $\frac{S_n}{n} \to 0$ a.s.

In fact given any $\{b_n : n \geq 1\} \subseteq (0, \infty)$ such that $b_n \uparrow \infty$ we have

$$\sum_{n=1}^{\infty} \frac{1}{b_n} < \infty \implies \frac{S_n}{b_n} \to 0 \text{ a.s.}$$

This is derived by first noticing that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}[X_n^2]}{b_n^2} < \infty$$

which then implies that $\sum_{n=1}^{\infty} \frac{X_n}{b_n}$ converges almost surely and so by Kronecker's Lemma this leads to $\frac{S_n}{b_n} \to 0$ a.s.

On the other hand if $\{X_n : n \geq 1\}$ is i.i.d. $N(0,1)$ random variables then

$$\frac{S_n}{\sqrt{n}} =: \hat{S}_n$$

is again a $N(0,1)$ random variable for any n. At least in this case $\hat{S}_n$ does not converge to any constant a.s.

In fact

$$\limsup_n \frac{S_n}{\sqrt{n}} = \infty \text{ a.s.}$$

and similarly for

$$\liminf_n \frac{S_n}{\sqrt{n}} = -\infty \text{ a.s.}$$

This is because for any $R > 0$ we have

$$\begin{aligned}
&\mathbb{P}(\limsup_n \hat{S}_n > R) \\
\geq &\limsup_n \mathbb{P}(\hat{S}_n > R) \\
= &\mathbb{P}(N(0,1) > R) > 0
\end{aligned}$$

And so since $\limsup \hat{S}_n \in m\mathcal{T}$ Kolmorgorov's 0-1 law gives us that $\mathbb{P}(\limsup_n \hat{S}_n > R) = 1$

This phenomenon that $\hat{S}_n \approx N(0,1)$ in fact applies to a much wider range of random variables. This is the central limit phenomenon.

First we want to get a better handle on the asymptotic behavior of S_n.

Theorem 1.1.1 (Law of the Iterated Logarithm(LIL)). *Let $\{X_n : n \geq 1\}$ be i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$ then define*

$$\forall n \geq 1, S_n = \sum_{j=1}^{n} X_j, \Lambda_n = \sqrt{2n \ln(\ln(n \vee 3))}$$

Then

$$\limsup_n \frac{S_n}{\Lambda_n} = 1 \text{ a.s. and } \liminf_n \frac{S_n}{\Lambda_n} = -1 \text{ a.s}$$

In fact $\forall c \in [-1, 1]$ for a.e. $\omega \in \Omega$ there exists a subsequence $\{n_k\}_\omega \subseteq \mathbb{N}$ such that

$$\limsup_n \frac{S_{n_k}(\omega)}{\Lambda_{n_k}(\omega)} = c.$$

Note that this means LIL $\implies$ SLLN.

Remark 1.1.2. For this topic we will denote $\overline{S_n} = \frac{S_n}{n}$ which has $\mathbb{E}[\overline{S_n}] = \mathbb{E}[X_1] = 0$. And we will denote $\hat{S_n} = \frac{S_n}{\sqrt{n}}$ where $\mathbb{E}[\hat{S_n}] = 0$ and $\mathbb{E}[(\hat{S_n})^2] = \frac{n\mathbb{E}[X_1^2]}{n} = 1$

Heuristically the second moment measures the amount of 'randomness' in the random variable and so CLT studies how the randomness is redistributed given that its total amount stays constant.

Remark 1.1.3. In fact by fixing the second moment like this we actually fix all higher moments of the distribution.

Proof. Assume $X_1 \in L^p \forall p \geq 1$ and we use induction. Assume that for $m \in \mathbb{N}$

$$L_j := \lim_{n\to\infty} \mathbb{E}[(\hat{S}_n)^j] \text{ exists } \forall 1 \leq j \leq m$$

We want to then prove this for the $(m + 1)$st moment if $\hat{S}_n$.

$$
\begin{aligned}
\mathbb{E}[S_n^{m+1}] &= \mathbb{E}[S_n \cdot S_n^m] = \sum_{j=1}^{n} \mathbb{E}[X_j(X_j + (S_n - X_j))] \\
&= \sum_{j=1}^{n} \sum_{k=0}^{m} \binom{m}{k} \mathbb{E}[X_j^{k+1}]\mathbb{E}[(S_n - X_j)^{m-k}] \\
&= n\left(\mathbb{E}[X_1]\mathbb{E}[S_{n-1}^m] + m\mathbb{E}[X_1^2]\mathbb{E}[S_{n-1}^{m-1}] + \sum_{k=2}^{m} \mathbb{E}[X_1^{k+1}]\mathbb{E}[S_{n-1}^{m-k}]\right) \\
&= n\left(0 + m \cdot 1 \cdot \mathbb{E}[S_{n-1}^{m-1}] + \sum_{k=2}^{m} \mathbb{E}[X_1^{k+1}]\mathbb{E}[S_{n-1}^{m-k}]\right) \\
&= n\left(m\mathbb{E}[S_{n-1}^{m-1}] + \sum_{k=2}^{m} \mathbb{E}[X_1^{k+1}]\mathbb{E}[S_{n-1}^{m-k}]\right) \\
\mathbb{E}[(\hat{S}_n)^{m+1}] &= n^{-(m+1)/2}\mathbb{E}[(S_n)^{m+1}] \\
&= n^{-(m-1)/2}\left(m\mathbb{E}[S_{n-1}^{m-1}] + \sum_{k=2}^{m} \mathbb{E}[X_1^{k+1}]\mathbb{E}[S_{n-1}^{m-k}]\right) \\
&= n^{-(m-1)/2}\left(m(n-1)^{(m-1)/2}\mathbb{E}[(\hat{S}_{n-1})^{m-1}] + \sum_{k=2}^{m}(n-1)^{(m-k)/2}\mathbb{E}[X_1^{k+1}]\mathbb{E}[(\hat{S}_{n-1})^{m-k}]\right) \\
&= m\left(\frac{n-1}{n}\right)^{(m-1)/2}\mathbb{E}[(\hat{S}_{n-1})^{m-1}] + \sum_{k=2}^{m} \frac{(n-1)^{(m-k)/2}}{n^{(m-1)/2}}\mathbb{E}[X_1^{k+1}]\mathbb{E}[(\hat{S}_{n-1})^{m-k}] \\
&\to m \cdot 1 \cdot L_{m-1} + \sum_{k=2}^{m} 0 \cdot \mathbb{E}[X_1^{k+1}]L_{m-k} = mL_{m-1}
\end{aligned}
$$

Allowed by independence of $X_j \in m\sigma(X_j)$ and $S_n - X_j \in m\sigma(\{X_i\} \setminus X_j)$

And so $L_{m+1} = m \cdot L_{m-1}$ with $L_{2m+1} = 0$ and $L_{2m} = (2m-1)!!$ for all m.

Note that these *exactly* match the standard gaussian. So in the limit our moments converge to the corresponding moments of the gaussian. □

Corollary 1.1.4. If ϕ is a polynomial then

$$\lim_{n\to\infty} \mathbb{E}[\phi(\hat{S}_n)] \to \mathbb{E}[\phi(N(0,1))] \tag{*}$$

1.2 First Central Limit Theorem

Theorem 1.2.1 (Lindeberg's Central Limit Theorem (CLT)). *Assume that $\{X_n : n \geq 1\}$ is a sequence of independent square-integrable random variables with $\mathbb{E}[X_n] = 0, \forall n$ and we set*

$$\sigma_n = \sqrt{\mathrm{Var}(X_n)} \quad \Sigma_n = \sqrt{\mathrm{Var}(S_n)} = \sqrt{\sum_{j=1}^{n} \sigma_j^2} \quad \hat{S}_n = \frac{S_n}{\Sigma_n}$$

So that $\mathbb{E}[\hat{S}_n] = 0$ and $\mathrm{Var}(\hat{S}_n) = 1$

As well as for any $\varepsilon > 0$

$$g_n(\varepsilon) := \frac{1}{\Sigma_n^2} \sum_{j=1}^{n} \mathbb{E}\left[X_j^2; |X_j| > \varepsilon \Sigma_n\right].$$

Then for every $\phi \in C^3(\mathbb{R})$ with ϕ'', ϕ''' being bounded we have

$$\forall \varepsilon, \left|\mathbb{E}\left[\phi(\hat{S}_n)\right] - \mathbb{E}[\phi(N(0,1))]\right| \leq \frac{1}{2}\left(\varepsilon + \sqrt{g_n(\varepsilon)}\right) \|\phi'''\|_\infty + g_n(\varepsilon) \|\phi''\|_\infty .$$

And in particular if $\forall \varepsilon > 0$ we have $\lim_{n\to\infty} g_n(\varepsilon) = 0$ (this is called Lindeberg's Condition) then $(*)$ holds, i.e.

$$\lim_{n\to\infty} \mathbb{E}[\phi(\hat{S}_n)] = \mathbb{E}[\phi(N(0,1))]$$

Remark 1.2.2. For our usual case of IID variables with $\mathbb{E}[X_1^2] = 1$ we have

$$\begin{aligned}
\forall \varepsilon > 0, g_n(\varepsilon) &= \frac{1}{\Sigma_n^2} \sum_{j=1}^{n} \mathbb{E}[X_j^2; |X_j| > \varepsilon \Sigma_n] \\
&= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2; |X_j| > \varepsilon \sqrt{n}] \\
&= \mathbb{E}[X_1^2; |X_1| > \varepsilon \sqrt{n}] \to 0
\end{aligned}$$

and so the condition always holds.

Proof. This entire proof is based on the insight that the *contribution* of each X_j becomes increasingly close to a $N(0, \sigma_j^2)$ random variable. First let $\{Y_n : n \geq 1\}$ be the Normal version of each X_j, i.e. $Y_i = N(0, \sigma_i^2)$ and each Y_i are independent. In that case we know that

$$\hat{T}_n := \frac{1}{\Sigma_n} \sum_{j=1}^{n} Y_j$$

is a $N(0,1)$ random variable for all n and so we can rewrite $\mathbb{E}[\phi(N(0,1))]$ as $\mathbb{E}[\phi(\hat{T}_n)]$. We also define

$$P_{m,n} := \frac{1}{\Sigma_n}\left(\sum_{j=1}^{m} X_j + \sum_{j=m+1}^{n} Y_j\right)$$

Now we compute

$$\begin{aligned}
\phi(\hat{S}_n) - \phi(\hat{T}_n) &= \phi(\hat{S}_n) - \phi(\hat{T}_n) + \sum_{j=1}^{n-1} \Big(\phi(P_{j,n}) - \phi(P_{j,n})\Big) \\
&= \phi(\hat{S}_n) - \phi(\hat{T}_n) + \sum_{j=1}^{n-1} \phi(P_{j,n}) - \sum_{j=1}^{n-1} \phi(P_{j,n}) \\
&= \sum_{j=0}^{n-1} \phi(P_{j,n}) - \sum_{j=1}^{n} \phi(P_{j,n}) \\
&= \sum_{j=0}^{n-1} \phi(P_{j,n}) - \sum_{j=0}^{n-1} \phi(P_{j+1,n}) \\
&= \sum_{j=0}^{n-1} \Big(\phi(P_{j,n}) - \phi(P_{j+1,n})\Big)
\end{aligned}$$

Now note that $P_{j,n} - P_{j+1,n} = (X_j - Y_j)/\Sigma_n$ is a small step size between the two functions as n grows large. For ease of computation we will fix n and define

$$U_j = P_{j,n} - \frac{X_j}{\Sigma_n}$$

giving us $P_{j,n} = U_j + X_j/\Sigma_n$ and $P_{j+1,n} = U_j + Y_j/\Sigma_n$. Importantly U_j is independent of both Y_j and X_j since it contains no terms with them.

We can thus use Taylor expansion to control change in output given change in input. First we know that we can write by Taylor's theorem

$$R(x+\delta) := \phi(x+\delta) - \left(\phi(x) + \delta \cdot \phi'(x) + \frac{\delta^2}{2} \cdot \phi''(x)\right) \implies R(x+\delta) = \frac{\phi'''(c)}{6} \cdot \delta^3 \tag{1}$$

for some $c \in (x, x+\delta)$. But another bound is given by the integral form

$$\begin{aligned}
R(x+\delta) &= \int_x^{x+\delta} \frac{\phi'''(t)}{2!}(x+\delta-t)^2 dt \le \int_x^{x+\delta} \frac{\phi'''(t)}{2!}(\delta)^2 dt = (\delta)^2 \int_x^{x+\delta} \frac{\phi'''(t)}{2!} \\
|R(x+\delta)| &= \left|(\delta)^2 \frac{1}{2} \left[\phi''(t)\right]_x^{x+\delta}\right| \le \delta^2 \left\|\phi''\right\|_\infty
\end{aligned} \tag{2}$$

And combining 1 and 2 gives us

$$R(x+\delta) \le \frac{\phi'''(c)}{6} \cdot \delta^3 \wedge \delta^2 \left\|\phi''\right\|_\infty \le \frac{\left\|\phi'''\right\|_\infty}{6} \cdot \delta^3 \wedge \delta^2 \left\|\phi''\right\|_\infty \tag{3}$$

Now plugging in $P_{j,n}$ gives us

$$
\begin{aligned}
\mathbb{E}\left[\phi\left(U_j + \frac{X_j}{\Sigma_n}\right)\right] &= \mathbb{E}[\phi(U_j)] + \mathbb{E}\left[\frac{X_j}{\Sigma_n}\phi'(U_j)\right] + \mathbb{E}\left[\frac{X_j^2}{2\Sigma_n^2}\phi''(U_j)\right] + \mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right)\right] \\
&= \mathbb{E}[\phi(U_j)] + \mathbb{E}\left[\frac{X_j}{\Sigma_n}\right]\mathbb{E}\left[\phi'(U_j)\right] + \mathbb{E}\left[\frac{X_j^2}{2\Sigma_n^2}\right]\mathbb{E}\left[\phi''(U_j)\right] + \mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right)\right] \\
&= \mathbb{E}[\phi(U_j)] + 0 + \frac{1}{2}\frac{\sigma_j^2}{\Sigma_n^2}\mathbb{E}\left[\phi''(U_j)\right] + \mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right)\right] \\
&= \mathbb{E}[\phi(U_j)] + \frac{1}{2}\frac{\sigma_j^2}{\Sigma_n^2}\mathbb{E}\left[\phi''(U_j)\right] + \mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right)\right]
\end{aligned}
$$

Similarly we can do the same thing for $P_{j+1,n}$ which gives us

$$
\mathbb{E}\left[\phi\left(U_j + \frac{Y_j}{\Sigma_n}\right)\right] = \mathbb{E}[\phi(U_j)] + \frac{1}{2}\frac{\sigma_j^2}{\Sigma_n^2}\mathbb{E}\left[\phi''(U_j)\right] + \mathbb{E}\left[R\right]
$$

this then implies that

$$
\mathbb{E}[\phi(P_{j,n}) - \phi(P_{j+1,n})] = \mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right) - R\left(U_j + \frac{Y_j}{\Sigma_n}\right)\right]
$$

but since we know that $\phi'''(c)$ is bounded then we get

$$
\begin{aligned}
\mathbb{E}\left[\phi(\hat{S}_n) - \phi(\hat{T}_n)\right] &= \left|\mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right) - R\left(U_j + \frac{Y_j}{\Sigma_n}\right)\right]\right| \\
&\leq \left|\mathbb{E}\left[R\left(U_j + \frac{X_j}{\Sigma_n}\right)\right]\right| + \left|\mathbb{E}\left[R\left(U_j + \frac{Y_j}{\Sigma_n}\right)\right]\right| \\
&\leq \mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty \wedge \frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty\right] + \mathbb{E}\left[\frac{1}{6}\frac{|Y_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty\right] \quad \text{by 3}
\end{aligned}
$$

Now for the first term we have

$$
\begin{aligned}
&\sum_{j=1}^{n}\mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty \wedge \frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty\right] = \sum_{j=1}^{n}\mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty \wedge \frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty(\mathbb{1}_{X_j\leq\varepsilon\Sigma_n} + \mathbb{1}_{X_j>\varepsilon\Sigma_n})\right] \\
&= \sum_{j=1}^{n}\mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty \wedge \frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty(\mathbb{1}_{X_j\leq\varepsilon\Sigma_n})\right] + \sum_{j=1}^{n}\mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty \wedge \frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty(\mathbb{1}_{X_j>\varepsilon\Sigma_n})\right] \\
&\leq \sum_{j=1}^{n}\mathbb{E}\left[\frac{1}{6}\frac{|X_j^3|}{\Sigma_n^3}\|\phi'''\|_\infty(\mathbb{1}_{X_j\leq\varepsilon\Sigma_n})\right] + \sum_{j=1}^{n}\mathbb{E}\left[\frac{|X_j^2|}{\Sigma_n^2}\|\phi''\|_\infty(\mathbb{1}_{X_j>\varepsilon\Sigma_n})\right] \\
&\leq \frac{1}{6}\|\phi'''\|_\infty\sum_{j=1}^{n}\mathbb{E}\left[\varepsilon\frac{|X_j^2|}{\Sigma_n^2}(\mathbb{1}_{X_j\leq\varepsilon\Sigma_n})\right] + \|\phi''\|_\infty\sum_{j=1}^{n}\mathbb{E}\left[\frac{|X_j^2|}{\Sigma_n^2}(\mathbb{1}_{X_j>\varepsilon\Sigma_n})\right] \\
&\leq \frac{1}{6}\|\phi'''\|_\infty\,\varepsilon\frac{\sum_{j=1}^{n}\sigma_j^2}{\Sigma_n^2} + \|\phi''\|_\infty\, g_n(\varepsilon) \leq \frac{\varepsilon}{6}\|\phi'''\|_\infty + \|\phi''\|_\infty\, g_n(\varepsilon)
\end{aligned}
$$

For the second term we have

$$\begin{aligned}
\sum_{j=1}^{n} \|\phi'''\|_\infty \, \mathbb{E}\left[\frac{1}{6}\frac{|Y_j^3|}{\Sigma_n^3}\right] &= \frac{1}{6}\,\|\phi'''\|_\infty \, \mathbb{E}[N(0,1)^3] \sum_{j=1}^{n} \frac{\sigma_j^3}{\Sigma_n^3} \\
&\leq \frac{2}{6}\,\|\phi'''\|_\infty \sum_{j=1}^{n} \frac{\sigma_j^3}{\Sigma_n^3} \qquad \text{since } \mathbb{E}[N(0,1)^3] \leq 2 \\
&\leq \frac{1}{3}\,\|\phi'''\|_\infty \max_{1\leq j\leq n} \frac{\sigma_j}{\Sigma_n} \sum_{j=1}^{n} \frac{\sigma_j^2}{\Sigma_n^2} \leq \frac{1}{3}\,\|\phi'''\|_\infty \max_{1\leq j\leq n} \frac{\sigma_j}{\Sigma_n}
\end{aligned}$$

Now we also know that

$$\begin{aligned}
\frac{\sigma_j^2}{\Sigma_n^2} = \frac{\mathbb{E}[X_j^2]}{\Sigma_n^2} &= \frac{\mathbb{E}[X_j^2 \mathbb{1}_{|X_j|\leq \varepsilon\Sigma_n}] + \mathbb{E}[X_j^2 \mathbb{1}_{|X_j|>\varepsilon\Sigma_n}]}{\Sigma_n^2} = \frac{\varepsilon^2\Sigma_n^2 + \mathbb{E}[X_j^2 \mathbb{1}_{|X_j|>\varepsilon\Sigma_n}]}{\Sigma_n^2} \\
&= \varepsilon^2 + \mathbb{E}\left[\frac{X_j^2}{\Sigma_n^2}\mathbb{1}_{|X_j|>\varepsilon\Sigma_n}\right] \leq \varepsilon^2 + \sum_{j=1}^{n} \mathbb{E}\left[\frac{X_j^2}{\Sigma_n^2}\mathbb{1}_{|X_j|>\varepsilon\Sigma_n}\right] = \varepsilon^2 + g_n(\varepsilon)
\end{aligned}$$

And so plugging this back in gives us

$$\|\phi'''\|_\infty \sum_{j=1}^{n} \mathbb{E}\left[\frac{1}{6}\frac{|Y_j^3|}{\Sigma_n^3}\right] \leq \frac{1}{3}\,\|\phi'''\|_\infty \left(\varepsilon + \sqrt{g_n(\varepsilon)}\right)$$

which we finally plug back in with the first term to get

$$\left|\mathbb{E}\left[\phi(\hat{S}_n)\right] - \mathbb{E}\left[\phi(\hat{T}_n)\right]\right| \leq \frac{\varepsilon}{6}\,\|\phi'''\|_\infty + \|\phi''\|_\infty \, g_n(\varepsilon) + \frac{1}{3}\,\|\phi'''\|_\infty \left(\varepsilon + \sqrt{g_n(\varepsilon)}\right)$$

□

1.3 Corollaries of LCLT

Corollary 1.3.1. Under the same conditions as 1.2.1 we have that for any compactly supported $\phi \in C^{\infty}(\mathbb{R})$

$$\lim_{n} \mathbb{E}\left[\phi\left(\hat{S_n}\right)\right] = \mathbb{E}[\phi(N(0,1))]$$

In particular

$$\lim_{n} \mathbb{P}(a < \hat{S_n} \leq b) = \int_a^b e^{-\frac{x^2}{2}} dx$$

First an important lemma we will use for study a study like this

Lemma 1.3.2 (Bump Functions). *For any set $[a,b]$ there exists a sequence of functions $\{\phi_k : k \geq 1\} \subseteq C^{\infty}(\mathbb{R})$ which are compactly supported such that $0 \leq \phi_k \leq 1, \forall k \geq 1$ and ϕ_k is decreasing in the limit to $\mathbb{1}_{[a,b]}$.*

Similarly for (a,b) we have the same story with ϕ_k' but in the limit they increase towards $\mathbb{1}_{(a,b)}$.

Proof. We will first start with the auxillary function Next we prove some important properties

$$f(x) := \begin{cases} e^{-\frac{1}{x}} & : x > 0 \\ 0 & : x \leq 0 \end{cases}$$

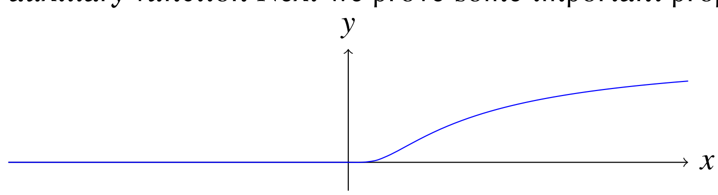

Figure 1: $f(x)$

of $f(x)$

Claim 1.3.3.

$$\frac{d^k}{dx^k} f(x) = \frac{p_k(x)}{q_k(x)} f(x)$$

for some polynomials p_k, q_k.

Proof. We prove by induction, its obviously true for $k = 0$ so assume its true for some k and we want to prove for $k = 1$.

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}} \exp\left[-\frac{1}{x}\right] = \frac{d}{dx}\frac{d^k}{dx^k} \exp\left[-\frac{1}{x}\right] = \frac{d}{dx}\frac{p_k(x)}{q_k(x)} \exp\left[-\frac{1}{x}\right] &= \left(\frac{d}{dx}\frac{p_k(x)}{q_k(x)}\right) \exp\left[-\frac{1}{x}\right] + \frac{p_k(x)}{q_k(x)}\left(\frac{d}{dx} \exp\left[-\frac{1}{x}\right]\right) \\
&= \left(\frac{q_k(x)p_k'(x) - q_k'(x)p_k(x)}{q_k^2(x)}\right) \exp\left[-\frac{1}{x}\right] + \frac{p_k(x)}{q_k(x)}\left(\frac{1}{x^2}\right) \exp\left[-\frac{1}{x}\right] \\
&= \left(\frac{p_k(x)q_k(x) + \left(q_k(x)p_k'(x) - q_k'(x)p_k(x)\right)x^2}{q_k^2(x)x^2}\right) \exp\left[-\frac{1}{x}\right]
\end{aligned}$$

These are clearly both polynomials and so we are done. □

Next we show the limiting properties of the derivatives of $f(x)$.

Claim 1.3.4.

$$\lim_{x\to 0} \frac{d^k}{dx^k} f(x) = 0$$

for all $k \geq 1$.

Proof. Simply note that as an exponential $f(x)$ converges to 0 extremely rapidly as $x \to 0$ and actually converges faster than any polynomial and so the polynomial form of the derivatives gives the limit. □

As a corollary $f(x)$ matches all derivatives for both its piece wise parts at $x = 0$ and thus is smooth.

Next we introduce our second auxillary function $g(x, a, b)$ are all obviously smooth as a

$$g(x, a, b) := \frac{f(x-a)}{f(x-a) + f(b-x)}$$

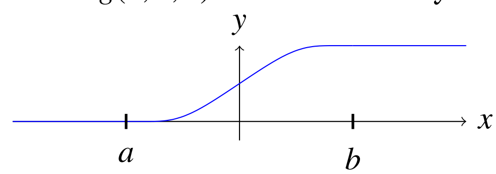

Figure 2: $g(x, a, b)$

quotients of sums of smooth functions.

And finally we can define the functions ϕ_k and ϕ'_k as

$$\phi_k := g\left(x, a - \frac{1}{k}, a\right) \cdot \left(1 - g\left(x, b, b + \frac{1}{k}\right)\right), \; \phi'_k := g\left(x, a, a + \frac{1}{k}\right) \cdot \left(1 - g\left(x, b - \frac{1}{k}, b\right)\right)$$

Compact support is obvious by the fact the fact that their support is contained in $[a-1, b+1]$. The increasing and decreasing properties are self evident from definition and the above plots. □

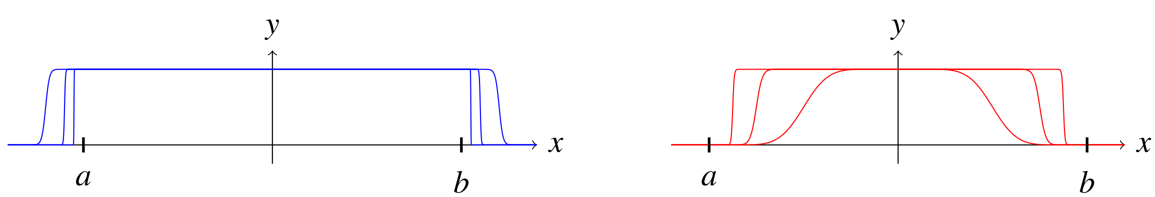

Figure 3: ϕ_k for $k = 5, 10, 20$ and ϕ'_k for $k = 2, 4, 8$

Proof of 1.3.1. Lemma 1.3.2 gives us that for the decreasing sequence that

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}(\hat{S}_n \in [a, b]) &\leq \lim_n \mathbb{E}[\phi_k(\hat{S}_n)] = \lim_n \mathbb{E}[\phi_k(N(0,1)] = \mathbb{E}[\mathbb{1}_{[a,b]}(N(0,1))] \\ &= \mathbb{P}(a \leq N(0,1) \leq B)\end{aligned}$$

Similarly for the increasing sequence we get

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}(\hat{S}_n \in (a, b)) &\leq \lim_n \mathbb{E}[\phi'_k(\hat{S}_n)] = \lim_n \mathbb{E}[\phi'_k(N(0,1)] = \mathbb{E}[\mathbb{1}_{(a,b)}(N(0,1))] \\ &= \mathbb{P}(a < N(0,1) < b)\end{aligned}$$

And since $\mathbb{P}(a < N(0,1) < b) = \mathbb{P}(a \leq N(0,1) \leq b)$ we get our desired results. □

Definition 1.3.5. *Let μ and ν be two probability measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ then we define the convolution $\mu * \nu$ to be the set function on $\mathfrak{B}(\mathbb{R}^d)$ given by*

$$\forall B \in \mathfrak{B}(\mathbb{R}^d),\ \mu * \nu = \int_{\mathbb{R}^d} \nu(B - x)\mu(dx)$$

The convolution satisfies the following properties

- It is well defined, i.e.
$$x \mapsto \nu(B - x)$$
is measurable.
- $\mu * \nu$ is a probability measure.
- $\mu * \nu = \nu * \mu$.
- $(\mu * \nu) * \rho = \mu * (\nu * \rho)$.

Proposition 1.3.6. Given any independent random variables X, Y with values in $\mathbb{R}^d$ with $\mathcal{L}_X = \mu$ and $\mathcal{L}_Y = \nu$ then $\mathcal{L}_{X+Y} = \mu * \nu$.

Proof. This is really easy to see as

$$\begin{aligned}\mathbb{P}(X + Y \in B) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_B(x + y)(\mu \times \nu)(dxdy) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbb{1}_B(x + y)\nu(dy) \right) \mu(dx) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbb{1}_{B-x}(y)\nu(dy) \right) \mu(dx) = \int_{\mathbb{R}^d} (\nu(B - x))\,\mu(dx) = \mu * \nu(B)\end{aligned}$$

□

We can also define convolution of functions as

Definition 1.3.7. *For any two functions f, g that are measurable we define*

$$f * g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy$$

1.4 Complex Valued Random Variables

For a complex function f we denote.