

Math 595: Geometric Analysis

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Abstract

My course notes for the Geometric Analysis course.

1 ABP and Basic Geometry

1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where B is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\begin{aligned}\Delta u &= c \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 1 \quad \text{on } \partial\Omega\end{aligned}$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.

For such a map we set $T = \nabla u$ to be the gradient map $\Omega \rightarrow \mathbb{R}^n$. We now want a characterization of the 'extremal' points of u as a graph, we define

$$\Gamma_u^- = \{x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega\}.$$

In other words Γ_u^- are the points of Ω where the tangent plane lies entirely below the graph of u .

This set is called the 'contact' set.

Remark 1.1.1. For any point x in the contact set we have $\nabla^2 u(x) \geq 0$ where ∇^2 is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

Claim 1.1.2 (ABP). For a solution u of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

Proof. Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so \tilde{u} cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.

To see that x is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

□

Claim 1.1.3. If a solution u to the above PDE exists then we have

$$\frac{|\partial \Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

Proof. Then

$$\begin{aligned} |B_1| &\leq |T(\Gamma_u^-)| \leq \int_{\Gamma_u^-} J_T = \int_{\Gamma_u^-} \det(\nabla^2 u) \\ &= \int_{\Gamma_u^-} \lambda_1 \lambda_2 \cdots \lambda_n \\ &\leq \int_{\Gamma_u^-} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n \quad \text{Since all the eigenvalues are positive.} \\ &\leq \int_{\Gamma_u^-} \left(\frac{\Delta u}{n} \right)^n \\ &\leq \int_{\Omega} \left(\frac{\Delta u}{n} \right)^n \\ &\leq \left(\frac{|\partial \Omega|}{n|\Omega|} \right)^n |\Omega| = \frac{|\partial \Omega|^n}{n^n |\Omega|^{n-1}} \end{aligned}$$

and since $|B| = \frac{1}{n} |\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial \Omega \end{aligned}$$

then a necessary condition for existence is

$$\int_{\Omega} f = \int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

Claim 1.1.4. The above condition is sufficient.

Proof. Assume first that $h = 0$. Thus the condition above becomes $\int_{\Omega} F = 0$. Then take the positive definite symmetric bilinear form $B(u, v) = \int_{\Omega} \nabla u \nabla v$ and notice

$$B(u, v) = (Lu, v)$$

and so L is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any g in the kernel we have by our boundary conditions

$$0 = \int_{\Omega} g Lg = \int_{\Omega} |\nabla g|^2$$

and so g is a constant function.

Thus we have a solution if and only if

$$\int_{\Omega} F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is C^2 then $\rho(x) = d(x, \partial\Omega)$ is C^2 in Ω near $\partial\Omega$, we then choose a cutoff function η satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is C^2 everywhere on Ω and as we approach the boundary we will have $\frac{\partial \gamma}{\partial \nu} = -1$.

Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for U exists if and only if

$$0 = \int_{\Omega} \Delta U = \int_{\Omega} \Delta u + \Delta(h\gamma) = \int_{\Omega} f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial \nu} = \int_{\Omega} f - \int_{\partial\Omega} h$$

and so we get our desired result. □

1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the ‘upper contact’ set, notice that we no longer require u to be differentiable. In conjunction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

Remark 1.2.1. If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

Remark 1.2.2. If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

Example 1.2.3. $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in \mathbb{R}^{n+1} .

We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R} \frac{x - z}{|x - z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a \left(1 - \frac{|y - z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y - z}{|y - z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Lemma 1.2.4.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma_u^+} |\det(\nabla^2 u)| \right)^{1/n}$$

Proof. Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).

Now consider Γ_v^+ , we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of Ω such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane P given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to x_0 , thus $v - l$ must achieve its maximum somewhere on the interior of Ω where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where \tilde{h} is given by

$$\tilde{h}(x) = v(x_0) \left(1 - \frac{x - x_0}{d}\right).$$

We can see this because \tilde{h} is just a cone with a wider base than h and thus its supporting hyperplanes must have smaller gradients.

But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right) \omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) \geq f \quad \text{in } \Omega$$

Lemma 1.2.5. *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}} \left(\int_{\Gamma_u^+} \left(\frac{-f}{\det^{1/n}(a_{ij})} \right)^n \right)^{1/n}$$

Remark 1.2.6. If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

Lemma 1.2.7. For symmetric positive matrices A, B we have

$$\det(A) \det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n} \right)^n$$

Proof. Left side is equal to product of all eigenvalues, $\lambda_1 \tilde{\lambda}_1 \cdots \lambda_n \tilde{\lambda}_n$.

$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1 \tilde{\lambda}_1 + \cdots + \lambda_n \tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. \square

Proof. Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_{ij}) > 0$ then

$$-f = -Lu = \operatorname{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_{ij}))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. \square

This lemma is sometimes called the weak maximum principle.

Remark 1.2.8. There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) + \sum_k b_k(x) u_k(x) + c(x) u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients b_k and c .

1.3 Introduction to Riemannian Geometry

Let M^n be an n -dimensional manifold, every point $p \in M^n$ has a tangent space $T_p M$, then a metric g on M^n is a choice of inner product on $T_p M$ for every $p \in M$ which varies smoothly in p . A manifold with a metric is called a Riemannian Manifold.

In any local coordinate chart (x_1, \dots, x_n) we define the ‘components’ of g to be

$$g_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle.$$

Then if at some point p we have two vectors

$$X = \sum_{j=1}^N a_j(x) \frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^N b_k(x) \frac{\partial}{\partial x_k}$$

then their inner product is given by

$$\begin{aligned}\langle X, Y \rangle_g &= \left\langle \sum_{j=1}^N a_j(x) \frac{\partial}{\partial x_j}, \sum_{k=1}^N b_k(x) \frac{\partial}{\partial x_k} \right\rangle = \sum_{j,k} a_j(x) b_k(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle \\ &= \sum_{j,k} a_j(x) b_k(x) g_{jk}(x)\end{aligned}$$

More formally, let dx_i be the dual frame to $\frac{\partial}{\partial x_i}$, as in

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_i^j,$$

then we can write the metric as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on M .

If $e_1, \dots, e_n \in T_p M$ is an orthonormal basis, that is $\langle e_i, e_j \rangle_g = \delta_{ij}$. Set $\omega_1, \dots, \omega_n$ to be its dual basis. We then get a top-form $\omega_1 \wedge \dots \wedge \omega_n$.

If

$$e_j = \sum_k a_j^k \frac{\partial}{\partial x_k}$$

where $A = a_j^k$ is a matrix, then by standard linear algebra we have that

$$\omega_1 \wedge \dots \wedge \omega_n = \det(A^{-1}) dx_1 \wedge \dots \wedge dx_n$$

Claim 1.3.1.

$$|\det(A^{-1})| = \sqrt{\det g}$$

Proof.

$$\delta_{ij} = (e_i, e_j) = a_j^k a_i^l g_{kl}$$

this implies that

$$I = A^T g A$$

where A is the transpose.

Thus

$$1 = \det(A^T g A) = \det(A^2) \det(g)$$

and so

$$\sqrt{\det(g)} = \det A^{-1}$$

□

Claim 1.3.2. The top-form $dV = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$ is coordinate change invariant.

Proof. Let us assume that $(\tilde{x}_1, \dots, \tilde{x}_n)$ are coordinates given by the transition function $\tilde{x}_\alpha = \phi(x_\alpha)$ with jacobian J_ϕ , we know that in these coordinates we have

$$\tilde{g} = \left(\frac{\partial(x_1, \dots, x_n)}{\partial(\tilde{x}_1, \dots, \tilde{x}_n)} \right)^T g \left(\frac{\partial(x_1, \dots, x_n)}{\partial(\tilde{x}_1, \dots, \tilde{x}_n)} \right) = (J_\phi^{-1})^T g (J_\phi^{-1})$$

and so

$$\sqrt{\det \tilde{g}} = \det J^{-1} \sqrt{\det g}.$$

On the other hand we have

$$d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J dx_1 \wedge \cdots \wedge dx_n$$

and so

$$\sqrt{\tilde{g}} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J^{-1} \sqrt{\det g} \det J dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$$

□

Definition 1.3.3. An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ satisfying the following properties for any smooth functions $f_1, f_2 \in C^\infty(M)$ and any smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$

- $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$
- $\nabla_X Z + Y = \nabla_X Z + \nabla_X Y$
- $\nabla_X f_1 Y = X(f_1)Y + f_1 \nabla_X Y$

Definition 1.3.4. A Levi-Civita connection is an affine connection which also satisfies

- *Symmetry:* $\nabla_X Y - \nabla_Y X = [X, Y]$
- *Compatability with g:* $X(\langle Y, Z \rangle_g) = \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g$

Remark 1.3.5. Compatability with g is essentially like the product rule.

Theorem 1.3.6 (Fundamental theorem of Riemannian Geometry). *For every Riemannian manifold there exists a unique Levi-Civita Connection.*

Proof. Take any smooth vector fields X, Y, Z , we know that the following are true

$$\begin{aligned} X(\langle Y, Z \rangle_g) &= \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g \\ Y(\langle Z, X \rangle_g) &= \langle \nabla_Y Z, X \rangle_g + \langle Z, \nabla_Y X \rangle_g \\ Z(\langle X, Y \rangle_g) &= \langle \nabla_Z X, Y \rangle_g + \langle X, \nabla_Z Y \rangle_g \end{aligned}$$

then by adding the first two equations and subtracting the third we get

$$\begin{aligned} X(\langle Y, Z \rangle_g) + Y(\langle Z, X \rangle_g) - Z(\langle X, Y \rangle_g) &= \langle Y, \nabla_X Z \rangle_g - \langle \nabla_Z X, Y \rangle_g \\ &\quad + \langle \nabla_Y Z, X \rangle_g - \langle X, \nabla_Z Y \rangle_g \\ &\quad + \langle \nabla_X Y, Z \rangle_g + \langle Z, \nabla_Y X \rangle_g \end{aligned}$$

using the symmetry of the connection we get

$$\begin{aligned} X(\langle Y, Z \rangle_g) + Y(\langle Z, X \rangle_g) - Z(\langle X, Y \rangle_g) &= \langle Y, [X, Z] \rangle_g + \langle [Y, Z], X \rangle_g + \langle [X, Y], Z \rangle_g \\ &\quad + 2 \langle Z, \nabla_Y X \rangle_g \end{aligned}$$

from here we can solve for $\langle Z, \nabla_Y X \rangle_g$ giving us the connection since as a vector, $\nabla_Y X$ is fully determined by its inner products with all other vectors. \square

One can check that in a coordinate chart that the Levi Civita connection has the form

$$\begin{aligned} \nabla_X Y &= \nabla_{\sum_i a_i(x) \frac{\partial}{\partial x_i}} \sum_j b_j(x) \frac{\partial}{\partial x_j} \\ &= \sum_i a_i(x) \left(\nabla_{\frac{\partial}{\partial x_i}} \sum_j b_j(x) \frac{\partial}{\partial x_j} \right) \\ &= \sum_{i,j} a_i(x) \left(\left(\frac{\partial}{\partial x_i} b_j(x) \right) \frac{\partial}{\partial x_j} + b_j(x) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right). \end{aligned}$$

Now we know that for some coefficients Γ_{ij}^k we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

and so

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g = \sum_k \Gamma_{ij}^k g_{k\ell}$$

Now by the previous proof and the fact that coordinate vector fields have vanishing brackets we have that

$$\begin{aligned} 2 \left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g &= \frac{\partial}{\partial x_j} \left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_\ell} \right\rangle_g \right) + \frac{\partial}{\partial x_i} \left(\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g \right) - \frac{\partial}{\partial x_\ell} \left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g \right) \\ &= \frac{\partial}{\partial x_j} (g_{i\ell}) + \frac{\partial}{\partial x_i} (g_{j\ell}) - \frac{\partial}{\partial x_\ell} (g_{ij}) \end{aligned}$$

and so by using the inverse of the metric we get

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial x_j} (g_{i\ell}) + \frac{\partial}{\partial x_i} (g_{j\ell}) - \frac{\partial}{\partial x_\ell} (g_{ij}) \right).$$

The coefficients Γ are often called the Christoffel Symbols of g in these coordinates.

Claim 1.3.7. At any point p there exists a local coordinate chart (x_1, \dots, x_n) such that

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial}{\partial x_i} (g_{jk})(p) = 0$$

Proof. We have $g_{ij}(x) = g_{ij}(0) + \sum_k a_{ij}^k x_k + O(|X|^2)$, we can always change variables so that $g_{ij}(0) = \delta_{ij}$. The tricky part is eliminating the first derivatives, for that we do a change of coordinates

$$y_\alpha = \phi(x_\alpha) = x_\alpha + \frac{1}{2} b_\alpha^{k\ell} x_k x_\ell + O(|X|^3).$$

The jacobian of this transformation is

$$J_{\phi^{-1}} = I - b_\alpha^{k\ell} x_\ell + O(|X|^3)$$

and so the new metric is

$$\begin{aligned} \tilde{g}_{\alpha\beta} &= J_{\phi^{-1}}^T g J_{\phi^{-1}} = (I - b_\alpha^{i\ell} x_\ell + O(|X|^3))^T (I + a_{ij}^m x_m) (I - b_\beta^{j\ell} x_\ell + O(|X|^3)) \\ &= I - 2b_\alpha^{i\ell} g_{i\beta} + a_{ij}^\ell x_\ell + O(|X|^2), \end{aligned}$$

then from here you can solve for b . □

1.4 Geometric constructions

We now have several natural constructions once we fix a metric on our manifold.

Consider a vector field X and a point p on a Riemannian manifold, the map $P : T_p(M) \rightarrow T_p(M)$, given by

$$v \mapsto \nabla_v X$$

is a linear map. We define its trace to be the divergence of X , denoted $\operatorname{div}(X)$.

In a local orthonormal chart at p , if we write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$, then

$$\begin{aligned} \operatorname{div}(X)_p &= \sum_i \left\langle \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \left\langle \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g \\ &= \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \delta_{ij} = \sum_i \nabla_{\frac{\partial}{\partial x_i}} a_i(x) \\ &= \sum_i \frac{\partial a_i(x)}{\partial x_i} \end{aligned}$$

Where we used the fact that in an orthonormal frame $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. We see then that in an orthonormal frame the divergence matches our ‘classical’ definition of the divergence.

Next consider a function $f \in C^\infty(M)$, we define the gradient to be a map $\operatorname{grad} : C^\infty(M) \rightarrow \mathfrak{X}(M)$ defined by

$$\langle \operatorname{grad} f, v \rangle_g = df(v)$$

for every tangent vector v .

In a local (not necessarily orthonormal) chart we have

$$\text{grad } f = \sum_j a_j(x) \frac{\partial}{\partial x_j}, df = \sum_k \frac{\partial f}{\partial x_k} dx_k,$$

then for any $v = \sum_\ell b_\ell \frac{\partial}{\partial x_\ell}$ we have

$$\langle \text{grad } f, v \rangle_g = \sum_{j,\ell} a_j g_{j\ell} b_\ell$$

but we also have

$$df(v) = \sum_{k,\ell} \frac{\partial f}{\partial x_k} b_\ell dx_k \left(\frac{\partial}{\partial x_\ell} \right) = \sum_k \frac{\partial f}{\partial x_k} b_k.$$

Now let's choose $b = (0, 0, \dots, 1, \dots, 0, 0)$ with a 1 in the m -th position then

$$\langle \text{grad } f, v \rangle_g = \sum_j a_j g_{jm}$$

and

$$df(v) = \frac{\partial f}{\partial x_m}$$

so since these are equal we can multiply both by the inverse of the metric g^{mi} to get

$$a_i = \sum_{j,m} a_j g_{jm} g^{mi} = \sum_m \frac{\partial f}{\partial x_m} g^{mi}$$

and thus

$$\text{grad } f = \sum_i a_i \frac{\partial}{\partial x_i} = \sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i}$$

Finally again for a function $f \in C^\infty(M)$, the hessian is defined as the map $\text{Hess} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ given by

$$X \mapsto \nabla_X(\text{grad } f)$$

Let us write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$ then we have by the previous results that in an orthonormal

chart around p

$$\begin{aligned}
\nabla_X(\text{grad } f) &= \nabla_X \left(\sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i} \right) \\
&= \nabla_X \left(\sum_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \right) \quad (\text{because } g^{mi} = \delta^{mi} \text{ at } p \text{ in orthonormal chart}) \\
&= \sum_{j,i} a_j(x) \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \right) \\
&= \sum_{j,i} a_j(x) \left(\left(\frac{\partial f}{\partial x_j \partial x_i} \right) \frac{\partial}{\partial x_i} \right) \quad (\text{because } \Gamma_{ij}^k = 0 \text{ at } p \text{ in orthonormal chart})
\end{aligned}$$

and so if $Y = \sum_\ell b_\ell(x) \frac{\partial}{\partial x_\ell}$ we have

$$\langle \nabla_X(\text{grad } f), Y \rangle_g = \sum_{j,\ell} a_j(x) \left(\frac{\partial f}{\partial x_j \partial x_i} \right) b_\ell(x).$$

Importantly notice that if we exchange a and b then this expression does not change and so $\langle \nabla_X(\text{grad } f), Y \rangle_g = \langle \nabla_Y(\text{grad } f), X \rangle_g$ and so as an operator Hess is symmetric. We also get that the form in orthonormal coordinates for the operator is the matrix

$$\frac{\partial f}{\partial x_j \partial x_i}$$

Now we consider the trace of the modified hessian operator, given by $\text{div}(h \cdot \text{grad } f)$. Notice that we have, in an orthonormal chart,

$$\text{div}(h \cdot \text{grad } f) = \sum_j \frac{\partial}{\partial x_j} E_j = \sum_j \frac{\partial}{\partial x_j} \left(h \sum_k g^{jk} \frac{\partial f}{\partial x_k} \right)$$

Claim 1.4.1. In a general local chart,

$$\text{div}(h \cdot \text{grad } f) = (\det g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x_j} \left(h(\det g)^{1/2} g^{ij} \frac{\partial f}{\partial x_i} \right)$$

Proof. It is enough to show that the expression on the right is coordinate invariant, since then plugging in an orthonormal chart gives us the desired result.

To see this consider a different chart $(\tilde{x}_1, \dots, \tilde{x}_n)$ and set

$$\begin{aligned}
Q &= (\det g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x_j} \left(h(\det g)^{1/2} g^{ij} \frac{\partial f}{\partial x_i} \right) \\
\tilde{Q} &= (\det \tilde{g})^{-1/2} \sum_{i,j} \frac{\partial}{\partial \tilde{x}_j} \left(h(\det \tilde{g})^{1/2} \tilde{g}^{ij} \frac{\partial f}{\partial \tilde{x}_i} \right)
\end{aligned}$$

then consider the set of functions η with support contained within both charts, if

$$\int_{\Omega} Q\eta dV = \int_{\Omega} \tilde{Q}\eta dV$$

then $Q = \tilde{Q}$.

Now we plug in our known expressions and get

$$\int_{\Omega} Q\eta dV = \int_{\Omega} \eta \sum_j \frac{\partial}{\partial x_j} \left(h(\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) dx_1 dx_2 \dots dx_n$$

then we notice that we have the a divergence term in the integral. Then by using integration by parts we can remove that divergence and instead take the gradient of η , the boundary term then dissapears by compactness of η . All together this gives us

$$\begin{aligned} \int_{\Omega} Q\eta dV &= - \int_{\Omega} \sum_j \left(\frac{\partial \eta}{\partial x_j} \right) \left(h(\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) dx_1 dx_2 \dots dx_n \\ &= - \int_{\Omega} h \sum_{j,i} \left(g^{ij} \frac{\partial \eta}{\partial x_j} \frac{\partial f}{\partial x_i} \right) (\det g)^{1/2} dx_1 dx_2 \dots dx_n \\ &= - \int_{\Omega} h \langle \text{grad } \eta, \text{grad } f \rangle_g dV \end{aligned}$$

now notice that the same calculation holds in the second chart, and so we get

$$\int_{\Omega} Q\eta dV = \int_{\Omega} Q'\eta dV$$

□

Theorem 1.4.2 (Divergence theorem). *Suppose that $\Omega \subseteq M$ is a compact domain with a smooth boundary $\partial\Omega$, then $\forall f, h \in C^\infty(M)$ we have*

$$\int_{\Omega} \text{div}(h \text{grad } f) dV = \int_{\partial\Omega} \langle h \text{grad } f, \nu \rangle_g d\tilde{V}$$

where ν is the normal vector and $d\tilde{V}$ is the induced volume form on the metric.

Proof. Find a partition of unity for some neighborhood of Ω , that is a collection of functions ρ_k with $\sum_k \rho_k = 1$ and the support of each ρ_k being contained in a single chart U_k . Now we have

$$\int_{\Omega} \text{div}(h \text{grad } f) dV = \sum_k \int_{\Omega} \text{div}(\rho_k h \text{grad } f) dV = \sum_k \int_{\Omega \cap U_k} \text{div}(\rho_k h \text{grad } f) dV$$

now for each of these smaller integrals we have

$$\int_{\Omega \cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \int_{\Omega \cap U_k} \sum_j \frac{\partial}{\partial x_j} \left(\rho_k h (\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) dx_1 dx_2 \dots dx_n.$$

We will now apply IBP to this integral, note that the interior term will contain a derivative of 1 and so will vanish, the boundary term will only be non-zero outside of the boundary of U_k , that is it will be non-zero only on $\partial\Omega \cap U_k$.

So this integral becomes

$$\int_{\partial\Omega \cap U_k} \sum_j \nu_j \left(\rho_k h (\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) d\tilde{x}_1 d\tilde{x}_2 \dots d\tilde{x}_{n-1},$$

which simplifies to

$$\int_{\partial\Omega \cap U_k} \langle \operatorname{grad} f, \nu \rangle_g (\rho_k h) d\tilde{V}.$$

This then gets summed up over k to give

$$\sum_k \int_{\partial\Omega \cap U_k} \langle \rho_k h \operatorname{grad} f, \nu \rangle_g d\tilde{V} = \sum_k \int_{\partial\Omega} \langle \rho_k h \operatorname{grad} f, \nu \rangle_g d\tilde{V} = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu \rangle_g d\tilde{V}$$

□

Theorem 1.4.3. $\forall h \in C^\infty(M)$ with $h > 0$ in Ω a compact connected open set, the system

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f && \in \Omega \\ \frac{\partial u}{\partial \nu} &= g && \in \partial\Omega \end{aligned}$$

is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} hg$.

Proof. We follow a similar proof to 1.1.4, first assume $g = 0$, then we have in the space of functions in $W^{2,1}(\Omega)$ that are zero on the boundary the image of the operator $u \mapsto \operatorname{div}(h \operatorname{grad} u)$ is orthogonal to its kernel. Now the set of functions in the kernel are those satisfying

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= 0 && \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \in \partial\Omega \end{aligned}$$

and so for those functions by the divergence theorem we get

$$0 = \int_\Omega u \operatorname{div}(h \operatorname{grad} u) dV = - \int_\Omega h |\operatorname{grad} u|^2 dV$$

and so since $h > 0$ we get $\text{grad } u = 0$ everywhere on Ω and so it is constant on Ω . Thus the image is those functions f that are orthogonal to constant functions, that is the system is solvable if and only if

$$\int_{\Omega} f dV = 0$$

Next we identically construct a function γ which is C^2 everywhere on Ω and satisfying $\frac{\partial \gamma}{\partial \nu} = -1$. We then define $U(x) = u(x) + \gamma(x)g(x)$ and notice that since

$$\text{div}(h \text{grad } U) = \text{div}(h \text{grad}(u(x) + \gamma(x)g(x))) = \text{div}(h \text{grad } u(x)) + \text{div}(h \text{grad}(\gamma(x)g(x)))$$

and

$$\frac{\partial U}{\partial \nu} = \frac{\partial u}{\partial \nu} + \frac{\partial(\gamma \cdot g)}{\partial \nu} = g - g = 0$$

then we have a solution U if and only if

$$\begin{aligned} 0 &= \int_{\Omega} \text{div}(h \text{grad } U) dV = \int_{\Omega} \text{div}(h \text{grad } u) dV + \int_{\Omega} \text{div}(h \text{grad}(\gamma(x)g(x))) dV \\ &= \int_{\Omega} f dV + \int_{\partial \Omega} h \frac{\partial(\gamma \cdot g)}{\partial \nu} d\tilde{V} = \int_{\Omega} f dV + \int_{\partial \Omega} -hg d\tilde{V} \end{aligned}$$

□

1.5 Extrinsic Geometry

Suppose we have an n -dimensional Riemannian Manifold (M^n, g) with $F : M^n \hookrightarrow N$ an immersion where N is an $n + m$ -dimensional Riemannian Manifold with metric \bar{g} . Every point $x \in M$ has a tangent space $T_x M$ and also after identifying x with $F(x)$ we have the larger tangent space $T_x N$ that contains $T_x M$. We say that M is isometrically immersed if for all $X, Y \in T_x M$ we have

$$\langle X, Y \rangle_g = \langle X, Y \rangle_{\bar{g}},$$

essentially \bar{g} extends g to a larger tangent space.

Recall that both g and \bar{g} induce connections ∇ and $\bar{\nabla}$ respectively.

Lemma 1.5.1. *Let vector fields $X, Y \in \mathfrak{X}(M)$ extend to vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(M)$. Then*

$$\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T,$$

where T is the orthogonal projection onto $T_x M$.

Proof. Define the connection $\tilde{\nabla}_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$, then by uniqueness of the Levi-Civita connection of if we have that $\tilde{\nabla}$ satisfies the axioms it must be equal to ∇ .

First we check metric compatability,

$$\langle \tilde{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle_g + \langle \bar{Y}, \tilde{\nabla}_{\bar{X}} \bar{Z} \rangle_g = \langle (\bar{\nabla}_{\bar{X}} \bar{Y})^T, \bar{Z} \rangle_g + \langle \bar{Y}, (\bar{\nabla}_{\bar{X}} \bar{Z})^T \rangle_g$$

then since all the terms are tangent to M we can replace g with \bar{g} .

$$\langle (\bar{\nabla}_{\bar{X}} \bar{Y})^T, \bar{Z} \rangle_g + \langle \bar{Y}, (\bar{\nabla}_{\bar{X}} \bar{Z})^T \rangle_g = \langle (\bar{\nabla}_{\bar{X}} \bar{Y})^T, \bar{Z} \rangle_{\bar{g}} + \langle \bar{Y}, (\bar{\nabla}_{\bar{X}} \bar{Z})^T \rangle_{\bar{g}}$$

then we can throw away the projections since taking inner product with a vector already tangent to $T_{\bar{X}}M$ implicitly projects onto that space.

$$\langle (\bar{\nabla}_{\bar{X}} \bar{Y})^T, \bar{Z} \rangle_{\bar{g}} + \langle \bar{Y}, (\bar{\nabla}_{\bar{X}} \bar{Z})^T \rangle_{\bar{g}} = \langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle_{\bar{g}} + \langle \bar{Y}, \bar{\nabla}_{\bar{X}} \bar{Z} \rangle_{\bar{g}}$$

by metric compatability of $\bar{\nabla}$ we have that

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle_{\bar{g}} + \langle \bar{Y}, \bar{\nabla}_{\bar{X}} \bar{Z} \rangle_{\bar{g}} = \bar{\nabla}_{\bar{X}} \langle \bar{Y}, \bar{Z} \rangle_{\bar{g}}$$

and then we get

$$\bar{\nabla}_{\bar{X}} \langle \bar{Y}, \bar{Z} \rangle_{\bar{g}} = \bar{X} \left(\langle \bar{Y}, \bar{Z} \rangle_{\bar{g}} \right) = X \left(\langle \bar{Y}, \bar{Z} \rangle_{\bar{g}} \right) = X \left(\langle Y, Z \rangle_{\bar{g}} \right) = X \left(\langle Y, Z \rangle_g \right)$$

For symmetry we need a small fact about Lie Brackets

Lemma 1.5.2. *If $X, Y \in \mathfrak{X}(M)$ and M is immersed in N then for any extension \bar{X}, \bar{Y} we have $[\bar{X}, \bar{Y}]_N = [X, Y]_M$.*

Proof. See Lee's smooth manifolds page 189. □

We then have that

$$\tilde{\nabla}_X^Y - \tilde{\nabla}_Y^X = (\bar{\nabla}_{\bar{X}}^{\bar{Y}})^T - (\bar{\nabla}_{\bar{Y}}^{\bar{X}})^T = ([\bar{X}, \bar{Y}]_N)^T = [X, Y]_M$$

□

Now we define the second fundamental form. For every point $p \in M$ we have $T_p N = T_p M \oplus T_p^\perp M$, this decomposition defines the normal bundle $NM = \{p \in M | T_p^\perp M\} \subseteq TN$.

For every smooth normal vector field $V \in NM$ and every vector $X \in T_p M$ we can define $\bar{\nabla}_X V \in T_p N$ we can define

$$\nabla_X^\perp V := (\bar{\nabla}_X V)^\top.$$

We now define the second fundamental form to be the map $A^W : T_p M \rightarrow T_p M$ parametrized by some vector field in NM which is defined through

$$A^W(X) = -(\bar{\nabla}_X W)^T$$

We want to check that this map is well defined, suppose $W \in NM$ is a normal vector field with two extensions \tilde{W}_1, \tilde{W}_2 . We want to check that $A^{\tilde{W}_1}(X) = A^{\tilde{W}_2}(X)$ for all vectors $X \in T_p M$.

To see this we check

$$\left\langle (\bar{\nabla}_X \tilde{W}_1)^T, Y \right\rangle_g - \left\langle (\bar{\nabla}_X \tilde{W}_2)^T, Y \right\rangle_g = \left\langle \bar{\nabla}_X \tilde{W}_1 - \bar{\nabla}_X \tilde{W}_2, Y \right\rangle_{\bar{g}}$$

and so we can apply the compatability of $\bar{\nabla}$ with the metric to get

$$\left\langle \bar{\nabla}_X \tilde{W}_1 - \bar{\nabla}_X \tilde{W}_2, Y \right\rangle_{\bar{g}} = \bar{\nabla}_X \left\langle \tilde{W}_1 - \tilde{W}_2, Y \right\rangle_{\bar{g}} - \left\langle \tilde{W}_1 - \tilde{W}_2, \bar{\nabla}_X Y \right\rangle_{\bar{g}}$$

and notice that the first term is trivially zero since both \tilde{W}_1 and \tilde{W}_2 are perpendicular to $T_p M$, and similarly the second term is also zero since at any point of M , $\tilde{W}_1 - \tilde{W}_2 = 0$.

Lemma 1.5.3. A^W is a symmetric map for any $W \in NM$.

Proof. We compute

$$\left\langle A^W(X), Y \right\rangle_g - \left\langle A^W(Y), X \right\rangle_g = \left\langle \bar{\nabla}_{\bar{Y}} W, \bar{X} \right\rangle_{\bar{g}} - \left\langle \bar{\nabla}_{\bar{X}} W, \bar{Y} \right\rangle_{\bar{g}}$$

and then apply compatability

$$\left\langle \bar{\nabla}_{\bar{Y}} W, \bar{X} \right\rangle_{\bar{g}} - \left\langle \bar{\nabla}_{\bar{X}} W, \bar{Y} \right\rangle_{\bar{g}} = \bar{\nabla}_{\bar{Y}} \left\langle W, \bar{X} \right\rangle_{\bar{g}} - \left\langle W, \bar{\nabla}_{\bar{Y}} \bar{X} \right\rangle_{\bar{g}} - \bar{\nabla}_{\bar{X}} \left\langle W, \bar{Y} \right\rangle_{\bar{g}} + \left\langle W, \bar{\nabla}_{\bar{X}} \bar{Y} \right\rangle_{\bar{g}}$$

and then clearly the first and third terms are zero and the second and fourth terms give

$$\left\langle W, [\bar{X}, \bar{Y}]_N \right\rangle_{\bar{g}}$$

which is also zero by the lemma from before. \square

From the second fundamental form we can define a mean curvature vector, consider the map $\mathbb{I} : T_p M \times T_p M \rightarrow N_p M$ defined to be the unique vector satisfying

$$\langle \mathbb{I}(X, Y), W \rangle_{\bar{g}} = \langle A^W(X), Y \rangle_g$$

for all X, Y . We then define the mean curvature vector to be the trace

$$\vec{H} = \sum_{i=1}^n \mathbb{I}(e_i, e_i)$$

where e_i is the frame of any orthonormal chart for M . One can check this definition is independent of which orthonormal chart you pick.

We now come back to the ABP setting, assume that M is isometrically embedded in \mathbb{R}^{n+m} .

Recall the PDE we were considering the solvability of,

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f && \in \Omega \\ \langle \nabla u, \nu \rangle &= 1 && \in \partial\Omega \end{aligned}$$

which is solvable if and only if $\int_{\Omega} f = \int_{\partial\Omega} h$.

We first define the sets,

$$\Omega^* = \{x \in \Omega \mid |\nabla u(x)| < 1\}, \quad \hat{\Omega} = \{(x, Y) \in N\Omega \mid |Y|^2 + |\nabla u(x)|^2 < 1\}$$

and then we define the contact set

$$\Gamma = \{(x, Y) \in N\hat{\Omega} \mid \operatorname{Hess}_u(X) - \langle \mathbb{I}, Y \rangle \geq 0\}$$

where the inequality is in terms of matrices, that is the map defined by $(v, w) \mapsto \operatorname{Hess}_u(X)(v, w) - \langle \mathbb{I}(v, w), Y \rangle$ is symmetric positive semidefinite.

Recall that $N\hat{\Omega} = \{(x, Y) \mid x \in \Omega, Y \in T_x^\perp M\}$. We now define the ABP map $\Phi : N\hat{\Omega} \rightarrow \mathbb{R}^{n+m}$ to be

$$\Phi(x, Y) = \nabla u(x) + Y$$

noting that $\nabla u(x)$ is orthogonal to Y since Y is in the normal bundle. We thus have by definition of $\hat{\Omega}$ that

$$|\Phi(x, Y)| = |Y|^2 + |\nabla u(x)|^2 < 1$$

and so $\Phi(N\hat{\Omega}) \subseteq B_1^{(n+m)}$.

Lemma 1.5.4. $\Phi(N\Gamma) \supseteq B_1^{(n+m)}$

Proof. Take some $\xi \in B_1^{(n+m)}$, that is $|\xi| < 1$, then define $w(x) = u(x) - \langle x, \xi \rangle$. Then there exists a unique minimum at $x_0 \in \bar{\Omega}$. Assume that $x_0 \notin \partial\Omega$, then $\nabla w(x_0) = 0$, thus $\nabla u(x_0) = \xi^T$ and $\operatorname{Hess}_w(x_0) \geq 0$. From there we get

$$\operatorname{Hess}_w(x_0) = \operatorname{Hess}_u(x_0) - \langle \nabla_{e_i e_j} x, \xi^\perp \rangle$$

and one can check that $\langle \nabla_{e_i e_j} x, \xi^\perp \rangle = \langle \mathbb{I}(e_i, e_j), \xi^\perp \rangle$ and so we get that $(x, y) = \xi^T + \xi^\perp$ is exactly in our contact set. \square

Lemma 1.5.5 (Jacobian Lemma). *The Jacobian J_Φ is given by*

$$J_\Phi = \det(D\Phi(x, Y)) = \det(\operatorname{Hess}_u(x) - \langle \mathbb{I}, Y \rangle)$$

and so by our lemma before we may apply the inequality and get

$$\det(\operatorname{Hess}_u(x) - \langle \mathbb{I}, Y \rangle) \leq \left(\frac{\operatorname{tr}(\operatorname{Hess}_u(x) - \langle \mathbb{I}, Y \rangle)}{n} \right)^n = \left(\frac{\Delta u(x) - \langle \vec{H}, Y \rangle}{n} \right)^n$$

Proof. Take any $(x_0, y_0) \in N\hat{\Omega}$ fixed, then fix a local orthonormal chart e_1, \dots, e_n . We can also find a nice frame for the normal bundle $\nu_1, \nu_2, \dots, \nu_m$. That is a frame satisfying

$$\langle \nu_i(x), \nu_j(x) \rangle = \delta_{ij}, \langle \nu_i(x), e_j(x) \rangle = 0,$$

we thus get local coordinates for $N\hat{\Omega}$, $(x_1, \dots, x_n, y_1, \dots, y_m)$.

Now compute

$$\left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \langle \bar{\nabla}_{e_i}(\nabla u), e_j \rangle + \sum_{\alpha=1}^m y_\alpha \langle \nabla_{e_i}(\nu_\alpha), e_j \rangle$$

and so since in the first term we are inner producting with e_j we may drop all normal components of $\bar{\nabla}$ and reduce it to the standard ∇ on M , this gives us the hessian, on the other hand for the second term we pick up exactly the expression for the second fundamental form. Thus we define

$$A := \left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \text{Hess}_u(e_i, e_j) - (\mathbb{I}(e_i, e_j), Y)$$

Next note that

$$\left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), \nu_j \right\rangle = \delta_{ij}, \quad \left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), e_j \right\rangle = 0$$

because Φ is the identity map on the Y component. We thus have that the Jacobian matrix takes the block form

$$\begin{bmatrix} A_{n \times n} & 0 \\ * & I_{m \times m} \end{bmatrix}$$

and so its determinant is just the determinant of A , proving the lemma. \square

1.6 Isoperimetric Inequality on Minimal Submanifolds

Suppose M^n is immersed in \mathbb{R}^{n+m} , with $\Omega \subseteq M$, $\bar{\Omega}$ compact and $\partial\Omega \in C^\infty$. We then have along the boundary μ the normal vector to $\partial\Omega$ and a collection of normal vectors $T_x^\perp M \subseteq T_x \mathbb{R}^{n+m}$.

We denote $|B_1^{(k)}|$ to be the volume of the ball of radius 1 in \mathbb{R}^k .

Last time we considered the PDE

$$\begin{aligned} \text{div}_g(f \nabla_g u) &= h \\ \langle \nabla_g u, \mu \rangle_g \Big|_{\partial\Omega} &= 1 \end{aligned}$$

which is solvable if and only if $\int_\Omega h = \int_{\partial\Omega} f$.

We will now prove the following Sobolev inequality, for any $f > 0$, $f \in C^\infty(M)$, we have

$$\int_{\Omega} \left(|\nabla f|^2 + f^2 |\vec{H}|^2 \right)^{1/2} + \int_{\partial\Omega} f \geq n \left(\frac{n+m}{n} \frac{|B_1^{(n+m)}|}{|B_1^{(m)}|} \right)^{1/n} \left(\int_{\Omega} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}.$$

If M is minimal ($\vec{H} = 0$) and $f = 1$ then we get $|\partial\Omega| \geq C_{n,m} |\Omega|^{\frac{n}{n-1}}$, the Isoperimetric inequality.

Proof. We assume that $m \geq 2$, if $m = 1$ then we can lift the surface one more dimension to make $m = 2$.

Our job now is to pick a special h to use the PDE. Note that the equation is scaling invariant, we can then see that by changing $f \rightarrow cf$ we get

$$\int_{\Omega} n(cf)^{\frac{n}{n-1}} = c^{\frac{n}{n-1}} \int_{\Omega} n f^{\frac{n}{n-1}}$$

and

$$\int_{\Omega} \sqrt{|\nabla cf|^2 + (cf)^2 |\vec{H}|^2} + \int_{\partial\Omega} cf = c \left(\int_{\Omega} \sqrt{|\nabla f|^2 + f^2 |\vec{H}|^2} + \int_{\partial\Omega} f \right)$$

so by rescaling we can make these two expressions equal for f . Then by setting $h = n f^{\frac{n}{n-1}} - \sqrt{|\nabla f|^2 + f^2 |\vec{H}|^2}$ then there is a solution u to the PDE with f, h .

Claim 1.6.1. $0 \leq \det(\text{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle) \leq f^{\frac{n}{n-1}}(x)$

For now we will assume the claim is true.

Then we have $B_1^{(n+m)} \subseteq \Phi(\hat{\Omega})$ and so

$$|B_1^{(n+m)}| \leq \int_{\hat{\Omega}} \det(J_{\Phi}) \leq \int_{x \in \Omega^*} \int_{T_x^\perp \Omega} \det(J_{\Phi})$$

we will now restrict the domain so that $|\Phi(\hat{\Omega})| \geq \delta$, then using the fact that $|\Phi(\hat{\Omega})|^2 = |\nabla u(x)|^2 + Y^2$ we get that $(\delta^2 - |\nabla u(x)|^2)_+ < Y^2 < 1 - |\nabla u(x)|^2$. Set B'_x to be the set of Y satisfying the above, we then get

$$(1 - \delta^{n+m}) |B_1^{(n+m)}| \leq \int_{x \in \Omega^*} \int_{B'_x} |\det(J_{\Phi})| dY dx$$

and substituting the determinant we get

$$\int_{x \in \Omega^*} \int_{B'_x} \det(\text{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle) dY dx$$

and using the claim we get that this is less than

$$\int_{x \in \Omega^*} \int_{B'_x} f^{\frac{n}{n-1}} dY dx.$$

Next we get

$$\int_{x \in \Omega^*} \int_{B'_x} f^{\frac{n}{n-1}} dY dx = \int_{x \in \Omega^*} f^{\frac{n}{n-1}} (1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+)^{m/2} |B_1^{(m)}| dx$$

then by the inequality $A^s - B^s \leq s(A - B)$ for $A \geq B \geq 0$ and $s \geq 1$ we get since $m \geq 2$ that

$$\begin{aligned} & \int_{x \in \Omega^*} f^{\frac{n}{n-1}} (1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+)^{m/2} |B_1^{(m)}| dx \\ & \leq \int_{x \in \Omega^*} f^{\frac{n}{n-1}} \frac{m}{2} (1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+) |B_1^{(m)}| dx \end{aligned}$$

then by checking both cases we can find that

$$1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+ \leq 1 - \delta^2$$

and so we can then get

$$(1 - \delta^{n+m})(|B_1^{(n+m)}|) \leq \frac{m}{2} (1 - \delta^2) |B_1^{(m)}| \int_{x \in \Omega^*} f^{\frac{n}{n-1}} dx.$$

Dividing this inequality by $1 - \delta$ we get

$$(1 + \delta + \delta^2 + \dots + \delta^{n+m-1})(|B_1^{(n+m)}|) \leq \frac{m}{2} (1 + \delta) |B_1^{(m)}| \int_{x \in \Omega^*} f^{\frac{n}{n-1}} dx$$

and then by letting $\delta \rightarrow 1$ we get

$$(n + m)(|B_1^{(n+m)}|) \leq m |B_1^{(m)}| \int_{x \in \Omega^*} f^{\frac{n}{n-1}} dx$$

now we can rewrite this as,

$$\left(\frac{n + m}{m} \frac{|B_1^{(n+m)}|}{|B_1^{(m)}|} \right)^{1/n} \leq \left(\int_{x \in \Omega^*} f^{\frac{n}{n-1}} dx \right)^{1/n}$$

and so we have

$$n \int_{\Omega} f^{\frac{n}{n-1}} = n \left(\int_{\Omega} f^{\frac{n}{n-1}} \right)^{\frac{1}{n}} \left(\int_{\Omega} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \geq n \left(\frac{n + m}{m} \frac{|B_1^{(n+m)}|}{|B_1^{(m)}|} \right)^{1/n} \left(\int_{\Omega} n f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

and so by our rescaling of f we get the desired result.

All that remains is to prove the claim, as we saw before in the determinant Lemma we have that

$$\det(\text{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle) \leq \left(\frac{\text{tr}(\text{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle)}{n} \right)^n = \left(\frac{\Delta_g u(x) - \langle \vec{H}, Y \rangle}{n} \right)^n.$$

From the PDE of u we get that $\operatorname{div}(f\nabla u) = nf^{\frac{1}{n}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$ and so by evaluating using divergence rules we get

$$\operatorname{div}(f\nabla u) = \langle \nabla f, \nabla u \rangle + f\Delta_g u = nf^{\frac{n}{n-1}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$$

and solving for $\Delta_g u$ we get

$$\Delta_g u = nf^{\frac{n}{n-1}-1} - f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} - \left\langle \frac{\nabla f}{f}, \nabla u \right\rangle.$$

Plugging this into the inequality we get

$$\det(\operatorname{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle) \leq \left(\frac{nf^{\frac{n}{n-1}-1} - f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} - \left\langle \frac{\nabla f}{f}, \nabla u \right\rangle - \langle \vec{H}, Y \rangle}{n} \right)^n.$$

We now use a Cauchy-Schwartz inequality, for any $a, A, b, B \in \mathbb{R}^n$ we have

$$|a \cdot A + b \cdot B| \leq \sqrt{A^2 + B^2} \sqrt{a^2 + b^2}$$

then we get

$$\left| \langle \nabla f, \nabla u \rangle + \nabla \langle f\vec{H}, Y \rangle \right| \leq \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} \sqrt{|\nabla u|^2 + Y^2} \leq \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$$

and so we get that

$$f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} + \left\langle \frac{\nabla f}{f}, \nabla u \right\rangle + \langle \vec{H}, Y \rangle \geq 0$$

and thus

$$\det(\operatorname{Hess}_u(x) - \langle \mathbb{I}(x), Y \rangle) \leq \left(\frac{nf^{\frac{1}{n-1}}}{n} \right)^n = f^{\frac{n}{n-1}}$$

□

1.7 Log-Sobolev inequality

Lemma 1.7.1 (ABP Lemma). *For a closed manifold $\Phi(NM) = \mathbb{R}^{n+m}$.*

Proof. Take $\xi \in \mathbb{R}^{n+m}$ then set $w(x) = u(x) - \langle x, \xi \rangle$. Since M is compact there exists a minimum of w at some points x_0 . We write $\xi = \xi^T + \xi^\perp$ then at x_0 we have $\nabla w(x) = 0$ so $\nabla u = \xi$. Then choosing $Y = \xi^\perp$ we get $\Phi(x_0, y) = \xi$. □

We now have the new estimate

Claim 1.7.2. $0 \leq \det J_\Phi(x, y) \leq f \exp \left(-\frac{|2\vec{H}(x)+y|^2}{4} - n \right)$

Proof. Take u the solution to our standard PDE with $h = f \log f - \frac{|\nabla f|^2}{f} - f|\vec{H}|^2$, note that we can always scale f so that $\int_M h = 0$.

Next we compute

$$\begin{aligned} \Delta u - \langle \vec{H}(x), Y \rangle &= -\frac{\nabla f \cdot \nabla u}{f} + \log f - \frac{|\nabla f|^2}{f^2} - |\vec{H}|^2 - \langle \vec{H}, y \rangle \\ &= \log f + \frac{|\nabla u|^2 + |Y|^2}{4} - \frac{|2\nabla f + f\nabla u|^2}{4f^2} - \frac{|2\vec{H} + Y|^2}{4} \\ &\leq \log f + \frac{|\nabla u|^2 + |Y|^2 - |2\vec{H} + Y|^2}{4} \end{aligned}$$

We thus get that

$$\begin{aligned} \left(\frac{\Delta u - \langle \vec{H}(x), Y \rangle}{n} \right)^n &\leq \left(\frac{\log f + \frac{|\nabla u|^2 + |Y|^2 - |2\vec{H} + Y|^2}{4}}{n} \right)^n \\ &\leq \left(f^{1/n} \exp \left(-\frac{|2\vec{H}(x) + y|^2}{4n} - 1 \right) \right)^n \\ &= f \exp \left(-\frac{|2\vec{H}(x) + y|^2}{4} - n \right). \end{aligned}$$

Where we employed the inequality $x \leq e^{x-1}$.

□

Theorem 1.7.3. Let $f > 0$ and $f \in C^\infty(M)$ then

$$\int_M f \left(\log f + n + \frac{n}{2} \log(4\pi) \right) - \int_M \frac{|\nabla f|^2}{f} - \int_M f |\vec{H}|^2 \leq \left(\int_M f \right) \log \left(\int_M f \right)$$

Proof. By standard calculus proof

$$1 = (4\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^{n+m}} e^{-\frac{|\xi|^2}{4}} d\xi$$

we can then use this to get

$$1 \leq (4\pi)^{-\frac{n+m}{2}} \int_{M \times \mathbb{R}^{n+m}} \exp \left(-\frac{|\Phi(x, Y)|^2}{4} \right) |\det J_\Phi| dy dV$$

then by the previous lemma we get

$$1 \leq (4\pi)^{-\frac{n+m}{2}} \int_M \int_{\mathbb{R}^{n+m}} \exp \left(-\frac{|\Phi(x, Y)|^2}{4} \right) f(x) \exp \left(-\frac{|2\vec{H}(x) + y|^2}{4} - n \right) dy dV$$

then $\exp\left(-\frac{|\Phi(x,Y)|^2}{4}\right) \leq 1$ and so we can rewrite this as

$$1 \leq (4\pi)^{-\frac{n+m}{2}} \int_M f(x) e^{-n} \int_{\mathbb{R}^{n+m}} \exp\left(-\frac{|2\vec{H}(x) + y|^2}{4}\right) dy dV$$

and by change of variables $z = 2\vec{H}(x) + y$ we get

$$1 \leq (4\pi)^{-\frac{n+m}{2}} \int_M f(x) e^{-n} \int_{\mathbb{R}^{n+m}} \exp\left(-\frac{|z|^2}{4}\right) dz dV = (4\pi)^{-\frac{n}{2}} \int_M f(x) e^{-n} dV$$

we thus end up with

$$((4\pi)^{1/2} e)^n = \int_M f(x)$$

We now have

$$0 = \int_M \left(f \log f - \frac{|\nabla f|^2}{f} - f |\vec{H}|^2 \right)$$

and so

$$\begin{aligned} \int_M f \left(\log f + n + \frac{n}{2} \log(4\pi) \right) - \int_M \frac{|\nabla f|^2}{f} - \int_M f |\vec{H}|^2 &= \int_M \left(f \left(n + \frac{n}{2} \log(4\pi) \right) \right) \\ &\leq \int_M f \log \left(\int_M f(x) \right) \end{aligned}$$

□

Corollary 1.7.4. For any φ we have

$$\int_M \varphi \log \varphi d\gamma - \int_M \frac{|\nabla \varphi|^2}{\varphi} d\gamma - \int_M \varphi \left| \vec{H} + \frac{x^\perp}{2} \right|^2 d\gamma \leq \left(\int_M \varphi \right) \log \left(\int_M \varphi d\gamma \right)$$

where $d\gamma$ is the Gaussian normalized measure.

2 Extrinsic Geometry

2.1 Curvature Constructions

We recall that the second fundamental form is a map

$$\mathbb{I} : T_p M \otimes T_p M \rightarrow T_p^\perp M$$

where $M \subseteq \mathbb{R}^{n+1}$ is a submanifold of \mathbb{R}^n . If $\bar{\nabla}$ is the connection on \mathbb{R}^n and ∇ is the connection on M then

$$\bar{\nabla}_Y X = \nabla_Y X + \mathbb{I}(X, Y)$$

Next assume that M is an n dimensional submanifold, also called a hypersurface. Then the normal bundle NM is one dimensional. Then at any point we can pick ν such that ν spans $T_p^\perp M$ and is of length 1. If M is orientable we can pick the ‘outer’ normal to globally define ν as a vector field.

Since $\mathbb{I}(X, Y)$ is in $T^\perp M$ then

$$\mathbb{I}(X, Y) = c\nu$$

for some constant depending on p, X, Y . We define h to be the bilinear form satisfying

$$\mathbb{I}(X, Y) = -h(X, Y)\nu.$$

Clearly from the properties of \mathbb{I} we get that h is a symmetric bilinear form.

We can also check that

$$h(X, Y) = \langle \bar{\nabla}_X \nu, Y \rangle = \langle \bar{\nabla}_Y \nu, X \rangle$$

Let $p \in M$ and e_1, \dots, e_n be some orthonormal frame at p . We set the ‘components’ h_{ij} to be $h(e_i, e_j)$.

Example 2.1.1. Set $M = S^n \subseteq \mathbb{R}^{n+1}$. Then parametrize as the graph $x_{n+1} = \sqrt{1 - \sum_i x_i^2}$. We then get

$$h(e_i, e_j) = \langle \nabla_{e_i} \nu, \nabla_{e_j} \nu \rangle = \langle e_i + c(p)\nu, \nabla_{e_j} \nu \rangle = \delta_{ij}$$

We now have a very important property unique to submanifolds of \mathbb{R}^n .

Claim 2.1.2 (Codazzi Property). Let $M^n \subseteq \mathbb{R}^{n+1}$ be a hypersurface with e_1, \dots, e_n a local orthonormal frame near p . Then

$$\nabla_{e_k} h_{ij} = \nabla_{e_i} h_{jk} = \nabla_{e_j} h_{ki}$$

Proof. We compute

$$\bar{\nabla}_X (\bar{\nabla}_Y Z) = \bar{\nabla}_X (\nabla_Y Z + \mathbb{I}(Y, Z)) = \nabla_X (\nabla_Y Z) + \mathbb{I}(X, \nabla_Y Z) + \bar{\nabla}_X (\mathbb{I}(Y, Z))$$

and similarly

$$\bar{\nabla}_Y (\bar{\nabla}_X Z) = \bar{\nabla}_Y (\nabla_X Z + \mathbb{I}(X, Z)) = \nabla_Y (\nabla_X Z) + \mathbb{I}(Y, \nabla_X Z) + \bar{\nabla}_Y (\mathbb{I}(X, Z))$$

finally we have

$$\bar{\nabla}_{[X, Y]} Z = \nabla_{[X, Y]} Z + \mathbb{I}([X, Y], Z) = \nabla_{[X, Y]} Z + \mathbb{I}(\nabla_X Y, Z) - \mathbb{I}(\nabla_Y X, Z).$$

We now compute the first equation, minus the second, minus the third.

This gives us on the left

$$\bar{\nabla}_X (\bar{\nabla}_Y Z) - \bar{\nabla}_Y (\bar{\nabla}_X Z) - \bar{\nabla}_{[X, Y]} Z$$

which is always zero because \mathbb{R}^{n+1} is flat (technically this comes from the Riemann tensor being equal to zero).

On the right we get

$$\begin{aligned}
& \nabla_Y (\nabla_X Z) + \mathbb{I}(Y, \nabla_X Z) + \bar{\nabla}_Y (\mathbb{I}(X, Z)) \\
& - \nabla_X (\nabla_Y Z) - \mathbb{I}(X, \nabla_Y Z) - \bar{\nabla}_X (\mathbb{I}(Y, Z)) \\
& - \nabla_{[X, Y]} Z - \mathbb{I}(\nabla_X Y, Z) + \mathbb{I}(\nabla_Y X, Z) \\
& = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X, Y]} Z \\
& + \bar{\nabla}_X (\mathbb{I}(Y, Z)) - \bar{\nabla}_Y (\mathbb{I}(X, Z)) - \mathbb{I}(\nabla_X Y, Z) + \mathbb{I}(\nabla_Y X, Z)
\end{aligned}$$

Now the first 3 terms here are all parallel to M , so when we next take inner product with ν they all disappear. We are thus left with

$$0 = \langle \bar{\nabla}_X (\mathbb{I}(Y, Z)), \nu \rangle - \langle \bar{\nabla}_Y (\mathbb{I}(X, Z)), \nu \rangle - \langle \mathbb{I}(\nabla_X Y, Z), \nu \rangle + \langle \mathbb{I}(\nabla_Y X, Z), \nu \rangle$$

Now by definition of h we have that this is equal to

$$0 = \langle \bar{\nabla}_X (h(Y, Z)\nu), \nu \rangle - \langle \bar{\nabla}_Y (h(X, Z)\nu), \nu \rangle - h(\nabla_X Y, Z) + h(\nabla_Y X, Z)$$

Now in the first term by product rule we will get $\nu \bar{\nabla}_X h(Y, Z) + h(Y, Z) \bar{\nabla}_X \nu$. Now $\bar{\nabla}_X \nu$ is orthogonal to ν because

$$0 = \bar{\nabla}_X 1 = \bar{\nabla}_X \langle \nu, \nu \rangle = 2 \langle \bar{\nabla}_X \nu, \nu \rangle$$

and so since we are taking the inner product with ν all those terms involving $\bar{\nabla}_X \nu$ disappear. We are thus left with

$$0 = \bar{\nabla}_X (h(Y, Z)) - \bar{\nabla}_Y (h(X, Z)) - h(\nabla_X Y, Z) + h(\nabla_Y X, Z)$$

Plugging in $X = e_i, Y = e_j, Z = e_k$ makes the covariant derivatives in the last two terms vanish and so we are left with

$$0 = \bar{\nabla}_{e_i} (h(e_j, e_k)) - \bar{\nabla}_{e_j} (h(e_i, e_k))$$

then the symmetry of h gives us the result. \square

Remark 2.1.3. 2-Tensors with the property above, where we can permute covariant derivatives with the indices of the tensor, are called Codazzi tensors.

Lemma 2.1.4. Set $\sigma_n(W) = \det(W)$, with W a symmetric tensor on M , if W is Codazzi then

$$\sum_j e_j \left(\frac{\partial \sigma_n(W)}{\partial W_{ij}} \right) = 0$$

Proof. At p we assume $\sigma_n(W) \neq 0$ then

$$\frac{\partial \sigma_n(W)}{\partial W_{ij}} = C^{ij}$$

where C^{ij} is the cofactor matrix defined by $C^{ij}W_{j\ell} = \delta_{i\ell}\sigma_n(W)$. We clearly have $C^{ij}/\sigma_n(W) = W^{-1}$ as well.

Then consider the identity $C^{ij}W_{j\ell} = \delta_{i\ell}\sigma_n(W)$ and differentiate it

$$(\sigma_n(W))_m = e_m(\delta_{i\ell}\sigma_n(W)) = e_m(C^{ij}W_{j\ell}) = (C_m^{ij}W_{j\ell} + C^{ij}W_{j\ell,m})$$

we now multiply this by the matrix $C^{m\ell}$ to get

$$(\sigma_n(W))_m C^{m\ell} = C^{m\ell}(C_m^{ij}W_{j\ell} + C^{ij}W_{j\ell,m}) = \left(C_m^{ij}\sigma_n(W)\delta_{jm} + C^{ij}\frac{\partial \sigma_n(W)}{\partial W_{m\ell}}W_{j\ell,m} \right)$$

Now since W is Codazzi and symmetric in the last copy of W we can permute the indices to get

$$\left(C_j^{ij}\sigma_n(W) + C^{ij}\frac{\partial \sigma_n(W)}{\partial W_{m\ell}}W_{m\ell,j} \right) = (C_j^{ij}\sigma_n(W) + C^{ij}(\sigma_n(W))_j)$$

We are thus left with

$$(\sigma_n(W))_m C^{m\ell} = C_j^{ij}\sigma_n(W) + C^{ij}(\sigma_n(W))_j$$

and so

$$C_j^{ij}\sigma_n(W) = 0$$

If $\sigma_n(W) = 0$ then exchange W for $W + tg$ and let $t \rightarrow 0$ and you will recover the same identity. \square

We now introduce the elementary symmetric functions

$$\sigma_k(W) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

where λ_i are the eigenvalues of W . For example for $k = 1$ this is just the sum of eigenvalues which is the trace.

Claim 2.1.5. $\sigma_n(W + tI) = \sum_{k=0}^n t^{n-k}\sigma_k(W)$

Proof. Pick a basis in which W is diagonal, then we have the eigenvalues $\tilde{\lambda}_i$ of $W + tI$ are given by $\tilde{\lambda}_i = \lambda_i + t$ and so we get

$$\sigma_n(W + tI) = \prod_i \tilde{\lambda}_i = \prod_i (\lambda_i + t) = \sum_{k=0}^n t^{n-k}\sigma_k(W)$$

\square

Lemma 2.1.6. *If W is Codazzi then $\sum_{j,k} e_j \left(\frac{\partial \sigma_k(W)}{\partial W_{ij}} \right) = 0$*

Proof. Set $t = 1$ then $\sigma_n(W + I) = \sum_{j,k} \sigma_k(W)$. Then we have

$$0 = \sum_j e_j \left(\frac{\partial \sigma_n(W + I)}{\partial W_{ij}} \right) = \sum_{j,k} e_j \left(\frac{\partial \sigma_k(W)}{\partial W_{ij}} \right)$$

□

Now let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain such that $\partial\Omega$ is compact with normal vector ν . Then we define the ‘expansion’ of Ω to be

$$\Omega_t = \Omega \cup \{x + s\nu | x \in \partial\Omega, s \in [0, t]\}$$

Then the boundary of Ω_t is

$$\partial\Omega_t = \{x + t\nu | x \in \partial\Omega\}.$$

For small t , $\partial\Omega_t$ is smooth.

Now it is easy to see that

$$V(\Omega_t) = V(\Omega) + \int_0^t A(\partial\Omega_s) ds$$

where V denotes the volume and $A(\partial\Omega_s)$ the surface area.

But we also know that

$$A(\partial\Omega_s) = \int_{\Omega} dV_s$$

now we can see that if $Y = X + s\nu_x$ is the position vector of some point on $\partial\Omega_s$ then we can choose coordinates around that point such that g and h_{ij} is diagonal, then if we set \tilde{g} to be the metric on $\partial\Omega_s$ then we can compute

$$\begin{aligned} \tilde{g}_{ij} &= \tilde{g}(Y_{e_i}, Y_{e_j}) = \langle (X + s\nu_x)_{e_i}, (X + s\nu_x)_{e_j} \rangle_{\tilde{g}} \\ &= \langle X_{e_i} + sh_{ii}e_i, X_{e_j} + sh_{jj}e_j \rangle_{\tilde{g}} = \langle e_i + sh_{ii}e_i, e_j + sh_{jj}e_j \rangle_{\tilde{g}} \\ &= g_{ij}(1 + sh_{ii})(1 + sh_{jj}). \end{aligned}$$

Since g_{ij} is diagonal then \tilde{g}_{ij} is too and it has volume coefficient

$$\sqrt{\det(\tilde{g})} = \sqrt{\prod_i \tilde{g}_{ii}} = \prod_i \sqrt{(1 + sh_{ii})^2 g_{ii}} = \prod_i (1 + sh_{ii}) \sqrt{\det(g)}$$

and so $dV_s = \sum_{k=0}^n s^k \sigma_k(h) dV_0$.

We then get

$$V(\Omega_t) = V(\Omega) + \int_0^t \sum_{k=0}^n s^k \int_{\partial\Omega} \sigma_k(h) dV_0 ds$$

The integrals $\int_{\partial\Omega} \sigma_k(h) dV_0$ are called quermassintegrals, for $k = 1$ this is Total Mean Curvature, and for $k = 2$ this is total Scalar curvature.