Math 595: Geometric Analysis

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October 31, 2023

Abstract

My course notes for the Geometric Analysis course taught by Pengfei Guan at McGill.

1 ABP and Basic Geometry

1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where B is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\Delta u = c \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.
For such a map we set $T = \nabla u$ to be the gradient map $\Omega \to \mathbb{R}^n$. We now want a characterization of the 'extremal' points of u as a graph, we define

$$\Gamma_u^- = \{x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega\}.$$

In other words Γ_u^- are the points of Ω where the tangent plane lies entirely below the graph of u.
This set is called the 'contact' set.

Remark 1.1.1. For any point x in the contact set we have $\nabla^2 u(x) \geq 0$ where ∇^2 is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

Claim 1.1.2 (ABP)**.** For a solution u of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

Proof. Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so $\tilde{u}$ cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.
To see that x is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

$\square$

Claim 1.1.3. If a solution u to the above PDE exists then we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

Proof. Then

$$\begin{aligned}
|B_1| \leq |T(\Gamma_u^-)| \leq \int_{\Gamma_u^-} J_T &= \int_{\Gamma_u^-} \det(\nabla^2 u) \\
&= \int_{\Gamma_u^-} \lambda_1\lambda_2\cdots\lambda_n \\
&\leq \int_{\Gamma_u^-} \left(\frac{\lambda_1+\cdots+\lambda_n}{n}\right)^n \qquad \text{Since all the eigenvalues are positive.} \\
&\leq \int_{\Gamma_u^-} \left(\frac{\Delta u}{n}\right)^n \\
&\leq \int_{\Omega} \left(\frac{\Delta u}{n}\right)^n \\
&\leq \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Omega| = \frac{|\partial\Omega|^n}{n^n|\Omega|^{n-1}}
\end{aligned}$$

and since $|B| = \frac{1}{n}|\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial\Omega
\end{aligned}$$

then a necessary condition for existence is

$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

Claim 1.1.4. The above condition is sufficient.

Proof. Assume first that $h = 0$. Thus the condition above becomes $\int_\Omega F = 0$. Then take the positive definite symmetric bilinear form $B(u,v) = \int_\Omega \nabla u \nabla v$ and notice

$$B(u,v) = (Lu, v)$$

and so L is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any g in the kernel we have by our boundary conditions

$$0 = \int_\Omega gLg = \int_\Omega |\nabla g|^2$$

and so g is a constant function.
Thus we have a solution if and only if

$$\int_\Omega F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is C^2 then $\rho(x) = d(x, \partial\Omega)$ is C^2 in Ω near $\partial\Omega$, we then choose a cutoff function η satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is C^2 everywhere on Ω and as we approach the boundary we will have $\frac{\partial\gamma}{\partial\nu} = -1$.
Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for U exists if and only if

$$0 = \int_\Omega \Delta U = \int_\Omega \Delta u + \Delta(h\gamma) = \int_\Omega f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial\nu} = \int_\Omega f - \int_{\partial\Omega} h$$

and so we get our desired result. □

1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the 'upper contact' set, notice that we no longer require u to be differentiable. In conjuction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

Remark 1.2.1. If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

Remark 1.2.2. If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

Example 1.2.3. $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in $\mathbb{R}^{n+1}$.

We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R}\frac{x-z}{|x-z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a\left(1 - \frac{|y-z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y-z}{|y-z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Lemma 1.2.4.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma_u^+} |\det(\nabla^2)| \right)^{1/n}$$

Proof. Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).

Now consider Γ_v^+, we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of Ω such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane P given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to x_0, thus $v - l$ must achieve its maximum somewhere on the interior of Ω where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where $\tilde{h}$ is given by

$$\tilde{h}(x) = v(x_0)\left(1 - \frac{x - x_0}{d}\right).$$

We can see this because $\tilde{h}$ is just a cone with a wider base than h and thus its supporting hyperplanes must have smaller gradients.
But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right)\omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) \geq f \quad \text{in } \Omega$$

Lemma 1.2.5. *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}}\left(\int_{\Gamma_u^+}\left(\frac{-f}{\det^{1/n}(a_{ij})}\right)^n\right)^{1/n}$$

Remark 1.2.6. If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

Lemma 1.2.7. *For symmetric positive matrices A, B we have*

$$\det(A)\det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n$$

Proof. Left side is equal to product of all eigenvalues, $\lambda_1\tilde{\lambda}_1 \cdots \lambda_n\tilde{\lambda}_n$.
$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1\tilde{\lambda}_1 + \cdots + \lambda_n\tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. □

Proof. Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_i j) > 0$ then

$$-f = -Lu = \text{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_i j))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. □

This lemma is sometimes called the weak maximum principle.

Remark 1.2.8. There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) + \sum_k b_k(x)u_k(x) + c(x)u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients b_k and c.

1.3 Introduction to Riemannian Geometry

Let M^n be an n-dimensional manifold, every point $p \in M^n$ has a tangent space T^pM, then a metric g on M^n is a choice of inner product on T_pM for every $p \in M$ which varies smoothly in p. A manifold with a metric is called a Riemannian Manifold.

In any local coordinate chart $(x_1, \dots, x_n)$ we define the 'components' of g to be

$$g_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle.$$

Then if at some point p we have two vectors

$$X = \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k}$$

then their inner product is given by

$$\begin{aligned}\langle X, Y \rangle_g &= \left\langle \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k} \right\rangle = \sum_{j,k} a_j(x)b_k(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle \\ &= \sum_{j,k} a_j(x)b_k(x)g_{jk}(x)\end{aligned}$$

More formally, let dx_i be the dual frame to $\frac{\partial}{\partial x_i}$, as in

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_i^j,$$

then we can write the metric as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on M.
If $e_1, \ldots, e_n \in T_pM$ is an orthonormal basis, that is $\langle e_i, e_j \rangle_g = \delta_{ij}$. Set $\omega_1, \ldots, \omega_n$ to be its dual basis. We then get a top-form $\omega_1 \wedge \cdots \wedge \omega_n$.
If

$$e_j = \sum_k a_j^k \frac{\partial}{\partial x_k}$$

where $A = a_j^k$ is a matrix, then by standard linear algebra we have that

$$\omega_1 \wedge \cdots \wedge \omega_n = \det(A^{-1}) dx_1 \wedge \cdots \wedge dx_n$$

Claim 1.3.1.

$$|\det(A^{-1})| = \sqrt{\det g}$$

Proof.

$$\delta_{ij} = (e_i, e_j) = a_j^k a_i^l g_{kl}$$

this implies that

$$I = A^T g A$$

where A is the transpose.
Thus

$$1 = \det(A^T g A) = \det(A^2) \det(g)$$

and so

$$\sqrt{\det(g)} = \det A^{-1}$$

□

Claim 1.3.2. The top-form $dV = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$ is coordinate change invariant.

Proof. Let us assume that $(\tilde{x}_1, \ldots, \tilde{x}_n)$ are coordinates given by the transition function $\tilde{x}_\alpha = \phi(x_\alpha)$ with jacobian J_ϕ, we know that in these coordinates we have

$$\tilde{g} = \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)} \right)^T g \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)} \right) = \left(J_\phi^{-1}\right)^T g \left(J_\phi^{-1}\right)$$

and so

$$\sqrt{\det \tilde{g}} = \det J^{-1} \sqrt{\det g}.$$

On the other hand we have

$$d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J dx_1 \wedge \cdots \wedge dx_n$$

and so

$$\sqrt{\tilde{g}}d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J^{-1}\sqrt{\det g}\det J dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$$

$\square$

Definition 1.3.3. *An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following properties for any smooth functions $f_1, f_2 \in C^\infty(M)$ and any smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$*

- $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$
- $\nabla_X Z + Y = \nabla_X Z + \nabla_X Y$
- $\nabla_X f_1 Y = X(f_1) Y + f \nabla_X Y$

Definition 1.3.4. *A Levi-Civita connection is an affine connection which also satisfies*

- *Symmetry:* $\nabla_X Y - \nabla_Y X = [X, Y]$
- *Compatability with g:* $X(\langle Y, Z\rangle_g) = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$

Remark 1.3.5. Compatability with g is essentially like the product rule.

Theorem 1.3.6 (Fundamental theorem of Riemannian Geometry)**.** *For every Riemannian manifold there exists a unique Levi-Civita Connection.*

Proof. Take any smooth vector fields X, Y, Z, we know that the following are true

$$\begin{aligned} X(\langle Y, Z\rangle_g) &= \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g \\ Y(\langle Z, X\rangle_g) &= \langle \nabla_Y Z, X\rangle_g + \langle Z, \nabla_Y X\rangle_g \\ Z(\langle X, Y\rangle_g) &= \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Z Y\rangle_g \end{aligned}$$

then by adding the first two equations and subtracting the third we get

$$\begin{aligned} X(\langle Y, Z\rangle_g) + Y(\langle Z, X\rangle_g) - Z(\langle X, Y\rangle_g) &= \langle Y, \nabla_X Z\rangle_g - \langle \nabla_Z X, Y\rangle_g \\ &+ \langle \nabla_Y Z, X\rangle_g - \langle X, \nabla_Z Y\rangle_g \\ &+ \langle \nabla_X Y, Z\rangle_g + \langle Z, \nabla_Y X\rangle_g \end{aligned}$$

using the symmetry of the connection we get

$$\begin{aligned} X(\langle Y, Z\rangle_g) + Y(\langle Z, X\rangle_g) - Z(\langle X, Y\rangle_g) &= \langle Y, [X, Z]\rangle_g + \langle [Y, Z], X\rangle_g + \langle [X, Y], Z\rangle_g \\ &+ 2\langle Z, \nabla_Y X\rangle_g \end{aligned}$$

from here we can solve for $\langle Z, \nabla_Y X\rangle_g$ giving us the connection since as a vector, $\nabla_Y X$ is fully determined by its inner products with all other vectors. $\square$

One can check that in a coordinate chart that the Levi Civita connection has the form

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_i a_i(x)\frac{\partial}{\partial x_i}} \sum_j b_j(x)\frac{\partial}{\partial x_j} \\
&= \sum_i a_i(x)\left(\nabla_{\frac{\partial}{\partial x_i}} \sum_j b_j(x)\frac{\partial}{\partial x_j}\right) \\
&= \sum_{i,j} a_i(x)\left(\left(\frac{\partial}{\partial x_i} b_j(x)\right)\frac{\partial}{\partial x_j} + b_j(x)\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right).
\end{aligned}$$

Now we know that for some coefficients Γ^k_{ij} we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial x_k}$$

and so

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}\right\rangle_g = \sum_k \Gamma^k_{ij} g_{k\ell}$$

Now by the previous proof and the fact that coordinate vector fields have vanishing brackets we have that

$$\begin{aligned}
2\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}\right\rangle_g &= \frac{\partial}{\partial x_j}\left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_\ell}\right\rangle_g\right) + \frac{\partial}{\partial x_i}\left(\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}\right\rangle_g\right) - \frac{\partial}{\partial x_\ell}\left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle_g\right) \\
&= \frac{\partial}{\partial x_j}(g_{i\ell}) + \frac{\partial}{\partial x_i}(g_{j\ell}) - \frac{\partial}{\partial x_\ell}(g_{ij})
\end{aligned}$$

and so by using the inverse of the metric we get

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x_j}(g_{i\ell}) + \frac{\partial}{\partial x_i}(g_{j\ell}) - \frac{\partial}{\partial x_\ell}(g_{ij})\right).$$

The coefficients Γ are often called the Christoffel Symbols of g in these coordinates.

Claim 1.3.7. At any point p there exists a local coordinate chart $(x_1, \dots, x_n)$ such that

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial}{\partial x_i}(g_{jk})(p) = 0$$

Proof. We have $g_{ij}(x) = g_{ij}(0) + \sum_k a^k_{ij} x_k + O(|X|^2)$, we can always change variables so that $g_{ij}(0) = \delta_{ij}$. The tricky part is eliminating the first derivatives, for that we do a change of coordinates

$$y_\alpha = \phi(x_\alpha) = x_\alpha + \frac{1}{2}b^{k\ell}_\alpha x_k x_\ell + O(|X|^3).$$

The jacobian of this transformation is

$$J_{\phi^{-1}} = I - b_{\alpha}^{k\ell} x_\ell + O(|X|^3)$$

and so the new metric is

$$\begin{aligned}\tilde{g}_{\alpha\beta} = J_{\phi^{-1}}^T g J_{\phi^{-1}} &= (I - b_\alpha^{i\ell} x_\ell + O(|X|^3))^T (I + a_{ij}^m x_m)(I - b_\beta^{j\ell} x_\ell + O(|X|^3)) \\ &= I - 2b_\alpha^{i\ell} g_{i\beta} + a_{ij}^\ell x_\ell + O(|X|)^2,\end{aligned}$$

then from here you can solve for b. □

1.4 Geometric constructions

We now have several natural constructions once we fix a metric on our manifold.

Consider a vector field X and a point p on a Riemannian manifold, the map $P : T_p(M) \to T_p(M)$, given by

$$v \mapsto \nabla_v X$$

is a linear map. We define its trace to be the divergence of X, denoted $\operatorname{div}(X)$.

In a local orthonormal chart at p, if we write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$, then

$$\begin{aligned}\operatorname{div}(X)_p &= \sum_i \left\langle \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \left\langle \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g \\ &= \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \delta_{ij} = \sum_i \nabla_{\frac{\partial}{\partial x_i}} a_i(x) \\ &= \sum_i \frac{\partial a_i(x)}{\partial x_i}\end{aligned}$$

Where we used the fact that in an orthonormal frame $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. We see then that in an orthonormal frame the divergence matches our 'classical' definition of the divergence.

Next consider a function $f \in C^\infty(M)$, we define the gradient to be a map $\operatorname{grad} : C^\infty(M) \to \mathfrak{X}(M)$ defined by

$$\langle \operatorname{grad} f, v \rangle_g = df(v)$$

for every tangent vector v.

In a local (not necessarily orthonormal) chart we have

$$\operatorname{grad} f = \sum_j a_j(x) \frac{\partial}{\partial x_j}, df = \sum_k \frac{\partial f}{\partial x_k} dx_k,$$

then for any $v = \sum_\ell b_\ell \frac{\partial}{\partial x_\ell}$ we have

$$\langle \operatorname{grad} f, v \rangle_g = \sum_{j,\ell} a_j g_{j\ell} b_\ell$$

but we also have

$$df(v) = \sum_{k,\ell} \frac{\partial f}{\partial x_k} b_\ell dx_k \left(\frac{\partial}{\partial x_\ell} \right) = \sum_k \frac{\partial f}{\partial x_k} b_k.$$

Now lets choose $b = (0, 0, \ldots, 1, \ldots, 0, 0)$ with a 1 in the m-th position then

$$\langle \operatorname{grad} f, v \rangle_g = \sum_j a_j g_{jm}$$

and

$$df(v) = \frac{\partial f}{\partial x_m}$$

so since these are equal we can multiply both by the inverse of the metric g^{mi} to get

$$a_i = \sum_{j,m} a_j g_{jm} g^{mi} = \sum_m \frac{\partial f}{\partial x_m} g^{mi}$$

and thus

$$\operatorname{grad} f = \sum_i a_i \frac{\partial}{\partial x_i} = \sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i}$$

Finally again for a function $f \in C^\infty(M)$, the hessian is defined as the map Hess : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$X \mapsto \nabla_X (\operatorname{grad} f)$$

Let us write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$ then we have by the previous results that in an orthonormal chart around p

$$\begin{aligned}
\nabla_X (\operatorname{grad} f) &= \nabla_X \left(\sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i} \right) \\
&= \nabla_X \left(\sum_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \right) \quad \text{(because } g^{mi} = \delta^{mi} \text{ at } p \text{ in orthonormal chart)} \\
&= \sum_{j,i} a_j(x) \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \Gamma^k_{ij} \frac{\partial}{\partial x_k} \right) \\
&= \sum_{j,i} a_j(x) \left(\left(\frac{\partial f}{\partial x_j \partial x_i} \right) \frac{\partial}{\partial x_i} \right) \quad \text{(because } \Gamma^k_{ij} = 0 \text{ at } p \text{ in orthonormal chart)}
\end{aligned}$$

and so if $Y = \sum_\ell b_\ell(x) \frac{\partial}{\partial x_\ell}$ we have

$$\langle \nabla_X (\operatorname{grad} f), Y \rangle_g = \sum_{j,\ell} a_j(x) \left(\frac{\partial f}{\partial x_j \partial x_i} \right) b_\ell(x).$$

Importantly notice that if we exchange a and b then this expression does not change and so $\langle \nabla_X(\operatorname{grad} f), Y\rangle_g = \langle \nabla_Y(\operatorname{grad} f), X\rangle_g$ and so as an operator Hess is symmetric. We also get that the form in orthonormal coordinates for the operator is the matrix

$$\frac{\partial f}{\partial x_j \partial x_i}$$

Now we consider the trace of the modified hessian operator, given by $\operatorname{div}(h \cdot \operatorname{grad} f)$. Notice that we have, in an orthonormal chart,

$$\operatorname{div}(h \cdot \operatorname{grad} f) = \sum_j \frac{\partial}{\partial x_j} E_j = \sum_j \frac{\partial}{\partial x_j}\left(h \sum_k g^{jk} \frac{\partial f}{\partial x_k}\right)$$

Claim 1.4.1. In a general local chart,

$$\operatorname{div}(h \cdot \operatorname{grad} f) = (\det g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x_j}\left(h (\det g)^{1/2} g^{ij} \frac{\partial f}{\partial x_i}\right)$$

Proof. It is enough to show that the expression on the right is coordinate invariant, since then plugging in an orthonormal chart gives us the desired result.
To see this consider a different chart $(\tilde{x}_1, \ldots, \tilde{x}_n)$ and set

$$Q = (\det g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x_j}\left(h (\det g)^{1/2} g^{ij} \frac{\partial f}{\partial x_i}\right)$$
$$\tilde{Q} = (\det \tilde{g})^{-1/2} \sum_{i,j} \frac{\partial}{\partial \tilde{x}_j}\left(h (\det \tilde{g})^{1/2} \tilde{g}^{ij} \frac{\partial f}{\partial \tilde{x}_i}\right)$$

then consider the set of functions η with support contained within both charts, if

$$\int_\Omega Q \eta dV = \int_\Omega \tilde{Q} \eta dV$$

then $Q = \tilde{Q}$.
Now we plug in our known expressions and get

$$\int_\Omega Q\eta dV = \int_\Omega \eta \sum_j \frac{\partial}{\partial x_j}\left(h (\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \ldots dx_n$$

then we notice that we have the a divergence term in the integral. Then by using integration by parts we can remove that divergence and instead take the gradient of η, the boundary term then dissapears by compactness of η. All together this gives us

$$\begin{aligned}\int_\Omega Q\eta dV &= -\int_\Omega \sum_j \left(\frac{\partial \eta}{\partial x_j}\right)\left(h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n \\ &= -\int_\Omega h\sum_{j,i}\left(g^{ij}\frac{\partial \eta}{\partial x_j}\frac{\partial f}{\partial x_i}\right)(\det g)^{1/2} dx_1 dx_2 \dots dx_n \\ &= -\int_\Omega h \left\langle \operatorname{grad}\eta, \operatorname{grad} f\right\rangle_g dV\end{aligned}$$

now notice that the same calculation holds in the second chart, and so we get

$$\int_\Omega Q\eta dV = \int_\Omega Q'\eta dV$$

$\square$

Theorem 1.4.2 (Divergence theorem). *Suppose that $\Omega \subseteq M$ is a compact domain with a smooth boundary $\partial\Omega$, then $\forall f, h \in C^\infty(M)$ we have*

$$\int_\Omega \operatorname{div}(h \operatorname{grad} f) dV = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu\rangle_g d\tilde{V}$$

where ν is the normal vector and $d\tilde{V}$ is the induced volume form on the metric.

Proof. Find a partition of unity for some neighborhood of Ω, that is a collection of functions ρ_k with $\sum_k \rho_k = 1$ and the support of each ρ_k being contained in a single chart U_k. Now we have

$$\int_\Omega \operatorname{div}(h \operatorname{grad} f) dV = \sum_k \int_\Omega \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \sum_k \int_{\Omega\cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV$$

now for each of these smaller integrals we have

$$\int_{\Omega\cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \int_{\Omega\cap U_k} \sum_j \frac{\partial}{\partial x_j}\left(\rho_k h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n.$$

We will now apply IBP to this integral, note that the interior term will contain a derivative of 1 and so will vanish, the boundary term will only be non-zero outside of the boundary of U_k, that is it will be non-zero only on $\partial\Omega \cap U_k$.
So this integral becomes

$$\int_{\partial\Omega\cap U_k} \sum_j \nu_j \left(\rho_k h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) d\tilde{x}_1 d\tilde{x}_2 \dots d\tilde{x}_{n-1},$$

which simplifies to

$$\int_{\partial\Omega\cap U_k} \langle \operatorname{grad} f, \nu\rangle_g\, (\rho_k h)\, d\tilde{V}.$$

This then gets summed up over k to give

$$\sum_k \int_{\partial\Omega\cap U_k} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g\, d\tilde{V} = \sum_k \int_{\partial\Omega} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g\, d\tilde{V} = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu\rangle_g\, d\tilde{V}$$

□

Theorem 1.4.3. *$\forall h \in C^\infty(M)$ with $h > 0$ in Ω a compact connected open set, the system*

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f \quad && \in \Omega \\ \frac{\partial u}{\partial \nu} &= g && \in \partial\Omega \end{aligned}$$

is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} hg$.

Proof. We follow a similar proof to 1.1.4, first assume $g = 0$, then we have in the space of functions in $W^{2,1}(\Omega)$ that are zero on the boundary the image of the operator $u \mapsto \operatorname{div}(h \operatorname{grad} u)$ is orthogonal to its kernel. Now the set of functions in the kernel are those satisfying

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= 0 \quad && \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \in \partial\Omega \end{aligned}$$

and so for those functions by the divergence theorem we get

$$0 = \int_\Omega u \operatorname{div}(h \operatorname{grad} u) dV = -\int_\Omega h |\operatorname{grad} u|^2 dV$$

and so since $h > 0$ we get $\operatorname{grad} u = 0$ everywhere on Ω and so it is constant on Ω. Thus the image is those functions f that are orthogonal to constant functions, that is the system is solvable if and only if

$$\int_\Omega f dV = 0$$

Next we identically construct a function γ which is C^2 everywhere on Ω and satisfying $\frac{\partial\gamma}{\partial\nu} = -1$. We then define $U(x) = u(x) + \gamma(x)g(x)$ and notice that since

$$\operatorname{div}(h \operatorname{grad} U) = \operatorname{div}(h \operatorname{grad}(u(x) + \gamma(x)g(x))) = \operatorname{div}(h \operatorname{grad} u(x)) + \operatorname{div}(h \operatorname{grad}(\gamma(x)g(x)))$$

and

$$\frac{\partial U}{\partial \nu} = \frac{\partial u}{\partial \nu} + \frac{\partial(\gamma \cdot g)}{\partial \nu} = g - g = 0$$

then we have a solution U if and only if

$$0 = \int_\Omega \operatorname{div}(h \operatorname{grad} U) dV = \int_\Omega \operatorname{div}(h \operatorname{grad} u) dV + \int_\Omega \operatorname{div}(h \operatorname{grad} (\gamma(x) g(x))) dV$$
$$= \int_\Omega f dV + \int_{\partial\Omega} h \frac{\partial(\gamma \cdot g)}{\partial \nu} d\tilde{V} = \int_\Omega f dV + \int_{\partial\Omega} -hg d\tilde{V}$$

□

1.5 Extrinsic Geometry

Suppose we have an n-dimensional Riemannian Manifold (M^n, g) with $F : M^n \hookrightarrow N$ an immersion where N is an $n+m$-dimensional Riemannian Manifold with metric $\overline{g}$. Every point $x \in M$ has a tangent space T_xM and also after identifying x with $F(x)$ we have the larger tangent space T_xN that contains T_xM. We say that M is isometrically immersed if for all $X, Y \in T_xM$ we have

$$\langle X, Y \rangle_g = \langle X, Y \rangle_{\overline{g}},$$

essentially $\overline{g}$ extends g to a larger tangent space.
Recall that both g and $\overline{g}$ induce connections ∇ and $\overline{\nabla}$ respectively.

Lemma 1.5.1. *Let vector fields $X, Y \in \mathfrak{X}(M)$ extend to vector fields $\overline{X}, \overline{Y} \in \mathfrak{X}(M)$. Then*

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T,$$

where T is the orthogonal projection onto T_xM.

Proof. Define the connection $\tilde{\nabla}_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$, then by uniqueness of the Levi-Civita connection of if we have that $\tilde{\nabla}$ satisfies the axioms it must be equal to ∇.
First we check metric compatability,

$$\left\langle \tilde{\nabla}_{\overline{X}} \overline{Y}, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, \tilde{\nabla}_{\overline{X}} \overline{Z} \right\rangle_g = \left\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^T, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}} \overline{Z})^T \right\rangle_g$$

then since all the terms are tangent to M we can replace g with $\overline{g}$.

$$\left\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^T, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}} \overline{Z})^T \right\rangle_g = \left\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^T, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}} \overline{Z})^T \right\rangle_{\overline{g}}$$

then we can throw away the projections since taking inner product with a vector already tangent to $T_{\overline{X}}M$ implicitly projects onto that space.

$$\left\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^T, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}} \overline{Z})^T \right\rangle_{\overline{g}} = \left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}} \overline{Z} \right\rangle_{\overline{g}}$$

by metric compatability of $\overline{\nabla}$ we have that

$$\left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}} \overline{Z} \right\rangle_{\overline{g}} = \overline{\nabla}_{\overline{X}} \left\langle \overline{Y}, \overline{Z} \right\rangle_{\overline{g}}$$

and then we get

$$\overline{\nabla}_{\overline{X}} \langle \overline{Y}, \overline{Z} \rangle_{\overline{g}} = \overline{X} \left(\langle \overline{Y}, \overline{Z} \rangle_{\overline{g}} \right) = X \left(\langle \overline{Y}, \overline{Z} \rangle_{\overline{g}} \right) = X \left(\langle Y, Z \rangle_{\overline{g}} \right) = X \left(\langle Y, Z \rangle_{g} \right)$$

For symmetry we need a small fact about Lie Brackets

Lemma 1.5.2. *If $X, Y \in \mathfrak{X}(M)$ and M is immersed in N then for any extension $\overline{X}, \overline{Y}$ we have $[\overline{X}, \overline{Y}]_N = [X, Y]_M$.*

Proof. See Lee's smooth manifolds page 189. □

We then have that

$$\tilde{\nabla}_X^Y - \tilde{\nabla}_Y^X = (\overline{\nabla}_{\overline{X}}^{\overline{Y}})^T - (\overline{\nabla}_{\overline{Y}}^{\overline{X}})^T = ([\overline{X}, \overline{Y}]_N) = [X, Y]_M$$

□

Now we define the second fundamental form. For every point $p \in M$ we have $T_pN = T_pM \oplus T_p^{\perp}M$, this decomposition defines the normal bundle $NM = \{p \in M | T_p^{\perp}M\} \subseteq TN$. For every smooth normal vector field $V \in NM$ and every vector $X \in T_pM$ we can define $\overline{\nabla}_X V \in T_pN$ we can define

$$\nabla_X^{\perp} V := \left(\overline{\nabla}_X V \right)^{\top}.$$

We now define the second fundamental form to be the map $A^W : T_pM \to T_pM$ parametrized by some vector field in NM which is defined through

$$A^W(X) = -(\overline{\nabla}_X W)^T$$

We want to check that this map is well defined, suppose $W \in NM$ is a normal vector field with two extensions $\tilde{W}_1, \tilde{W}_2$. We want to check that $A^{\tilde{W}_1}(X) = A^{\tilde{W}_2}(X)$ for all vectors $X \in T_pM$.
To see this we check

$$\left\langle (\overline{\nabla}_X \tilde{W}_1)^T, Y \right\rangle_g - \left\langle (\overline{\nabla}_X \tilde{W}_2)^T, Y \right\rangle_g = \left\langle \overline{\nabla}_X \tilde{W}_1 - \overline{\nabla}_X \tilde{W}_2, Y \right\rangle_{\overline{g}}$$

and so we can apply the compatability of $\overline{\nabla}$ with the metric to get

$$\left\langle \overline{\nabla}_X \tilde{W}_1 - \overline{\nabla}_X \tilde{W}_2, Y \right\rangle_{\overline{g}} = \overline{\nabla}_X \left\langle \tilde{W}_1 - \tilde{W}_2, Y \right\rangle_{\overline{g}} - \left\langle \tilde{W}_1 - \tilde{W}_2, \overline{\nabla}_X Y \right\rangle_{\overline{g}}$$

and notice that the first term is trivially zero since both $\tilde{W}_1$ and $\tilde{W}_2$ are perpendicular to T_pM, and similarly the second term is also zero since at any point of M, $\tilde{W}_1 - \tilde{W}_2 = 0$.

Lemma 1.5.3. *A^W is a symmetric map for any $W \in NM$.*

Proof. We compute

$$\left\langle A^W(X), Y\right\rangle_g - \left\langle A^W(Y), X\right\rangle_g = \left\langle \overline{\nabla}_{\overline{Y}} W, \overline{X}\right\rangle_{\overline{g}} - \left\langle \overline{\nabla}_{\overline{X}} W, \overline{Y}\right\rangle_{\overline{g}}$$

and then apply compatability

$$\left\langle \overline{\nabla}_{\overline{Y}} W, \overline{X}\right\rangle_{\overline{g}} - \left\langle \overline{\nabla}_{\overline{X}} W, \overline{Y}\right\rangle_{\overline{g}} = \overline{\nabla}_{\overline{Y}} \left\langle W, \overline{X}\right\rangle_{\overline{g}} - \left\langle W, \overline{\nabla}_{\overline{Y}} \overline{X}\right\rangle_{\overline{g}} - \overline{\nabla}_{\overline{X}} \left\langle W, \overline{Y}\right\rangle_{\overline{g}} + \left\langle W, \overline{\nabla}_{\overline{X}} \overline{Y}\right\rangle_{\overline{g}}$$

and then clearly the first and third terms are zero and the second and fourth terms give

$$\left\langle W, [\overline{X}, \overline{Y}]_N\right\rangle_{\overline{g}}$$

which is also zero by the lemma from before. □

From the second fundamental form we can define a mean curvature vector, consider the map $\mathrm{I\!I} : T_pM \times T_pM \to N_pM$ defined to be the unique vector satisfying

$$\langle \mathrm{I\!I}(X, Y), W\rangle_{\overline{g}} = \left\langle A^W(X), Y\right\rangle_g$$

for all X, Y. We then define the mean curvature vector to be the trace

$$\vec{H} = \sum_{i=1}^{n} \mathrm{I\!I}(e_i, e_i)$$

where e_i is the frame of any orthonormal chart for M. One can check this definition is independent of which orthonormal chart you pick.

We now come back to the ABP setting, assume that M is isometrically embedded in $\mathbb{R}^{n+m}$.

Recall the PDE we were considering the solvability of,

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f \quad &\in \Omega \\ \langle \nabla u, \nu\rangle &= 1 \quad &\in \partial\Omega \end{aligned}$$

which is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} h$.

We first define the sets,

$$\Omega^* = \{x \in \Omega || \nabla u(x)| < 1\}, \quad \hat{\Omega} = \{(x, Y) \in N\Omega |, |Y|^2 + |\nabla u(x)|^2 < 1\}$$

and then we define the contact set

$$\Gamma = \{(x, Y) \in N\hat{\Omega} | \operatorname{Hess}_u(X) - (\mathrm{I\!I}, Y) \geq 0\}$$

where the inequality is in terms of matrices, that is the map defined by $(v, w) \mapsto \operatorname{Hess}_u(X)(v, w) - (\mathrm{I\!I}(v, w), Y)$ is symmetric positive semidefinite.

Recall that $N\hat{\Omega} = \{(x, Y) | x \in \Omega, Y \in T_x^{\perp} M\}$. We now define the ABP map $\Phi : N\hat{\Omega} \to \mathbb{R}^{n+m}$ to be

$$\Phi(x, Y) = \nabla u(x) + Y$$

noting that $\nabla u(x)$ is orthogonal to Y since Y is in the normal bundle. We thus have by definition of $\hat{\Omega}$ that

$$|\Phi(x, Y)| = |Y|^2 + |\nabla u(x)|^2 < 1$$

and so $\Phi(N\hat{\Omega}) \subseteq B_1^{(n+m)}$.

Lemma 1.5.4. $\Phi(N\Gamma) \supseteq B_1^{(n+m)}$

Proof. Take some $\xi \in B_1^{(n+m)}$, that is $|\xi| < 1$, then define $w(x) = u(x) - \langle x, \xi \rangle$. Then there exists a unique minimum at $x_0 \in \overline{\Omega}$. Assume that $x_0 \notin \partial\Omega$, then $\nabla w(x_0) = 0$, thus $\nabla u(x_0) = \xi^T$ and $\mathrm{Hess}_w(x_0) \geq 0$. From there we get

$$\mathrm{Hess}_w(x_0) = \mathrm{Hess}_u(x_0) - \left\langle \nabla_{e_i e_j} x, \xi^{\perp} \right\rangle$$

and one can check that $\left\langle \nabla_{e_i e_j} x, \xi^{\perp} \right\rangle = \left\langle \mathrm{I\!I}(e_i, e_j), \xi^{\perp} \right\rangle$ and so we get that $(x, y) = \xi^T + \xi^{\perp}$ is exactly in our contact set. $\square$

Lemma 1.5.5 (Jacobian Lemma)**.** *The Jacobian J_Φ is given by*

$$J_\Phi = \det(D\Phi(x, Y)) = \det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle)$$

and so by our lemma before we may apply the inequality and get

$$\det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle) \leq \left(\frac{\mathrm{tr}\,(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle)}{n} \right)^n = \left(\frac{\Delta u(x) - \left\langle \vec{H}, Y \right\rangle}{n} \right)^n$$

Proof. Take any $(x_0, y_0) \in N\hat{\Omega}$ fixed, then fix a local orthonormal chart $e_1, \ldots, e_n$. We can also find a nice frame for the normal bundle $\nu_1, \nu_2, \ldots, \nu_m$. That is a frame satisfying

$$\langle \nu_i(x), \nu_j(x) \rangle = \partial_{ij}, \langle \nu_i(x), e_j(x) \rangle = 0,$$

we thus get local coordinates for $N\hat{\Omega}$, $(x_1, \ldots, x_n, y_1, \ldots, y_m)$.

Now compute

$$\left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \left\langle \overline{\nabla}_{e_i}(\nabla u), e_j \right\rangle + \sum_{\alpha=1}^{m} y_\alpha \left\langle \nabla_{e_i}(\nu_\alpha), e_j \right\rangle$$

and so since in the first term we are inner producting with e_j we may drop all normal components of $\overline{\nabla}$ and reduce it to the standard ∇ on M, this gives us the hessian, on the other hand for for the second term we pick up exactly the expression for the second fundamental form. Thus we define

$$A := \left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \mathrm{Hess}_u(e_i, e_j) - (\mathrm{II}(e_i, e_j), Y)$$

Next note that

$$\left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), \nu_j \right\rangle = \delta_{ij}, \quad \left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), e_j \right\rangle = 0$$

because Φ is the identity map on the Y component. We thus have that the Jacobian matrix takes the block form

$$\begin{bmatrix} A_{n\times n} & 0 \\ * & I_{m\times m} \end{bmatrix}$$

and so its determinant is just the determinant of A, proving the lemma. □

1.6 Isoperimetric Inequality on Minimal Submanifolds

Suppose M^n is immersed in $\mathbb{R}^{n+m}$, with $\Omega \subseteq M$, $\overline{\Omega}$ compact and $\partial\Omega \in C^\infty$. We then have along the boundary μ the normal vector to $\partial\Omega$ and a collection of normal vectors $T_x^\perp M \subseteq T_x\mathbb{R}^{n+m}$.

We denote $|B_1^{(k)}|$ to be the volume of the ball of radius 1 in $\mathbb{R}^k$.

Last time we considered the PDE

$$\begin{aligned} \mathrm{div}_g(f\nabla_g u) &= h \\ \langle \nabla_g u, \mu\rangle_g\Big|_{\partial\Omega} &= 1 \end{aligned}$$

which is solvable if and only if $\int_\Omega h = \int_{\partial\Omega} f$.

We will now prove the following Sobolev inequality, for any $f > 0, f \in C^\infty(M)$, we have

$$\int_\Omega \left(|\nabla f|^2 + f^2|\vec{H}|^2\right)^{1/2} + \int_{\partial\Omega} f \geq n\left(\frac{n+m}{n}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n}\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

If M is minimal ($\vec{H} = 0$) and $f = 1$ then we get $|\partial\Omega| \geq C_{n,m}|\Omega|^{\frac{n}{n-1}}$, the Isoperimetric inequality.

Proof. We assume that $m \geq 2$, if $m = 1$ then we can lift the surface one more dimension to make $m = 2$.

Our job now is to pick a special h to use the PDE. Note that the equation is scaling invariant, we can then see that by changing $f \to cf$ we get

$$\int_\Omega n(cf)^{\frac{n}{n-1}} = c^{\frac{n}{n-1}}\int_\Omega nf^{\frac{n}{n-1}}$$

and

$$\int_\Omega \sqrt{|\nabla cf|^2 + (cf)^2|\vec{H}|^2} + \int_{\partial\Omega} cf = c\left(\int_\Omega \sqrt{|\nabla f|^2 + (f)^2|\vec{H}|^2} + \int_{\partial\Omega} f\right)$$

so by rescaling we can make these two expressions equal for f. Then by setting $h = nf^{\frac{n}{n-1}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$ then there is a solution u to the PDE with f, h.

Claim 1.6.1. $0 \leq \det(\mathrm{Hess}_u(x) - \langle \mathrm{II}(x), Y\rangle) \leq f^{\frac{n}{n-1}}(x)$

For now we will assume the claim is true.
Then we have $B_1^{(n+m)} \subseteq \Phi(\hat{\Omega})$ and so

$$|B_1^{(n+m)}| \leq \int_{\tilde{\Omega}} \det(J_\Phi) \leq \int_{x\in\Omega^*}\int_{T_x^\perp\Omega} \det(J_\Phi)$$

we will now restrict the domain so that $|\Phi(\hat{\Omega})| \geq \delta$, then using the fact that $|\Phi(\hat{\Omega})|^2 = |\nabla u(x)|^2 + Y^2$ we get that $(\delta^2 - |\nabla u(x)|^2)_+ < Y^2 < 1 - |\nabla u(x)|^2$. Set B'_x to be the set of Y satisfying the above, we then get

$$(1-\delta^{n+m})|B_1^{(n+m)}| \leq \int_{x\in\Omega^*}\int_{B'_x} |\det(J_\Phi)| dY\, dx$$

and substituting the determinant we get

$$\int_{x\in\Omega^*}\int_{B'_x} \det(\mathrm{Hess}_u(x) - \langle \mathrm{II}(x), Y\rangle) dY\, dx$$

and using the claim we get that this is less than

$$\int_{x\in\Omega^*}\int_{B'_x} f^{\frac{n}{n-1}} dY\, dx.$$

Next we get

$$\int_{x\in\Omega^*}\int_{B'_x} f^{\frac{n}{n-1}} dY\, dx = \int_{x\in\Omega^*} f^{\frac{n}{n-1}}(1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+)^{m/2}|B_1^{(m)}| dx$$

then by the inequality $A^s - B^s \leq s(A-B)$ for $A \geq B \geq 0$ and $s \geq 1$ we get since $m \geq 2$ that

$$\begin{aligned}&\int_{x\in\Omega^*} f^{\frac{n}{n-1}}(1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+)^{m/2}|B_1^{(m)}| dx \\ &\leq \int_{x\in\Omega^*} f^{\frac{n}{n-1}}\frac{m}{2}(1 - |\nabla u(x)|^2 - (\delta^2 - |\nabla u(x)|^2)_+)|B_1^{(m)}| dx\end{aligned}$$

then by checking both cases we can find that

$$1-|\nabla u(x)|^2-(\delta^2-|\nabla u(x)|^2)_+\leq 1-\delta^2$$

and so we can then get

$$(1-\delta^{n+m})(|B_1^{(n+m)}|)\leq \frac{m}{2}(1-\delta^2)|B_1^{(m)}|\int_{x\in\Omega^*}f^{\frac{n}{n-1}}dx.$$

Dividing this inequality by $1-\delta$ we get

$$(1+\delta^1+\delta^2+\cdots+\delta^{n+m-1})(|B_1^{(n+m)}|)\leq \frac{m}{2}(1+\delta)|B_1^{(m)}|\int_{x\in\Omega^*}f^{\frac{n}{n-1}}dx$$

and then by letting $\delta\to 1$ we get

$$(n+m)(|B_1^{(n+m)}|)\leq m|B_1^{(m)}|\int_{x\in\Omega^*}f^{\frac{n}{n-1}}dx$$

now we can rewrite this as,

$$\left(\frac{n+m}{m}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n}\leq\left(\int_{x\in\Omega^*}f^{\frac{n}{n-1}}dx\right)^{1/n}$$

and so we have

$$n\int_\Omega f^{\frac{n}{n-1}}=n\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{1}{n}}\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\geq n\left(\frac{n+m}{m}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n}\left(\int_\Omega nf^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

and so by our rescaling of f we get the desired result.
All that remains is to prove the claim, as we saw before in the determinant Lemma we have that

$$\det(\mathrm{Hess}_u(x)-\langle \mathrm{I\!I}(x),Y\rangle)\leq\left(\frac{\mathrm{tr}(\mathrm{Hess}_u(x)-\langle \mathrm{I\!I}(x),Y\rangle)}{n}\right)^n=\left(\frac{\Delta_g u(x)-\left\langle \vec{H},Y\right\rangle}{n}\right)^n.$$

From the PDE of u we get that $\div(f\nabla u)=nf^{\frac{1}{n}}-\sqrt{|\nabla f|^2+f^2|\vec{H}|^2}$ and so by evaluating using divergence rules we get

$$\mathrm{div}(f\nabla u)=\langle\nabla f,\nabla u\rangle+f\Delta_g u=nf^{\frac{n}{n-1}}-\sqrt{|\nabla f|^2+f^2|\vec{H}|^2}$$

and solving for $\Delta_g u$ we get

$$\Delta_g u=nf^{\frac{n}{n-1}-1}-f^{-1}\sqrt{|\nabla f|^2+f^2|\vec{H}|^2}-\left\langle\frac{\nabla f}{f},\nabla u\right\rangle.$$

Plugging this into the inequality we get

$$\det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) \le \left(\frac{n f^{\frac{n}{n-1}-1} - f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} - \left\langle \frac{\nabla f}{f}, \nabla u \right\rangle - \left\langle \vec{H}, Y \right\rangle}{n} \right)^n .$$

We now use a Cauchy-Schwartz inequality, for any $a, A, b, B \in \mathbb{R}^n$ we have

$$|a \cdot A + b \cdot B| \le \sqrt{A^2 + B^2}\sqrt{a^2 + b^2}$$

then we get

$$\left| \langle \nabla f, \nabla u \rangle + \nabla \left\langle f\vec{H}, Y \right\rangle \right| \le \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}\sqrt{|\nabla u|^2 + Y^2} \le \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$$

and so we get that

$$f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} + \left\langle \frac{\nabla f}{f}, \nabla u \right\rangle + \left\langle \vec{H}, Y \right\rangle \ge 0$$

and thus

$$\det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) \le \left(\frac{n f^{\frac{1}{n-1}}}{n} \right)^n = f^{\frac{n}{n-1}}$$

□

1.7 Log-Sobolev inequality

Lemma 1.7.1 (ABP Lemma)**.** *For a closed manifold* $\Phi(NM) = \mathbb{R}^{n+m}$.

Proof. Take $\xi \in \mathbb{R}^{n+m}$ then set $w(x) = u(x) - \langle x, \xi \rangle$. Since M is compact there exists a minimum of w at some points x_0. We write $\xi = \xi^T + \xi^\perp$ then at x_0 we have $\nabla w(x) = 0$ so $\nabla u = \xi$. Then choosing $Y = \xi^\perp$ we get $\Phi(x_0, y) = \xi$. □

We now have the new estimate

Claim 1.7.2. $0 \le \det J_\Phi(x, y) \le f \exp\left(-\frac{|2\vec{H}(x)+y|^2}{4} - n \right)$

Proof. Take u the solution to our standard PDE with $h = f \log f - \frac{|\nabla f|^2}{f} - f|\vec{H}|^2$, note that we can always scale f so that $\int_M h = 0$.

Next we compute

$$\begin{aligned} \Delta u - \left\langle \vec{H}(x), Y \right\rangle &= -\frac{\nabla f \cdot \nabla u}{f} + \log f - \frac{|\nabla f|^2}{f^2} - |\vec{H}|^2 - \left\langle \vec{H}, y \right\rangle \\ &= \log f + \frac{|\nabla u|^2 + |Y|^2}{4} - \frac{|2\nabla f + f\nabla u|^2}{4f^2} - \frac{|2\vec{H} + Y|^2}{4} \\ &\le \log f + \frac{|\nabla u|^2 + |Y|^2 - |2\vec{H} + Y|^2}{4} \end{aligned}$$

We thus get that

$$\begin{aligned}\left(\frac{\Delta u-\left\langle\vec{H}(x),Y\right\rangle}{n}\right)^n &\le \left(\frac{\log f+\frac{|\nabla u|^2+|Y|^2-|2\vec{H}+Y|^2}{4}}{n}\right)^n\\ &\le \left(f^{1/n}\exp\left(-\frac{|2\vec{H}(x)+y|^2}{4n}-1\right)\right)\\ &= f\exp\left(-\frac{|2\vec{H}(x)+y|^2}{4}-n\right).\end{aligned}$$

Where we employed the inequality $x \le e^{x-1}$.

$\square$

Theorem 1.7.3. *Let $f>0$ and $f\in C^\infty(M)$ then*

$$\int_M f\left(\log f+n+\frac{n}{2}\log(4\pi)\right)-\int_M\frac{|\nabla f|^2}{f}-\int_M f|\vec{H}|^2 \le \left(\int_M f\right)\log\left(\int_M f\right)$$

Proof. By standard calculus proof

$$1=(4\pi)^{-\frac{n+m}{2}}\int_{\mathbb{R}^{n+m}} e^{-\frac{|\xi|^2}{4}}d\xi$$

we can then use this to get

$$1\le(4\pi)^{-\frac{n+m}{2}}\int_{M\times\mathbb{R}^{n+m}}\exp\left(-\frac{|\Phi(x,Y)|^2}{4}\right)|\det J_\Phi|dydV$$

then by the previous lemma we get

$$1\le(4\pi)^{-\frac{n+m}{2}}\int_M\int_{\mathbb{R}^{n+m}}\exp\left(-\frac{|\Phi(x,Y)|^2}{4}\right)f(x)\exp\left(-\frac{|2\vec{H}(x)+y|^2}{4}-n\right)dydV$$

then $\exp\left(-\frac{|\Phi(x,Y)|^2}{4}\right)\le 1$ and so we can rewrite this as

$$1\le(4\pi)^{-\frac{n+m}{2}}\int_M f(x)e^{-n}\int_{\mathbb{R}^{n+m}}\exp\left(-\frac{|2\vec{H}(x)+y|^2}{4}\right)dydV$$

and by change of variables $z=2\vec{H}(x)+y$ we get

$$1\le(4\pi)^{-\frac{n+m}{2}}\int_M f(x)e^{-n}\int_{\mathbb{R}^{n+m}}\exp\left(-\frac{|z|^2}{4}\right)dzdV=(4\pi)^{-\frac{n}{2}}\int_M f(x)e^{-n}dV$$

we thus end up with

$$\left((4\pi)^{1/2}e\right)^n = \int_M f(x)$$

We now have

$$0 = \int_M \left(f \log f - \frac{|\nabla f|^2}{f} - f|\vec{H}|^2 \right)$$

and so

$$\begin{aligned}\int_M f\left(\log f + n + \frac{n}{2}\log(4\pi)\right) - \int_M \frac{|\nabla f|^2}{f} - \int_M f|\vec{H}|^2 &= \int_M \left(f(n + \frac{n}{2}\log(4\pi))\right) \\ &\leq \int_M f \log\left(\int_M f(x)\right)\end{aligned}$$

$\square$

Corollary 1.7.4. For any φ we have

$$\int_M \varphi \log \varphi d\gamma - \int_M \frac{|\nabla \varphi|^2}{\varphi} d\gamma - \int_M \varphi|\vec{H} + \frac{x^\perp}{2}|^2 d\gamma \leq \left(\int_M \varphi\right) \log\left(\int_M \varphi d\gamma\right)$$

where $d\gamma$ is the Gaussian normalized measure.

2 Extrinsic Geometry and Elliptic PDEs

2.1 Curvature Constructions

We recall that the second fundamental form is a map

$$\mathrm{II} : T_pM \otimes T_pM \to T_p^{\perp}M$$

where $M \subseteq \mathbb{R}^{n+1}$ is a submanifold of $\mathbb{R}^n$. If $\overline{\nabla}$ is the connection on $\mathbb{R}^n$ and ∇ is the connection on M then

$$\overline{\nabla}_Y X = \nabla_Y X + \mathrm{II}(X, Y)$$

Next assume that M is an an n dimensional submanifold, also called a hypersurface. Then the normal bundle NM is one dimensional. Then at any point we can pick ν such that ν spans $T_p^{\perp}M$ and is of length 1. If M is orientable we can pick the 'outer' normal to globally define ν as a vector field.

Since $\mathrm{II}(X, Y)$ is in $T^{\perp}M$ then

$$\mathrm{II}(X, Y) = c\nu$$

for some constant depending on p, X, Y. We define h to be the bilinear form satisfying

$$\mathrm{II}(X, Y) = -h(X, Y)\nu.$$

Clearly from the properties of II we get that h is a symmetric bilinear form.

We can also check that

$$h(X, Y) = \left\langle \overline{\nabla}_X \nu, Y \right\rangle = \left\langle \overline{\nabla}_Y \nu, X \right\rangle$$

Let $p \in M$ and $e_1, \ldots, e_n$ be some orthonormal frame at p. We set the 'components' h_{ij} to be $h(e_i, e_j)$.

Example 2.1.1. Set $M = S^n \subseteq \mathbb{R}^{n+1}$. Then parametrize as the graph $x_{n+1} = \sqrt{1 - \sum_i x_i^2}$. We then get

$$h(e_i, e_j) = \left\langle \nabla_{e_i} \nu, \nabla_{e_j} \right\rangle = \left\langle e_i + c(p)\nu, \nabla_{e_j} \right\rangle = \delta_{ij}$$

We now have a very important property unique to submanifolds of $\mathbb{R}^n$.

Claim 2.1.2 (Codazzi Property)**.** Let $M^n \subseteq \mathbb{R}^{n+1}$ be a hypersurface with $e_1, \ldots, e_n$ a local orthonormal frame near p. Then

$$\nabla_{e_k} h_{ij} = \nabla_{e_i} h_{jk} = \nabla_{e_j} h_{ki}$$

Proof. We compute

$$\overline{\nabla}_X \left(\overline{\nabla}_Y Z \right) = \overline{\nabla}_X \left(\nabla_Y Z + \mathrm{II}(Y, Z) \right) = \nabla_X \left(\nabla_Y Z \right) + \mathrm{II}(X, \nabla_Y Z) + \overline{\nabla}_X \left(\mathrm{II}(Y, Z) \right)$$

and similarly

$$\overline{\nabla}_Y\left(\overline{\nabla}_X Z\right) = \overline{\nabla}_Y\left(\nabla_X Z + \mathrm{II}(X,Z)\right) = \nabla_Y\left(\nabla_X Z\right) + \mathrm{II}(Y,\nabla_X Z) + \overline{\nabla}_Y\left(\mathrm{II}(X,Z)\right)$$

finally we have

$$\overline{\nabla}_{[X,Y]} Z = \nabla_{[X,Y]} Z + \mathrm{II}([X,Y],Z) = \nabla_{[X,Y]} Z + \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(\nabla_Y X, Z).$$

We now compute the first equation, minus the second, minus the third.
This gives us on the left

$$\overline{\nabla}_X\left(\overline{\nabla}_Y Z\right) - \overline{\nabla}_Y\left(\overline{\nabla}_X Z\right) - \overline{\nabla}_{[X,Y]} Z$$

which is always zero because $\mathbb{R}^{n+1}$ is flat (technically this comes from the Riemann tensor being equal to zero).
On the right we get

$$\begin{aligned}
&\nabla_Y\left(\nabla_X Z\right) + \mathrm{II}(Y,\nabla_X Z) + \overline{\nabla}_Y\left(\mathrm{II}(X,Z)\right)\\
&-\nabla_X\left(\nabla_Y Z\right) - \mathrm{II}(X,\nabla_Y Z) - \overline{\nabla}_X\left(\mathrm{II}(Y,Z)\right)\\
&-\nabla_{[X,Y]} Z - \mathrm{II}(\nabla_X Y, Z) + \mathrm{II}(\nabla_Y X, Z)\\
=&\nabla_X\left(\nabla_Y Z\right) - \nabla_Y\left(\nabla_X Z\right) - \nabla_{[X,Y]} Z\\
&+\overline{\nabla}_X\left(\mathrm{II}(Y,Z)\right) - \overline{\nabla}_Y\left(\mathrm{II}(X,Z)\right) - \mathrm{II}(\nabla_X Y, Z) + \mathrm{II}(\nabla_Y X, Z)
\end{aligned}$$

Now the first 3 terms here are all parallel to M, so when we next take inner product with ν they all dissapear. We are thus left with

$$0 = \left\langle \overline{\nabla}_X\left(\mathrm{II}(Y,Z)\right), \nu\right\rangle - \left\langle \overline{\nabla}_Y\left(\mathrm{II}(X,Z)\right), \nu\right\rangle - \langle \mathrm{II}(\nabla_X Y, Z), \nu\rangle + \langle \mathrm{II}(\nabla_Y X, Z), \nu\rangle$$

Now by definition of h we have that this is equal to

$$0 = \left\langle \overline{\nabla}_X(h(Y,Z)\nu), \nu\right\rangle - \left\langle \overline{\nabla}_Y(h(X,Z)\nu), \nu\right\rangle - h\left(\nabla_X Y, Z\right) + h\left(\nabla_Y X, Z\right)$$

Now in the first term by product rule we will get $\nu\overline{\nabla}_X h(Y,Z) + h(Y,Z)\overline{\nabla}_X\nu$. Now $\overline{\nabla}_X\nu$ is orthogonal to to ν because

$$0 = \overline{\nabla}_X 1 = \overline{\nabla}_X\left\langle \nu, \nu\right\rangle = 2\left\langle \overline{\nabla}_X \nu, \nu\right\rangle$$

and so since we are taking the inner product with ν all those terms involving $\overline{\nabla}_X\nu$ dissapear.
We are thus left with

$$0 = \overline{\nabla}_X(h(Y,Z)) - \overline{\nabla}_Y(h(X,Z)) - h\left(\nabla_X Y, Z\right) + h\left(\nabla_Y X, Z\right)$$

Plugging in $X = e_i, Y = e_j, Z = e_k$ makes the covariant derivatives in the last two terms vanish and so we are left with

$$0 = \overline{\nabla}_{e_i}(h(e_j,e_k)) - \overline{\nabla}_{e_j}(h(e_i,e_k))$$

then the symmetry of h gives us the result. □

Remark 2.1.3. 2-Tensors with the property above, where we can permute covariant derivatives with the indices of the tensor, are called Codazzi tensors.

Lemma 2.1.4. *Set $\sigma_n(W) = \det(W)$, with W a symmetric tensor on M, if W is Codazzi then*

$$\sum_j e_j \left(\frac{\partial \sigma_n(W)}{\partial W_{ij}} \right) = 0$$

Proof. At p we assume $\sigma_n(W) \neq 0$ then

$$\frac{\partial \sigma_n(W)}{\partial W_{ij}} = C^{ij}$$

where C^{ij} is the cofactor matrix defined by $C^{ij} W_{j\ell} = \delta_{i\ell}\sigma_n(W)$. We clearly have $C^{ij}/\sigma_n(W) = W^{-1}$ as well.
Then consider the identity $C^{ij} W_{j\ell} = \delta_{i\ell}\sigma_n(W)$ and differentiate it

$$(\sigma_n(W))_m = e_m(\delta_{i\ell}\sigma_n(W)) = e_m(C^{ij} W_{j\ell}) = (C^{ij}_m W_{j\ell} + C^{ij} W_{j\ell,m})$$

we now multiply this by the matrix $C^{m\ell}$ to get

$$(\sigma_n(W))_m C^{m\ell} = C^{m\ell}(C^{ij}_m W_{j\ell} + C^{ij} W_{j\ell,m}) = \left(C^{ij}_m \sigma_n(W)\delta_{jm} + C^{ij} \frac{\partial \sigma_n(W)}{\partial W_{m\ell}} W_{j\ell,m} \right)$$

Now since W is Codazzi and symmetric in the last copy of W we can permute the indices to get

$$\left(C^{ij}_j \sigma_n(W) + C^{ij} \frac{\partial \sigma_n(W)}{\partial W_{m\ell}} W_{m\ell,j} \right) = \left(C^{ij}_j \sigma_n(W) + C^{ij}(\sigma_n(W))_j \right)$$

We are thus left with

$$(\sigma_n(W))_m C^{m\ell} = C^{ij}_j \sigma_n(W) + C^{ij}(\sigma_n(W))_j$$

and so

$$C^{ij}_j \sigma_n(W) = 0$$

If $\sigma_n(W) = 0$ then exchange W for $W + tg$ and let $t \to 0$ and you will recover the same identity. ☐

We now introduce the elementary symmetric functions

$$\sigma_k(W) = \sum_{i_i < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

where λ_i are the eigenvalues of W. For example for $k = 1$ this is just the sum of eigenvalues which is the trace.

Claim 2.1.5. $\sigma_n(W+tI) = \sum_{k=0}^{n} t^{n-k}\sigma_k(W)$

Proof. Pick a basis in which W is diagonal, then we have the eigenvalues $\tilde{\lambda}_i$ of $W+tI$ are given by $\tilde{\lambda}_i = \lambda_i + t$ and so we get

$$\sigma_n(W+tI) = \prod_i \tilde{\lambda}_i = \prod_i (\lambda_i + t) = \sum_{k=0}^{n} t^{n-k}\sigma_k(W)$$

□

Lemma 2.1.6. *If W is Codazzi then* $\sum_{j,k} e_j\left(\frac{\partial\sigma_k(W)}{\partial W_{ij}}\right) = 0$

Proof. Set $t=1$ then $\sigma_n(W+I) = \sum_{j,k}\sigma_k(W)$. Then we have

$$0 = \sum_j e_j\left(\frac{\partial\sigma_n(W+I)}{\partial W_{ij}}\right) = \sum_{j,k} e_j\left(\frac{\partial\sigma_k(W)}{\partial W_{ij}}\right)$$

□

Now let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain such that $\partial\Omega$ is compact with normal vector ν. Then we define the 'expansion' of Ω to be

$$\Omega_t = \Omega \cup \{x + s\nu | x \in \partial\Omega, s \in [0,t]\}$$

Then the boundary of Ω_t is

$$\partial\Omega_t = \{x + t\nu | x \in \partial\Omega\}.$$

For small t, $\partial\Omega_t$ is smooth.
Now it is easy to see that

$$V(\Omega_t) = V(\Omega) + \int_0^t A(\partial\Omega_s)ds$$

where V denotes the volume and $A(\partial\Omega_s)$ the surface area.
But we also know that

$$A(\partial\Omega_s) = \int_\Omega dV_s$$

now we can see that if $Y = X + s\nu_x$ is the position vector of some point on $\partial\Omega_s$ then we can choose coordinates around that point such that g and h_{ij} is diagonal, then if we set $\tilde{g}$ to be the metric on $\partial\Omega_s$ then we can compute

$$\begin{aligned}
\tilde{g}_{ij} &= \tilde{g}(Y_{e_i}, Y_{e_j}) = \langle (X + s\nu_x)_{e_i}, (X+s\nu_x)e_j\rangle_{\overline{g}} \\
&= \left\langle X_{e_i} + sh_{ii}e_i, X_{e_j} + sh_{jj}e_j\right\rangle_{\overline{g}} = \langle e_i + sh_{ii}e_i, e_j + sh_{jj}e_j\rangle_{\overline{g}} \\
&= g_{ij}(1+sh_{ii})(1+sh_{jj}).
\end{aligned}$$

Since g_{ij} is diagonal then $\tilde{g}_{ij}$ is too and it has volume coefficient

$$\sqrt{\det(\tilde{g})} = \sqrt{\prod_i \tilde{g}_{ii}} = \prod_i \sqrt{(1+sh_{ii})^2 g_{ii}} = \prod_i (1+sh_{ii})\sqrt{\det(g)}$$

and so $dV_s = \sum_{k=0}^n s^k \sigma_k(h) dV_0$.

We then get

$$V(\Omega_t) = V(\Omega) + \int_0^t \sum_{k=0}^n s^k \int_{\partial\Omega} \sigma_k(h) dV_0 ds$$

The integrals $\int_{\partial\Omega} \sigma_k(h) dV_0$ are called quermassintegrals, for $k=0$ this Surface area, for $k=1$ this is Total Mean Curvature, and for $k=2$ this is total Scalar curvature.

There is also a sense in which $k=-1$ corresponds to Volume, we will see this later.

2.2 Variation Formule

We define a variation vector field η to be equal to

$$\eta(t) = tf\nu$$

where f is some function defined on $\partial\Omega$ called the 'speed' function.

Under this variation we get a time parametrized boundary

$$M^t = \{x + tf\nu(x) | x \in M\} = \partial\Omega^t$$

now we want to get a handle on

$$\left.\frac{d}{dt}\right|_{t=0} |\Omega^+|, \quad \left.\frac{d}{dt}\right|_{t=0} |\partial\Omega^+|$$

called the 1st variation of volume and surface area, respectively.

When t is small enough we can parametrize M^t by M and so we can use Fubini's theorem with some extra effort to get

$$|\Omega^+| = \int_0^t \int_M f dV$$

giving us that

$$\left.\frac{d}{dt}\right|_{t=0} |\Omega^+| = \int_M f dV$$

Another way to prove the same result is through the divergence theorem. Set X to be the position vector field defined by $X(p) = p$ seen as a vector in $\mathbb{R}^{n+1}$. Then $X(p)$ has divergence $(n+1)$ and so by divergence theorem

$$\int_M \left\langle X^t, \nu \right\rangle dV_{M^t} = \int_{\Omega^t} (n+1) dV_{\Omega^t} = (n+1)|\Omega^t|$$

analyzing how this integral changes over the boundary is enough to calculate the same equation.

We now want to consider higher quermassintegrals.

At some time t we have $X^t = X + tf(x)\nu(x)$. We now have $e_j = \nabla_{e_j} X$ for all j so we can set

$$e_i^t = \nabla_{e_i} X^t = e_i + tf_i(x)\nu + tf(x)h_{ii}e_i = (1 + tfh_{ii})e_i + tf_i(x)\nu$$

and so we have

$$\begin{aligned} g_{ij}^t &= \left\langle e_i^t, e_j^t \right\rangle_{ij} = \langle (1 + tfh_{ii})e_i + tf_i(x)\nu, (1 + tfh_{jj})e_j + tf_j(x)\nu \rangle \\ &= \delta_{ij}(1 + tfh_{ii})^2 + t^2 f_i(x)f_j(x) = \delta_{ij}(1 + tfh_{ii})^2 + O(t^2) \end{aligned}$$

we then have that

$$\sqrt{\det(g^t)} = \prod_{i=1}^{n} (1 + tfh_{ii}) + O(t^2)$$

So now we have

$$\frac{d}{dt}|M^t| = \frac{d}{dt}\int_M dV_{M^t} = \frac{d}{dt}\int_M \prod_{i=1}^{n}(1 + tfh_{ii})dV_m$$

then after taking the derivative and setting $t = 0$ so only the terms containing one copy of f and one copy of h_{ii} survive and we get

$$\int_M f \sum_i h_{ii} dV_M = \int_M fH dV_m$$

Claim 2.2.1.

$$\frac{d}{dt}\int_{M^t} \sigma_k(h^t) dV_{M^t} = (k + 1)\int_M f\sigma_{k+1}(h)dV_M$$

Proof. First we want to compute how other geometric quantities change over time. First we deal with ν. By Gram-Schmidt process we can orthogonalize ν with respect to e_i^t to get

$$\begin{aligned} \nu^t &= \nu - t\sum_i \frac{f_i(x)}{1 + tf(x)h_{ii}} e_i + O(t^2) \\ &= \nu - t\sum_i f_i(x)e_i(1 - tf(x)h_{ii} + O(t^2)) + O(t^2) \\ &= \nu - t\sum_i f_i(x)e_i + O(t^2) = \nu - t\nabla f + O(t^2) \end{aligned}$$

From here we can compute the second fundamental form,

$$\begin{aligned} h_{ij}^t &= \left\langle \overline{\nabla}_{e_i}\nu^t, e_j^t \right\rangle = -\left\langle \nu^t, \overline{\nabla}_{e_i} e_j^t \right\rangle = -\left\langle \nu - t\nabla f, \nabla_{e_i} e_j^t \right\rangle + O(t^2) \\ &= -\left\langle \nu - t\nabla f, \nabla_{e_i}(1 + tfh_{ii})e_j + \nabla_{e_i} tf_j\nu \right\rangle + O(t^2) \\ &= -\left\langle \nu, \nabla_{e_i}(1 + tfh_{ii})e_j \right\rangle + t\left\langle \nabla f, \nabla_{e_i}(1 + tfh_{ii})e_j \right\rangle - t\left\langle \nu, \nabla_{e_i} f_j\nu \right\rangle + O(t^2) \end{aligned}$$

For the first term we have

$$-\langle v, e_j \nabla_{e_i}(1+tfh_{ii})\rangle - \langle v, (1+tfh_{ii})(-h_{ij}\nu)\rangle = 0 + (h_{ij} + tfh_{ij}h_{ii})$$

and for the second term we have

$$t\langle \nabla f, \nabla_{e_i} e_j\rangle + O(t^2) = t\langle \nabla f, -h_{ij}\nu\rangle + O(t^2) = O(t^2)$$

and finally for the third term we have

$$-t\langle \nu, \nabla_{e_i} f_j \nu\rangle = -t\langle \nu, \nu\nabla_{e_i} f_j\rangle - t\langle \nu, f_j \nabla_{e_i}\nu\rangle = -t\,\mathrm{Hess}(f) - 0$$

where we get the zero because ν is always orthogonal to $\nabla\nu$ because its a unit a vector. We are thus left with

$$h_{ij}^t = h_{ij} + tfh^2 - t\,\mathrm{Hess}(f)$$

We then get

$$\int_{M^t} \sigma_k(h^t) dV_{M^t} = \int_M \sigma_k(h - tfh^2 - t\,\mathrm{Hess}(f))(1+tfH)dV_m + O(t^2)$$

and so

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \int_{M^t} \sigma_k(h^t) dV_{M^t} &= \frac{d}{dt}\bigg|_{t=0} \int_M \sigma_k(h - tfh^2 - t\,\mathrm{Hess}(f))(1+tfH)dV_m + O(t^2)\\ &= \int_M -\frac{\partial\sigma_k}{\partial W_{ij}}(h)(fh_{i\ell}h_{\ell j} + f_{ij}) + \sigma_k(h)fHdV_M\end{aligned}$$

We now set $\vec{F} = \sum_{i=1}^n \frac{\partial\sigma_k}{\partial W_{ij}} f_i e_j$ then

$$\mathrm{div}(\vec{F}) = \mathrm{div}\left(\frac{\partial\sigma_k}{\partial W_{ij}} f_i e_j\right) = \sum_{i,j}\left(\frac{\partial\sigma_k}{\partial W_{ij}}\right)_j f_i + \frac{\partial\sigma_k}{\partial W_{ij}} f_{ij}$$

now the first term vanishes by the Codazzi relation from before. We thus get

$$\int_M \frac{\partial\sigma_k}{\partial W_{ij}} f_{ij} dV = \int_{\partial M} \vec{F}\cdot\tilde{\nu}dV$$

which is zero because M is closed and has no boundary.

We thus can get rid of the f_{ij} term in the integral for the variation. Next since h is diagonal we get the following simplification

$$\sigma_k(h) = \sum_{i_1<\cdots<i_k} h_{i_1 i_1}\cdots h_{i_k i_k}$$

giving us

$$\frac{\partial \sigma_k(h)}{\partial h_{ii}} = \sum_{i_1<\cdots<i_k, i_\ell \neq i} h_{i_1 i_1} \cdots h_{i_k i_k} = \sigma_\ell(h|i)$$

where $(h|i)$ denotes the matrix h with the i-th row and column removed. So now if we fix i we get

$$h_{ii}\sigma_\ell(h) = h_{ii}\sigma_\ell(h|i) + h_{ii}^2 \sigma_{\ell-1}(h|i)$$

Then notice that σ_k is homogeneous degree k, by which we mean $\sigma_k(sh) = s^k \sigma_k(h)$. Then first by the derivative identity above we have

$$h_{ii}\sigma_\ell(h|i) = h_{ii} \frac{\partial \sigma_{\ell+1}(h)}{\partial h_{ii}}$$

as well as

$$h_{ii}^2 \sigma_{\ell-1}(h|i) = h_{ii}^2 \frac{\partial \sigma_\ell(h)}{\partial h_{ii}}$$

we thus get

$$h_{ii}\sigma_\ell(h) = h_{ii} \frac{\partial \sigma_{\ell+1}(h)}{\partial h_{ii}} + h_{ii}^2 \frac{\partial \sigma_\ell(h)}{\partial h_{ii}}$$

then summing over i and using Euler's homogeneous function theorem we get

$$\sum_i h_{ii}^2 \frac{\partial \sigma_\ell(h)}{\partial h_{ii}} = H\sigma_\ell(h) - (\ell+1)\sigma_{\ell+1}(h)$$

Plugging this back into the integral gives

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \int_{M^t} \sigma_k(h^t) dV_{M^t} &= \int_M -\frac{\partial \sigma_k}{\partial W_{ij}}(h)(f h_{ii}^2) + \sigma_k(h) f H dV_M \\ &= \int_M -f(H\sigma_k(h) - (k+1)\sigma_{k+1}(h)) + \sigma_k(h) f H dV_M \\ &= (k+1) \int_M f \sigma_{k+1}(h) dV_M \end{aligned}$$

Which finishes the proof. $\square$

Finally we have the Minkowski identity

Lemma 2.2.2 (Minkowski identity)**.**

$$k \int_M u \sigma_k(h) dV_M = (n-k+1) \int_M \sigma_{k-1}(h) dV_M$$

Proof. Let X be the position vector of M then set $\Phi = \frac{|X|^2}{2}$ which gives us

$$\nabla_{e_i}\Phi = \langle X, X_{e_i}\rangle = \langle X, e_i\rangle$$

and so

$$\nabla_{e_j}\nabla_{e_i}\Phi = \left\langle X_{e_j}, e_i\right\rangle + \left\langle X, \nabla_{e_j} e_i\right\rangle = g - h_{ij}\langle X, \nu\rangle = g - h_{ij}u$$

then we set $\sigma_k^{ij} = \frac{\partial \sigma_k}{\partial W_{ij}}$ and contract it with the hessian of Φ to get

$$\sigma_k^{ij}\Phi_{ij} = \sigma_k^{ij} g_{ij} - u\sigma_k^{ij} h_{ij} = \sum_{i=1}^{n} \sigma_{k-1}(h|i) - ku\sigma_k(h)$$

Note that every product of eigenvalues in $\sigma_{k-1}(h)$ is gonna appear exactly $n-(k-1)$ times in the sum $\sum_{i=1}^{n}\sigma_{k-1}(h|i)$ and so we get

$$\sum_{i=1}^{n}\sigma_{k-1}(h|i) - ku\sigma_k(h) = (n-k+1)\sigma_k(h) - ku\sigma_k(h)$$

Finally we have that

$$\sigma_k^{ij}\Phi_{ij} = \operatorname{div}\left(\sum_j \sigma_k^{ij}\phi^i e_j\right)$$

by the same Codazzi relation as before and so we get that

$$\int_M \sigma_k^{ij}\Phi_{ij} dV_M = 0$$

giving us the desired result. □

2.3 Garding's Theory of Hyperbolic Polynomials

Next we will develop some tools to analyze a certain type of polynomial. The full importance of this theory will not become apparant until later and this section can be skipped and accepted as fact if needed.

Let V be an n-dim vector space and P be a polynomial in V of degree m. Fix some $\theta \in V$ then we say that P is *hyperbolic* at θ if

- $P(\theta) \neq 0$
- $P(x + t\theta)$ has only real roots t roots.

It is further called *complete hyperbolic* if $P(x+ty) = P(x), \forall x, t \implies y = 0$.

Proposition 2.3.1. Let P be hyperbolic at θ and define $\tilde{\Gamma}$ to be the connected component of $\Gamma = \{x : P(x) \neq 0\}$ which contains θ, we call $\tilde{\Gamma}$ the Garding cone at θ. If $\tilde{\Gamma}$ is convex then

$$P(x + ty) = 0$$

only for real t if $x, y \in \tilde{\Gamma}$. Additionally we have that $\frac{P(x)}{P(\theta)} > 0$ for all $x \in \tilde{\Gamma}$ and $\left(\frac{P(x)}{P(\theta)}\right)^{1/m}$ is concave in $\tilde{\Gamma}$.

Proof. We may assume by rescaling that $P(\theta) = 1$ then since it is hyperbolic

$$P(x + t\theta) = (t - t_1) \cdots (t - t_m)$$

then

$$P(x) = \prod_{i=1}^{m} (-t_i)$$

Next define $\Gamma_\theta = \{x \in V | P(x + t\theta) \neq 0, \forall t \geq 0\}$. Notice that this condition is equivalent to $P(\mu x + \lambda\theta) \neq 0$ for $\mu > 0, \lambda \geq 0, \mu + \lambda = 1$, since

$$P(x + t\theta) = (t + 1)^n P\left(\frac{1}{t+1}x + \frac{t}{t+1}\theta\right) \neq 0.$$

Then if this condition holds for some x it is equivalent to P not being zero anywhere along the line between θ so then we can cover that line with finitely many balls in which P is not zero. So for sufficiently close y to x the line between y and θ is also contained within those balls so the condition holds for y. Thus Γ_θ is open.

Now Γ_θ is also closed inside Γ, to see this let y be a limit point in Γ then let y_n be a sequence in Γ_θ then since P is hyperbolic all the zeros of $P(y_n + t\theta)$ are negative, then we know that the zeros change continuously in y then all the zeros of $P(y + t\theta)$ are non positive. But since $y \in \Gamma$, $t = 0$ is not a zero so all the zeros of $P(y + t\theta)$ are negative and so $y \in \Gamma_\theta$.

Clearly we have $\theta \in \tilde{\Gamma}$ since $P(\theta + t\theta) = (1 + t)^m P(\theta)$ which is zero only if $t = -1$.

Thus Γ_θ is clopen and so it contains $\tilde{\Gamma}$. It is also contained in $\tilde{\Gamma}$ since if some point y is in a connected component other than $\tilde{\Gamma}$ then the line between θ and y must contain at least one zero of P otherwise they would be in the same component. But then since that map contains a zero y cannot be in $\tilde{\Gamma}$.

By the alternative characterization of Γ_θ we get that it is starshaped.

Now take some $y \in \Gamma_\theta$, $\delta > 0$ both fixed and define

$$E_{y,\delta} = \{x \in V | P(x + i\delta\theta + isy) \neq 0, \Re(s) \geq 0\}$$

which we also see is an open set for the same reason. Now If $s \neq 0$ then if

$$0 = P(i\delta\theta + isy) = (is)^m P\left(\frac{\delta\theta}{s} + y\right)$$

then hyperbolicity gives us that $s < 0$ so $0 \in E_{\delta,y}$.

Now take $x \in \overline{E}_{y,\delta}$ then $\Re(s) > 0 \implies P(x + i\delta\theta + isy) \neq 0$. By an identical argument to that of Γ_θ we see that $E_{\delta,y}$ is closed and so since its open and closed in V so it all of V.

We thus get that

$$P(x + i(\delta\theta + y)) \neq 0, \quad \forall x \in V, y \in \Gamma_\theta, \delta > 0$$

then Γ is open so for small enough δ we have $y - \delta\theta \in \Gamma$ so this is also true for $t \geq 0$.

Now the equation $P(x + ty) = 0$ has only real roots. To see this assume t is a root of $P(x + ty) = 0$ then $t = t_1 + it_2$ so if we assume $t_2 \neq 0$ then by homogeneity

$$P(x + ty) = t_2^m P\left(\frac{x + t_1 y}{t_2} + iy\right) = 0$$

which is a contradiction to the previous statement.

We can thus treat any $y \in \Gamma_\theta$ as our θ and so we get that $\Gamma_\theta = \Gamma_y$ for any $y \in \Gamma_\theta$ so since Γ_y is star shaped with respect to y then $\tilde{\Gamma} = \Gamma_\theta = \Gamma_y$ is convex. Since P has no roots in Γ then $\frac{P(x)}{P(\theta)}$ is positive on Γ.

Finally it is left to prove concavity of $\left(\frac{P(x)}{P(\theta)}\right)^{1/m}$. Take $P(x + ty)$, it has only real roots that depend on x so we have

$$P(x + ty) = a \sum_{i=1}^{m} (t - t_i(x))$$

so we have

$$a = \lim_{t\to\infty} \frac{a \sum_{i=1}^{m} (t - t_i(x))}{t^m} = \lim_{t\to\infty} \frac{P(x + ty)}{t^m} = \lim_{t\to\infty} P\left(\frac{1}{t} + y\right) = P(y)$$

We similarly have

$$P(sx + y) = s^m P\left(x + \frac{y}{s}\right) = P(y) \sum_{i=1}^{m} (1 - st_i(x))$$

So if $sx + y$ then $(1 - st_j) > 0$ for all $j > 0$.

Next set $f(s) = \log(P(sx + y))$. We then have

$$f'(s) = \sum_{i=1}^{m} \frac{-t_i(x)}{1 - s_{t_i(x)}}$$

and

$$f''(s) = \sum_{i=1}^{m} \frac{-t_i^2(x)}{(1 - s_{t_i(x)})^2}.$$

We now have

$$m^2 e^{-\frac{f(s)}{m}} \left(\frac{\partial^2}{\partial s^2} e^{\frac{f(s)}{m}} \right) = f'(s)^2 + m f''(s) = \left(\sum_{i=1}^{m} \frac{-t_i(x)}{1 - s_{t_i(x)}} \right)^2 - m \left(\sum_{i=1}^{m} \frac{-t_i^2(x)}{(1 - s_{t_i(x)})^2} \right)$$

which is negative by Cauchy-Schwartz so $\frac{\partial^2}{\partial s^2} e^{\frac{f(s)}{m}}$ is non-positive which is equivalent to the concavity of what we want. $\square$

We now define the concept of polarization of a homogeneous polynomial m. Set $X^\ell = (x_1^\ell, \ldots, x_n^\ell)$ be a collection of m vectors. Then we denote

$$\left\langle X^\ell, \frac{\partial}{\partial x} \right\rangle = \sum_{j=1}^{n} X_j^\ell \frac{\partial}{\partial x^j}$$

We now define the complete polarization to be

$$\tilde{P}(X^1, X^2, \ldots, X^m) = \frac{1}{m!} \left\langle X^1, \frac{\partial}{\partial x} \right\rangle \cdots \left\langle X^n, \frac{\partial}{\partial x} \right\rangle P(x)$$

then $\tilde{P}(X, X, X, \ldots, X) = P(X)$ by Euler's theorem.
Now we have

$$P(t_1 X^1 + t_2 X^2 + \cdots + t_m X^m) = m! t_1 \cdots t_m \tilde{P}(X^1, \ldots, X^m) + o\left(\sum t_i^2 \right)$$

by Taylor expansion so.

Lemma 2.3.2. *If P is hyperbolic at θ and $m > 1$ then for all $y \in \Gamma$ we have*

$$Q(x) = \sum_{j=1}^{m} y_j \frac{\partial}{\partial x_j} P(x)$$

is also hyperbolic at θ.
In general, for any ℓ with $\ell < m$ we have

$$\tilde{Q}_\ell(x) = \tilde{P}(x^1, \ldots, x^\ell, x, \ldots, x)$$

is also hyperbolic at θ.

Proof. Note that

$$\tilde{Q}_\ell(x) = \frac{1}{m!} D_{x^1} D_{x^2} \cdots D_{x^\ell} P(x)$$

where D_{x^i} denotes the directional derivative in the direction of x^i. It is thus enough to prove that a single directional derivative preserves hyperbolicity.

To get this we apply Rolle's theorem, note that $D_{x^1}P(x)$ is homogeneuous of degree $m-1$ and so since for non-negative s

$$(D_{x^1}P(x))(x+s\theta) = \frac{\partial}{\partial t}P(x+s\theta+tx^1)$$

we get that since P is hyperbolic at $x+s\theta$ then $P(x+s\theta+tx^1)$ has only negative roots for $x^1 \in \tilde{\Gamma}$ and so the roots of its derivative are interspaced between those roots by Rolle's theorem. Thus we get that all the roots of $(D_{x^1}P(x))(x+s\theta+tx^1)$ are negative, in particular t is never a root and so all the roots of $(D_{x^1}P(x))(x+s\theta)$ must be negative. $\square$

Proposition 2.3.3. The following polynomials are hyperbolic.

- $P = x_1^- x_2^2 - \ldots x_n^2$ at $(1, 0, \ldots, 0)$.
- $P = x_1 x_2 \cdots x_n$ is complete hyperbolic at all θ where $P(\theta) \neq 0$. For $\theta^* = (1, \ldots, 1)$ we have $\Gamma_n = \{x | x_j > 0, \forall j\}$.
- $\sigma_k(x)$ is complete hyperbolic at θ^* we have $\Gamma_k = \{\sigma_\ell(x) > 0 | \ell \leq k\}$.

Proof. We only prove the second and third case since the first is trivial.

First the second case. To see it is hyperbolic let θ be any vector then

$$P(x+t\theta) = (x_1+t\theta_1)(x_2+t\theta_2)\cdots(x_n+t\theta_n)$$

then t being a root implies that for some i, $x_i + t\theta_i = 0$. But then $P(\theta) \neq 0$ implies $\theta_i \neq 0$ so $t = \frac{x_i}{\theta_i}$ and thus t must be real.

To see it is complete note that if $\frac{\max |x_i|}{\min \theta_i} \ll t$ we can make $P(x+t\theta)$ arbitrarily high in absolute value, thus the condition is never true.

Next we fix $\theta^* = (1, \ldots, 1)$ and consider its Garding cone Γ_{θ^*}. If $x \in \Gamma_{\theta^*}$ then

$$P(x+t\theta^*) = (x_1+t)(x_2+t)\cdots(x_n+t)$$

then t being a root implies $t = -x_i$ for some i and so if this is only the case for $t < 0$ we have $x_i > 0$ for all i.

Next the third case. We recall from previous results that if $\theta = \theta^*$ then

$$P(x+t\theta) = \sum_{k=0}^{n} t^{n-k}\sigma_k(x)$$

and so

$$\tilde{Q}_k(x) = \tilde{P}(\underbrace{\theta^*, \ldots, \theta^*}_{n-k}, \underbrace{x, \ldots, x}_{k}) = c_{k,n}\sigma_k(x)$$

is hyperbolic.

It is also complete for a similar reason as before. $\square$

Corollary 2.3.4. Let $S = \{M \in M_{n\times n}(\mathbb{R}) | M \text{ is symmetric}\}$ then $\sigma_k(W)$ is complete hyperbolic at identity with $\Gamma_k = \{W \in S | \sigma_\ell(w) > 0, \ell \leq k\}$.

2.4 Newton MacLawrin Inequality

We can now reap the rewards of the theory we previously set up to prove powerful facts about hyperbolic polynomials which we can use in PDE theory.

To start let us prove the hyperbolic variant of Cauchy-Schwartz inequality.

Lemma 2.4.1. *Let P be a hyperbolic polynomial of degree m at θ, $P(\Theta) > 0$, with $\Gamma = \Gamma_\theta$ its Garding cone. For all $X^1, \ldots, X^n \in \Gamma$ we have*

$$\left(\tilde{P}(X^1, X^2, X^3, \ldots, X^m)\right)^2 \geq \left(\tilde{P}(X^1, X^1, X^3, \ldots, X^m)\right)\left(\tilde{P}(X^2, X^2, X^3, \ldots, X^m)\right),$$

furthermore, if P is complete then equality implies all X^i are colinear.

Remark 2.4.2. Note that this is in opposition to the Cauchy-Schwartz we are used to, for if $P' = \sum_i x_i^2$ is our polynomial (importantly not hyperbolic) then its polarization $\tilde{P}'$ is exactly the standard dot product in these coordinates and so we have

$$\tilde{P}'(X^1, X^2)^2 = (X^1 \cdot X^2)^2 \leq (X^1 \cdot X^1)(X^2 \cdot X^2) = \tilde{P}'(X^1, X^1)\tilde{P}'(X^2, X^2)$$

Proof. First note that $P(X, X, X^2, \cdot, X^n)$ is a hyperbolic polynomial of degree 2 in X and so we can reduce the above lemma to the result we will now show.

We will crucially use the fact that $P^{1/m}$ is concave in Γ in particular this means that $\eta(t) = P^{1/m}(X + tY)$ is concave for $t \geq 0$ and for any $X, Y \in \Gamma$. Thus we have $\eta''(t) \leq 0$ for all $t \leq 0$, and in particular at $t = 0$. Now recall that

$$\tilde{P}(Y, X, X, \ldots, X) = \frac{1}{m}\left\langle \frac{\partial P(X + tY)}{\partial X}, Y \right\rangle$$

this lets us compute the following

$$\begin{aligned}\eta'(t) &= \frac{1}{m}P^{1/m-1}(X + tY)\left\langle \frac{\partial P(X + tY)}{\partial X}, Y \right\rangle \\ &= P^{1/m-1}(X + tY)\tilde{P}(\underbrace{X + tY, X + tY, \ldots, X + tY}_{m-1}, Y)\end{aligned}$$

which then gives

$$\begin{aligned}n''(0) = &\left(\frac{1}{m} - 1\right) m\ P^{1/m-2}(X + tY)\tilde{P}(\underbrace{X + tY, X + tY, \ldots, X + tY}_{m-1}, Y)^2\Bigg|_{t=0} \\ &+ (m-1)\tilde{P}(\underbrace{X, \ldots, X}_{m-2}, Y, Y)\tilde{P}^{1/m-1}(\underbrace{X, \ldots, X}_{m}) \\ &= (m-1)P^{1/m-1}(X)\left(\tilde{P}(X, \ldots, X, Y, Y)\tilde{P}(X, \ldots, X) - \tilde{P}(X, \ldots, X, Y)^2\right)\end{aligned}$$

which shows that

$$\tilde{P}(X,\ldots,X,Y)^2 \geq \tilde{P}(X,\ldots,X,Y,Y)\tilde{P}(X,\ldots,X)$$

In the case of equality for $m \geq 3$ we define $Q(x) = \tilde{P}(Y,X,\ldots,X)$ and it is enough to show that it is complete as well. Suppose that $Q(x) = Q(x+tz)\forall x,t$ then by replacing t with $\frac{1}{s}$ we and x with y we get

$$Q(sy) = Q(sy+z)$$

but then the derivative of $P(sy+z) - P(sy)$ is zero and so it is constant equal to some a. But then the zeros of $P(sy+z) = s^m P(y) + a$ are all real and so a must be zero (otherwise since $m \geq 3$ there is a non-real zero). Thus $P(y+sz) = P(y) \neq 0$ for all all s, and so it follows that $y + sz \in \Gamma$ and so

$$\frac{sx+y+sz}{s+1} \in \Gamma, \forall x \in \Gamma, s > 0$$

by convexity of Γ and so by letting $s \to \infty$ we get $x + z \in \overline{\Gamma}$ for all $x \in \Gamma$ thus we also get $x + z \in \Gamma$. Then by replacing z with tz for any $t \neq 0$ and so $x + tz \in \Gamma$. But then $P(z+sx)$ can not have any zeros s that are non-zero and so $P(z+sx) = s^m P(x)$ where the factor $P(x)$ comes from taking $s \to \infty$. But then $P(tz+x) = P(x)$ for all t and all $x \in \Gamma$. Since P is analytic then since they are equal on an open set they are equal everyhwere and thus $P(tz+x) = P(x)$ for all t and all $x \in V$. □

Theorem 2.4.3. *Under the same conditions as the lemma above we have*

$$\tilde{P}(X^1,\ldots,X^m) \geq P^{1/m}(X^1)\cdots P^{1/m}(X^m)$$

if P is complete then equality holds if and only if X^j are proportional.

Proof. Let us denote $Q_i(X,Y) = \tilde{P}(\underbrace{X,\ldots,X}_{m-i},\underbrace{Y,\ldots,Y}_{i})$. We have shown that

$$Q_1^2(X,Y) \geq Q_0(X,Y)Q_2(X,Y),$$

or in other words

$$\frac{Q_1(X,Y)}{Q_0(X,Y)} \geq \frac{Q_2(X,Y)}{Q_1(X,Y)}.$$

Now if we fix Y, $Q_i(X,Y)$ is again a hyperbolic polynomial at θ and so we get in general

$$\frac{Q_{i+1}(X,Y)}{Q_i(X,Y)} \geq \frac{Q_{i+2}(X,Y)}{Q_{i+1}(X,Y)}.$$

We thus get

$$\frac{P(Y)}{P(X)} = \frac{Q_m(X,Y)}{Q_0(X,Y)} = \prod_{i=1}^{m} \frac{Q_i(X,Y)}{Q_{i-1}(X,Y)} \leq \prod_{i=1}^{m} \frac{Q_1(X,Y)}{Q_0(X,Y)} = \frac{Q_1^m(X,Y)}{Q_0^m(X,Y)}$$

which then becomes

$$\tilde{P}^m(Y, \underbrace{X, \ldots, X}_{m-1}) \geq P(Y)P^{m-1}(X)$$

We now by induction get

$$\tilde{P}^m(X^1, X^2, \ldots, X^m) \geq P(X^1)P(X^2)P(X^3)\cdots P(X^m)$$

giving us the desired result. $\square$

Corollary 2.4.4. For $P(x) = \sigma_k(x)$ then

$$\sigma_k^2(X^1, X^2, X^3, \ldots, X^n) \geq \sigma_k(X^1, X^1, X^3, \ldots, X^n)\sigma_k(X^2, X^2, X^3, \ldots, X^n)$$

as long as $X^i \in \Gamma_k$.

This results in the following extermely important result regarding specifically the symmetric functions.

Theorem 2.4.5. *Newton MacLawrin Inequality,* $\forall X \in \Gamma_k$

$$\sigma_k^2(X) \geq c_{n,k}\sigma_{n,k+1}(X)\sigma_{n,k-1}(X), \quad c_{n,k} = \frac{\sigma_{k+1}(1, \ldots, 1)\sigma_{k_1}(1, \ldots, 1)}{\sigma_k^2(1, \ldots, 1)}$$

and equality is equivalent to $x = t(1, \ldots, 1)$ *for* $t > 0$*.*

Proof. If $x \notin \Gamma_{k+1}$ then the left side is positive and the right side is negative so this result is trivial.
Else we have σ_{k+1} is complete hyperbolic so

$$\sigma_k^2(X^1, X^2, \ldots, X^k, X^{k+1}) \geq \sigma_{k+1}(X^1, X^1, \ldots, X^k, X^{k+1})\sigma_{k-1}(X^2, X^2, \ldots, X^k, X^{k+1})$$

substituting $X^1 = I$ and $X^2 = \cdots = X^{k+1} = X$ gives

$$\sigma_k^2(X) \geq \sigma_{k-1}(X)\sigma_{k+1}(X)$$

with equality if and only if X is proportional to I. $\square$

This result can be directly applied to PDE theory.

Corollary 2.4.6. The matrix

$$\frac{\partial \sigma_k}{\partial W_{ij}}$$

is positive definite for all $W \in \Gamma_k$. And also if $\lambda \in \Gamma_k$ then set $(\lambda|i)$ to be the vector λ with the i-th entry set to 0. Then $(\lambda|i) \in \Gamma_{k-1}$.

Proof. For all $A \in \Gamma_k$, $\left\langle A, \frac{\partial}{\partial W_{ij}} \right\rangle \sigma_k(W) > 0$.
Then we have

$$\left\langle A, \frac{\partial}{\partial W_{ij}} \right\rangle \sigma_k(W) = A_{ij} \frac{\partial \sigma_k(W)}{\partial W_{ij}}$$

and so by diagonalization and choosing $A = I$ we get that all the eigenvalues of $\frac{\partial \sigma_k(W)}{\partial W_{ij}}$ are positive.
We also get that

$$0 < \frac{\partial \sigma_k}{\partial \lambda_i} = \sigma_{k-1}(\lambda|i)$$

and so we get the second part. $\square$

Consider the quotients

$$\frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)}$$

for $\lambda \in \Gamma_k$. Then if we define $Q_j(\lambda) = \frac{\sigma_j(\lambda)}{\sigma_{j-1}(\lambda)}$, we get

$$\frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)} = \prod_{j=i+1}^{k} Q_j(\lambda)$$

Lemma 2.4.7. *We have* $\left(\frac{\partial}{\partial W_{ij}} \frac{\sigma_k}{\sigma_\ell}(W)\right) \geq 0$, *for all* $W \in \Gamma_k$. *Additionally* $\left(\frac{\sigma_k}{\sigma_l}(W)\right)^{\frac{1}{k-\ell}}$ *is concave in* Γ_k.

Proof. We have

$$\frac{\partial}{\partial W_{ij}} \left(\frac{\sigma_k}{\sigma_\ell}\right) = \frac{\frac{\partial \sigma_k}{\partial W_{ij}}(W)\sigma_\ell(W) - \sigma_k(W)\frac{\partial \sigma_\ell}{\partial W_{ij}}(W)}{\sigma_\ell^2(W)}$$

Assuming that W is diagonal we get that

$$\begin{aligned}
\frac{\partial}{\partial \lambda_i} \left(\frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)}\right) &= \frac{\sigma_{k-1}(\lambda|i)\sigma_\ell(\lambda) - \sigma_{\ell-1}(\lambda|i)\sigma_k(\lambda)}{\sigma_\ell^2(\lambda)} \\
&= \frac{\sigma_{k-1}(\lambda|i)\left(\sigma_\ell(\lambda|i) + \lambda_i\sigma_{\ell-1}(\lambda|i)\right) - \sigma_{\ell-1}(\lambda|i)\left(\sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i)\right)}{\sigma_\ell^2(\lambda)} \\
&= \frac{\sigma_{k-1}(\lambda|i)\sigma_\ell(\lambda|i) - \sigma_{\ell-1}(\lambda|i)\sigma_k(\lambda|i)}{\sigma_\ell^2(\lambda)}
\end{aligned}$$

Now note that by setting $\lambda = k-1$ we get that this is non-negative. We then get the same result for all $\ell < k$ by product rule of derivatives.
Next we will use the following fact, if $f = f_1^\alpha \cdot f_2^\beta$ with $\alpha + \beta = 1$, f_1 and f_2 being concave implies f being concave.

Then concavity of Q_k implies concavity of

$$\left(\frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)}\right)^{\frac{1}{k-\ell}} = \prod_{i=\ell+1}^{k} \left(Q_k(\lambda)\right)^{\frac{1}{k-\ell}}.$$

Now I did not understand the proof of concavity of Q_k from class, so I will adapt a proof from "Second Order Parabolic Differential Equations (1996)" by Gary M. Lieberman (note their g_k is our Q_k).

We recall from a previous proof that we have

$$k\sigma_k(\lambda) = \sigma_1(\lambda)\sigma_{k-1}(\lambda) - \sum_i \lambda_i^2 \sigma_{k-2}(\lambda|i)$$

From here from by dividing we get that

$$kQ_k = S_1(\lambda) - \sum_i \frac{\lambda_i^2 \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}(\lambda)}$$

now we also have

$$\sigma_{k-1}(\lambda) = \sigma_{k-1}(\lambda|i) + \lambda_i \sigma_{k-2}(\lambda|i)$$

and so if we set $Q_{k,i}(\lambda) = \frac{\sigma_k(\lambda|i)}{\sigma_{k-1}(\lambda|i)}$ then we get

$$\frac{\sigma_{k-1}(\lambda)}{\sigma_{k-2}\lambda} = Q_{k-1,i}(\lambda) + \lambda_i.$$

This then gives us

$$kQ_k = S_1(\lambda) - \sum_i \frac{\lambda_i^2}{Q_{k-1,i}(\lambda) + \lambda_i}.$$

Now by continuity it is enough to check convexity at the midpoints, thus we want

$$\phi_k(\lambda,\mu) = 2Q_k\left(\frac{\lambda+\mu}{2}\right) - Q_k(\lambda) - Q_k(\mu) = Q_k(\lambda+\mu) - Q_k(\lambda) - Q_k(\mu) \geq 0$$

where the 2 gets absorbed since the numerator of Q_k is homogeneus of degree k and the denominator is of degree $k-1$.

Now we can easily see from the calculations right above that

$$\phi_k(\lambda,\mu) = \frac{1}{k}\sum_i \left(\frac{\lambda_i^2}{\lambda_i + Q_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + Q_{k-1,i}(\mu)} - \frac{(\lambda_i+\mu_i)^2}{\lambda_i+\mu_i+Q_{k-1,i}(\lambda+\mu)}\right)$$

Now $Q_{k-1,i}(\lambda+\mu)$ is equal to $Q_{k-1}(\lambda+\mu)$ after setting $\lambda_i = \mu_i = 0$. But then by induction we get $Q_{k-1}(\lambda+\mu) \geq Q_{k-1}(\lambda) + Q_{k-1}(\mu)$, so then we have

$$\phi_k(\lambda,\mu) \geq \frac{1}{k}\sum_i \left(\frac{\lambda_i^2}{\lambda_i + Q_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + Q_{k-1,i}(\mu)} - \frac{(\lambda_i+\mu_i)^2}{\lambda_i+\mu_i+Q_{k-1,i}(\lambda)+Q_{k-1,i}(\mu)}\right)$$

then for each i we have

$$\frac{\lambda_i^2}{\lambda_i + Q_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + Q_{k-1,i}(\mu)} - \frac{(\lambda_i + \mu_i)^2}{\lambda_i + \mu_i + Q_{k-1,i}(\lambda) + Q_{k-1,i}(\mu)}$$

which has the following numerator

$$\begin{aligned}
&\lambda_i^2(\lambda_i + \mu_i + Q_{k-1,i}(\lambda) + Q_{k-1,i}(\mu))(\mu_i + Q_{k-1,i}(\mu)) \\
&+ \mu_i^2(\lambda_i + \mu_i + Q_{k-1,i}(\lambda) + Q_{k-1,i}(\mu))(\lambda_i + Q_{k-1,i}(\lambda)) \\
&- (\lambda_i + \mu_i)^2(\mu_i + Q_{k-1,i}(\mu))(\lambda_i + Q_{k-1,i}(\lambda)) \\
&= \lambda_i^2(\mu_i + Q_{k-i,i}(\mu))^2 + \mu_i^2(\lambda_i + Q_{k-1,i}(\lambda))^2 - 2\lambda_i\mu_i(\mu_i + Q_{k-1,i}(\mu))(\lambda_i + Q_{k-1,i}(\lambda)) \\
&= \lambda_i^2(Q_{k-i,i}(\mu))^2 + \mu_i^2(Q_{k-1,i}(\lambda))^2 - 2\lambda_i\mu_i Q_{k-1,i}(\mu)Q_{k-1,i}(\lambda) \\
&= (\lambda_i Q_{k-1,i}(\mu) - \mu_i Q_{k-1,i}(\lambda))^2
\end{aligned}$$

Then this is clearly positive and so we are done. □

2.5 Support function and Mixed Volumes

We now return to some geometric constructions. Let Ω be a domain in $\mathbb{R}^{n+1}$ with $\partial\Omega \in C^2$ then the second fundamental form h is symmetric. The eigenvalues of h are called its principal curvatures. $h > 0$ implies Ω is convex.

Definition 2.5.1. *We define the support funtion u to be $\langle X, \nu \rangle$ where ν is the unit outward pointing normal and $X = (x_1, x_2, \ldots, x_{n+1})$ is the position vector field.*

Recall that in an orthonormal frame e_i we have

$$\nabla_{e_i}\nu = h_{j\ell}e_\ell$$

and so the Gauss map $x \mapsto \nu(x)$ is non-degenerate and thus is injective, is surjective by a tangent plane argument, and so is invertible.

We thus may parametrize $\partial\Omega$ by S^n using $\nu^{-1}(x)$, for $x \in S^n$. Next we get $u(x) = \nu^{-1}(x) \cdot x$ and so we can define u on S^n. Then we extend $u(x)$ to a function on $\mathbb{R}^{n+1}$ by making it homogeneus of degree 1, as in

$$u(z) = z \cdot \nu^{-1}\left(\frac{z}{\|z\|}\right)$$

Then $\partial\Omega = \{\operatorname{grad} u(x) | x \in \mathbb{R}^{n+1}\}$. To see this let $Z = \nu^{-1}(x)$ then for any fixed $x \in S^n$ we have

$$\nabla_{e_j}\nu^{-1}(x) \perp \nu = 0 \implies \nabla_{e_j}\nu^{-1}(x) \perp x = 0$$

and so we have

$$\nabla(u(z)) = \nu^{-1}(x)\nabla x + x\nabla\nu^{-1}(x) = \nu^{-1}(x)\nabla x = \nu^{-1}(x)$$

Now if e_i is an orthonormal frame on S^n with ω_i its dual frame then set e_{n+1} to be the normal vector to S^n then we have

$$\nabla e_{n+1} = \sum_{i=1}^{n} e_i \omega_i$$

and we also have

$$\nabla u(x) = \sum_{i=1}^{n+1} u_i e_i = u e_{n+1} + \sum_{i=1}^{n} u_i e_i$$

but then

$$dZ(x) = \sum_{i,j} (u_{ij} + u\delta_{ij}) e_i \otimes \omega_j$$

We call $W := u_{ij} + u\delta_{ij}$ the spherical Hessian of u.

We note that the Jacobian of ν is h as we saw before, and the Jacobian of ν^{-1} is h^{-1}, but the Jacobian is also W so $h^{-1} = W$.

Now we go back to the Quermass integrals, we recall

$$V_k(\Omega) = \int_{\partial\Omega} \sigma_k(h) d\mu_{\partial\Omega}$$

but then if $h > 0$ and our Ω is convex then $\partial\Omega = \nu^{-1}(S_n)$ so we can use reparametrize these integrals over S^n. To that end we need to somehow factor our a Jacobian determinant from the expression in the integral, we do this as follows. First let k_i be the eigenvalues of h, the principal curvatures, then set λ_i to be the eigenvalues of h^{-1}, namely $\frac{1}{k_i}$.

Now we have

$$\sigma_k(k_1, \ldots, k_n) = \sigma_k\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) = \frac{\sigma_{n-k}(\lambda)}{\sigma_n(\lambda)}.$$

So then we have

$$\begin{aligned} V_k(\Omega) &= \int_{\partial\Omega} \frac{\sigma_{n-k}(\lambda)}{\sigma_n(\lambda)} d\mu_{\partial\Omega} = \int_{\partial\Omega} \sigma_{n-k}(\lambda)\sigma_n(h) d\mu_{\partial\Omega} = \int_{S^n} \sigma_{n-k}(W) d\mu_{S^n} \\ &= C_{n,n-k-1} \int_{S^n} u\sigma_{n-k-1}(W) d\mu_{S^n} \end{aligned}$$

where the second equality comes from the fact that the determinant of the inverse is the inverse of the determinant.

Now let $u^1, \ldots, u^{n+1} \in C^2(S^n)$. We define the following mixed volume form

Definition 2.5.2. *Let Z^j denote the spherical Hessian of u^j, namely*

$$Z^j = \sum_{i=1}^{n} u_i^j e_i + u^j e_{n+1}$$

then define

$$\Omega(u^1,\ldots,u^{n+1}) = \det(Z^1, dZ^2, dZ^3, \ldots, dZ^{n+1})$$

which is a top form and so is of the form

$$A(Z^1,\ldots,Z^{n+1})d\mu S^n$$

for some linear function A.
We also define

$$V(u^1,\ldots,u^{n+1}) = \int_{S^n} \Omega(u^1,u^2,\ldots,u^{n+1})d\mu_{S^n}$$

called the Minkowski Mixed Volume.

If $u^1 = u^2 = \cdots = u^{n+1} = u$ where u is the support function of Ω then we get

$$\begin{aligned} V(u,\ldots,u) &= \int_{S^n} u \det(u_{ij} + u\delta_{ij})d\mu_{S^n} = \int_{\partial\Omega} ud\mu_{\partial\Omega} = \int_{\partial\Omega} \nu \cdot X d\mu_{\partial\Omega} \\ &= \int_{\Omega}(n+1)dx = (n+1)V(\Omega) \end{aligned}$$

where the third to last equality comes from divergence theorem

Lemma 2.5.3. *$V(u^1,\ldots,u^{n+1})$ is symmetric in $u^1,\ldots,u^{n+1}$.*

Proof. First if u^1 is fixed then V is clearly symmetric in $u^2,\ldots,u^{n+1}$.
It is thus enouigh to show that

$$V(u^1,u^2,u^3,\ldots,u^{n+1}) = V(u^2,u^1,u^3,\ldots,u^{n+1})$$

now define

$$\omega(u^1,\ldots,u^{n+1}) = \det(Z^1, Z^2, dZ^3, \ldots, dZ^{n+1})$$

then we have by the product rule for wedges of one forms (see for example Lee's smooth manifolds) we get that

$$d\omega(u^1,\ldots,u^{n+1}) = -\Omega(u^2,u^1,u^3,\ldots,u^{n+1}) + \Omega(u^1,u^2,u^3,\ldots,u^{n+1}),$$

this is because wherever we apply d hits the third term and higher it hits dZ^i which becomes zero since $d \circ d = 0$.
From stokes theorem we then get that

$$0 = \int_{S^n} d\omega = \int_{S^n} \Omega(u^1,u^2,u^3,\ldots,u^{n+1}) - \int_{S^n} \Omega(u^2,u^1,u^3,\ldots,u^{n+1})$$

□

Lemma 2.5.4. *Take some $u, \tilde{u} \in C^2(S^n)$ then denote $W = u_{ij} + u\delta_{ij}, \tilde{W} = \tilde{u}_{ij} + \tilde{u}\delta_{ij}$ their spherical hessian. Then*

$$\int_{S^n} u\sigma_k(\underbrace{\tilde{W}, \dots, \tilde{W}}_{k})d\mu_{S^n} = \int_{S^n} \tilde{u}\sigma_k(W, \underbrace{\tilde{W}, \dots, \tilde{W}}_{k-1})d\mu_{S^n}$$

as well as

$$\int_{S^n} u\sigma_k(\underbrace{W, \dots, W}_{k-1}, \tilde{W})d\mu_{S^n} = \int_{S^n} \tilde{u}\sigma_k(\underbrace{W, \dots, W}_{k})d\mu_{S^n}$$

Proof. Follows from symmetry of V after setting $u^1 = u$, $u^2, \dots, u^{k+1} = \tilde{u}$ as well as $u^{k+2}, \dots, u^{k+1} = 1$. □

2.6 Elliptic operators

First we will start with a different point of view on the generalized volume form we saw last time

Claim 2.6.1.

$$V(u^1, \dots, u^{n+1}) = \int_{S^n} u^1\sigma_n(W_{u^2}, \dots, W_{u^{n+1}})$$

where $W_v = v_{ij} + v\delta_{ij}$.

Proof. First we will need a small fact, for all $u \in C^2(S^n), W_{ij} = u_{ij} + u\delta_{ij}$ is Codazzi.
To see this we compute

$$u_{ij,k} = \left\langle \nabla_{e_k}\nabla_{e_j}\nabla u, e_i \right\rangle$$
$$u_{ik,j} = \left\langle \nabla_{e_j}\nabla_{e_k}\nabla u, e_i \right\rangle$$

and so if we empose $[e_i, e_j] = 0$ we get

$$u_{ij,k} - u_{ik,j} = \left\langle (\nabla_{e_k}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_k})\nabla u, e_i \right\rangle = \langle R(e_k, e_j)\nabla u, e_i\rangle = \sum_\ell u_\ell \langle R(e_k, e_j)e_\ell, e_i\rangle$$

then a quick exercise shows that

$$\langle R(e_k, e_j)e_\ell, e_i\rangle = h_{i\ell}h_{ki} - h_{k\ell}h_{ij}$$

and since we are on the sphere the second fundamental form is just δ_{ij}. We thus get

$$u_{ij,k} - u_{ik,j} = \sum_\ell u_\ell\left(\delta_{i\ell}\delta_{ki} - \delta_{k\ell}\delta_{ij}\right) = \delta_{ki}u_j - u_k\delta_{ij}$$

and so we get exactly $W_{ij,k} = W_{ik,j}$.

But now we have

$$\begin{aligned}\int_{S^n} v\sigma_k(W_u,\dots,W_u) &= \int_{S^n} v\frac{\partial\sigma_k}{\partial W_{ij}}(u_{ij}+u\delta_{ij}) = \int_{S^n} uv\frac{\partial\sigma_k}{\partial W_{ij}}\delta_{ij} + \int_{S^n} v\left(\frac{\partial\sigma_k}{\partial W_{ij}}u_{ij}\right)\\ &= \int_{S^n} u\frac{\partial\sigma_k}{\partial W_{ij}}(v\delta_{ij}) + \int_{S^n} u\frac{\partial\sigma_k}{\partial W_{ij}}v_{ij} = \int_{S^n} u\frac{\partial\sigma_k}{\partial W_{ij}}(W_v)_{ij}\end{aligned}$$

where crucially we used the fact that W is codazzi.

Finally it remains to show that

$$\int_{S^n} v\sigma_k(W_{u^1},\dots,W_{u^k}) = \int_{S^n} u^1\sigma_k(W_v,\dots,W_{u^k}).$$

To see this for all $t^1,\dots,t^k$ we set $u_t = t_1u^1+\dots+t_ku^k$. Then we have

$$\int_{S^n} v\sigma_k(W_{u_t},\dots,W_{u_t}) = \int_{S^n} u_t\sigma_k(W_v,\dots,W_{u_t})$$

and so we can taylor expand both sides and compare the coefficients for the $t_{i_1}t_{i_2}\dots t_{i_k}$ to get

$$\int_{S^n} v\sigma_k(W_{u^{i_1}},\dots,W_{u^{i_k}})t_{i_1}\cdots t_{i_k} = \int_{S^n} u^{i_1}\sigma_k(W_v,\dots,W_{u^{i_k}})t_{i_1}\cdots t_{i_k}$$

$\square$

We see from the above that we have

$$\int_{S^n}(u\sigma_k(W_v,\dots,W_v) - v\sigma_k(W_u,W_v,\dots W_v)) = 0$$

and by symmetry we also have

$$\int_{S^n}(u\sigma_k(W_u,\dots,W_u,W_v) - v\sigma_k(W_u,\dots,W_u)) = 0$$

but then we get

$$\begin{aligned}&2\int_{S^n} u[\sigma_k(W_v,\dots,W_v) - \sigma_k(W_u,\dots,W_u,W_v)]\\ &= \int_{S^n} v[\sigma_k(W_u,W_v,\dots,W_v) - \sigma_k(W_u,\dots,W_u)]\\ &- \int_{S^n} u[\sigma_k(W_u,\dots,W_u,W_v) - \sigma_k(W_v,\dots,W_v)]\end{aligned}$$

now the right hand side here is anticommutative in u,v and so the left hand sign is as well. But then we have

$$\begin{aligned}&\int_{S^n} u[\sigma_k(W_v,\dots,W_v) - \sigma_k(W_u,\dots,W_u,W_v)]\\ &+ \int_{S^n} v[\sigma_k(W_u,\dots,W_u) - \sigma_k(W_v,\dots,W_v,W_u)] = 0\end{aligned}$$

We can thus get the following powerful tool

Corollary 2.6.2. $u, v \geq 0$, $\sigma_k(W_u) = \sigma_k(W_v)$ everywhere on S^n implies that $u - v \in \mathrm{Span}\{x_1, \ldots, x_{n+1}\}$ if $W_u, W_v \in \Gamma_k$.

Proof. By Garding's inequality we have

$$\sigma_k(W_u, \ldots, W_u, W_v) \geq \sigma_k^{\frac{k-1}{k}}(W_u, \ldots, W_u)\sigma_k^{\frac{1}{k}}(W_v, \ldots, W_v) = \sigma_k(W_v, \ldots, W_v)$$

and similarly

$$\sigma_k(W_v, \ldots, W_v, W_u) \geq \sigma_k^{\frac{k-1}{k}}(W_v, \ldots, W_v)\sigma_k^{\frac{1}{k}}(W_u, \ldots, W_u) = \sigma_k(W_u, \ldots, W_u)$$

and so both functions

$$u[\sigma_k(W_v, \ldots, W_v) - \sigma_k(W_u, \ldots, W_u, W_v)]$$

and

$$v[\sigma_k(W_u, \ldots, W_u) - \sigma_k(W_v, \ldots, W_v, W_u)]$$

are non positive, but their integrals sum to zero so these functions must be equal to zero. But then by Garding's inequality if we have equality we have W_u, W_v are colinear.
We can thus enforce

$$(u - v)_{ij} + (u - v)\delta_{ij} = 0$$

after rescaling.
But now for what functions f can we have $W_f = 0$ on S^n? If we define

$$Z_f = \sum_{i=1}^{n} f_i e_i + f e_{n+1}$$

if f is extended to $\mathbb{R}^{n+1}$ so that it is homogeneus of degree 1 then $Z_f = \nabla_{\mathbb{R}^{n+1}} f$. But then for $i \neq n+1$ we have

$$\nabla_{e_i} Z_f = \sum_{j=1}^{n} (f_{ij} + f\delta_{ij})e_j$$

and so by assumption Z_f is constant. But if it is constant then f must be a linear polynomial. □

Let us introduce new notation to summarize these results, we define the linearized operator of $\sigma_k(W_u)$, assuming $W_u \in \Gamma_k$, to be

$$\mathcal{L}_u(\rho) = \sum_{ij} \frac{\partial \sigma_k}{\partial W_{ij}}(W_u)(\rho_{ij} + \rho\delta_{ij})$$

By the previous result we get that for $W_u \in \Gamma_k$ and $u > 0$ then $\ker(\mathcal{L}_u) = \mathrm{Span}\{x_1, \ldots, x_{n+1}\}$.

Proposition 2.6.3. If we assume $u^1, \dots, u^k \in C^2(S^n)$ with $W_{u^i} \in \Gamma_k$ and assume $u^2 > 0$. We define another linear operator

$$\mathcal{L}(\rho) = \Omega(1, \rho, u^2, \dots, u^k, 1, \dots, 1)$$

then $\ker(L) = \operatorname{Span}\{x_1, \dots, x_{n+1}\}$.

Proof. Suppose that $\mathcal{L}(\rho) = 0$ on S^n then $\sigma_k(W_\rho, W_{u^2}, \dots, W_{u^k}) \equiv 0$. Then we claim that

$$\sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) \leq 0$$

If we know the claim then

$$\int_{S^n} u^2 \sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) \leq 0$$

but

$$\int_{S^n} u^2 \sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) = \int_{S^n} \rho \sigma_k(W_\rho, W_{u^2}, W_{u^3}, \dots, W_{u^k}) = \int_{S^n} \rho \mathcal{L}(\rho) = 0$$

and so we get

$$\sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) \equiv 0$$

Now set $u_t = u_2 + t\rho$ then

$$W_{u_t} = W_{u^2} + tW_\rho$$

and so

$$\begin{aligned}\sigma_k(W_{u_t}, W_{u_t}, W_{u^3}, \dots, W_{u^k}) = \sigma_k(W_{u^2}, W_{u^2}, W_{u^3}, \dots, W_{u^k}) + 2t\sigma_k(W_\rho, W_{u^2}, W_{u^3}, \dots, W_{u^k}) \\ + t^2 \sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k})\end{aligned}$$

but the second term here goes to zero by assunption. We then have

$$(W_{u_t}, W_{u_t}, W_{u^3}, \dots, W_{u^k}) = \sigma_k(W_{u^2}, W_{u^2}, W_{u^3}, \dots, W_{u^k}) + t^2 \sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k})$$

but we know by hyperbolicity that the roots here in t are all real and so since the first term is clearly positive the second term must be negative and so

$$\sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) \leq 0$$

which also proves the claim. But then combining this with the claim we get

$$\sigma_k(W_\rho, W_\rho, W_{u^3}, \dots, W_{u^k}) = 0$$

but by completeness we get that $W_\rho = 0$ and thus we get the desired result. $\square$

2.7 Mixed Convex Domains

We saw how we can define a mixed volume using $n + 1$ functions with positive spherical Hessian. We now study what happens when we plug $n + 1$ support functions into this volume.

Let $\Omega_1, \ldots, \Omega_{n+1}$ be $n+1$ convex bounded domains and $u_1, \ldots, u^{n+1}$ their respective support functions. We have the following notation

$$V(\Omega_1, \ldots, \Omega^{n+1}) = V(u_1, \ldots, u^{n+1})$$

Theorem 2.7.1. *For $1 \leq k \leq n$ if we assume $W_{u^1}, \ldots, W_{u^k} \in \Gamma_k$ as well as $v \in C^2(S^n)$ and $u^1, u^2 \geq 0$, we have*

$$V_{k+1}^2(v, u^1, \ldots, u^k) \geq V_{k+1}(v, v, \ldots, u^k) V_{k+1}(u^1, u^{\cdot} \ldots, u^k)$$

and equality holds if and only if $V = cu^1 + \sum_i \alpha_i x_i$ or equivelantly $cW_{u^1} = W_v$.

Proof. It is clear that if the theorem is true then it is necessary that the claim

$$V_{k+1}(\tilde{v}, u^1, \ldots, u^k) = 0 \implies V_{k+1}(\tilde{v}, \tilde{v}, \ldots, u^k) = 0$$

holds for any v. We will now show that the claim is also sufficient.
First by rescaling u^k we my assume

$$V_{k+1}(u^1, u^1, \ldots, u^k) = 1$$

then if $V_{k+1}(v, v, \ldots, u^k) \leq 0$ the statement is trivial so we may assume $V_{k+1}(v, v, \ldots, u^k) > 0$ and so if the claim holds then $V_{k+1}(v, u^1, \ldots, u^k) > 0$. We then rescale v so that $V_{k+1}(v, u^1, \ldots, u^k) = 1$. But then by linearity if $\tilde{v} = u^1 - v$ we have

$$V_{k+1}(\tilde{v}, u^1, \ldots, u^k) = 0$$

and so by the claim

$$V_{k+1}(\tilde{v}, \tilde{v}, \ldots, u^k) \leq 0$$

which is also equal to

$$V_{k+1}(u^1, u^1, \ldots, u^k) - 2V_{k+1}(v, u^1, \ldots, u^k) + V_{k+1}(v, v, \ldots, u^k) \leq 0$$

and so by our rescaling the first term is 1 and the second is -2 and so we get $V_{k+1}(v, v, \ldots, u^k) \leq 1$. But then we have

$$V_{k+1}^2(v, u^1, \ldots, u^k) = 1 \geq 1 \cdot 1 \geq V_{k+1}(v, v, \ldots, u^k) \cdot V_{k+1}(u^1, u^1, \ldots, u^k)$$

We now need to prove this claim, assume that $V_{k+1(\tilde{v}, u^1, \ldots, u^k)} = 0$. Recall the linear operator $\mathcal{L}(\tilde{v})$ defined as

$$\mathcal{L}(\tilde{v}) = \sum_{i,j}^{n} \frac{\partial \sigma_k}{\partial W_{ij}} (W_{u^2}, \ldots, W_{u^k})(\tilde{v}_{ij} + \tilde{v}\delta_{ij})$$

with $u^2 \geq 0$. As we saw this operator's kernel is just linear functions.

Consider now the weighted eigenvalue problem,

$$\mathcal{L}(\eta) = \lambda\rho\eta$$

where $\rho = \frac{\sigma_k(W_{u^1},\ldots,W_{u^k})}{u^1}$. We claim then, that this problem has only one positive eigenvalue 1 with eigenfunction $\eta = u^1$, linear functions being the zero eigenvalue, and all other eigenvalues being negative.

If we assume such a fact then the fact follows since we can decompose $\tilde{v}$ according to the eigenfunctions decomposition

$$\tilde{v} = a^1 u^1 + \sum_{j=1}^{n+1} a_j x_i + \sum_{\ell} a^\ell \xi_\ell$$

where ξ_ℓ are all negative eigenvalues and so the statement $V_{k+1}(\tilde{v}, u^1, \ldots, u^k) = 0$ is equivalent to $a^1 = 0$. But then $V_{k+1}(\tilde{v}, \tilde{v}, \ldots, u^k)$ is gonna decompose as

$$V_{k+1}(\tilde{v}, u^1, \ldots, u^k) = \sum_{\ell} (a^\ell)^2 \lambda_\ell \left\|\xi_\ell\right\|_2^2$$

where all the λ_ℓ are the negative eigenvalues. But this sum must then also be negative and so we are done.

It remains to prove the eigenvalue fact. For this we will use method of continuity, define the following time parametrization

$$u_t^i = tu^i + (1-t)$$

one can easily show that $W_{u_t^i} = tW_{u^i} + (1-t)I$. Now at $t = 0$ we have all the functions are constant and so we get that their spherical Hessians are all identity and so

$$\mathcal{L}(\tilde{v}) = \sum_{i,j}^{n} \frac{\partial \sigma_k}{\partial W_{ij}}(I, I, \ldots, I)(\tilde{v}_{ij} + \tilde{v}\delta_{ij}) = \sum_{i,j}^{n} C_{n,k}\delta_{ij}(\tilde{v}_{ij} + \tilde{v}\delta_{ij}) = C_{n,k}(\Delta\tilde{v} + n\tilde{v}).$$

Additionally ρ becomes just $C_{n,k}$ and so we are left with the problem of

$$C_{n,k}(\Delta\tilde{v} + n\tilde{v}) = C_{n,k} n\lambda\tilde{v} \implies \Delta\tilde{v} + n\tilde{V} = n\lambda\tilde{v}$$

which turns out to only have eigenvalues 1,0 and negatives.

Now as we vary t the 0 eigenspace remains fixed and so since the eigenvalues vary continuously we cannot have them cross zero. Thus we will always have 1 positive eigenvalue which we can easily check corresponds to the u eigenvector with eigenvalue 1. $\square$

2.8 Mixed Convex Domains

We have proven a very strong theorem last time and we can now use it to prove inequalities relating convex domains. In particular let $u^1 = \cdots = u^n = u$ be the support function of a convex body, then recall that

$$V_{n+1}^2(v, u, \ldots, u) \geq V_{n+1}(v, v, u, \ldots, u) V_{n+1}(u, \ldots, u)$$

Consider two convex domains Ω_0, Ω_1 in $\mathbb{R}^{n+1}$, define

$$\Omega_t = (1-t)\Omega_0 + t\Omega_1 = \big\{(1-t)x + ty | x \in \Omega_0, y \in \Omega_1\big\}$$

Let u^0 and u^1 be the support functions of Ω_0 and Ω_1 respectively, we want to get a handle on the support of Ω_t.

For this we use a useful alternative definition of the support function for convex bodies, which in fact was historically discovered first.

Proposition 2.8.1. For a convex body we have

$$u(x) = \nu^{-1}(x) \cdot x = \sup_{z \in \Omega} z \cdot x$$

Using this we can calculate

$$\begin{aligned} u^t(p) &= \sup_{z \in \Omega^t} z \cdot p = \sup_{x \in \Omega^0, y \in \Omega_1} ((1-t)x + ty) \cdot p \\ &= (1-t) \sup_{x \in \Omega^0} x \cdot p + t \sup_{y \in \Omega_1} y \cdot p = (1-t)u^0 + tu^1 \end{aligned}$$

From this we know that the volume can be rewritten using divergence formula as

$$V(\Omega_t) = \frac{1}{n+1} \int_{\partial \Omega_t} u_t d\mu_{\partial \Omega_t} = \frac{1}{n+1} \int_{S^n} u_t \sigma_n(W_{u_t}) d\mu_{S^n} = \frac{1}{n+1} V_{n+1}(u_t, u_t, \ldots, u_t).$$

Using this we can prove the Brunn-Minkowski inequality.

Lemma 2.8.2.

$$\left(\frac{\partial}{\partial t}\right)^2 \left(V^{\frac{1}{n+1}}(\Omega_t)\right) \leq 0$$

Proof. Using the previous calculation it is enough to show that statement for

$$V_{n+1}^{1/(n+1)}(u_t, u_t, \ldots, u_t)$$

Now set $v = u^1 - u^0$, we compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[V_{n+1}^{1/(n+1)}(u_t,\ldots,u_t)\right] \\
&= V_{n+1}^{-n/(n+1)}(u_t,\ldots,u_t)\int_{S^n}\left[v\sigma_n(W_{u_t}) + u_t\frac{\partial \sigma_n}{\partial W_{ij}}(W_{u_t})\,(v_{ij}+v\delta_{ij})\right]d\mu_{S^n} \\
&= V_{n+1}^{-n/(n+1)}(u_t,\ldots,u_t)\left[V_{n+1}(v,u_t,\ldots,u_t) + nV_{n+1}(u_t,v,u^t,\ldots,u^t)\right] \\
&= (n+1)V_{n+1}^{-n/(n+1)}(u_t,\ldots,u_t)V_{n+1}(v,u_t,\ldots,u_t)
\end{aligned}$$

And now we differentiate again

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left[V_{n+1}^{1/(n+1)}(u_t,\ldots,u_t)\right] \\
&= (n+1)\frac{\partial}{\partial t}\left[V_{n+1}^{-n/(n+1)}(u_t,\ldots,u_t)V_{n+1}(v,u_t,\ldots,u_t)\right] \\
&= (n+1)\Big[\frac{-n}{n+1}V_{n+1}^{-n/(n+1)-1}(u_t,u_t,\ldots,u_t)(n+1)V_{n+1}^2(v,u_t,\ldots,u_t) \\
&+ nV_{n+1}^{-n/(n+1)}(u_t,\ldots,u_t)V_{n+1}(v,v,u_t,\ldots,u_t)\Big] \\
&= n(n+1)V_{n+1}^{-n/(n+1)-1}(u_t,u_t,\ldots,u_t)\Big[-V_{n+1}^2(v,u_t,\ldots,u_t) \\
&\qquad + V_{n+1}(u_t,\ldots,u_t)V_{n+1}(v,v,u_t,\ldots,u_t)\Big] \\
&\leq 0
\end{aligned}$$

□

For another consequence of last times theorem we set $v = 1$ to get

$$\begin{aligned}
V_k^2(u,\ldots,u) &= C_{n,k}V_{k+1}^2(1,u,\ldots,u) \geq C_{n,k}V_{k+1}(u,u,\ldots,u)V_{k+1}(1,1,\ldots,u) \\
&= \tilde{C}_{n,k}V_{k-1}(u,\ldots,u)V_{k+1}(u,\ldots,u)
\end{aligned}$$

From this we get

$$\frac{V_{k+1}(u,\ldots,u)}{V_k(u,\ldots,u)} \geq B_{n,k}\frac{V_k(u,\ldots,u)}{V_{k-1}(u,\ldots,u)}$$

Which we can now use to show

$$\left(\frac{V_{k+1}}{V_k}\right)^{k+1} = \prod_{i=0}^{k}\frac{V_{k+1}}{V_k} \leq \prod_{i=0}^{k}B_{n,k}\frac{V_{i+1}}{V_i} = \tilde{B}_{n,k}V_{k+1}$$

From here we get

$$V_{k+1}(u,\ldots,u) \leq A_{n,k}V_k^{\frac{k+1}{k}}(u,\ldots,u).$$

Using this fact in our quermass integrals we get that since we have

$$A_k(\Omega)\int_{\partial\Omega}\sigma_k(h)d\mu_{\partial\Omega} = V_{n-k}(u,\ldots,u)$$

then we get

$$A_k(\Omega)^{\frac{1}{n-k}} \leq C_{n,k} A_{k+1}(\Omega)^{\frac{1}{n-k-1}}$$

which is sometimes called the Alexandrov Fenchel(AF) inequalities in special form.

For $k = 0$ we get the special form

$$|\partial\Omega| \leq C_{n,1} \left(\int_{\partial\Omega} H d\mu_{\partial\Omega} \right)^{\frac{n}{n-1}}$$

called the Minkowski inequality.

All of these inequalities assume convexity of Ω, so we want to try and weaken that assumption. Since we want the right side of the special form inequality to be positive it is natural to ask whether it holds under the weaker assumption of $h \in \Gamma_{k+1}$, but this is still open.

3 Geometric Flows

This section of the class will be spent trying to weaken the assumptions of theorems we have already seen. The general approach will be the flow approach, which we will now outline.

Let us assume that we have a sequence of $(k+1)$-convex domains Ω_t paramterized by $t \in [0, \infty)$ which satisfy

- $\frac{\partial}{\partial t}(A_{k+1}(\Omega_t)) \leq 0$.
- $A_k(\Omega_t) \equiv A_k(\Omega_0)$.
- $\Omega_t \to B$ as $t \to \infty$ where B is a ball.

Then together these imply the special AF inequality for Ω_0.

We will now try to design a normal flow, that is a flow of the form $X_t = f\nu$ that satisfies the above.

Recall that under such a flow we have

$$\frac{\partial}{\partial t} A_\ell(\Omega_t)|_{t=t_0} = (\ell+1) \int_{\partial\Omega_t} f\sigma_{\ell+1}(h) d\mu_{\partial\Omega_t}$$

since we want this to be equal to zero for $\ell = k$ then we want

$$\int_{\partial\Omega_t} f\sigma_{k+1}(h) d\mu_{\partial\Omega_t} = 0$$

We then recall the following Minkowski identity (not to be confused with the Minkowski inequality)

$$\int_{\partial\Omega} u\sigma_{\ell+1}(h) d\mu_{\partial\Omega} = C_{n,\ell} \int_{\partial\Omega} \sigma_\ell(h) d\mu_{\partial\Omega}$$

This suggests some ratio of σ's might be useful. If we now try $f = \frac{\sigma_k}{\sigma_{k+1}} - \eta$ for some η then we get

$$\int_{\partial\Omega_t} \left(\frac{\sigma_k}{\sigma_{k+1}} - \eta \right) \sigma_{k+1} = \int_{\partial\Omega_t} (\sigma_k - \eta\sigma_{k+1})$$

Now if $\eta = \frac{1}{C_{n,k}} u$ then we get that this is zero as we want. Let us now see if this helps with the other condition,

$$\frac{\partial}{\partial t} A_{k+1}(\Omega_t) \leq 0$$

We calculate

$$\begin{aligned}
\frac{\partial}{\partial t} A_{k+1}(\Omega_t) &= (k+2) \int_{\Omega_t} \left(\frac{\sigma_k}{\sigma_{k+1}} - C_{n,k}^{-1} u \right) \sigma_{k+2} \\
&= (k+2) \left[-C_{n,k}^{-1} C_{n,k+1} \int_{\partial\Omega_t} \sigma_{k+1} + \int_{\partial\Omega_t} \frac{\sigma_k \sigma_{k+2}}{\sigma_{k+1}} \right]
\end{aligned}$$

By Newton-Maclaurine inequality that we have shown in the previous sections we have $\sigma_k \sigma_{k+2} \leq A_{n,k} \sigma_{k+1}^2$ with equality if and only if h is a multiple of the identity. Then we have

$$\begin{aligned}
&= (k+2) \left[-C_{n,k}^{-1} C_{n,k+1} \int_{\partial\Omega_t} \sigma_{k+1} + \int_{\partial\Omega_t} \frac{\sigma_k \sigma_{k+2}}{\sigma_{k+1}} \right] \\
&= (k+2) \left[-C_{n,k}^{-1} C_{n,k+1} \int_{\partial\Omega_t} \sigma_{k+1} + \int_{\partial\Omega_t} A_{n,k} \sigma_{k+1} \right] \\
&= (k+2) \int_{\partial\Omega_t} \sigma_{k+1} \left[-C_{n,k}^{-1} C_{n,k+1} + A_{n,k} \right]
\end{aligned}$$

and an explicit calculation using the σ function of a ball shows that $-C_{n,k}^{-1} C_{n,k+1} + A_{n,k}$ is exactly zero.

These conditions are unfortunately the easy conditions, the hard work is to show that the flow exists for all time and that it converges to a sphere.