Math 595: Geometric Analysis

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Abstract

My course notes for the Geometric Analysis course.

1 ABP and Basic Geometry

1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where B is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\Delta u = c \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.

For such a map we set $T = \nabla u$ to be the gradient map $\Omega \to \mathbb{R}^n$. We now want a characterization of the 'extremal' points of u as a graph, we define

$$\Gamma_u^- = \{x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega\}.$$

In other words Γ_u^- are the points of Ω where the tangent plane lies entirely below the graph of u.

This set is called the 'contact' set.

Remark 1.1.1. For any point x in the contact set we have $\nabla^2 u(x) \geq 0$ where ∇^2 is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

Claim 1.1.2 (ABP)**.** For a solution u of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

Proof. Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so $\tilde{u}$ cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.
To see that x is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

□

Claim 1.1.3. If a solution u to the above PDE exists then we have

$$\frac{|\partial \Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

Proof. Then

$$\begin{aligned}
|B_1| \leq |T(\Gamma_u^-)| \leq \int_{\Gamma_u^-} J_T = \int_{\Gamma_u^-} \det(\nabla^2 u) \\
&= \int_{\Gamma_u^-} \lambda_1 \lambda_2 \cdots \lambda_n \\
&\leq \int_{\Gamma_u^-} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n \quad \text{Since all the eigenvalues are positive.} \\
&\leq \int_{\Gamma_u^-} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \int_{\Omega} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \left(\frac{|\partial \Omega|}{n |\Omega|} \right)^n |\Omega| = \frac{|\partial \Omega|^n}{n^n |\Omega|^{n-1}}
\end{aligned}$$

and since $|B| = \frac{1}{n} |\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial \Omega
\end{aligned}$$

then a necessary condition for existence is

$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

Claim 1.1.4. The above condition is sufficient.

Proof. Assume first that $h = 0$. Thus the condition above becomes $\int_\Omega F = 0$. Then take the positive definite symmetric bilinear form $B(u, v) = \int_\Omega \nabla u \nabla v$ and notice

$$B(u, v) = (Lu, v)$$

and so L is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any g in the kernel we have by our boundary conditions

$$0 = \int_\Omega gLg = \int_\Omega |\nabla g|^2$$

and so g is a constant function.

Thus we have a solution if and only if

$$\int_\Omega F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is C^2 then $\rho(x) = d(x, \partial\Omega)$ is C^2 in Ω near $\partial\Omega$, we then choose a cutoff function η satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is C^2 everywhere on Ω and as we approach the boundary we will have $\frac{\partial \gamma}{\partial \nu} = -1$.

Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for U exists if and only if

$$0 = \int_\Omega \Delta U = \int_\Omega \Delta u + \Delta(h\gamma) = \int_\Omega f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial \nu} = \int_\Omega f - \int_{\partial\Omega} h$$

and so we get our desired result. □

1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the 'upper contact' set, notice that we no longer require u to be differentiable. In conjuction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

Remark 1.2.1. If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

Remark 1.2.2. If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

Example 1.2.3. $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in $\mathbb{R}^{n+1}$.
We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R}\frac{x - z}{|x - z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a\left(1 - \frac{|y - z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y - z}{|y - z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Lemma 1.2.4.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma_u^+} |\det(\nabla^2)| \right)^{1/n}$$

Proof. Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).
Now consider Γ_v^+, we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of Ω such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane P given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to x_0, thus $v - l$ must achieve its maximum somewhere on the interior of Ω where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where $\tilde{h}$ is given by

$$\tilde{h}(x) = v(x_0)\left(1 - \frac{x - x_0}{d}\right).$$

We can see this because $\tilde{h}$ is just a cone with a wider base than h and thus its supporting hyperplanes must have smaller gradients.

But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right)\omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) \geq f \quad \text{in } \Omega$$

Lemma 1.2.5. *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}} \left(\int_{\Gamma_u^+} \left(\frac{-f}{\det^{1/n}(a_{ij})}\right)^n\right)^{1/n}$$

Remark 1.2.6. If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

Lemma 1.2.7. *For symmetric positive matrices A, B we have*

$$\det(A)\det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n$$

Proof. Left side is equal to product of all eigenvalues, $\lambda_1\tilde{\lambda}_1 \cdots \lambda_n\tilde{\lambda}_n$.
$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1\tilde{\lambda}_1 + \cdots + \lambda_n\tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. □

Proof. Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_i j) > 0$ then

$$-f = -Lu = \operatorname{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_i j))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. □

This lemma is sometimes called the weak maximum principle.

Remark 1.2.8. There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) + \sum_k b_k(x)u_k(x) + c(x)u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients b_k and c.

1.3 Introduction to Riemannian Geometry

Let M^n be an n-dimensional manifold, every point $p \in M^n$ has a tangent space $T^p M$, then a metric g on M^n is a choice of inner product on $T_p M$ for every $p \in M$ which varies smoothly in p. A manifold with a metric is called a Riemannian Manifold.
In any local coordinate chart $(x_1, \ldots, x_n)$ we define the 'components' of g to be

$$g_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle.$$

Then if at some point p we have two vectors

$$X = \sum_{j=1}^N a_j(x)\frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^N b_k(x)\frac{\partial}{\partial x_k}$$

then their inner product is given by

$$\langle X, Y\rangle_g = \left\langle \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k} \right\rangle = \sum_{j,k} a_j(x)b_k(x)\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle$$
$$= \sum_{j,k} a_j(x)b_k(x)g_{jk}(x)$$

More formally, let dx_i be the dual frame to $\frac{\partial}{\partial x_i}$, as in

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_i^j,$$

then we can write the metric as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on M.
If $e_1, \ldots, e_n \in T_pM$ is an orthonormal basis, that is $\langle e_i, e_j\rangle_g = \delta_{ij}$. Set $\omega_1, \ldots, \omega_n$ to be its dual basis. We then get a top-form $\omega_1 \wedge \cdots \wedge \omega_n$.
If

$$e_j = \sum_k a_j^k \frac{\partial}{\partial x_k}$$

where $A = a_j^k$ is a matrix, then by standard linear algebra we have that

$$\omega_1 \wedge \cdots \wedge \omega_n = \det(A^{-1}) dx_1 \wedge \cdots \wedge dx_n$$

Claim 1.3.1.

$$|\det(A^{-1})| = \sqrt{\det g}$$

Proof.

$$\delta_{ij} = (e_i, e_j) = a_j^k a_i^l g_{kl}$$

this implies that

$$I = A^T g A$$

where A is the transpose.
Thus

$$1 = \det(A^T g A) = \det(A^2)\det(g)$$

and so

$$\sqrt{\det(g)} = \det A^{-1}$$

□

Claim 1.3.2. The top-form $dV = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$ is coordinate change invariant.

Proof. Let us assume that $(\tilde{x}_1, \ldots, \tilde{x}_n)$ are coordinates given by the transition function $\tilde{x}_\alpha = \phi(x_\alpha)$ with jacobian J_ϕ, we know that in these coordinates we have

$$\tilde{g} = \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right)^T g \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right) = \left(J_\phi^{-1}\right)^T g \left(J_\phi^{-1}\right)$$

and so

$$\sqrt{\det \tilde{g}} = \det J^{-1} \sqrt{\det g}.$$

On the other hand we have

$$d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J dx_1 \wedge \cdots \wedge dx_n$$

and so

$$\sqrt{\tilde{g}} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J^{-1} \sqrt{\det g} \det J dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$$

□

Definition 1.3.3. *An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following properties for any smooth functions $f_1, f_2 \in C^\infty(M)$ and any smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$*

- $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$
- $\nabla_X Z + Y = \nabla_X Z + \nabla_X Y$
- $\nabla_X f_1 Y = X(f_1) Y + f \nabla_X Y$

Definition 1.3.4. *A Levi-Civita connection is an affine connection which also satisfies*

- *Symmetry:* $\nabla_X Y - \nabla_Y X = [X, Y]$
- *Compatability with g:* $X(\langle Y, Z\rangle_g) = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$

Remark 1.3.5. Compatability with g is essentially like the product rule.

Theorem 1.3.6 (Fundamental theorem of Riemannian Geometry)**.** *For every Riemannian manifold there exists a unique Levi-Civita Connection.*

Proof. Take any smooth vector fields X, Y, Z, we know that the following are true

$$\begin{aligned} X(\langle Y, Z\rangle_g) &= \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g \\ Y(\langle Z, X\rangle_g) &= \langle \nabla_Y Z, X\rangle_g + \langle Z, \nabla_Y X\rangle_g \\ Z(\langle X, Y\rangle_g) &= \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Z Y\rangle_g \end{aligned}$$

then by adding the first two equations and subtracting the third we get

$$\begin{aligned}X(\langle Y,Z\rangle_g)+Y(\langle Z,X\rangle_g)-Z(\langle X,Y\rangle_g) &= \langle Y,\nabla_X Z\rangle_g-\langle \nabla_Z X,Y\rangle_g\\ &\quad+\langle \nabla_Y Z,X\rangle_g-\langle X,\nabla_Z Y\rangle_g\\ &\quad+\langle \nabla_X Y,Z\rangle_g+\langle Z,\nabla_Y X\rangle_g\end{aligned}$$

using the symmetry of the connection we get

$$\begin{aligned}X(\langle Y,Z\rangle_g)+Y(\langle Z,X\rangle_g)-Z(\langle X,Y\rangle_g) &= \langle Y,[X,Z]\rangle_g+\langle [Y,Z],X\rangle_g+\langle [X,Y],Z\rangle_g\\ &\quad+2\langle Z,\nabla_Y X\rangle_g\end{aligned}$$

from here we can solve for $\langle Z,\nabla_Y X\rangle_g$ giving us the connection since as a vector, $\nabla_Y X$ is fully determined by its inner products with all other vectors. □

One can check that in a coordinate chart that the Levi Civita connection has the form

$$\begin{aligned}\nabla_X Y &= \nabla_{\sum_i a_i(x)\frac{\partial}{\partial x_i}}\sum_j b_j(x)\frac{\partial}{\partial x_j}\\ &= \sum_i a_i(x)\left(\nabla_{\frac{\partial}{\partial x_i}}\sum_j b_j(x)\frac{\partial}{\partial x_j}\right)\\ &= \sum_{i,j} a_i(x)\left(\left(\frac{\partial}{\partial x_i}b_j(x)\right)\frac{\partial}{\partial x_j}+b_j(x)\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right).\end{aligned}$$

Now we know that for some coefficients Γ_{ij}^k we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\sum_k \Gamma_{ij}^k\frac{\partial}{\partial x_k}$$

and so

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g=\sum_k \Gamma_{ij}^k g_{k\ell}$$

Now by the previous proof and the fact that coordinate vector fields have vanishing brackets we have that

$$\begin{aligned}2\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g &= \frac{\partial}{\partial x_j}\left(\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_\ell}\right\rangle_g\right)+\frac{\partial}{\partial x_i}\left(\left\langle\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g\right)-\frac{\partial}{\partial x_\ell}\left(\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right\rangle_g\right)\\ &= \frac{\partial}{\partial x_j}(g_{i\ell})+\frac{\partial}{\partial x_i}(g_{j\ell})-\frac{\partial}{\partial x_\ell}(g_{ij})\end{aligned}$$

and so by using the inverse of the metric we get

$$\Gamma_{ij}^k=\frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x_j}(g_{i\ell})+\frac{\partial}{\partial x_i}(g_{j\ell})-\frac{\partial}{\partial x_\ell}(g_{ij})\right).$$

The coefficients Γ are often called the Christoffel Symbols of g in these coordinates.

Claim 1.3.7. At any point p there exists a local coordinate chart $(x_1, \ldots, x_n)$ such that

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial}{\partial x_i}(g_{jk})(p) = 0$$

Proof. We have $g_{ij}(x) = g_{ij}(0) + \sum_k a_{ij}^k x_k + O(|X|^2)$, we can always change variables so that $g_{ij}(0) = \delta_{ij}$. The tricky part is eliminating the first derivatives, for that we do a change of coordinates

$$y_\alpha = \phi(x_\alpha) = x_\alpha + \frac{1}{2} b_\alpha^{k\ell} x_k x_\ell + O(|X|^3).$$

The jacobian of this transformation is

$$J_{\phi^{-1}} = I - b_\alpha^{k\ell} x_\ell + O(|X|^3)$$

and so the new metric is

$$\begin{aligned}\tilde{g}_{\alpha\beta} = J_{\phi^{-1}}^T g J_{\phi^{-1}} &= (I - b_\alpha^{i\ell} x_\ell + O(|X|^3))^T (I + a_{ij}^m x_m)(I - b_\beta^{j\ell} x_\ell + O(|X|^3)) \\ &= I - 2b_\alpha^{i\ell} g_{i\beta} + a_{ij}^\ell x_\ell + O(|X|)^2,\end{aligned}$$

then from here you can solve for b. □

Geometric constructions

We now have several natural constructions once we fix a metric on our manifold.

Consider a vector field X and a point p on a Riemannian manifold, the map $P : T_p(M) \to T_p(M)$, given by

$$v \mapsto \nabla_v X$$

is a linear map. We define its trace to be the divergence of X, denoted $\operatorname{div}(X)$.

In a local orthonormal chart at p, if we write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$, then

$$\begin{aligned}\operatorname{div}(X)_p &= \sum_i \left\langle \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \left\langle \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g \\ &= \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \delta_{ij} = \sum_i \nabla_{\frac{\partial}{\partial x_i}} a_i(x) \\ &= \sum_i \frac{\partial a_i(x)}{\partial x_i}\end{aligned}$$

Where we used the fact that in an orthonormal frame $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. We see then that in an orthonormal frame the divergence matches our 'classical' definition of the divergence.

Next consider a function $f \in C^\infty(M)$, we define the gradient to be a map $\operatorname{grad} : C^\infty(M) \to \mathfrak{X}(M)$ defined by

$$\langle \operatorname{grad} f, v \rangle_g = df(v)$$

for every tangent vector v.

In a local (not necessarily orthonormal) chart we have

$$\operatorname{grad} f = \sum_j a_j(x) \frac{\partial}{\partial x_j}, df = \sum_k \frac{\partial f}{\partial x_k} dx_k,$$

then for any $v = \sum_\ell b_\ell \frac{\partial}{\partial x_\ell}$ we have

$$\langle \operatorname{grad} f, v \rangle_g = \sum_{j,\ell} a_j g_{j\ell} b_\ell$$

but we also have

$$df(v) = \sum_{k,\ell} \frac{\partial f}{\partial x_k} b_\ell dx_k \left(\frac{\partial}{\partial x_\ell} \right) = \sum_k \frac{\partial f}{\partial x_k} b_k.$$

Now lets choose $b = (0, 0, \ldots, 1, \ldots, 0, 0)$ with a 1 in the m-th position then

$$\langle \operatorname{grad} f, v \rangle_g = \sum_j a_j g_{jm}$$

and

$$df(v) = \frac{\partial f}{\partial x_m}$$

so since these are equal we can multiply both by the inverse of the metric g^{mi} to get

$$a_i = \sum_{j,m} a_j g_{jm} g^{mi} = \sum_m \frac{\partial f}{\partial x_m} g^{mi}$$

and thus

$$\operatorname{grad} f = \sum_i a_i \frac{\partial}{\partial x_i} = \sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i}$$

Finally again for a function $f \in C^\infty(M)$, the hessian is defined as the map Hess : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$X \mapsto \nabla_X(\operatorname{grad} f)$$

Let us write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$ then we have by the previous results that in an orthonormal chart around p

$$\begin{aligned}
\nabla_X(\operatorname{grad} f) &= \nabla_X\left(\sum_{m,i}\frac{\partial f}{\partial x_m}g^{mi}\frac{\partial}{\partial x_i}\right)\\
&= \nabla_X\left(\sum_{i}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}\right) \quad \text{(because } g^{mi}=\delta^{mi} \text{ at } p \text{ in orthonormal chart)}\\
&= \sum_{j,i} a_j(x)\left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\Gamma^k_{ij}\frac{\partial}{\partial x_k}\right)\\
&= \sum_{j,i} a_j(x)\left(\left(\frac{\partial f}{\partial x_j \partial x_i}\right)\frac{\partial}{\partial x_i}\right) \quad \text{(because } \Gamma^k_{ij}=0 \text{ at } p \text{ in orthonormal chart)}
\end{aligned}$$

and so if $Y=\sum_\ell b_\ell(x)\frac{\partial}{\partial x_\ell}$ we have

$$\langle \nabla_X(\operatorname{grad} f), Y\rangle_g = \sum_{j,\ell} a_j(x)\left(\frac{\partial f}{\partial x_j \partial x_i}\right) b_\ell(x).$$

Importantly notice that if we exchange a and b then this expression does not change and so $\langle \nabla_X(\operatorname{grad} f), Y\rangle_g = \langle \nabla_Y(\operatorname{grad} f), X\rangle_g$ and so as an operator Hess is symmetric. We also get that the form in orthonormal coordinates for the operator is the matrix

$$\frac{\partial f}{\partial x_j \partial x_i}$$

Now we consider the trace of the modified hessian operator, given by $\operatorname{div}(h\cdot \operatorname{grad} f)$. Notice that we have, in an orthonormal chart,

$$\operatorname{div}(h\cdot \operatorname{grad} f) = \sum_j \frac{\partial}{\partial x_j}E_j = \sum_j \frac{\partial}{\partial x_j}\left(h\sum_k g^{jk}\frac{\partial f}{\partial x_k}\right)$$

Claim 1.3.8. In a general local chart,

$$\operatorname{div}(h\cdot \operatorname{grad} f) = (\det g)^{-1/2}\sum_{i,j}\frac{\partial}{\partial x_j}\left(h(\det g)^{1/2}g^{ij}\frac{\partial f}{\partial x_i}\right)$$

Proof. It is enough to show that the expression on the right is coordinate invariant, since then plugging in an orthonormal chart gives us the desired result.

To see this consider a different chart $(\tilde{x}_1,\ldots,\tilde{x}_n)$ and set

$$\begin{aligned}
Q &= (\det g)^{-1/2}\sum_{i,j}\frac{\partial}{\partial x_j}\left(h(\det g)^{1/2}g^{ij}\frac{\partial f}{\partial x_i}\right)\\
\tilde{Q} &= (\det \tilde{g})^{-1/2}\sum_{i,j}\frac{\partial}{\partial \tilde{x}_j}\left(h(\det \tilde{g})^{1/2}\tilde{g}^{ij}\frac{\partial f}{\partial \tilde{x}_i}\right)
\end{aligned}$$

then consider the set of functions η with support contained within both charts, if

$$\int_\Omega Q\eta dV = \int_\Omega \tilde{Q}\eta dV$$

then $Q = \tilde{Q}$.
Now we plug in our known expressions and get

$$\int_\Omega Q\eta dV = \int_\Omega \eta \sum_j \frac{\partial}{\partial x_j}\left(h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n$$

then we notice that we have the a divergence term in the integral. Then by using integration by parts we can remove that divergence and instead take the gradient of η, the boundary term then dissapears by compactness of η. All together this gives us

$$\begin{aligned}
\int_\Omega Q\eta dV &= -\int_\Omega \sum_j \left(\frac{\partial \eta}{\partial x_j}\right)\left(h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n \\
&= -\int_\Omega h \sum_{j,i}\left(g^{ij}\frac{\partial \eta}{\partial x_j}\frac{\partial f}{\partial x_i}\right)(\det g)^{1/2} dx_1 dx_2 \dots dx_n \\
&= -\int_\Omega h \left\langle \operatorname{grad}\eta, \operatorname{grad} f\right\rangle_g dV
\end{aligned}$$

now notice that the same calculation holds in the second chart, and so we get

$$\int_\Omega Q\eta dV = \int_\Omega Q'\eta dV$$

□

Theorem 1.3.9 (Divergence theorem)**.** *Suppose that $\Omega \subseteq M$ is a compact domain with a smooth boundary $\partial\Omega$, then $\forall f, h \in C^\infty(M)$ we have*

$$\int_\Omega \operatorname{div}(h \operatorname{grad} f) dV = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu\rangle_g d\tilde{V}$$

where ν is the normal vector and $d\tilde{V}$ is the induced volume form on the metric.

Proof. Find a partition of unity for some neighborhood of Ω, that is a collection of functions ρ_k with $\sum_k \rho_k = 1$ and the support of each ρ_k being contained in a single chart U_k. Now we have

$$\int_\Omega \operatorname{div}(h \operatorname{grad} f) dV = \sum_k \int_\Omega \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \sum_k \int_{\Omega\cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV$$

now for each of these smaller integrals we have

$$\int_{\Omega\cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \int_{\Omega\cap U_k} \sum_j \frac{\partial}{\partial x_j}\left(\rho_k h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n.$$

We will now apply IBP to this integral, note that the interior term will contain a derivative of 1 and so will vanish, the boundary term will only be non-zero outside of the boundary of U_k, that is it will be non-zero only on $\partial\Omega \cap U_k$.
So this integral becomes

$$\int_{\partial\Omega\cap U_k} \sum_j \nu_j \left(\rho_k h(\det g)^{1/2}\sum_i g^{ij}\frac{\partial f}{\partial x_i}\right) d\tilde{x}_1 d\tilde{x}_2 \dots d\tilde{x}_{n-1},$$

which simplifies to

$$\int_{\partial\Omega\cap U_k} \langle \operatorname{grad} f, \nu\rangle_g (\rho_k h)\, d\tilde{V}.$$

This then gets summed up over k to give

$$\sum_k \int_{\partial\Omega\cap U_k} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g d\tilde{V} = \sum_k \int_{\partial\Omega} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g d\tilde{V} = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu\rangle_g d\tilde{V}$$

□

Theorem 1.3.10. *$\forall h \in C^\infty(M)$ with $h > 0$ in Ω a compact connected open set, the system*

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f \quad &\in \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad &\in \partial\Omega \end{aligned}$$

is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} hg$.

Proof. We follow a similar proof to 1.1.4, first assume $g = 0$, then we have in the space of functions in $W^{2,1}(\Omega)$ that are zero on the boundary the image of the operator $u \mapsto \operatorname{div}(h \operatorname{grad} u)$ is orthogonal to its kernel. Now the set of functions in the kernel are those satisfying

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= 0 \quad &\in \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \quad &\in \partial\Omega \end{aligned}$$

and so for those functions by the divergence theorem we get

$$0 = \int_\Omega u \operatorname{div}(h \operatorname{grad} u) dV = -\int_\Omega h |\operatorname{grad} u|^2 dV$$

and so since $h > 0$ we get $\operatorname{grad} u = 0$ everywhere on Ω and so it is constant on Ω. Thus the image is those functions f that are orthogonal to constant functions, that is the system is solvable if and only if

$$\int_\Omega f dV = 0$$

Next we identically construct a function γ which is C^2 everywhere on Ω and satisfying $\frac{\partial \gamma}{\partial \nu} = -1$. We then define $U(x) = u(x) + \gamma(x) g(x)$ and notice that since

$$\operatorname{div}(h \operatorname{grad} U) = \operatorname{div}(h \operatorname{grad}(u(x) + \gamma(x)g(x))) = \operatorname{div}(h \operatorname{grad} u(x)) + \operatorname{div}(h \operatorname{grad}(\gamma(x)g(x)))$$

and

$$\frac{\partial U}{\partial \nu} = \frac{\partial u}{\partial \nu} + \frac{\partial(\gamma \cdot g)}{\partial \nu} = g - g = 0$$

then we have a solution U if and only if

$$\begin{aligned} 0 = \int_\Omega \operatorname{div}(h \operatorname{grad} U) dV &= \int_\Omega \operatorname{div}(h \operatorname{grad} u) dV + \int_\Omega \operatorname{div}(h \operatorname{grad}(\gamma(x)g(x))) dV \\ = \int_\Omega f dV + \int_{\partial\Omega} h \frac{\partial(\gamma \cdot g)}{\partial \nu} d\tilde{V} &= \int_\Omega f dV + \int_{\partial\Omega} -hg d\tilde{V} \end{aligned}$$

$\square$