Math 595: Geometric Analysis

Jacob Reznikov

September 10, 2023

Abstract

My course notes for the Geometric Analysis course.

1 ABP and Basic Geometry

1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where B is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\Delta u = c \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.
For such a map we set $T = \nabla u$ to be the gradient map $\Omega \to \mathbb{R}^n$. We now want a characterization of the 'extremal' points of u as a graph, we define

$$\Gamma_u^- = \left\{ x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega \right\}.$$

In other words Γ_u^- are the points of Ω where the tangent plane lies entirely below the graph of u.
This set is called the 'contact' set.

Remark 1.1.1. For any point x in the contact set we have $\nabla^2 u(x) \geq 0$ where ∇^2 is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

Claim 1.1.2 (ABP)**.** For a solution u of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

Proof. Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so $\tilde{u}$ cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.
To see that x is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

□

Claim 1.1.3. If a solution u to the above PDE exists then we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

Proof. Then

$$\begin{aligned}
|B_1| \leq |T(\Gamma_u^-)| &\leq \int_{\Gamma_u^-} J_T = \int_{\Gamma_u^-} \det(\nabla^2 u) \\
&= \int_{\Gamma_u^-} \lambda_1 \lambda_2 \cdots \lambda_n \\
&\leq \int_{\Gamma_u^-} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n \quad \text{Since all the eigenvalues are positive.} \\
&\leq \int_{\Gamma_u^-} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \int_{\Omega} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Omega| = \frac{|\partial\Omega|^n}{n^n |\Omega|^{n-1}}
\end{aligned}$$

and since $|B| = \frac{1}{n}|\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial\Omega
\end{aligned}$$

then a necessary condition for existence is

$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

Claim 1.1.4. The above condition is sufficient.

Proof. Assume first that $h = 0$. Thus the condition above becomes $\int_\Omega F = 0$. Then take the positive definite symmetric bilinear form $B(u, v) = \int_\Omega \nabla u \nabla v$ and notice

$$B(u, v) = (Lu, v)$$

and so L is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any g in the kernel we have by our boundary conditions

$$0 = \int_\Omega gLg = \int_\Omega |\nabla g|^2$$

and so g is a constant function.

Thus we have a solution if and only if

$$\int_\Omega F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is C^2 then $\rho(x) = d(x, \partial\Omega)$ is C^2 in Ω near $\partial\Omega$, we then choose a cutoff function η satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is C^2 everywhere on Ω and as we approach the boundary we will have $\frac{\partial \gamma}{\partial \nu} = -1$.

Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for U exists if and only if

$$0 = \int_\Omega \Delta U = \int_\Omega \Delta u + \Delta(h\gamma) = \int_\Omega f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial \nu} = \int_\Omega f - \int_{\partial\Omega} h$$

and so we get our desired result. □

1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the 'upper contact' set, notice that we no longer require u to be differentiable. In conjuction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

Remark 1.2.1. If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

Remark 1.2.2. If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

Example 1.2.3. $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in $\mathbb{R}^{n+1}$.
We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R}\frac{x - z}{|x - z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a\left(1 - \frac{|y - z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y - z}{|y - z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Lemma 1.2.4.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma_u^+} |\det(\nabla^2)| \right)^{1/n}$$

Proof. Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).
Now consider Γ_v^+, we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of Ω such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane P given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to x_0, thus $v - l$ must achieve its maximum somewhere on the interior of Ω where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where $\tilde{h}$ is given by

$$\tilde{h}(x) = v(x_0)\left(1 - \frac{x - x_0}{d}\right).$$

We can see this because $\tilde{h}$ is just a cone with a wider base than h and thus its supporting hyperplanes must have smaller gradients.

But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right)\omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) \geq f \quad \text{in } \Omega$$

Lemma 1.2.5. *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}}\left(\int_{\Gamma_u^+}\left(\frac{-f}{\det^{1/n}(a_{ij})}\right)^n\right)^{1/n}$$

Remark 1.2.6. If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

Lemma 1.2.7. *For symmetric positive matrices A, B we have*

$$\det(A)\det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n$$

Proof. Left side is equal to product of all eigenvalues, $\lambda_1\tilde{\lambda}_1 \cdots \lambda_n\tilde{\lambda}_n$.
$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1\tilde{\lambda}_1 + \cdots + \lambda_n\tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. □

Proof. Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_i j) > 0$ then

$$-f = -Lu = \operatorname{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_i j))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. □

This lemma is sometimes called the weak maximum principle.

Remark 1.2.8. There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) + \sum_k b_k(x)u_k(x) + c(x)u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients b_k and c.