# Math 595: Geometric Analysis

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September 12, 2023

**Abstract**

My course notes for the Geometric Analysis course.

# 1 ABP and Basic Geometry

## 1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where $B$ is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\Delta u = c \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.
For such a map we set $T = \nabla u$ to be the gradient map $\Omega \to \mathbb{R}^n$. We now want a characterization of the 'extremal' points of $u$ as a graph, we define

$$\Gamma_u^- = \{x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega\}.$$

In other words $\Gamma_u^-$ are the points of $\Omega$ where the tangent plane lies entirely below the graph of $u$.
This set is called the 'contact' set.

**Remark 1.1.1.** For any point $x$ in the contact set we have $\nabla^2 u(x) \geq 0$ where $\nabla^2$ is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

**Claim 1.1.2** (ABP)**.** For a solution $u$ of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

*Proof.* Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so $\tilde{u}$ cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.
To see that $x$ is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

□

**Claim 1.1.3.** If a solution $u$ to the above PDE exists then we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

*Proof.* Then

$$\begin{aligned}
|B_1| \leq |T(\Gamma_u^-)| \leq \int_{\Gamma_u^-} J_T &= \int_{\Gamma_u^-} \det(\nabla^2 u) \\
&= \int_{\Gamma_u^-} \lambda_1 \lambda_2 \cdots \lambda_n \\
&\leq \int_{\Gamma_u^-} \left( \frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n \quad \text{Since all the eigenvalues are positive.} \\
&\leq \int_{\Gamma_u^-} \left( \frac{\Delta u}{n} \right)^n \\
&\leq \int_{\Omega} \left( \frac{\Delta u}{n} \right)^n \\
&\leq \left( \frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Omega| = \frac{|\partial\Omega|^n}{n^n |\Omega|^{n-1}}
\end{aligned}$$

and since $|B| = \frac{1}{n}|\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial\Omega
\end{aligned}$$

then a necessary condition for existence is

$$\int_{\Omega} f = \int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

**Claim 1.1.4.** The above condition is sufficient.

*Proof.* Assume first that $h = 0$. Thus the condition above becomes $\int_{\Omega} F = 0$. Then take the positive definite symmetric bilinear form $B(u, v) = \int_{\Omega} \nabla u \nabla v$ and notice

$$B(u, v) = (Lu, v)$$

and so $L$ is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any $g$ in the kernel we have by our boundary conditions

$$0 = \int_{\Omega} gLg = \int_{\Omega} |\nabla g|^2$$

and so $g$ is a constant function.

Thus we have a solution if and only if

$$\int_{\Omega} F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is $C^2$ then $\rho(x) = d(x, \partial\Omega)$ is $C^2$ in $\Omega$ near $\partial\Omega$, we then choose a cutoff function $\eta$ satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is $C^2$ everywhere on $\Omega$ and as we approach the boundary we will have $\frac{\partial \gamma}{\partial \nu} = -1$.

Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for $U$ exists if and only if

$$0 = \int_{\Omega} \Delta U = \int_{\Omega} \Delta u + \Delta(h\gamma) = \int_{\Omega} f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial \nu} = \int_{\Omega} f - \int_{\partial\Omega} h$$

and so we get our desired result. □

## 1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the ‘upper contact’ set, notice that we no longer require $u$ to be differentiable. In conjuction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

**Remark 1.2.1.** If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

**Remark 1.2.2.** If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

**Example 1.2.3.** $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in $\mathbb{R}^{n+1}$.
We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R}\frac{x - z}{|x - z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a\left(1 - \frac{|y - z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y - z}{|y - z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

**Lemma 1.2.4.**

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d(\Omega)}{\omega_n^{1/n}} \left( \int_{\Gamma_u^+} |\det(\nabla^2)| \right)^{1/n}$$

*Proof.* Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).
Now consider $\Gamma_v^+$, we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of $\Omega$ such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane $P$ given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to $x_0$, thus $v - l$ must achieve its maximum somewhere on the interior of $\Omega$ where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where $\tilde{h}$ is given by

$$\tilde{h}(x) = v(x_0)\left(1 - \frac{x - x_0}{d}\right).$$

We can see this because $\tilde{h}$ is just a cone with a wider base than $h$ and thus its supporting hyperplanes must have smaller gradients.

But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right) \omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) \geq f \quad \text{in } \Omega$$

**Lemma 1.2.5.** *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}} \left(\int_{\Gamma_u^+} \left(\frac{-f}{\det^{1/n}(a_{ij})}\right)^n\right)^{1/n}$$

**Remark 1.2.6.** If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

**Lemma 1.2.7.** *For symmetric positive matrices $A, B$ we have*

$$\det(A)\det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n$$

*Proof.* Left side is equal to product of all eigenvalues, $\lambda_1\tilde{\lambda}_1 \cdots \lambda_n\tilde{\lambda}_n$.
$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1\tilde{\lambda}_1 + \cdots + \lambda_n\tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. □

*Proof.* Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_i j) > 0$ then

$$-f = -Lu = \operatorname{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_i j))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. □

This lemma is sometimes called the weak maximum principle.

**Remark 1.2.8.** There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) + \sum_k b_k(x)u_k(x) + c(x)u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients $b_k$ and $c$.

## 1.3 Introduction to Riemannian Geometry

Let $M^n$ be an $n$-dimensional manifold, every point $p \in M^n$ has a tangent space $T^pM$, then a metric $g$ on $M^n$ is a choice of inner product on $T_pM$ for every $p \in M$ which varies smoothly in $p$. A manifold with a metric is called a Riemannian Manifold.
In any local coordinate chart $(x_1, \ldots, x_n)$ we define the 'components' of $g$ to be

$$g_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle.$$

Then if at some point $p$ we have two vectors

$$X = \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k}$$

then their inner product is given by

$$\langle X, Y \rangle_g = \left\langle \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j}, \sum_{k=1}^{N} b_k(x) \frac{\partial}{\partial x_k} \right\rangle = \sum_{j,k} a_j(x) b_k(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle$$
$$= \sum_{j,k} a_j(x) b_k(x) g_{jk}(x)$$

More formally, let $dx_i$ be the dual frame to $\frac{\partial}{\partial x_i}$, as in

$$dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_i^j,$$

then we can write the metric as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on $M$.
If $e_1, \ldots, e_n \in T_pM$ is an orthonormal basis, that is $\langle e_i, e_j \rangle_g = \delta_{ij}$. Set $\omega_1, \ldots, \omega_n$ to be its dual basis. We then get a top-form $\omega_1 \wedge \cdots \wedge \omega_n$.
If

$$e_j = \sum_k a_j^k \frac{\partial}{\partial x_k}$$

where $A = a_j^k$ is a matrix, then by standard linear algebra we have that

$$\omega_1 \wedge \cdots \wedge \omega_n = \det(A^{-1}) dx_1 \wedge \cdots \wedge dx_n$$

**Claim 1.3.1.**

$$|\det(A^{-1})| = \sqrt{\det g}$$

*Proof.*

$$\delta_{ij} = (e_i, e_j) = a_j^k a_i^l g_{kl}$$

this implies that

$$I = A^T g A$$

where $A$ is the transpose.
Thus

$$1 = \det(A^T g A) = \det(A^2) \det(g)$$

and so

$$\sqrt{\det(g)} = \det A^{-1}$$

□

**Claim 1.3.2.** The top-form $dV = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$ is coordinate change invariant.

*Proof.* Let us assume that $(\tilde{x}_1, \ldots, \tilde{x}_n)$ are coordinates given by the transition function $\tilde{x}_\alpha = \phi(x_\alpha)$ with jacobian $J_\phi$, we know that in these coordinates we have

$$\tilde{g} = \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right)^T g \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right) = \left(J_\phi^{-1}\right)^T g \left(J_\phi^{-1}\right)$$

and so

$$\sqrt{\det \tilde{g}} = \det J^{-1} \sqrt{\det g}.$$

On the other hand we have

$$d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J dx_1 \wedge \cdots \wedge dx_n$$

and so

$$\sqrt{\tilde{g}} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J^{-1} \sqrt{\det g} \det J dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$$

□

**Definition 1.3.3.** *An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following properties for any smooth functions $f_1, f_2 \in C^\infty(M)$ and any smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$*

- $$\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$$
- $$\nabla_X Z + Y = \nabla_X Z + \nabla_X Y$$
- $$\nabla_X f_1 Y = X(f_1) Y + f \nabla_X Y$$

**Definition 1.3.4.** *A Levi-Civita connection is an affine connection which also satisfies*

- *Symmetry*
$$\nabla_X Y - \nabla_Y X = [X, Y]$$
- *Compatability with $g$*
$$X(\langle Y, Z \rangle_g) = \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g$$

**Theorem 1.3.5** (Fundamental theorem of Riemannian Geometry)**.** *For every Riemannian manifold there exists a unique Levi-Civita Connection.*

*Proof.* Take any smooth vector fields $X, Y, Z$, we know that the following are true

$$\begin{aligned}
X(\langle Y, Z\rangle_g) &= \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g \\
Y(\langle Z, X\rangle_g) &= \langle \nabla_Y Z, X\rangle_g + \langle Z, \nabla_Y X\rangle_g \\
Z(\langle X, Y\rangle_g) &= \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Z Y\rangle_g
\end{aligned}$$

then by adding the first two equations and subtracting the third we get

$$\begin{aligned}
X(\langle Y, Z\rangle_g) + Y(\langle Z, X\rangle_g) - Z(\langle X, Y\rangle_g) &= \langle Y, \nabla_X Z\rangle_g - \langle \nabla_Z X, Y\rangle_g \\
&\quad + \langle \nabla_Y Z, X\rangle_g - \langle X, \nabla_Z Y\rangle_g \\
&\quad + \langle \nabla_X Y, Z\rangle_g + \langle Z, \nabla_Y X\rangle_g
\end{aligned}$$

using the symmetry of the connection we get

$$\begin{aligned}
X(\langle Y, Z\rangle_g) + Y(\langle Z, X\rangle_g) - Z(\langle X, Y\rangle_g) &= \langle Y, [X, Z]\rangle_g + \langle [Y, Z], X\rangle_g + \langle [X, Y], Z\rangle_g \\
&\quad + 2\langle Z, \nabla_Y X\rangle_g
\end{aligned}$$

from here we can solve for $\langle Z, \nabla_Y X\rangle_g$ giving us the connection since as a vector, $\nabla_Y X$ is fully determined by its inner products with all other vectors. □

One can check that in a coordinate chart that the Levi Civita connection has the form

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_i a_i(x)\frac{\partial}{\partial x_i}} \sum_j b_j(x) \frac{\partial}{\partial x_j} \\
&= \sum_i a_i(x) \left( \nabla_{\frac{\partial}{\partial x_i}} \sum_j b_j(x) \frac{\partial}{\partial x_j} \right) \\
&= \sum_{i,j} a_i(x) \left( \left( \frac{\partial}{\partial x_i} b_j(x) \right) \frac{\partial}{\partial x_j} + b_j(x) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right).
\end{aligned}$$

Now we know that for some coefficients $\Gamma_{ij}^k$ we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

and so

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g = \sum_k \Gamma_{ij}^k g_{k\ell}$$

Now by the previous proof and the fact that coordinate vector fields have vanishing brackets we have that

$$\begin{aligned}
2\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g &= \frac{\partial}{\partial x_j}\left( \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_\ell} \right\rangle_g \right) + \frac{\partial}{\partial x_i}\left( \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell} \right\rangle_g \right) - \frac{\partial}{\partial x_\ell}\left( \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g \right) \\
&= \frac{\partial}{\partial x_j}(g_{i\ell}) + \frac{\partial}{\partial x_i}(g_{j\ell}) - \frac{\partial}{\partial x_\ell}(g_{ij})
\end{aligned}$$

and so by using the inverse of the metric we get

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell} \left( \frac{\partial}{\partial x_j} (g_{i\ell}) + \frac{\partial}{\partial x_i} (g_{j\ell}) - \frac{\partial}{\partial x_\ell} (g_{ij}) \right).$$

The coefficients $\Gamma$ are often called the Christoffel Symbols of $g$ in these coordinates.

**Claim 1.3.6.** At any point $p$ there exists a local coordinate chart $(x_1, \ldots, x_n)$ such that

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial}{\partial x_i} (g_{jk}) (p) = 0$$

*Proof.* We have $g_{ij}(x) = g_{ij}(0) + \sum_k a_{ij}^k x_k + O(|X|^2)$, we can always change variables so that $g_{ij}(0) = \delta_{ij}$. The tricky part is eliminating the first derivatives, for that we do a change of coordinates

$$y_\alpha = \phi(x_\alpha) = x_\alpha + \frac{1}{2} b_\alpha^{k\ell} x_k x_\ell + O(|X|^3).$$

The jacobian of this transformation is

$$J_{\phi^{-1}} = I - b_\alpha^{k\ell} x_\ell + O(|X|^3)$$

and so the new metric is

$$\begin{aligned} \tilde{g}_{\alpha\beta} = J_{\phi^{-1}}^T g J_{\phi^{-1}} &= (I - b_\alpha^{i\ell} x_\ell + O(|X|^3))^T (I + a_{ij}^m x_m)(I - b_\beta^{j\ell} x_\ell + O(|X|^3)) \\ &= I - 2 b_\alpha^{i\ell} g_{i\beta} + a_{ij}^\ell x_\ell + O(|X|)^2, \end{aligned}$$

then from here you can solve for $b$. $\square$