Math 577: Geometry And Topology 2

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February 8, 2022

Abstract

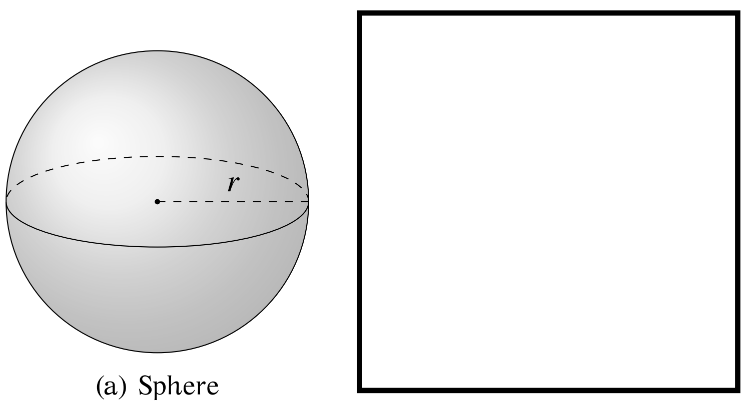
My course notes for Geometry And Topology 2

1 Smooth Manifolds

1.1 Starting Definitions

Smooth Manifolds are topological spaces that look locally like $\mathbb{R}^n$ where we can do calculus.

Some examples of these are



(a) Sphere

(b) $\mathbb{R}^n$

Figure 1: Different Manifolds

Before we get to smoothness though, we need to first discuss a simpler type of manifold.

Definition 1.1.1. *A* ***Topological Manifold*** *of dimension n is a topological space M with the following properties:*

- *M is Locally-Euclidean, i.e. every point $p \in M$ has a neighborhood U which is homeomorphic to an open set in $\mathbb{R}^n$*

- *M is Hausdorff, i.e. any two distinct points p, q lie in disjoint open neighborhoods.*

- *M is second countable, i.e. has a countable basis of open sets.*

Remark 1.1.2. Any subspace of $\mathbb{R}^n$ automatically satisfies the second and third conditions.

Example 1.1.3. $R^0 \simeq \{p\}$ is a Topological Manifold of dimension 0.

Example 1.1.4. $M = \mathbb{R}^n$ is a Topological Manifold of dimension n

Example 1.1.5. V is an n-dimensional $\mathbb{R}$-vector space with the topology induced by a norm (all norms induce the same topology), then $V \simeq \mathbb{R}^n$ and so it is also a Topological Manifold of dimension n.

Example 1.1.6. For any open set $U \subseteq \mathbb{R}^n$ and continuous function $f : U \to \mathbb{R}$ the graph of the function defined by

$$\Gamma(f) = \{(x, f(x)) : x \in U\}$$

is a Topological manifold of dimension n by the projection to U.

Example 1.1.7. The sphere S^n is always a Topological n-Manifold, it gets the second and third properties from being a subset of $\mathbb{R}^{n+1}$. On the other hand if we consider any point on the sphere then by rotation we can assume it is the north pole of the sphere and then the upper hemisphere of the sphere at that point can be described by the graph

$$S^n_+ := \{p = (x^0, \dots, x^n) | \, ||x|| = 1, x^n > 0\} = \Gamma\left((x^0, \dots, x^{n-1}) \mapsto \sqrt{1 - (x^0)^2 - \cdots - (x^{n-1})^2}\right)$$

Example 1.1.8. Similarly Balls B^n are also Topological n-Manifolds.

Proposition 1.1.9. Let M be a topological manifold. Then the following statements hold:

- M is normal, i.e. any two disjoint closed subsets have disjoint open neighborhoods.

- M is locally connected, i.e. every point has a connected neighborhood.

- M is locally compact, i.e. every point has a neighborhood whose closure is compact.

- M is paracompact (see later).

- M is metrizable, i.e. there exists a metric d on M that induces the topology.

Definition 1.1.10. *A **Chart** of an n-Manifold M is an open set U along with a map $\phi : U \to \mathbb{R}^n$ such that $\phi(U)$ is an open set and ϕ is homeomorphic onto its image.*

Remark 1.1.11. A Chart (ϕ, U) is an identification of every point $p \in U$ with a point

$$\phi(p) = (\phi(p)^0, \dots, \phi(p)^n)$$

in $\mathbb{R}^n$.

Example 1.1.12. The Cartesian coordinates on $\mathbb{R}^2$ are a chart with $U = \mathbb{R}^2$ and ϕ being the identity function.

Example 1.1.13. Similarly the Polar coordinates are also a chart with

$$U = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ if } y = 0\} = \mathbb{R}^2 \setminus (-\infty, 0] \times 0$$

$\phi : (x, y) \to (r, \theta)$.

We then have that $\phi(U) = [0, \infty) \times (-\pi, \pi)$

Remark 1.1.14. Given a continuous function $f : U \to X$ on the domain of a chart (ϕ, U) we can identify points of U with points of $\phi(U)$ giving us the coordinate form

$$f \circ \phi^{-1} : \phi(U) \to X, f : (\phi(p)^0, \dots, \phi(p)^n) \mapsto f(p)$$

Definition 1.1.15. *Given two charts (ϕ_1, U_1) and (ϕ_2, U_2) we define the **transition function** between the two charts as*

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \text{ and } \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

These are both

1.2 Transition functions and Smooth Structures

Example 1.2.1. Let us consider the transition function between cartesian coordinates $(\mathrm{Id}, \mathbb{R}^2)$ and polar coordinates $(\phi, \mathbb{R}^2 \setminus (-\infty, 0] \times [0])$ then the transition map from polar coordinates to cartesian is

$$\mathrm{Id} \circ \phi^{-1} = (r\cos(\theta), r\sin(\theta))$$

and the other transition map is

$$\phi \circ \mathrm{Id}^{-1} = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)$$

Example 1.2.2. Now consider the 2 sphere S^2 the first chart being spherical coordinates $(\phi_1, S^2 \setminus \{x \leq 0, y = 0, z \in \mathbb{R}\})$ and the stereographic projection $(\phi_2, S^2 \setminus \{(0, 0, 1)\})$ defined by projecting the sphere down to a plane below it.

The transition function $\phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is given by

$$(\phi, \theta) \mapsto \left(\frac{\cos(\phi)\sin(\theta)}{1 - \cos(\theta)}, \frac{\sin(\phi)\sin(\theta)}{1 - \cos(\theta)}\right)$$

Example 1.2.3. $M = \mathbb{R}$ define the following charts: $\left\{(\phi_n : t \mapsto t^{2n+1}, \mathbb{R}) : n \in \mathbb{N}\right\}$ the transition maps between them is then $\phi_m \circ \phi_n^{-1}(x) = x^{m/n}$ note then that the transition maps are not differentiable at $x = 0$ unless $n | m$.

This points to a problem if we want to do calculus on this manifold since a function can be differentiable in one coordinate system but not differentiable in another coordinate system. The fix is to require our transition functions to be differentiable.

Definition 1.2.4. *Let M be a Topological Manifold, we say that two charts (ϕ_1, U_1) and (ϕ_2, U_2) are* $\mathbf{C}^\infty$-***compatible*** *if the transition maps are C^∞ differentiable as maps between open sets in $\mathbb{R}^n$.*

Remark 1.2.5. We can also replace C^∞ in the above definition by C^r (r times differentiable), C^ω (real analytic) or holomorphic by replacing $\mathbb{R}^n$ with $\mathbb{C}^n$.

Definition 1.2.6. *A C^∞-atlas for a topological manifold M is a possible infinite collection $\mathcal{A} = \{(\phi_i, U_i)\}_{i\in I}$ of charts on M satisfying:*

- *$\{u_i\}_{i\in I}$ is an open cover of M*
- *The charts are pairwise C^∞ compatible.*

An atlas is further more a ***Maximal*** *atlas if it cannot be enlarged by adding more charts while remaining an atlas.*

Lemma 1.2.7. *Every C^∞ atlas is contained in a unique maximal atlas.*

Definition 1.2.8. *A C^∞-structure on a Topological n-Manifold is a choice of a maximal C^∞ atlas. A* ***Smooth Manifold*** *is a Topological Manifold equipped with a C^∞ structure.*

Example 1.2.9. If $(M, \mathcal{A})$ is a manifold then for every open set $V \subseteq M$ we have that V inherits the smooth structure defined by

$$\mathcal{A}_v := \{(U \cap V, \phi|_{U\cap V}) : (U, \phi) \in \mathcal{A}\}$$

Example 1.2.10. If $(M, \mathcal{A}_M)$, $(N, \mathcal{A}_N)$ are smooth manifolds then $M \times N$ has a canonical smooth structure with atlas

$$\mathcal{A} = \{(U \times V, \phi \times \psi) | (U, \phi) \in \mathcal{A}_m, (V, \psi) \in \mathcal{A}_n\}$$

then if M is m dimensional and n is n dimensional then $(M \times N, A)$ is an $m + n$ dimensional smooth manifold.

Example 1.2.11. The torus $T^n := S^1 \times \cdots \times S^1$ is the product of n circles is also an n-dimensional manifold.

Example 1.2.12. $GL_n(R) := \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\}$ is the preimage of the open set $\mathbb{R} \setminus \{0\}$ under the continuous function det. It thus has a canonical smooth structure with a global chart given by the inclusion $GL_n(\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$.

Remark 1.2.13. In many cases there exist preferred smooth structures on a given Manifold, but they are almost never unique.

Further more there exists Topological Manifolds that do not admit ANY Smooth structure.

Remark 1.2.14. Let $(N, \mathcal{A})$ be a Smooth manifold with $f : M \to N$ continuous then the pullback atlas defined by

$$f^*\mathcal{A} := \left\{(f^{-1}(u), \phi \circ f|_{f^{-1}(u)})|(U, \phi) \in \mathcal{A}\right\}$$

is a C^∞ atlas for M.

Definition 1.2.15. *Two smooth manifolds $(M, \mathcal{A}_m)$, $(N, \mathcal{A}_n)$ are equivalent ("diffeomorphic") if there exists a homeomorphism $f : M \to N$ such that $f^*\mathcal{A}_n = \mathcal{A}_m$.*

Remark 1.2.16. There is an equivalent but nicer definition for diffeomorphism that we will give later.

1.3 Projective Spaces

Example 1.3.1. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be some underlying field then $\mathbb{K}P^n$ is called the projective space and is the set of one-dimensional $\mathbb{K}$-linear subspaces of $\mathbb{K}^{n+1}$.

For every non zero vector $v \in \mathbb{K}^{n+1}$ that vector spans a one dimensional subspace $\mathbb{K}v$ and every one dimensional subspace can be constructed using some such vector. We also find that two such vectors v, w create the same subspace if and only if they are multiples of each other.

This allows us to write the projective space as

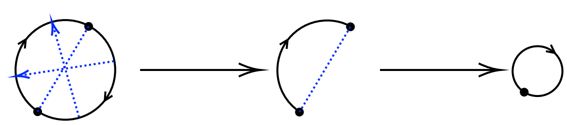
$$\mathbb{K}P^n \cong \frac{\mathbb{K}^{n+1} \setminus \{0\}}{\mathbb{K}^\times}$$

i.e the quotient of the nonzero vectors by the group of units of the field.

As a quotient this gives $\mathbb{K}P^n$ a natural topology inherited from $\mathbb{K}^{n+1}$. Also note that since $S^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\mathbb{R}_+}$ this then implies that

$$\mathbb{R}P^n \cong \frac{S^n}{\{-1, 1\}}$$

which gives us a more geometric way to imagine $\mathbb{R}P^n$.



$\mathbb{R}P^1 \cong S^1$

We now want to give $\mathbb{K}P^n$ a smooth manifold structure, for this we need to have some charts we can work with.

Definition 1.3.2. *Let $v \in \mathbb{K}^{n+1}$ be some non-zero vector with coordinates $(x^0, \ldots, x^n)$ then we denote its equivalence class in $\mathbb{K}P^n$ as*

$$[x^0 : \cdots : x^n]$$

Now we can define our charts over $\mathbb{K}P^n$ as follows:

$$U_i := \left\{[x^0 : \cdots : x^n] \in \mathbb{K}P^n | x^i \neq 0\right\}$$

then clearly its open as a preimage under the projection map of the open set $\mathbb{K} \setminus 0$. We then define our chart map as

$$\phi_i : U_i \to \mathbb{K}^n \cong \begin{cases} \mathbb{R}^n & : \mathbb{K} = \mathbb{R} \\ \mathbb{R}^{2n} & : \mathbb{K} = \mathbb{C} \end{cases}, \quad [x^0 : \cdots : x^n] \mapsto \left(\frac{x^0}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^n}{x^i}\right)$$

we are simply mapping our one dimensional subspace to the ratios of coordinates in that subspace which is obviously well defined over that equivalence class. This map is clearly continuous and it also has inverse by

$$\phi^{-1} : (x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \mapsto [x^0 : \cdots : x^{i-1} : 1 : x^{i+1} : x^n]$$

it is then easy to see that the transition functions are all smooth (they are simply quotients of non-zero coordinates).

Note that charts of this form obviously cover the entire the entire projective space since a point not covered would have to have all its coordinates be exactly zero which is not in the projective space. Because of that these charts uniquely determine a smooth structure of this manifold who's dimension is clearly n for $\mathbb{K} = \mathbb{R}$ and $2n$ for $\mathbb{K} = \mathbb{C}$.

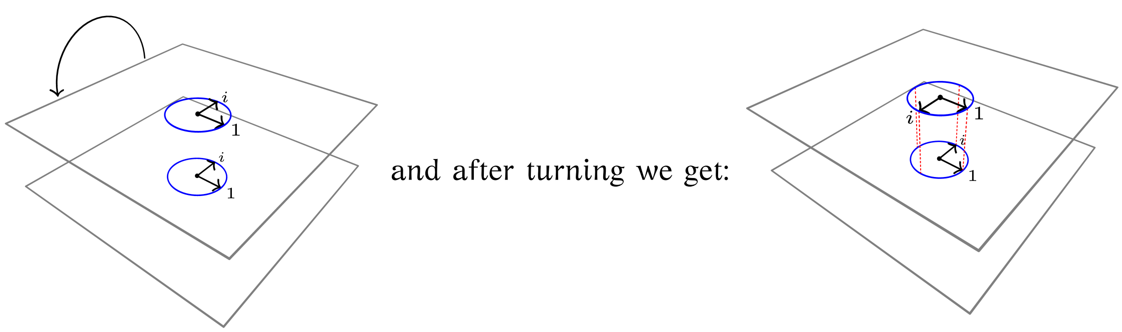
There is also a the special case of $\mathbb{C}P^1$. In this case our two open sets are

$$U_1 = \{[1 : w] : w \in \mathbb{C}\}, \; U_2 = \{[z : 1] : z \in \mathbb{C}\}$$

with their transition functions being $w = \frac{1}{z}$ on their intersection

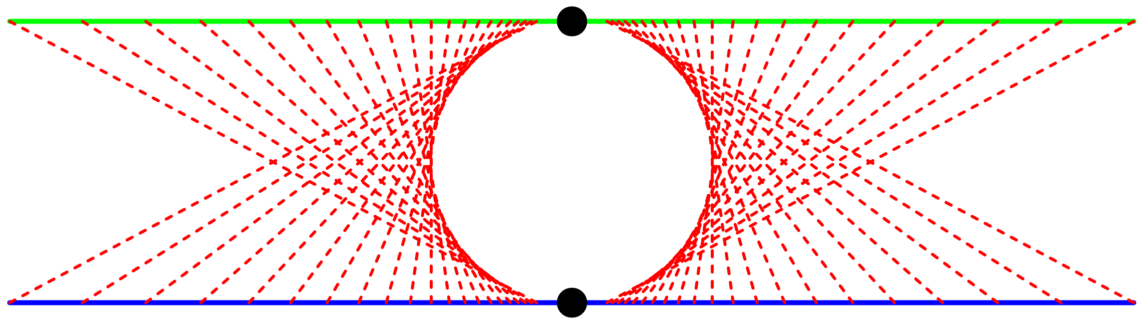
$$U_1 \cap U_2 = \{[1 : w] : w \in \mathbb{C}^\times\}$$

we thus have two copies of $\mathbb{C}$ glued along the relation $w = \frac{1}{z}$. We know that inversion on the complex plane is a composition of conjugation and circle inversion. First to deal with the conjugation we flip one of the planes as follows



Note that this matches up all the points on the unit circle as we expect.

Now we have rotational symmetry so we can focus on one slice of the plane.



we see that the lines representing the equivalence relation outline a circle and so wrapping the lines around this circle using stereographic projection gives us the wrapped real lines we see on the right.

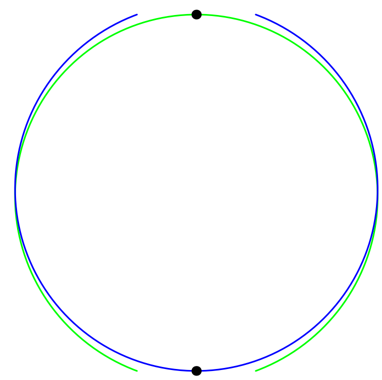


Figure 2: Wrapped real lines.

We then rotate those around the z axis to get the Riemann sphere. We thus find that $\mathbb{C}P^1$ is homeomorphic to the Riemann Sphere.

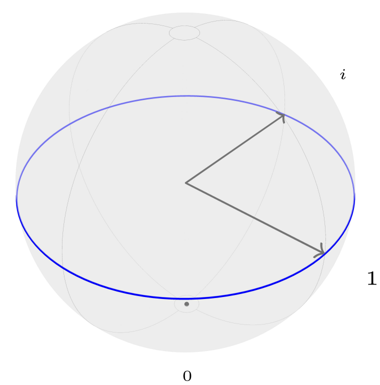


Figure 3: Riemann Sphere.

Glueing

We can actually expand on this idea of glueing more.

Definition 1.3.3. *Suppose that M, N are two smooth manifolds and $M_0 \subseteq M, N_0 \subseteq N$ are some open sets in those manifolds. Then assume that M_0 is diffeomorphic to N_0 under some diffeomorphism ϕ.*

Then we define the glueing $M \sqcup_\phi N$ as the quotient

$$M \sqcup_\phi N = \frac{M \sqcup N}{\phi(p) \sim p}$$

We are basically taking our two manifolds and identifying two open sets with each other which basically glues them together.

Is is easy to see that the result of such a glueing is always locally Euclidean and always 2nd countable but it might not be Hausdorff. As a counter example consider the glueing of $\mathbb{R}$ to $\mathbb{R}$ along the open sets $\mathbb{R} \setminus \{0\}$ glued with the identity map. This results in a structure that can be visualized as so this is also sometimes called the line with two origins.

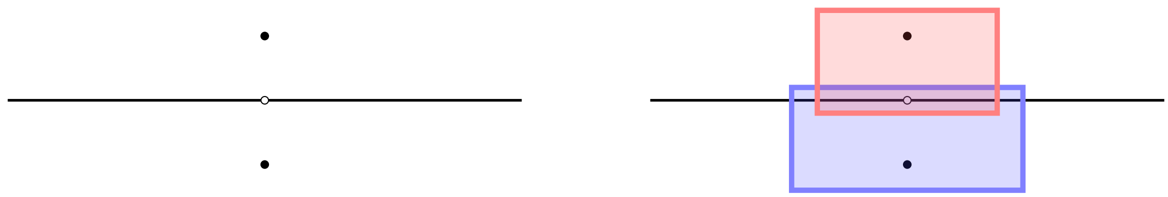


Figure 4: $\mathbb{R} \sqcup_{\mathrm{Id}|_{\mathbb{R}\setminus\{0\}}} \mathbb{R}$

The problem here is that any neighborhood of the top origin intersects any neighborhood of the bottom origin which violates Hausdorff-ness, as an example take the blue neighborhood of the bottom origin and the red neighborhood of the top origin.

It turns out, however, that this is the only obstacle that prevents a glueing from being a manifold. In particular:

Theorem 1.3.4. *Let $M \sqcup_\phi N$ be an arbitrary glueing of smooth manifolds M, N, then if $M \sqcup_\phi N$ is a Hausdorff then it has a unique smooth structure such that the inclusion maps*

$$\iota_M : M \hookrightarrow M \sqcup N \twoheadrightarrow M \sqcup_\phi N \quad \text{and} \quad \iota_N : N \hookrightarrow M \sqcup N \twoheadrightarrow M \sqcup_\phi$$

are diffeomorphisms onto their images.

Proof. Clearly $M \sqcup_\phi N$ is a topological manifold. Now take the atlas of $M \sqcup_\phi N$ defined by the union of the two maximal atlases of M and N projected onto $M \sqcup_\phi N$, i.e.

$$\mathcal{A}_M^* = \left\{ (\iota_M(U), \phi \circ \iota_M^{-1}|_{\iota_M(M)}) : (U, \phi) \in \mathcal{A}_M \right\}$$

and similarly for $\mathcal{A}_N$. Then clearly the union of the two projected atlases is an open cover of $M \sqcup_\phi N$ and so we only need to check smooth compatibility. Now clearly the charts are compatible with other charts in the same original atlas and so we must only check compatibility with between the atlases. Now by intersecting and restricting to $V = \iota_M(M_0) = \iota_N(N_0)$ we may assume that the two charts have their domains contained in V.

Now inside V we have that the transition functions take the form

$$\left(\psi \circ \iota_M^{-1}|_{\iota_M(M)}\right) \circ \left(\varphi \circ \iota_N^{-1}|_{\iota_N(N)}\right)^{-1}$$

and by how we defined the glueing we have

$$\left(\psi \circ \iota_M^{-1}|_{\iota_M(M)}\right) \circ \left(\varphi \circ \iota_N^{-1}|_{\iota_N(N)}\right)^{-1} = \psi \circ \iota_M^{-1}|_{\iota_M(M)} \circ \iota_N \circ \varphi^{-1} = \psi \circ \phi \circ \varphi^{-1}$$

which is a composition of 3 diffeomorphisms and so also a diffeomorphism. □

A specific case of a glueing is very important in topology and geometry, and that is the case of a connected sum.

Definition 1.3.5. *Let M be a manifold with some closed set B then B is called a closed ball in M if there exists a chart (U, φ) such that $B \subseteq U$ and $\phi(B) = \mathbb{B}^n$ where $\mathbb{B}^n$ is the standard unit closed ball in $\mathbb{R}^n$.*

Lemma 1.3.6. $\mathbb{B}^n \setminus \{0\}$ *is diffeomorphic to* $S^{n-1} \times [0,1]$

Proof. We know that $\partial\mathbb{B}^n = S^{n-1}$ and so we can simply identify a point with its projection from the center onto the boundary and its radius.

Visually this will look like so:

□

Definition 1.3.7.