Math 455: Honors Analysis 4

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Abstract

My course notes for Math 455

1 Hausdorff Dimension and Fractals

1.1 Fractals in nature

1.2 Mathematical examples

Take the interval $[0, 1]$ and remove the middle third to get $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. If we call this set C_1 we notice that C_1 has two scaled down copies of the original interval unioned together.
Since they are disjoint we can apply the same steps to them to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

We can this continue this for any $n \in \mathbb{N}$ to get the corresponding C_n

Definition 1.2.1. *We define the cantor set* $C := \bigcap_{i=1}^{\infty} C_i$

Now we want to try and calculate the dimension of the this set. We can try and replicate the intuition of dimension from our every day life. If you scale up a 3D cube by a factor of $a = 2$ we increase the volume by $a^3 = 2^3 = 8$, where the 3 comes from the dimension of the box.
Now if we could narrow down the a (scaling factor) and the 8 in that equation without including the 3 we could reverse engineer the 3 factor and thus get the dimension.
Note. From now on m denotes the mass factor, which is the 8 in the previous example.
We could use the Lebegue measure of the set to try and emulate the m, however this will breakdown for any set of measure 0.
We instead use a trick and cover the set in question with a countable collection of sets of a certain maximum diameter. We can then decrease that maximum diameter to get more detail on the set and count the amount of sets used to get a sense of scale.

Before we get into an example we need to standardize what we mean by a diameter.

Definition 1.2.2. *For any $S \subseteq \mathbb{R}^n$ we define the diameter*

$$D(S) := \sup_{x,y \in S} |x - y|$$

We can now use this to check a covering in the following way

Definition 1.2.3. *For any $S \subseteq \mathbb{R}^n$ we define a covering of S, $\{E_i : i \in \mathbb{N}\}$ to be bounded in radius by δ if*

$$\forall i \in \mathbb{N} : D(E_i) \leq \delta$$

2 Point Set Topology

2.1 Basic Definitions

Definition 2.1.1. *A topological space is a set X together with a collection of subsets of X, $\mathcal{O}$ where $\mathcal{O}$ satisfies*

- $\forall i \in I, E_i \in \mathcal{O} \implies \bigcup_{i \in I} E_i \in \mathcal{O}$
- $\forall i \in \mathbb{N} : 1 \leq i \leq n, E_i \in \mathcal{O} \implies \bigcap_{i=1}^{n} E_i \in \mathcal{O}$
- $\emptyset, X \in \mathcal{O}$

This definition then endows a dual definition of a closed set

Definition 2.1.2. *A subset $C \subseteq X$ is called closed iff $X \setminus C \in \mathcal{O}$*

We can then use De'Morgan's Laws to create new axioms for a topology in terms of its closed sets

Definition 2.1.3. *A topological space is a set X together with a collection of subsets of X, $\mathcal{C}$ where $\mathcal{C}$ satisfies*

- $\forall i \in I, E_i \in \mathcal{C} \implies \bigcap_{i \in I} E_i \in \mathcal{C}$
- $\forall i \in \mathbb{N} : 1 \leq i \leq n, E_i \in \mathcal{C} \implies \bigcup_{i=1}^{n} E_i \in \mathcal{C}$
- $\emptyset, X \in \mathcal{C}$

It also turns out that we can generate these collections of open/closed sets using the much more concrete ideas of metrics that we can almost always apply.

Definition 2.1.4. *A space X along with a function $d : X \times X \longrightarrow [0, \infty]$ is called a metric space if*

- $d(x, y) \geq 0$ *with equality iff* $x = y$
- $d(x, y) = d(y, x), \forall x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$ *(often called the traingle inequality)*

This then allows us to define

Definition 2.1.5. *Let X be a metric space and $x \in X$ then define the ball of radius r centered at x*

$$B_r(x) := \{y \in X : d(x, y) < r\}$$

Where we then let an open set be any arbitrary union of these balls.

We have multiple examples of these metrics that we are familiar with, but the large majority we use are classified under the label p-metrics.

Example 2.1.6. Let $X = \mathbb{R}^n$ then we take $p \geq 1$ and $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \mathbb{R}^n$ define

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

It is not trivial to prove that this is a metric, the difficult axiom being the triangle inequality

2.2 Topological properties & concepts

Definition 2.2.1. *Let $\mathcal{O}_1, \mathcal{O}_2$ be two topologies on X then suppose $\mathcal{O}_1 \subseteq \mathcal{O}_2$ we then say that $\mathcal{O}_2$ is a **finer** topology than $\mathcal{O}_1$ while $\mathcal{O}_1$ is **coarser** than $\mathcal{O}_2$*

We can tell from this definition that $\{\emptyset, X\}$ is the coursest topology on X while $\mathcal{P}(X)$ (power set) is the finest topology on X.

Now take $x \in X$ and a subset $E \subseteq X$, we can conclude from basic set theory that 1 and only 1 of the following conditions hold at a time

(1) $\exists O \in \mathcal{O} : x \in O \subseteq E$

(2) $\exists O \in \mathcal{O} : x \in O \subseteq X \setminus E$

(3) $\forall O \in \mathcal{O} : x \in O, O \cap E \neq \emptyset \wedge O \cap X \setminus E \neq \emptyset$

Notice that the first 2 conditions contradict each other and the 3rd is the negative of the first 2 proving our previous conclusion.
This then allows us to partition X into 3 sets of different topology

Definition 2.2.2. *Let X be a topological space and $E \subseteq X$ then define the interior of E to be*

$$\text{Int}(E) := \{x \in X : x \text{ satisfies condition } (1)\}$$

Definition 2.2.3. *Let X be a topological space and $E \subseteq X$ then define the boundary of E*

$$\partial E := \{x \in X : x \textit{ satisfies condition } (2)\}$$

We can then see that

$$\mathrm{Int}(X \setminus E) := \{x \in X : x \text{ satisfies condition } (3)\}$$

And so our partition is complete.
Similarly to a closed set this concept of interior has a dual concept, the **closure**.

Definition 2.2.4. *Let X be a topological space and $E \subseteq X$ then define the closure of E to be*

$$\overline{E} = \mathrm{Int}(E) \cup \partial E$$

Now similarly we take any $E \in X$ and we get some properties:

- $\mathrm{Int}(E)$ is open
- $\overline{E}$ is closed
- $X \setminus \mathrm{Int}(E) = \overline{X \setminus E}$
- A is open iff $A = \mathrm{Int}(A)$
- A is closed iff $\overline{A} = A$

2.3 Basis for a topology

Definition 2.3.1. *Let $\mathcal{O}$ be the collection of open sets in a topology of X, then $\mathbb{B}$ is a* ***basis*** *for the topology if every set in $\mathcal{O}$ is a union of sets of $\mathbb{B}$*

Example 2.3.2. The set of open intervals in $\mathbb{R}$ is a basis for the topology

Remark 2.3.3. Basis are not unique!

Proposition 2.3.4. $\mathbb{B}$ is a basis iff the following conditions hold

(1) Every $x \in X$ is in some $B \in \mathbb{B}$

(2) For any $B_1, B_2 \in \mathbb{B}$ and $B_1 \cap B_2 \neq \emptyset$ then for any $x \in B_1 \cap B_2$ there $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. First we prove that $\mathbb{B}$ is a basis if (1) and (2) holds.
Let

$$O = \{\text{all subsets of } X \text{ that are unions of sets in } \mathbb{B}\}.$$

Then if $E_\alpha \in O, a \in I$ then

$$\bigcup_{\alpha \in I} E_\alpha \in O$$

For finite $\cap$, enough to show that if $E_1 \in O, E_2 \in O$ then $E_1 \cap E_2 \in O$ if $E_1 \cap E_2 = \emptyset$ then its the union of 0 sets and thus in O.
If $E_1 \cap E_2 \neq \emptyset$ then for any $x \in E_1 \cap E_2$, by property (2) we have

$$\forall x \in E_1 \cap E_2, \exists B_x \in O : x \in B_x \subseteq E_1 \cap E_2$$

And so

$$E_1 \cap E_2 = \bigcup_{x \in E_1 \cap E_2} B_x \in O$$

□

Definition 2.3.5. *A set $A \subseteq X$ is a neighberhood of $x \in X$, if there exists E open such that $x \in E \subseteq A$*

Remark 2.3.6. This is equivalent to $x \in \mathrm{Int}(A)$

Proposition 2.3.7. x-limit point of A iff every open neighberhood of X intersects A

Proposition 2.3.8. In a metric space, a collection of open balls forms a basis for topology

Proof. Check property (2) □

Definition 2.3.9. *Let $A \subseteq X$. Then the subspace topology on A are open sets of the form $O \cap A$ where O are open in X.*

Proposition 2.3.10. If $\mathbb{B}$ is a basis for X then $\{B \cap A : B \in \mathbb{B}\}$ is a basis for A.

Proposition 2.3.11. If $A \subseteq X$ metric space, then A with the metric of X restricted to A is a metric and the topology generated by this metric matches with the subspace topology.

Definition 2.3.12. *A discrete subspace of X is a subspace A where the subspace topology is the discrete topology*

2.4 Continuous Functions

Definition 2.4.1. *$f : X \longrightarrow Y$ is continuous iff $\forall U \in Y$ such that U is open $f^{-1}(U)$ is open.*

Proposition 2.4.2. Since the set of open sets is generated by the basis through unions we only need to verify the condition for a basis of Y.

Proposition 2.4.3. If $f : X \longrightarrow Y$ continuous and $g : Y \longrightarrow Z$ is continuous then $g \circ f : X \longrightarrow Z$ is also continuous

Proof. Take $U \subseteq Z$ open, then $g^{-1}(U)$ is open by continuity of g and so $f^{-1}(g^{-1}(U))$ is also open and so $(g \circ f)^{-1}(U)$ is open and so $g \circ f$ is continuous. □

Proposition 2.4.4. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$ is a subspace of X then $f|_A$ is also continuous

Definition 2.4.5. *$f : X \longrightarrow Y$ is a **homeomorphism** if f is continuous, injective, surjective. Otherwise said, a continuous bijection, and f^{-1} is also continuous.*

Example 2.4.6. The stereographic projection from a circle to the real line is a homeomorphism

2.5 Product Spaces

Definition 2.5.1. *$X \times Y$ is a **product space** of X and Y where the basis is given by $\mathbb{B}_X \times \mathbb{B}_Y$*

This gives us projection functions

$$p_X : X \times Y \longrightarrow X$$

$$(x, y) \mapsto x$$

And

$$p_Y : X \times Y \longrightarrow Y$$

$$(x, y) \mapsto y$$

Proposition 2.5.2. Let $f : Z \longrightarrow X \times Y$ then $F(z) = (f_X(z), f_Y(z))$ where we can write

$$f_X(z) = (p_X \circ F)(z)$$

$$f_Y(z) = (p_Y \circ F)(z)$$