Math 595: Geometric Analysis

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Abstract

My course notes for the Geometric Analysis course.

1 ABP and Basic Geometry

1.1 Classic Isoperimetric inequality

The classic Isoperimetric inequality states that for any domain $\Omega \in \mathbb{R}^n$ we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

where B is the unit ball. We want to prove this inequality using the help of PDEs, we thus set up the following PDE

$$\Delta u = c \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega$$

If this is true then by divergence theorem we have

$$|\partial\Omega| = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = \int_{\Omega} c = c|\Omega|$$

and so we set $c = \frac{|\partial\Omega|}{|\Omega|}$.
For such a map we set $T = \nabla u$ to be the gradient map $\Omega \to \mathbb{R}^n$. We now want a characterization of the 'extremal' points of u as a graph, we define

$$\Gamma_u^- = \left\{x \in \Omega : u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \forall y \in \Omega\right\}.$$

In other words Γ_u^- are the points of Ω where the tangent plane lies entirely below the graph of u.
This set is called the 'contact' set.

Remark 1.1.1. For any point x in the contact set we have $\nabla^2 u(x) \geq 0$ where ∇^2 is the Hessian, if some eigenvalue of the Hessian was negative then in a small enough neighborhood the condition for being in the contact set would be violated.

Claim 1.1.2 (ABP)**.** For a solution u of the PDE above, we have $T(\Gamma_u^-)$ (the collection of all gradients at all contact points) contains $B_1 \setminus \partial B_1$

Proof. Take a vector $v \in B_1 \setminus \partial B_1$ and consider the function $\tilde{u} = u - v \cdot x$. We have that since $\frac{\partial \tilde{u}}{\partial \nu} > 1 - |v| > 0$ and so $\tilde{u}$ cannot attain its minimum on the boundary. Thus it attains it on its interior where then must have $\nabla \tilde{u}(x) = 0$ and so $\nabla u(x) = v$.

To see that x is a contact point we calculate

$$\tilde{u}(y) \geq \tilde{u}(x) \implies u(y) - v \cdot y \geq u(x) - v \cdot x \implies u(y) \geq u(x) + v \cdot (y - x)$$

□

Claim 1.1.3. If a solution u to the above PDE exists then we have

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}}$$

Proof. Then

$$\begin{aligned}
|B_1| \leq |T(\Gamma_u^-)| \leq \int_{\Gamma_u^-} J_T &= \int_{\Gamma_u^-} \det(\nabla^2 u) \\
&= \int_{\Gamma_u^-} \lambda_1 \lambda_2 \cdots \lambda_n \\
&\leq \int_{\Gamma_u^-} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n \quad \text{Since all the eigenvalues are positive.} \\
&\leq \int_{\Gamma_u^-} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \int_{\Omega} \left(\frac{\Delta u}{n} \right)^n \\
&\leq \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Omega| = \frac{|\partial\Omega|^n}{n^n |\Omega|^{n-1}}
\end{aligned}$$

and since $|B| = \frac{1}{n}|\partial B|$ we get the desired result. □

We now want to show existence of a solution to the PDE above. Let us consider the general linear PDE

$$\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial\Omega
\end{aligned}$$

then a necessary condition for existence is

$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} h.$$

Claim 1.1.4. The above condition is sufficient.

Proof. Assume first that $h = 0$. Thus the condition above becomes $\int_\Omega F = 0$. Then take the positive definite symmetric bilinear form $B(u, v) = \int_\Omega \nabla u \nabla v$ and notice

$$B(u, v) = (Lu, v)$$

and so L is a self-adjoint operator. Now in $W^{2,1}(\Omega)$ we know that the range of a self-adjoint operator is orthogonal to its kernel, so the PDE is solvable iff $F \perp \ker L$.

Now we know that for any g in the kernel we have by our boundary conditions

$$0 = \int_\Omega gLg = \int_\Omega |\nabla g|^2$$

and so g is a constant function.

Thus we have a solution if and only if

$$\int_\Omega F = 0$$

Now for $h \neq 0$ assume that $\partial\Omega$ is C^2 then $\rho(x) = d(x, \partial\Omega)$ is C^2 in Ω near $\partial\Omega$, we then choose a cutoff function η satisfying $\eta(x) = 1$ if $\rho(x) \leq \frac{\varepsilon}{4}$ and $\eta(x) = 0$ if $\rho(x) \geq \frac{\varepsilon}{2}$. Then $\gamma = \eta \cdot \rho$ is C^2 everywhere on Ω and as we approach the boundary we will have $\frac{\partial \gamma}{\partial \nu} = -1$.

Now define $U(x) := u(x) + h(x)\gamma(x)$, we have $\frac{\partial U}{\partial \nu} = 0$ and $\Delta U = \Delta u + \Delta(h\gamma)$. We then see that a solution for U exists if and only if

$$0 = \int_\Omega \Delta U = \int_\Omega \Delta u + \Delta(h\gamma) = \int_\Omega f + \int_{\partial\Omega} \frac{\partial(h\gamma)}{\partial \nu} = \int_\Omega f - \int_{\partial\Omega} h$$

and so we get our desired result. □

1.2 Deeper into ABP

Last class we used ABP method to prove

$$\frac{|\Omega|^{n-1}}{|\partial\Omega|^n} \leq \frac{|B|^{n-1}}{|\partial B|^n}$$

Now we study ABP further, if $u \in C(\Omega)$ then we set

$$\Gamma_u^+ = \{x \in \Omega | u(y) \leq u(x) + P \cdot (y - x) | \exists P \in \mathbb{R}^n, \forall y \in \Omega\},$$

we call this the 'upper contact' set, notice that we no longer require u to be differentiable. In conjuction with the upper contact set we have generalized gradient map given by

$$T_u(x) = \{p \in \mathbb{R}^n | u(y) \leq u(x) + p \cdot (y - x), \forall x \in \Omega\}.$$

Essentially it is the set of gradients of hyperplanes that touch our function from above.

Remark 1.2.1. If $u \in C^1$ then we can only have $T_u(x) = \nabla u$.

Remark 1.2.2. If $u \in C^2$ and $x \in \Gamma_u^+$ then $\nabla^2 u(x) \leq 0$.

Example 1.2.3. $z \in \mathbb{R}^n$, $R > 0$, $a > 0$ then $u(x) = a(1 - \frac{|x-z|}{R})$. This is the graph of a cone in $\mathbb{R}^{n+1}$.
We then have for all $x \neq z$ that the function is differentiable and so

$$T_u(x) = \nabla u(x) = -\frac{a}{R}\frac{x - z}{|x - z|}.$$

For $x = z$ we have

$$\begin{aligned} u(y) &\leq u(z) + P \cdot (y - z) \\ a\left(1 - \frac{|y - z|}{R}\right) &\leq a + P \cdot (y - z) \\ -\frac{a}{R} &\leq P \cdot \frac{y - z}{|y - z|} \end{aligned}$$

But we know that $\frac{y-z}{|y-z|}$ is a unit vector and so this is equivalent to

$$|P| \leq \frac{a}{R}$$

and so

$$T_u(x) = B_{a/R}(0)$$

Now let us assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Lemma 1.2.4.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma_u^+} |\det(\nabla^2)|\right)^{1/n}$$

Proof. Set $v = u - \sup_{\partial\Omega} u$ and suppose $\max_{\overline{\Omega}} v = v(x_0)$ with $v(x_0) \geq 0$ (if $v(x_0) < 0$ then the statement follows trivially).
Now consider Γ_v^+, we have

$$T(\Gamma_v^+) \leq \int_{\Gamma_v^+} |J_{\nabla v}| = \int_{\Gamma_v^+} |\det(\nabla^2 v)|.$$

Now let $h(x)$ be defined of Ω such that $(x, h(x))$ be the cone with vertex at $(x_0, v(x_0))$ and base $\partial\Omega$. Then we must have $T_v(\Omega) \supseteq T_h(\Omega)$. to see this take a hyperplane P given by a function $l(x)$ that touches this cone, then it is easy to see that it must touch it at $(x, v(x_0))$, it is easy to see that on the boundary we have $v(x) = h(x) \leq l(x)$. We then have $v(x) - l(x) \leq 0$ on the boundary.

On the other hand we have $\nabla(v - l)(x_0) \neq 0$ so $v - l$ must be positive at some point close to x_0, thus $v - l$ must achieve its maximum somewhere on the interior of Ω where we would then have $\nabla v = \nabla l$.

Next we have $T_h(\Omega) \supseteq T_{\tilde{h}}(B_d(x_0))$ where $\tilde{h}$ is given by

$$\tilde{h}(x) = v(x_0)\left(1 - \frac{x - x_0}{d}\right).$$

We can see this because $\tilde{h}$ is just a cone with a wider base than h and thus its supporting hyperplanes must have smaller gradients.

But we know from the example above that

$$T_{\tilde{h}}(B_d(x_0)) = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

and so we get

$$|T_v(\Omega)| \geq |T_{\tilde{h}}(B_d(x_0))| = \left(\frac{v(x_0)}{d}\right)^n \omega_n$$

which then gives us

$$\left(\frac{v(x_0)}{d}\right)\omega_n^{1/n} \leq |T_v(\Gamma_v^+)|^{\frac{1}{n}} \leq \left(\int_{\Gamma_v^+} |\det(\nabla^2 u)|\right)^{1/n}$$

□

Now we move on to more general elliptic equations, lets say we have $\lambda I \leq a_{ij}(x) \leq \Lambda I$ with $0 < \lambda < \Lambda < \infty$ and

$$Lu = \sum_{i,j} a_{ij}(x) u_{ij}(x) \geq f \quad \text{in } \Omega$$

Lemma 1.2.5. *Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the above, then*

$$u(x) \leq \sup_{\partial\Omega} u + \frac{d(\Omega)}{n\omega_n^{1/n}} \left(\int_{\Gamma_u^+} \left(\frac{-f}{\det^{1/n}(a_{ij})}\right)^n\right)^{1/n}$$

Remark 1.2.6. If $x \in \Gamma_u^+$ then $-(\nabla^2 u) \geq 0$ and so $0 \leq -Lu \leq -f$.

We need a small linear algebra lemma to prove the results.

Lemma 1.2.7. *For symmetric positive matrices A, B we have*

$$\det(A)\det(B) \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n$$

Proof. Left side is equal to product of all eigenvalues, $\lambda_1\tilde{\lambda}_1 \cdots \lambda_n\tilde{\lambda}_n$.
$\operatorname{tr}(AB)$ is equal to sum of products of eigenvalues, $\lambda_1\tilde{\lambda}_1 + \cdots + \lambda_n\tilde{\lambda}_n$. Then by arithmetic-geometric mean inequality we get the desired result. □

Proof. Now to prove the main lemma, set $B = -\nabla^2 u \geq 0$ and $A = (a_ij) > 0$ then

$$-f = -Lu = \operatorname{tr}(AB) \geq n(\det(A))^{\frac{1}{n}}(\det(B))^{\frac{1}{n}} = n(\det(a_ij))^{1/n}(\det(-\nabla^2 u))^{1/n}$$

which then gives us

$$\det(-\nabla^2 u) \leq \frac{-f}{\det^{1/n}(a_{ij})}$$

which together with the lemma before that gives us the desired result. □

This lemma is sometimes called the weak maximum principle.

Remark 1.2.8. There is a more general result with more general elliptic operators

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij}(x) + \sum_k b_k(x)u_k(x) + c(x)u(x)$$

the only thing that changes is that the constant in front of the integral now depends on the coefficients b_k and c.

1.3 Introduction to Riemannian Geometry

Let M^n be an n-dimensional manifold, every point $p \in M^n$ has a tangent space T^pM, then a metric g on M^n is a choice of inner product on T_pM for every $p \in M$ which varies smoothly in p. A manifold with a metric is called a Riemannian Manifold.
In any local coordinate chart $(x_1, \ldots, x_n)$ we define the 'components' of g to be

$$g_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle.$$

Then if at some point p we have two vectors

$$X = \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k}$$

then their inner product is given by

$$\langle X, Y\rangle_g = \left\langle \sum_{j=1}^{N} a_j(x)\frac{\partial}{\partial x_j}, \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k} \right\rangle = \sum_{j,k} a_j(x)b_k(x)\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle$$
$$= \sum_{j,k} a_j(x)b_k(x)g_{jk}(x)$$

More formally, let dx_i be the dual frame to $\frac{\partial}{\partial x_i}$, as in

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_i^j,$$

then we can write the metric as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on M.
If $e_1, \dots, e_n \in T_pM$ is an orthonormal basis, that is $\langle e_i, e_j\rangle_g = \delta_{ij}$. Set $\omega_1, \dots, \omega_n$ to be its dual basis. We then get a top-form $\omega_1 \wedge \cdots \wedge \omega_n$.
If

$$e_j = \sum_k a_j^k \frac{\partial}{\partial x_k}$$

where $A = a_j^k$ is a matrix, then by standard linear algebra we have that

$$\omega_1 \wedge \cdots \wedge \omega_n = \det(A^{-1}) dx_1 \wedge \cdots \wedge dx_n$$

Claim 1.3.1.

$$|\det(A^{-1})| = \sqrt{\det g}$$

Proof.

$$\delta_{ij} = (e_i, e_j) = a_j^k a_i^l g_{kl}$$

this implies that

$$I = A^T g A$$

where A is the transpose.
Thus

$$1 = \det(A^T g A) = \det(A^2)\det(g)$$

and so

$$\sqrt{\det(g)} = \det A^{-1}$$

□

Claim 1.3.2. The top-form $dV = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$ is coordinate change invariant.

Proof. Let us assume that $(\tilde{x}_1, \ldots, \tilde{x}_n)$ are coordinates given by the transition function $\tilde{x}_\alpha = \phi(x_\alpha)$ with jacobian J_ϕ, we know that in these coordinates we have

$$\tilde{g} = \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right)^T g \left(\frac{\partial(x_1, \ldots, x_n)}{\partial(\tilde{x}_1, \ldots, \tilde{x}_n)}\right) = \left(J_\phi^{-1}\right)^T g \left(J_\phi^{-1}\right)$$

and so

$$\sqrt{\det \tilde{g}} = \det J^{-1} \sqrt{\det g}.$$

On the other hand we have

$$d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J dx_1 \wedge \cdots \wedge dx_n$$

and so

$$\sqrt{\tilde{g}} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n = \det J^{-1} \sqrt{\det g} \det J dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$$

□

Definition 1.3.3. *An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following properties for any smooth functions $f_1, f_2 \in C^\infty(M)$ and any smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$*

- $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$
- $\nabla_X Z + Y = \nabla_X Z + \nabla_X Y$
- $\nabla_X f_1 Y = X(f_1) Y + f \nabla_X Y$

Definition 1.3.4. *A Levi-Civita connection is an affine connection which also satisfies*

- *Symmetry:* $\nabla_X Y - \nabla_Y X = [X, Y]$
- *Compatability with g:* $X(\langle Y, Z \rangle_g) = \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g$

Remark 1.3.5. Compatability with g is essentially like the product rule.

Theorem 1.3.6 (Fundamental theorem of Riemannian Geometry)**.** *For every Riemannian manifold there exists a unique Levi-Civita Connection.*

Proof. Take any smooth vector fields X, Y, Z, we know that the following are true

$$\begin{aligned} X(\langle Y, Z \rangle_g) &= \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g \\ Y(\langle Z, X \rangle_g) &= \langle \nabla_Y Z, X \rangle_g + \langle Z, \nabla_Y X \rangle_g \\ Z(\langle X, Y \rangle_g) &= \langle \nabla_Z X, Y \rangle_g + \langle X, \nabla_Z Y \rangle_g \end{aligned}$$

then by adding the first two equations and subtracting the third we get

$$\begin{aligned}X(\langle Y,Z\rangle_g)+Y(\langle Z,X\rangle_g)-Z(\langle X,Y\rangle_g) &= \langle Y,\nabla_X Z\rangle_g-\langle \nabla_Z X,Y\rangle_g\\ &\quad+\langle \nabla_Y Z,X\rangle_g-\langle X,\nabla_Z Y\rangle_g\\ &\quad+\langle \nabla_X Y,Z\rangle_g+\langle Z,\nabla_Y X\rangle_g\end{aligned}$$

using the symmetry of the connection we get

$$\begin{aligned}X(\langle Y,Z\rangle_g)+Y(\langle Z,X\rangle_g)-Z(\langle X,Y\rangle_g) &= \langle Y,[X,Z]\rangle_g+\langle [Y,Z],X\rangle_g+\langle [X,Y],Z\rangle_g\\ &\quad+2\,\langle Z,\nabla_Y X\rangle_g\end{aligned}$$

from here we can solve for $\langle Z,\nabla_Y X\rangle_g$ giving us the connection since as a vector, $\nabla_Y X$ is fully determined by its inner products with all other vectors. $\square$

One can check that in a coordinate chart that the Levi Civita connection has the form

$$\begin{aligned}\nabla_X Y &= \nabla_{\sum_i a_i(x)\frac{\partial}{\partial x_i}}\sum_j b_j(x)\frac{\partial}{\partial x_j}\\ &= \sum_i a_i(x)\left(\nabla_{\frac{\partial}{\partial x_i}}\sum_j b_j(x)\frac{\partial}{\partial x_j}\right)\\ &= \sum_{i,j} a_i(x)\left(\left(\frac{\partial}{\partial x_i}b_j(x)\right)\frac{\partial}{\partial x_j}+b_j(x)\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right).\end{aligned}$$

Now we know that for some coefficients Γ_{ij}^k we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\sum_k \Gamma_{ij}^k\frac{\partial}{\partial x_k}$$

and so

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g=\sum_k \Gamma_{ij}^k g_{k\ell}$$

Now by the previous proof and the fact that coordinate vector fields have vanishing brackets we have that

$$\begin{aligned}2\left\langle \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g &= \frac{\partial}{\partial x_j}\left(\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_\ell}\right\rangle_g\right)+\frac{\partial}{\partial x_i}\left(\left\langle\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_\ell}\right\rangle_g\right)-\frac{\partial}{\partial x_\ell}\left(\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right\rangle_g\right)\\ &= \frac{\partial}{\partial x_j}(g_{i\ell})+\frac{\partial}{\partial x_i}(g_{j\ell})-\frac{\partial}{\partial x_\ell}(g_{ij})\end{aligned}$$

and so by using the inverse of the metric we get

$$\Gamma_{ij}^k=\frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x_j}(g_{i\ell})+\frac{\partial}{\partial x_i}(g_{j\ell})-\frac{\partial}{\partial x_\ell}(g_{ij})\right).$$

The coefficients Γ are often called the Christoffel Symbols of g in these coordinates.

Claim 1.3.7. At any point p there exists a local coordinate chart $(x_1, \ldots, x_n)$ such that

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial}{\partial x_i}(g_{jk})(p) = 0$$

Proof. We have $g_{ij}(x) = g_{ij}(0) + \sum_k a_{ij}^k x_k + O(|X|^2)$, we can always change variables so that $g_{ij}(0) = \delta_{ij}$. The tricky part is eliminating the first derivatives, for that we do a change of coordinates

$$y_\alpha = \phi(x_\alpha) = x_\alpha + \frac{1}{2} b_\alpha^{k\ell} x_k x_\ell + O(|X|^3).$$

The jacobian of this transformation is

$$J_{\phi^{-1}} = I - b_\alpha^{k\ell} x_\ell + O(|X|^3)$$

and so the new metric is

$$\begin{aligned}\tilde{g}_{\alpha\beta} = J_{\phi^{-1}}^T g J_{\phi^{-1}} &= (I - b_\alpha^{i\ell} x_\ell + O(|X|^3))^T (I + a_{ij}^m x_m)(I - b_\beta^{j\ell} x_\ell + O(|X|^3)) \\ &= I - 2b_\alpha^{i\ell} g_{i\beta} + a_{ij}^\ell x_\ell + O(|X|)^2,\end{aligned}$$

then from here you can solve for b. $\square$

1.4 Geometric constructions

We now have several natural constructions once we fix a metric on our manifold.

Consider a vector field X and a point p on a Riemannian manifold, the map $P : T_p(M) \to T_p(M)$, given by

$$v \mapsto \nabla_v X$$

is a linear map. We define its trace to be the divergence of X, denoted $\operatorname{div}(X)$.

In a local orthonormal chart at p, if we write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$, then

$$\begin{aligned}\operatorname{div}(X)_p &= \sum_i \left\langle \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \left\langle \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g \\ &= \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle_g = \sum_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} a_j(x) \delta_{ij} = \sum_i \nabla_{\frac{\partial}{\partial x_i}} a_i(x) \\ &= \sum_i \frac{\partial a_i(x)}{\partial x_i}\end{aligned}$$

Where we used the fact that in an orthonormal frame $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. We see then that in an orthonormal frame the divergence matches our 'classical' definition of the divergence.

Next consider a function $f \in C^\infty(M)$, we define the gradient to be a map $\operatorname{grad} : C^\infty(M) \to \mathfrak{X}(M)$ defined by

$$\langle \operatorname{grad} f, v \rangle_g = df(v)$$

for every tangent vector v.

In a local (not necessarily orthonormal) chart we have

$$\operatorname{grad} f = \sum_j a_j(x) \frac{\partial}{\partial x_j}, df = \sum_k \frac{\partial f}{\partial x_k} dx_k,$$

then for any $v = \sum_\ell b_\ell \frac{\partial}{\partial x_\ell}$ we have

$$\langle \operatorname{grad} f, v \rangle_g = \sum_{j,\ell} a_j g_{j\ell} b_\ell$$

but we also have

$$df(v) = \sum_{k,\ell} \frac{\partial f}{\partial x_k} b_\ell dx_k \left(\frac{\partial}{\partial x_\ell} \right) = \sum_k \frac{\partial f}{\partial x_k} b_k.$$

Now lets choose $b = (0, 0, \ldots, 1, \ldots, 0, 0)$ with a 1 in the m-th position then

$$\langle \operatorname{grad} f, v \rangle_g = \sum_j a_j g_{jm}$$

and

$$df(v) = \frac{\partial f}{\partial x_m}$$

so since these are equal we can multiply both by the inverse of the metric g^{mi} to get

$$a_i = \sum_{j,m} a_j g_{jm} g^{mi} = \sum_m \frac{\partial f}{\partial x_m} g^{mi}$$

and thus

$$\operatorname{grad} f = \sum_i a_i \frac{\partial}{\partial x_i} = \sum_{m,i} \frac{\partial f}{\partial x_m} g^{mi} \frac{\partial}{\partial x_i}$$

Finally again for a function $f \in C^\infty(M)$, the hessian is defined as the map Hess : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$X \mapsto \nabla_X(\operatorname{grad} f)$$

Let us write $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$ then we have by the previous results that in an orthonormal chart around p

$$\begin{aligned}
\nabla_X(\operatorname{grad} f) &= \nabla_X\left(\sum_{m,i}\frac{\partial f}{\partial x_m}g^{mi}\frac{\partial}{\partial x_i}\right)\\
&= \nabla_X\left(\sum_{i}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}\right) \quad \text{(because } g^{mi}=\delta^{mi} \text{ at } p \text{ in orthonormal chart)}\\
&= \sum_{j,i} a_j(x)\left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\frac{\partial}{\partial x_i}+\frac{\partial f}{\partial x_i}\Gamma^k_{ij}\frac{\partial}{\partial x_k}\right)\\
&= \sum_{j,i} a_j(x)\left(\left(\frac{\partial f}{\partial x_j\partial x_i}\right)\frac{\partial}{\partial x_i}\right) \quad \text{(because } \Gamma^k_{ij}=0 \text{ at } p \text{ in orthonormal chart)}
\end{aligned}$$

and so if $Y=\sum_\ell b_\ell(x)\frac{\partial}{\partial x_\ell}$ we have

$$\langle\nabla_X(\operatorname{grad} f),Y\rangle_g=\sum_{j,\ell}a_j(x)\left(\frac{\partial f}{\partial x_j\partial x_i}\right)b_\ell(x).$$

Importantly notice that if we exchange a and b then this expression does not change and so $\langle\nabla_X(\operatorname{grad} f),Y\rangle_g=\langle\nabla_Y(\operatorname{grad} f),X\rangle_g$ and so as an operator Hess is symmetric. We also get that the form in orthonormal coordinates for the operator is the matrix

$$\frac{\partial f}{\partial x_j\partial x_i}$$

Now we consider the trace of the modified hessian operator, given by $\operatorname{div}(h\cdot\operatorname{grad} f)$. Notice that we have, in an orthonormal chart,

$$\operatorname{div}(h\cdot\operatorname{grad} f)=\sum_j\frac{\partial}{\partial x_j}E_j=\sum_j\frac{\partial}{\partial x_j}\left(h\sum_k g^{jk}\frac{\partial f}{\partial x_k}\right)$$

Claim 1.4.1. In a general local chart,

$$\operatorname{div}(h\cdot\operatorname{grad} f)=(\det g)^{-1/2}\sum_{i,j}\frac{\partial}{\partial x_j}\left(h(\det g)^{1/2}g^{ij}\frac{\partial f}{\partial x_i}\right)$$

Proof. It is enough to show that the expression on the right is coordinate invariant, since then plugging in an orthonormal chart gives us the desired result.

To see this consider a different chart $(\tilde{x}_1,\dots,\tilde{x}_n)$ and set

$$\begin{aligned}
Q &= (\det g)^{-1/2}\sum_{i,j}\frac{\partial}{\partial x_j}\left(h(\det g)^{1/2}g^{ij}\frac{\partial f}{\partial x_i}\right)\\
\tilde{Q} &= (\det \tilde{g})^{-1/2}\sum_{i,j}\frac{\partial}{\partial \tilde{x}_j}\left(h(\det \tilde{g})^{1/2}\tilde{g}^{ij}\frac{\partial f}{\partial \tilde{x}_i}\right)
\end{aligned}$$

then consider the set of functions η with support contained within both charts, if

$$\int_{\Omega} Q\eta dV = \int_{\Omega} \tilde{Q}\eta dV$$

then $Q = \tilde{Q}$.
Now we plug in our known expressions and get

$$\int_{\Omega} Q\eta dV = \int_{\Omega} \eta \sum_j \frac{\partial}{\partial x_j} \left(h(\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) dx_1 dx_2 \dots dx_n$$

then we notice that we have the a divergence term in the integral. Then by using integration by parts we can remove that divergence and instead take the gradient of η, the boundary term then dissapears by compactness of η. All together this gives us

$$\begin{aligned}
\int_{\Omega} Q\eta dV &= -\int_{\Omega} \sum_j \left(\frac{\partial \eta}{\partial x_j} \right) \left(h(\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i} \right) dx_1 dx_2 \dots dx_n \\
&= -\int_{\Omega} h \sum_{j,i} \left(g^{ij} \frac{\partial \eta}{\partial x_j} \frac{\partial f}{\partial x_i} \right) (\det g)^{1/2} dx_1 dx_2 \dots dx_n \\
&= -\int_{\Omega} h \left\langle \operatorname{grad} \eta, \operatorname{grad} f \right\rangle_g dV
\end{aligned}$$

now notice that the same calculation holds in the second chart, and so we get

$$\int_{\Omega} Q\eta dV = \int_{\Omega} Q'\eta dV$$

□

Theorem 1.4.2 (Divergence theorem)**.** *Suppose that $\Omega \subseteq M$ is a compact domain with a smooth boundary $\partial\Omega$, then $\forall f, h \in C^{\infty}(M)$ we have*

$$\int_{\Omega} \operatorname{div}(h \operatorname{grad} f) dV = \int_{\partial\Omega} \left\langle h \operatorname{grad} f, \nu \right\rangle_g d\tilde{V}$$

where ν is the normal vector and $d\tilde{V}$ is the induced volume form on the metric.

Proof. Find a partition of unity for some neighborhood of Ω, that is a collection of functions ρ_k with $\sum_k \rho_k = 1$ and the support of each ρ_k being contained in a single chart U_k. Now we have

$$\int_{\Omega} \operatorname{div}(h \operatorname{grad} f) dV = \sum_k \int_{\Omega} \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \sum_k \int_{\Omega \cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV$$

now for each of these smaller integrals we have

$$\int_{\Omega\cap U_k} \operatorname{div}(\rho_k h \operatorname{grad} f) dV = \int_{\Omega\cap U_k} \sum_j \frac{\partial}{\partial x_j}\left(\rho_k h (\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i}\right) dx_1 dx_2 \dots dx_n.$$

We will now apply IBP to this integral, note that the interior term will contain a derivative of 1 and so will vanish, the boundary term will only be non-zero outside of the boundary of U_k, that is it will be non-zero only on $\partial\Omega \cap U_k$.
So this integral becomes

$$\int_{\partial\Omega\cap U_k} \sum_j \nu_j \left(\rho_k h (\det g)^{1/2} \sum_i g^{ij} \frac{\partial f}{\partial x_i}\right) d\tilde{x}_1 d\tilde{x}_2 \dots d\tilde{x}_{n-1},$$

which simplifies to

$$\int_{\partial\Omega\cap U_k} \langle \operatorname{grad} f, \nu\rangle_g (\rho_k h)\, d\tilde{V}.$$

This then gets summed up over k to give

$$\sum_k \int_{\partial\Omega\cap U_k} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g d\tilde{V} = \sum_k \int_{\partial\Omega} \langle \rho_k h \operatorname{grad} f, \nu\rangle_g d\tilde{V} = \int_{\partial\Omega} \langle h \operatorname{grad} f, \nu\rangle_g d\tilde{V}$$

□

Theorem 1.4.3. *$\forall h \in C^\infty(M)$ with $h > 0$ in Ω a compact connected open set, the system*

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f \quad &\in \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad &\in \partial\Omega \end{aligned}$$

is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} hg$.

Proof. We follow a similar proof to 1.1.4, first assume $g = 0$, then we have in the space of functions in $W^{2,1}(\Omega)$ that are zero on the boundary the image of the operator $u \mapsto \operatorname{div}(h \operatorname{grad} u)$ is orthogonal to its kernel. Now the set of functions in the kernel are those satisfying

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= 0 \quad &\in \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \quad &\in \partial\Omega \end{aligned}$$

and so for those functions by the divergence theorem we get

$$0 = \int_\Omega u \operatorname{div}(h \operatorname{grad} u) dV = -\int_\Omega h |\operatorname{grad} u|^2 dV$$

and so since $h > 0$ we get $\operatorname{grad} u = 0$ everywhere on Ω and so it is constant on Ω. Thus the image is those functions f that are orthogonal to constant functions, that is the system is solvable if and only if

$$\int_{\Omega} f dV = 0$$

Next we identically construct a function γ which is C^2 everywhere on Ω and satisfying $\frac{\partial \gamma}{\partial \nu} = -1$. We then define $U(x) = u(x) + \gamma(x)g(x)$ and notice that since

$$\operatorname{div}(h \operatorname{grad} U) = \operatorname{div}(h \operatorname{grad}(u(x) + \gamma(x)g(x))) = \operatorname{div}(h \operatorname{grad} u(x)) + \operatorname{div}(h \operatorname{grad}(\gamma(x)g(x)))$$

and

$$\frac{\partial U}{\partial \nu} = \frac{\partial u}{\partial \nu} + \frac{\partial(\gamma \cdot g)}{\partial \nu} = g - g = 0$$

then we have a solution U if and only if

$$\begin{aligned} 0 &= \int_{\Omega} \operatorname{div}(h \operatorname{grad} U) dV = \int_{\Omega} \operatorname{div}(h \operatorname{grad} u) dV + \int_{\Omega} \operatorname{div}(h \operatorname{grad}(\gamma(x)g(x))) dV \\ &= \int_{\Omega} f dV + \int_{\partial\Omega} h \frac{\partial(\gamma \cdot g)}{\partial \nu} d\tilde{V} = \int_{\Omega} f dV + \int_{\partial\Omega} -hg d\tilde{V} \end{aligned}$$

□

1.5 Extrinsic Geometry

Suppose we have an n-dimensional Riemannian Manifold (M^n, g) with $F : M^n \hookrightarrow N$ an immersion where N is an $n+m$-dimensional Riemannian Manifold with metric $\overline{g}$. Every point $x \in M$ has a tangent space T_xM and also after identifying x with $F(x)$ we have the larger tangent space T_xN that contains T_xM. We say that M is isometrically immersed if for all $X, Y \in T_xM$ we have

$$\langle X, Y \rangle_g = \langle X, Y \rangle_{\overline{g}},$$

essentially $\overline{g}$ extends g to a larger tangent space.

Recall that both g and $\overline{g}$ induce connections ∇ and $\overline{\nabla}$ respectively.

Lemma 1.5.1. *Let vector fields $X, Y \in \mathfrak{X}(M)$ extend to vector fields $\overline{X}, \overline{Y} \in \mathfrak{X}(M)$. Then*

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T,$$

where T is the orthogonal projection onto T_xM.

Proof. Define the connection $\tilde{\nabla}_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$, then by uniqueness of the Levi-Civita connection of if we have that $\tilde{\nabla}$ satisfies the axioms it must be equal to ∇.

First we check metric compatability,

$$\left\langle \tilde{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, \tilde{\nabla}_{\overline{X}}\overline{Z} \right\rangle_g = \left\langle (\overline{\nabla}_{\overline{X}}\overline{Y})^T, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}}\overline{Z})^T \right\rangle_g$$

then since all the terms are tangent to M we can replace g with $\overline{g}$.

$$\left\langle (\overline{\nabla}_{\overline{X}}\overline{Y})^T, \overline{Z} \right\rangle_g + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}}\overline{Z})^T \right\rangle_g = \left\langle (\overline{\nabla}_{\overline{X}}\overline{Y})^T, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}}\overline{Z})^T \right\rangle_{\overline{g}}$$

then we can throw away the projections since taking inner product with a vector already tangent to $T_{\overline{X}}M$ implicitly projects onto that space.

$$\left\langle (\overline{\nabla}_{\overline{X}}\overline{Y})^T, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, (\overline{\nabla}_{\overline{X}}\overline{Z})^T \right\rangle_{\overline{g}} = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z} \right\rangle_{\overline{g}}$$

by metric compatability of $\overline{\nabla}$ we have that

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} \right\rangle_{\overline{g}} + \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z} \right\rangle_{\overline{g}} = \overline{\nabla}_{\overline{X}} \left\langle \overline{Y}, \overline{Z} \right\rangle_{\overline{g}}$$

and then we get

$$\overline{\nabla}_{\overline{X}} \left\langle \overline{Y}, \overline{Z} \right\rangle_{\overline{g}} = \overline{X}\left(\left\langle \overline{Y}, \overline{Z} \right\rangle_{\overline{g}} \right) = X\left(\left\langle \overline{Y}, \overline{Z} \right\rangle_{\overline{g}} \right) = X\left(\left\langle Y, Z \right\rangle_{\overline{g}} \right) = X\left(\left\langle Y, Z \right\rangle_{g} \right)$$

For symmetry we need a small fact about Lie Brackets

Lemma 1.5.2. *If $X, Y \in \mathfrak{X}(M)$ and M is immersed in N then for any extension $\overline{X}, \overline{Y}$ we have $[\overline{X}, \overline{Y}]_N = [X, Y]_M$.*

Proof. See Lee's smooth manifolds page 189. □

We then have that

$$\tilde{\nabla}^Y_X - \tilde{\nabla}^X_Y = (\overline{\nabla}^{\overline{Y}}_{\overline{X}})^T - (\overline{\nabla}^{\overline{X}}_{\overline{Y}})^T = ([\overline{X}, \overline{Y}]_N) = [X, Y]_M$$

□

Now we define the second fundamental form. For every point $p \in M$ we have $T_pN = T_pM \oplus T_p^{\perp}M$, this decomposition defines the normal bundle $NM = \{p \in M | T_p^{\perp}M\} \subseteq TN$. For every smooth normal vector field $V \in NM$ and every vector $X \in T_pM$ we can define $\overline{\nabla}_X V \in T_pN$ we can define

$$\nabla^{\perp}_X V := \left(\overline{\nabla}_X V \right)^{\top}.$$

We now define the second fundamental form to be the map $A^W : T_pM \to T_pM$ parametrized by some vector field in NM which is defined through

$$A^W(X) = -(\overline{\nabla}_X W)^T$$

We want to check that this map is well defined, suppose $W \in NM$ is a normal vector field with two extensions $\tilde{W}_1, \tilde{W}_2$. We want to check that $A^{\tilde{W}_1}(X) = A^{\tilde{W}_2}(X)$ for all vectors $X \in T_pM$.
To see this we check

$$\left\langle (\overline{\nabla}_X \tilde{W}_1)^T, Y \right\rangle_g - \left\langle (\overline{\nabla}_X \tilde{W}_2)^T, Y \right\rangle_g = \left\langle \overline{\nabla}_X \tilde{W}_1 - \overline{\nabla}_X \tilde{W}_2, Y \right\rangle_{\overline{g}}$$

and so we can apply the compatability of $\overline{\nabla}$ with the metric to get

$$\left\langle \overline{\nabla}_X \tilde{W}_1 - \overline{\nabla}_X \tilde{W}_2, Y \right\rangle_{\overline{g}} = \overline{\nabla}_X \left\langle \tilde{W}_1 - \tilde{W}_2, Y \right\rangle_{\overline{g}} - \left\langle \tilde{W}_1 - \tilde{W}_2, \overline{\nabla}_X Y \right\rangle_{\overline{g}}$$

and notice that the first term is trivially zero since both $\tilde{W}_1$ and $\tilde{W}_2$ are perpendicular to T_pM, and similarly the second term is also zero since at any point of M, $\tilde{W}_1 - \tilde{W}_2 = 0$.

Lemma 1.5.3. *A^W is a symmetric map for any $W \in NM$.*

Proof. We compute

$$\left\langle A^W(X), Y \right\rangle_g - \left\langle A^W(Y), X \right\rangle_g = \left\langle \overline{\nabla}_{\overline{Y}} W, \overline{X} \right\rangle_{\overline{g}} - \left\langle \overline{\nabla}_{\overline{X}} W, \overline{Y} \right\rangle_{\overline{g}}$$

and then apply compatability

$$\left\langle \overline{\nabla}_{\overline{Y}} W, \overline{X} \right\rangle_{\overline{g}} - \left\langle \overline{\nabla}_{\overline{X}} W, \overline{Y} \right\rangle_{\overline{g}} = \overline{\nabla}_{\overline{Y}} \left\langle W, \overline{X} \right\rangle_{\overline{g}} - \left\langle W, \overline{\nabla}_{\overline{Y}} \overline{X} \right\rangle_{\overline{g}} - \overline{\nabla}_{\overline{X}} \left\langle W, \overline{Y} \right\rangle_{\overline{g}} + \left\langle W, \overline{\nabla}_{\overline{X}} \overline{Y} \right\rangle_{\overline{g}}$$

and then clearly the first and third terms are zero and the second and fourth terms give

$$\left\langle W, [\overline{X}, \overline{Y}]_N \right\rangle_{\overline{g}}$$

which is also zero by the lemma from before. □

From the second fundamental form we can define a mean curvature vector, consider the map $\mathrm{I\!I} : T_pM \times T_pM \to N_pM$ defined to be the unique vector satisfying

$$\langle \mathrm{I\!I}(X, Y), W \rangle_{\overline{g}} = \left\langle A^W(X), Y \right\rangle_g$$

for all X, Y. We then define the mean curvature vector to be the trace

$$\vec{H} = \sum_{i=1}^{n} \mathrm{I\!I}(e_i, e_i)$$

where e_i is the frame of any orthonormal chart for M. One can check this definition is independent of which orthonormal chart you pick.
We now come back to the ABP setting, assume that M is isometrically embedded in $\mathbb{R}^{n+m}$.

Recall the PDE we were considering the solvability of,

$$\begin{aligned} \operatorname{div}(h \operatorname{grad} u) &= f \quad \in \Omega \\ \langle \nabla u, \nu \rangle &= 1 \quad \in \partial\Omega \end{aligned}$$

which is solvable if and only if $\int_\Omega f = \int_{\partial\Omega} h$.
We first define the sets,

$$\Omega^* = \{x \in \Omega || \nabla u(x)| < 1\}, \quad \hat{\Omega} = \{(x, Y) \in N\Omega |, |Y|^2 + |\nabla u(x)|^2 < 1\}$$

and then we define the contact set

$$\Gamma = \{(x, Y) \in N\hat{\Omega} | \operatorname{Hess}_u(X) - (\mathrm{I\!I}, Y) \geq 0\}$$

where the inequality is in terms of matrices, that is the map defined by $(v, w) \mapsto \operatorname{Hess}_u(X)(v, w) - (\mathrm{I\!I}(v, w), Y)$ is symmetric positive semidefinite.
Recall that $N\hat{\Omega} = \{(x, Y) | x \in \Omega, Y \in T_x^\perp M\}$. We now define the ABP map $\Phi : N\hat{\Omega} \to \mathbb{R}^{n+m}$ to be

$$\Phi(x, Y) = \nabla u(x) + Y$$

noting that $\nabla u(x)$ is orthogonal to Y since Y is in the normal bundle. We thus have by definition of $\hat{\Omega}$ that

$$|\Phi(x, Y)| = |Y|^2 + |\nabla u(x)|^2 < 1$$

and so $\Phi(N\hat{\Omega}) \subseteq B_1^{(n+m)}$.

Lemma 1.5.4. $\Phi(N\Gamma) \supseteq B_1^{(n+m)}$

Proof. Take some $\xi \in B_1^{(n+m)}$, that is $\underline{|\xi|} < 1$, then define $w(x) = u(x) - \langle x, \xi \rangle$. Then there exists a unique minimum at $x_0 \in \overline{\Omega}$. Assume that $x_0 \notin \partial\Omega$, then $\nabla w(x_0) = 0$, thus $\nabla u(x_0) = \xi^T$ and $\operatorname{Hess}_w(x_0) \geq 0$. From there we get

$$\operatorname{Hess}_w(x_0) = \operatorname{Hess}_u(x_0) - \left\langle \nabla_{e_i e_j} x, \xi^\perp \right\rangle$$

and one can check that $\left\langle \nabla_{e_i e_j} x, \xi^\perp \right\rangle = \left\langle \mathrm{I\!I}(e_i, e_j), \xi^\perp \right\rangle$ and so we get that $(x, y) = \xi^T + \xi^\perp$ is exactly in our contact set. $\square$

Lemma 1.5.5 (Jacobian Lemma). *The Jacobian J_Φ is given by*

$$J_\Phi = \det(D\Phi(x, Y)) = \det(\operatorname{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle)$$

and so by our lemma before we may apply the inequality and get

$$\det(\operatorname{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle) \leq \left(\frac{\operatorname{tr}\left(\operatorname{Hess}_u(x) - \langle \mathrm{I\!I}, Y \rangle\right)}{n} \right)^n = \left(\frac{\Delta u(x) - \left\langle \vec{H}, Y \right\rangle}{n} \right)^n$$

Proof. Take any $(x_0, y_0) \in N\hat{\Omega}$ fixed, then fix a local orthonormal chart $e_1, \ldots, e_n$. We can also find a nice frame for the normal bundle $\nu_1, \nu_2, \ldots, \nu_m$. That is a frame satisfying

$$\langle \nu_i(x), \nu_j(x) \rangle = \partial_{ij}, \langle \nu_i(x), e_j(x) \rangle = 0,$$

we thus get local coordinates for $N\hat{\Omega}$, $(x_1, \ldots, x_n, y_1, \ldots, y_m)$.
Now compute

$$\left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \left\langle \overline{\nabla}_{e_i}(\nabla u), e_j \right\rangle + \sum_{\alpha=1}^{m} y_\alpha \left\langle \nabla_{e_i}(\nu_\alpha), e_j \right\rangle$$

and so since in the first term we are inner producting with e_j we may drop all normal components of $\overline{\nabla}$ and reduce it to the standard ∇ on M, this gives us the hessian, on the other hand for for the second term we pick up exactly the expression for the second fundamental form. Thus we define

$$A := \left\langle \frac{\partial \Phi}{\partial x_i}(x_0, y_0), e_j \right\rangle = \mathrm{Hess}_u(e_i, e_j) - (\mathrm{I\!I}(e_i, e_j), Y)$$

Next note that

$$\left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), \nu_j \right\rangle = \delta_{ij}, \quad \left\langle \frac{\partial \Phi}{\partial \nu_i}(x_0, y_0), e_j \right\rangle = 0$$

because Φ is the identity map on the Y component. We thus have that the Jacobian matrix takes the block form

$$\begin{bmatrix} A_{n\times n} & 0 \\ * & I_{m\times m} \end{bmatrix}$$

and so its determinant is just the determinant of A, proving the lemma. □

1.6 Isoperimetric Inequality on Minimal Submanifolds

Suppose M^n is immersed in $\mathbb{R}^{n+m}$, with $\Omega \subseteq M$, $\overline{\Omega}$ compact and $\partial\Omega \in C^\infty$. We then have along the boundary μ the normal vector to $\partial\Omega$ and a collection of normal vectors $T_x^\perp M \subseteq T_x\mathbb{R}^{n+m}$.
We denote $|B_1^{(k)}|$ to be the volume of the ball of radius 1 in $\mathbb{R}^k$.
Last time we considered the PDE

$$\mathrm{div}_g(f\nabla_g u) = h$$
$$\langle \nabla_g u, \mu \rangle_g \Big|_{\partial\Omega} = 1$$

which is solvable if and only if $\int_\Omega h = \int_{\partial\Omega} f$.

We will now prove the following Sobolev inequality, for any $f > 0, f \in C^\infty(M)$, we have

$$\int_\Omega \left(|\nabla f|^2 + f^2|\vec{H}|^2\right)^{1/2} + \int_{\partial\Omega} f \geq n\left(\frac{n+m}{n}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n}\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

If M is minimal ($\vec{H} = 0$) and $f = 1$ then we get $|\partial\Omega| \geq C_{n,m}|\Omega|^{\frac{n}{n-1}}$, the Isoperimetric inequality.

Proof. We assume that $m \geq 2$, if $m = 1$ then we can lift the surface one more dimension to make $m = 2$.

Our job now is to pick a special h to use the PDE. Note that the equation is scaling invariant, we can then see that by changing $f \to cf$ we get

$$\int_\Omega n(cf)^{\frac{n}{n-1}} = c^{\frac{n}{n-1}}\int_\Omega nf^{\frac{n}{n-1}}$$

and

$$\int_\Omega \sqrt{|\nabla cf|^2 + (cf)^2|\vec{H}|^2} + \int_{\partial\Omega} cf = c\left(\int_\Omega \sqrt{|\nabla f|^2 + (f)^2|\vec{H}|^2} + \int_{\partial\Omega} f\right)$$

so by rescaling we can make these two expressions equal for f. Then by setting $h = nf^{\frac{n}{n-1}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$ then there is a solution u to the PDE with f, h.

Claim 1.6.1. $0 \leq \det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) \leq f^{\frac{n}{n-1}}(x)$

For now we will assume the claim is true.

Then we have $B_1^{(n+m)} \subseteq \Phi(\hat{\Omega})$ and so

$$|B_1^{(n+m)}| \leq \int_{\tilde{\Omega}} \det(J_\Phi) \leq \int_{x\in\Omega^*}\int_{T_x^\perp\Omega} \det(J_\Phi)$$

we will now restrict the domain so that $|\Phi(\hat{\Omega})| \geq \delta$, then using the fact that $|\Phi(\hat{\Omega})|^2 = |\nabla u(x)|^2 + Y^2$ we get that $(\delta^2 - |\nabla u(x)|^2)_+ < Y^2 < 1 - |\nabla u(x)|^2$. Set B'_x to be the set of Y satisfying the above, we then get

$$(1 - \delta^{n+m})|B_1^{(n+m)}| \leq \int_{x\in\Omega^*}\int_{B'_x} |\det(J_\Phi)| dY\, dx$$

and substituting the determinant we get

$$\int_{x\in\Omega^*}\int_{B'_x} \det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) dY\, dx$$

and using the claim we get that this is less than

$$\int_{x\in\Omega^*}\int_{B'_x} f^{\frac{n}{n-1}} dY\, dx.$$

Next we get

$$\int_{x\in\Omega^*}\int_{B'_x} f^{\frac{n}{n-1}}\,dY\,dx = \int_{x\in\Omega^*} f^{\frac{n}{n-1}}(1-|\nabla u(x)|^2-(\delta^2-|\nabla u(x)|^2)_+)^{m/2}|B_1^{(m)}|dx$$

then by the inequality $A^s - B^s \le s(A-B)$ for $A \ge B \ge 0$ and $s \ge 1$ we get since $m \ge 2$ that

$$\begin{aligned}&\int_{x\in\Omega^*} f^{\frac{n}{n-1}}(1-|\nabla u(x)|^2-(\delta^2-|\nabla u(x)|^2)_+)^{m/2}|B_1^{(m)}|dx\\ &\le \int_{x\in\Omega^*} f^{\frac{n}{n-1}}\frac{m}{2}(1-|\nabla u(x)|^2-(\delta^2-|\nabla u(x)|^2)_+)|B_1^{(m)}|dx\end{aligned}$$

then by checking both cases we can find that

$$1-|\nabla u(x)|^2-(\delta^2-|\nabla u(x)|^2)_+ \le 1-\delta^2$$

and so we can then get

$$(1-\delta^{n+m})(|B_1^{(n+m)}|) \le \frac{m}{2}(1-\delta^2)|B_1^{(m)}|\int_{x\in\Omega^*} f^{\frac{n}{n-1}}\,dx.$$

Dividing this inequality by $1-\delta$ we get

$$(1+\delta^1+\delta^2+\cdots+\delta^{n+m-1})(|B_1^{(n+m)}|) \le \frac{m}{2}(1+\delta)|B_1^{(m)}|\int_{x\in\Omega^*} f^{\frac{n}{n-1}}\,dx$$

and then by letting $\delta \to 1$ we get

$$(n+m)(|B_1^{(n+m)}|) \le m|B_1^{(m)}|\int_{x\in\Omega^*} f^{\frac{n}{n-1}}\,dx$$

now we can rewrite this as,

$$\left(\frac{n+m}{m}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n} \le \left(\int_{x\in\Omega^*} f^{\frac{n}{n-1}}\,dx\right)^{1/n}$$

and so we have

$$n\int_\Omega f^{\frac{n}{n-1}} = n\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{1}{n}}\left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \ge n\left(\frac{n+m}{m}\frac{|B_1^{(n+m)}|}{|B_1^{(m)}|}\right)^{1/n}\left(\int_\Omega nf^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

and so by our rescaling of f we get the desired result.

All that remains is to prove the claim, as we saw before in the determinant Lemma we have that

$$\det(\mathrm{Hess}_u(x)-\langle \mathrm{I\!I}(x),Y\rangle) \le \left(\frac{\mathrm{tr}(\mathrm{Hess}_u(x)-\langle \mathrm{I\!I}(x),Y\rangle)}{n}\right)^n = \left(\frac{\Delta_g u(x)-\left\langle \vec{H},Y\right\rangle}{n}\right)^n.$$

From the PDE of u we get that $\div(f\nabla u) = nf^{\frac{1}{n}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$ and so by evaluating using divergence rules we get

$$\div(f\nabla u) = \langle \nabla f, \nabla u\rangle + f\Delta_g u = nf^{\frac{n}{n-1}} - \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$$

and solving for $\Delta_g u$ we get

$$\Delta_g u = nf^{\frac{n}{n-1}-1} - f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} - \left\langle \frac{\nabla f}{f}, \nabla u\right\rangle.$$

Plugging this into the inequality we get

$$\det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) \leq \left(\frac{nf^{\frac{n}{n-1}-1} - f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} - \left\langle \frac{\nabla f}{f}, \nabla u\right\rangle - \left\langle \vec{H}, Y\right\rangle}{n}\right)^n.$$

We now use a Cauchy-Schwartz inequality, for any $a, A, b, B \in \mathbb{R}^n$ we have

$$|a\cdot A + b \cdot B| \leq \sqrt{A^2 + B^2}\sqrt{a^2 + b^2}$$

then we get

$$\left|\langle \nabla f, \nabla u\rangle + \nabla\left\langle f\vec{H}, Y\right\rangle\right| \leq \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}\sqrt{|\nabla u|^2 + Y^2} \leq \sqrt{|\nabla f|^2 + f^2|\vec{H}|^2}$$

and so we get that

$$f^{-1}\sqrt{|\nabla f|^2 + f^2|\vec{H}|^2} + \left\langle \frac{\nabla f}{f}, \nabla u\right\rangle + \left\langle \vec{H}, Y\right\rangle \geq 0$$

and thus

$$\det(\mathrm{Hess}_u(x) - \langle \mathrm{I\!I}(x), Y\rangle) \leq \left(\frac{nf^{\frac{1}{n-1}}}{n}\right)^n = f^{\frac{n}{n-1}}$$

$\square$