1 Elementary Algebraic Topology

These notes are based on first chapter of the phenomenal book by Hatcher (2002) [1].

1.1 Homotopy

For the rest of these notes, all maps are continuous.

Definition 1.1.1. *A family of maps $f_t : X \to Y$ parameterized by $t \in [0, 1] = I$ is called a* ***homotopy*** *if the companion map $F : I \times X \to Y$ defined by*

$$F(t, x) = f_t(x)$$

is continuous under the product topology of $I \times X$.

If $f_0 = f$ and $f_1 = g$ we call f_t a ***homotopy between*** *f and g.*

A homotopy between f and g captures the intuitive notion of deforming a function f into another function g continuously.

Definition 1.1.2. *A map $f : I \to X$ is called a* ***path*** *in X, if $f(0) = f(1)$ it is also a* ***loop****.*

Definition 1.1.3. *Given two paths f and g such that $f(0) = g(0)$ and $f(1) = g(1)$, A* ***homotopy of paths*** *between f and g is a homotopy h_t between f and g with the additional property*

$$h_t(0) = f(0) \wedge h_t(1) = f(1), \forall t \in I.$$

If f and g are both loops it is also a ***homotopy of loops****.*

Two paths or loops are called ***homotopy equivalent*** *if there exists a homotopy of paths or loops respectively between them.*

We denote this as $f \simeq g$ or $f \simeq_{h_t} g$.

A homotopy equivalence captures the intuitive notion that two loops can be deformed into one another continuously while keeping their endpoints constant. We can then use this idea to organize all possible loops between two fixed points into equivalence classes based on their homotopy.

Lemma 1.1.4. *Let $x_0, x_1 \in X$, then $\simeq$ is an equivalence relation on the set*

$$\{f : f \text{ is a path} \wedge f(0) = x_0 \wedge f(1) = x_1\}$$

Proof. See [1, p. 26]. □

1.2 Fundamental group

Since $\simeq$ is an equivalence relation between paths between any two points we can now reasonable discuss the equivalence classes $[f]$ under this relation.

We now can define a way to combine equivalence classes in order to get other equivalence classes in a group fashion

Definition 1.2.1. *Given two paths f, g with $f(1) = g(0)$ we define $g \circ f$ to be*

$$(g \circ f)(t) = \begin{cases} f(2t) & : t \in [0, \frac{1}{2}] \\ g(2t-1) & : t \in [\frac{1}{2}, 1] \end{cases}$$

This is essentially going along one path at twice the speed and then along the second path at twice the speed, which stays continuous since their endpoints match.

Theorem 1.2.2. *Composition of paths is invariant under $\simeq$*

Proof. Let f, g be paths such that $f(1) = g(0)$ as well as $f \simeq_{f_t} \tilde{f}$ and $g \simeq_{g_t} \tilde{g}$, we have that

$$f_t(1) = f(1) = g(0) = g_t(0),\ \forall t.$$

Thus the composition $f_t \circ g_t$ is valid. Now since

$$(f_t \circ g_t)(0) = f_t(0) = f(0) = (f \circ g)(0),\ \forall t,$$

And

$$(f_t \circ g_t)(1) = g_t(1) = g(1) = (f \circ g)(1),\ \forall t,$$

We have that $f_t \circ g_t$ is a homotopy of paths from $f \circ g$ to $f_1 \circ g_1$. But note that $f_1 \circ g_1 = \tilde{f} \circ \tilde{g}$ and thus

$$f \circ g \simeq_{f_t \circ g_t} \tilde{f} \circ \tilde{g}.$$ □

Because of this we can reasonably talk about the composition of equivalence classes as now that operation is well defined.

Now consider any point $x_0 \in X$ and consider the set of loops that start and end at x_0, we denote this set

$$\Pi_1(X, x_0) = \{f : I \to X | f(0) = f(1) = x_0\}.$$

Now we know that $\simeq$ is an equivalence relation on paths and so we define

$$\pi_1(X, x_0) = \Pi_1(X, x_0)/\simeq.$$

Proposition 1.2.3. $(\pi_1(X, x_0), \circ)$ is a group, called the Fundamental group of the space X at the base point x_0.

Proof. See [1, p. 26] □

Remark 1.2.4. If X is path-connected, the group structure does not depend on the point x_0. Namely for any $x_0, x_1 \in X$ we have $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ [1, p. 28].

For such spaces we write the group as $\pi_1(X)$.

The Fundamental group allows us to study properties of a topological space from the lens e of algebra and get more information about a space.

One of the best ways to extract the Fundamental group of a space is by building it up from multiple components and then using the van Kampen Theorem

Theorem 1.2.5 (van Kampen Theorem). *If X is the union of path-connected open sets A_α each containing a base point x_0 and if all intersections of the form $A_\alpha \cap A_\beta \cap A_\gamma$ are path connected, then the the fundamental group $\pi_1(X, x_0)$ is the free product $*_\alpha \pi_1(A_\alpha, x_0)$ quotiented by the set*

$$\{w_\alpha w_\beta^{-1} | w_\alpha \in \pi_1(A_\alpha, x_0) \wedge w_\beta \in \pi_1(A_\beta, x_0) \wedge w_\alpha = w_\beta\}$$

Proof. See [1, p. 44] □

We can then immediately use this theorem to characterize the Fundamental group for graphs (viewed topologically as 1-dimensional CW complexes).

Proposition 1.2.6. The Fundamental group of a connected graph $G(V, E)$ when considered a topological space is the free group on n letters where $n = |E| - |V| + 1$ [1, p. 43].

Proof. Take any spanning tree of G, this tree must have n vertices and $n - 1$ edges, call its interior T. T obviously has no nontrivial loops and so $\pi_1(T) = 0$.

Now take any edge $(v_1, v_2) \in G \setminus T$. Then attaching this closed segment to the tree always produces a loop because of the Euler characteristic, thus since the rest of the tree cannot have extra loops we get that $\pi_1(T \cup \{(v_1, v_2)\}) = \mathbb{Z}$.

Now to for each such subgraph we attach some small neighborhood to each point to make it open, this will not change the Fundamental group if the neighborhood is small enough. Thus we get an open cover of G since every edge is in either the tree or outside of it.

Now by van Kampen's theorem we have that

$$\pi_1(G) = *_{e \in G \setminus T}(\pi_1(T \cup \{e\})) * \pi_1(T) = *_{e \in G \setminus T} \mathbb{Z} * 0 = *_{1 \le i \le |G \setminus T|} \mathbb{Z}$$

Which is the free group over $|G \setminus T|$ letters. Now there are $|V| - 1$ edges in T and so $|G \setminus T| = |E| - (|V| - 1) = |E| - |V| + 1$. □

1.3 Problems

Problem 1.3.1. *Let $f_0 \circ g_0 \simeq f_1 \circ g_1$ and $g_0 \circ g_1$, then $f_0 \circ f_1$*

Proof. By assumption there exist homotopy h_t and g_t such that $f_0 \circ g_0 \simeq_{h_t} f_1 \circ g_1$ and $g_0 \circ_{g_t} g_1$. Now construct the following homotopy, $h_t \circ \overline{g_t}$ where $\overline{g_t}$ is the path that goes backwards along g_t, which can be defined as

$$\overline{g_t}(k) = g_t(1-k)$$

First we confirm that this homotopy is a continuous map since it is continuous on $[0, \frac{1}{2}] \times I$ and $[\frac{1}{2}, 1] \times I$.

Next we have that

$$(h_t \circ \overline{g_t})(0) = \overline{g_t}(0) = g_t(1) = g_0(1) = f_0(0)$$

as well as

$$(h_t \circ \overline{g_t})(1) = h_t(1) = h_0(1) = (f_0 \circ g_0)(1) = f_0(1)$$

and by the same argument matches the endpoints of f_1.

Finally we have

$$h_0 \circ \overline{g_0} = f_0 \circ g_0 \circ \overline{g_0}$$

And

$$h_1 \circ \overline{g_1} = f_1 \circ g_1 \circ \overline{g_1}$$

Thus

$$f_0 \circ g_0 \circ \overline{g_0} \simeq f_1 \circ g_1 \circ \overline{g_1}$$

But we know that

$$f_0 \circ g_0 \circ \overline{g_0} \simeq f_0, \; f_1 \circ g_1 \circ \overline{g_1} \simeq f_1,$$

And so by transitivity of the equivalence relation we get

$$f_0 \simeq f_1$$

□

Problem 1.3.2. *For a given space X, the following are equivalent:*

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$

Proof. Since a map $S^1 \to X$ is simply a loop (because S^1 is homeomorphic to I with its ends identified) then assuming that all such maps are homotopic to a point is equivalent to saying that there is only 1 homotopy class of loops at any given point and thus the fundamental group is trivial at any point. Thus we have $(a) \iff (c)$.

Take any map $f : S^1 \to X$, and assume that there exists a homotopy between it and the constant map, we have a function

$$F(t, s) : I \times S^1 \to X$$

with $F(1, s) = x_0, \forall s$ but then by using radial coordinates we can parameterize D^2 by $I \times S^1$ with the points $(0, s)$ being identified, and so combining these we get a map $\tilde{f} : D^2 \to X$ which is an extension of f since $\tilde{f}|_{S^1} = F(0, s) = f(s)$.

Similarly we can do the exact opposite construction by making a map $f : D^2 \to X$ into a homotopy between $f|_{S^1}$ and the constant map on S^1. Thus we have $(a) \iff (b)$. □

Definition 1.3.3. *Let $\phi : X \to Y$ be a map taking x_0 to y_0, then ϕ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined as*

$$\phi_* : [f] \mapsto [\phi f]$$

Remark 1.3.4. This map well defined, see [1][p. 34].

Lemma 1.3.5. *If $\phi_t : X \to Y$ is a homotopy and h is the path $\phi_t(x_0)$ traced out by the point x_0, then*

$$(\phi_0)_* = \beta_h(\phi_1)_*$$

Where β_h is a map from $\pi_1(X, \phi_0(x_0))$ to $\pi_1(X, \phi_1(x_0))$ defined by

$$\beta_h f = h \circ f \circ \overline{h}$$

Proof. See [1][p. 37] □

Problem 1.3.6. *Let $f_t : X \to X$ be a homotopy such that f_0 and f_1 are both the identity map. Then for any $x_0 \in X$, the loop $h := f_t(x_0) \in Z(\pi_1(X, x_0))$ where $Z(G)$ is the center of the group G.*

Proof. By the previous lemma since f_0 and f_1 are both identity, we have that the induced homomorphism is also identity, thus

$$\beta_h f = f, \forall f.$$

But note that since h is a loop in this case, starting and ending at x_0, thus $\beta_h = h \circ f \circ \overline{h}$ is in fact

$$\beta_h = h \circ f \circ h^{-1}.$$

Which is a conjugation by the loop h.

But we know from basic group theory that if conjugation by an element is the identity map that element must be inside the center of the group, thus h is in the center of the group. □

2 Conformally Covariant Operators And Conformal Invariants on Weighted Graphs

2.1 Basic Graph constructions

We now switch our attention to Weighted Graphs and discuss how they are discrete analogs to Manifolds as described in [2][3]. Let $G(V, E)$ be a simple graph, we define a weight function on G to be $w : E \to \mathbb{R}^+$. Analogous to a Riemannian metric on a smooth manifold these weights represent distance between points, we refer to a pair of a graph and a weight function on it as (G, w).

Definition 2.1.1. *Given a weighted graph (G, w) a **differential operator** on (G, w) is a linear endomorphism on either $\mathrm{Hom}(E, \mathbb{R})$ or $\mathrm{Hom}(V, \mathbb{R})$*

Now it is often useful to define a differential operator in terms of the weight function of a graph (see examples below), thus we need to extend our definitions to match that.

Definition 2.1.2. *Given a graph G let $\mathcal{W}(G)$ be the space of all weight functions on G. A **weighted differential operator** is a differential operator indexed by $w \in \mathcal{W}(G)$.*

Example 2.1.3. The **adjacency matrix** A_w of a weighted graph (G, w) is a linear endomorphism on $\mathrm{Hom}(V, \mathbb{R})$ defined by its matrix representation

$$[A_w]_{ij} = \begin{cases} w(v_i, v_j) & : (v_i, v_j) \in E, \\ 0 & : (v_i, v_j) \notin E \end{cases}$$

Example 2.1.4. The **degree matrix** D_w of a weighted graph (G, w) is a similarly defined as

$$[D_w]_{ii} = \sum_{j=1}^{n} [A_w]_{ij}$$

This then also defines

$$\Delta_w := D_w - A_w$$

Called the **vertex Laplacian**.

2.2 Conformal weights and operators

We now define the most important construction, inspired by conformal transformations on manifolds as in [2] and [3].

Definition 2.2.1. *Two weights $w, \tilde{w} \in \mathcal{W}(G)$ are called **conformally equivalent** if there exists a function $u \in \mathrm{Hom}(V, \mathbb{R})$ such that*

$$\tilde{w}(v_i, v_j) = w(v_i, v_j) e^{u(v_i)+u(v_j)},$$

*we call u the **conformal factor**.*

It is easily shown that this is an equivalence relation which we denote $\sim_c$, then given a $w \in \mathcal{W}(G)$ we denote its equivalence class as $[w]$. The set of all equivalences classes is denoted $\mathcal{M}(G) := \mathcal{W}(G)/\sim_c$.

We can treat $\mathcal{W}(G)$ as a vector space by taking the log of all the weights. This then gives us the equation

$$\log \tilde{w}(v_i, v_j) - \log w(v_i, v_j) = u(v_i) + u(v_j)$$

for conformal equivalence.

That then means that we can write this as a linear map $\mathrm{Hom}(V, \mathbb{R}) \to [w]$. This map happens to be represented by the vertex-edge incidence matrix.

Definition 2.2.2. *For a graph G define B to be the $|E| \times |V|$ matrix given by*

$$[B]_{ij} = \begin{cases} 1 & : v_j \textit{ is an endpoint of } e_i \\ 0 & : \textit{otherwise} \end{cases}.$$

We call this the unsigned vertex-edge incidence matrix.

Remark 2.2.3. As shown by [2] we have that

$$\dim \mathcal{M}(G) = |E| - \mathrm{rank}(B) = |E| - |V| + \omega_0$$

where ω_0 is the number of bipartite components of G.

We can also generalize the definition of conformal operators to graphs as in [2].

Definition 2.2.4. *A weighted differential operator S_w is called* ***Conformally covariant*** *if for any weight $\tilde{w} \in [w]$ there exist diagonal matrices D_α, D_β with positive entries (which only depend on the conformal factors between $\tilde{w}$ and w) such that*

$$S_{\tilde{w}} = D_\alpha S_w D_\beta$$

Example 2.2.5. A_w is conformally covariant, this can be easily seen as if $\tilde{w} \in [w]$ we have that

$$[A_{\tilde{w}}]_{ij} = \begin{cases} \tilde{w}(v_i, v_j) & : (v_i, v_j) \in E, \\ 0 & : (v_i, v_j) \notin E \end{cases} = \begin{cases} w(v_i, v_j) e^{u(v_i)} e^{u(v_j)} & : (v_i, v_j) \in E, \\ 0 & : (v_i, v_j) \notin E \end{cases} = D_u A_w D_u$$

Where $D_u = \mathrm{diag}\left(e^{u(v_1)}, \ldots, e^{u(v_n)}\right)$

Example 2.2.6. We have the following generalization of the incidence operator presented earlier, first orient each edge in E in any fashion to get $e_i = (e_i^-, e_i^+)$. We then define

$$[M_w]_{ij} = \begin{cases} \sqrt{w(v_i, v_j)} & : v_i = e_j^+, \\ -\sqrt{w(v_i, v_j)} & : v_i = e_j^- \\ 0 & : \text{otherwise} \end{cases}$$

Proposition 2.2.7. M_w is a conformally covariant.

Proof. Let $\tilde{w} \in [w]$ with conformal factor $u(v_i)$ then

$$\begin{aligned}
[M_{\tilde{w}}]_{ij} &= \begin{cases} \sqrt{\tilde{w}(v_i, v_j)} & : v_i = e_j^+, \\ -\sqrt{\tilde{w}(v_i, v_j)} & : v_i = e_j^- \\ 0 & : \text{otherwise} \end{cases} \\
&= \begin{cases} \sqrt{w(v_i, v_j)} e^{\frac{1}{2}(u(v_i)+u(v_j))} & : v_i = e_j^+, \\ -\sqrt{w(v_i, v_j)} e^{\frac{1}{2}(u(v_i)+u(v_j))} & : v_i = e_j^- \\ 0 & : \text{otherwise} \end{cases} \\
&= [M_w]_{ij} e^{\frac{1}{2}(u(v_i)+u(v_j))}
\end{aligned}$$

Thus we have

$$M_{\tilde{w}} = M_w D_u$$

Where

$$(D_u)_{ii} = e^{\frac{1}{2}(u(e_i^-)+u(e_i^+))}$$

□

Example 2.2.8. We can now also define $\Delta'_w = M_w^T M_w$ to be the **Weighted Edge Laplacian**. We also note that by the transformation rule for the incidence matrix we get that

$$\Delta'_{\tilde{w}} = D_u \Delta'_w D_u.$$

Thus, this operator is also conformally covariant.

2.3 Important properties

Conformally covariant operators carry with them very powerful properties proven by [2].

Proposition 2.3.1. Let S_w be a conformally covariant operator, $\ker(S_w)$ is conformally invariant up to isomorphism, in particular, the dimension of $\ker(S_w)$ and the multiplicity of 0 as an eigenvalue are also conformally invariant.

Definition 2.3.2. *Let S be a linear operator over a real vector space with only real eigenvalues, the signature of S is defined as (N_-, N_0, N_+) where N_- is the number of negative eigenvalues, N_0 is the multiplicity of the 0 eigenvalue, N_+ is the number of positive eigenvalues.*

Theorem 2.3.3. *Let S_w be a conformally covariant operator depending continuously on w with only real eigenvalues, its signature is a conformal invariant.*

First we need to prove a small lemma

Lemma 2.3.4. *A continuous path starting in one path-connected closed set and ending in another must go through their intersection.*

Proof of lemma. Let C_1, C_2 be the closed sets and $f : [0,1] \to C_1 \cup C_2$ be the path we have. Assume then that $f([0,1]) \cap C_1 \cap C_2 = \varnothing$, then f is a path in the subspace topology of $(C_1 \cup C_2) \setminus (C_1 \cap C_2)$ connecting $(C_1 \cup C_2) \setminus C_1$ and $(C_1 \cup C_2) \setminus C_2$. Now we know that $(C_1 \cup C_2) \setminus C_1$ and $(C_1 \cup C_2) \setminus C_2$ are both complements of closed sets and so are both open in the subspace topology. But they also disjoint and union to the entire subspace and so are both closed. Thus they are both disjoint unions of connected components and so there can be no path between them. Thus we get a contradiction and so any path must go through the intersection. $\square$

Now we can prove the theorem.

Proof of theorem 2.3.3. We already know that N_0 is conformally invariant. Now assume that the signature changes from w to $\tilde{w} \in [w]$ then we can connect w to $\tilde{w}$ with a line $w(t) : [0,1] \to [w]$. Then along this line we can parameterize the operator as $S_{w(t)}$ and by Theorem 5.2 of [4, p. 109] we can select continuous functions $\lambda_1(t), \ldots, \lambda_n(t)$ such that $\lambda_i(t)$ is an eigenvalue for $S_{w(t)}$ for any i, t.

Now let $I = (i_1, \ldots, i_{N_0})$ be an increasing multi-index of length N_0 and let $C_I = \{x \in \mathbb{R}^n : x_i = 0 \forall i \in I\}$, then note that if we consider our λ functions as one vector valued function to $\mathbb{R}^n$ then there exists exactly one multi-index of length N_0 such that $\lambda(0) \in C_I$.

Now assume WLOG that $\lambda_1(t)$ switches sign along $w(t)$, then by intermediate value theorem applied to $\lambda_1(t)$ we get that there exists a c such that $\lambda_1(c) = 0$. Now since the number of eigenvalues with value 0 is a constant along this path we know that at c there exists a unique multi-index J of length N_0 with $\lambda(c) \in C_J$ with I and J being distinct.

Now it is clear that C_I is closed and path-connected (it is isomorphic to $\mathbb{R}^{n-N_0}$) for any multi-index I and so by lemma 2.3.4 since the path λ goes from C_I at $t = 0$ to C_J at $t = c$ the path must cross through their intersection. But since I and J are distinct the intersection C_I and C_J must have at least $N_0 + 1$ coordinates with value 0, but we know this is not possible as that would imply there are more than N_0 eigenvalues with value 0. Thus we have a contradiction and so $\lambda_1(t)$ cannot change sign from $t = 0$ to $t = 1$. But this applies to all the eigenvalues and so none of the eigenvalues can change sign. $\square$

Remark 2.3.5. The sign of the smallest and largest eigenvalues of a such an operator S_w are well-determined by the signature of S_w, and so are also conformally invariant.

Remark 2.3.6. For the adjacency matrix operator as well as the vertex and edge Laplacian, they are all self-adjoint, thus, all their eigenvalues are real and so the theorem applies to them.

3 Conformal Invariants

Let S_w be a conformally covariant operator on a domain Y, then let $f_1, \ldots, f_m$ be a basis for $\ker(S_w)$. Now let

$$X_w = \bigcap_{i=1}^{m} \ker(f_i).$$

We then define $\Phi_w : Y \setminus X_w \to \mathbb{RP}^{m-1}$ by

$$\Phi_w(y) = (f_1(y) : \ldots : f_m(y)) \quad \forall y \in Y \setminus X_w.$$

Proposition 3.0.1. Φ_w is a conformal invariant.

Proof. We recall that there is an isomorphism between $\ker(S_w)$ and $\ker(S_{\tilde{w}})$ given by $\ker(S_{\tilde{w}}) = D_\beta^{-1} \cdot \ker(S_w)$ and so

$$\Phi_{\tilde{w}}(y) = (D_\beta^{-1}(y) f_1(y) : \ldots : D_\beta^{-1}(y) f_m(y)) = (f_1(y) : \ldots : f_m(y)) \quad \forall y \in Y \setminus X_w.$$

□

We also have invariants related to the changing of signs across edges

Definition 3.0.2. *Let $H : V(G) \to \mathbb{R}$ be a function on vertices. The nodal set $\mathcal{N}(H)$ is the set of all edges $e = (v_1, v_2) \in E(G)$ such that $H(v_1)H(V_2) < 0$, i.e H changes sign across e, along with the vertices v such that $H(v) = 0$.*

Proposition 3.0.3. Let S_w be a conformally covariant operator and $H \in \ker S_w$. Then H is related to $\tilde{H} \in \ker S_{\tilde{w}}$ by D_β^{-1}, thus $\mathcal{N}(H) = \mathcal{N}(\tilde{H})$.

4 Null Adjacency Weights

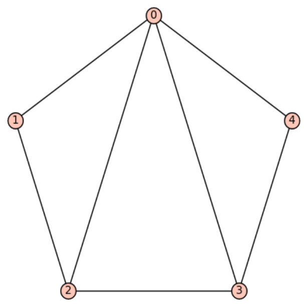
Definition 4.0.1. *Let A_w be the adjacency matrix of a weighted graph. We define*

$$\mathcal{D} = \{w \in \mathcal{M}(G) | \det(A_w) = 0\}$$

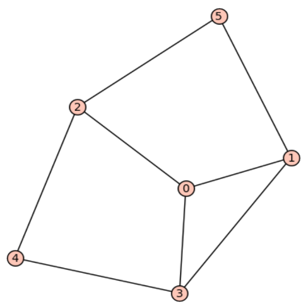
Lemma 4.0.2. *The pre-image of 0 under the determinant is conformally invariant and so we can pass $\mathcal{D}$ through the quotient while staying well defined and so we define a subset of $\mathcal{M}(G)$ given by*

$$\mathcal{D}_c = \{[w] \in \mathcal{M}(G) | \det(A_w) = 0\}$$

Here are some graphs and the defining functions of their subsets



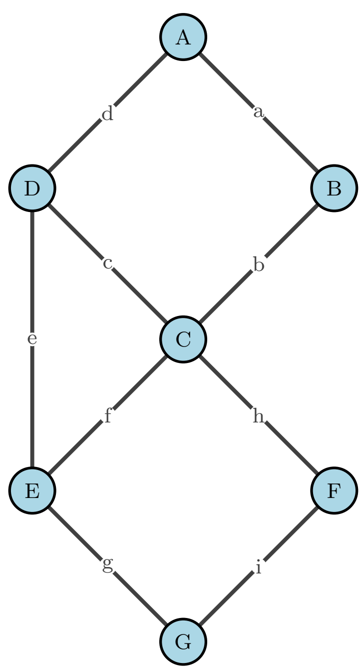
$$-\frac{2\left(x_0^3x_1^4 - x_0^4 + x_1\right)}{x_0^3x_1} = 0$$



$$-\frac{x_0^{12}x_1^6 - 2\,x_0^9x_1^7 + x_0^6x_1^8 + 2\,x_0^6x_1^3 - 2\,x_0^3x_1^4 + 1}{x_0^6x_1^4} = 0$$

We provide the code to generate these equations in the Appendix.

Remark 4.0.3. Neither $\mathcal{D}$ nor $\mathcal{D}_c$ are in general manifolds, as an example consider G as shown below:



We now consider the following 2 smooth curves through $\mathcal{M}(G)$, represented by the corresponding adjacency matrix:

$$f(t) = \begin{bmatrix} 0 & e^t & 0 & e^{-t} & 0 & 0 & 0 \\ e^t & 0 & e^{-t} & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & e^t & 1 & 1 & 0 \\ e^{-t} & 0 & e^t & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad g(t) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & e^t & e^{-t} & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & e^t & 1 & 0 & 0 & e^{-t} \\ 0 & 0 & e^{-t} & 0 & 0 & 0 & e^t \\ 0 & 0 & 0 & 0 & e^{-t} & e^t & 0 \end{bmatrix}$$

It is clear that these curves are smooth by the smoothness of e^x and so we can use the known identification between vectors in the tangent space of p and smooth curves at p.

We can also use a parameterization of $\mathcal{M}(G)$ by edges c and f to write these curves in global coordinates as simply $f(t) = (e^t, 1)$ and $g(t) = (1, e^t)$, this then gives us that $f'(t)|_0 = (1, 0)$ and $g'(t)|_0 = (0, 1)$. Thus these two curves span the ambient tangent space at $(1, 1) \in \mathcal{M}(G)$ and so since these curves are contained in $\mathcal{D}_c$, then the tangent space inside $\mathcal{D}_c$ also has dimension 2.

But now if we assume that $\mathcal{D}_c$ is a submanifold then it must have dimension 2, however, clearly $\mathcal{M}(G)$ also has dimension 2 since $|E| - |V| = 2$ and G has no bipartite components. But we know by [5, p. 99], that a submanifold with codimension 0 is an open submanifold. However, clearly as the pre-image of 0, $\mathcal{D}$ is closed. It is also not all of $\mathcal{W}(G)$ since the weights

$$\det \begin{bmatrix} 0 & e^1 & 0 & e^{-1} & 0 & 0 & 0 \\ e^1 & 0 & e^{-1} & 0 & 0 & 0 & 0 \\ 0 & e^{-1} & 0 & e^1 & e^1 & e^{-1} & 0 \\ e^{-1} & 0 & e^1 & 0 & 1 & 0 & 0 \\ 0 & 0 & e^1 & 1 & 0 & 0 & e^{-1} \\ 0 & 0 & e^{-1} & 0 & 0 & 0 & e^1 \\ 0 & 0 & 0 & 0 & e^{-1} & e^1 & 0 \end{bmatrix} = \frac{(e^4 - 1)^2}{e^2}$$

are not in $\mathcal{D}$. Thus $\mathcal{D}$ cannot be open, and so when passed to the quotient, $\mathcal{D}_c$ also cannot be open, this is a contradiction and so $\mathcal{D}_c$ cannot be a submanifold.

Finally since $\mathcal{D}$ is conformally invariant then if it were a submanifold of $\mathcal{W}(G)$ then $\mathcal{D}_c$ would be a submanifold of $\mathcal{M}(G)$ and so since this is not the case we must conclude that $\mathcal{D}$ is not a submanifold either.

On the other hand we do know that as a set of solutions to a polynomial, $\mathcal{D}$ and $\mathcal{D}_c$ are both Algebraic Varieties and so locally are submanifolds around any non singular point.

We also know that if at least one full rank weighting exists on G then the polynomial must be non-zero, thus we also know that as a set of solutions to a non-zero polynomial, both of $\mathcal{D}$ and $\mathcal{D}_c$ have measure 0, as explored in [6]. Thus the existence of such a weighting gives us information about the geometry of $\mathcal{C}$, so it seems interesting to study when such weightings exist.

5 Existence of Full Rank Weights

As explored in [7], there is a strong connection between $\{1,2\}$-factors and the determinant of the Adjacency Matrix.

Definition 5.0.1. *Let M be a square $n\times n$ matrix, a* ***transversal*** *of M is a product*

$$\prod_{k=1}^{n}(M)_{i_k j_k}$$

that has exactly one entry from every column and exactly one entry from every row.

By definition i_k and j_k are permutations in S_n and so $j_k i_k^{-1}$ is also a permutations in S_n. One can very easily verify that this is an identification between permutations in S_n and transversals of an $n\times n$ matrix given by mapping each column to the unique row who's intersection with the column was chosen as an entry. We denote this identification with $t_M(\sigma)$ for the transversal of M at permutation σ.

It is clear from the identification that there are $n!$ different transversals of any $n\times n$ matrix.

It is also known from the Leibniz formula for determinants that

$$\det(M)=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)t_M(\sigma).$$

It is clear that a transversal only contributes to this sum if the product is non-zero, this only happens when every entry chosen is non-zero, we call this type of transversal a non-zero transversal.

Definition 5.0.2. *Given a graph G, a spanning subgraph H such that H is a disjoint union of cycles C_n and K_2 is called a $\{1,2\}$-factor of G.*

Lemma 5.0.3. *Let A_w be the adjacency matrix of a weighted graph (G,w), each non-zero transversal of A_w corresponds to a unique $\{1,2\}$-factor of G, each $\{1,2\}$-factor H of G corresponds to $2^{c(H)}$ different non-zero transversals where $c(H)$ is the number of cycles in H.*

Proof. Let $t_{A_w}(\sigma)$ be a non-zero transversal, since it is a bijective map from columns to rows then entry k is the weight of the edge between v_k and $v_\sigma(k)$, since this is a non-zero transversal $(v_k,v_\sigma(k))$ is an edge in E for all k. Thus, since the graph is simple, σ has no fixed points. It thus decomposes as a product of disjoint cycles and transpositions in S_n with total length n.

Let (ij) be a transposition in the decomposition of σ, in the product it corresponds to the weight product $w((v_i,v_j))\cdot w((v_j,v_i))$, thus $(v_i,v_j)\in E$ which is a copy of K_2 in E. Now let $(i_1 i_2\dots i_l)$ be a cycle in the decomposition of σ, it corresponds to the product

$$\left(\prod_{m=1}^{l-1}w((v_{i_m},v_{i_{m+1}}))\right)\cdot w((v_{i_l},v_{i_1}))$$

And since it is non-zero, all the edges $(v_{i_m}, v_{i_{m+1}})$ and (v_{i_l}, v_{i_1}) are all in E. This corresponds to cycle of length l in G.

Now note that by construction each vertex v_i is in either a transposition, or a longer cycle, which are all disjoint. Thus this decomposition of σ corresponds exactly to a subgraph H of G which is a union of disjoint copies of K_2 and cycles. Since every vertex appears in one of the transpositions or cycles, H is a spanning subgraph and so it is a $\{1,2\}$-factor of G.

Now let H be any $\{1,2\}$-factor of G. We can construct a permutation of vertex indices by multiplying disjoint transpositions and cycles together. For every copy of K_2 in H, it has the form (v_i, v_j), we multiply by (ij). For every cycle C_l in H, it has the form $\{v_{i_1}, v_{i_2}, \dots, v_{i_l}\}$, we now make a choice and either multiply by $(i_1 i_2 \dots i_l)$ or multiply by $(i_1 i_l i_{l-1} \dots i_2)$. By construction this results in a permutation σ in which $(v_k, v_{\sigma(k)})$ is an edge for any k. Thus since edge weights cannot be 0, $t_{A_w}(\sigma)$ is a non-zero transversal of A_w. We made 2 choices for every cycle and so there are exactly $2^{c(H)}$ transversals we could have made. $\square$

It is also important to note that by the constructions in the proof, the correspondence is such that for a corresponding pair of a transversal $t_{A_w}(\sigma)$ and a $\{1,2\}$-factor H of G we have

$$t_{A_w}(\sigma) = \prod_{e \in E(H)} w(e).$$

Furthermore, since the sign of a permutation is determined only by its cycle structure, which is determined only by the number of cycles in H, each transversal corresponding to H has exactly the same sign. We can thus define this sign as a property of H as follows

Definition 5.0.4. *Let H be a $\{1,2\}$-factor of G with k connected components. Let h_i be the i-th connected component of H. We define the sign of H to be*

$$\operatorname{sgn}(H) := \prod_{i=1}^{k} (-1)^{|V(h_i)|-1} = (-1)^{n-k}$$

Note that this definition of a sign matches the sign of the corresponding transversals to the $\{1,2\}$-factor.

Lemma 5.0.5. *If G has a $\{1,2\}$-factor then G has a $\{1,2\}$-factor H' such that for any other $\{1,2\}$-factor H we have $E(H) \not\subseteq E(H')$.*

Proof. $E(H) \subseteq E(H')$ is a partial ordering on the set of $\{1,2\}$-factors of G and since this is a finite non-empty set it has a minimal element. $\square$

5.1 Main Theorems

Theorem 5.1.1. *Let G be a graph, G has a $\{1,2\}$-factor if and only if there exists a weight w on G such that A_w has full rank.*

Proof. Assume that G has a $\{1,2\}$-factor H, we can assume that H is minimal, then construct a weight w_ε as following

$$w_\varepsilon(e) = \begin{cases} 1 & e \in E(H) \\ \varepsilon & e \notin E(H) \end{cases}.$$

Now by assumption for any other $\{1,2\}$-factor H' there exists an edge e_0 such that $e_0 \in E(H')$ but $e_0 \notin E(H)$. Thus for any transversal $t_{A_{w_\varepsilon}}(\sigma')$ corresponding to H' we have

$$\begin{aligned} t_{A_{w_\varepsilon}}(\sigma') &= \prod_{e \in E(H')} w_\varepsilon(e) \\ &= w(e_0) \cdot \left(\prod_{e \in E(H') \setminus \{e_0\}} w_\varepsilon(e) \right) \\ &= \varepsilon \cdot \left(\prod_{e \in E(H') \setminus \{e_0\}} w_\varepsilon(e) \right) \\ &\leq \varepsilon \cdot \left(\prod_{e \in E(H') \setminus \{e_0\}} 1 \right) \\ &= \varepsilon \end{aligned}$$

Now fix $\frac{1}{2(n!)} > \varepsilon > 0$, and denote S_H the set of $\sigma \in S_n$ such that $t_{A_{w_\varepsilon}}(\sigma)$ corresponds to H.

We can now compute the determinant $\det(A_w)$ as follows:

$$\begin{aligned}
\frac{\det(A_{w_\varepsilon})}{\operatorname{sgn}(H)} &= \frac{1}{\operatorname{sgn}(H)} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) t_{A_{w_\varepsilon}}(\sigma) \\
&= \sum_{\sigma\in S_H} \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(H)} t_{A_{w_\varepsilon}}(\sigma) + \sum_{\sigma\in S_n\setminus S_H} \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(H)} t_{A_{w_\varepsilon}}(\sigma) \\
&\geq \sum_{\sigma\in S_H} \frac{\operatorname{sgn}(H)}{\operatorname{sgn}(H)} t_{A_{w_\varepsilon}}(\sigma) - \left| \sum_{\sigma\in S_n\setminus S_H} \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(H)} t_{A_{w_\varepsilon}}(\sigma) \right| \\
&\geq \sum_{\sigma\in S_H} t_{A_{w_\varepsilon}}(\sigma) - \left| \sum_{\sigma\in S_n\setminus S_H} \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(H)} t_{A_{w_\varepsilon}}(\sigma) \right| \\
&\geq \sum_{\sigma\in S_H} 1 - \sum_{\sigma\in S_n\setminus S_H} \left| \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(H)} \right| \left| t_{A_{w_\varepsilon}}(\sigma) \right| \\
&= 2^{c(H)} - \sum_{\sigma\in S_n\setminus S_H} \left| t_{A_{w_\varepsilon}}(\sigma) \right| \\
&\geq 2^{c(H)} - n!\varepsilon \\
&\geq 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

And so $\det(A_w) \neq 0$ and thus A_w is full rank.

Assume then that G has no $\{1,2\}$-factor, then any non-zero transversal would imply the existence of a factor and thus no such transversals exist. Thus we get that

$$\det(A_w) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) t_{A_{w_\varepsilon}}(\sigma) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) 0 = 0,$$

for any w. Thus there exists no weight such that A_w is full rank. $\square$

We can use a similar technique to decern whether a graph has a weight without full rank.

Theorem 5.1.2. *Let G be a graph with at least one $\{1,2\}$-factor, then there exists a weight w on G such that $\det(A_w) = 0$ if and only if G has at least two $\{1,2\}$-factors with differing signs.*

Proof. Let G be a graph with a weight w such that $\det(A_w) = 0$. Assume for a contradiction that G only has $\{1,2\}$-factors of a single sign, we call that sign $\operatorname{sgn}(G)$, let F_G denote the set of $\{1,2\}$-factors of G. We then compute

$$\begin{aligned}
\det(A_w) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) t_{A_{w_\varepsilon}}(\sigma) \\
&= \sum_{H \in F_G} \sum_{\sigma \in S_H} \operatorname{sgn}(\sigma) t_{A_{w_\varepsilon}}(\sigma) \\
&= \sum_{H \in F_G} \sum_{\sigma \in S_H} \operatorname{sgn}(H) t_{A_{w_\varepsilon}}(\sigma) \\
&= \sum_{H \in F_G} \operatorname{sgn}(H) \sum_{\sigma \in S_H} t_{A_{w_\varepsilon}}(\sigma) \\
&= \operatorname{sgn}(G) \sum_{H \in F_G} \sum_{\sigma \in S_H} t_{A_{w_\varepsilon}}(\sigma) \\
&= \operatorname{sgn}(G) \sum_{\sigma \in S_n} t_{A_{w_\varepsilon}}(\sigma).
\end{aligned}$$

But now note that all of the non-zero transversals of G are a product of strictly positive edge weights and thus strictly positive. But then this sum is also strictly positive and since the sign is non-zero the determinant cannot be 0. This is a contradiction and thus G must have $\{1,2\}$-factors of differing signs.

Assume that G has such factors of differing signs, let H be the factor of sign 1 and H' be the factor of sign -1. Similarly to the previous theorem we define

$$w_{H,\varepsilon}(e) = \begin{cases} 1 & e \in E(H) \\ \varepsilon & e \notin E(H) \end{cases} \text{ and } w_{H',\varepsilon}(e) = \begin{cases} 1 & e \in E(H') \\ \varepsilon & e \notin E(H') \end{cases}.$$

Now by the same computation we get that with correctly chosen ε we have

$$\frac{\det(A_{w_{H,\varepsilon}})}{\operatorname{sgn}(H)} \geq \frac{1}{2} \implies \det(A_{w_{H,\varepsilon}}) \geq \frac{1}{2}.$$

Meanwhile for H' we get

$$\frac{\det(A_{w_{H',\varepsilon}})}{\operatorname{sgn}(H')} \geq \frac{1}{2} \implies \det(A_{w_{H',\varepsilon}}) \leq -\frac{1}{2}.$$

But now we know that both the sets $P_G = \{w : \det(A_w) > 0\}$ and $N_G = \{w : \det(A_w) < 0\}$ are open by continuity of det. By the previous construction we know that they are both non-empty, thus by connectivity of $\mathcal{W}(G)$ we have that $P_G \cup N_G \neq \mathcal{W}(G)$. But since we know that $\mathcal{W}(G) \setminus (P_G \cup N_G) = \mathcal{D}$ we get that $\mathcal{D}$ is non empty and so there must exist a weight such that $w \in \mathcal{D}$ and A_w is not full rank. $\square$

5.2 Examples

Example 5.2.1. A hamiltonian cycle is a $\{1, 2\}$-factor consisting of only one cycle, thus a graph G with one hamiltonian cycle always has a weight of full rank.

Example 5.2.2. An odd cycle C_{2k+1} has no way to pair up vertices with edges and thus only has one $\{1, 2\}$-factor, thus all its weights are full rank.

Example 5.2.3. An even cycle C_{2k} can pair up vertices and is thus has a $\{1, 2\}$-factor consisting of k copies of K_2, we call this factor H_{2k}. C_{2k} is also a $\{1, 2\}$-factor of itself, note then that

$$\operatorname{sgn}(C_{2k}) = (-1)^{n-1} = (-1)^{2k-1} = -1,$$

on the other hand

$$\operatorname{sgn}(H_{2k}) = (-1)^{n-k} = (-1)^{2k-k} = (-1)^k.$$

From this we can gather that C_{2k} always has a weight of full rank, but it only has a null weight if $k \neq 0 \pmod 2$ and so C_n has a null weight if and only if $n = 0 \pmod 4$.

6 The directed case

Ideally all of our ideas would transfer over to the case of a directed graph, however, most of the operators we deal with are no longer self-adjoint in this case, thus we must loosen our conditions. Let $G(V, E)$ be a simple directed graph, we define a weight function on G to be $w : E \to \mathbb{C}^*$. Again we define $\mathcal{W}_{\mathbb{C}}(G)$ to be the set of all possible weights on G.

Definition 6.0.1. *Two weights $w, \tilde{w} \in \mathcal{W}_{\mathbb{C}}(G)$ are called **conformally equivalent** if there exists a function $u \in \operatorname{Hom}(V, \mathbb{C})$ such that*

$$\tilde{w}(v_i, v_j) = w(v_i, v_j)e^{u(v_i)+u(v_j)},$$

*we call u the **conformal factor**.*

Again this is an equivalence relationship denoted $\sim_u$. We then similarly quotient by this relationship to get the set of equivalence classes $\mathcal{M}_{\mathbb{C}}(G) := \mathcal{W}_{\mathbb{C}}(G)/\sim_u$.

We can also treat $\mathcal{W}_{\mathbb{C}}(G)$ as a complex vector space again by taking the log of the weights. Applying that to the conformal equivalence equation gives us

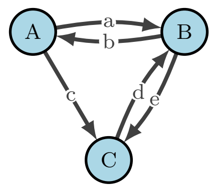
$$\log \tilde{w}(v_i, v_j) - \log w(v_i, v_j) = u(v_i) + u(v_j)$$

which in the exact same way gives us a map from $\operatorname{Hom}(V, \mathbb{C}) \to [w]$.

Corollary 6.0.2. $\mathcal{M}_{\mathbb{C}}(G)$ also has dimension $|E| - |V| + \omega_0$ where ω_0 is the number of bipartite components of the underlying undirected graph of G.

We can define Conformally Covariant operators identically to the undirected case. The adjacency matrix operator as well as the incidence and edge Laplacian operators are all still Conformally Covariant. However, now that they are no longer symmetric they are also no longer self adjoint and so no longer preserve signature.

Example 6.0.3. Let us consider a simple graph such as the one shown below:

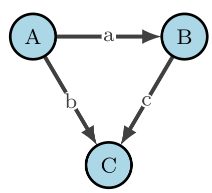


One finds that the adjacency matrix is

$$\begin{bmatrix} 0 & a & c \\ b & 0 & e \\ 0 & d & 0 \end{bmatrix}.$$

Clearly by setting $u(v_i) = \frac{\pi}{2}i$ one can reverse the signature of A_w inside a conformal class and so it is not invariant. It seems interesting to check which, if any, operators do still conserve signature.

Example 6.0.4. Adjusting the the graph slightly we can also get



With adjacency matrix

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

Which is nilpotent. Note that a nilpotent adjacency matrix was not possible in the undirected case. It seems interesting to check whether nilpotency is an invariant or not.

6.1 Full Rank and Null Weights

We also have a simple generalization of the existence theorem 5.1.1.

Definition 6.1.1. *Let G be a directed graph, a spanning subgraph H such that each node in H has in degree one and out degree one is called a directed $\{1,2\}$-factor of G.*

With this definition we again get that a single non-zero transversal of the matrix A_w corresponds exactly to a single directed $\{1,2\}$-factor.

This leads us to the generalization:

Theorem 6.1.2. *Let G be a directed graph, G has a directed $\{1,2\}$-factor if and only if G has a weight $w \in \mathcal{W}_{\mathbb{C}}(G)$ such that $\det(A_w) \neq 0$.*

Proof. If no directed $\{1,2\}$-factors exist for graph G then it has non-zero transversals and thus A_w is identically 0 for any weight.

On the other hand if a directed $\{1,2\}$-factor H exists we can again assume that it is a minimal such factor and thus assign weights as

$$w_\varepsilon(e) = \begin{cases} 1 & e \in E(H) \\ \varepsilon & e \notin E(H) \end{cases}.$$

We can then use the same line of reasoning as in 5.1.1 by letting $\varepsilon \to 0$ and get a non-zero determinant. □

For the second theorem we actually have a small simplification

Theorem 6.1.3. *Let G be a directed graph with at least one directed $\{1,2\}$-factor, G has a weight with $\det(A_w) = 0$ if and only if it has at least two distinct directed $\{1,2\}$-factors.*

Proof. Assume first that G only has one such factor H. Since the factor corresponds to a exactly one non-zero transversal of A_w we find that

$$\det(A_w) = \prod_{e \in E(H)} w(e).$$

This product cannot be 0 since every term is in $\mathbb{C}^*$, thus G has no weighting with $\det(A_w) = 0$.

On the other hand assume that G has at least two distinct factors. This implies that there is an edge e_0 which is in some factors but not in others, by the correspondence of factors to non-zero transversals we can rewrite the determinant as

$$\det(A_w) = w(e_0) f_G(w') + g_G(w'),$$

where f_G and g_G are polynomials and w' are the weights on all the edges beside e_0.
Since e_0 is in some factors but not others we know that f_G and g_G are both non-zero, thus the sets

$$\{w' : f_G(w') = 0\} \text{ and } \{w' : g_G(w') = 0\}$$

both have measure 0, thus there must exist a weighting w' such that neither are 0. Then we let

$$w(e) = \begin{cases} w'(e) & : e \neq e_0 \\ -\frac{g_G(w')}{f_G(w')} & : e = e_0 \end{cases}$$

and so since none of the weights are 0 this is a valid weighting of G and if we plug back into the previous equation we get

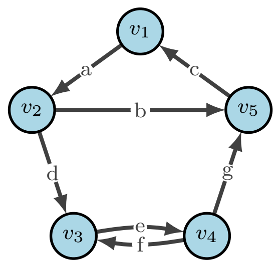
$$\det(A_w) = w(e_0)f_G(w') + g_G(w') = -\frac{g_G(w')}{f_G(w')}f_G(w') + g_G(w') = g_G(w') - g_G(w') = 0$$

$\square$

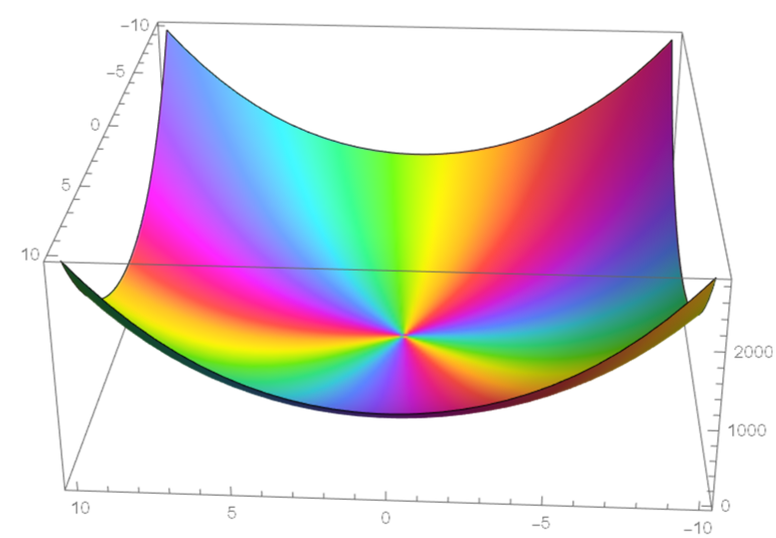
6.2 Examples

Example 6.2.1. A directed hamiltonian cycle is a $\{1, 2\}$-factor consisting of only one cycle, thus a directed graph G with one hamiltonian cycle always has a weight of full rank. Similarly a directed graph G with at least two hamiltonian cycles always has a null weight.

Example 6.2.2. Let C_5 be a circuit (a directed path that ends on the same vertex it begins) and add the directed edges (v_4, v_3) and (v_2, v_5) as shown below:



We call this graph G and we find that $\mathcal{M}_{\mathbb{C}}(G) \cong (\mathbb{C} \setminus \{0\})^2$ which can be parametrized by the weights on edges f and b as (x, y). Checking the determinant of the adjacency matrix reveals that (x, y) has determinant 0 if $x^3 = y$. We can then see that the null weights can be represented by the following complex graph.

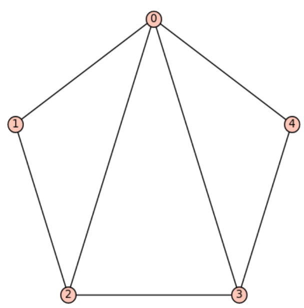


7 Appendix

7.1 Manifold Code

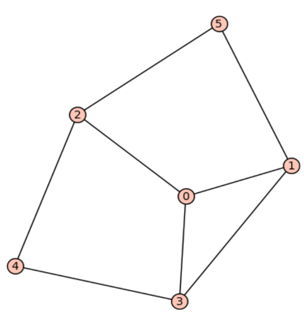
The library used for this code is hosted here

[1]:
```
from WeightManifold import WeightManifold
G1 = graphs.CycleGraph(5)
G1.add_edge((0,2))
G1.add_edge((0,3))
show(G1)
ManifoldG1 = WeightManifold(G1)
show(ManifoldG1.submanifold_equation())
```



$$-\frac{2\left(x_0^3x_1^4 - x_0^4 + x_1\right)}{x_0^3x_1} = 0$$

[4]:
```
G = graphs.RandomGNP(6, 0.5)
G.weighted(True)
show(G)
ManifoldG = WeightManifold(G)
show(ManifoldG.submanifold_equation())
```



$$-\frac{x_0^{12}x_1^6 - 2\,x_0^9x_1^7 + x_0^6x_1^8 + 2\,x_0^6x_1^3 - 2\,x_0^3x_1^4 + 1}{x_0^6x_1^4} = 0$$

References

[1] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2000.

[2] Dmitry Jakobson, Thomas Ng, Matthew Stevenson, and Mashbat Suzuki. Conformally covariant operators and conformal invariants on weighted graphs. *Geometriae Dedicata*, 174, 03 2014.

[3] Alexander I Bobenko, Ulrich Pinkall, and Boris A Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. *Geometry & Topology*, 19(4):2155–2215, Jul 2015.

[4] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer Berlin Heidelberg, 2012.

[5] J. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 2012.

[6] Boris Mityagin. The zero set of a real analytic function, 2015.

[7] S. Akbari, A. Ghafari, K. Kazemian, and M. Nahvi. Some criteria for a signed graph to have full rank, 2017.