Mathematical Logic

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Contents

1. Models and Languages

Definition 1.1: A *model* or *structure* is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i\in I}, (R_j)_{j\in J}, (c_k)_{k\in K}\right)$$

where
- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

Remark: For a model $\mathcal{M}$ we will sometimes use $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe. Sometimes we will also use $\mathcal{M}$ to refer to the underlying universe, but only when it is clear from context what we are referring to.

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.2: A *language* (or *signature*) is a tuple

$$L = \left(\left(\underline{f_i}\right)_{i\in I'}, \left(\underline{R_j}\right)_{j\in J'}, \left(\underline{c_k}\right)_{k\in K'}\right)$$

where now the f_i are function *symbols* with arity $a_i' \in \mathbb{N}$, each R_j are relation *symbols* with arity $a_j' \in \mathbb{N}$, and c_j are constant *symbols*.

Definition 1.3: A model $\mathcal{M}$ is an *L-structure* if

$$I = I', J = J', K = K', a_i = a'_i, a_j = a'_j.$$

If $\mathcal{M}$ is an L-structure then the *interpretations* of the symbols of the language are defined as

$$\underline{f_i}^{\mathcal{M}} = f_i, \underline{R_j}^{\mathcal{M}} = R_j, \underline{c_k}^{\mathcal{M}} = c_k$$

We have defined the symbols of L, but how do we speak it? We will need the following

- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: $x, y, z, v, ...$
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Remark: We will often use $\overline{a}$ to denote the ordered collection $(a_1, ..., a_n)$ where n will be clear from context. We will also write $\overline{a}\overline{b}, a\overline{b}$ etc to mean tuple concatenation.

We will also often use the notation $\overline{a}A$ for some subset A to denote $A \cup \{a_1, ..., a_n\}$, and if A is finite we will use it to mean the tuple $(a_1, ..., a_n, x_1, ..., x_m)$ where x_i are the ordered elements of A.

Definition 1.4: *L-terms* are defined inductively as follows

- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, ..., \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, ..., \tau_n)$$

is a term.

An L-term is said to be *constant* if it does not contain any variables.

A term is what we would usually call a mathematical *expression*.

Example: $f(1 + 2)$, $3 + x$, $\sin(e^{-15y})$ are all terms in appropriate languages.

Definition 1.5: If $\mathcal{M}$ is an L-structure and τ is a constant L-term then the *interpretation* of τ, $\tau^{\mathcal{M}}$, is also defined inductively

- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in |\mathcal{M}|$

Example: $L = (+, \cdot, 0, 1)$ then $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0.

Definition 1.6: An *L-formula* is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

$$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

$$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called *atomic* L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x(x = 1),$$
$$(\exists x(x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Formulas are our bread and butter, they form the language (no pun intended) of first order logic and it is through formulas that we will express and prove all nearly all results in this course.

Now this is all first order logic, but one might wonder, what makes it "first"? This comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in |\mathcal{M}|$, in *second* order logic we can quantify over subsets $S \subseteq |\mathcal{M}|$ like for example all relations. We can also see this as $S \in \mathcal{P}(|\mathcal{M}|)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(|\mathcal{M}|))$, and so on.

In this course, however, we will only be looking at first order logic.

Definition 1.7: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are *bound* in φ', and we say that φ is the *range* of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is *free* if it is not bound in φ.

An *L-sentence* is an L-formula with no free variables.

Definition 1.8: Let φ be a formula containing x (which we will follow denote as $\varphi(x)$), $\varphi(\tau \,/\, x)$ will denote the formula obtained by replacing every free occurrence of x by τ.

Now one would expect that substitution should never change the meaning of a logical formula, but in fact, this is not quite right. Consider the case $\varphi = \forall y(y = x)$, the substitution

$\varphi(y\,/\,x)$ changes the meaning of the statement from "all y are equal to x" to "all y are equal to themselves". We want to avoid this outcome, which we can formalize as follows.

Definition 1.9: A substitution $\varphi(\tau\,/\,x)$ is called *correct* if no free variable of τ becomes bound in $\varphi(\tau\,/\,x)$

Definition 1.10: If $A \subseteq |\mathcal{M}|$ and $\mathcal{M}$ is an L-structure then $L(A)$ is the language

$$L \cup \{\underline{a} : a \in A\}$$

We extend our definition of interpretation of terms to terms of $L(\mathcal{M})$ by setting $\underline{a}^{\mathcal{M}} = a$

Definition 1.11: Let $\mathcal{M}$ be an L-structure and σ an $L(\mathcal{M})$-sentence. We say that σ is *true* in $\mathcal{M}$ or $\mathcal{M}$ *satisfies* σ, if $\mathcal{M} \vDash \sigma$, which we define inductively as follows:

- If σ is of the form $\tau_1 = \tau_2$ then $M \vDash \sigma$ if and only if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$
- If σ is of the form $\underline{R}_j(\tau_1, ..., \tau_n)$, then $\mathcal{M} \vDash \sigma$ if and only if $\left(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}\right) \in R_j^{\mathcal{M}}$
- If σ is of the form $\sigma_1 \wedge \sigma_2$ then $\mathcal{M} \vDash \sigma_1 \wedge \sigma_2$ if and only if $\mathcal{M} \vDash \sigma_1$ and $\mathcal{M} \vDash \sigma_2$. A similar definition follows for the other logical connectives.
- If σ is of the form $\exists x\, \varphi$ then $\mathcal{M} \vDash \sigma$ if there exists $a \in |\mathcal{M}|$ with $\mathcal{M} \vDash \varphi(\underline{a}\,/\,x)$. Similarly for $\forall x\, \varphi$.

Note that while the first step may look circular, the first equality is in the space of *terms* while the second is in the universe $|\mathcal{M}|$.

An important consequence of this definition is that every sentence of $L(\mathcal{M})$ is either true or false in $\mathcal{M}$, hence either it or its negation are true in $\mathcal{M}$. We will formalize this very soon.

2. Model equivalences

Definition 2.1: Let $\mathcal{M}$ be a model in the language L. The *theory* of $\mathcal{M}$ is defined as

$$\mathrm{Th}_L(\mathcal{M}) = \{\sigma \text{ is an } L\text{-sentence} : \mathcal{M} \vDash \sigma\}.$$

If L is clear from context we will often omit it.

We say that two L-structures, $\mathcal{M}$ and $\mathcal{N}$, are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Definition 2.2: We write $\mathcal{M} \cong \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* if there is a bijection $g : \mathcal{M} \to \mathcal{N}$ with

$$g\left(\underline{c_k}^{\mathcal{M}}\right) = \underline{c_k}^{\mathcal{N}}$$

$$(a_1, ..., a_n) \in \underline{R_j}^{\mathcal{M}} \Leftrightarrow (g(a_1), ..., g(a_n)) \in \underline{R_j}^{\mathcal{N}}$$

$$g\left(\underline{f_i}^{\mathcal{M}}(a_1, ..., a_n)\right) = \underline{f_i}^{\mathcal{N}}(g(a_1), ..., g(a_n))$$

A lot of these definitions might look similar but they are of very different flavor, they all describe model equivalence but of different *resolutions*.

Elementarily equivalence means that the two models agree on all L-sentences, while isomorphism implies that they agree on all *L-formulas* on *all inputs* (after replacing the inputs with their images under g).

Definition 2.3: We write that $\mathcal{M} \subseteq \mathcal{N}$ to mean that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, meaning that

$$|\mathcal{M}| \subseteq |\mathcal{N}|, \underline{f_i}^{\mathcal{M}} \subseteq \underline{f_i}^{\mathcal{N}}, \underline{R_j}^{\mathcal{M}} = \underline{R_j}^{\mathcal{N}} \cap |\mathcal{M}|^{a_j}, \text{ and } \underline{c_k}^{\mathcal{M}} = \underline{c_k}^{\mathcal{N}}$$

intuitively this means that the structure of $\mathcal{M}$ is induced from that of $\mathcal{N}$.

Definition 2.4: A map $f : A \to \mathcal{M}$ between two L-models is called an *embedding* if it is an injective map who's image is a substructure of $\mathcal{M}$.

A map $f : \mathcal{M} \to \mathcal{N}$ between two L-models is called an *elementary embedding* if for every tuple $\overline{a} \in \mathcal{M}^n$ and every L-formula we have

$$\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \varphi(f(\overline{a})).$$

We write this as $f : \mathcal{M} \hookrightarrow \mathcal{N}$.

Definition 2.5: We say $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ if $\mathcal{M} \subseteq \mathcal{N}$ and the inclusion map $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$ is an elementary embedding. We write this as $\mathcal{M} \prec \mathcal{N}$.

These two definitions also deal with concept of 'substructure' on different resolutions. A standard substructure is a very weak property, substructures $\mathcal{M}$ of $\mathcal{N}$ could have radically different behaviour (we will see many examples of this).

However, an *elementary substructure* must agree with its superstructure on *all formulas* involving the smaller structures inputs, this is a much stronger condition and is akin to a subfield in algebra.

Proposition 2.1: If $\mathcal{M} \prec \mathcal{N}$ then $\mathcal{M}$ is elementary equivalent to $\mathcal{N}$.

Proof: This is immediate from the definition of elementary equivalence. □

Proposition 2.2: If $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{N} \prec \mathcal{M}$ then $\mathcal{M} \cong \mathcal{N}$.

Proof: This is also immediate. □

Theorem 2.3 (Tarski-Vaught test): Suppose $\mathcal{M}$ is an L-structure, $A \subseteq |\mathcal{M}|$, then A is the universe of an elementary substructure iff the following condition holds, called the Tarski-Vaught test (T-V test)

> For every formula $\varphi(x, \overline{y})$ in L and every $\overline{a} \subseteq A$, if $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ then there exists $b \in A$ such that $\mathcal{M} \vDash \varphi(b, \overline{a})$

Proof: First the backwards direction, assume that the T-V test holds, then we need to show that A is a substructure. First we use $\varphi = (x = c)$ to show that A contains all constants of $\mathcal{M}$, then $\varphi = (x = f_i(\overline{a}))$ for $\overline{a} \subseteq A$ to show it contains the images of all functions, and we define the interpretation of $\underline{R_j}$ to be exactly $\underline{R_j}^{\mathcal{M}} \cap A^{a_j}$ to make it a substructure.

Now A being a substructure is equivalent to

$$A \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{M} \vDash \varphi(\overline{a})$$

for all $\overline{a} \subseteq A$ and φ being an *atomic* formula. So now we only need to prove this is true for the other formula types, which we do by induction on the structure of the formula.

- The connective types are immediate.
- For the quantifiers we can assume that the quantifier is $\exists$ as otherwise we can use negation to change $\forall$ to an $\exists$. Hence let us assume $\varphi(\overline{x}) = \exists y\, \psi(y, \overline{x})$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ iff $\mathcal{M} \vDash \exists y\, \psi(y, \overline{a})$ iff there exists $b \in A$ with $\mathcal{M} \vDash \psi(b, \overline{a})$. But by definition this last form is equivalent to $A \vDash \exists y\, \psi(y, \overline{a})$

Assume, on the other hand, that A is the universe of an elementary substructure $\mathcal{A}$, then we need to prove the T-V test holds. Assume then that for some formula $\varphi(x, \overline{y})$ in L and some $\overline{a} \subseteq A$ we have $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$, then since $\mathcal{A}$ it is an elementary substructure we also have that $\mathcal{A} \vDash \exists x\, \varphi(x, \overline{a})$ and so we must have some $x \in A$ such that $\varphi(x, \overline{a})$ holds. $\square$

Theorem 2.4 (Löwenheim-Skolem-Skolem downwards Theorem): Let L be a language, for any L-structure $\mathcal{M}$ and every $A \subseteq |\mathcal{M}|$, there exists an elementary substructure $\mathcal{N} \prec \mathcal{M}$ with $A \subseteq |\mathcal{N}|$

$$\|\mathcal{N}\| = |A| + |L| + \aleph_0$$

Proof: Set $\kappa = |A| + |L| + \aleph_0$, using induction we will define a sequence A_n of subsets of $\mathcal{M}$, where at each step n we will try to satisfy all existential statements in $\mathrm{Th}_{L(A_{n-1})}(\mathcal{M})$, we will then set $|\mathcal{N}| = \bigcup_n A_n$.

First we set $A_0 = A$, then at step $n > 0$, we will consider all formulas in $L(A_{n-1})$ (there are $|\kappa \times \mathbb{N}| = |\kappa|$ many of them, see Proposition A.4) and for each formula $\varphi(\overline{x})$ we will pick some collection of elements $\overline{a} \subseteq |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(\overline{a})$, then we will add $\overline{a}$ to A_{n-1}, adding these elements for each formula gives us A_n.

Now we can use Theorem 2.3 to check that $\mathcal{N} \prec \mathcal{M}$. Let $\varphi(\overline{a}) = \exists x(\psi(x), \overline{a})$ be a formula in $\mathrm{Th}_{L(\mathcal{N})}(\mathcal{M})$, then $\overline{a} \in |\mathcal{N}|$ and so $\overline{a} \in A_n$ for some n and thus for some $b \in A_{n+1}$ we have $\mathcal{N} \vDash \psi(b, \overline{a})$ and thus $\mathcal{N} \vDash \varphi(\overline{a})$. $\square$

Remark (Skolem's Paradox): Let $ZFC^* \subseteq ZFC$ be a finite theory which proves Cantor's theorem. Let $V \vDash ZFC^*$. By the previous theorem we can find a countable $\mathcal{M} \prec V$ for which $\mathcal{M} \vDash ZFC^*$ and $\mathcal{M} \vDash$ "exists an uncountable set".

Definition 2.6: In first order logic we have the concept of a *proof system*, consisting of two parts. *Axioms* which are formulas which are declared to always be true, and *proofs* which is a finite sequence of L-formulas such that every step is either an axiom or follows from the previous steps using an inference rule.

Example: An example proof system has the following 4 types of axioms.
- All instances of propositional tautologies are axioms.
- $[\forall x\, (\varphi \to \psi)] \to [\varphi \to \forall \psi]$ as long as x is not free in φ.
- $\forall x \to \varphi(t \,/\, s)$ where t is any L-term where the substitution is correct.
- $x = x$,
  $x = y \to t(..., x, ...) = t(..., y, ...)$ for any L-term t,
  $x = y \to (\varphi(..., x, ...) \to \varphi(..., y, ...))$ for any formula φ.

And the following inference rules.
- If φ and $\varphi \to \psi$ then ψ.
- If φ then $\forall x\, \varphi$.

We will use the notation $\Gamma \vdash \varphi$ to mean "Γ proves φ" and define it as the existence of a proof whose final step is φ and every step is either an axiom or an element of Γ or follows from a previous step or by an inference in φ.

Definition 2.7: We say that Γ is *consistent* if there exists φ such that $\Gamma \nvdash \varphi$.
We say that Γ *has a model* if there is a model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ for all $\varphi \in \Gamma$.

By a famous theorem of Gödel that we will not prove in this class we can actually not care about any proof system details.

Theorem 2.5 (Gödel's completeness theorem): Let Γ be a set of sentences in L then Γ is consistent if and only if Γ has a model.

We will not prove this theorem in this class but we will use an important corollary of it.

Corollary 2.5.1 (Compactness Theorem): Let Γ be a set of L-sentences, Γ has a model if and only if every finite subset of Γ has a model.

Proof: The forward direction is immediate, the hard part is the backward direction. By Gödel's completeness theorem, we can replace "having a model" with "is consistent".

We now prove this by contrapositive, assume that Γ is inconsistent, then we have $\Gamma \vdash \exists x\, (x = x) \wedge (\neg(x = x))$, now this proof consists of finitely many steps and thus can only use finitely many statements in Γ, let Γ_0 be that subset of statements. Since we can prove a contradiction using Γ_0 it must also be inconsistent, thus one of the finite subsets of Γ is inconsistent. $\square$

As an example use we have the following theorem.

Theorem 2.6 (Löwenheim-Skolem upwards Theorem): If $\mathcal{M}$ is an infinite L-structure where L is countably infinite then $\forall k > \|\mathcal{M}\|$ there exists a model $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| = k$

Proof: Let us consider the language $L' = L(\mathcal{M}) \cup \{c_\alpha : \alpha < \kappa\}$ where c_α are new constants. Now set

$$\Gamma = \mathrm{Th}_{L'}(\mathcal{M}) \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta < \kappa\}$$

We want to show now that Γ is consistent, to see this we use compactness and take an arbitrary finite subset Γ_0. Let $\alpha_1, ..., \alpha_n$ be such that

$$\Gamma_0 \subseteq \mathrm{Th}_{L'}(\mathcal{M}) \cup \left\{c_{\alpha_i} \neq c_{\alpha_j} : i \neq j\right\}$$

choose then any $a_1, ..., a_n \in |\mathcal{M}|$ which are distinct and interpret c_{α_i} as a_i to get a model of Γ_0, hence Γ_0 is consistent.

Now we have by Theorem 2.5 that there exists a model $\mathcal{N}$ such that $\mathcal{N} \vDash \Gamma$, then by construction we have $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| \geq \kappa$ and so by downwards theorem we can now decrease the cardinality until we reach κ. $\square$

Corollary 2.6.1: If $\mathcal{M}$ is infinite then there exists $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \ncong \mathcal{N}$.

Proof: We simply pick some $\kappa > \|\mathcal{M}\|$ and then use the upwards theorem to get a model $\mathcal{N}$ with $\mathcal{M} \prec \mathcal{N}$ with $\|\mathcal{N}\| = \kappa$, now there can't exist a bijection between the two since they have different cardinalities. $\square$

Definition 2.8: A *theory* is a set Γ of sentences such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$. A theory T is *complete* if for every sentence φ either $\varphi \in T$ or $\neg\varphi \in T$.

Remark:

- For any model $\mathcal{M}$ the theory $\mathrm{Th}(\mathcal{M})$ is complete.
- For any theory T which is complete and consistent, there exists a model $\mathcal{M}$ with $T = \mathrm{Th}(\mathcal{M})$.

3. Categoricity

One can consider a theory T to be the 'resolution' of mathematical subject, the set of statement which it can prove entirely within itself. From that point of view, it is a natural question to ask, exactly how high is this resolution. That is, to what degree does a theory **uniquely determine a model**.

Now Corollary 2.6.1 unfortunately disallows a theory to directly determine a model, its cardinality will always be a free variable we can tune. But this allows us to ask a more specific question, if we fix the cardinality, can we then uniquely determine a model?

Definition 3.1: Let κ be an infinite cardinal, a theory T is κ-categorical if it has infinitely many models but exactly one model (up to isomorphism) of size κ.

Proposition 3.1: If T is κ-categorical, then T is complete.

Proof: Suppose that T is not complete, let σ be such that $\sigma \notin T$ and $\neg\sigma \notin T$, then let $T_1 = T \cup \{\sigma\}$ and $T_2 = T \cup \{\neg\sigma\}$. Both are consistent, and thus have models of size κ which are both models of T, but the models are not isomorphic. This contradicts the fact that there is only one model of this size. □

Example: Consider the language $L = (<)$, a dense linear order (DLO_0) is the theory generated by the additional axioms: $<$ is total, dense and has no endpoints.

- Total means $\forall x\, \forall y(x = y \vee x < y \vee y < x)$
- Dense means $\forall x\, \forall y(x < y \Rightarrow \exists z x < z < y)$
- No endpoints means $\neg(\exists z\, \forall x(x \neq z \Rightarrow x < z))$ for the max endpoint and similarly for the min endpoint.

Examples of such a structure include $\mathbb{Q}$, $\mathbb{R}$ and many others.

It turns out, however, that the only countable such structure is $\mathbb{Q}$, up to isomorphism.

Theorem 3.2 (Cantor): DLO_0 is $\aleph_0$-categorical.

In order to prove this result we will need to use a specific technique, since it appears all over model theory we will write up the main idea here for future reference.

Technique 3.1 (Back and Forth method): Let $\mathcal{M}$ and $\mathcal{N}$ be two countable models between which we want to construct an isomorphism $\mathcal{M} \to \mathcal{N}$.

The method involves us enumerating the two models as $\mathcal{M} = \{a_0, a_1, ...\}$ and $\mathcal{N} = \{b_0, b_1, ...\}$, it does not matter what in what order this enumeration happens. Our goal will be to construct a sequence of functions $f_n : A_n \to B_n$, where A_n and B_n are substructures of $\mathcal{M}$ and $\mathcal{N}$ respectively, f_n is an isomorphism between said substructures, and f_{n+1} is an extension of f_n. Additionally we will construct f_n's such that every a_i and b_i will eventually appear in the domain / codomain of some f_n. Our goal then is to take the function

$$f := \bigcup_{n \in \mathbb{N}} f_n,$$

one can easily check that this will be an isomorphism if we indeed have such a sequence.

We construct f_n inductively, usually starting with either $A_0 = \{a_0\}$ if there is a natural b_i to map it to, or with $A_0 = \varnothing$ otherwise. We then assume that f_n is constructed and try to construct f_{n+1} by adding a single element mapping, this depends on the parity of n. If n is even we pick the element in $\mathcal{M}$ with smallest index that has not yet been picked, let say a_i, and try to find a b_j that has not yet been picked such that the extended function,

$$f_{n+1} \coloneqq f_n \cup \{(a_i, b_j)\},$$

is again an isomorphism of $A \cup \{a_i\}$ to $B \cup \{b_i\}$. If n is odd we do the exact same thing, but instead pick b_i with smallest index that has not yet been picked. Doing this means that eventually each a_i and b_i will eventually be mapped and thus f is indeed an isomorphism.

Lets see an example of this.

Proof (of Theorem 3.2): Let $\mathcal{M} = (M, <)$ and $\mathcal{N} = (N, <)$ be two countable models of DLO_0, we enumerate them $M = \{a_0, a_1, ...\}$ and $N = \{b_0, b_1, ...\}$.

We use Technique 3.1 to construct an isomorphism, we start by with $f_0 : \{a_0\} \to \{b_0\}$. As explained above we now only need to describe how we add one element from $\mathcal{M}$ and $\mathcal{N}$. Assume then that $f_n : A \to B$ is an isomorphism, then assume that we are on an even step and so we are adding some element a_i, then a_i has some ordering compared to A. If a_i is less than every element in A, then since $\mathcal{N}$ has no end points there is a $b_j \in \mathcal{N} \setminus B$ which is smaller than every element in B. Similarly if a_i is larger than every element in A then there exists a b_j larger than every element of B. If a_i is in between elements $x, y \in B$, then because of density know that there is an element b_j which is between $f(x), f(y)$.

In all 3 cases we will map a_i to b_j and take that to be f_{n+1}. One can easily check that f_{n+1} remains an isomorphism and thus the back and forth method gives us an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 3.2.1: $\mathrm{DLO}_0 = \mathrm{Th}(\mathbb{Q}, <)$, and so is complete.

Example: ACF_p is the theory generated by the axioms of an algebraically closed field of characteristic p.

The key question for any theory is, "is this theory complete?". We want to use our previous method and show that ACF_p is categorical for some cardinal, but it turns out that it is not $\aleph_0$-categorical. To see this we note that $\hat{\mathbb{Q}}, \widehat{\mathbb{Q}[a]}, \widehat{\mathbb{Q}[a, b]}, ...$ are all non-isomorphic algebraically closed fields, where a, b are transcendental and $\widehat{\ }$ denotes algebraic closure.

Proposition 3.3: ACF_p is κ-categorical for every uncountable κ.

Proof: Let K, L be models of ACF_p of size $\kappa \geq \aleph_1$. The transcendental degree, the size of a field's transcendental basis, will also be equal to κ, then any bijection between transcendental bases will extend to an isomorphism between K and L. □

Corollary 3.3.1: ACF_p is complete.

We now want to discuss how to check that two models are elementarily equivalent.

Definition 3.2: Given a formula φ its *quantifier depth* qd is defined by induction,
- If φ is atomic $\mathrm{qd}(\varphi) = 0$.
- If φ is a formula of the form $\varphi_1 \vee \varphi_2$ then $\mathrm{qd}(\varphi) = \max(\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2))$
- If φ is a formula of the form $\exists x\, \varphi'$ then $\mathrm{qd}(\varphi) = \mathrm{qd}(\varphi') + 1$, similarly for $\forall$.

We write $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ to mean "$\mathcal{M}$ is equivalent to $\mathcal{N}$ up to order n" if for every sentence σ with $\mathrm{qd}(\sigma) \leq n$ we have $\mathcal{M} \vDash \sigma \Leftrightarrow \mathcal{N} \vDash \sigma$.

We now define a tool for proving such partial equivalences.

Definition 3.3 (Ehreufeucht-Fraisse (E-F) Games): Let L be finite relational, $\Gamma(\mathcal{M}, \mathcal{N})$ is a two player game where player I is called the Spoiler and player II is called the Prover. Together they will construct a function $f : A \to B$ where $A \subseteq |\mathcal{M}|$ and $B \subseteq |\mathcal{N}|$.

Spoiler plays first and either plays an element of $m \in |\mathcal{M}|$, challenging Prover to put m in the domain of f, or they play an element $n \in |\mathcal{N}|$ challenging Prover to put it in the range of f. Prover then plays the corresponding pairing for whatever Spoiler played. Then Spoiler starts again and they continue forever. Prover wins if the resulting f is an isomorphism of the induced structures on A and B, and Spoiler wins otherwise.

We will also define a finite version of this game which we will denote $\Gamma(\mathcal{M}, \mathcal{N})_n$, it is the same as the regular game except that it ends at step n and Prover wins if when it ends it is a finite partial isomorphism.

Theorem 3.4: Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures where L is a finite relational language. TFAE
- $\mathcal{M} \equiv \mathcal{N}$
- The Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$ for every n.

To prove this we will need a lemma first.

Lemma 3.5: We say that formulas $\varphi(\overline{x}), \psi(\overline{x})$ are equivalent if $\forall \overline{x}\, \varphi(\overline{x}) \Leftrightarrow \psi(\overline{x})$ is true in every model. Equivalently if $\forall \overline{x}\, \varphi \Leftrightarrow \psi$ is provable from the empty set of axioms.

If L is finite relational then for each n, ℓ there exists a finite list $\Phi_1, ..., \Phi_k$ of formulas with $\mathrm{qd}(n)$ in ℓ variables such that every formula φ with $\mathrm{qd}(\varphi) \leq n$ in ℓ variables is equivalent to φ_i for some $i \leq k$.

Proof: We induct on n, $n = 0$, there are finitely many atomic formulas so we are done. If φ is quantifier free, then it is a boolean combination of formulas $\tau_1, ..., \tau_m$ then φ is equivalent to

$$\bigvee_{X \in S} \left(\bigwedge_{i \in X} \sigma_i \bigwedge_{i \notin X} (\neg \sigma_i) \right)$$

where S is a collection of subsets of $\{1, ..., m\}$, this case then follows from the fact that S is finite. Now assume this holds for quantifier depth at most n, if φ is of quantifier depth at most $n + 1$, then φ is equivalent to a disjunction of conjunctions of formulas of the form $\exists x \, \varphi'$ or $\forall x \, \varphi'$, where $\mathrm{qd}(\varphi') \leq n$. By inductive hypothesis we then have φ' is equivalent to one of finitely many formulas Φ'_k, then $\exists x \, \varphi'$ is equivalent to $\exists x \, \Phi'_k$ and similarly for $\forall$. □

We will now use this lemma to prove a slightly weaker statement that will then use to prove the main theorem.

Lemma 3.6: TFAE
- $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$
- Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$.

Proof: We show equivalence by induction on n. For $n = 0$ this is obvious since the two conditions are empty. For $n > 0$ we know that one of the two players has a winning strategy since its a finite length game.

Assume then that $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$, we want to show the Prover has a winning strategy. Suppose Spoiler plays $a \in M$, by the previous lemma there exists a formula $\varphi(x)$ of quantifier depth at most $n - 1$ such that $\mathcal{M} \vDash \varphi(a)$ where

$$\mathcal{N} \vDash \varphi(b) \Leftrightarrow (\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b).$$

Since $\mathcal{M} \vDash \exists x \, \varphi(x)$, the quantifier depth of $\exists x \, \varphi(x) \leq n$, and by our assumption $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ we have that $\mathcal{N} \vDash \exists x \, \varphi(x)$ so there is some b such that $\mathcal{N} \vDash \varphi(b)$. Our strategy is to just play b and then continue with whatever strategy we have for the $n - 1$ step game.

Now assume that $\mathcal{M} \underset{n}{\not\equiv} \mathcal{N}$, but that the duplicator has a winning strategy, so there exists a formula $\exists x \, \varphi(x)$ where the quantifier depth of φ is at most $n - 1$ such that

$$\mathcal{M} \vDash \exists x \, \varphi(x) \text{ but } \mathcal{N} \nvDash \exists x \, \varphi(x)$$

Choose $a \in |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(a)$ and make a the first move of the Spoiler. Let b be the response of the duplicator, then in $\Gamma_{n-1}(\mathcal{M}(a), \mathcal{N}(b))$ the Prover still has a winning strategy so by inductive hypothesis $(\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b)$ which contradicts the above assertion. □

Proposition 3.7: If $\mathcal{M}$ and $\mathcal{N}$ are countable then we also have

$$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \text{The Prover has a winning strategy in } \Gamma(\mathcal{M}, \mathcal{N})$$

Proof: Assume $\mathcal{M} \cong \mathcal{N}$, then the Prover wins trivially by just following the isomorphism.

On the other hand assume Prover has a winning strategy, then we can play the role of the Spoiler to force Prover to construct an isomorphism. First enumerate the models

$$|\mathcal{M}| = \{m_0, m_1, ...\}, \quad |\mathcal{N}| = \{n_0, n_1, ...\}$$

on the first turn we pick m_0 and let Prover map it to some element of $|\mathcal{N}|$. On the second turn we pick the smallest index element of $|\mathcal{N}|$ that has not been picked before and force Prover to map it. We continue this, on odd turns we pick the smallest index element of $|\mathcal{M}|$ that has not been picked before, and on even turns we pick the smallest index element of $|\mathcal{N}|$ that has not been picked before. This essentially forces Prover to use the back-and-forth method. Since every element of both models will eventually be mapped and since Prover has to win this game, the resulting map $\bigcup_i f_i$ will be an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

4. Ultrafilters and Ultraproducts

Definition 4.1: A family $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a filter if it is non empty, does not contain the empty set and satisfies the two conditions

- $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in F$.

Example:

- The collection of cofinite subsets of $\mathbb{N}$
- The set of neighborhoods of any point in a topological space
- The set of subsets containing a fixed element in any set.

This last example is called a principal filter.

Definition 4.2: A filter is called an *ultrafilter* if it is not strictly contained in any other filter.

Remark: By Zorn's lemma every filter is contained in at least one ultrafilter. Since the collection of cofinite subsets is not contained in the principal filter this proves that every infinite set admits a non-principal ultrafilter (assuming ZFC).

Proposition 4.1: Let $\mathcal{U}$ be a filter over I. TFAE

- $\mathcal{U}$ is an ultrafilter
- For any $A \subseteq I$ we have either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, but not both.

Proof: Assume that $\mathcal{U}$ is an ultrafilter, then clearly for every A we cannot have both A and $I \setminus A$ be in $\mathcal{U}$. Now take some $A \notin \mathcal{U}$, then

$$\mathcal{U}' = \{Y' \subseteq I : Y \setminus A \subseteq Y' \text{ for some } Y \in \mathcal{U}\}$$

this is a filter since

$$Y_1 \setminus A \subseteq Y_{1'} \text{ and } Y_2 \setminus A \subseteq Y_{2'} \Rightarrow (Y_1 \cap Y_2) \setminus A = (Y_1 \setminus A) \cap (Y_2 \setminus A) \subseteq Y_{1'} \cap Y_{2'}$$

and is obviously upwards closed. Now $\mathcal{U} \subseteq \mathcal{U}'$ since for every $Y \in \mathcal{U}$ we have $Y \setminus X \subseteq Y$ and so since $\mathcal{U}$ is an ultrafilter then $\mathcal{U} = \mathcal{U}'$. But note that $I \in \mathcal{U}$ so $I \setminus A \in \mathcal{U}'$ and so $I \setminus A \in \mathcal{U}$.

On the other hand assume that the second condition holds, then let F be a filter containing $\mathcal{U}$, then if F contains a subset $A \notin \mathcal{U}$ then $I \setminus A \in \mathcal{U}$ and so $I \setminus A \in F$. But then $A \cap (I \setminus A) = \varnothing \in F$ which contradicts the definition of a filter. $\square$

Corollary 4.1.1: If $\mathcal{U}$ is an ultrafilter

$$A \cup B \in \mathcal{U} \Leftrightarrow A \in \mathcal{U} \vee B \in \mathcal{U}$$

Remark: An Ultrafilter has a very natural description as a finitely additive measure on I, who's only values are 0 and 1. The measure is defined by $\mu(A) = 1 \Leftrightarrow A \in I$.

In this context, if $p(x)$ holds on all $x \in A$ for some $A \in \mathcal{U}$, then we can think of this as $p(x)$ holding almost everywhere. It is through this lens that we will often think of ultrafilters, so keep this in mind as you read the rest of this section.

Definition 4.3: If $(\mathcal{M}_i)_{i \in I}$ are L-structures we can define $\prod_{i \in I} \mathcal{M}_i$ to be an L-structure with the natural pointwise interpretation of all the constants, relations, and functions.

This definition is not really satisfying from the point of view of model theory since it rarely preserves any structure. For example the product of two fields is not a field. However, we can take the quotient of the product by a maximal ideal to get a field, this is the approach we will try to mimic with model theory and ultrafilters.

Definition 4.4: Let I be a set. Let $(\mathcal{M}_i : i \in I)$ be a sequence of L-structures. Let $\mathcal{U}$ be an ultrafilter on I, the *ultraproduct*

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U}$$

is defined as follows.

On $\prod_{i \in I} |\mathcal{M}_i|$ we define the equivalence relation $\underset{\mathcal{U}}{\sim}$ by

$$(a_i) \underset{\mathcal{U}}{\sim} (b_i) \text{ if } \{i \in I : a_i = b_i\} \in \mathcal{U}$$

one can easily show that this is indeed an equivalence relation.

The universe of $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is just this infinite Cartesian product quotiented by this equivalence relation. The constants are interpreted as just the sequence of interpretations on each $\mathcal{M}_i$. Functions are interpreted pointwise as one would expect. Relations are interpreted as

$$R^{\prod_{i \in I} \mathcal{M}_i / \mathcal{U}}\left([(a_i^1)]_{\underset{\mathcal{U}}{\sim}}, ..., [(a_i^k)]_{\underset{\mathcal{U}}{\sim}} \right) \text{ if } \left\{i \in I : \mathcal{M}_i \vDash R\left(a_i^1, ..., a_i^k\right)\right\} \in \mathcal{U}$$

Remark: One needs to check that the last two interpretations are well defined, but this is easy to do by the definition of an ultrafilter.

Remark: If $\mathcal{U}$ is the principal ultrafilter generated by $i_0 \in I$ then

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U} \cong \mathcal{M}_{i_0}$$

Theorem 4.2 (Łoś's theorem): Let $\prod \mathcal{M}_i / \mathcal{U}$ be an ultraproduct, fix any formula $\varphi(x_1, ..., x_n)$ and $(a_i^1), ..., (a_i^n) \in \prod \mathcal{M}_i$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi([(a_i^1)], ..., [(a_i^n)]) \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi(a_i^1, ..., a_i^n)\} \in \mathcal{U}$$

Proof: The atomic case is covered by the definition of an ultraproduct.

We now induce on the complexity of φ,

- For $\varphi = \varphi_1 \wedge \varphi_2$ we have by definition

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2$$

now set

$$A = \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \quad B = \{i \in I : \mathcal{M}_i \vDash \varphi_2\}$$

then we know that for any A, B we have

$$A \in \mathcal{U}, B \in \mathcal{U} \Leftrightarrow A \cap B \in \mathcal{U}$$

now by inductive hypothesis we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2 \Leftrightarrow A \in \mathcal{U} \text{ and } B \in \mathcal{U}$$

and so combined this gives us exactly what we want.

- For $\varphi = \neg\varphi_1$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \nvDash \varphi_1$$

but since $\mathcal{U}$ is an ultrafilter then by Proposition 4.1 we have that

$$\{i \in I : \mathcal{M}_i \vDash \varphi\} \in \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi\}^c \notin \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \notin \mathcal{U}$$

which is exactly what we want. This also gives us the disjunction case.

- For $\varphi = \exists\psi$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \exists(a_i) \in \prod \mathcal{M}_i : \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \psi([a_i])$$

but by inductive hypothesis this is equivalent to

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \in \mathcal{U}$$

and so we have

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \subseteq \{i \in I : \mathcal{M}_i \vDash \exists x\, \psi(x)\}$$

and thus the right set here is also in $\mathcal{U}$ which proves what we wanted to show. $\square$

Corollary 4.2.1: If the $\mathcal{M}_i$ are all elementarily equivalent then

$$\operatorname{Th}\left(\prod \mathcal{M}_i \middle/ \mathcal{U}\right) = \operatorname{Th}(\mathcal{M}_i)$$

Definition 4.5: If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then $\prod \mathcal{M}_i / \mathcal{U}$ is called the *ultrapower* of $\mathcal{M}$.

Corollary 4.2.2: Let T be a set of sentences, T has a model iff every finite subset of T has a model.

Proof: Assume that L is countable and T is countable and enumerate $T = \{\sigma_1, \sigma_2, ...\}$. Then set T_n to be the truncation of T, that is $T_n = \{\sigma_1, ..., \sigma_n\}$. By assumption we have the existence of some models $\mathcal{M}_n$ with $\mathcal{M}_n \vDash T_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$.

Set $\mathcal{M} = \prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$, then

$$\mathcal{M} \vDash \sigma \Leftrightarrow \{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma\} \in \mathcal{U}$$

Now for a fixed σ_i we have $\mathcal{M}_n \vDash \sigma_i$ if $n \geq i$ so

$$\{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma_i\} \in \mathcal{U}$$

because it is cofinite and a non-principal ultrafilter contains all cofinite sets. Thus

$$\prod \mathcal{M}_i \middle/ \mathcal{U} \vDash \sigma_i$$

The uncountable case is a bit more complicated, we start with defining

$$F = \{\Delta \subseteq T : \Delta \text{ is finite}\}.$$

Now set $X_\Delta = \{Y \in F : \Delta \subseteq Y\}$, then I claim that the set

$$D = \{Y \subseteq F : X_\Delta \subseteq Y \text{ for some } \Delta\}$$

is a filter. This is easy to see by just checking the definition. Now since it is a filter it is contained in some maximal ultrafilter $\mathcal{U}$. Now for each finite subset $\Delta \in F$ we have some model $\mathcal{M}_\Delta \vDash \Delta$ so we can consider $\mathcal{M} = \prod_{\Delta \in F} \mathcal{M}_\Delta / \mathcal{U}$. Now for a fixed $\sigma \in T$ we have that

$$\{\Delta \in F : \mathcal{M}_\Delta \vDash \sigma\} \supseteq X_{\{\sigma\}} \in \mathcal{U},$$

and so $\mathcal{M} \vDash \sigma$. $\square$

5. Types and Definable Sets

We will now develop more tools to use with models, first of these is the **type**, in short, a type is to formulas what a satisfiable theory is to sentences. For this chapter we will assume that L is a countable language unless stated otherwise.

Definition 5.1: Let T be a complete L-theory. Let $\mathcal{M} \vDash T$ then for $a \in |\mathcal{M}|$ we say that the *type* of a is

$$\operatorname{tp}^{\mathcal{M}}(a) = \{\varphi(x) : \mathcal{M} \vDash \varphi(a)\}.$$

If two elements a, b have the same type then we cannot distinguish a, b with first order formulas.

More generally, if $\overline{a}$ is a tuple of elements of $\mathcal{M}$ then the type of $\overline{a}$ is

$$\operatorname{tp}^{\mathcal{M}}(\overline{a}) = \{\varphi(x) : \mathcal{M} \vDash \varphi(\overline{a})\}.$$

We will also sometimes use the following notation

$$F_L(\overline{x}) = \{\text{formulas with variables } \overline{x}\}$$

if $\varphi(\overline{a}) \in F_L(\overline{x})$ and $\mathcal{M}$ is a model

$$\varphi(\mathcal{M}) = \{\overline{a} \in \mathcal{M} : \mathcal{M} \vDash \varphi(\overline{a})\}$$

Definition 5.2: $\varphi(\overline{x})$ is T-*consistent* if $T \vdash \exists \overline{x} \varphi(\overline{x})$ or equivalently $\varphi(\mathcal{M}) \neq \varnothing$.

Definition 5.3: A set of formulas $p(\overline{x}) \subseteq F_L(\overline{x})$ is T-*consistent* if for every finite subset $p_0(\overline{x}) \subseteq p(\overline{x})$ we have

$$T \vdash \exists \overline{x} \left(\bigwedge_{\varphi \in p_0} \varphi(\overline{x}) \right)$$

Definition 5.4: A *type in* T is a set of formulas $p(\overline{x})$ which is T-consistent, we call it a 1-type if $\overline{x} = x$ and an n-type if $\overline{x} = (x_1, ..., x_n)$

Definition 5.5: A type $p(\overline{x})$ is *complete* if for every formula $\varphi(\overline{x}) \in F_L(\overline{x})$ either $\varphi(\overline{x}) \in p$ or $\neg\varphi(\overline{x}) \in p$

Example: $\operatorname{tp}^{\mathcal{M}}(\overline{x})$ is always a complete type

Remark: If $\mathcal{M} \prec \mathcal{N}$, and $\overline{a} \in \mathcal{M}$ then $\operatorname{tp}^{\mathcal{M}}(\overline{a}) = \operatorname{tp}^{\mathcal{N}}(\overline{a})$.

Slightly generalizing the concept of a type we have the following

Definition 5.6: For a set of parameters $A \subseteq |\mathcal{M}|$ we define

$$T(A) = \mathrm{Th}_{L(A)}(\mathcal{M}),$$

that is all the true $L(A)$-sentences in $\mathcal{M}$.

A type *over* A is a type in $T(A)$.

We then have the generalization of the notation,

$$F_{L(A)}(\overline{x}) = \{\varphi(\overline{x}, \overline{a}) : \overline{a} \in A, \varphi(\overline{x}, \overline{y}) \in F_L(\overline{x}, \overline{y})\}$$

and

$$\mathrm{tp}^{\mathcal{M}}\left(\overline{b} \,/\, A\right) = \left\{\varphi(\overline{x}, \overline{a}) : \mathcal{M} \vDash \varphi\left(\overline{b}, \overline{a}\right)\right\}$$

as well as

$$S_n^T(A) = \{\text{all complete n-types in } T \text{ on } A\}.$$

We will often drop the superscript when it is clear what theory we are working on.

Proposition 5.1: If A is a finite set there is a natural injection $S_n^T(A) \to S_{n+|A|}^T(\varnothing)$.

Proof: Enumerate A as $A = \{a_1, ..., a_m\}$, then consider any type $p(\overline{x}) \in S_n^T(A)$. Write

$$p(\overline{x}) = \{\varphi_\alpha(\overline{x}) : \alpha \in I\}$$

for some index set I. For each $\varphi_\alpha(\overline{x})$ we can write it as $\varphi'_\alpha(\overline{x}, \overline{a})$ where $\overline{a}$ are some parameters from A. Then we can define the set of formulas in $n + m$ variables:

$$q(\overline{x}, \overline{y}) = \{\varphi'_\alpha(\overline{x}, \overline{y}))\}.$$

Since p is consistent then q is also consistent since any finite fragment can be realized by plugging back elements of A, so q is a type which can be completed into a complete type in $S_{n+m}^T(\varnothing)$, this defines a map $f : S_n^T(A) \to S_{n+|A|}^T(\varnothing)$.

To see that this is an injection note that if $p \neq p'$ then there is some formula $\varphi \in p$ with $\neg\varphi \in p'$. Then $\varphi' \in f(p)$ and $\neg\varphi' \in f(p')$ so these cannot be equal types. □

Definition 5.7: A type $p(\overline{x})$ is realized in a model $\mathcal{M}$ if there exists $\overline{a} \in \mathcal{M}$ with $p(\overline{x}) \subseteq \mathrm{tp}^{\mathcal{M}}(\overline{a})$.

Example: If $T = \mathrm{DLO}_0$ and $\mathcal{M} = \mathbb{Q}$ then

$$p(x) = \left\{s < x, x < r : s < \sqrt{2} < r\right\}$$

is not realized in $\mathbb{Q}$.

Types have several basic properties that we will use quite often.

Proposition 5.2: If $p(\overline{x})$ is a type over $A \subseteq |\mathcal{M}|$ then there exists $\mathcal{M} \prec \mathcal{N}$ such that $p(\overline{x})$ is realized in $\mathcal{N}$.

Proof: Let $\overline{c}$ be new constants, define

$$T' = \{\varphi(\overline{c}) : \varphi(\overline{x}) \in p(\overline{x})\} \cup \mathrm{Th}_{L(M)}(M)$$

and model of T' will realize p because the interpretation of $\overline{c}$ will realize p.

Since $\mathrm{Th}_{L(M)}(M) \subseteq T'$ any model of T' will be an elementary extension of $\mathcal{M}$. It is thus enough to show that T' is consistent.

By assumption every finite subset of $p(\overline{x})$ will be consistent with $\mathrm{Th}_{L(M)}(M)$ and thus by compactness T' is consistent. □

Corollary 5.2.1: Every type is a subset of a complete type since if p is realized by $\overline{b} \in \mathcal{N}$ then $p \subseteq \mathrm{tp}^{\mathcal{N}}\left(\overline{b} \,/\, A\right)$

We can also prove the above corollary in a different way, using Zorn's lemma. Next we will need will need some more notation.

Definition 5.8: A subset $F \subseteq \mathbb{B} \setminus \{0\}$, where $\mathbb{B}$ is a Boolean algebra, is a *filter* if

- If $a, b \in F$ then $a \cdot b \in F$.
- If $a \in F$ and $a \leq b$ then $b \in F$

An *ultrafilter* is a maximal filter with respect to inclusion.

Example: We say an ultrafilter $\mathcal{U}$ on $\mathbb{B}$ is principal if $\mathcal{U} = \{a \in \mathbb{B} : a \geq a_0\}$ for some atom a_0.

Definition 5.9: If $\mathbb{B}$ is a Boolean algebra then $S(\mathbb{B})$ is the set of all ultrafilters over $\mathbb{B}$, we can give it a topology generated by

$$[a] = \{\mathcal{U} \in S(\mathbb{B}) : a \in \mathcal{U}\}$$

Proposition 5.3:
(1) $\{[a] : a \in \mathbb{B}\}$ is indeed a basis of a topology.
(2) $[a]^c = [-a]$
(3) $[a + b] = [a] \cup [b]$
(4) $[a \cdot b] = [a] \cap [b]$
(5) The topology defined above is Hausdorff and compact.

Proof:
(1) This will follow from 4.
(2) For any ultrafilter $\mathcal{U}$ that does not contain a we must have $-a \in \mathcal{U}$ and so

$$\mathcal{U} \in [a] \Leftrightarrow \mathcal{U} \notin [-a]$$

(3) Since $a, b \leq a + b$ then

$$(\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b]) \Rightarrow (a+b) \in \mathcal{U} \Rightarrow \mathcal{U} \in [a+b]$$

on the other hand $a+b \in \mathcal{U} \Rightarrow (a \in \mathcal{U}) \vee (b \in \mathcal{U})$ and so

$$\mathcal{U} \in [a+b] \Rightarrow (\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b])$$

(4) Since $a \cdot b \leq a, b$ then almost by definition

$$(a \in \mathcal{U}) \wedge (b \in \mathcal{U}) \Leftrightarrow a \cdot b \in \mathcal{U}$$

(5) For any two distinct ultrafilters $\mathcal{U}, \mathcal{U}'$, then for some x we have $x \in \mathcal{U}$ and $x \notin \mathcal{U}'$. Then $\mathcal{U} \in [x], \mathcal{U}' \notin [x]$ as well as $\mathcal{U} \notin [-x], \mathcal{U}' \in [-x]$ and so the topology is Hausdorff. To show compactness let $\bigcup_i [a_i] = S(\mathbb{B})$, then $\{-a_i : i \in I\}$ cannot be a subset of any ultrafilter $\mathcal{U}$, for then

$$-a_i \in \mathcal{U}, \forall i \in I \Rightarrow a_i \notin \mathcal{U}, \forall i \in I \Rightarrow \mathcal{U} \notin [a_i], \forall i \in I \Rightarrow \mathcal{U} \notin \bigcup_i [a_i].$$

Thus, some finite subset of $-a_i$'s must have product zero since otherwise $\{-a_i : i \in I\}$ satisfies the finite intersection property and thus is contained in some ultrafilter. But then if $\left\{-a_{i_1}, ..., -a_{i_k}\right\}$ has zero product then any ultrafilter cannot contain all of them, thus any ultrafilter $\mathcal{U}$ has to contain some a_{i_j} and so $\bigcup_k \left[a_{i_k}\right] = S(\mathbb{B})$.

□

Theorem 5.4 (Stone's Theorem): For every Boolean algebra $\mathbb{B}$ there exists a set I with $\mathbb{B} \subseteq \mathcal{P}$

Proof: Set $I = S(\mathbb{B})$, then the map $a \mapsto [a]$ is clearly a homomorphism by the above proposition, to see it is 1 to 1 we use the proof for Hausdorffness above to see that $[a] \neq [b]$ if $[a] \neq [b]$. □

Proposition 5.5: Let $\mathcal{U}$ be an ultrafilter, $\mathcal{U}$ is principal iff it is isolated in $S(\mathbb{B})$.

Proof: Assume that $\{\mathcal{U}\}$ is an open set, then $\{\mathcal{U}\} = [a]$ for some a. Now if a is not atomic then $0 < b < a$ for some b and so $[a] = [a \cdot b] \cup [a \cdot (-b)]$ but $[a \cdot b], [a \cdot (-b)]$ are both non-empty and not equal since they both contain the ultrafilters generated by the filter

$$\{Y \in \mathbb{B} : a \cdot b \leq Y\} \text{ and } \{Y \in \mathbb{B} : a \cdot (-b) \leq Y\}$$

this contradicts the fact that $[a]$ contains only one element. Thus a is an atom and so the principal ultrafilter of a is in $[a]$. Since $[a] = \{\mathcal{U}\}$ we have that U must be the principal ultrafilter of a.

On the other hand if $\mathcal{U}$ is principal then $\mathcal{U} \in [a]$ for some atom a but since its atomic anything in $[a]$ must be the principal ultrafilter of a. Thus $[a] = \{\mathcal{U}\}$ and so $\mathcal{U}$ is isolated. □

Definition 5.10: Let T be a complete theory and $\mathcal{M} \models T$ then

$$\operatorname{Def}(\mathcal{M}) = \{\varphi(\mathcal{M}) : \varphi \in F_L(x)\}$$

is a Boolean algebra of subsets of $\mathcal{M}$ called the algebra of *definable* subsets of $\mathcal{M}$.

Proposition 5.6: The map $\iota : F_L(\overline{x}) \to \operatorname{Def}(\mathcal{M})$ given by

$$\iota : \varphi \mapsto \varphi(\mathcal{M})$$

is a homomorphism.

Remark: $\ker(\iota) = \{\varphi : \varphi(\mathcal{M}) = \varnothing\}$ is the set of T-inconsistent formulas.

We have then by Isomorphism theorem for rings

$$F_L(\overline{x}) / \ker(\iota) \cong \operatorname{Def}(\mathcal{M})$$

We can also identify $S_n^T(\varnothing)$ with $S(F_L(\overline{x}))$ which makes it a compact set with basic open sets $[\varphi(\overline{x})] = \{p \in S_n^T(\varnothing) : \varphi(\overline{x}) \in p\}$.

Proposition 5.7: If L is countable then $S_n^T(\varnothing)$ is homeomorphic to a closed subset of the Cantor space.

Proof: To see this we will turn $S_n^T(\varnothing)$ into an infinite binary tree, first enumerate $F_L(\overline{x}) = \{\varphi_1, ...\}$ then for every type $p \in S_n^T(\varnothing)$ we have either $\varphi_1 \in p$ or $\neg\varphi_1 \in p$. This gives a splitting of $S_n^T(\varnothing)$ into two open subsets, we then split again on φ_2 and get 4 open subsets. Continuing this construction, we get that the complete types will be infinite branches in this tree, and it is well known that such an infinite binary tree is isomorphic to the Cantor space. □

Remark: This construction can also be done with L uncountable, we then get a homomorphism to $2^{|L|}$ seen as a product space.

The space $S_n^T(\varnothing)$ carries a clopen basis of the form $[\varphi(\overline{x})] = \{q \in S_n^T(\varnothing) : \varphi \subseteq q\}$.

All of these also hold if we change $S_n^T(\varnothing)$ to $S_n^T(A)$

Definition 5.11: If $\mathcal{M}$ is a model of T and $\kappa \geq \aleph_0$ is an infinite cardinal, we say that $\mathcal{M}$ is *κ-saturated* if for every subset $A \subseteq |\mathcal{M}|$ of size less than κ every type in $S_n^T(A)$ is realized in $\mathcal{M}$.

$\mathcal{M}$ is *saturated* if $\mathcal{M}$ is $|\mathcal{M}|$-saturated.

Remark: $\{x \neq a : a \in \mathcal{M}\}$ is not realized in any model $\mathcal{M}$, so no model is κ-saturated for any $\kappa > |\mathcal{M}|$.

We will next show how to construct saturated models, to complete this we will need a lemma.

Lemma 5.8: If $(\mathcal{N}_\alpha)_{\alpha<\kappa}$ is an elementary chain, that is $\mathcal{N}_\alpha \prec \mathcal{N}_\beta$ for $\alpha < \beta$. Then if $\mathcal{N} = \bigcup_{\alpha=0}^{\kappa} \mathcal{N}_\alpha$ we have $\mathcal{N}_\alpha \prec \mathcal{N}$ for all α.

Proof: We will use Theorem 2.3 to prove this, by structural induction on the formula φ. Assume that $\overline{a} \subseteq \mathcal{N}_\alpha$ for some α and $\mathcal{N} \vDash \exists x\, \varphi(x, \overline{a})$. Then for some $\overline{b} \in \mathcal{N}$ we have $\mathcal{N} \vDash \varphi(\overline{b}, \overline{a})$. Now since it is a finite tuple we also have that $\overline{b} \in \mathcal{N}_\beta$ for some β, if $\beta \leq \alpha$ then $\overline{b} \in \mathcal{N}_\alpha$ so we are done, hence we assume that $\beta > \alpha$. Then we have $\mathcal{M} \vDash \varphi(\overline{b}, \overline{a})$ so by induction we know that $\mathcal{N}_\beta \vDash \varphi(\overline{b}, \overline{a})$. But then since $\beta > \alpha$ we know that $\mathcal{N}_\alpha \prec \mathcal{N}_\beta$ and thus $\mathcal{N}_\alpha \vDash \varphi(\overline{b}, \overline{a})$ and so the test holds by induction. □

Theorem 5.9: For every κ, for every $\mathcal{M}$, there exists a model $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ and $\mathcal{N}$ is κ-saturated.

If κ is *weakly inaccessible*, that is $\lambda < \kappa \Rightarrow 2^\lambda \leq \kappa$ (note that such cardinals cannot be proved to exist in ZFC) then for every $\mathcal{M}$ with $|\mathcal{M}| \leq \kappa$ there exists $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ saturated with size κ.

Proof: Since L is countable, then $S_n^T(A) \leq 2^{|A|\, + \aleph_0}$ by Proposition 5.7. Let $\mu = 2^\kappa$, note that $\operatorname{cf}(\mu) > \kappa$ by Theorem A.5.

We will now construct a sequence of models $(\mathcal{M}_\alpha)_{\alpha<\mu}$. We set $\mathcal{M}_0 = \mathcal{M}$, and at limit α we have $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$, we will assume that $|\mathcal{M}_\alpha| < \mu$.

At successor steps $\alpha = \beta + 1$, we want to find $\mathcal{M}_\alpha$ with $\mathcal{M}_\beta \prec \mathcal{M}_\alpha$ such that for all $A \subseteq \mathcal{M}_\beta$ with $|A| < \kappa$, every type in $S_n^T(A)$ is realized in $\mathcal{M}_\alpha$. Now we know that for every single type $p(\overline{x})$ by Proposition 5.2 we can add a realization of that type, and then by Theorem 2.4 we can get that realization with size at most μ, so we just need to do induction again to add every type.

Let us count how many types we need to add, we know that for any fixed A we have $|S_n^T(A)| \leq 2^{\kappa+\aleph_0} = \mu$. Now for any cardinality β we have that the number of subsets A of a set of cardinality μ with size $|A| = \beta$ is

$$\mu^\beta = (2^\kappa)^\beta = 2^{\kappa\times\beta} = 2^\kappa = \mu$$

so in total we have $\sum_{\lambda<\kappa} \mu^\lambda = \kappa\mu = \mu$ steps and so our final model $\mathcal{M}_{\alpha+1}$ is also of size at most μ which completes the induction. □

Example: There are strange consequences to this theorem, for example there are models of Piano Arithmetic that satisfy a statement encoding "PA is inconsistent".

We can see that the process of adding types is not very difficult, in model theory we have a saying about this: "Any fool can realize a type, but it takes a model theorist to omit one". We have not yet looked at omitting types, but the definition is exactly what you would expect.

Definition 5.12: For a complete theory T, a model $\mathcal{M} \vDash T$ and a type $p(\overline{x})$. We say that $\mathcal{M}$ *omits* $p(\overline{x})$ if it does not realize it, i.e. $p(\mathcal{M}) = \varnothing$.

Now the difficulty in omitting types is that some types can **never** be omitted.

Example: If c is a constant of a language L then the type of the interpretation of c can never be omitted.

And yet other types can be omitted

Example: The type of a transcendental number in ACF_p is distinct from that of an algebraic number, and can be omitted, for example in $\hat{\mathbb{Q}}$.

The first example here is an important one to keep in mind since all the properties of that type can be proved from the single formula $\varphi(x) = (x = c)$.

Definition 5.13: A type $p(\overline{x})$ is isolated if there exists a formula $\varphi(\overline{x}) \in p(\overline{x})$ such that for every $\psi(\overline{x}) \in p(\overline{x})$ we have

$$T \vdash (\varphi(\overline{x}) \Rightarrow \psi(\overline{x}))$$

Proposition 5.10: $p(\overline{x}) \in S_n(A)$ is isolated iff $\{p\}$ is open in $S_n(A)$.

Proof: $\{p\}$ being open is equivalent to p being an isolated point in the topological sense in $S_n(A)$. By Proposition 5.5 we then know that this is equivalent to p being a principal ultrafilter. Now for formulas φ, ψ we have

$$\varphi \leq \psi \Leftrightarrow T \vdash (\psi \wedge \varphi \Leftrightarrow \varphi) \Leftrightarrow T \vdash (\varphi \Rightarrow \psi)$$

and hence p being a principal ultrafilter is equivalent to $p = \{\psi \text{ formula} : T \vdash \varphi \Rightarrow \psi\}$ which is exactly the definition of φ isolating p. □

Proposition 5.11: If $p(\overline{x})$ is isolated, then p cannot be omitted.

Proof: Let $\varphi(\overline{x})$ be the generating formula for p, then

$$\exists x \varphi(\overline{x})$$

is a true sentence in T and thus any witness of this sentence is a realization of the type. Hence the type is realized in any model of T. □

Now a priori we would not expect this converse to hold since it feels like being isolated is quite the strong condition, but in fact the converse does hold, which is shown in this theorem.

Theorem 5.12: If $p(\overline{x})$ is not isolated, then there exists $\mathcal{M} \vDash T$ which omits $p(\overline{x})$.

There are many proofs of this theorem but we will use one called **Henkin's construction**. This proof method is also the modern method for proving Theorem 2.5.

Proof: Let L be a countable language and let $\{c_n\}_{n\in\mathbb{N}}$ be a family of new constants not in L, enumerate all formulas in $L \cup \{c_n\}_{n\in\mathbb{N}}$ as φ_n. Let $f : \mathbb{N} \to \mathbb{N}$ be increasing such that $c_{f(n)}$ does not appear in $\varphi_0, ..., \varphi_n$.

We define the **Henkin axioms**

$$H_i = (\exists x \varphi_i(x)) \Rightarrow \varphi_i\left(c_{f(i)}\right).$$

We now construct a sequence of sets of sentences $T_0 = T \subseteq T_1 \subseteq T_2 \subseteq ...$ such that

$$T_{2n+1} = T_{2n} \cup \{H_n\} \quad \text{and} \quad T_{2n+2} = T_{2n+1} \cup \{\neg\varphi_n(c_n)\} \text{ for some } \varphi_n(\overline{x}) \in p(\overline{x})$$

Then by taking the union of these sets, we will get an axiomization of a consistent theory. We can then use Zorn's lemma to get a complete theory containing it and then if we set our universe to be the set of constants quotiented by the relation

$$c_i = c_j \text{ as elements if } \left(c_i = c_j\right) \text{ as a formula is in } T$$

Now a model satisfying this theory will not realize the type $p(\overline{x})$ since if it did then some constant would realize it which would contradict the fact that our theory contains $\neg\varphi(c_n)$ for every n.

All that is left to do is to check that at every odd step these sentences are indeed consistent and that at even steps we can pick specific φ_n to make the set of sentences consistent.

For the even steps assume that T_{2n+1} is consistent but for every $\psi(\overline{x}) \in p(\overline{x})$ we have that $T_{2n+1} \cup \{\neg\psi(c_n)\}$ is inconsistent. Then T_{2n+1} is T where we added some finitely many sentences, so we can write $T_{2n+1} = T \cup \left\{\psi_j(\overline{c}, c_n) : j < k\right\}$ for some k and ψ_j.

Now set

$$\varphi(\overline{y}, x) = \bigwedge_{j<k} \psi_j(\overline{y}, x)$$

then for every $\psi(\overline{x}) \in p(\overline{x})$ we have $T \cup \{\varphi(\overline{c}, c_n)\} \cup \{\neg\psi(c_n)\}$ is inconsistent so

$$T \vdash (\varphi(\overline{c}, c_n) \Rightarrow \psi(c_n))$$

But now since the T does not contain c_n as a constant we can replace all instances of c_n with x and all instances of $\overline{c}$ with $\overline{y}$ in the proof and get that

$$T \vdash (\varphi(\overline{y}, x)) \Rightarrow \psi(x))$$

but then this means that

$$T \vdash \forall\overline{y}(\varphi(\overline{y}, x) \Rightarrow \psi(x))$$

but we have that

$$\begin{aligned}\forall\overline{y}(\varphi(\overline{y}, x) \Rightarrow \psi(x)) &= \forall\overline{y}(\neg\varphi(\overline{y}, x) \vee \psi(x)) = \neg\exists\overline{y}(\varphi(\overline{y}, x) \wedge \neg\psi(x)) \\ &= \neg(\exists\overline{y}\varphi(\overline{y}, x) \wedge \neg\psi(x)) = (\exists\overline{y}(\varphi(\overline{y}, x))) \Rightarrow \psi(x)\end{aligned}$$

then $\exists \overline{y}(\varphi(\overline{y}, x))$ implies every ψ in the type $p(\overline{x})$, but also $\exists \overline{y}(\varphi(\overline{y}, x))$ is true in T_{2n+1} and thus is consistent with T and thus is in the type p. This contradicts our assumption that $p(\overline{x})$ is not isolated. □

We now have a powerful way to think about and use types in proofs.

Corollary 5.12.1: In a complete theory T,

$$p \text{ is isolated} \Leftrightarrow \text{every model of } T \text{ realizes } p$$

Now that we have the tools to omit types, we can use them to characterize the $\aleph_0$-categorical theories.

Theorem 5.13 (Ryll-Nardzewski): Let T be a complete theory over a countable language L, the following are equivalent.
(1) T is $\aleph_0$-categorical.
(2) $\forall n, S_n^T(\varnothing)$ is finite.

Proof: (1) ⇒ (2). Suppose that $S_n^T(\varnothing)$ is infinite, we know that it is always a closed subset of the Cantor set. As an infinite compact space, $S_n^T(\varnothing)$ has a non isolated point, corresponding to a non isolated type p. By the omitting types theorem, there exists a model which omits p, since it is a type there is another model which realizes p, those two models then cannot be isomorphic. We can then make them both countable by Theorem 2.4 which completes this side of the proof.

(2) ⇒ (1). We assume that $S_n^T(\varnothing)$ is finite. This implies that if $A \subseteq \mathcal{M} \vDash T$, with A being finite, then $S_n^T(A)$ is also finite by Proposition 5.1. Hence $S_n^T(A)$ is a finite Hausdorff space, so every type in $S_n^T(A)$ is isolated.

Now let $\mathcal{M}, \mathcal{N} \vDash T$ be countable models, enumerate them as $\mathcal{M} = \{a_0, a_1, ...\}$ and $\mathcal{N} = \{b_0, b_1, ...\}$. We will now do a back and forth construction, at step n we have a partial isomorphism $f_n : A_n \to B_n$. Define the tuples $\overline{a} = (a_1, ..., a_n), \overline{b} = (b_1, ..., b_n)$ containing all elements of A_n and B_n respectively. From the fact that it is a partial isomorphism we will know that $\mathrm{tp}_n^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_n^{\mathcal{N}}\left(\overline{b}\right)$.

Now let us create the construct the maps by induction, at step 0 we pick some $a \in \mathcal{M}$, then by the discussion above $\mathrm{tp}_n^{\mathcal{M}}(a)$ is isolated. Since it is isolated every model of T realizes this type and so in particular there is an element $b \in \mathcal{N}$ that realizes $\mathrm{tp}_n^{\mathcal{M}}(a)$ and so we map a to it.

At the inductive even steps we will pick some $a_{n+1} \in \mathcal{M}$ and note that $\mathrm{tp}_n^{\mathcal{M}(A)}(a_{n+1})$ is again isolated so again there is some element $b_{n+1} \in \mathcal{N}$ such that $\mathrm{tp}_n^{\mathcal{N}(B)}(b_{n+1}) = \mathrm{tp}_n^{\mathcal{M}(A)}(a_{n+1})$ and so we can map a_{n+1} to b_{n+1}. At the odd steps we do the same thing as above but pick $b_{n+1} \in \mathcal{N}$ first. □

Example: In ACF_p we have that the type of any root of an irreducible polynomial is isolated while the type of the transcendental number is not isolated.

6. Automorphism groups

In algebra for some algebraic structure an important role is played by the automorphism groups of these structures. As model theory is a sort of algebra without fields we will also use automorphism groups.

Definition 6.1: Let $\mathcal{M}$ be a countable structure of a countable language L. We define the automorphism group $\operatorname{Aut}(\mathcal{M})$ to be

$$\operatorname{Aut}(\mathcal{M}) := \{F : \mathcal{M} \hookrightarrow \mathcal{M} : F \text{ is an automorphism}\}$$

$\operatorname{Aut}(\mathcal{M})$ acts on $\mathcal{M}^n$ for all n, and is in fact a Polish topological group.

Proposition 6.1: $\operatorname{Aut}(\mathcal{M})$ is a Polish group, that is separable, infinite, and admits a complete metric.

Proof: Given $f \in \operatorname{Aut}(\mathcal{M})$, neighborhoods of f are

$$U^f_{a_1,\ldots,a_n} = \{g \in \operatorname{Aut}(\mathcal{M}) : g(a_1) = f(a_1), \ldots, g(a_n) = f(a_n)\}$$

Define the sets $[A, B]$ for finite tuples $A, B \subseteq \mathcal{M}$ by

$$[A, B] = \{f \in \operatorname{Aut}(\mathcal{M}) : f(A) = B\}$$

A complete metric can be defined as

$$d(f, g) := \exp_2(-\min\{n : f(n) \neq g(n) \text{ or } f^{-1}(n) \neq g^{-1}(n)\}),$$

where n □

Theorem 6.2: $\operatorname{Th}(\mathcal{M})$ is $\aleph_0$-categorical if and only if for all n, $\operatorname{Aut}(\mathcal{M})$ acts on $\mathcal{M}^n$ with finitely many orbits.

Proof: Every tuple in an orbit has the same type since $\operatorname{Aut}(\mathcal{M})$ consists of automorphisms. Hence we have that if there are finitely many orbits, each $S_n(\varnothing)$ is finite, making T $\aleph_0$-categorical by Theorem 5.13.

On the other hand if it is $\aleph_0$-categorical, every $S_n(\varnothing)$ is finite, and so by Proposition 5.1 so is every $S_n(A)$. Now let $\overline{a}, \overline{b}$ be two tuples of the same type and consider $T(\overline{a})$. In this theory we have $S_n^{T(\overline{a})}(\varnothing) = S_n^T(\overline{a})$ and thus each of its type spaces are finite.

Now again through Theorem 5.13 we know that $T(\overline{a})$ is $\aleph_0$-categorical so since we can interpret $\mathcal{M}$ as an $T(\overline{a})$ model by interpreting $\overline{a}$ as $\overline{b}$, so we have an isomorphism mapping $\overline{a}$ to $\overline{b}$, hence they are in the same orbit. Hence there at most as many orbits as there are n-types, so there are finitely many orbits. □

7. Infinitary Logic and Scott Analysis

We now want to take a short look at different types of logic.

$\mathcal{L}_{\omega_1,\omega}$ is the extension of finite order logic over a countable language L, where in formulas we allow infinite countable $\bigvee, \bigwedge$.

More precisely,

(1) The atomic formulas of $\mathcal{L}_{\omega_1,\omega}$ are the same as in first order logic.
(2) If φ_k is a countable set of formulas then

$$\bigwedge_{k\in\omega} \varphi_k \text{ and } \bigvee_{k\in\omega} \varphi_k$$

are both in $\mathcal{L}_{\omega_1,\omega}$
(3) If φ is in $\mathcal{L}_{\omega_1,\omega}$ then $\exists x(\varphi(x))$ and $\forall y(\varphi(y))$ are both in $\mathcal{L}_{\omega_1,\omega}$.

Now notice that, in ordinary logic, for a finite model $\mathcal{M}$, there exists a sentence σ with $\mathcal{M} \vDash \sigma$ and

$$(\mathcal{N} \vDash \sigma) \Rightarrow \mathcal{M} \cong \mathcal{N}$$

Our goal now is to generalize this using our new type of logic to the case of countable models.

Definition 7.1: Let $\mathcal{M}$ be a countable structure. Define $\equiv_\alpha$ on $\mathcal{M}^n$ for α an ordinal, n a natural number, by transfinite induction. For the base case

$$\overline{a} \equiv_0 \overline{b} \quad \text{if} \quad \mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}\left(\overline{b}\right),$$

in the limit case

$$\overline{a} \equiv_\gamma \overline{b} \quad \text{if } \overline{a} \equiv_\beta \overline{b}, \forall \beta < \gamma,$$

and in the successor case

$$\begin{aligned} \overline{a} \equiv_{\alpha+1} \overline{b} \quad \text{if} \quad & \forall c \in \mathcal{M}, \exists d \in \mathcal{M}\ (\overline{a}, c) \equiv_\alpha \left(\overline{b}, d\right) \\ \text{and} \quad & \forall d \in \mathcal{M}, \exists c \in \mathcal{M}\ (\overline{a}, c) \equiv_\alpha \left(\overline{b}, d\right) \end{aligned}$$

We record here some important properties of these relations

Proposition 7.1:
(1) $\equiv_\alpha$ is an equivalence relation on $\mathcal{M}^n$ for all $n \in \mathbb{N}$.
(2) If $\alpha < \beta$ and $\overline{a} \equiv_\beta \overline{b}$ then $\overline{a} \equiv_\alpha \overline{b}$.

Essentially we are saying that the equivalence classes of these relations form a decreasing sequence in alpha which stabilizes at some countable ordinal.

Proof: (1) is trivial.

(2). We prove by induction on α that for any $\overline{a}, \overline{b}, c, d$ we have

$$(\overline{a}, c) \equiv_\alpha \left(\overline{b}, d\right) \Rightarrow \overline{a} \equiv_\alpha \overline{b}.$$

For base case this is immediate, as is true for the limiting case. Assume then that this is true for α, then if $(\overline{a}, c) \equiv_{\alpha+1} \left(\overline{b}, d\right)$ then

$$\forall c' \in \mathcal{M}, \exists d \in \mathcal{M}\ (\overline{a}, c, c') \equiv_{\alpha} \left(\overline{b}, d, d'\right),$$

and so by induction

$$\forall c' \in \mathcal{M}, \exists d \in \mathcal{M}\ (\overline{a}, c) \equiv_{\alpha} \left(\overline{b}, d\right),$$

and a similar working out goes for the other part of the successor case. Then by definition we get

$$\overline{a} \equiv_{\alpha+1} \overline{b} \Rightarrow \exists c, d \in \mathcal{M}\ (\overline{a}, c) \equiv_{\alpha} \left(\overline{b}, d\right) \Rightarrow \overline{a} \equiv_{\alpha} \overline{b}.$$

□

We have a strong 'stabilization' property for these equivalence relations.

Proposition 7.2: There exists $\alpha < \omega_1$, such that $\equiv_\alpha$ is the same equivalence relation as $\equiv_\beta$ for all $\beta \geq \alpha$.

Proof: First note that if $\equiv_\alpha$ is the same relation as $\equiv_{\alpha+1}$, then $\equiv_\beta$ is the same as $\equiv_{\beta+1}$ for all $\beta > \alpha$, one can see this directly from definition. □

This proposition motivates the following definition.

Definition 7.2: The *Scott height* (or *rank*) of a countable structure $\mathcal{M}$ is defined as

$$\mathrm{SH}(\mathcal{M}) = \min\left\{\alpha < \omega_1 : \equiv_\alpha \text{ is the same as } \equiv_{\alpha+1}\right\}$$

We now want to use these tools to work towards our characterizing sentence for countable structures. We now define an equivalence on models that mirrors Definition 7.1.

Definition 7.3: We define $\equiv_\alpha$ on countable L structures through transfinite induction. For the base case

$$\mathcal{M} \equiv_0 \mathcal{N} \text{ if } \mathcal{M} \equiv \mathcal{N},$$

for the limit case

$$\mathcal{M} \equiv_\gamma \mathcal{N} \text{ if } \mathcal{M} \equiv_\beta \mathcal{N} \text{ for all } \beta < \gamma,$$

and for the successor step

$$\begin{aligned} \mathcal{M} \equiv_{\alpha+1} \mathcal{N} \text{ if } \quad & \forall a \in \mathcal{M}, \exists b \in \mathcal{N}, (\mathcal{M}, a) \equiv_\alpha (\mathcal{N}, b) \\ \text{and} \quad & \forall b \in \mathcal{N}, \exists a \in \mathcal{M}, (\mathcal{M}, a) \equiv_\alpha (\mathcal{N}, b) \end{aligned}$$

We can see that this definition in fact generalizes Definition 7.1.

Proposition 7.3: $\overline{a} \equiv_\alpha \overline{b}$ if and only if $(\mathcal{M}, \overline{a}) \equiv_\alpha \left(\mathcal{M}, \overline{b}\right)$.

Proof: We prove by induction on α, for $\alpha = 0$ we know that $\overline{a} \equiv_0 \overline{b}$ if and only if they have the same type. But notice that the types of $\overline{a}$ and of $\overline{b}$ are exactly the theories of $(\mathcal{M}, \overline{a})$ and $\left(\mathcal{M}, \overline{b}\right)$ respectively. Then since $(\mathcal{M}, \overline{a}) \equiv_0 \left(\mathcal{M}, \overline{b}\right)$ if and only if the two theories are equal we see that the relations are equivalent.

In the case of limit α this is trivial, so we consider the successor case. Assume that this is the case for α, then we have

$$\begin{aligned} \overline{a} \equiv_{\alpha+1} \overline{b} \quad \text{if} \quad & \forall c \in \mathcal{M}, \exists d \in \mathcal{M}\ (\overline{a}, c) \equiv_\alpha \left(\overline{b}, d\right) \\ \text{and} \quad & \forall d \in \mathcal{M}, \exists c \in \mathcal{M}\ (\overline{a}, c) \equiv_\alpha \left(\overline{b}, d\right). \end{aligned}$$

But by induction we know that this is equivalent to

$$\begin{aligned} \overline{a} \equiv_{\alpha+1} \overline{b} \quad \text{if} \quad & \forall c \in \mathcal{M}, \exists d \in \mathcal{M}\ (\mathcal{M}, \overline{a}, c) \equiv_\alpha \left(\mathcal{M}, \overline{b}, d\right) \\ \text{and} \quad & \forall d \in \mathcal{M}, \exists c \in \mathcal{M}\ (\mathcal{M}, \overline{a}, c) \equiv_\alpha \left(\mathcal{M}, \overline{b}, d\right). \end{aligned}$$

which is exactly the definition of $(\mathcal{M}, \overline{a}) \equiv_{\alpha+1} \left(\mathcal{M}, \overline{b}\right)$. □

Now with this definition we can start to construct some characterizing sentences.

Lemma 7.4: $\forall \alpha < \omega_1$, $\forall \overline{a} \in \mathcal{M}, \exists \varphi_\alpha^{\mathcal{M},\overline{a}}(x) \in \mathcal{L}_{\omega_1,\omega}$, such that $\forall \mathcal{N}, \forall \overline{b} \in \mathcal{N}$,

$$(\mathcal{M}, \overline{a}) \equiv_\alpha \left(\mathcal{N}, \overline{b}\right) \Leftrightarrow \mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\overline{a}}\left(\overline{b}\right)$$

Proof: We prove by induction on α, in the case of $\alpha = 0$

$$\varphi_0(\overline{x}) = \bigwedge_{\varphi \in \operatorname{tp}^{\mathcal{M}}(\overline{a})} \varphi(\overline{x}).$$

If α is a limit ordinal then

$$\varphi_\alpha^{\mathcal{M},\overline{a}}(\overline{x}) = \bigwedge_{\beta<\alpha} \varphi_\beta^{\mathcal{M},\overline{a}}(\overline{x}).$$

Finally for $\alpha + 1$ we have

$$\varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(\overline{x}) = \left(\bigwedge_{b \in \mathcal{M}} \exists y \varphi_\alpha^{\mathcal{M},(\overline{a},b)}(\overline{x}, y)\right) \wedge \left(\forall y \bigvee_{b \in \mathcal{M}} \varphi_\alpha^{\mathcal{M},(\overline{a},b)}(\overline{x}, y)\right)$$

□

Unfortunately, the sentences are not exactly what we want, they only guarantee isomorphic models under a fairly strong assumption.

Theorem 7.5 (Scott): Let $\mathcal{M}, \mathcal{N}$ be countable structures with

$$\mathrm{SH}(\mathcal{M}) = \mathrm{SH}(\mathcal{N}) = \alpha,$$

if $\mathcal{M} \equiv_{\alpha+\omega} \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.

Proof: We construct the isomorphism by back and forth, by induction we will construct maps $f_n : A_n \to B_n$ such that $(\mathcal{M}, A_n) \equiv_{\alpha+1} (\mathcal{N}, B_n)$. For the case $n = 0$ we set $A_0 = B_0 = \varnothing$ and indeed $\mathcal{M} \equiv_{\alpha+1} \mathcal{N}$, so assume for the induction step that we have the map f_n.

Now we want to add a specific element $a \in \mathcal{M}$ to A_n, so by definition of $\equiv_{\alpha+1}$ there is some element $b \in \mathcal{N}$ such that $(\mathcal{M}, A_n, a) \equiv_\alpha (\mathcal{N}, B_n, b)$. Now we also know that $\mathcal{M} \equiv_{\alpha+n+2} \mathcal{N}$ so again by definition we can pick elements $c_i \in \mathcal{N}$ such that, if we define $C_n = \{c_i : i \leq n\}$ then we have

$$(\mathcal{M}, A_n, a) \equiv_{\alpha+1} (\mathcal{N}, C_n, c_{n+1}).$$

Now notice that

$$(\mathcal{N}, B_n, b) \equiv_\alpha (\mathcal{M}, A_n, a) \equiv_\alpha (\mathcal{N}, C_n, c_{n+1}),$$

so we have that $(\mathcal{N}, B_n, b) \equiv_\alpha (\mathcal{N}, C_n, c_{n+1})$. But since the Scott rank of $\mathcal{N}$ is α then that also means that $(\mathcal{N}, B_n, b) \equiv_{\alpha+1} (\mathcal{N}, C_n, c_{n+1})$.

We thus have

$$(\mathcal{N}, B_n, b) \equiv_{\alpha+1} (\mathcal{N}, C_n, c_{n+1}) \equiv_{\alpha+1} (\mathcal{M}, A_n, a)$$

so $(\mathcal{N}, B_n, b) \equiv_{\alpha+1} (\mathcal{M}, A_n, a)$ and thus we have constructed f_{n+1} with the desired property by setting $f_{n+1}(a) = b$. Then taking $f = \bigcup_{n \in \mathbb{N}} f_n$ gives us an isomorphism. □

We also have a partial converse to this result.

Proposition 7.6: Suppose that $\mathrm{SH}(\mathcal{M}) = \alpha$ and $\mathcal{M} \equiv_{\alpha+\omega} \mathcal{N}$, then $\mathrm{SH}(\mathcal{N}) = \alpha$.

Proof: First we want to show that $\mathrm{SH}(\mathcal{N}) \leq \alpha$. Choose $\overline{a}, \overline{b} \in \mathcal{N}^n$ and suppose that $\overline{a} \equiv_\alpha \overline{b}$. We want to show that $\overline{a} \equiv_{\alpha+1} \overline{b}$ using $\mathcal{N} \equiv_{\alpha+\omega} \mathcal{M}$. Find $\overline{c}, \overline{d} \in \mathcal{M}^n$ such that

$$(\mathcal{M}, \overline{c}) \equiv_{\alpha+1} (\mathcal{N}, \overline{a}) \text{ and } \left(\mathcal{M}, \overline{d}\right) \equiv_{\alpha+1} \left(\mathcal{N}, \overline{b}\right)$$

then we also have

$$\left(\mathcal{N}, \overline{b}\right) \equiv_{\alpha+1} \left(\mathcal{M}, \overline{d}\right) \equiv_{\alpha+1} (\mathcal{M}, \overline{c}) \equiv_{\alpha+1} (\mathcal{N}, \overline{a})$$

and thus $\mathrm{SH}(\mathcal{N}) \leq \alpha$.

For the other inequality we just swap $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 7.6.1: Let $\mathcal{M}$ be a countable structure, there exists $\alpha < \omega_1$ such that for every countable structure $\mathcal{N}$

$$\mathcal{N} \cong \mathcal{M} \Leftrightarrow \mathcal{N} \equiv_\alpha \mathcal{M}$$

However, with a bit of trickery, we can define a sentence which does uniquely classify our countable model.

Definition 7.4: Let $\mathcal{M}$ be an L structure, $\alpha = \mathrm{SH}(\mathcal{M})$. We define the *Scott Sentence* of $\mathcal{M}$ as

$$\phi = \varphi_\alpha^{\mathcal{M},\varnothing} \wedge \bigwedge_{n=0}^{\infty} \bigwedge_{\overline{a} \in \mathcal{M}} \left[\forall \overline{x}\left(\varphi_\alpha^{\mathcal{M},\overline{a}}(\overline{x}) \Rightarrow \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(\overline{x})\right)\right]$$

Theorem 7.7 (Scott Isomorphism Theorem): Let $\mathcal{M}$ be a countable structure for every countable structure $\mathcal{N}$,

$$\mathcal{N} \cong \mathcal{M} \Leftrightarrow \mathcal{N} \vDash \phi^{\mathcal{M}}$$

Proof: The forward direction is simple, if the two models are isomorphic $\mathcal{N}$ satisfies the sentence of $\mathcal{M}$ since they have the same sentences.

For the backwards direction we want to use back and forth, we will use induction and assume we have some tuple $\overline{a}$ and a partial isomorphism $f_n : \mathcal{M} \to \mathcal{N}$, in the sense that $(\mathcal{M}, \overline{a}) \equiv_\alpha (\mathcal{N}, f_n(\overline{a}))$.

For $n = 0$ we have $\mathcal{M} \equiv_\alpha \mathcal{N}$ since $\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\varnothing}$. Now assume that we have constructed the map for n, then we have $(\mathcal{M}, \overline{a}) \equiv_\alpha (\mathcal{N}, f_n(\overline{a}))$, then since $\mathcal{N} \vDash \varphi^{\mathcal{M}}$ then we get

$$\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\overline{a}}(f_n(\overline{a})) \Rightarrow \mathcal{N} \vDash \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(f_n(\overline{a}))$$

but we know that

$$\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\overline{a}}(f_n(\overline{a}))$$

so we must have

$$\mathcal{N} \vDash \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(f_n(\overline{a}))$$

and so

$$(\mathcal{M}, \overline{a}) \equiv_{\alpha+1} (\mathcal{N}, f_n(\overline{a})).$$

Now by Definition 7.1 we get that for any element in $a \in \mathcal{M}$ we can pick an element $b \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, a) \equiv_\alpha (\mathcal{N}, f_n(\overline{a}), b)$ and so we set f_{n+1} to be the extension of f_n with $f_{n+1}(a) = b$.

This describes how we do the odd steps, on even steps we just swap $\mathcal{N}$ and $\mathcal{M}$. □

8. Quantifier Elimination

Definition 8.1: A theory T has *quantifier elimination*, if for every formula $\varphi(\overline{x})$ there exists a quantifier free formula $\psi(\overline{x})$ such that

$$T \vdash \forall \overline{x} \, (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

At face value this seems like a hopelessly strong property and almost no models should satisfy it, but in fact we can make any theory have quantifier elimination if we expand our language. This is called *Skolemization*.

Definition 8.2: A theory T has *Skolem functions*, if for every formula $\varphi(\overline{x}, y)$ there exists a term $t_\varphi(\overline{x})$ such that

$$T \vdash \left[\exists y(\varphi(\overline{x}, y)) \Rightarrow \varphi\big(\overline{x}, t_\varphi(\overline{x})\big)\right]$$

Proposition 8.1: If T has Skolem functions then it has quantifier elimination.

Proof: We prove by induction on the complexity of a formula $\varphi(\overline{x})$, for atomic formulas this is trivial. For conjunctions, disjunctions and negations this is also trivial. Now if $\varphi(\overline{x}) = \exists y(\psi(\overline{x}, y))$ then through Skolem functions we get a term t_ψ, such that

$$T \vdash \exists y \, \psi(\overline{x}, y) \Rightarrow \psi\big(\overline{x}, t_\psi(\overline{x})\big).$$

Then we clearly also have that

$$T \vdash \psi\big(\overline{x}, t_\psi(\overline{x})\big) \Rightarrow \exists y \, \psi(\overline{x}, y),$$

since $t_\psi(\overline{x})$ is exactly a witness of $\psi(\overline{x}, y)$. So we get

$$T \vdash \exists y \, \psi(\overline{x}, y) \Leftrightarrow \psi\big(\overline{x}, t_\psi(\overline{x})\big).$$

But now by induction we can assume that ψ is equivalent to a quantifier free formula ϕ. This means that

$$T \vdash \varphi(\overline{x}) \Leftrightarrow \exists y \, \psi(\overline{x}, y) \Leftrightarrow \psi\big(\overline{x}, t_\psi(\overline{x})\big) \Leftrightarrow \phi\big(\overline{x}, t_\psi(\overline{x})\big)$$

and so φ is equivalent to a quantifier free formula and so by induction T has quantifier elimination. □

If T has Skolem functions and $\mathcal{M} \vDash T$ with $A \subseteq \mathcal{M}$, we can define $\mathrm{Sc}(A)$ to be the closure of A under all Skolem functions, sometimes called the Skolem hull of A.

Proposition 8.2: $\mathrm{Sc}(A) \prec \mathcal{M}$

Proof: Proof is trivial by Theorem 2.3. □

Let T be a theory in L, we can add enough Skolem functions in steps. In a single step, we do as follows:

- We replace L with L' with new added function symbols.
- We replace T with $T' = T \cup \left\{\exists y(\varphi(\overline{x}, y)) \to \varphi\left(\overline{x}, f_\varphi(\overline{x})\right) : \varphi\right\}$
- We replace $\mathcal{M}$ with $\mathcal{M}'$ where we interpret the functions using the witnesses we know exist.

We now use induction to iterate this process, we set
- $L^{n+1} = (L^n)'$
- $T^{n+1} = (T^n)'$
- $\mathcal{M}^{n+1} = (\mathcal{M}^n)'$

then in the limit we have
- $L^s = \bigcup_{n<\omega} L^n$
- $T^s = \bigcup_{n<\omega} T^n$
- $M^s = \bigcup_{n<\omega} M^n$

Proposition 8.3:
- $\mathcal{M}^s \vDash T^s$, and T^s has Skolem function.
- T^s is a conservative extension, in the sense that

$$T \vdash \sigma \Leftrightarrow T^s \vdash \sigma$$

Proof: Exercise. □

Now using Skolem functions is a very crude way to add quantifier elimination, and its main issue is that it forces us to add functions to our language. Very often we want to keep our language as is, so we need other ways to prove quantifier elimination. Luckily, this is often possible.

Proposition 8.4: DLO_0 has quantifier elimination.

Proof: We induct on the logical structure of $\varphi(\overline{x})$, we show that there exists a quantifier free formula $\psi(\overline{x})$ such that

$$\mathrm{DLO}_0 \vdash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})).$$

For atomic formulas this is trivial, for logical connectives this is also trivial, now assume that $\varphi(\overline{y}) = \exists x\, \varphi'(x, \overline{y})$. Using induction we know that $\varphi'(x, \overline{y})$ is equivalent to a quantifier free formula so we can WLOG assume it is quantifier free. Next since $\exists x(\alpha \vee \beta) \leftrightarrow \exists x(\alpha) \vee \exists x \beta$, we can assume that φ' is in normal form, that is

$$\varphi' \leftrightarrow \bigvee_i \exists x \bigwedge_j a^i_j$$

where a^i_j are atomic or negations of atomic formulas. Then WLOG $\varphi' = \alpha_1 \wedge \ldots \wedge \alpha_n$ with α_i atomic or a negation of an atomic formula.

Now write $c_1(\overline{y}), \ldots, c_m(\overline{y})$ to be all of the quantifier free formulas which describe a total order on $y_1, \ldots, y_n$ which possibly identifies some of them. Now for each $i \leq m$ if

$$\mathbb{Q} \vDash c_i\left(\overline{b}\right) \wedge c_i\left(\overline{b}'\right),$$

then there exists an automorphism of $(\mathbb{Q}, \leq)$ mapping $\overline{b}$ to $\overline{b}'$. For each $i \leq m$ let $\overline{b}_i$ such that $\mathbb{Q} \vDash c_i\left(\overline{b}_i\right)$, then consider the index set

$$I = \left\{i \leq m : \mathbb{Q} \vDash \exists \overline{x}\, \varphi'\left(x, \overline{b}_i\right)\right\}$$

then we have

$$\mathbb{Q} \vDash \left(\exists x\, \varphi(x, \overline{y}) \leftrightarrow \bigwedge_{j \in I} c_j(\overline{y})\right)$$

because if $\overline{y}$ satisfies the left formula then it has some ordering and so we can use the automorphisms to map $\overline{y}$ to some $\overline{b}_i$ and then $i \in I$ and thus the right side also holds. Similarly we can go the other way.

Now since DLO_0 is complete we can lift the above sentence from $\mathbb{Q}$ to DLO_0 and get our result. □

What we see in this proof is that quantifier elimination is intimately related to the type structure for finite tuples. We can make this relation more precise.

Proposition 8.5: Let $p \in S_n(\varnothing)$, write p_0 for $\{\varphi \in p : \varphi \text{ is quantifier free}\}$.

A complete theory has quantifier elimination if and only if

$$\forall p \in S_n(\varnothing), \quad T \cup p_0 \vdash p$$

Proof: The forward direction is trivial, we just take any $\varphi(\overline{x}) \in p$ and use quantifier elimination to get an equivalent quantifier free version which must also lie in p and thus lie in p_0, then by equivalence we get the result.

For the backwards direction assume the condition above holds, then let $\varphi(\overline{x})$ be a formula and $[\varphi] \subseteq S_n(\varnothing)$ be the corresponding open set. For every $p \in [\varphi]$ we have $T \cup p_0 \vdash \varphi$ so by compactness, for some finite collection $\psi_i^p(\overline{x})$ of quantifier free formulas we have $T \cup \left\{\varphi_i^p(\overline{x}) : i \leq n\right\} \vdash \varphi(\overline{x})$. Then set

$$\psi^p(\overline{x}) = \bigwedge_i \psi_i^p(\overline{x})$$

and note that $[\psi^p] \subseteq [\varphi]$. Now since open sets of the form $[\psi^p]$ cover $[\varphi]$ which is compact, we can take a finite subcollection p_j such that $[\psi^{p_j}]$ cover $[\varphi]$, then $[\varphi] = \bigcup_j^k [\psi^{p_j}]$ and then

$$T \vdash \varphi(\overline{x}) \leftrightarrow \bigvee_{j=1}^k \psi^{p_j}(\overline{x})$$

□

Proposition 8.6: ACF_p has quantifier elimination.

Proof: Let $p \in S_n(\varnothing)$, we need $T \cup p_0 \vdash p$. Choose a large algebraically closed field K and let $\overline{a}, \overline{b} \in K$ such that both realize p_0. We will show that there exists $\varphi \in \mathrm{Aut}(K)$ such that $\varphi(\overline{a}) = \overline{b}$, this will then imply that $\mathrm{tp}(\overline{a}) = \mathrm{tp}\left(\overline{b}\right)$ which proves what we want since that would imply any realization of p_0 has the same type p.

Recall that $\langle\overline{a}\rangle, \left\langle\overline{b}\right\rangle$ are the subrings generated by $\overline{a}$ and $\overline{b}$ respectively. Now map $a_i \mapsto b_i$, we want to extend this map to $\langle\overline{a}\rangle \to \left\langle\overline{b}\right\rangle$. Recall that elements of $\langle\overline{a}\rangle$ are are of the form $P(\overline{a})$ where $P \in \mathbb{Z}[\overline{x}]$, which we can also write as $\tau_1(\overline{a}) - \tau_2(\overline{a})$ where τ_1, τ_2 are two terms. We now map

$$\tau_1(\overline{a}) - \tau_2(\overline{a}) \to \tau_1\left(\overline{b}\right) - \tau_2\left(\overline{b}\right)$$

one can easily check that this is a well defined map.

We now extend the isomorphism to the field of fractions for $\langle\overline{a}\rangle$ and $\left\langle\overline{b}\right\rangle$ in exactly the same way, by mapping

$$\frac{\tau_1(\overline{a})}{\tau_2(\overline{a})} \to \frac{\tau_2\left(\overline{b}\right)}{\tau_2\left(\overline{b}\right)}$$

and then once again we can extend to the algebraic closure of this field of fractions.

Finally we have a map of countable algebraically closed subfields $L \to L$, we can extend this map to all of K since K has a transcendental basis of L and so we can permute this transcendental basis whichever way we like to extend this map. $\square$

9. Algebraic Geometry

Now that we have quantifier elimination of ACF_p we can use it to very quickly prove the foundations of algebraic geometry

Theorem 9.1 (Lefchetz's principle): Let σ be a sentence in the language of fields. TFAE

(1) σ is true in every algebraically closed field of characteristic 0.
(2) σ is true in every algebraically closed field of characteristic p for all but finitely many p.
(3) σ is true in every algebraically closed field of characteristic p for infinitely many p

Proof: Recall that for finite p,

$$ACF_p = ACF \cup \{\text{characteristic} = p\},$$

and

$$ACF_0 = ACF \cup \{\text{characteristic} \neq p : p \text{ prime}\}$$

Now (2) $\Rightarrow$ (3) is trivial, so we prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

For (1) $\Rightarrow$ (2), assume that $ACF_0 \vdash \sigma$, then there is a finite subcollection of sentences $\text{characteristic} \neq p$ such that

$$ACF \cup \{\text{characteristic} \neq p_i : i \leq n\} \vdash \sigma$$

so we are done.

For (3) $\Rightarrow$ (1), suppose that $ACF_0 \nvdash \sigma$ and $ACF_p \vdash \sigma$ for infinitely many p. Then by completeness $ACF_0 \vdash \neg\sigma$ so by (1) $\Rightarrow$ (2) there exists a prime p_0 such that for all prime numbers $p \geq p_0$ we have $ACF_p \vdash \neg\sigma$ and so we get a contradiction. □

There are some fun consequences of this theorem.

Theorem 9.2 (Ax): If $f : \mathbb{C}^N \to \mathbb{C}^N$ is a map where every coordinate is a polynomial, then if it is injective, then it is surjective.

Proof: Let $\hat{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. We claim that every injective polynomial map $f : \hat{\mathbb{F}}_p^N \to \hat{\mathbb{F}}_p^N$ is surjective. If we have this then by Theorem 9.1 we can transfer this result to $\mathbb{C}$.

Now to prove the claim first note that every polynomial has finitely many coefficients and that

$$\hat{\mathbb{F}}_p = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$$

has the property that every finite subset generates a finite field. Then WLOG we may assume that all the coefficients of f are in $\mathbb{F}_{p^m}$ for some fixed m. Then we get that f induced a map $f_{(n)} : \mathbb{F}_{p^n}^N \to \mathbb{F}_{p^n}^N$ for $n \geq m$. By assumption all $f_{(n)}$ are injective and since these are finite fields they must also be surjective. Hence since $f = \bigcup_{n=m}^{\infty} f_{(n)}$ we get that f is also surjective.

Now to transfer this result to $\mathbb{C}$, fix $d \in \mathbb{N}$ the degree of the polynomial and $N \in \mathbb{N}$ the number of variables. We write $g(\overline{x}, \overline{a})$ the degree d polynomial in $\overline{x}$ with coefficients $\overline{a}$. Then consider the sentence

$$\forall \overline{a}((\forall \overline{x} \forall \overline{y}(g(\overline{x}, \overline{a}) = g(\overline{y}, \overline{a}) \Rightarrow \overline{x} = \overline{y}) \Rightarrow \forall \overline{y} \exists \overline{x}(f(\overline{x}, \overline{c}) = \overline{y}))$$

this sentence encodes exactly the statement of the theorem for polynomials of degree $\leq d$. Hence by Theorem 9.1 since these sentences are true in $\hat{\mathbb{F}}_p$ then they are also true in $\mathbb{C}$. □

Coming back to quantifier elimination, we have an assortment of corollaries stemming from Proposition 8.6.

Corollary 9.1: Let $K < L$ both be algebraically closed fields, if $F(\overline{x})$ is a system of polynomial equations and inequalities with coefficients from K with a solution in L, then the system also has a solution in K.

Proof: Let $\varphi(\overline{y}) = \exists \overline{x} F(\overline{x}, \overline{y})$ where $\overline{y}$ are the coefficients of the polynomials.

By quantifier elimination we have that $\varphi(\overline{y})$ is equivalent to a quantifier free formula $\psi(\overline{y})$. Then if for some choice of coefficients $L \vDash \psi(\overline{c})$ then $K \vDash \psi(\overline{c})$ and so we are done. □

Corollary 9.2 (Weak Hilbert Nullstellensatz): Let K be an algebraically closed field, $f_1, ..., f_n \in K[\overline{x}]$. f_i have a common zero in K^n if and only if $1 \notin (f_1, ..., f_n)$.

Proof: The forward direction is very easy, if they have a common zero then everything in the ideal has that same common zero, so $1 \notin (f_1, ..., f_n)$.

On the other hand if 1 is not in the ideal, let I be a maximal ideal containing $(f_1, ..., f_n)$ then set

$$Z = K[\overline{x}] \,/\, I \qquad L = \hat{Z}.$$

Clearly L is an algebraically closed field containing K. Now in L there are common roots, they are the variables $x_1, ..., x_n$. Hence by Corollary 9.1 we get the desired result. □

We can now apply this to some basic algebraic geometry.

Definition 9.1:

(1) If $S \subseteq K[\overline{x}]$ we set

$$V(S) = \{\overline{a} \in K^n : f(\overline{a}) = 0, \forall f \in S\}$$

(2) If $Y \subseteq K^n$ we set

$$I(Y) = \{f \in K[\overline{x}] : f(\overline{a}), \forall \overline{a} \in Y\}$$

We call a subset V of K^n *Zariski-closed* if $V = V(S)$ for some $S \in K[\overline{x}]$. An ideal is *radical* if it is closed under taking roots.

Proposition 9.3: For all $X, Y \subseteq K^n$

(1) $I(Y)$ is a radical ideal

(2) If X is Zariski-closed, then $X = V(I(X))$.

(3) If $X \subsetneq Y$ and X, Y are Zariski-closed, then $I(Y) \subsetneq I(X)$.

(4) The Zariski-closed sets form a topology, that is they are closed under finite unions and arbitrary intersections. In particular if X, Y are Zariski-closed then

$$X \cup Y = V(I(X) \cap I(Y))$$

and

$$X \cap Y = V(I(X) + I(Y)).$$

Proof: Exercise. □

Theorem 9.4 (Hilbert basis theorem): If K is a field, then $K[\overline{x}]$ is a Noetherian ring. That is, there is no infinite increasing chain of ideals. In particular, every ideal is finitely generated.

Corollary 9.4.1: If K is a field, then there is no infinite decreasing sequence of Zariski-closed sets.

Proof: We apply Theorem 9.4 along with (3) of Proposition 9.3. □

Definition 9.2: An ideal I in a ring is *prime* if

$$a \cdot b \in I \Rightarrow a \in I \text{ or } b \in I$$

Clearly every prime ideal is radical, and we have a sort of converse.

Theorem 9.5 (Primary decomposition): If $I \subseteq K[\overline{x}]$ is a radical ideal, then there are prime ideals $J_1, ..., J_n$ such that

$$I = J_1 \cap ... \cap J_n$$

We can now prove the strong form of Corollary 9.2.

Theorem 9.6 (Hilbert Nullstellensatz strong form): Let K be algebraically closed, if $I \subsetneq J$ and both are radical in $K[\overline{x}]$, then

$$V(J) \subsetneq V(I)$$

Proof: Note that the non-strict inclusion is trivial, the hard part is to prove the strict inclusion. That is, we want to find a common root of I which is not a common root of J. Let $p \in J \setminus I$, we want want to find a point $\overline{b}$ which is a common root of I but $p(\overline{b}) \neq 0$.

We decompose $I = I_1 \cap ... \cap I_n$ into prime ideals and let i be such that $p \notin I_i$. By Theorem 9.4 we have $I_i = (f_1, ..., f_n)$ so we want to find a root of $f_1, ..., f_n$ which is not a root of p. Let $R = K[\overline{x}] / I_i$ then R is an integral domain since I_i is prime, let R_0 be the field of fractions of R and $L = \widehat{R_0}$.

In L consider the system

$$\begin{cases} f_i = 0 : 1 \leq i \leq m \\ p \neq 0 \end{cases}$$

it has a solution in L since $p \neq 0$ in L, and thus by Corollary 9.1 it also has a solution in K and so we are done. □

Corollary 9.6.1: If I is a radical ideal then $I = I(V(I))$.

Proof: Apply Theorem 9.6 to $J = I(V(I))$. □

Lemma 9.7: Let K be a field,

(1) A subset of K^n is definable over K by an atomic formula if and only if it is of the form $V(p)$ for some $p \in K[x_1, ..., x_n]$.

(2) A subset of K^n is definable over K by a quantifier free formula if and only if it is a Boolean combinations of Zariski closed sets.

Proof: (1) is straight forward, as is the forward direction for (2).

For the backward direction of (2), assume that X is a Boolean combination of Z_i for some Zariski-closed family Z_i, then by definition we have

$$Z_i = V(p_1^i) \cap ... \cap V\left(p_{n_i}^i\right)$$

then immediately we have that X is a boolean combination of $V\left(p_j^i\right)$, which are all Zariski closed sets. □

Definition 9.3: A set in K^n is constructible if it satisfies (2) in Lemma 9.7.

Theorem 9.8 (Chevalley): Let K be algebraically closed, the images of constructible sets by polynomial maps are constructible.

Proof: Let $X \subseteq K^n$ be constructible, $p : K^n \to K^m$ be a polynomial map, its image is given by

$$p(X) = \{\overline{y} \in K^n : q(\overline{y}) = \exists \overline{x}(\overline{x} \in X \land \overline{y} = p(\overline{x}))\}.$$

Then since $q(\overline{x})$ is a formula this image is a definable subset of K^m. Since the theory of K has quantifier elimination, $p(\overline{x})$ is definable by a quantifier free formula and thus is constructible. □

10. Homogeneous Structures

Definition 10.1: $\mathcal{M}$ is *κ-homogeneous* if for every subset $A \subseteq \mathcal{M}$ with $|A| < \kappa$, every elementary embedding $f : A \hookrightarrow \mathcal{M}$ and every element $a \in \mathcal{M}$ there is an extension $g : A \cup \{a\} \hookrightarrow \mathcal{M}$ which is also an elementary embedding.

$\mathcal{M}$ is called *homogeneous* if it is $\|\mathcal{M}\|$-homogeneous. $\mathcal{M}$ is *strongly κ-homogeneous* if we have an extension $g : \mathcal{M} \hookrightarrow \mathcal{M}$ of f instead.

One might wonder why we do not similarly define $\mathcal{M}$ to be strongly homogeneous if it is strongly $\|\mathcal{M}\|$-homogeneous. This is explained by the following proposition.

Proposition 10.1: $\mathcal{M}$ is homogeneous if and only if it is strongly $\|\mathcal{M}\|$-homogeneous.

Proof: The backwards direction is immediate, so we prove the forward direction. Assume $f : A \hookrightarrow \mathcal{M}$ is an elementary embedding, then we construct a sequence of maps $f_\alpha : A_\alpha \hookrightarrow \mathcal{M}$. We do this by setting $f_0 = f$ and then taking unions in the limit step and adding the elements of $\mathcal{M} \setminus A$ one by one using homogeneity in the successor step. Then $g \coloneqq \bigcup_\alpha f_\alpha$ will be a map $g : \mathcal{M} \hookrightarrow \mathcal{M}$ as desired. □

Proposition 10.2: If $\mathcal{M} \equiv \mathcal{N}$ are saturated and are of the same cardinality then $\mathcal{M} \cong \mathcal{N}$.

Proof: We prove, as expected, by back and forth. Set $\kappa = \|\mathcal{M}\| = \|\mathcal{N}\|$ and enumerate both models as

$$\mathcal{M} = \{a_\alpha : \alpha < \kappa\} \text{ and } \mathcal{N} = \{b_\alpha : \alpha < \kappa\}.$$

We will construct a partial map $f_\alpha : A_\alpha \to B_\alpha$ with $|f| \leq 2|\alpha|$ such that $f_\alpha \subseteq f_{\alpha+1}$ and $a_\alpha \in A_\alpha, b_\alpha \in B_\alpha$.

We start with the base case of $\alpha = 0$ where $f_\alpha = \varnothing$. For the limit case suppose that f_β is constructed for $\beta < \alpha$, we write

$$f'_\alpha = \bigcup_{\beta < \alpha} f_\beta$$

and we look at the type $p = \operatorname{tp}^{\mathcal{M}}(a_\alpha \,/\, A_\alpha)$. Notice that for any formula $\varphi \in p$ we can replace all the parameters in A_α with their image under f_α, so we can define $f'_\alpha(p)$ which is then a complete type over B_α and thus $f'_\alpha(p) \in S_n(B_\alpha)$. Then set b to be the realization of this type, then we define the extension $f''_\alpha(a_\alpha) = b$. We similarly do the same for the backwards direction, we take a type in $\mathcal{N}$ and map it back to $\mathcal{M}$. Then we set $f_\alpha = f''_\alpha$. □

Theorem 10.3: Suppose $\mathcal{M} \equiv \mathcal{N}$ are homogeneous of the same cardinality, then if $\mathcal{M}, \mathcal{N}$ realize the same complete n-types over the empty set for each n, then $\mathcal{M} \cong \mathcal{N}$.

Before we prove this we need a small lemma

Lemma 10.4: Under the same conditions as Theorem 10.3, for any $A \subseteq \mathcal{M}$, there is some elementary embedding $A \hookrightarrow \mathcal{N}$.

Proof: Induction on $|A|$. If A is finite then since $\mathcal{M}$ and $\mathcal{N}$ realize the same types this is immediate.

If $|A| = \mu \geq \aleph_0$, then we can enumerate $A = \{a_\alpha : \alpha < \mu\}$ and so by a sub-induction on α we construct $f(a_\alpha)$. Suppose that for some fixed α we have constructed $f(a_\beta)$ for $\beta < \alpha$. Then let $A_\alpha = \{a_\beta : \beta \leq \alpha\}$ then by our outer induction hypothesis there exists an elementary embedding $g : A_\alpha \hookrightarrow \mathcal{N}$. Note that we are not done since the g could be incompatible with f, but notice that that $f \circ g^{-1}$ is an embedding $g(A_\alpha \setminus \{a_\alpha\}) \hookrightarrow \mathcal{N}$, so then by homogeneity we can extend this to an elementary embedding $h : g(A_\alpha) \hookrightarrow \mathcal{N}$ and then we set $f(a_\alpha) = h(g(a_\alpha))$. This is the desired extension of f to a_α. □

The intuitive explanation for this proof is that by induction we get a sequence of maps f_α, and then by homogeneity we can arrange the images of the maps so that they sit on top of each other for increasing α, which is enough to construct a limit map.

Now to prove the theorem.

Proof (of Theorem 10.3): We now use a back and forth argument to prove the theorem. We will not delve into the full details here but simply mention that when we want to add an element to the partial isomorphism f_α, we use the above lemma to get a new map g with an extended domain. But then to make that map compatible with the previous maps we can use homogeneity again to align the images so that the image of g sits on top of the images of f_α, then we use that as our extension. □

Definition 10.2: A model $\mathcal{M}$ is called *κ-universal* if for every $\mathcal{N} \equiv \mathcal{M}$ with $\|\mathcal{N}\| \leq \kappa$ there exists an elementary embedding $f : \mathcal{N} \hookrightarrow \mathcal{M}$.

$\mathcal{M}$ is called *universal* if it is $\|\mathcal{M}\|$-universal.

$\mathcal{M}$ is $< \aleph_0$-universal if for every n, $\mathcal{M}$ realizes all types in $S_n^{\mathrm{Th}(\mathcal{M})}(\varnothing)$.

It turns out that we can think of homogeneity and universality as the two halves that together form saturatedness.

Theorem 10.5: If $\mathcal{M}$ is κ-saturated then $\mathcal{M}$ is κ-homogeneous and κ-universal.

Proof: For κ-homogeneity if we have $|A| < \kappa$ and an elementary embedding $f : A \hookrightarrow \mathcal{M}$, then pick any $a \in \mathcal{M}$. We can take the type $p = \mathrm{tp}(a \,/\, A)$ and map it to $q = f(p)$ and define $f(a)$ to be the element that realizes this type in $\mathcal{M}$, which always exists by saturation.

For κ-universality we let $\mathcal{N} \equiv \mathcal{M}$, and $\|\mathcal{N}\| \leq \kappa$. Then we enumerate $\mathcal{N} = \{a_\alpha : \alpha < \kappa\}$, and we construct $f(a_\alpha)$ by induction. We set $p = \mathrm{tp}(a_\alpha \,/ \{a_\beta : \beta < \alpha\})$ and then $q = f(p)$ and so again we just set $f(a_\alpha)$ to be any element which realizes q. □

Theorem 10.6: If $\mathcal{M}$ is κ-homogeneous and $< \aleph_0$-universal then $\mathcal{M}$ is κ-saturated.

Proof: Let $A \subseteq \mathcal{M}$ with $|A| < \kappa$, let $p \in S(A)$, we want to show that $p(M) \neq \varnothing$, we prove this by induction on $|A|$.

First assume that $|A|$ is finite, then let $\mathcal{N}$ be an extension $\mathcal{M} \prec \mathcal{N}$ which realizes the type p through some element c. Then consider the type $q = \mathrm{tp}^{\mathcal{N}}(Ac)$, by $< \aleph_0$ universality we get that $\mathcal{M}$ realizes q through some set A' and element b'. But then by homogeneity since A and A' have the same type, $f : A' \hookrightarrow \mathcal{M}$ defined by $f(A') = A$ is an elementary embedding in $\mathcal{M}$. Then homogeneity gives us that we can extend this to an elementary embedding $g : A' \cup \{b'\} \hookrightarrow \mathcal{M}$, then the image of b' under this map must have $\mathrm{tp}(g(b') \,/\, A) = \mathrm{tp}(b' \,/\, A') = p$.

Next for $|A|$ infinite we use induction, assume that the statement holds for all A' with $|A'| < \mu$ for some cardinal μ, we want to show it holds for $|A| = \mu$. Enumerate $\{a_\alpha : \alpha < \mu\}$, then let p_0 be all the formulas in p that do not use any of the constants in A. Since $\mathcal{M}$ realizes p_0 let b' be a witness of p_0, let $\mathcal{N}$ again be an extension of $\mathcal{M}$ which realizes p with c as a witness.

Now b' and c have the same type over the empty set, so if we consider $\mathrm{tp}^{\mathcal{N}}(a_0 \,/\, c)$ we can replace c by b' in every formula and obtain a type over b' in $\mathcal{M}$. By inductive hypothesis this type will be witnessed by an element a'_0 in $\mathcal{M}$. We then repeat this by induction, assuming we found a'_β for $\beta < \alpha$, then we can consider the type $\mathrm{tp}^{\mathcal{N}}(a_\alpha \,/\, a_{<\alpha} c)$, we again replace $a_{<\alpha} c$ in the parameters by $a'_{<\alpha} b'$ and then we get the element a'_α in $\mathcal{M}$.

We thus obtain $A' = \{a'_\alpha : \alpha < \mu\}$ such that b' satisfies the same formula over A' as c satisfies over A. We then can use homogeneity to map A', b' into $\mathcal{M}$ so that the image of A' is A, then the image of b' is an element b which is a witness to the type p. □

11. Fraïssé Theory

Definition 11.1: Let $\mathcal{M}$ be a countable structure in a countable language L. The *age* of $\mathcal{M}$, written $\mathrm{Age}(\mathcal{M})$, is the family of finitely generated submodels of $\mathcal{M}$. Alternatively $\mathrm{Age}(\mathcal{M})$ is the set of Isomorphism classes of finitely generated L-models that can be embedded into $\mathcal{M}$.

Proposition 11.1:
(1) Hereditary Property (HP) - If $A \in \mathrm{Age}(\mathcal{M})$ and B embeds into A then $B \in \mathrm{Age}(\mathcal{M})$.
(2) Joint Embedding Property (JEP) - If $A, B \in \mathrm{Age}(\mathcal{M})$ then there exists $C \in \mathrm{Age}(\mathcal{M})$, such that A, B both embed into C.

Proof: (1). This is immediate since if $f : B \to A$ is an embedding and $g : A \to \mathcal{M}$ is an embedding then $g \circ f$ is also an embedding.

(2). Let a_i be the generators of A and b_i be the generators of B, then both A and B embed into the model generated by $\{a_i : i \leq n\} \cup \{b_i : i \leq m\}$. □

Proposition 11.2: Let K be a countable class of finitely generated L-structures, if K satisfies HP and JEP as above, then $K = \mathrm{Age}(\mathcal{M})$ for some countable model $\mathcal{M}$.

Proof: Enumerate $K = \{B_1, B_2, ...\}$. By induction construct a sequence $A_n \in K$ such that $A_1 \subseteq A_2 \subseteq ...$ as follows. Start with $A_0 = B_0$, now given A_n we let $A_{n+1} \in K$ be an element such that A_n, B_n both embed into A_{n+1} (using JEP). Now at the end we take

$$\mathcal{M} = \bigcup_{n=1}^{\infty} A_n.$$

It is clear that $\mathrm{Age}(\mathcal{M})$ is contained in K since every finitely generated submodel of $\mathcal{M}$ is a submodel of some A_n and thus is in K. On the other hand for each $B_i \in K$ then it embeds into A_{i+1} and thus into $\mathcal{M}$. □

Definition 11.2: A class K has the amalgamation property (AP) if for every $A, B, C \in K$ such that A embeds into both B and C, then there exists a $D \in K$ such that the following commutative-diagram of embeddings holds.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, dashed] \\ C \arrow[r, dashed] & D \end{tikzcd}$$

D is sometimes called an *amalgam* of B, C over A.

Definition 11.3: A countable structure $\mathcal{M}$ is called *ultrahomogeneous* if every partial isomorphism $A \to B$ between finitely generated substructures $A, B \subseteq \mathcal{M}$ extends to an automorphism $\mathcal{M} \to \mathcal{M}$.

Proposition 11.3: If $\mathcal{M}$ is ultrahomogeneous then $\mathrm{Age}(\mathcal{M})$ satisfies AP.

Proof: Assume we have the following diagram in $\mathrm{Age}(\mathcal{M})$.

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "f_2"] & A \\ B & \end{tikzcd}$$

Both A and B are submodels of $\mathcal{M}$ so let g_1, g_2 be their respective embeddings. Now consider $C_1 = g_1 \circ f_1(C)$ and $C_2 = g_2 \circ f_2(C)$, both are finitely generated submodels of $\mathcal{M}$ isomorphic to each other, and thus by ultrahomogeneity we get a map $h : \mathcal{M} \to \mathcal{M}$ with $h \circ g_1 \circ f_1 = g_2 \circ f_2$.

Now consider $A' = h \circ g_1(A)$ and $B' = g_2(B)$, we have that $D = \langle A', B' \rangle$ is finitely generated and thus is in $\mathrm{Age}(\mathcal{M})$. We now have the diagram

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "f_2"] & A \arrow[d, "h \circ g_1"] \\ B \arrow[r, "g_2"] & D \end{tikzcd}$$

By construction this diagram commutes and so we have the amalgamation property. □

It turns out that the converse to this result is also true.

Theorem 11.4 (Fraïssé): If K satisfies HP, JEP, AP then there exists a unique ultrahomogeneous $\mathcal{M}$ such that $K = \mathrm{Age}(\mathcal{M})$. This is often denoted as $M = \lim K$, and called the Fraïssé limit of K.

Before we start with the proof we need to introduce a bit of theory.

Definition 11.4: A structure $\mathcal{M}$ is weakly homogeneous, if for all finitely generated $A, B \subseteq \mathcal{M}$ with $A \subseteq B$ and all embeddings $f : A \to \mathcal{M}$ we have an extension $g : B \to \mathcal{M}$. Equivalently we have that the following commutative diagram holds

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{M} \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Proposition 11.5: If $\mathcal{M}$ is ultrahomogeneous then $\mathcal{M}$ is weakly homogeneous.

Proof: Let $A \subseteq B$ be as above, then for any embedding $f : A \to \mathcal{M}$, f descends to an isomorphism $f : A \to f(A)$, with both A and $f(A)$ finitely generated. By ultrahomogeneity we then know that f extends to a map $g : \mathcal{M} \to \mathcal{M}$ and then $g|_B$ is an extension of f to a map $B \to \mathcal{M}$. □

One can notice that the definition of weak homogeneity is ideal for extending back and forth maps, as is confirmed in the next proposition.

Lemma 11.6: For $\mathcal{M}, \mathcal{N}$ weakly homogeneous with $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$, every isomorphism $f : A \to B$ between finitely generated substructures $A \subseteq \mathcal{M}$, $B \subseteq \mathcal{N}$, extends to a full isomorphism $g : \mathcal{M} \to \mathcal{N}$.

Proof: We use back and forth, as usual we will attempt to build a map $f_n : A_n \to B_n$, starting with $f_0 : A \to B$. On even induction steps we try to extend the domain of f_n from A_n to A_{n+1} where $A_{n+1} = \langle A_n \cup \{a\} \rangle$ for some element a. Notice that A_{n+1} is in $\mathrm{Age}(\mathcal{M})$ and thus by assumption also in $\mathrm{Age}(\mathcal{N})$. Then by weak homogeneity we have that $f_n : A_n \to \mathcal{N}$ extends into $f_n' : A_{n+1} \to \mathcal{N}$, we then call $B_{n+1} = f_n'(A_{n+1})$ and $f_{n+1} = f_n'$. On odd steps we do the same thing but swap $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 11.6.1: For $\mathcal{M}$ countable, weakly homogeneous is equivalent to ultrahomogeneous.

Proof (of Theorem 11.4): Uniqueness is easily shown since ultrahomogeneity implies weak homogeneity which allows us to apply Lemma 11.6 to the empty isomorphism.

For existence, by the corollary above, it is enough to find a weakly homogeneous countable model $\mathcal{M}$ such that $\mathrm{Age}(\mathcal{M}) = \boldsymbol{K}$.

Enumerate $\boldsymbol{K} = \{B_n : n \in \mathbb{N}\}$, and all pairs of embedding $f_n : B_{k_n} \to B_{\ell_n}$. We want to construct a sequence $A_n \subseteq A_{n+1}$ such that $A_n \in \boldsymbol{K}$ for all n, with the additional property that if we have an embedding $B_{k_n} \to A_m$ for some $m \leq n$ then we also have an extension $B_{\ell_n} \to A_{n+1}$. That is, we want the following diagram to commute

$$\begin{tikzcd} B_{k_n} \arrow[r] \arrow[d] & A_n \arrow[d, dashed] \\ B_{\ell_n} \arrow[r, dashed] & A_{n+1} \end{tikzcd}$$

But this is exactly the commutative diagram for AP so we can find such an A_{n+1}. The union $\bigcup_{n \in \mathbb{N}} A_n$ is then the desired model. □

12. Monster Model

Let κ be a big cardinal (not too large, something of the order $2^{2^{\cdot^{\cdot^{\omega}}}}$). Ideally we would like a saturated model of size κ, but as we saw in Theorem 5.9 this is not possibly to guarantee in ZFC. Instead, in practice, we often use a κ-saturated model which is κ-strongly homogeneous.

Theorem 12.1 (Monster Model): For $\kappa \geq \aleph_0$, T complete and L countable, there exists a model $\mathfrak{C} \vDash T$ which is κ-saturated and κ-strongly homogeneous.

Before we prove this we will need a tiny lemma.

Lemma 12.2: For all $\mathcal{N} \vDash T$ there exists an elementary extension $\mathcal{N} \prec \mathcal{N}'$, such that

- For all $A \subseteq \mathcal{N}$ with $|A| < \kappa$ all of $S(A)$ are realized in $\mathcal{N}'$
- For all $f : A \hookrightarrow B$ elementary embedding between two subsets $A, B \subseteq \mathcal{N}$ with $|A|, |B| < \|\mathcal{N}\|$, f can be extended to $f' : A' \hookrightarrow B'$ also an elementary embedding with $A \cup \mathcal{N} \subseteq A'$ and $B \cup \mathcal{N} \subseteq B'$.

Proof: Let $\mu = \|\mathcal{N}\|$, then for the first condition we simply pick $\mathcal{N}'$ which is μ^+-saturated through Theorem 5.9.

Now assume that we have an embedding $f : A \hookrightarrow B$, since $\mathcal{N}'$ is μ-saturated we can, by a simple inductive argument, construct an extension $g : \mathcal{N} \hookrightarrow \mathcal{N}'$. Now the issue here is that $g(\mathcal{N})$ might contain $\mathcal{N}$ which is required by the lemma.

To fix this set $\mathcal{N}_0 = g(\mathcal{N})$, and construct $h : \mathcal{N}_0 \cup \mathcal{N} \hookrightarrow \mathcal{N}'$ extending g^{-1}. We then can set $A' = h(\mathcal{N}_0 \cup \mathcal{N})$ and $B' = \mathcal{N}_0 \cup \mathcal{N}$. Then $h^{-1} : A' \hookrightarrow B'$ is an extension of f as desired. □

Proof (of Theorem 12.1): We will construct an elementary chain $\mathcal{N}_\alpha$ with $\alpha < \kappa^+$, $\mathcal{N}_\alpha \vDash T$.

- $\mathcal{N}_0$ can be arbitrary
- For limit α we will have $\mathcal{N}_\alpha = \bigcup_{\gamma < \alpha} \mathcal{N}_\gamma$
- For $\alpha + 1$ we will have $\mathcal{N}_{\alpha+1}$ be the extension of $\mathcal{N}_\alpha$ provided by Lemma 12.2.

We then set $\mathfrak{C} = \bigcup_{\alpha < \kappa^+} \mathcal{N}_\alpha$, note that since κ^+ is regular then $\mathrm{cf}(\kappa^+) = \kappa^+$.

Now we check κ-saturation, if $A \subseteq \mathcal{M}$, $|A| < \kappa$, $p \in S(A)$, we know that $A \subseteq \mathcal{N}_\alpha$ for some α by regularity of κ. So we also know that p is realized in $\mathcal{N}_{\alpha+1}$ and thus since this is an elementary chain p is realized through the same element in $\mathfrak{C}$.

Next we check κ-homogeneity, if $A, B \in \mathfrak{C}$, $|A|, |B| < \kappa$, $f : A \hookrightarrow B$, there is α such that $A, B \subseteq \mathcal{N}_\alpha$ again by regularity. We then fix $f = f_\alpha$ and use Lemma 12.2 to extend f_α to $f_{\alpha+1}$, we repeat this for successor steps and for limit steps we union. We can continue doing this for all α to extend f to an automorphism $g : \mathfrak{C} \hookrightarrow \mathfrak{C}$. □

Remark: Since $\mathfrak{C}$ has so many automorphisms, it is often useful to consider, for any subset $A \subseteq \mathfrak{C}$ with $|A| \subseteq \kappa$, the group $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ consisting of automorphisms of $\mathfrak{C}$ that fix A. This group naturally acts on $\mathfrak{C}^n$.

One can easily show that for any two tuples $\overline{x}, \overline{y}$, they are in the same orbit of $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ if and only if $\mathrm{tp}(\overline{x} \,/\, A) = \mathrm{tp}(\overline{y} \,/\, A)$. So orbits of $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ are equivalent to realizations of $S(A)$.

13. Indiscernibles

Definition 13.1: Let $(I, \leq)$ be a linear order, a set $A = \{a_i : i \subseteq I\} \subseteq \mathcal{M}$ is called *order-indiscernible* if for all formulas $\varphi(x_1, ..., x_n)$ and linear suborders $\forall i_i < ... < i_n, j_1 < ... < j_n \in I$ we have

$$\mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right) \Leftrightarrow \mathcal{M} \vDash \varphi\left(a_{j_1}, ..., a_{j_n}\right)$$

In other words, the type of a finite subtuple of elements a_k depends only on their order.

Example: DLO_0 with $A = \mathcal{M} = I = \mathbb{Q}$ is order indiscernible.

If $K > L$ are algebraically closed fields with K transcendental over L, then a transcendental basis of K over L is also an example.

Any basis of an infinite dimensional vector space.

Theorem 13.1 (Ehrenfeucht–Mostowski): Let T be a theory with infinite models, I arbitrary, there exists a model $\mathcal{M} \vDash T$ with an order-indiscernible sequence $(a_i : i \in I)$ of infinite cardinality.

Proof: Let $(c_i : i \in I)$ be new constants and set $L' = L \cup \{c_i : i \in I\}$. We want to show that the augmented theory

$$T' \coloneqq T \cup \left\{\varphi\left(c_{i_1}, ..., c_{i_n}\right) \leftrightarrow \varphi\left(c_{j_1}, ..., c_{j_n}\right)\right\} : \varphi \in F_L, i_1 < ... < i_n, j_1 < ... < j_n\}$$

is consistent.

We use compactness, let $S \subseteq F_L$ be a finite subset of formulas, then we define the theory

$$T'' = T \cup \left\{\varphi\left(c_{i_1}, ..., c_{i_n}\right) \leftrightarrow \varphi\left(c_{j_1}, ..., c_{j_n}\right)\right\} : \varphi \in S, i_1 < ... < i_n, j_1 < ... < j_n\}$$

and we will construct a model of T''.

Lemma 13.2: If $A \subseteq \mathbb{N}$, $\{a_n : n \in A\} \subseteq \mathcal{M}$, and φ a formula. There exists $B \subseteq A$ infinite such that for any sequences $i_1 < ... < i_n, j_1 < ... < j_n \in B$ we have

$$\mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right) \Leftrightarrow \mathcal{M} \vDash \varphi\left(a_{j_1}, ..., a_{j_n}\right)$$

Proof: Ramsey's theorem says that if we color ordered n-tuples of A infinite, then there exists an infinite subset $B \subseteq A$ such that all ordered tuples of B have the same color.

We define the coloring

$$C\left(a_{i_1}, ..., a_{i_n}\right) = \begin{cases} 1 \text{ if } \mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right), \\ 0 \text{ otherwise} \end{cases}$$

then the infinite set given by Ramsey's theorem is exactly the subset B we want.□

Applying the Lemma for each formula in S we get that any model of T satisfies T'' and so T'' is consistent and so by compactness so is T'. $\square$

Corollary 13.2.1: Let T be a theory with infinite models, then there is a model $\mathcal{M} \vDash T$ with

$$\mathrm{Aut}(\mathbb{Q}, <) < \mathrm{Aut}(\mathcal{M})$$

where the second $<$ means subgroup.

Proof: Consider the Skolemization T^S of T, use Theorem 13.1 to get a model $\mathcal{M}^S \vDash T^S$ with an order-indiscernible sequence $\left(a_q : q \in \mathbb{Q}\right)$.

Set $\mathcal{N}^S$ be the smallest submodel containing $\left(a_q : q \in \mathbb{Q}\right)$ then $\mathcal{N}^S \prec \mathcal{M}^S$ since with Skolemization we have quantifier elimination. Let $\mathcal{N}$ be a reduct of $\mathcal{N}^S$ to L.

If $\varphi \in \mathrm{Aut}(\mathbb{Q}, <)$ we want to define $\overline{\varphi} \in \mathrm{Aut}(\mathcal{N})$. Since everything in $\mathcal{N}$ has the form of a Skolem term in the elements a_q we can define

$$\overline{\varphi}\left(a_q\right) = a_{\varphi(q)}$$

on the generating elements, and

$$\overline{\varphi}\left(t\left(a_{i_1}, ..., a_{i_n}\right)\right) = t\left(a_{\varphi(i_1)}, ..., a_{\varphi(i_n)}\right)$$

on terms.

The fact that $\overline{\varphi}$ is well defined and an automorphism will follow from the fact that $\left(a_q : q \in \mathbb{Q}\right)$ is order-indiscernible.

One can then easily check that the map $\varphi \mapsto \overline{\varphi}$ is an embedding

$$\mathrm{Aut}(\mathbb{Q}, <) \to \mathrm{Aut}(\mathcal{N}),$$

which gives us the subgroup relation. $\square$

Definition 13.2: A model $\mathcal{M}$ is an *Ehrenfeucht–Mostowski model* if there exists an infinite order-indiscernible sequence $(a_i : i \in I) \subseteq \mathcal{M}$ such that $\mathcal{M}$ is generated by $(a_i : i \in I)$. This generating sequence is called the *spine* of $\mathcal{M}$.

Theorem 13.3: Let $\mathcal{M}$ be an Ehrenfeucht-Mostowski model with theory $T = \mathrm{Th}(\mathcal{M})$ over a countable language L.
- The number of types in $S_n^T(\varnothing)$ realized in $\mathcal{M}$ is countable.
- If I is well-ordered, then for all $A \subseteq \mathcal{M}$ the number of types in $S_n^T(A)$ realized in $\mathcal{M}$ is at most $|A| + \aleph_0$.

Proof: For simplicity we restrict to $n = 1$, the argument for other n is nearly exactly the same. Now $\forall a \in \mathcal{M}$ we can write a as a term

$$a = t\left(a_{i_1}, ..., a_{i_n}\right), \quad i_1 < ... < i_n.$$

Assume then that two elements a and a' are assigned the same term but with different input sequences a_{i_j} and $a_{i'_j}$ (though with the same order), then the assumption that a_i are order-indiscernible gives us that $\mathrm{tp}^{\mathcal{M}}(a) = \mathrm{tp}^{\mathcal{M}}(a')$. Thus we have a well-defined surjective map

$$\text{terms} \to \text{types realized}$$

so since our language is countable the number of terms is countable and hence we get the first statement of the theorem.

For the second statement, we again assume $n = 1$, and we notice that for any term $t\left(a_{i_1}, ..., a_{i_n}\right)$ if we we have a, a' both equal to this term for different sequences of inputs, then they have the same type over A only if we have an automorphism taking the arguments of a to the arguments of a' while fixing A.

Now since every element of A can be written as a term $t\left(a_{i_1}, ..., a_{i_n}\right)$ for some a_i we can replace the parameters in any formulas with said terms and hence assume that the set of parameters A is a subset of $\{a_i : i \in I\}$.

With that assumption the desired automorphism exists if the arguments a_{i_k} and $a_{i'_k}$ have the same relative positions to A. But since by assumption I is well ordered, so is A, so a relative position to A can be entirely encoded by the minimal element larger than that position. Hence the cardinality of relative positions to A is at most the cardinality of A (if A is infinite). □

Definition 13.3: For T a complete theory, L-countable and κ a cardinal.
- T is called *κ-stable* if for any model $\mathcal{M} \vDash T$ and all subsets $A \subseteq \mathcal{M}$, $|A| \leq \kappa$, we have $|S_1(A)| \leq \kappa$.
- T is stable if it is κ-stable for some κ.
- T is *totally transcendental* (TT) if it is κ-stable for *all* κ.

Example: DLO_o is not $\aleph_0$-stable since $S_1(\mathbb{Q}) \cong \mathbb{R}$ which has larger cardinality than $\aleph_0$.

By applying Theorem 13.1 we get that any theory T has an Ehrenfeuchet-Mostowski model, and by Theorem 13.3 we get that that model is totally transcendental.

Theorem 13.4: T is totally transcendental if and only if it is $\aleph_0$-stable.

Proof: One direction is obvious, for the other direction we prove by contrapositive. Suppose T is not κ stable for some κ, that is, there is a model $\mathcal{M} \vDash T$ with a subset $A \subseteq \mathcal{M}$, $|A| = \kappa$ satisfying $|S_1(A)| > \kappa$.

We now want to find a countable subset $A_0 \subseteq A$ such that $S_1(A_0) \geq 2^{\aleph_0}$. For the rest of this proof we will call a formula φ **big** if the neighborhood $[\varphi] \subseteq S_1(A)$ has size $|[\varphi]| > \kappa$.

Lemma 13.5: If φ is big, then there exists ψ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are big.

Proof: Assume for a contradiction that for each ψ either $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ are not big, since φ is big one of the two must be big and then WLOG we can assume $\varphi \wedge \psi$ is not big and $\varphi \wedge \neg\psi$ is big.

Consider then the set of formulas,

$$p = \{\varphi \wedge \psi : \varphi \wedge \psi \text{ is big}\} \in S_1(A),$$

For any finite set of formulas ψ_i such that $\varphi \wedge \psi_i$ are all in p we know that $\varphi \wedge (\neg\psi_i)$ are all small so $|[\varphi \wedge \neg\psi_i]| \leq \kappa$ hence

$$\bigcup_{i\leq n}[\varphi \wedge \neg\psi_i] = \left[\bigvee_{i\leq n} \varphi \wedge (\neg\psi_i)\right] = \left[\varphi \wedge \neg \bigwedge_{i\leq n} \psi_i\right]$$

is also of cardinality at most κ so $\neg \bigwedge_{i\leq n} \psi_i$ is not big and so $\bigwedge_{i\leq n} \psi_i$ is big. In particular this set is never empty and thus p is finitely satisfiable, and corresponds to a type p.

Now $[\varphi] \setminus \{p\}$ is an open set in $S_1(A)$ and thus is the union of its open subsets. These open subsets are exactly formulas that imply φ but are not in p, that is

$$[\varphi] \setminus \{p\} = \bigcup_{\psi \notin p}[\psi] = \bigcup_{\psi : \varphi \wedge \psi \text{ is not big}} \psi.$$

We now can write

$$[\varphi] = \bigcup_{\psi : \varphi \wedge \psi \text{ is not big}} [\varphi \wedge \neg\psi] \cup \{p\},$$

and so since this is a union of countable subsets each of cardinality at most κ, then $[\varphi]$ has size κ, which is a contradiction since we assumed that φ is big. □

Now using this lemma we can keep splitting $S_1(A)$ into a tree of formulas φ_s such that at each level all the formulas φ_s are big (see Proposition B.1 for detail).

Now let A_0 be the set of parameters in the formulas φ_s, this set is countable since there are countably many formulas. For every branch $x \in 2^\omega$ the set of we get by following a branch down the tree gives a type which we can complete to a type in $S_1(A_0)$. These types are not equal for different branches, and there are $2^{\aleph_0}$ many branches so $S_1(A_0)$ has cardinality at least $2^{\aleph_0}$. □

14. Ranks in Topological Spaces

Definition 14.1: We define the *Cantor-Bendixson derivative* as

$$X' = X \setminus \{\text{isolated points of } X\}.$$

We then define X^α by induction on α,
- $X^{\alpha+1} = (X^\alpha)'$.
- $X^\gamma = \bigcap_{\beta<\gamma} X^\beta$.

Definition 14.2: If X is separable then $\exists \alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$. The minimal α at which this stabilizes is called the *Cantor-Bendixson rank*, often written as

$$\mathrm{CB}(X) = \min\{\alpha : X^\alpha = X^{\alpha+1}\}.$$

The left over after removing these isolated points is called the *perfect kernel*, written as

$$X^\infty = \bigcap_{\alpha < \mathrm{CB}(X)+1} X^\alpha$$

and it is, as the name suggests, perfect, as in it does not have any isolated points.

Definition 14.3: As a bit of abuse of notation we also define the function

$$\mathrm{CB} : X \to \mathrm{Ord} \cup \{\infty\}$$

as

$$\mathrm{CB}(p) = \begin{cases} \min(\alpha : p \notin X^\alpha) & : p \notin X^\infty \\ \infty & : p \in X^\infty \end{cases}$$

Definition 14.4: Assume that X is a 0 dimensional space, that is

$$\mathrm{Clop}(X) = \{U \subseteq X : U \text{ is clopen}\}$$

forms a basis.

We define (again with abuse of notation) $\mathrm{CB} : \mathrm{Clop}(X) \to \mathrm{Ord} \cup \{\infty\}$, by induction

- $\mathrm{CB}(U) \geq 0$ if $U \neq \varnothing$ and $\mathrm{CB}(\varnothing) = -1$.
- $\mathrm{CB}(U) \geq \alpha + 1$ if $\forall n$ we can find $V_1, ..., V_n$ disjoint clopen subsets of of U with $\mathrm{CB}(V_i) \geq \alpha$.
- $\mathrm{CB}(U) \geq \gamma$ if $\forall \beta < \gamma$ we have $\mathrm{CB}(U) \geq \beta$.

Example: $X = \alpha + 1$ for α some ordinal, we can define a topology by setting $(\gamma, \beta]$ to be the open basis.

Write α in its Cantor normal form

$$\alpha = \omega^{\alpha_0} k_0 + ... + \omega^{\alpha_\ell} k_\ell$$

then $\mathrm{CB}(\alpha + 1) = \alpha_0$.

Proposition 14.1: For X being 0-dimensional compact space and U, V clopen subsets.

(1) $\mathrm{CB}(U) = 0$ if and only if U is finite.
(2) If $U \subseteq V$ then $\mathrm{CB}(U) \leq \mathrm{CB}(V)$.
(3) $\mathrm{CB}(U \cup V) = \max\{\mathrm{CB}(U), \mathrm{CB}(V)\}$.
(4) $\mathrm{CB}(U \geq \alpha + 1)$ if and only if there is an infinite sequence $V_n \subseteq U$ which is open and disjoint with $\mathrm{CB}(V_i) \geq \alpha$.
(5) If $p \in X$ then $\mathrm{CB}(p) = \min\{\mathrm{CB}(U) : p \in U \text{ with } U \text{ clopen}\}$.
(6) $\mathrm{CB}(U) = \max\{\mathrm{CB}(p) : p \in U\}$.

Proof:
(1) Exercise.
(2) Exercise.
(3) One direction is clear by 2, for the other we prove by induction that if $\mathrm{CB}(U \cup V) \geq \alpha$ then either $\mathrm{CB}(U) \geq \alpha$ or $\mathrm{CB}(V) \geq \alpha$, we leave induction step as exercise.
(4) One direction is again clear, for the other we assume that $\mathrm{CB}(U) \geq \alpha + 1$ then we can find two disjoint U_1, U_2 clopen subsets of U with $\mathrm{CB}(U_1), \mathrm{CB}(U_2) \geq \alpha$. We can then enlarge U_2 to $U \setminus U_1$ and then by 3 we know that one of U_1, U_2 has $\mathrm{CB}(U_i) = \alpha + 1$ so we can repeat this splitting again on that U_i. Doing this by induction we get a sequence of U_i with $\mathrm{CB}(U_i) \geq \alpha$ as desired.
(5) Exercise.
(6) Exercise.

□

With these topological preliminaries out of the way we can apply them to Model Theory, namely noticing that $S_n(A)$ for any A is always a 0-dimensional compact Hausdorff space.

15. Morley Rank

Let T be a complete theory and $\mathcal{M} \vDash T$ an $\aleph_0$-saturated model.

Definition 15.1: We define the *Morley Rank* as a function

$$\mathrm{RM} : \mathrm{Def}_{\mathcal{M}(\mathcal{M})} \to \mathrm{Ord} \cup \{\infty\}$$

where $\mathrm{Def}_{\mathcal{M}(\mathcal{M})}$ are the definable sets with $\mathcal{M}$ as parameters, we define it through

$$\mathrm{RM}(\varphi) = \mathrm{CB}([\varphi] \cap S_1(\mathcal{M}))$$

Proposition 15.1:
(1) $\mathrm{RM}(\varphi) = 0$ if and only if $\varphi(\mathcal{M})$ is finite (we sometimes call this φ being *algebraic*).
(2) If $\varphi \vdash \psi$ then $\mathrm{RM}(\varphi) \leq \mathrm{RM}(\psi)$.
(3) $\mathrm{RM}(\varphi \vee \psi) = \max(\mathrm{RM}(\varphi), \mathrm{RM}(\psi))$.

Proof: Follows from Proposition 14.1 with a little effort, details left as Exercise. □

Definition 15.2: For a type $p \in S_1(\mathcal{M})$ we define

$$\mathrm{RM}(p) = \min\{\mathrm{RM}(\varphi) : \varphi \in p\}$$

Proposition 15.2: $\mathrm{RM}(p) = \mathrm{CB}(p)$ where we see p as a point in $S_1(\mathcal{M})$.

Proof: Directly from Proposition 14.1 point 5. □

Since the definitions heavily depend on $\mathcal{M}$ it is natural to ask whether we can say anything about how these properties change when we change the model. In fact we can, and this is formalized in the following proposition.

Proposition 15.3: If φ is a formula with parameters in $\mathcal{M}$, $\mathcal{M}$ is $\aleph_0$ saturated and $\mathcal{M} \prec \mathcal{N}$ is an $\aleph_0$-saturated extension then

$$\mathrm{RM}^{\mathcal{M}}(\varphi) = \mathrm{RM}^{\mathcal{N}}(\varphi)$$

Proof: Exercise on assignment, will add when due date is passed. □

By convention we usually define RM inside the Monster Model, since we can easily embed other models into it.

Notice that we can extend this definition to not complete types, we do this through

$$\mathrm{RM}(p) = \min\{\mathrm{RM}(\varphi) : \varphi \in p\}.$$

Proposition 15.4: If p is a type over A, then there is a complete q extending p with

$$\mathrm{RM}(p) = \mathrm{RM}(q)$$

Proof: We know that p corresponds to a closed set of $S_1(A)$, we then consider the collection of formulas

$$q_0 = \{\neg\varphi : \mathrm{RM}(p \cup \{\varphi\}) < \mathrm{RM}(p)\}$$

one can check that q_0 is also a type which extend p.

Any q completing q_0 is an extension of p with correct rank. □

Theorem 15.5: Let T be a complete theory, T is totally transcendental if and only if $\mathrm{RM}(x = x) < \infty$.

Remark: This is actually equivalent to $\mathrm{RM}(x = x) < \omega_1$, this is left as an exercise.

Proof (of Theorem 15.5): Suppose that $\mathrm{RM}(x = x) = \infty$, then there is some ordinal β such if $\mathrm{RM}(\varphi) > \beta$ then $\mathrm{RM}(\varphi) = \infty$ (since we can never have arbitrarily large ranks).

We will now construct a tree of formulas φ_n indexed by $n \in 2^{<\omega}$, we start with $\varphi_\varnothing = (x = x)$ and continue by noticing that $\mathrm{RM}(\varphi_\varnothing) > \beta + 1$ implies that we can find two formulas $\varphi_0$and φ_1 with $\mathrm{RM}(\varphi_0), \mathrm{RM}(\varphi_1) > \beta$ and hence we also have $\mathrm{RM}(\varphi_0) = \mathrm{RM}(\varphi_1) = \infty$. We then continue this and keep splitting formulas φ_n to get a tree of non empty formulas.

Morally the construct of this tree is using the fact that Moraly rank 'stabilizes' in a very similar way as Scott rank.

There are then at least $2^{\aleph_0}$ leaves in this tree which correspond to at least $2^{\aleph_0}$ types over the set of parameters of all φ_n which is a countable set.

For the other direction assume that T is not transcendental and that $\mathrm{RM}(x = x) < \infty$, then we can construct a similar tree as in the proof of Theorem 13.4. Let $\alpha = \inf(\mathrm{RM}(\varphi_n) : 2^{<\omega})$, then if $\mathrm{RM}(\varphi_n) = \alpha$ then we can expand the tree starting from φ_n to get arbitrarily large collections of disjoint formulas φ_i that each have rank at least α, this then implies that $\mathrm{RM}(\varphi_n) = \alpha + 1$, and so the infimum is actually at least $\alpha + 1$, which is a contradiction. □

Definition 15.3: $\{\overline{a}_i : i \in I\} \subseteq \mathcal{M}$ is called *indiscernible*, if for any two sequences of tuples $i_1 \neq \dots \neq i_n \subseteq I$ and $j_1 \neq \dots \neq j_n \subseteq I$ we have

$$\mathrm{tp}\left(\overline{a}_{i_1}, \dots, \overline{a}_{i_n}\right) = \mathrm{tp}\left(\overline{a}_{j_1}, \dots, \overline{a}_{j_n}\right)$$

Theorem 15.6: If T is stable then every order indiscernible sequence is indiscernible.

Proof: Let κ be such that T is κ-stable, then assuming, aiming for a contradiction, that $(a_i : i \in I)$ is order indiscernible but not indiscernible. WLOG we may assume that I has a dense subset J of size $\leq \kappa$ and that every non-empty interval has size at least κ.

By assumption we have a finite sequence $1, \dots, n \in I$ (we will write them as integer for simplicity) and a permutation σ such that

$$\mathrm{tp}(a_1, \dots, a_n) \neq \mathrm{tp}\left(a_{\sigma(1)}, \dots, a_{\sigma(n)}\right),$$

namely for some formula φ we have

$$\vDash \varphi(a_1, \dots, a_n) \text{ and} \vDash \neg\varphi\left(a_{\sigma(1)}, \dots, a_{\sigma(n)}\right)$$

then by writing $\sigma = \tau_1 \dots \tau_m$ where τ_i are each transpositions of consecutive integers, we notice that by considering the partial products $\tau_1 \dots \tau_j$ we know that at some j φ flips from being true to not true and hence we can reduce this to the case of one such transposition. That is

$$\vDash \varphi(a_1, \dots, a_{i-1}, a_i, a_{i+1}, a_{i+2} \dots, a_n)$$

and

$$\vDash \neg\varphi(a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2} \dots, a_n).$$

Now let $A = \left\{a_j : j \in J\right\} \cup \left\{a_1, \dots, a_{i-1}, a_{i+2}, \dots, a_n\right\}$, we can now show that for any $i' < i'' \in (i, i+1)$ (interval inside the ordering of I not in the integers) we have

$$\mathrm{tp}(a_{i'} \,/\, A) \neq \mathrm{tp}(a_{i''} \,/\, A)$$

Let $j \in J$ be such that $i' < j < i''$, then consider the formula

$$\chi(x, y) = \varphi(a_1, ..., a_{i-1}, x, y, a_{i+2}, ..., a_n)$$

then we have by assumption $\vDash \chi(a_i, a_{i+1})$ and $\vDash \neg\chi(a_{i+1}, a_i)$ so by order indiscernibility we have

$$\vDash \chi(i', j), \vDash \neg\chi(i'', j)$$

□

Proposition 15.7: If X is a compact space and $U \subseteq X$ with $\mathrm{CB}(U) = \alpha$, then there exists n such that if $U_1, ..., U_k \subseteq U$ which are disjoint with $\mathrm{CB}(U_i) \geq \alpha$ then $k \leq n$.

Proof: Directly by definition. □

Definition 15.4: We call the minimal such n in Proposition 15.7 the CB-degree of U.

Definition 15.5: If φ is a formula over A with $A \subseteq \mathfrak{C}$. The *Morley degree* of φ is the CB-degree of $[\varphi] \subseteq S(\mathcal{M})$, where $\mathcal{M}$ is any $\aleph_0$-saturated model containing A.

We denote it $\deg_{\mathcal{M}}(\varphi)$.

Proposition 15.8: $\deg_{\mathcal{M}}(\varphi)$ does not depend on $\mathcal{M}$.

Proposition 15.9: If φ has Morley degree n, then there exists formulas $\varphi_1, ..., \varphi_n$ all with $\deg(\varphi) = 1$ such that

$$[\varphi] = [\varphi_1] \cup ... \cup [\varphi_n]$$

Proposition 15.10:
- If $\mathrm{RM}(\varphi_1) = \mathrm{RM}(\varphi_2) < \infty$ and $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \varnothing$ then
$$\deg(\varphi_1 \vee \varphi_2) = \deg(\varphi_1) + \deg(\varphi_2)$$
- If $\mathrm{RM}(\varphi_1) < \mathrm{RM}(\varphi_2)$ then
$$\deg(\varphi_1 \vee \varphi_2) = \deg(\varphi_2)$$

Proof: Exercise. □

As usual we extend these definitions to types

Definition 15.6: If p is a type then its *degree* is defined as

$$\deg(p) = \min\{\deg(\varphi) : p \vdash \varphi \wedge \mathrm{RM}(\varphi) = \mathrm{RM}(p)\}$$

A type is stationary if $\deg(p) = 1$.

16. Algebraic and Definable Closure

Definition 16.1: We say that a formula $\varphi(\overline{x}, \overline{a})$ is *algebraic* if $\varphi(\mathfrak{C})$ is finite.

We say a type p is *algebraic* if $p(\mathfrak{C})$ is finite.

The *algebraic closure* $\mathrm{acl}(A)$ of a set A is the set of tuples $\overline{a}$ such that $\mathrm{tp}(\overline{a} \,/\, A)$ is algebraic. We sometimes think of it as a subset of the model $\mathfrak{C}$.

The *definable closure* $\mathrm{dcl}(A)$ of a set A is the set of tuples $\overline{a}$ such that $\mathrm{tp}(\overline{a} \,/\, A)$ has a unique realization.

Remark: We do not call formulas or types with a unique realization "definable formulas" or "definable types" because those terms are already in use in model theory for something else.

Proposition 16.1:
(1) p is algebraic if and only if there is a formula implied by p which is algebraic.
(2) $|p(\mathfrak{C})| = 1$ if and only if there is a formula implied by p such that $|\varphi(\mathfrak{C})| = 1$.

Proof: The backwards direction is immediate since the any realization of p is also a realization of φ.

On the other hand assume that p is algebraic, then we know that the collection of formulas

$$\{\overline{x}_1 \neq \overline{x}_2 : m \neq n\} \cup \bigcup_{n=1}^{\infty} \{\varphi(\overline{x}_n) : \varphi \in p\}.$$

is not realizable, since that would contradict the fact that it is algebraic. Hence by compactness some finite subcollection of these formulas is also not realizable. But then we have some formulas $\varphi_1, ..., \varphi_n$ which are not consistent with $\{\overline{x}_1 \neq \overline{x}_2 : m \neq n\}$, we then have

$$\Phi := \varphi_1(\overline{x}) \wedge ... \wedge \varphi_n(\overline{x})$$

is also not consistent with $\{\overline{x}_1 \neq \overline{x}_2 : m \neq n\}$ and so Φ is formula implied by p which has finitely many realizations.

For 2 we do the exact same thing. $\square$

Proposition 16.2: If $\mathcal{M} \prec \mathfrak{C}$ arbitrary with $\varphi(\overline{x})$ a formula over $\mathcal{M}$. φ is algebraic if and only if $\varphi(\mathcal{M})$ is finite.

Proof: Forward direction is obvious.

Assume that $\mathcal{M}$ has at most n many realizations of φ, then we can write the formula

$$\exists \overline{x}_1 \overline{x}_2 ... \overline{x}_n \forall y \left(\bigwedge_{i=1}^{n} y \neq x \right) \Rightarrow \neg\varphi(y)$$

and since it is not true in $\mathcal{M}$ it is not true in $\mathfrak{C}$, hence $\mathfrak{C}$ also has at most n many realizations. □

Proposition 16.3: $\operatorname{tp}(ab \,/\, A)$ is algebraic if and only if $\operatorname{tp}(a \,/\, A)$ and $\operatorname{tp}(b \,/\, Aa)$ are algebraic.

Proof: Let $p = \operatorname{tp}(ab \,/\, A)$, $q = \operatorname{tp}(a \,/\, A)$, $p_a = \operatorname{tp}(b \,/\, Aa)$.

Assume that p is algebraic, we want to show that q is algebraic, we have a map $p(\mathfrak{C}) \to q(\mathfrak{C})$ defined by forgetting the second coordinate of the tuple. It is onto because if $a' \in q(\mathfrak{C})$ then by homogeneity of $\mathfrak{C}$ we get that there is a b' such that $a'b'$ has the same type as ab and so $a'b' \in p(\mathfrak{C})$ and projects onto a'. Hence $q(\mathfrak{C})$ is finite.

We also see that $p_a(\mathfrak{C})$ is finite since if there were infinitely many realizations of it then the pairs (a, b_i) would all be unique in $p(\mathfrak{C})$ which would contradict the fact that p is algebraic.

On the other hand assume that q, p_a are algebraic, then by homogeneity we know that $|p_{a'}(\mathfrak{C})| = |p_a(\mathfrak{C})|$ and so since

$$p(\mathfrak{C}) = \cup_{a' \in q(\mathfrak{C})} \{a'\} \times p_{a'}(\mathfrak{C})$$

we get that $p(\mathfrak{C})$ is finite. □

Proposition 16.4:
1. $A \subseteq\subseteq \operatorname{acl}(A)$.
2. If $A \subset B$ then $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$.
3. $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$.
4. $\operatorname{acl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ A_0 \text{ finite}}} \operatorname{acl}(A_0)$.
5. $A \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$.
6. If A is a subset of B then $\operatorname{dcl}(A) \subseteq \operatorname{dcl}(B)$.
7. $\operatorname{dcl}(\operatorname{dcl}(A)) = \operatorname{dcl}(A)$
8. $\operatorname{dcl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ A_0 \text{ finite}}} \operatorname{dcl}(A_0)$.

Properties 1,2,3 and 5,6,7 are sometimes shortened to "acl/dcl is a monotone idempotent operator".

Proof: All are trivial apart from 3,7.

For 3 first let $a \in \operatorname{acl}(\operatorname{acl}(A))$, by Proposition 16.1 we get that there is a formula $\varphi\left(\overline{x}, \overline{b}\right)$ with parameters $\overline{b} \in \operatorname{acl}(A)$ such that $|\varphi\left(x, \overline{b}\right) < \infty$ and so that $\vDash \varphi\left(a, \overline{b}\right)$.

$\operatorname{tp}\left(a \,/\, A\overline{b}\right)$ is algebraic and $\operatorname{tp}\left(\overline{b} \,/\, A\right)$ is algebraic so by Proposition 16.3 we get that $\operatorname{tp}\left(a\overline{b} \,/\, A\right)$ is algebraic so by Proposition 16.3 again $\operatorname{tp}(a \,/\, A)$ is algebraic. □

Example: Let $T = ACF_p$, for a set A, (A) denotes the field generated by A, we then have.

$$\operatorname{acl}(A) = \text{algebraic closure of } (A).$$

We have $a \in \operatorname{acl}(A)$ if and only if $I(a / \langle A \rangle) \neq 0$.

The defined closure is more interesting, it is

$$\operatorname{dcl}(A) = \begin{cases} (A) \text{ if } p = 0 \\ \widehat{(A)}^{\text{rad}} \text{ if } p > 0 \end{cases}$$

one would expect it to always be (A) but in positive characteristic we can also take p-th roots because of the properties of the Frobenious map.

Often in math, a closure operator has another interesting property which we have not talked about before, called the exchange property. A familiar example of this are vector spaces, where two basis for the same subset must have the same cardinality. We now start exploring how we can extend this to our model theoretic setting.

Definition 16.2: A definable (over $\mathcal{M}$) set $D \subseteq \mathcal{M}$ is *minimal* if D is infinite and every definable subset of D is either finite or co-finite.

Definition 16.3: A formula $\varphi(x)$ is strongly minimal if $\varphi(\mathcal{N})$ is minimal for each elementary extension $\mathcal{M} \prec \mathcal{N}$.

Proposition 16.5: If $\mathcal{M}$ is $\aleph_0$-saturated then $\varphi(\mathcal{M})$ is minimal if and only if φ is strongly minimal.

Definition 16.4: T is strongly minimal if $x = x$ is strongly minimal, or equivalently all models $\mathcal{M} \vDash T$ are minimal.

Example: ACF_p is strongly minimal.

Proposition 16.6: If T is $\aleph_0$-stable then there exists a formula $\varphi(x)$ over $\mathfrak{C}$ which is strongly minimal.

Proof: We know from Theorem 15.5 that $\mathrm{RM}(x = x) < \infty$ and since $x = x$ has infinitely many realizations we have $\mathrm{RM}(x = x) > 0$, so we can find a formula φ with $\mathrm{RM}(\varphi) = 1$.

We can then pick a formula ψ with $[\psi] \subseteq [\varphi]$, $\mathrm{RM}(\psi) = 1$ and $\deg(\psi) = 1$. One can then check that the formula ψ is strongly-minimal. □

Theorem 16.7 (Exchange Property): Suppose that $D \subseteq \mathcal{M}$ is minimal, $A \subseteq D$, $a, b \in D$.

If $a \in \operatorname{acl}(Ab) \setminus \operatorname{acl}(A)$, then $b \in \operatorname{acl}(Aa)$.

Proof: Since $a \in \operatorname{acl}(Ab)$, there is a formula $\varphi\left(x, \overline{b}\right)$ with $\overline{b} \in A$ such that

$$\mathcal{M} \vDash \varphi\left(a, \overline{b}\right) \text{ and } |\left\{x \in D : \mathcal{M} \vDash \varphi\left(x, \overline{b}\right)\right\}| = n$$

for some $n \in \mathbb{N}$. Let $\psi(y)$ be the formula encoding

$$|\{x \in D : \varphi(x, y)\}| = n.$$

Now $\psi(y)$ is either finite or co-finite in D by assumption of strong minimality. If $\psi(D)$ is finite then $b \in \operatorname{acl}(A)$ so $a \in \operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$ which contradicts our assumption.

Hence $\varphi(D)$ is co-finite and so the complement is finite, we now let

$$E = D \cap \{y : \varphi(a, y) \wedge \psi(y)\},$$

if E is finite then $b \in \operatorname{acl}(Aa)$ and so we are done, we thus assume that E is co-finite.

Assume that $D \setminus E$ has ℓ elements, let $\chi(x)$ be the formula that says

$$|D \setminus \{y : \varphi(x, y) \wedge \psi(y)\}| = \ell.$$

If $\chi(\mathcal{M}) \cap D$ is finite then $a \in \operatorname{acl}(A)$ which is again a contradiction so $\chi(\mathcal{M}) \cap D$ is co-finite.

Pick $n + 1$ elements $a_1, ..., a_{n+1}$ in $\chi(\mathcal{M}) \cap D$, we then have that

$$D \cap \bigcap_{i=1}^{n+1} \{y : \varphi(a_i, y) \wedge \psi(y)\}$$

is an intersection of co-finite sets in D and thus is also co-finite in D, hence we can pick an element b' in this set. Now b' satisfies ψ, which contradicts the fact that each a_i is in $\{x \in D : \varphi(x, y)\}$. $\square$

We now know that acl is an operator which is monotone idempotent with the exchange property, operators with these properties are called a *pregeometry* or a *matroid*.

Definition 16.5: Let $D \subseteq \mathcal{M}$ be minimal, a set $A \subseteq D$ is *independent* if for all $a \in A$ we have

$$a \notin \operatorname{acl}(A \setminus \{a\}).$$

If we have some other $C \subseteq D$ then A is *independent over* C if instead $a \notin \operatorname{acl}(A \cup C \setminus \{a\})$

Definition 16.6: A is a basis for $Y \subseteq D$ if $A \subseteq Y$, A is independent and $\operatorname{acl}(A) = \operatorname{acl}(Y)$.

The following is standard.

Lemma 16.8: Let $A, B \subseteq D$ be independent, $A \subseteq \operatorname{acl}(B) = Y$.

(1) If $A_0 \subseteq A$, B_0, B such that $A_0 \cup B_0$ is a basis for Y, for each $a \in A \setminus A_0$ there exists $b \in B_0$ such that $A_0 \cup \{a\} \cup B_0 \setminus \{b\}$ is a basis for Y.
(2) $|A| \leq |B|$.

Proof: Since $a \in \operatorname{acl}(A_0 \cup B_0)$ there is a finite subset B' of B_0 such that $a \in \operatorname{acl}(A_0 \cup B')$, then we can pick a minimal such B'.

Pick $b \in B'$ and by applying Theorem 16.7 we know that

$$b \in \operatorname{acl}(A_0 \cup \{a\} \cup B_0 \setminus \{b\}).$$

This then gives us that $\operatorname{acl}(A_0 \cup \{a\} \cup B_0 \setminus \{b\}) \supseteq Y$.

Now $A_0 \cup \{a\} \cup B_0 \setminus \{b\}$ is independent since if $a \in \operatorname{acl}(A_0 \cup B_0 \setminus \{b\})$ then $b \in \operatorname{acl}(A_0 \cup B_0 \setminus \{b\})$ which contradicts the fact that $A_0 \cup B_0$ is a basis.

Now for the second result we know that if $|B| \geq \aleph_0$ then $|\operatorname{acl}(B)| \leq |B|$ since our language is countable and so the number of formulas is countable, so we have

$$|A| \leq |\operatorname{acl}(B)| \leq |B|.$$

On the other hand assume that B is finite, then we start with $A_0 = \varnothing$, $B_0 = B$ and add elements to A_i while removing from B_i and keeping $A_i \cup B_i$ a basis, since we can keep doing this until $A_i = A$ then we must have at least $|A|$ elements in B and so $|A| \leq |B|$. $\square$

Corollary 16.8.1: If A is also a basis then $|A| = |B|$.

Definition 16.7: Let $\mathcal{M}$ be a model with $A \subseteq \mathcal{M}$, the *dimension* of a strongly-minimal formula φ over A is the cardinality of any basis of $\varphi(\mathcal{M})$ (which is well defined by the work done above).

Theorem 16.9: Suppose that $\mathcal{N}_1, \mathcal{N}_2 \succ \mathcal{M}$ (or $\mathcal{M} = \varnothing$) are all theories of T, and that φ is a strongly-minimal formula with parameters in $A \subseteq \mathcal{M}$. If $a_1, ... \in \varphi(\mathcal{N}_1), b_1, ... \in \varphi(\mathcal{N}_2)$ are independent sets then

$$\operatorname{tp}(\overline{a} \,/\, A) = \operatorname{tp}\left(\overline{b} \,/\, A\right)$$

where $\overline{a}, \overline{b}$ are any tuples of the same length of distinct elements of a_i, b_i respectively.

Proof: We induct on n, for $n = 1$ suppose that $\mathcal{N}_1 \vDash \psi(a)$ then we want to show that $\mathcal{N}_2 \vDash \psi(b)$, since a is independent it cannot be algebraic so $\psi(\mathcal{N}_1) \cap \varphi(\mathcal{N}_1)$ cannot be finite.

Then since φ is strongly-minimal we have that $\varphi(\mathcal{N}_1) \cap \psi(\mathcal{N}_1)$ is co-finite in $\varphi(\mathcal{N}_1)$ and so $\mathcal{N}_1 \vDash |\varphi(\mathcal{N}_1) \setminus \psi(\mathcal{N}_1)| = m$ for some m. But then $\mathcal{N}_2 \vDash |\varphi(\mathcal{N}_2) \setminus \psi(\mathcal{N}_2)| = m$, so since b is not algebraic we cannot have $b \in \varphi(\mathcal{N}_2) \setminus \psi(\mathcal{N}_2)$ and so $b \in \varphi(\mathcal{N}_2) \cap \psi(\mathcal{N}_2)$ and thus $\mathcal{N}_2 \vDash \psi(b)$.

Now for the inductive step assume $\overline{a} = a_1 ... a_{j+1}$ and $\overline{b} = b_1 ... b_{k+1}$ and write $\overline{a}' = a_1 ... a_k, \overline{b}' = b_1 ... b_k$. By inductive hypothesis we have $\operatorname{tp}(\overline{a}' \,/\, A) = \operatorname{tp}\left(\overline{b}' \,/\, A\right)$ and so suppose that $\mathcal{N}_1 \vDash \psi(a_{k+1}, \overline{a}')$ and we want to show that $\mathcal{N}_2 \vDash \psi\left(b_{k+1}, \overline{b}'\right)$.

In $\mathcal{N}_1$ we have by the same argument that a_{k+1} is not algebraic over $\overline{a}' A$ and so

$$\mathcal{N}_1 \vDash |\varphi(\mathcal{N}_1) \setminus \psi(\mathcal{N}_1, \overline{a}')| = n$$

hence

$$\mathcal{N}_2 \vDash |\varphi(\mathcal{N}_2) \setminus \psi\left(\mathcal{N}_2, \overline{b}'\right)| = n$$

and by the same argument $\mathcal{N}_2 \vDash \psi\left(b_{n+1}, \overline{b}\right)$. □

Theorem 16.10: Suppose that $\mathcal{N}_1, \mathcal{N}_2 \succ \mathcal{M}$ (or $\mathcal{M} = \varnothing$) are all theories of T, and that φ is a strongly-minimal formula with parameters in $A \subseteq \mathcal{N}_1 \cap \mathcal{N}_2$. If

$$\dim(\varphi(\mathcal{N}_1)) = \dim(\varphi(\mathcal{N}_2))$$

then there exists a partial elementary map $f : \varphi(\mathcal{N}_1) \twoheadrightarrow \varphi(\mathcal{N}_2)$.

Proof: First we set f' to be identity on A, then we pick bases $(a_\alpha)_{\alpha \in I}, (b_\alpha)_{\alpha \in I}$. Then by mapping $a_\alpha \mapsto b_\alpha$ we know that by Theorem 16.9 this remains a partial embedding. We now use Zorn's lemma to pick a maximal partial embedding f with respect to inclusion that contains f', and we want to show that the domain of f is $\varphi(\mathcal{N}_1)$ and the range is $\varphi(\mathcal{N}_2)$.

To see this assume that we have $x \in \varphi(\mathcal{N}_1) \setminus \operatorname{dom}(f)$, then we know that x is algebraic over $(a_\alpha)_{\alpha \in I}$ so we know by assignment that it is isolated. Hence we can find $y \in \varphi(\mathcal{N}_2)$ with $\operatorname{tp}(x \,/\, \operatorname{dom}(f)) = \operatorname{tp}(y \,/\, \operatorname{rng}(f))$, hence g which extends f by mapping x to y is also elementary which contradicts the fact that f was maximal. □

Corollary 16.10.1: If T is strongly-minimal then $\mathcal{N}_1 \cong \mathcal{N}_2$ if and only if $\dim(\mathcal{N}_1) = \dim(\mathcal{N}_2)$.

Corollary 16.10.2: If T is strongly-minimal, then T is κ-categorical for every $\kappa \geq \aleph_1$.

Proof: Let $\mathcal{N}_1, \mathcal{N}_2 \vDash T$ with $\|\mathcal{N}_1\| = \|\mathcal{N}_2\| = \kappa$ and let $I_1 \subseteq \mathcal{N}_1, I_2 \subseteq \mathcal{N}_2$ be bases, since L is countable we have

$$\|I_1\| = \|\operatorname{acl}(I_1)\| = \|\mathcal{N}_1\| = \|\mathcal{N}_2| = \|\operatorname{acl}(I_2)\| = \|I_2\|$$

and so by Corollary 16.10.1 we get that $\mathcal{N}_1 \cong \mathcal{N}_2$. □

17. Prime Model Extensions

Let $A \subseteq \mathfrak{C}$.

Definition 17.1: A model $\mathcal{M} \prec \mathfrak{C}$ such that $A \subseteq \mathcal{M}$ is *prime over* A if for every other model $\mathcal{N}$ with $A \subseteq \mathcal{N} \prec \mathfrak{C}$ there is an elementary embedding $\mathcal{M} \prec \mathcal{N}$ which restricts to the identity on A.

Theorem 17.1: If T is $\aleph_0$-stable then for every $A \subseteq \mathfrak{C}$ there exists an $\mathcal{M} \prec \mathfrak{C}$ which is prime over A.

Proof: The strategy is quite simple, pick δ with $\delta \leq \|\mathfrak{C}\|$ and we construct $(a_\alpha : \alpha < \delta)$ such that if there exists $a \in \mathfrak{C}$ such that $\operatorname{tp}(a \,/\, A \cup a_{<\alpha})$ is isolated and who's type is not realized in $A \cup (a_\alpha : \alpha < \delta)$ then $a_\alpha = a$ for one such a. At some point we stop and are left with $\mathcal{M} = A \cup \{a_\alpha : \alpha < \delta\}$. Now we need to show is that $\mathcal{M}$ is an elementary sub model and that it is prime over A.

Let us denote $A_\alpha = A \cup \{a_\alpha : \alpha < \beta\}$, then we will use Theorem 2.3 to prove $\mathcal{M}$ is an elementary submodel, assume that in $\mathfrak{C}$ we have

$$\mathfrak{C} \vDash \exists x \varphi(x, \overline{a})$$

for $\overline{a} \in \mathcal{M}$, since $\mathcal{M} = \bigcup_{\alpha<\delta} A_\alpha$ we may assume that $\overline{a} \in A_\alpha$ for some α. By $\aleph_0$-stability we know that isolated types in $S(A_\alpha)$ are dense, assume otherwise, then there is a neighborhood in $S(A_\alpha)$ without any isolated points. Then since we have no isolated points and the space is Hausdorff we can repeatedly split it in half to construct a tree of formulas $\varphi_\sigma, \sigma \in 2^{<\omega}$ such that $\varphi_{\sigma 0}, \varphi_{\sigma 1}$ are inconsistent and both imply φ_σ, this tree contradicts $\aleph_0$-stability as can be seen in the proof of Theorem 13.4.

Now since the isolated types in $S(A_\alpha)$ are dense, we choose an isolated type $p \in S(A_\alpha)$ such that $\varphi(x, \overline{a}) \in p$, so let ψ be the formula isolating p, since ψ has finitely many parameters and $\mathfrak{C}$ is $\aleph_0$ saturated we can find an element realizing ψ and hence p. Let $a \in \mathfrak{C}$ with $a \vDash p$ then it will be added at some point to A_α and hence will be in $\mathcal{M}$.

Now to show $\mathcal{M}$ is prime we use transfinite induction on α, the base case and limit case as trivial, then for the successor step we use the fact that if $\operatorname{tp}(a \,/\, B \cup A)$ is isolated and for any $b \in B$ $\operatorname{tp}(b \,/\, A)$ is isolated then $\operatorname{tp}(a \,/\, A)$ is isolated. Now from this we get that $\mathcal{M}$ is prime over A since we can inductively construct an embedding $\mathcal{M} \to \mathcal{N}$. □

Remark: This is a stronger condition than the isolated types being dense, which is what is needed to guarantee a prime model if A is empty.

Corollary 17.1.1: If T is $\aleph_0$-stable then $\forall A \subseteq \mathfrak{C}$ there exists $\mathcal{M}$ prime over A such that for all elements $a \in \mathcal{M}$ we have $\operatorname{tp}(a \,/\, A)$ is isolated.

18. Categoricity theorem

We now start working towards Categoricity Theorem, the main result of this course.

Definition 18.1: Let $\kappa > \lambda \geq \aleph_0$, an L-theory T is said to have a (κ, λ)-*model* if there exists a model $\mathcal{M} \vDash T$ with $|\mathcal{M}| = \kappa$ and an $L(\mathcal{M})$-formula, $\varphi(x)$ such that $|\varphi(\mathcal{M})| = \lambda$.

Definition 18.2: Let $\mathcal{M} \prec \mathcal{N} \vDash T$. Then $(\mathcal{N}, \mathcal{M})$ is a *Vaughtian pair* if there is an L -formula $\varphi(x)$ such that $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ are both infinite.

Lemma 18.1: If T has a (κ, λ)-model, then it has a Vaughtian pair $(\mathcal{N}, \mathcal{M})$ such that $|\mathcal{N}| < \kappa$ and $|\mathcal{M}| = \lambda$.

Proof: We use Theorem 2.4, we let $\mathcal{N} \vDash T$ of size κ and φ be such that $|\varphi(\mathcal{N})| = \lambda$, we can then find $\mathcal{M}$ such that $\varphi(\mathcal{N}) \subseteq \mathcal{M}$ and $\mathcal{M} \prec \mathcal{N}$. Then we have $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ by elementarity. $\square$

Definition 18.3: Suppose $\mathcal{U}$ is a unary predicate. Let $\varphi(x)$ be an L-formula. The *relativization* of φ to $\mathcal{U}$, denoted $\varphi^{\mathcal{U}}$, is defined as follows:

$$\varphi^{\mathcal{U}}(\overline{x}) := \mathcal{U}(x_1) \wedge ... \wedge \mathcal{U}(x_n) \wedge \varphi(\overline{x})$$

for φ atomic, and

$$\varphi^{\mathcal{U}}(\overline{x}) := \exists y\big(y \wedge \psi^{\mathcal{U}}(y, z)\big)$$

Proposition 18.2: If $\mathcal{N}$ is an L-model with an added unary predicate $\mathcal{U}$, then $\mathcal{U}(\mathcal{N})$ induced an elementary L-submodel if and only if for any tuple $\overline{a} \in \mathcal{M}$ and any L-formula φ we have

$$\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N}^{L'} \vDash \varphi^{\mathcal{U}}(\overline{a}).$$

Proof: Exercise, prove by induction. $\square$

If $(\mathcal{N}_i, \mathcal{M}_i) \vDash T$ are Vaughtian pairs we write $(\mathcal{N}_1, \mathcal{M}_1) \prec (\mathcal{N}_2, \mathcal{N}_2)$ to mean that they have the same L-formula in their definition, that $\mathcal{N}_1 \prec \mathcal{N}_2, \mathcal{M}_1 \prec \mathcal{M}_2$, and that $\varphi(\mathcal{N}_1) = \varphi(\mathcal{N}_2), \varphi(\mathcal{M}_1) = \varphi(\mathcal{M}_2)$.

Lemma 18.3: If T has a Vaughtian pair $(\mathcal{N}, \mathcal{M})$, then it also has a Vaughtian pair $(\mathcal{N}_0, \mathcal{M}_0)$ with $\|\mathcal{N}_0\| = \|\mathcal{M}_0\| = \aleph_0$.

Proof: Set $L' = L$ cal $\{\mathcal{U}\}$ and interpret $\mathcal{N}$ as an L' model by setting $\mathcal{U}(x) \Leftrightarrow x \in \mathcal{M}$, then consider for any L-formula φ the L-sentence

$$\sigma_\varphi := \forall \overline{z}\left(\left(\bigwedge_{i \leq n} \mathcal{U}(x_i) \wedge \varphi(\overline{x})\right) \to \varphi^{\mathcal{U}}(\overline{x})\right)$$

since $\mathcal{M} \prec \mathcal{N}$ then we have for any tuple $\overline{a} \in \mathcal{N}$

$$\begin{aligned}\mathcal{N} \vDash \bigwedge_{i \leq n} \mathcal{U}(a_i) \wedge \varphi(\overline{a}) \Leftrightarrow (\mathcal{N} \vDash \varphi(\overline{a})) \wedge \overline{a} \in \mathcal{M} \Rightarrow \mathcal{M} \vDash \varphi(\overline{a}) \\ \Rightarrow \mathcal{N} \vDash \varphi^{\mathcal{U}}(\overline{a})\end{aligned}$$

and so $\mathcal{N} \vDash \sigma_\varphi$.

Now we use Theorem 2.4 to construct a countable L'-model $\mathcal{N}_0$ with $\mathcal{N}_0 \prec \mathcal{N}$, then in $\mathcal{N}_0$ we also have $\mathcal{N}_0 \vDash \sigma_\varphi$ for all φ. Then $\mathcal{U}(\mathcal{N}_0)$ is exactly an elementary submodel exactly by Proposition 18.2, so we will define $\mathcal{M}_0 := \mathcal{U}(\mathcal{N}_0)$. To check that $(\mathcal{N}_0, \mathcal{M}_0)$ is a Vaughtian pair we apply Proposition 18.2 specifically with φ being the formula that defines the infinite subset shared between $\mathcal{M}$ and $\mathcal{N}$. All its properties can be encoded as L'-sentences so are shared for $\mathcal{N}_0$. □

Lemma 18.4: Suppose that $\mathcal{M} \prec \mathcal{N} \vDash T$, then there is a pair $\mathcal{M}_0 \prec \mathcal{N}_0$ such that $(\mathcal{N}, \mathcal{M}) \succ (\mathcal{N}_0, \mathcal{M}_0)$ and such that $\mathcal{M}_0, \mathcal{N}_0$ are both countable, homogeneous, and realize the same types in $S_n(T)$.

Proof: Fix an L-formula φ and set $L' = L \cup \{\mathcal{U}\}$.

Claim 18.5: If $\overline{a} \in \mathcal{M}_0$, and $p \in S_n(\overline{a})$ is realized in $\mathcal{N}_0$, then there exists countable extensions $(\mathcal{N}', \mathcal{M}') \succ (\mathcal{M}_0, \mathcal{N}_0)$ such that p is realized in $\mathcal{M}'$.

Proof: Let $L'' = L'(\mathcal{N}_0)$, c a new constant, and let $T := \mathrm{Th}_{L''}(\mathcal{N}_0, \mathcal{M}_0) \cup \{\varphi^{\mathcal{U}}(c, \overline{a} : \varphi \in p\}$. T is finitely satisfiable since for any formulas φ_i we have that since p is realized in $\mathcal{N}_0$

$$\mathcal{N}_0 \vDash \exists \overline{x} \bigwedge_i \varphi_i(\overline{x})$$

and so by elementarity so does $\mathcal{N}_0$.

Hence let $\mathcal{N}' \vDash T$ and set $\mathcal{M}' = \mathcal{U}(\mathcal{N}')$. □

Claim 18.6: If $\overline{b} \in \mathcal{N}_0$, $p \in S_n\left(\overline{b}\right)$, then there exists two countable models $(\mathcal{N}'', \mathcal{M}'') \succ (\mathcal{N}_0, \mathcal{M}_0)$ such that p is realized in $\mathcal{N}''$.

Proof: Almost exactly the same. □

With these two claims we can now construct what we want, we will build sequences of countable models $\mathcal{M}_i, \mathcal{N}_i$ such that $\left(\mathcal{N}_{i+1}, \mathcal{M}_{i+1}\right) \succ (\mathcal{N}_i, \mathcal{M}_i)$ and

- $i = 3i$: Any type $p \in S_n(\varnothing)$ that is realized in $\mathcal{N}_i$ is realized in $\mathcal{M}_{i+1}$. We use the first claim for this.
- $i = 3i+1$: If $\overline{a}, \overline{b}, \overline{c} \in \mathcal{N}_i$ are such that $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{c})$ then there exists a $d \in \mathcal{N}_{i+1}$ such that $\mathrm{tp}(\overline{ac}) = \mathrm{tp}\left(\overline{bd}\right)$. We use claim 2 for this.
- $i = 3i+2$: If $\overline{a}, \overline{b}, \overline{c} \in \mathcal{M}_i$ are such that $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{c})$ then there exists a $d \in \mathcal{M}_{i+1}$ such that $\mathrm{tp}(\overline{ac}) = \mathrm{tp}\left(\overline{bd}\right)$. We use claim 1 for this.

Then it is easy to check that $\mathcal{N} := \bigcup_i \mathcal{N}_i$ and $\mathcal{M} := \bigcup_i \mathcal{M}$ work. □

Theorem 18.7 (Vaught's Two Cardinal Theorem): Let T be an L-theory. The following are equivalent:

(1) There exist cardinals $\kappa > \lambda \geq \aleph_0$ such that T has a (κ, λ)-model.
(2) T has a Vaughtian pair.
(3) T has an $(\aleph_1, \aleph_0)$-model.

Proof: We already proved (1) ⇒ (2), (3) ⇒ (1) is immediate, so we just need to show (2) ⇒ (3). Assume then, that T has a Vaughtian pair $(\mathcal{N}, \mathcal{M})$ with $\mathcal{M}, \mathcal{N}$ countable.

We now construct a sequence of models, $\mathcal{M}_\alpha$ such that $(\mathcal{M}_{i+1}, \mathcal{M}_i) \equiv (\mathcal{M}_1, \mathcal{M}_0)$, we start with $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_1 = \mathcal{N}$, we do the successor step using Lemma 18.4 and in limit steps we just take unions. We then set $\mathcal{N}' := \bigcup_{i<\alpha} \mathcal{N}_i$, this is model of size $\aleph_1$ but we still have that $(\mathcal{N}', \mathcal{M})$ is a Vaughtian pair so since we have a formula φ given to use by the pair, we have that $\varphi(\mathcal{N}') \subseteq \mathcal{M}$ and thus $|\varphi(\mathcal{N}')| = \aleph_0$ and so this $\mathcal{N}'$ is a $(\aleph_1, \aleph_0)$ model. □

Lemma 18.8: Suppose T is $\aleph_0$-stable, $\mathcal{M} \vDash T$, $\|\mathcal{M}\| \geq \aleph_1$. There exists $\mathcal{N}$ with $\mathcal{M} \prec \mathcal{N}$ such that $\mathcal{N}$ and $\mathcal{M}$ realize the same types over countable subsets of $\mathcal{M}$.

Proof: We start with a claim that finds an 'unsplittable' formula.

Claim 18.9: There exists an $L(\mathcal{M})$-formula $\varphi(x)$ such that $|\varphi(\mathcal{M})| \geq \aleph_1$ and for every formula $\psi(x)$ in $L(\mathcal{M})$ we have $|\varphi \wedge \psi(\mathcal{M})| \leq \aleph_0$ or $|\varphi \wedge (\neg\psi)(\mathcal{M})| \leq \aleph_0$.

Proof: Suppose such a formula does not exist, we can then construct a tree of formulas φ_σ for $\sigma \in 2^{<\omega}$ such that $|\varphi_\sigma(\mathcal{M})| \geq \aleph_1$, $\varphi_{\sigma^0}, \varphi_{\sigma^1}$ are inconsistent and both imply φ_σ.

If A is the set of parameters of φ_σ's then $|S_1(A)| \geq 2^{\aleph_0}$, which contradicts $\aleph_0$ stability. □

Let φ be as in the claim above, we define the type.

$$p = \{\psi(x) : \psi \text{ is an } L(\mathcal{M}) \text{ formula with } |\psi \wedge \varphi(\mathcal{M})| \geq \aleph_1\}.$$

Then p is a complete type in $S(\mathcal{M})$ due to the defining property of φ, then let $\mathcal{M}' \succ \mathcal{M}$ be the extension that realizes p, and set c to be a witness.

We now take $\mathcal{N}$ to be the prime model over $\mathcal{M} \cup \{c\}$ (Definition 17.1), then every $b \in \mathcal{N}$ has isolated type over $\mathcal{M} \cup \{c\}$. Clearly $\mathcal{N}$ contains $\mathcal{M}$ and thus all the types of $\mathcal{M}$, so it is enough to show that $\mathcal{M}$ contains all the types of $\mathcal{N}$. Let $\Gamma(w)$ be a countable type over $\mathcal{M}$ that is realized in $\mathcal{N}$, we show that is also realized in $\mathcal{M}$. Let $b \vDash \Gamma$, then $\operatorname{tp}(b \,/\, \mathcal{M}c)$ is isolated by some formula $\theta(w, c)$.

Now we know, since c realizes p and since $\exists w \theta(\omega, c)$ is true then $\exists w \theta(\omega, x) \in p$. We also know that

$$\mathcal{N} \vDash \forall w(\theta(\omega, c) \to \gamma(w))$$

for all formulas $\gamma \in \Gamma$, by definition of isolated type. We then choose to look at the set

$$\Delta = \{\exists w \theta(w, x)\} \cup \{\forall w(\theta(w, x) \to \gamma(w)) : \gamma \in \Gamma\}$$

this set is countable and if it has a realization in $\mathcal{M}$, say by c', then if b' is the witness of $\exists\theta(w, c')$ in $\mathcal{M}$ then it satisfies the type Γ.

Claim 18.10: Δ has a realization in $\mathcal{M}$.

Proof: For each $\delta \in \Delta$ we have that

$$\delta \wedge \varphi(\mathcal{M}) \text{ is co-countable in } \varphi(\mathcal{M})$$

so

$$\bigwedge_{\delta\in\Delta} \delta \wedge \varphi(\mathcal{M}) \text{ is non empty.}$$

□

□

Proposition 18.11: Let T be κ-categorical for some $\kappa \geq \aleph_1$, then T is $\aleph_0$-stable.

Proof: If T is not $\aleph_0$ stable, then it has a model $\|\mathcal{M}\| = \kappa$, with a countable subset A such that $\mathcal{M}$ realizes uncountably many types over A. Let T^s be the Skolemization of T and $(I, <)$ an ordered set order isomorphic to κ. Let $\mathcal{N}$ be an EM-Model (Definition 13.2) generated by an order-indiscernible sequence modeled after $(I, <)$.

For every $A \subseteq \mathcal{N}$, $\mathcal{N}$ realizes at most $|A| + \aleph_0$ types over A so $\mathcal{M} \ncong \mathcal{N}$ which contradicts categoricity. □

Proposition 18.12: Let κ be uncountable and T a complete theory, there exists $\mathcal{M} \vDash T$ with $\|\mathcal{M}\| = \kappa$ and any $L(\mathcal{M})$-definable subset of $\mathcal{M}$ is either finite or has size κ.

Proof: Exercise, use compactness. □

Corollary 18.12.1: If T is κ-categorical with $\kappa \geq \aleph_1$ then T has no Vaughtian pair.

Proof: Let $\mathcal{M}$ be as in Proposition 18.12, let $\mathcal{N}$ be the $(\kappa, \aleph_0)$-model $\mathcal{N}$ that we proved exists in Theorem 18.7. Clearly $\mathcal{M} \ncong \mathcal{N}$ which contradicts categoricity. □

Before we jump into the proof let us slightly generalize Proposition 16.6.

Lemma 18.13: Let T be an $\aleph_0$-stable theory, $\mathcal{M} \vDash T$, there exists an $L(\mathcal{M})$-formula which is minimal in $\mathcal{M}$.

Proof: We repeat the same tree trick we keep using, if such a formula does not exist we can start with $\varphi_\varnothing = (x = x)$ and keep splitting into two 'large' parts to generate a tree of formulas $\varphi_\sigma, \sigma \in 2^{<\omega}$. Let A be the number of parameters of each formula φ_σ, then $|S_1(A)| \geq 2^{\aleph_0}$. □

Since minimality seems weak one might question the usefulness of this lemma, but that worry should disappear given the next lemma.

Lemma 18.14: Suppose T has no Vaughtian pair, let $\mathcal{M}$ be a model of T, and $\varphi(x, \overline{y})$ be an L-formula. There is some number n such that for all $\overline{a} \in \mathcal{M}$,

$$|\varphi(\mathcal{M}, \overline{a})| > n \Rightarrow \varphi(\mathcal{M}, \overline{a}) \text{ infinite}$$

Proof: Suppose not, then for each $n \in \mathbb{N}$ we have some tuple $\overline{a}_n$ such that $\varphi(\mathcal{M}, \overline{a}_n)$ is finite of size at least n. Let $\mathcal{U}$ be a unary predicate, $L' = L \cup \{\mathcal{U}\}$, let $p(\overline{y})$ be the L'-type consisting of the formulas

(1) $\mathcal{U}(\mathcal{M})$ defined a proper elementary submodel, this is done by adding $\varphi^{\mathcal{U}}$ for every $\varphi \in \mathrm{Th}(\mathcal{M})$.
(2) $\mathcal{U}(y_1) \wedge ... \wedge \mathcal{U}(y_k)$.
(3) For each n the formulas $|\varphi(\mathcal{M}, \overline{y})| > n$.
(4) $\forall x(\varphi(x, \overline{y} \to \mathcal{U}(x)))$.

Claim 18.15: p is consistent.

Proof: We can, by compactness, only show that p' is consistent where we picked only finitely many formulas from (3) and keep the rest the same. To that end let $\mathcal{N}$ be an arbitrary proper supermodel with $\mathcal{M} \prec \mathcal{N}$. Interpret $\mathcal{U}$ as $\mathcal{M}$ and $\overline{y}$ as $\overline{a}_n$ where n is the largest that we picked in (3).

Then $\varphi(\mathcal{M}, \overline{a}_n)$ is finite implies that $\varphi(\mathcal{M}, \overline{a}_n) = \varphi(\mathcal{N}, \overline{a}_n)$ and so p' is realized. □

A realization of p is a Vaughtian pair, which is our contradiction. □

Corollary 18.15.1: If T has no Vaughtian pair, $\mathcal{M}$ is a model of T and φ is a minimal $L(\mathcal{M})$-formula, then it is also strongly minimal.

Proof: Suppose that $\mathcal{N} \succ \mathcal{M}$ and φ is not minimal in $\mathcal{N}$. Let $\psi(x, y)$, $\overline{a} \in \mathcal{N}$ be such that

$$\varphi \wedge \psi(x, \overline{a}) \quad \varphi \wedge \neg\psi(x, \overline{a})$$

are both infinite. Let n be as in Lemma 18.14 for both $\varphi \wedge \psi, \varphi \wedge \neg\psi$ (take max).

Then $\mathcal{N}$ satisfies

$$\exists \overline{y}(|\varphi \wedge \psi(x, \overline{y})| > n) \wedge (|\varphi \wedge \neg\psi(x, \overline{a})| > n),$$

so let $\overline{a}' \in \mathcal{M}$ be such that $\varphi \wedge \psi(\mathcal{M}, \overline{a}')$ and $\varphi \wedge \neg\psi\left(\mathcal{M}, \overline{a'}\right)$ are both infinite. This contradicts the fact that φ is minimal in $\mathcal{M}$. □

We finally have enough tools to prove the main theorem.

Theorem 18.16 (Morley): The following are equivalent.
(1) T is categorical in some uncountable cardinal κ.
(2) T is categorical in all uncountable cardinals.

Proof: (2) $\Rightarrow$ (1) is trivial. Assume then that T is γ-categorical for some uncountable cardinal γ, then it is $\aleph_0$-stable and has no Vaughtian pair. Let κ be some arbitrary cardinal, Let $\mathcal{M}_0$ be a prime model of T, this is possible to find because the isolated types are dense in $\mathcal{M}$ (because $S(\varnothing)$ is countable). Let $\mathcal{M}, \mathcal{N} \vDash T$ with $\|\mathcal{M}\| = \|\mathcal{N}\| = \kappa$, then since $\mathcal{M}_0$ is prime $\mathcal{M}_0 \prec \mathcal{M}$ and $\mathcal{M}_0 \prec \mathcal{M}$.

There exists a minimal $L(\mathcal{M}_0)$-formula φ, by Corollary 18.15.1 we get that φ is also minimal in $\mathcal{M}$ and in $\mathcal{N}$. Then $|\varphi(\mathcal{M})| = |\varphi(\mathcal{N})| = \kappa$ because we have no Vaughtian pairs. We then have $\dim(\varphi(\mathcal{M})) = \dim(\varphi(\mathcal{N}))$.

Let I be a basis for $\varphi(\mathcal{M})$, and J be a basis for $\varphi(\mathcal{N})$, take any bijection $f : I \to J$. This map extends to elementary maps $f' : \varphi(\mathcal{M}) \to \varphi(\mathcal{N})$. Now take $\mathcal{M}' \succ \mathcal{M}$ prime over $\varphi(\mathcal{M})$, since every element of $\mathcal{M}'$ realizes an isolated type over $\varphi(\mathcal{M})$ then we can extend f' to $f'' : \mathcal{M}' \to \operatorname{rng}(f'')$.

But since we have no Vaughtian pair we know that $\mathcal{M}' = \mathcal{M}$ and $\operatorname{rng}(f'') = \mathcal{N}$ and so $f'' : \mathcal{M} \to \mathcal{N}$ is an isomorphism and so we are done. □

This subject, of course, goes a lot deeper than this theorem. Here is a selection of results for the interested reader to look into.

Theorem 18.17 (Balduin-Lachlan): The following are equivalent.
(1) T is uncountably categorical.
(2) T is $\aleph_0$-stable and has no Vaughtian pairs.

Theorem 18.18 (Balduin-Lachlan): If T is $\aleph_1$-categorical, not $\aleph_0$-categorical, then it has $\aleph_0$ countably many models of size $\aleph_0$.

Appendix A: Transfinite induction and Cardinal Arithmetic

In model theory we very often want to count things, but natural numbers are often not enough since we deal with truly massive sets, this is where ordinals, which extend the counting of natural numbers, are very useful.

Definition A.1: An *ordinal* is a set α such that the relation $\in$ is a well ordering on α, that is a linear order where every subset $S \subseteq \alpha$ has a minimal element.

Equivalently α has no infinite strictly decreasing sequence with respect to $\in$.

We will not go into the details of ordinal theory, that is the job of Set Theory class. We will, however, list their key properties here.

Proposition A.1: Assuming ZFC,

(1) Every well ordered set is isomorphic to some ordinal.
(2) Any collection (not necessarily a set) of ordinals has a minimal ordinal with respect to $\in$.
(3) Every ordinal takes one of 3 forms
  - Zero/Empty Set: $\{\}$.
  - Successor ordinal: $\mathrm{suc}(\alpha) \coloneqq \alpha \cup \{\alpha\}$ for some ordinal α, sometimes denoted α^+ or $\alpha + 1$.
  - Limit ordinal: $\bigcup_{\gamma \in X} \gamma$ for some set X of ordinals.

Ordinals are important because of their ability to extend induction.

Theorem A.2: Let $p(\alpha)$ be a boolean property defined on all ordinals α. If

- $p(\{\})$ is true.
- $p(\alpha) \Rightarrow p(\mathrm{suc}(\alpha))$.
- $(\forall \gamma \in X, p(\gamma)) \Rightarrow p\left(\bigcup_{\gamma \in X} \gamma\right)$.

Then p is true for all ordinals.

We can also use induction for constructions.

Theorem A.3: Let x_α be variables indexed by ordinals α and $p_\beta(x_{\leq \beta})$ be a property depending on all variables x_α with $\alpha \leq \beta$. If the following conditions hold

- There is an object a_0 such that $p_0(a_0)$ is true.
- For any objects a_α for $\alpha \leq \beta$ such that $p_\beta(x_{\leq \beta})$ is true there is an object $a_{\beta+1}$ such that $p_{\beta+1}(x_{\leq \beta+1})$ is true.
- For any limit ordinal γ and any objects a_α for $\alpha < \gamma$ such that $p_\beta(x_{\leq \beta})$ is true for all $\beta < \gamma$ there is an object a_γ such that we also have $p_\gamma(x_{\leq \gamma})$.

Then there is an assignment of x_α such that p_β is true for all ordinals β.

Here is an example of this

Definition A.2: For any ordinal α, we define inductively

(1) $\alpha + 0 = \alpha$.
(2) $\alpha + \mathrm{suc}(\beta) = \mathrm{suc}(\alpha + \beta)$.
(3) If γ is a limit ordinal then $\alpha + \gamma = \bigcup_{\beta<\gamma}(\alpha + \beta)$.

We can similarly define multiplication and exponentiation of ordinals.

Now ordinals generalize counting, but it turns out that we can use them to get a generalized notion of size.

Definition A.3: Let α, β be two sets, we say that α and β are *equal in cardinality* and write $|\alpha| = |\beta|$ to mean that there is a bijection between α and β.

Now one can easily check that this is an equivalence relation when restricted to ordinals, and it thus partitions ordinals into equivalence classes.

Definition A.4: Let S be an equivalence class of cardinality in the ordinals, as a collection of ordinals it has a minimal element κ, all such minimal elements are called *cardinals*.

Because of axiom of choice every set X has a well ordering and thus is in bijection with some ordinal α, hence is also in bijection with exactly one cardinal κ. We say that κ is the *cardinality* of X and denote it as $\kappa = |X|$.

We can index the cardinals in increasing order using ordinals, all the natural numbers are cardinals, $|\mathbb{N}|$ is the next ordinal which we denote $\aleph_0$, the next ordinal after that is $\aleph_1$ and so on.

For cardinals we define operations differently.

Definition A.5: We define for any two cardinals α, β

$$\alpha + \beta = |\alpha \sqcup \beta|$$
$$\alpha \times \beta = |\alpha \times \beta|$$
$$\alpha^\beta = |\{f : \beta \to \alpha\}|$$

These operations are not as interesting as those of ordinals, which we see in the following proposition.

Proposition A.4: For any two cardinals α, β, if $\alpha \geq \aleph_0$ or $\beta \geq \aleph_0$ we have

$$\alpha + \beta = \beta + \alpha = \alpha \times \beta = \beta \times \alpha = \max(\alpha, \beta)$$

Definition A.6: The continuum hypothesis is the statement

$$2^{\aleph_0} = \aleph_1.$$

The generalized continuum hypothesis is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all ordinals α.

It turns out that the continuum hypothesis is independent of ZFC.

Definition A.7: For a cardinal γ, $\mathrm{cf}(\gamma)$ is called the *co-finality* of γ and is the cardinality of the shortest unbounded sequence in γ.

Equivalently, $\mathrm{cf}(\gamma)$ is the largest cardinal such that for every sequence of cardinals κ_α smaller than γ which has length at most $\mathrm{cf}(\gamma)$, has $\bigcup_{\alpha < \mathrm{cf}(\gamma)} \kappa_\alpha \leq \gamma$.

Theorem A.5 (König's theorem): For a cardinal γ, $\mathrm{cf}(2^\gamma) > \gamma$.

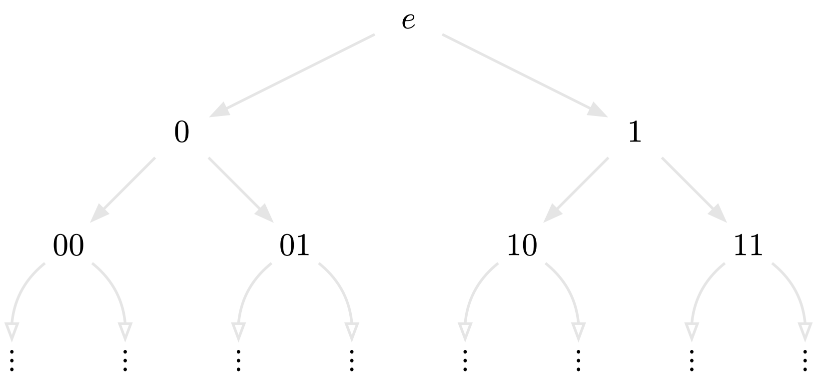
Appendix B: Infinite Trees

Near the end of this course we will deal a lot with infinite binary trees so we describe their basic properties and notation here.

We use 2^ω to denote the set of countable sequences of $\{0,1\}$, we use $2^{<\omega}$ to denote the set of finite sequences of $\{0,1\}$, including the empty sequence e. For any element $\sigma \in 2^\omega$, its truncation $\sigma|_n$ is the finite sequence we get by only considering the sequence for indices $1 \leq i \leq n$. For any finite sequence $\delta \in 2^{<\omega}$ with length n, we use $[\delta]$ to denote the set of countable sequences who's n'th truncation is δ, that is

$$[\delta] = \{\sigma \in 2^\omega : \sigma|_n = \delta\}.$$

Conceptually, we will think of elements of 2^ω as infinite **branches** in an infinite tree where elements of $2^{<\omega}$ are the nodes. The mental picture we will looks something like this.



Here an element of 2^ω corresponds to an infinite path down this tree.

Now in practice, trees like these come up naturally when we want to split up sets while preserving a sense of mass. Let us try to formalize this.

Definition B.1: Let X be any set and $\mathcal{F} \subseteq \mathcal{P}(X)$ be any family of sets. We will call the family $\mathcal{F}$ and its elements *big* if

- $X \in \mathcal{F}$.
- For any $Y \in \mathcal{F}$ there exist non-empty disjoint subsets $W, W' \subseteq Y$ such that $W, W' \in Y$.

Proposition B.1: If X is a set and $\mathcal{F}$ is a family of big subsets then there exists a function $2^{<\omega} \to \mathcal{F}$ such that

- The image of e is X.
- If Y is the image of a finite sequence δ, then the images of $\delta 0$ and $\delta 1$ (concatenation) are disjoint and subsets of Y.

Proof: Proof is nearly immediate since the definition of a big family is tailor made for this result. We define the map inductively, we set $X_e = X$ and then for each δ we set $X_{\delta 0}$ and $X_{\delta 1}$ to be the big subsets given to us by the definition. $\square$

Lets see a classic result that follows almost immediately from this construction.

Theorem B.2: Let X be a 0-dimensional, compact, Hausdorff topological space with no isolated points, then X has cardinality at least $2^{|\aleph_0|}$.

Proof: First we demonstrate that it is enough to show that any clopen basis of X forms a big family. If this is the case then by Proposition B.1 we get a tree X_δ of clopen sets, then every branch σ corresponds to a decreasing sequence of non-empty closed subsets of X. But then by compactness the intersection of this sequence is non-empty and thus contains at least one point p_σ.

This defines a map $f : 2^\omega \to X$ which must be injective since any two different branches σ and σ' will eventually diverge from each other down the tree, which corresponds to p_σ and $p_{\sigma'}$ being contained in disjoint open sets. Thus $|X| \geq |2^\omega| = 2^{|\aleph_0|}$.

We now show the clopen basis does indeed form a big family. Clearly X is a clopen basis element so then it is enough to show that any clopen basis set Y contains 2 disjoint clopen basis sets. To see this note that again Y cannot be finite so it contains at least 2 points, name them x, y. Then since X is Hausdorff we can pick open sets Z_0, Z_1 that are disjoint and contain x and y respectively. Then $Z_0 \cap Y, Z_1 \cap Y$ are disjoint open subsets of Y so $Z_0 \cap Y$ is a superset of a basis element W and $Z_1 \cap Y$ is a superset of a basis element W'. Then W, W' are clopen basis elements which are disjoint and subsets of Y, which proves that this is indeed a big family. □