Question 1

Statement

Consider a physical system in which a rocket's total energy is constant, given by

$$E = \frac{1}{2}mv^2e^{kmv}$$

where m is the mass of the rocket, v is its velocity, and k is a constant equal to $4\mathrm{s\,kg^{-1}\,m^{-1}}$.

Assume that at a certain snapshot in time, $t = 0$, we measure the mass of the rocket to be 1kg, its velocity to be $1\mathrm{m\,s^{-1}}$ and its acceleration to be $\frac{\partial v}{\partial t}|_{t=0} = 5\mathrm{m\,s^{-2}}$. Determine the rate at which it is losing mass, in other words, what is $\frac{\partial m}{\partial t}|_{t=0}$?

(a) $-4\mathrm{kg\,s^{-1}}$ (b) $-5\mathrm{kg\,s^{-1}}$ (c) $-6\mathrm{kg\,s^{-1}}$ (d) $-7\mathrm{kg\,s^{-1}}$ (e) None of the above

Solution

This is an implicit differentiation question since we are solving for the change of one variable when some implicit equation of them holds. Since E is constant we have that

$$\begin{aligned}
0 = \frac{\partial E}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{2}mv^2e^{kmv} \\
&= \frac{1}{2}v^2e^{kmv}\frac{\partial m}{\partial t} + mve^{kmv}\frac{\partial v}{\partial t} + \frac{1}{2}mv^2e^{kmv}(kv)\frac{\partial m}{\partial t} + \frac{1}{2}mv^2e^{kmv}(km)\frac{\partial v}{\partial t} \\
&= (1+mkv)\frac{1}{2}v^2e^{kmv}\frac{\partial m}{\partial t} + (2+mkv)\frac{1}{2}mve^{kmv}\frac{\partial v}{\partial t}
\end{aligned}$$

We now have the values at $t = 0$ of all variables here except the one we want to know. We thus get

$$\begin{aligned}
0 &= (1+4)\frac{1}{2}e^4\frac{\partial m}{\partial t}\mathrm{m^2\,s^{-2}} + (2+4)\frac{1}{2}e^4(5)\mathrm{kg\,m^2\,s^{-3}} \\
&= 5\frac{\partial m}{\partial t} + 30\mathrm{kg\,s^{-1}}
\end{aligned}$$

giving us $\frac{\partial m}{\partial t} = -6\mathrm{kg\,s^{-1}}$

Question 2

Statement

Consider the region $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$. For which of the following functions is the point (0.8, 0.6) an absolute maximum over R?

(a) $f(x, y) = x^3 - x + y$ (b) $f(x, y) = (4x + 3y - 1)^2$ (c) $f(x, y) = e^{xy}$ (d) $f(x, y) = 4x + 3y$ (e) none of the above

Solution

This one was the hardest question for the most students, the key is that the the point is on the boundary of R and so we can immediately use Lagrange multipliers to eliminate some options.

The gradient of the constraint function is easily seen to be $(2x, 2y) = (1.6, 1.2)$ and so we need to compute the gradients of the options, these are

$$\begin{aligned} \nabla(x^3 - x + y) &= \left(3(0.8)^2 - 1, 1\right) = (1.92, 1) \\ \nabla\left((4x + 3y - 1)^2\right) &= 2(4 \cdot (0.8) + 3 \cdot (0.6) - 1)(4, 3) = (32, 24) \\ \nabla(e^{xy}) &= e^{0.8 \cdot 0.6}(0.6, 0.8) \\ \nabla(4x + 3y) &= (4, 3) \end{aligned}$$

We can quickly see that both options (a) and (c) cannot be correct since the gradient is not colinear with $(1.6, 1.2)$. Options (b) and (d) are both on the table. We now explicitly use Lagrange multipliers on those two options. Let us first try for (b), this gives us

$$\lambda(2x, 2y) = 2(4x + 3y - 1)(4, 3)$$

which then gives us

$$\lambda x = 16x + 12y - 4, \quad \lambda y = 12x + 9y - 3$$

we can multiply the left equation by 3 and the right equation by 4 and then subtract to get

$$3\lambda x - 4\lambda y = 0$$

and so (x, y) must lie on the line $y = \frac{3}{4}x$. Since it must also be on the circle $x^2 + y^2 = 1$ and so that leaves us with the points $(-0.8, -0.6)$ as well as $(0.8, 0.6)$. We notice that the value at $(-0.8, -0.6)$ is 36 which is higher than the value at $(0.8, 0.6)$ which is 16 so $(0.8, 0.6)$ this is not true for (b).

Next we have (d), we get

$$\lambda(2x, 2y) = (4, 3)$$

and so we again have that (x, y) lies on the line $y = \frac{3}{4}x$. We then get that the points are again $(-0.8, -0.6)$ and $(0.8, 0.6)$.

But now the value at $(-0.8, -0.6)$ is -5 and the value at $(0.8, 0.6)$ is 5 and so it could be a maximum. All that remains to check is that there are no other critical points in the interior of R. But this is immediate from the fact that the gradient $(4, 3)$ is never zero and so we get that (d) is the right answer.

Question 3

Statement

Consider the triangle T with corners $(0, 0), (0, 2), (4, 0)$ and constant density function 1. Which point is its center of mass?

(a) $\left(1, \frac{2}{3}\right)$ (b) $\left(\frac{4}{3}, \frac{2}{3}\right)$ (c) $\left(\frac{4}{3}, \frac{1}{2}\right)$ (d) $\left(1, \frac{1}{2}\right)$ (e) None of the above

Solution

First let us compute the mass of the triangle, we can easily find that the region enclosed by the triangle is given by $T = \left\{(x, y) \mid 0 \le x \le 4, 0 \le y \le 2 - \frac{x}{2}\right\}$

$$\int_0^4 \int_0^{2-\frac{x}{2}} 1 \, \mathrm{d}y \, \mathrm{d}x = \int_0^4 \left(2 - \frac{x}{2}\right) \mathrm{d}x = \left[2x - \frac{x^2}{4}\right]_0^4 = 8 - 4 = 4$$

Next we want to calculate all the moments of the triangle T

$$\int_0^4 \int_0^{2-\frac{x}{2}} x \, \mathrm{d}y \, \mathrm{d}x = \int_0^4 x\left(2 - \frac{x}{2}\right) \mathrm{d}x = \left[x^2 - \frac{x^3}{6}\right]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}$$

and also

$$\begin{aligned}\int_0^4 \int_0^{2-\frac{x}{2}} y \, \mathrm{d}y \, \mathrm{d}x &= \int_0^4 \frac{1}{2}\left(2 - \frac{x}{2}\right)^2 \mathrm{d}x = \int_0^4 2 - x + \frac{x^2}{8} \, \mathrm{d}x \\ &= \left[2x - \frac{x^2}{2} + \frac{x^3}{24}\right]_0^4 = 8 - 8 + \frac{8}{3} = \frac{8}{3}\end{aligned}$$

So now we divide to get that the center of mass is $\left(\frac{4}{3}, \frac{2}{3}\right)$.

Question 4

Statement

In class we found the following estimate $\pi \leq \iint_D e^{x^2+y^2} \leq e\pi$ where D is the region $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Compute the exact value of this integral. The answer is

(a) $\pi(e-1)$ (b) $2e$ (c) $\frac{\pi e}{2}$ (d) 2π (e) None of the above

Solution

We use polar coordinates to compute this integral

$$\int_0^{2\pi}\int_0^1 e^{r^2} r \,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}e^u \,\mathrm{d}u\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}(e-1)\,\mathrm{d}u\,\mathrm{d}\theta = 2\pi\frac{1}{2}(e-1) = \pi(e-1)$$

and so a is the right answer.

Question 5

Statement

Find the dimensions of a rectangular tile with maximal area subject to the constraint that the total perimeter of the tile is 64cm^2. Please show all your work. (Hint: set this question up as a Lagrange multiper problem).

Solution

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