Mathematical Logic

Jacob Reznikov

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1. Basic Definitions and Concepts

1.1. Models and Languages

Definition 1.1.1: A *model* or *structure* is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\right)$$

where
- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.1.2: A *language* (or *signature*) is a tuple

$$L = \left(\left(\underline{f_i}\right)_{i \in I'}, \left(\underline{R_j}\right)_{j \in J'}, \left(\underline{c_k}\right)_{k \in K'}\right)$$

where now the f_i are function *symbols* with arity $a_i' \in \mathbb{N}$, each R_j are relation *symbols* with arity $a_j' \in \mathbb{N}$, and c_j are constant *symbols*.

A model $\mathcal{M}$ is an L-structure if

$$I = I', J = J', K = K', a_i = a_i', a_j = a_j'$$

If $\mathcal{M}$ is an L-structure then the *interpretations* of the symbols of the language are defined as

$$\underline{f_i}^{\mathcal{M}} = f_i, \underline{R_j}^{\mathcal{M}} = R_j, \underline{c_k}^{\mathcal{M}} = c_k$$

Remark: For a model $\mathcal{M}$ we will sometimes denote $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe.

We have defined the symbols of L, but how do we speak it? We will need the following

- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: $x, y, z, v, \ldots$
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Remark: We will often use $\overline{a}$ to denote the ordered collection $(a_1, \ldots, a_n)$ where n will be clear from context.

Definition 1.1.3: *L-terms* are defined inductively as follows

- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, \ldots, \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, \ldots, \tau_n)$$

is a term.

An L-term is said to be *constant* if it does not contain any variables.

Definition 1.1.4: If $\mathcal{M}$ is an L-structure and τ is a constant L-term then the *interpretation* of τ, $\tau^{\mathcal{M}}$, is defined equivalently
- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in |\mathcal{M}|$

Example: $L = (+, \cdot, 0, 1)$ then $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0

Definition 1.1.5: An *L-formula* is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

  $$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

  $$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called *atomic* L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x(x = 1),$$
$$(\exists x(x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Now this is all first order logic, but one might wonder, what makes it “first”? This comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in |\mathcal{M}|$, in *second* order logic we can quantify over subsets $S \subseteq |\mathcal{M}|$ like all relations for example. We can also see this as $S \in \mathcal{P}(|\mathcal{M}|)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(|\mathcal{M}|))$, and so on.

In this course, however, we will only be looking at first order logic.

Definition 1.1.6: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are *bound* in φ', and we say that φ is the *range* of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is *free* if it is not bound in φ.

An L-*sentence* is an L-formula with no free variables.

Definition 1.1.7: Let φ be a formula containing x (which we will follow denote as $\varphi(x)$), $\varphi(\tau \,/\, x)$ will denote the formula obtained by replacing every free occurrence of x by τ.

Now one would expect that substitution should never change the meaning of a logical statement, but in fact, this is not quite right. Consider the case $\varphi = \forall y (y = x)$, the substitution $\varphi(y \,/\, x)$ is changes the meaning of the statement from "all y are equal to x" to "all y are equal to themselves". We want to avoid this outcome, which we can formalize as follows.

Definition 1.1.8: A substitution $\varphi(\tau \,/\, x)$ is called *correct* if no free variable of τ becomes bound in $\varphi(\tau \,/\, x)$

Definition 1.1.9: If $A \subseteq |\mathcal{M}|$ and $\mathcal{M}$ is an L-structure then $L(A)$ is the language

$$L \cup \{a : a \in A\}$$

We extend our definition of interpretation of terms to terms of $L(|\mathcal{M}|)$ by setting $\underline{a}^{\mathcal{M}} = a$

Definition 1.1.10: Let $\mathcal{M}$ be an L-structure and σ an $L(|\mathcal{M}|)$-sentence. We say that σ is true in $\mathcal{M}$, and write $\mathcal{M} \vDash \sigma$ if

- If σ is of the form $\tau_1 = \tau_2$ then $M \vDash \sigma$ if and only if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ (note that while this may look circular, the first equality is in the space of *terms* while the second is in the universe $|\mathcal{M}|$)
- If σ is of the form $\underline{R}_j(\tau_1, ..., \tau_n)$, then $\mathcal{M} \vDash \sigma$ if and only if $\left(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}\right) \in R_j$
- If σ is of the form $\sigma_1 \wedge \sigma_2$ then $\mathcal{M} \vDash \sigma_1 \wedge \sigma_2$ if $\mathcal{M} \vDash \sigma_1$ and $\mathcal{M} \vDash \sigma_2$. A similar definition follows for the other logical connectives.
- If σ is of the form $\exists x\, \varphi$ then $\mathcal{M} \vDash \sigma$ if there exists $a \in |\mathcal{M}|$ with $\mathcal{M} \vDash \varphi(\underline{a} \,/\, x)$. Similarly for $\forall x\, \varphi$.

1.2. Model equivalences

Definition 1.2.1: Let $\mathcal{M}$ be a model. The *theory* of $\mathcal{M}$ is defined also

$$\mathrm{Th}(\mathcal{M}) = \{\sigma \text{ is an } L\text{-sentence} : \mathcal{M} \vDash \sigma\}$$

We say that two L-structures, $\mathcal{M}$ and $\mathcal{N}$, are elementarily equivalent, and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

We write that $\mathcal{M} \subseteq \mathcal{N}$ to mean that $\mathcal{M}$ is a substructure of $\mathcal{N}$, meaning that

$$|\mathcal{M}| \subseteq |\mathcal{N}|, \underline{f_i}^{\mathcal{M}} \subseteq \underline{f_i}^{\mathcal{N}}, \underline{R_j}^{\mathcal{M}} = \underline{R_j}^{\mathcal{N}} \cap |\mathcal{M}|^{a_j}, \text{ and } \underline{c_k}^{\mathcal{M}} = \underline{c_k}^{\mathcal{N}}$$

We write $\mathcal{M} \simeq \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there is a bijection g with

$$g\left(\underline{c_k}^{\mathcal{M}}\right) = \underline{c_k}^{\mathcal{N}}$$
$$(a_1, ..., a_n) \in \underline{R_j}^{\mathcal{M}} \Leftrightarrow (g(a_1), ..., g(a_n)) \in \underline{R_j}^{\mathcal{N}}$$
$$g\left(\underline{f_i}^{\mathcal{M}}(a_1, ..., a_n)\right) = \underline{f_i}^{\mathcal{N}}(g(a_1), ..., g(a_n))$$

We write $\mathcal{M} \prec (\preccurlyeq) \mathcal{N}$ to mean $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is true if $\mathcal{M} \subseteq \mathcal{N}$ and for every formula $\varphi(\overline{x})$ and for every $\overline{a} \subseteq |\mathcal{M}|$ we have

$$\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \varphi(\overline{a})$$

Theorem 1.2.1 (Tarski-Vaught test): Suppose $\mathcal{M}$ is an L-structure, $A \subseteq |\mathcal{M}|$, then A is the universe of an elementary substructure iff the following condition holds, called the Tarski-Vaught test

> For every formula $\varphi(x, \overline{y})$ in L and every $\overline{a} \subseteq A$, if $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ then there exists $b \in A$ such that $\mathcal{M} \vDash \varphi(b, \overline{a})$

Proof: First the $\Leftarrow$ direction, assume that the T-V test holds, then we need to show that A is a substructure. First we use $\varphi = (x = c)$ to show that A contains all constants of $\mathcal{M}$, then $\varphi = (x = \varphi_i(\overline{a}))$ for $\overline{a} \subseteq A$, and we define the interpretation of $\underline{R_j}$ to be exactly $\underline{R_j}^{\mathcal{M}} \cap A^{a_j}$ to make it a substructure.

Now A being a substructure is equivalent to

$$A \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{M} \vDash \varphi(\overline{a})$$

for all $\overline{a} \subseteq A$ and φ being an *atomic* formula. So now we only need to prove this is true for the other formula types.

- The connective types are immediate.
- Let us assume $\varphi(\overline{x}) = \exists y\, \psi(y, \overline{x})$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ iff $\mathcal{M} \vDash \exists y\, \psi(y, \overline{a})$ iff there exists $b \in A$ with $\mathcal{M} \vDash \psi(b, \overline{a})$. But by definition this last form is equivalent to $A \vDash \exists y\, \psi(y, \overline{a})$

Assume, on the other hand, that A is the universe of an elementary substructure $\mathcal{A}$, then we need to prove the T-V test holds, assume then that for some formula $\varphi(x, \overline{y})$ in

L and some $\overline{a} \subseteq A$ we have $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ and so since it is an elementary substructure we also have that $\mathcal{A} \vDash \exists x\, \varphi(x, \overline{a})$ and so we must have some $x \in A$ such that $\varphi(x, \overline{a})$ holds. □

Theorem 1.2.2 (Lowenheim-Skolem downwards Theorem): Let L be a language, for any L-structure $\mathcal{M}$ and every $A \subseteq |\mathcal{M}|$, there exists an elementary substructure $\mathcal{N} \prec \mathcal{M}$ with $A \subseteq |\mathcal{N}|$

$$\|\mathcal{N}\| = |A| + |L| + \aleph_0$$

Proof: Set $\kappa = |A| + |L| + \aleph_0$, using induction we will define a sequence A_n of subsets of $\mathcal{M}$, where at each step n we will try to satisfy all existential statements in $\mathrm{Th}_{L(A_{n-1})}(\mathcal{M})$, we will then set $|\mathcal{N}| = \bigcup_n A_n$.

First we set $A_0 = A$, then at step $n > 0$, we will consider all formulas in $L(A_{n-1})$ (there are $|\kappa \times \mathbb{N}| = |\kappa|$ many of them) and for each formula $\varphi(\overline{x})$ we will pick some collection of elements $\overline{a} \subseteq |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(\overline{a})$, then we will add $\overline{a}$ to A_{n-1}, adding these elements for each formula gives us A_n.

Now we can use Theorem 1.2.1 to check that $\mathcal{N} \prec \mathcal{M}$. Let $\varphi(\overline{a}) = \exists x(\psi(x), \overline{a})$ be a formula in $\mathrm{Th}_{L(\mathcal{N})}(\mathcal{M})$, then $\overline{a} \in |\mathcal{N}|$ and so $\overline{a} \in A_n$ for some n and thus for some $b \in A_{n+1}$ we have $\mathcal{N} \vDash \psi(b, \overline{a})$ and thus $\mathcal{N} \vDash \varphi(\overline{a})$. □

Remark (Skolem's Paradox): Let $ZFC^* \subseteq ZFC$ be a finite substructure which proves Cantor's theorem. Let $V \vDash ZFC^*$. By the previous theorem we can find a countable $\mathcal{M} \prec V$ for which $\mathcal{M} \vDash ZFC^*$ and $\mathcal{M} \vDash$ "exists an uncountable set".

Definition 1.2.2: In FOL we have the concept of a *proof system*, consisting of two parts. *Axioms*, and *proofs* which is a finite sequence of L-formulas such that every step is either an axiom of follows from the previous steps using an inference rule.

Example: An example proof system has the following 4 types of axioms.
- All instances of propositional tautologies are axioms.
- $[\forall x\, \varphi \to \psi] \to [\varphi \to \forall \psi]$ as long as x is not free in φ.
- $\forall x \to \varphi(t\,/\,s)$ where t is any L-term where the substitution is correct.
- $x = x$,
  $x = y \to t(..., x, ...) = t(..., y, ...)$ for any L-term,
  $x = y \to (\varphi(..., x, ...) \to \varphi(..., y, ...))$

And the following inference rules.
- If φ and $\varphi \to \psi$ then ψ.
- If φ then $\forall x\, \varphi$.

We will use the notation $\Gamma \vdash \varphi$ to mean "Γ proves φ" and define it as the existence of a proof whose final step is φ and every step is either an axiom or an element of Γ or follows from a previous step or by an inference in φ.

Definition 1.2.3: We say that Γ is consistent if there exists φ such that $\Gamma \nvdash \varphi$.

By a famous theorem of Gödel that we will not prove in this class we can actually not care about any proof system details.

Theorem 1.2.3 (Gödel's completeness theorem): Let Γ be a set of sentences in L then Γ is consistent if and only if Γ has a model.

We will not prove this theorem in this class but we will use an important corollary of it.

Corollary 1.2.3.1 (Compactness Theorem): Let Γ be a set of L-sentences, Γ has a model if and only if every finite subset of Γ has a model.

Proof: The $\Rightarrow$ direction is immediate, the hard part is the $\Leftarrow$ direction. By Gödel's completeness theorem, we can replace "having a model" with "is consistent".

We now prove this by contrapositive, assume that Γ is inconsistent, then we have $\Gamma \vdash \exists x\,(x = x) \wedge (\neg(x = x))$, now this proof consists of finitely many steps and thus can only use finitely many statements in Γ, let Γ_0 be that subset of statements. Since we can prove a contradiction using Γ_0 it must also be inconsistent, thus one of the finite subsets of Γ is inconsistent. $\square$

As an example use we have the following theorem.

Theorem 1.2.4 (Lowenheim-Skolem upwards Theorem): If $\mathcal{M}$ is an infinite L-structure where L is countably infinity then $\forall k > \|\mathcal{M}\|$ there exists a model $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| = k$

Proof: Let us consider the language $L' = L(\mathcal{M}) \cup \{c_\alpha : \alpha < \kappa\}$ where c_α are new constants. Now set

$$\Gamma = \mathrm{Th}(\mathcal{M}) \cup \left\{c_\alpha \neq c_\beta : \alpha \neq \beta < \kappa\right\}$$

We want to show now that Γ is consistent, to see this we use compactness and take an arbitrary finite subset Γ_0. Let $\alpha_1, ..., \alpha_n$ be such that

$$\Gamma_0 \subseteq \mathrm{Th}(\mathcal{M}) \cup \left\{c_{\alpha_i} \neq c_{\alpha_j} : i \neq j\right\}$$

choose then any $a_1, ..., a_n \in |\mathcal{M}|$ which are distinct and interpret c_{α_i} as a_i to get a model of Γ_0, hence Γ_0 is consistent.

Now we have by Gödel's completeness theorem that there exists a model $\mathcal{N}$ such that $\mathcal{N} \vDash \Gamma$ then by construction we have $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| \geq \kappa$ and so by downwards theorem we can now decrease the cardinality until we reach κ. $\square$

Corollary 1.2.4.1: If $\mathcal{M}$ is infinite then there exists $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$.

Proof: We simply pick some $\kappa > \|\mathcal{M}\|$ and then use the upwards theorem to get a model $\mathcal{N}$ with $\mathcal{M} \prec \mathcal{N}$ with $\|\mathcal{N}\| = \kappa$, now there can't exist a bijection between the two since they have different cardinalities. □

Definition 1.2.4: A *theory* is a set Γ of sentences such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$. A theory T is *complete* if for every sentence φ either $\varphi \in T$ or $\neg\varphi \in T$.

Remark:

- For any model $\mathcal{M}$ the theory $\mathrm{Th}(\mathcal{M})$ is complete.
- For any theory T which is complete and consistent, there exists a model $\mathcal{M}$ with $T = \mathrm{Th}(\mathcal{M})$.

1.3. Categoricity

Definition 1.3.1: Let κ be an infinite cardinal, a theory T is κ-categorical if it has infinitely many models but exactly one model (up to isomorphism) of size κ.

Proposition 1.3.1: If T is κ-categorical, then T is complete.

Proof: Suppose that T is not complete, let σ be such that $\sigma \notin T$ and $\neg\sigma \notin T$, then let $T_1 = T \cup \{\sigma\}$ and $T_2 = T \cup \{\neg\sigma\}$. Both are consistent, and thus have models of size κ which are both models of T, but the models are not isomorphic. This contradicts the fact that there is only one model of this size. □

Example: Consider the language $L = (<)$, a dense linear order (DLO_0) is the theory generated by the additional axioms: $<$ is total, dense and has no endpoints.

- Total means $\forall x\, \forall y (x = y \lor x < y \lor y < x)$
- Dense means $\forall x\, \forall y (x < y \Rightarrow \exists z x < z < y)$
- No endpoints means $\neg(\exists z\, \forall x (x \neq z \Rightarrow x < z))$ for the max endpoint and similarly for the min endpoint.

Examples of such a structure include $\mathbb{Q}$, $\mathbb{R}$ and many others.

It turns out, however, that the only countable such structure is $\mathbb{Q}$, up to isomorphism.

Proposition 1.3.2 (Cantor): DLO_0 is $\aleph_0$-categorical.

Proof: Let $(A, <)$ and $(B, <)$ be two countable models of DLO_0, we enumerate them $A = \{a_0, a_1, ...\}$ and $B = \{b_0, b_1, ...\}$.

We now use the "back and forth" method, which essentially incrementally pairs up elements of A with elements of B, and in the limit this will give us a bijection which will be our isomorphism.

More formally we will construct a sequence $\varphi_n : A_n \to B_n$ where $A_n \subseteq A, B_n \subseteq B$ where each φ_n is monotone increasing, and so that at step $2n$ we have $a_n \in A_n$ and $b_n \in B_n$.

For the base case we take a_0 and pair it with anything in B, lets say b_{20}, now we look at the smallest (in the sense of the enumeration) element b_i in B (in this case b_0), and try to map it to something in A. Now b_i will be somehow related to b_{20}, we can now use the density and the lack of endpoints to always find an element in A that has the same relations as b_i so we can always find a proper pairing.

□

Corollary 1.3.2.1: $DLO_0 = \mathrm{Th}(\mathbb{Q}, <)$, and so is complete.

Example: ACF_p is the theory generated by the axioms of an algebraically closed field of characteristic p.

The key question for any theory is, "is this theory complete?". We want to use our previous method and show that ACF_p is categorical for some cardinal, but it turns out that it is not $\aleph_0$-categorical. To see this we note that $\hat{\mathbb{Q}}, \widehat{\mathbb{Q}[a]}, \widehat{\mathbb{Q}[a, b]}, ...$ are all non-isomorphic algebraically closed fields, where a, b are transcendental and $\widehat{}$ denotes algebraic closure.

Proposition 1.3.3: ACF_p is κ-categorical for every uncountable κ.

Proof: If $K, L \vDash ACF_p$ of size κ. The transcendental degree, the size of a field's transcendental basis, will also be equal to κ, then any bijection between transcendental bases will extend to an isomorphism between K and L. □

Corollary 1.3.3.1: ACF_p is complete.

We now want to discuss how to check that two models are elementarily equivalent.

Definition 1.3.2: Given a formula φ its *quantifier depth* qd is defined by induction,
- If φ is atomic $\mathrm{qd}(\varphi) = 0$.
- If φ is a formula of the form $\varphi_1 \vee \varphi_2$ then $\mathrm{qd}(\varphi) = \max(\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2))$
- If φ is a formula of the form $\exists x\, \varphi'$ then $\mathrm{qd}(\varphi) = \mathrm{qd}(\varphi') + 1$, similarly for $\forall$.

We write $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ to mean "$\mathcal{M}$ is equivalent to $\mathcal{N}$ up to order n" if for every sentence σ of quantifier depth less than n we have $\mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{N} \vDash \varphi$.

We now define a tool for proving such partial equivalences.

Definition 1.3.3 (Ehreufeucht-Fraisse (E-F) Games): Let L be finite relational, $\Gamma(\mathcal{M}, \mathcal{N})$ is a two player game where player I is called the Spoiler and player II is called the Prover. Together they will construct a function $f : A \to B$ where $A \subseteq |\mathcal{M}|$ and $B \subseteq |\mathcal{N}|$.

Spoiler plays first and either plays an element of $m \in |\mathcal{M}|$, challenging Prover to put m in the domain of f, or they play an element $n \in |\mathcal{N}|$ challenging Prover to put it in the range of f. Prover then plays the corresponding pairing for whatever Spoiler played. Then Spoiler starts again and they continue forever. Prover wins if the resulting f is an isomorphism of the induced structures on A and B, and Spoiler wins otherwise.

We will also define a finite version of this game which we will denote $\Gamma(\mathcal{M}, \mathcal{N})_n$, it is the same as the regular game except that it ends at step n and Prover wins if when it ends it is a finite partial isomorphism.

Theorem 1.3.4: Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures where L is a finite relational language. TFAE
- $\mathcal{M} \equiv \mathcal{N}$
- The Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$ for every n.

To prove this we will need a lemma first.

Lemma 1.3.5: We say that formulas $\varphi(\overline{x}), \psi(\overline{x})$ are equivalent if $\forall \overline{x}\, \varphi(\overline{x}) \Leftrightarrow \psi(\overline{x})$ is true in every model. Equivalently if $\forall \overline{x}\, \varphi \Leftrightarrow \psi$ is provable from the empty set of axioms.

For each n, ℓ there exists a finite list $\Phi_1, ..., \Phi_k$ of formulas with $\mathrm{qd}(n)$ in ℓ variables such that every formula φ with $\mathrm{qd}(\varphi) \leq n$ in ℓ variables is equivalent to φ_i for some $i \leq k$.

Proof: We induct on n, $n = 0$, there are finitely many atomic formulas so we are done. If φ is quantifier free, then it is a boolean combination of formulas $\tau_1, ..., \tau_m$ then φ is equivalent to

$$\bigvee_{X \in S} \left(\bigwedge_{i \in X} \sigma_i \bigwedge_{i \notin X} (\neg \sigma_i) \right)$$

where S is a collection of subsets of $\{1, ..., m\}$, this case then follows from the fact that S is finite. Now assume this holds for quantifier depth at most n, if φ is of quantifier depth at most $n+1$, then φ is equivalent to a disjunction of conjunctions of formulas of the form $\exists x\, \varphi'$ or $\forall x\, \varphi'$, where $\operatorname{qd}(\varphi') \leq n$. By inductive hypothesis we then have φ' is equivalent to one of finitely many formulas Φ'_k, then $\exists x\, \varphi'$ is equivalent to $\exists x\, \Phi'_k$ and similarly for $\forall$. □

We will now use this lemma to prove a slightly weaker statement that will then use to prove the main theorem.

Lemma 1.3.6: TFAE
- $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$
- Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$.

Proof: We show equivalence by induction on n. For $n = 0$ this is obvious since the two conditions are empty. For $n > 0$ we know that one of the two players has a winning strategy since its a finite length game.

Assume then that $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$, we want to show the Prover has a winning strategy. Suppose Spoiler plays $a \in M$, by the previous lemma there exists a formula $\varphi(x)$ of quantifier depth at most $n-1$ such that $\mathcal{M} \vDash \varphi(a)$ where

$$\mathcal{N} \vDash \varphi(b) \Leftrightarrow (\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b).$$

Since $\mathcal{M} \vDash \exists x\, \varphi(x)$, the quantifier depth of $\exists x\, \varphi(x) \leq n$, and by our assumption $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ we have that $\mathcal{N} \vDash \exists x\, \varphi(x)$ so there is some b such that $\mathcal{N} \vDash \varphi(b)$. Our strategy is to just play b and then continue with whatever strategy we have for the $n-1$ step game.

Now assume that $\mathcal{M} \underset{n}{\not\equiv} \mathcal{N}$, but that the duplicator has a winning strategy, so there exists a formula $\exists x\, \varphi(x)$ where the quantifier depth of φ is at most $n-1$ such that

$$\mathcal{M} \vDash \exists x\, \varphi(x) \text{ but } \mathcal{N} \nvDash \exists x\, \varphi(x)$$

Choose $a \in |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(a)$ and make a the first move of the Spoiler. Let b be the response of the duplicator, then in $\Gamma_{n-1}(\mathcal{M}(a), \mathcal{N}(b))$ the Prover still has a winning strategy so by inductive hypothesis $(\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b)$ which contradicts the above assertion. □

Proposition 1.3.7: If $\mathcal{M}$ and $\mathcal{N}$ are countable then we also have

$$\mathcal{M} \simeq \mathcal{N} \Leftrightarrow \text{The Prover has a winning strategy in } \Gamma(\mathcal{M}, \mathcal{N})$$

Proof: Exercise, will add proof after deadline. □

1.4. Ultrafilters and Ultraproducts

Definition 1.4.1: A family $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a filter if it is non empty, does not contain the empty set and satisfies the two conditions
- $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in F$.

Example:
- The collection of cofinite subsets of $\mathbb{N}$
- The set of neighborhoods of any point in a topological space
- The set of subsets containing a fixed element in any set.

This last example is called a principal filter.

Definition 1.4.2: A filter is called an *ultrafilter* if it is not strictly contained in any other filter.

Remark: By Zorn's lemma every filter is contained in at least one ultrafilter. Since the collection of cofinite subsets is not contained in the principal filter this proves that every infinite set admits a non-principal ultrafilter (assuming ZFC).

Proposition 1.4.1: Let $\mathcal{U}$ be a filter over I. TFAE
- $\mathcal{U}$ is an ultrafilter
- For any $A \subseteq I$ we have either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, but not both.

Proof: Assume that $\mathcal{U}$ is an ultrafilter, then clearly for every A we cannot have both A and $I \setminus A$ be in $\mathcal{U}$. Now take some $A \notin \mathcal{U}$, then

$$\mathcal{U}' = \{Y' \subseteq I : Y \setminus A \subseteq Y' \text{ for some } Y \in \mathcal{U}\}$$

this is a filter since

$$Y_1 \setminus A \subseteq Y_{1'} \text{ and } Y_2 \setminus A \subseteq Y_{2'} \Rightarrow (Y_1 \cap Y_2) \setminus A = (Y_1 \setminus A) \cap (Y_2 \setminus A) \subseteq Y_{1'} \cap Y_{2'}$$

and is obviously upwards closed. Now $\mathcal{U} \subseteq \mathcal{U}'$ since for every $Y \in \mathcal{U}$ we have $Y \setminus X \subseteq Y$ and so since $\mathcal{U}$ is an ultrafilter then $\mathcal{U} = \mathcal{U}'$. But note that $I \in \mathcal{U}$ so $I \setminus A \in \mathcal{U}'$ and so $I \setminus A \in \mathcal{U}$.

On the other hand assume that the second condition holds, then let F be a filter containing $\mathcal{U}$, then if F contains a subset $A \notin \mathcal{U}$ then $I \setminus A \in \mathcal{U}$ and so $I \setminus A \in F$. But then $A \cap (I \setminus A) = \varnothing \in F$ which contradicts the definition of a filter. $\square$

Corollary 1.4.1.1: If $\mathcal{U}$ is an ultrafilter

$$A \cup B \in \mathcal{U} \Leftrightarrow A \in \mathcal{U} \vee B \in \mathcal{U}$$

Remark: An Ultrafilter has a very natural description as a finitely additive measure on I, who's only values are 0 and 1. The measure is defined by $\mu(A) = 1 \Leftrightarrow A \in I$.

In this context, if $p(x)$ holds on all $x \in A$ for some $A \in \mathcal{U}$, then we can think of this as $p(x)$ holding almost everywhere. It is through this lens that we will often think of ultrafilters, so keep this in mind as you read the rest of this section.

Definition 1.4.3: If $(\mathcal{M}_i)_{i \in I}$ are L-structures we can define $\prod_{i \in I} \mathcal{M}_i$ to be an L-structure with the natural pointwise interpretation of all the constants, relations, and functions.

This definition is not really satisfying from the point of view of model theory since it rarely preserves any structure. For example the product of two fields is not a field. However, we can take the quotient of the product by a maximal ideal to get a field, this is the approach we will try to mimic with model theory and ultrafilters.

Definition 1.4.4: Let I be a set. Let $(\mathcal{M}_i : i \in I)$ be a sequence of L-structures. Let $\mathcal{U}$ be an ultrafilter on I, the *ultraproduct*

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U}$$

is defined as follows.

On $\prod_{i \in I} |\mathcal{M}_i|$ we define the equivalence relation $\underset{\mathcal{U}}{\sim}$ by

$$(a_i) \underset{\mathcal{U}}{\sim} (b_i) \text{ if } \{i \in I : a_i = b_i\} \in \mathcal{U}$$

one can easily show that this is indeed an equivalence relation.

The universe of $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is just this infinite Cartesian product quotiented by this equivalence relation. The constants are interpreted as just the sequence of interpretations on each $\mathcal{M}_i$. Functions are interpreted pointwise as one would expect. Relations are interpreted as

$$R^{\prod_{i \in I} \mathcal{M}_i / \mathcal{U}}\left([(a_i^1)]_{\underset{\mathcal{U}}{\sim}}, ..., [(a_i^k)]_{\underset{\mathcal{U}}{\sim}} \right) \text{ if } \left\{ i \in I : \mathcal{M}_i \vDash R(a_i^1, ..., a_i^k) \right\} \in \mathcal{U}$$

Remark: One needs to check that the last two interpretations are well defined, but this is easy to do by the definition of an ultrafilter.

Remark: If $\mathcal{U}$ is the principal ultrafilter generated by $i_0 \in I$ then

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U} \simeq \mathcal{M}_{i_0}$$

Theorem 1.4.2 (Łoś's theorem): Let $\prod \mathcal{M}_i / \mathcal{U}$ be an ultraproduct, fix any formula $\varphi(x_1, ..., x_n)$ and $(a_i^1), ..., (a_i^n) \in \prod \mathcal{M}_i$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi([(a_i^1)], ..., [(a_i^n)]) \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi(a_i^1, ..., a_i^n)\} \in \mathcal{U}$$

Proof: The atomic case is covered by the definition of an ultraproduct.

We now induce on the complexity of φ,

- For $\varphi = \varphi_1 \wedge \varphi_2$ we have by definition

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2$$

now set

$$A = \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \quad B = \{i \in I : \mathcal{M}_i \vDash \varphi_2\}$$

then we know that for any A, B we have

$$A \in \mathcal{U}, B \in \mathcal{U} \Leftrightarrow A \cap B \in \mathcal{U}$$

now by inductive hypothesis we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2 \Leftrightarrow A \in \mathcal{U} \text{ and } B \in \mathcal{U}$$

and so combined this gives us exactly what we want.

- For $\varphi = \neg \varphi_1$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \nvDash \varphi_1$$

but since $\mathcal{U}$ is an ultrafilter then by Proposition 1.4.1 we have that

$$\{i \in I : \mathcal{M}_i \vDash \varphi\} \in \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi\}^c \notin \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \notin \mathcal{U}$$

which is exactly what we want. This also gives us the disjunction case.

- For $\varphi = \exists \psi$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \exists (a_i) \in \prod \mathcal{M}_i : \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \psi([a_i])$$

but by inductive hypothesis this is equivalent to

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \in \mathcal{U}$$

and so we have

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \subseteq \{i \in I : \mathcal{M}_i \vDash \exists x \, \psi(x)\}$$

and thus the right set here is also in $\mathcal{U}$ which proves what we wanted to show. □

Corollary 1.4.2.1: If the $\mathcal{M}_i$ are all elementarily equivalent then

$$\mathrm{Th}\left(\prod \mathcal{M}_i \Big/ \mathcal{U}\right) = \mathrm{Th}(\mathcal{M}_i)$$

Definition 1.4.5: If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then $\prod \mathcal{M}_i / \mathcal{U}$ is called the *ultrapower* of $\mathcal{M}$.

Corollary 1.4.2.2: Let T be a set of sentences T has a model iff every finite subset of T has a model.

Proof: Assume that L is countable and T is countable and enumerate $T = \{\sigma_1, \sigma_2, ...\}$. Then set T_n to be the truncation of T, that is $T_n = \{\sigma_1, ..., \sigma_n\}$. By assumption we have the existence of some models $\mathcal{M}_n$ with $\mathcal{M}_n \vDash T_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$.

Set $\mathcal{M} = \prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$, then

$$\mathcal{M} \vDash \sigma \Leftrightarrow \{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma\} \in \mathcal{U}$$

Now for a fixed σ_i we have $\mathcal{M}_n \vDash \sigma_i$ if $n \geq i$ so

$$\{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma_i\} \in \mathcal{U}$$

because it is cofinite and a non-principal ultrafilter contains all cofinite sets. Thus

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \sigma_i$$

The uncountable case is a bit more complicated, we start with defining

$$F = \{\Delta \subseteq T : \Delta \text{ is finite}\}.$$

Now set $X_\Delta = \{Y \in F : \Delta \subseteq Y\}$, then I claim that the set

$$D = \{Y \subseteq F : X_\Delta \subseteq Y \text{ for some } \Delta\}$$

is a filter. This is easy to see by just checking the definition. Now since it is a filter it is contained in some maximal ultrafilter $\mathcal{U}$. Now for each finite subset $\Delta \in F$ we have some model $\mathcal{M}_\Delta \vDash \Delta$ so we can consider $\mathcal{M} = \prod_{\Delta \in F} \mathcal{M}_\Delta / \mathcal{U}$. Now for a fixed $\sigma \in T$ we have that

$$\{\Delta \in F : \mathcal{M}_\Delta \vDash \sigma\} \supseteq X_{\{\sigma\}} \in \mathcal{U},$$

and so $\mathcal{M} \vDash \sigma$. $\square$

1.5. Types and Definable Sets

We will now develop more tools to use with models, first of these is the **type**, in short, a type is to formulas what a satisfiable theory is to sentences.

Definition 1.5.1: Let L be countable, and T a complete L-theory. Let $\mathcal{M} \vDash T$ then for $a \in |\mathcal{M}|$ we say that the *type* of a is

$$\mathrm{tp}^{\mathcal{M}}(a) = \{\varphi(x) : \mathcal{M} \vDash \varphi(a)\}.$$

If two elements a, b have the same type then we cannot distinguish a, b with first order formulas.

More generally, if $\overline{a}$ is a tuple of elements of $\mathcal{M}$ then the *type* of $\overline{a}$ is

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \{\varphi(x) : \mathcal{M} \vDash \varphi(\overline{a})\}.$$

We will also use the following notation

$$F_L(\overline{x}) = \{\text{formulas with variables } \overline{x}\}$$

if $\varphi(\overline{a}) \in F_L(\overline{x})$ and $\mathcal{M}$ is a model

$$\varphi(\mathcal{M}) = \{\overline{a} \in \mathcal{M} : \mathcal{M} \vDash \varphi(\overline{a})\}$$

Definition 1.5.2: $\varphi(\overline{x})$ is T-*consistent* if $T \vdash \exists \overline{x} \varphi(\overline{x})$ or equivalently $\varphi(\mathcal{M}) \neq \emptyset$.

Definition 1.5.3: A set of formulas $p(\overline{x}) \subseteq F_L(\overline{x})$ is T-*consistent* if for every finite subset $p_0(\overline{x}) \subseteq p(\overline{x})$ we have

$$T \vdash \exists \overline{x} \left(\bigwedge_{\varphi \in p_0} \varphi(\overline{x}) \right)$$

Definition 1.5.4: A *type in* T is a set of formulas $p(\overline{x})$ which is T-consistent, we call it a 1-type if $\overline{x} = x$ and an n-type if $\overline{x} = (x_1, ..., x_n)$

Definition 1.5.5: A type $p(\overline{x})$ is *complete* if for every formula $\varphi(\overline{x}) \in F_L(\overline{x})$ either $\varphi(\overline{x}) \in p$ or $\neg\varphi(\overline{x}) \in p$

Example: $\mathrm{tp}^{\mathcal{M}}(\overline{x})$ is always a complete type

Remark: If $\mathcal{M} \prec \mathcal{N}$, and $\overline{a} \in \mathcal{M}$ then $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{a})$.

Slightly generalizing the concept of a type we have the following

Definition 1.5.6: For a set of parameters $A \subseteq |\mathcal{M}|$ we define

$$T(A) = \mathrm{Th}_{L(A)}(\mathcal{M}),$$

that is all the true $L(A)$-sentences in $\mathcal{M}$.

A type *over* A is a type in $T(A)$.

We then have the generalization of the notation,

$$F_{L(A)}(\overline{x}) = \{\varphi(\overline{x}, \overline{a}) : \overline{a} \in A, \varphi(\overline{x}, \overline{y}) \in F_L(\overline{x}, \overline{y})\}$$

and

$$\mathrm{tp}^{\mathcal{M}}\left(\overline{b} \,/\, A\right) = \left\{\varphi(\overline{x}, \overline{a}) : \mathcal{M} \vDash \varphi\left(\overline{b}, \overline{a}\right)\right\}$$

as well as

$$S_n^T(A) = \{\text{all complete n-types in } T \text{ on } A\}$$

Definition 1.5.7: A type $p(\overline{x})$ is realized in a model $\mathcal{M}$ if there exists $\overline{a} \in \mathcal{M}$ with $p(\overline{x}) \subseteq \mathrm{tp}^{\mathcal{M}}(\overline{a})$.

Example: If $T = DLO_0$ and $\mathcal{M} = \mathbb{Q}$ then

$$p(x) = \left\{s < x, x < r : s < \sqrt{2} < r\right\}$$

is not realized in $\mathbb{Q}$.

Types have several basic properties that we will use quite often.

Proposition 1.5.1: If $p(\overline{x})$ is a type over $A \subseteq |\mathcal{M}|$ then there exists $\mathcal{M} \prec \mathcal{N}$ such that $p(\overline{x})$ is realized in $\mathcal{N}$.

Proof: Let $\overline{c}$ be new constants, define

$$T' = \{\varphi(\overline{c}) : \varphi(\overline{x}) \in p(\overline{x})\} \cup \mathrm{Th}_{L(M)}(M)$$

and model of T' will realize p because the interpretation of $\overline{c}$ will realize p.

Since $\mathrm{Th}_{L(M)}(M) \subseteq T'$ any model of T' will be an elementary extension of $\mathcal{M}$. It is thus enough to show that T' is consistent.

By assumption every finite subset of $p(\overline{x})$ will be consistent with $\mathrm{Th}_{L(M)}(M)$ and thus by compactness T' is consistent. $\square$

Corollary 1.5.1.1: Every type is a subset of a complete type since if p is realized by $\overline{b} \in \mathcal{N}$ then $p \subseteq \mathrm{tp}^{\mathcal{N}}\left(\overline{b} \,/\, A\right)$

We can also prove the above corollary in a different way, using Zorn's lemma, we will need some more notation.

Definition 1.5.8: A subset $F \subseteq \mathbb{B} \setminus \{0\}$, where $\mathbb{B}$ is a Boolean algebra, is a *filter* if
- If $a, b \in F$ then $a \cdot b \in F$.
- If $a \in F$ and $a \leq b$ then $b \in F$

An *ultrafilter* is a maximal filter with respect to inclusion.

Example: The principal ultrafilter $\mathcal{U}$ on $\mathbb{B}$ if $\mathcal{U} = \{a \in \mathbb{B} : a \geq a_0\}$ for some atom a_0.

Definition 1.5.9: If $\mathbb{B}$ is a Boolean algebra then $S(\mathbb{B})$ is the set of all ultrafilters over $\mathbb{B}$, we can give it a topology generated by

$$[a] = \{\mathcal{U} \in S(\mathbb{B}) : a \in \mathcal{U}\}$$

Proposition 1.5.2:
1. $\{[a] : a \in \mathbb{B}\}$ is indeed a basis of a topology.
2. $[a]^c = [-a]$
3. $[a + b] = [a] \cup [b]$
4. $[a \cdot b] = [a] \cap [b]$
5. The topology defined above is Hausdorff and compact.

Proof:
1. This will follow from 4.
2. For any ultrafilter $\mathcal{U}$ that does not contain a we must have $-a \in \mathcal{U}$ and so
$$\mathcal{U} \in [a] \Leftrightarrow \mathcal{U} \notin [-a]$$
3. Since $a, b \leq a + b$ then
$$(\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b]) \Rightarrow (a + b) \in \mathcal{U} \Rightarrow \mathcal{U} \in [a + b]$$
on the other hand $a \cup b \in \mathcal{U} \Rightarrow (a \in \mathcal{U}) \vee (b \in \mathcal{U})$ and so
$$\mathcal{U} \in [a + b] \Rightarrow (\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b])$$
4. Since $a \cdot b \leq a, b$ then almost by definition

$$(a \in \mathcal{U}) \wedge (b \in \mathcal{U}) \Leftrightarrow a \cdot b \in \mathcal{U}$$

5. For any two distinct ultrafilters $\mathcal{U}, \mathcal{U}'$, then for some x we have $x \in \mathcal{U}$ and $x \notin \mathcal{U}'$. Then $\mathcal{U} \in [x], \mathcal{U}' \notin [x]$ as well as $\mathcal{U} \notin [-x], \mathcal{U}' \in [-x]$ and so the topology is Hausdorff. To show compactness let $\bigcup_i [a_i] = S(\mathbb{B})$, then $\{-a_i\}$ cannot be a subset of any ultrafilter $\mathcal{U}$ since it would not be in the union, thus some finite subset of $-a_i$'s must have product zero. But then if $\left\{-a_{i_1}, ..., -a_{i_k}\right\}$ has zero product then any ultrafilter cannot contain all of them, thus any ultrafilter $\mathcal{U}$ has to contain some a_{i_j} and so $\bigcup_k \left[a_{i_k}\right] = S(\mathbb{B})$.

□

Theorem 1.5.3 (Stone's Theorem): For every Boolean algebra $\mathbb{B}$ there exists a set I with $\mathbb{B} \subseteq \mathcal{P}$

Proof: Set $I = S(\mathbb{B})$ and the map $a \mapsto [a]$ is clearly a homomorphism by the above proposition, to see it is 1 to 1 we use the proof for Hausdorffness above to see that $[a] \neq [b]$. □

Proposition 1.5.4: Let $\mathcal{U}$ be an ultrafilter, $\mathcal{U}$ is principal iff it is isolated in $S(\mathbb{B})$.

Proof: Assume that $\{\mathcal{U}\}$ is an open set, then $\{\mathcal{U}\} = [a]$ for some a. Now if a is not atomic then $0 < b < a$ for some b and so $[a] = [a \cdot b] \cup [a \cdot (-b)]$ but $[a \cdot b], [a \cdot (-b)]$ are both non-empty and not equal since they both contain the ultrafilters generated by the filter

$$\{Y \in \mathbb{B} : a \cdot b \leq Y\} \text{ and } \{Y \in \mathbb{B} : a \cdot (-b) \leq Y\}$$

this contradicts the fact that $[a]$ contains only one element. Thus a is an atom and so the principal ultrafilter of a is in $[a]$. Since $[a] = \{\mathcal{U}\}$ we have that U must be the principal ultrafilter of a.

On the other hand if $\mathcal{U}$ is principal then $\mathcal{U} \in [a]$ for some atom a but since its atomic anything in $[a]$ must be the principal ultrafilter of a. Thus $[a] = \{\mathcal{U}\}$ and so $\mathcal{U}$ is isolated. □

Definition 1.5.10: Let T be a complete theory and $\mathcal{M} \vDash T$ then

$$\operatorname{Def}(\mathcal{M}) = \{\varphi(\mathcal{M}) : \varphi \in F_L(x)\}$$

is a Boolean algebra of subsets of $\mathcal{M}$ called the algebra of *definable* subsets of $\mathcal{M}$.

Proposition 1.5.5: The map $\iota : F_L(\overline{x}) \to \mathrm{Def}(\mathcal{M})$ given by

$$\iota : \varphi \mapsto \varphi(\mathcal{M})$$

is a homomorphism.

Remark: $\ker(\iota) = \{\varphi : \varphi(\mathcal{M}) = \emptyset\}$ is the set of T-inconsistent formulas.

We have then by Isomorphism theorem for rings

$$F_L(\overline{x}) / \ker(\iota) = \mathrm{Def}(\mathcal{M})$$

We can also identify $S_n^T(\emptyset)$ with $S(F_L(\overline{x}))$ which makes it a compact set with basic open sets $[\varphi(\overline{x})] = \{p \in S_n^T(\emptyset) : \varphi(\overline{x}) \in p\}$.

Proposition 1.5.6: If L is countable then $S_n^T(\emptyset)$ is homeomorphic to a closed subset of the Cantor space.

Proof: To see this we will turn $S_n^T(\emptyset)$ into an infinite binary tree, first enumerate $F_L(\overline{x}) = \{\varphi_1, ...\}$ then for every type $p \in S_n^T(\emptyset)$ we have either $\varphi_1 \in p$ or $\neg\varphi_1 \in p$. This gives a splitting of $S_n^T(\emptyset)$ into two open subsets, we then split again on φ_2 and get 4 open subsets. Continuing this construction, we get that the complete types will be infinite branches in this tree, and it is well known that such an infinite binary tree is isomorphic to the Cantor space. □

Remark: This construction can also be done with L uncountable, we then get a homomorphism to $2^{|L|}$ seen as a product space.

The closed sets of $S_n^T(\emptyset)$ are of the form $[p(\overline{x})] = \{q \in S_n^T(\emptyset) : p \subseteq q\}$.

All of these also hold if we change $S_n^T(\emptyset)$ to $S_n^T(A)$

Definition 1.5.11: If $\mathcal{M}$ is a model of T and $\kappa \geq \aleph_0$ is an infinite cardinal, we say that $\mathcal{M}$ is *κ-saturated* if for every subset $A \subseteq |\mathcal{M}|$ of size less than κ every type in $S_n^T(A)$ is realized in $\mathcal{M}$.

$\mathcal{M}$ is *saturated* if $\mathcal{M}$ is $|\mathcal{M}|$-saturated.

Remark: $\{x \neq a : a \in \mathcal{M}\}$ is not realized in any model $\mathcal{M}$, so no model is κ-saturated for any $\kappa > |\mathcal{M}|$.

We will next show how to construct saturated models, to complete this we will need a lemma.

Definition 1.5.12: For a cardinal γ, $\mathrm{cf}(\gamma)$ is called the co-finality of γ and is the cardinality of the shortest unbounded sequence in γ.

Theorem 1.5.7 (König's theorem): For a cardinal γ, $\mathrm{cf}(2^{\gamma}) > \gamma$.

Lemma 1.5.8: If $(\mathcal{N}_\alpha)_{\alpha<\kappa}$ is an elementary chain, that is $\mathcal{N}_\alpha \prec \mathcal{N}_\beta$ for $\alpha < \beta$. Then if $\mathcal{N} = \bigcup_{\alpha=0}^{\kappa} \mathcal{N}_\alpha$ we have $\mathcal{N}_\alpha \prec \mathcal{N}$ for all α.

Proof: Let $\varphi(\overline{a})$ be a formula, we show that $\mathcal{N}_\alpha \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N}_\alpha$ for all alpha by induction. Since every $\mathcal{N}_\alpha$ is contained in $\mathcal{N}$ then this is true for all atomic formula φ. Now we induct on the structure of φ, for logical connectives this is trivial. Now assume that $\varphi = \exists x\, \psi(x, \overline{a})$, then certainly $\mathcal{N}_i \vDash \varphi \Rightarrow \mathcal{N} \vDash \varphi$, now if $\mathcal{N} \vDash \varphi(\overline{a})$ then there is some $j \geq i$ such that $b \in |\mathcal{N}_j|$ and so $\mathcal{N}_j \vDash \psi(b, \overline{a})$ so $\mathcal{N}_j \vDash \varphi(\overline{a})$ and so $\mathcal{N}_i \vDash \varphi(\overline{a})$. $\square$

Theorem 1.5.9: For every κ, for every $\mathcal{M}$, there exists a model $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ and $\mathcal{N}$ is κ-saturated.

If κ is *weakly inaccessible*, that is $\lambda < \kappa \Rightarrow 2^{\lambda} \leq \kappa$ (note that such cardinals cannot be proved to exist in ZFC) then for every $\mathcal{M}$ with $|\mathcal{M}| \leq \kappa$ there exists $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ saturated with size κ.

Proof: Assume that L is countable, then $S_n^T(A) \leq 2^{|A|\ +\aleph_0}$ by Proposition 1.5.6. Let $\mu = 2^{\kappa}$, note that $\mathrm{cf}(\mu) > \kappa$ by Theorem 1.5.7.

We will now construct a sequence of models $(\mathcal{M}_\alpha)_{\alpha<\mu}$ with $\mathcal{M}_0 = \mathcal{M}$ and at limit α we have $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$, we will assume that $|\mathcal{M}_\alpha| < \mu$. At successor steps $\alpha = \beta + 1$, we want to find $\mathcal{M}_\alpha$ with $\mathcal{M}_\beta \prec \mathcal{M}_\alpha$ such that for all $A \subseteq \mathcal{M}_\beta$ with $|A| < \kappa$, every type in $S_n^T(A)$ is realized in $\mathcal{M}_\alpha$. Now we know that for every single type $p(\overline{x})$ by Proposition 1.5.1 we can add a realization of that type, and then by Theorem 1.2.2 we can get that realization with size at most μ, so we just need to do induction again to add every type.

Let us count how many types we need to add, we know that for any fixed A we have $|S_n^T(A)| \leq 2^{\kappa+\aleph_0} = \mu$. Now for any cardinality β we have that the number of subsets A with $|A| = \beta$ is

$$\mu^{\beta} = (2^{\kappa})^{\beta} = 2^{\kappa\times\beta} = 2^{\kappa} = \mu$$

so in total we have $\sum_{\lambda<\kappa} \mu^{\lambda} = \kappa\mu = \mu$ steps and so our final model $\mathcal{M}_{\alpha+1}$ is also of size at most μ which completes the induction. $\square$

Example: There are strange consequences to this theorem, for example there are models of Piano Arithmetic that satisfy a statement encoding "PA is inconsistent".

We can see that the process of adding types is not very difficult, in model theory we have a saying about this: "Any fool can realize a type, but it takes a model theorist to omit one". We have not yet looked at omitting types, but the definition is exactly what you would expect.

Definition 1.5.13: For a complete theory T, a model $\mathcal{M} \vDash T$ and a type $p(\overline{x})$. We say that $\mathcal{M}$ *omits* $p(\overline{x})$ if it does not realize it, i.e. $p(\mathcal{M}) = \varnothing$.

Now the difficulty in omitting types is that some types can **never** be omitted.

Example: If c is a constant of a language L then the type of the interpretation of c can never be omitted.

But some types can be omitted

Example: The type of a transcendental number in ACF_p is distinct from that of an algebraic number, and can be omitted, for example in $\hat{Q}$.

The first example here is an important one to keep in mind since all the properties of that type can be proved from the single formula $x = c$.

Definition 1.5.14: A type $p(\overline{x})$ is isolated if there exists a formula $\varphi(\overline{x}) \in p(\overline{x})$ such that for every $\psi(\overline{x}) \in p(\overline{x})$ we have

$$T \vdash (\varphi(\overline{x}) \Rightarrow \psi(\overline{x}))$$

Proposition 1.5.10: $p(\overline{x}) \in S_{n(A)}$ is isolated iff $\{p\}$ is open in $S_{n(A)}$.

Proof: Exercise □

This characterization is important due to the following fact.

Proposition 1.5.11: If $p(\overline{x})$ is isolated, then p cannot be omitted.

Proof: Let $\varphi(\overline{x})$ be the generating formula for p, then

$$\exists x \varphi(\overline{x})$$

is a true sentence in T and thus any witness of this sentence is a realization of the type. □

Now apriori we would not expect this converse to hold since it feels like being isolated is quite the strong condition, but in fact the converse does hold, which is shown in this theorem.

Theorem 1.5.12: If $p(\overline{x})$ is not isolated, then there exists $\mathcal{M} \vDash T$ which omits $p(\overline{x})$.

There are many proofs of this theorem but we will use one called **Henkin's construction**. This proof method is also the modern method for proving Theorem 1.2.3.

Proof: Let L be a countable language and let $\{c_n\}_{n\in\mathbb{N}}$ be a family of new constants not in L, enumerate all formulas in $L \cup \{c_n\}_{n\in\mathbb{N}}$ as φ_n. Let $f : \mathbb{N} \to \mathbb{N}$ be increasing such that $c_{f(n)}$ does not appear in $\varphi_0, ..., \varphi_n$.

We define the **Henkin axioms**

$$H_i = (\exists x \varphi_i(x)) \to \varphi_i\left(c_{f(i)}\right).$$

We now construct a sequence of sets of sentences $T_0 = T \subseteq T_1 \subseteq T_2 \subseteq ...$ such that

$$T_{2n+1} = T_{2n} \cup \{H_n\} \quad \text{and} \quad T_{2n+2} = T_{2n+1} \cup \left\{\neg\varphi_{n(c_n)}\right\} \text{ for some } \varphi_{n(\overline{x})} \in p(\overline{x})$$

Then taking the union of these sets we will get an axiomization of a consistent theory. We can then use Zorn's lemma to get a complete theory containing it and then if we set our universe to be the set of constants quotiented by the relation

$$c_i = c_j \text{ as elements if } \left(c_i = c_j\right) \text{ as a formula is in } T$$

Now a model satisfying this theory will not realize the type $p(\overline{x})$ since if it did then some constant would realize it which would contradict the fact that our theory contains $\neg\varphi(c_n)$ for every n.

All that is left to do is to check that at every odd step these sentences are indeed consistent and that at even steps we can pick specific φ_n to make the set of sentences consistent.

For the even steps assume that T_{2n+1} is consistent but for every $\psi(\overline{x}) \in p(\overline{x})$ we have that $T_{2n+1} \cup \{\neg\psi(c_n)\}$ is inconsistent. Then T_{2n+1} is T where we added some finitely many sentences, so we can write $T_{2n+1} = T \cup \left\{\psi_j(\overline{c}, c_n) : j < k\right\}$ for some k and ψ_j.

Now set

$$\varphi(\overline{y}, x) = \bigwedge_{j<k} \psi_j(\overline{y}, x)$$

then for every $\psi(\overline{x}) \in p(\overline{x})$ we have $T \cup \{\varphi(\overline{c}, c_n)\} \cup \{\neg\psi(c_n)\}$ is inconsistent so

$$T \vdash (\varphi(\overline{c}, c_n) \to \psi(c_n))$$

But now since the T does not contain c_n as a constant we can replace all instances of c_n with x and all instances of $\overline{c}$ with $\overline{y}$ in the proof and get that

$$T \vdash (\varphi(\overline{y}, x)) \to \psi(x))$$

but then this means that

$$T \vdash \forall\overline{y}(\varphi(\overline{y}, x) \to \psi(x))$$

but we have that

$$\begin{aligned}\forall \overline{y}(\varphi(\overline{y}, x) \to \psi(x)) &= \forall \overline{y}(\neg\varphi(\overline{y}, x) \vee \psi(x)) = \neg\exists \overline{y}(\varphi(\overline{y}, x) \wedge \neg\psi(x)) \\ &= \neg(\exists \overline{y}\varphi(\overline{y}, x) \wedge \neg\psi(x)) = (\exists \overline{y}(\varphi(\overline{y}, x))) \to \psi(x)\end{aligned}$$

then $\exists \overline{y}(\varphi(\overline{y}, x))$ implies every ψ in the type $p(\overline{x})$ which contradicts our assumption that $p(\overline{x})$ is not isolated. $\square$