Mathematical Logic

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1. Basic Definitions and Concepts

1.1. Models and Languages

Definition 1.1.1: A *model* or *structure* is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\right)$$

where

- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.1.2: A *language* (or *signature*) is a tuple

$$L = \left(\left(\underline{f_i} \right)_{i \in I'}, \left(\underline{R_j} \right)_{j \in J'}, \left(\underline{c_k} \right)_{k \in K'} \right)$$

where now the f_i are function *symbols* with arity $a_i' \in \mathbb{N}$, each R_j are relation *symbols* with arity $a_j' \in \mathbb{N}$, and c_j are constant *symbols*.

A model $\mathcal{M}$ is an L-structure if

$$I = I', J = J', K = K', a_i = a_i', a_j = a_j'$$

If $\mathcal{M}$ is an L-structure then the *interpretations* of the symbols of the language are defined as

$$\underline{f_i}^{\mathcal{M}} = f_i, \underline{R_j}^{\mathcal{M}} = R_j, \underline{c_k}^{\mathcal{M}} = c_k$$

Remark: For a model $\mathcal{M}$ we will sometimes denote $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe.

We have defined the symbols of L, but how do we speak it? We will need the following

- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: $x, y, z, v, \ldots$
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Remark: We will often use $\overline{a}$ to denote the ordered collection $(a_1, \ldots, a_n)$ where n will be clear from context.

Definition 1.1.3: *L-terms* are defined inductively as follows

- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, \ldots, \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, \ldots, \tau_n)$$

is a term.

An L-term is said to be *constant* if it does not contain any variables.

Definition 1.1.4: If $\mathcal{M}$ is an L-structure and τ is a constant L-term then the *interpretation* of τ, $\tau^{\mathcal{M}}$, is defined equivalently
- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in |\mathcal{M}|$

Example: $L = (+, \cdot, 0, 1)$ then $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0

Definition 1.1.5: An *L-formula* is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

  $$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

  $$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called *atomic* L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x (x = 1),$$
$$(\exists x (x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Now this is all first order logic, but one might wonder, what makes it “first”? This comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in |\mathcal{M}|$, in *second* order logic we can quantify over subsets $S \subseteq |\mathcal{M}|$ like all relations for example. We can also see this as $S \in \mathcal{P}(|\mathcal{M}|)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(|\mathcal{M}|))$, and so on.

In this course, however, we will only be looking at first order logic.

Definition 1.1.6: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are *bound* in φ', and we say that φ is the *range* of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is *free* if it is not bound in φ.

An *L-sentence* is an L-formula with no free variables.

Definition 1.1.7: Let φ be a formula containing x (which we will follow denote as $\varphi(x)$), $\varphi(\tau \,/\, x)$ will denote the formula obtained by replacing every free occurrence of x by τ.

Now one would expect that substitution should never change the meaning of a logical statement, but in fact, this is not quite right. Consider the case $\varphi = \forall y(y = x)$, the substitution $\varphi(y \,/\, x)$ is changes the meaning of the statement from "all y are equal to x" to "all y are equal to themselves". We want to avoid this outcome, which we can formalize as follows.

Definition 1.1.8: A substitution $\varphi(\tau \,/\, x)$ is called *correct* if no free variable of τ becomes bound in $\varphi(\tau \,/\, x)$

Definition 1.1.9: If $A \subseteq |\mathcal{M}|$ and $\mathcal{M}$ is an L-structure then $L(A)$ is the language

$$L \cup \{\underline{a} : a \in A\}$$

We extend our definition of interpretation of terms to terms of $L(\mathcal{M})$ by setting $\underline{a}^{\mathcal{M}} = a$

Definition 1.1.10: Let $\mathcal{M}$ be an L-structure and σ an $L(\mathcal{M})$-sentence. We say that σ is true in $\mathcal{M}$, and write $\mathcal{M} \vDash \sigma$ if

- If σ is of the form $\tau_1 = \tau_2$ then $M \vDash \sigma$ if and only if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ (note that while this may look circular, the first equality is in the space of *terms* while the second is in the universe $|\mathcal{M}|$)
- If σ is of the form $\underline{R}_j(\tau_1, ..., \tau_n)$, then $\mathcal{M} \vDash \sigma$ if and only if $\left(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}\right) \in R_j$
- If σ is of the form $\sigma_1 \wedge \sigma_2$ then $\mathcal{M} \vDash \sigma_1 \wedge \sigma_2$ if $\mathcal{M} \vDash \sigma_1$ and $\mathcal{M} \vDash \sigma_2$. A similar definition follows for the other logical connectives.
- If σ is of the form $\exists x\, \varphi$ then $\mathcal{M} \vDash \sigma$ if there exists $a \in |\mathcal{M}|$ with $\mathcal{M} \vDash \varphi(\underline{a} \,/\, x)$. Similarly for $\forall x\, \varphi$.

1.2. Model equivalences

Definition 1.2.1: Let $\mathcal{M}$ be a model. The *theory* of $\mathcal{M}$ is defined also

$$\mathrm{Th}(\mathcal{M}) = \{\sigma \text{ is an } L\text{-sentence} : \mathcal{M} \vDash \sigma\}$$

We say that two L-structures, $\mathcal{M}$ and $\mathcal{N}$, are elementarily equivalent, and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

We write that $\mathcal{M} \subseteq \mathcal{N}$ to mean that $\mathcal{M}$ is a substructure of $\mathcal{N}$, meaning that

$$|\mathcal{M}| \subseteq |\mathcal{N}|, \underline{f_i}^{\mathcal{M}} \subseteq \underline{f_i}^{\mathcal{N}}, \underline{R_j}^{\mathcal{M}} = \underline{R_j}^{\mathcal{N}} \cap |\mathcal{M}|^{a_j}, \text{ and } \underline{c_k}^{\mathcal{M}} = \underline{c_k}^{\mathcal{N}}$$

We write $\mathcal{M} \simeq \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there is a bijection g with

$$g\left(\underline{c_k}^{\mathcal{M}}\right) = \underline{c_k}^{\mathcal{N}}$$
$$(a_1, ..., a_n) \in \underline{R_j}^{\mathcal{M}} \Leftrightarrow (g(a_1), ..., g(a_n)) \in \underline{R_j}^{\mathcal{N}}$$
$$g\left(\underline{f_i}^{\mathcal{M}}(a_1, ..., a_n)\right) = \underline{f_i}^{\mathcal{N}}(g(a_1), ..., g(a_n))$$

We write $\mathcal{M} \prec (\preccurlyeq) \mathcal{N}$ to mean $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is true if $\mathcal{M} \subseteq \mathcal{N}$ and for every formula $\varphi(\overline{x})$ and for every $\overline{a} \subseteq |\mathcal{M}|$ we have

$$\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \varphi(\overline{a})$$

Theorem 1.2.1 (Tarski-Vaught test): Suppose $\mathcal{M}$ is an L-structure, $A \subseteq |\mathcal{M}|$, then A is the universe of an elementary substructure iff the following condition holds, called the Tarski-Vaught test

> For every formula $\varphi(x, \overline{y})$ in L and every $\overline{a} \subseteq A$, if $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ then there exists $b \in A$ such that $\mathcal{M} \vDash \varphi(b, \overline{a})$

Proof: First the $\Leftarrow$ direction, assume that the T-V test holds, then we need to show that A is a substructure. First we use $\varphi = (x = c)$ to show that A contains all constants of $\mathcal{M}$, then $\varphi = (x = \varphi_i(\overline{a}))$ for $\overline{a} \subseteq A$, and we define the interpretation of $\underline{R_j}$ to be exactly $\underline{R_j}^{\mathcal{M}} \cap A^{a_j}$ to make it a substructure.

Now A being a substructure is equivalent to

$$A \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{M} \vDash \varphi(\overline{a})$$

for all $\overline{a} \subseteq A$ and φ being an *atomic* formula. So now we only need to prove this is true for the other formula types.

- The connective types are immediate.
- Let us assume $\varphi(\overline{x}) = \exists y\, \psi(y, \overline{x})$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ iff $\mathcal{M} \vDash \exists y\, \psi(y, \overline{a})$ iff there exists $b \in A$ with $\mathcal{M} \vDash \psi(b, \overline{a})$. But by definition this last form is equivalent to $A \vDash \exists y\, \psi(y, \overline{a})$

Assume, on the other hand, that A is the universe of an elementary substructure $\mathcal{A}$, then we need to prove the T-V test holds, assume then that for some formula $\varphi(x, \overline{y})$ in

L and some $\overline{a} \subseteq A$ we have $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ and so since it is an elementary substructure we also have that $\mathcal{A} \vDash \exists x\, \varphi(x, \overline{a})$ and so we must have some $x \in A$ such that $\varphi(x, \overline{a})$ holds. $\square$

Theorem 1.2.2 (Lowenheim-Skolem downwards Theorem): Let L be a language, for any L-structure $\mathcal{M}$ and every $A \subseteq |\mathcal{M}|$, there exists an elementary substructure $\mathcal{N} \prec \mathcal{M}$ with $A \subseteq |\mathcal{N}|$

$$\|\mathcal{N}\| = |A| + |L| + \aleph_0$$

Proof: Set $\kappa = |A| + |L| + \aleph_0$, using induction we will define a sequence A_n of subsets of $\mathcal{M}$, where at each step n we will try to satisfy all existential statements in $\mathrm{Th}_{L(A_{n-1})}(\mathcal{M})$, we will then set $|\mathcal{N}| = \bigcup_n A_n$.

First we set $A_0 = A$, then at step $n > 0$, we will consider all formulas in $L(A_{n-1})$ (there are $|\kappa \times \mathbb{N}| = |\kappa|$ many of them) and for each formula $\varphi(\overline{x})$ we will pick some collection of elements $\overline{a} \subseteq |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(\overline{a})$, then we will add $\overline{a}$ to A_{n-1}, adding these elements for each formula gives us A_n.

Now we can use Theorem 1.2.1 to check that $\mathcal{N} \prec \mathcal{M}$. Let $\varphi(\overline{a}) = \exists x(\psi(x), \overline{a})$ be a formula in $\mathrm{Th}_{L(\mathcal{N})}(\mathcal{M})$, then $\overline{a} \in |\mathcal{N}|$ and so $\overline{a} \in A_n$ for some n and thus for some $b \in A_{n+1}$ we have $\mathcal{N} \vDash \psi(b, \overline{a})$ and thus $\mathcal{N} \vDash \varphi(\overline{a})$. $\square$

Remark (Skolem's Paradox): Let $ZFC^* \subseteq ZFC$ be a finite substructure which proves Cantor's theorem. Let $V \vDash ZFC^*$. By the previous theorem we can find a countable $\mathcal{M} \prec V$ for which $\mathcal{M} \vDash ZFC^*$ and $\mathcal{M} \vDash$ "exists an uncountable set".

Definition 1.2.2: In FOL we have the concept of a *proof system*, consisting of two parts. *Axioms*, and *proofs* which is a finite sequence of L-formulas such that every step is either an axiom of follows from the previous steps using an inference rule.

Example: An example proof system has the following 4 types of axioms.
- All instances of propositional tautologies are axioms.
- $[\forall x\, \varphi \to \psi] \to [\varphi \to \forall \psi]$ as long as x is not free in φ.
- $\forall x \to \varphi(t \,/\, s)$ where t is any L-term where the substitution is correct.
- $x = x$,
  $x = y \to t(..., x, ...) = t(..., y, ...)$ for any L-term,
  $x = y \to (\varphi(..., x, ...) \to \varphi(..., y, ...))$

And the following inference rules.
- If φ and $\varphi \to \psi$ then ψ.
- If φ then $\forall x\, \varphi$.

We will use the notation $\Gamma \vdash \varphi$ to mean "Γ proves φ" and define it as the existence of a proof whose final step is φ and every step is either an axiom or an element of Γ or follows from a previous step or by an inference in φ.

Definition 1.2.3: We say that Γ is consistent if there exists φ such that $\Gamma \nvdash \varphi$.

By a famous theorem of Gödel that we will not prove in this class we can actually not care about any proof system details.

Theorem 1.2.3 (Gödel's completeness theorem): Let Γ be a set of sentences in L then Γ is consistent if and only if Γ has a model.

We will not prove this theorem in this class but we will use an important corollary of it.

Corollary 1.2.3.1 (Compactness Theorem): Let Γ be a set of L-sentences, Γ has a model if and only if every finite subset of Γ has a model.

Proof: The $\Rightarrow$ direction is immediate, the hard part is the $\Leftarrow$ direction. By Gödel's completeness theorem, we can replace "having a model" with "is consistent".

We now prove this by contrapositive, assume that Γ is inconsistent, then we have $\Gamma \vdash \exists x\,(x = x) \wedge (\neg(x = x))$, now this proof consists of finitely many steps and thus can only use finitely many statements in Γ, let Γ_0 be that subset of statements. Since we can prove a contradiction using Γ_0 it must also be inconsistent, thus one of the finite subsets of Γ is inconsistent. $\square$

As an example use we have the following theorem.

Theorem 1.2.4 (Lowenheim-Skolem upwards Theorem): If $\mathcal{M}$ is an infinite L-structure where L is countably infinity then $\forall k > \|\mathcal{M}\|$ there exists a model $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| = k$

Proof: Let us consider the language $L' = L(\mathcal{M}) \cup \{c_\alpha : \alpha < \kappa\}$ where c_α are new constants. Now set

$$\Gamma = \mathrm{Th}(\mathcal{M}) \cup \left\{c_\alpha \neq c_\beta : \alpha \neq \beta < \kappa\right\}$$

We want to show now that Γ is consistent, to see this we use compactness and take an arbitrary finite subset Γ_0. Let $\alpha_1, ..., \alpha_n$ be such that

$$\Gamma_0 \subseteq \mathrm{Th}(\mathcal{M}) \cup \left\{c_{\alpha_i} \neq c_{\alpha_j} : i \neq j\right\}$$

choose then any $a_1, ..., a_n \in |\mathcal{M}|$ which are distinct and interpret c_{α_i} as a_i to get a model of Γ_0, hence Γ_0 is consistent.

Now we have by Gödel's completeness theorem that there exists a model $\mathcal{N}$ such that $\mathcal{N} \vDash \Gamma$ then by construction we have $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| \geq \kappa$ and so by downwards theorem we can now decrease the cardinality until we reach κ. $\square$

Corollary 1.2.4.1: If $\mathcal{M}$ is infinite then there exists $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$.

Proof: We simply pick some $\kappa > \|\mathcal{M}\|$ and then use the upwards theorem to get a model $\mathcal{N}$ with $\mathcal{M} \prec \mathcal{N}$ with $\|\mathcal{N}\| = \kappa$, now there can't exist a bijection between the two since they have different cardinalities. □

Definition 1.2.4: A *theory* is a set Γ of sentences such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$. A theory T is *complete* if for every sentence φ either $\varphi \in T$ or $\neg\varphi \in T$.

Remark:

- For any model $\mathcal{M}$ the theory $\mathrm{Th}(\mathcal{M})$ is complete.
- For any theory T which is complete and consistent, there exists a model $\mathcal{M}$ with $T = \mathrm{Th}(\mathcal{M})$.

1.3. Categoricity

Definition 1.3.1: Let κ be an infinite cardinal, a theory T is κ-categorical if it has infinitely many models but exactly one model (up to isomorphism) of size κ.

Proposition 1.3.1: If T is κ-categorical, then T is complete.

Proof: Suppose that T is not complete, let σ be such that $\sigma \notin T$ and $\neg\sigma \notin T$, then let $T_1 = T \cup \{\sigma\}$ and $T_2 = T \cup \{\neg\sigma\}$. Both are consistent, and thus have models of size κ which are both models of T, but the models are not isomorphic. This contradicts the fact that there is only one model of this size. □

Example: Consider the language $L = (<)$, a dense linear order (DLO_0) is the theory generated by the additional axioms: $<$ is total, dense and has no endpoints.

- Total means $\forall x\, \forall y (x = y \vee x < y \vee y < x)$
- Dense means $\forall x\, \forall y (x < y \Rightarrow \exists z x < z < y)$
- No endpoints means $\neg(\exists z\, \forall x (x \neq z \Rightarrow x < z))$ for the max endpoint and similarly for the min endpoint.

Examples of such a structure include $\mathbb{Q}$, $\mathbb{R}$ and many others.

It turns out, however, that the only countable such structure is $\mathbb{Q}$, up to isomorphism.

Proposition 1.3.2 (Cantor): DLO_0 is $\aleph_0$-categorical.

Proof: Let $(A, <)$ and $(B, <)$ be two countable models of DLO_0, we enumerate them $A = \{a_0, a_1, ...\}$ and $B = \{b_0, b_1, ...\}$.

We now use the "back and forth" method, which essentially incrementally pairs up elements of A with elements of B, and in the limit this will give us a bijection which will be our isomorphism.

More formally we will construct a sequence $\varphi_n : A_n \to B_n$ where $A_n \subseteq A, B_n \subseteq B$ where each φ_n is monotone increasing, and so that at step $2n$ we have $a_n \in A_n$ and $b_n \in B_n$.

For the base case we take a_0 and pair it with anything in B, lets say b_{20}, now we look at the smallest (in the sense of the enumeration) element b_i in B (in this case b_0), and try to map it to something in A. Now b_i will be somehow related to b_{20}, we can now use the density and the lack of endpoints to always find an element in A that has the same relations as b_i so we can always find a proper pairing.

□

Corollary 1.3.2.1: $DLO_0 = \mathrm{Th}(\mathbb{Q}, <)$, and so is complete.

Example: ACF_p is the theory generated by the axioms of an algebraically closed field of characteristic p.

The key question for any theory is, "is this theory complete?". We want to use our previous method and show that ACF_p is categorical for some cardinal, but it turns out that it is not $\aleph_0$-categorical. To see this we note that $\hat{\mathbb{Q}}, \widehat{\mathbb{Q}[a]}, \widehat{\mathbb{Q}[a, b]}, ...$ are all non-isomorphic algebraically closed fields, where a, b are transcendental and $\widehat{\ }$ denotes algebraic closure.

Proposition 1.3.3: ACF_p is κ-categorical for every uncountable κ.

Proof: If $K, L \vDash ACF_p$ of size κ. The transcendental degree, the size of a field's transcendental basis, will also be equal to κ, then any bijection between transcendental bases will extend to an isomorphism between K and L. □

Corollary 1.3.3.1: ACF_p is complete.

We now want to discuss how to check that two models are elementarily equivalent.

Definition 1.3.2: Given a formula φ its *quantifier depth* qd is defined by induction,
- If φ is atomic $\mathrm{qd}(\varphi) = 0$.
- If φ is a formula of the form $\varphi_1 \vee \varphi_2$ then $\mathrm{qd}(\varphi) = \max(\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2))$
- If φ is a formula of the form $\exists x\, \varphi'$ then $\mathrm{qd}(\varphi) = \mathrm{qd}(\varphi') + 1$, similarly for $\forall$.

We write $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ to mean "$\mathcal{M}$ is equivalent to $\mathcal{N}$ up to order n" if for every sentence σ of quantifier depth less than n we have $\mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{N} \vDash \varphi$.

We now define a tool for proving such partial equivalences.

Definition 1.3.3 (Ehreufeucht-Fraisse (E-F) Games): Let L be finite relational, $\Gamma(\mathcal{M}, \mathcal{N})$ is a two player game where player I is called the Spoiler and player II is called the Prover. Together they will construct a function $f : A \to B$ where $A \subseteq |\mathcal{M}|$ and $B \subseteq |\mathcal{N}|$.

Spoiler plays first and either plays an element of $m \in |\mathcal{M}|$, challenging Prover to put m in the domain of f, or they play an element $n \in |\mathcal{N}|$ challenging Prover to put it in the range of f. Prover then plays the corresponding pairing for whatever Spoiler played. Then Spoiler starts again and they continue forever. Prover wins if the resulting f is an isomorphism of the induced structures on A and B, and Spoiler wins otherwise.

We will also define a finite version of this game which we will denote $\Gamma(\mathcal{M}, \mathcal{N})_n$, it is the same as the regular game except that it ends at step n and Prover wins if when it ends it is a finite partial isomorphism.

Theorem 1.3.4: Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures where L is a finite relational language. TFAE
- $\mathcal{M} \equiv \mathcal{N}$
- The Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$ for every n.

To prove this we will need a lemma first.

Lemma 1.3.5: We say that formulas $\varphi(\overline{x}), \psi(\overline{x})$ are equivalent if $\forall \overline{x}\, \varphi(\overline{x}) \Leftrightarrow \psi(\overline{x})$ is true in every model. Equivalently if $\forall \overline{x}\, \varphi \Leftrightarrow \psi$ is provable from the empty set of axioms.

For each n, ℓ there exists a finite list $\Phi_1, ..., \Phi_k$ of formulas with $\mathrm{qd}(n)$ in ℓ variables such that every formula φ with $\mathrm{qd}(\varphi) \leq n$ in ℓ variables is equivalent to φ_i for some $i \leq k$.

Proof: We induct on n, $n = 0$, there are finitely many atomic formulas so we are done. If φ is quantifier free, then it is a boolean combination of formulas $\tau_1, ..., \tau_m$ then φ is equivalent to

$$\bigvee_{X \in S} \left(\bigwedge_{i \in X} \sigma_i \bigwedge_{i \notin X} (\neg \sigma_i) \right)$$

where S is a collection of subsets of $\{1, ..., m\}$, this case then follows from the fact that S is finite. Now assume this holds for quantifier depth at most n, if φ is of quantifier depth at most $n + 1$, then φ is equivalent to a disjunction of conjunctions of formulas of the form $\exists x\, \varphi'$ or $\forall x\, \varphi'$, where $\mathrm{qd}(\varphi') \leq n$. By inductive hypothesis we then have φ' is equivalent to one of finitely many formulas Φ'_k, then $\exists x\, \varphi'$ is equivalent to $\exists x\, \Phi'_k$ and similarly for $\forall$. $\square$

We will now use this lemma to prove a slightly weaker statement that will then use to prove the main theorem.

Lemma 1.3.6: TFAE
- $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$
- Prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})_n$.

Proof: We show equivalence by induction on n. For $n = 0$ this is obvious since the two conditions are empty. For $n > 0$ we know that one of the two players has a winning strategy since its a finite length game.

Assume then that $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$, we want to show the Prover has a winning strategy. Suppose Spoiler plays $a \in M$, by the previous lemma there exists a formula $\varphi(x)$ of quantifier depth at most $n - 1$ such that $\mathcal{M} \vDash \varphi(a)$ where

$$\mathcal{N} \vDash \varphi(b) \Leftrightarrow (\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b).$$

Since $\mathcal{M} \vDash \exists x\, \varphi(x)$, the quantifier depth of $\exists x\, \varphi(x) \leq n$, and by our assumption $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ we have that $\mathcal{N} \vDash \exists x\, \varphi(x)$ so there is some b such that $\mathcal{N} \vDash \varphi(b)$. Our strategy is to just play b and then continue with whatever strategy we have for the $n - 1$ step game.

Now assume that $\mathcal{M} \underset{n}{\not\equiv} \mathcal{N}$, but that the duplicator has a winning strategy, so there exists a formula $\exists x\, \varphi(x)$ where the quantifier depth of φ is at most $n - 1$ such that

$$\mathcal{M} \vDash \exists x\, \varphi(x) \text{ but } \mathcal{N} \nvDash \exists x\, \varphi(x)$$

Choose $a \in |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(a)$ and make a the first move of the Spoiler. Let b be the response of the duplicator, then in $\Gamma_{n-1}(\mathcal{M}(a), \mathcal{N}(b))$ the Prover still has a winning strategy so by inductive hypothesis $(\mathcal{M}, a) \underset{n-1}{\equiv} (\mathcal{N}, b)$ which contradicts the above assertion. $\square$

Proposition 1.3.7: If $\mathcal{M}$ and $\mathcal{N}$ are countable then we also have

$$\mathcal{M} \simeq \mathcal{N} \Leftrightarrow \text{The Prover has a winning strategy in } \Gamma(\mathcal{M}, \mathcal{N})$$

Proof: Assume $\mathcal{M} \simeq \mathcal{N}$, then the Prover wins trivially by just following the isomorphism.

On the other hand assume Prover has a winning strategy, then we can play the role of the Spoiler to force Prover to construct an isomorphism. First enumerate the models

$$|\mathcal{M}| = \{m_0, m_1, ...\}, \quad |\mathcal{N}| = \{n_0, n_1, ...\}$$

on the first turn we pick m_0 and let Prover map it to some element of $|\mathcal{N}|$. On the second turn we pick the smallest index element of $|\mathcal{N}|$ that has not been picked before and force Prover to map it. We continue this, on odd turns we pick the smallest index element of $|\mathcal{M}|$ that has not been picked before, and on even turns we pick the smallest index element of $|\mathcal{N}|$ that has not been picked before. This essentially forces Prover to use the back-and-forth method. Since every element of both models will eventually be mapped and since Prover has to win this game, the resulting map $\bigcup_i f_i$ will be an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

1.4. Ultrafilters and Ultraproducts

Definition 1.4.1: A family $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a filter if it is non empty, does not contain the empty set and satisfies the two conditions
- $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in F$.

Example:
- The collection of cofinite subsets of $\mathbb{N}$
- The set of neighborhoods of any point in a topological space
- The set of subsets containing a fixed element in any set.

This last example is called a principal filter.

Definition 1.4.2: A filter is called an *ultrafilter* if it is not strictly contained in any other filter.

Remark: By Zorn's lemma every filter is contained in at least one ultrafilter. Since the collection of cofinite subsets is not contained in the principal filter this proves that every infinite set admits a non-principal ultrafilter (assuming ZFC).

Proposition 1.4.1: Let $\mathcal{U}$ be a filter over I. TFAE
- $\mathcal{U}$ is an ultrafilter
- For any $A \subseteq I$ we have either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, but not both.

Proof: Assume that $\mathcal{U}$ is an ultrafilter, then clearly for every A we cannot have both A and $I \setminus A$ be in $\mathcal{U}$. Now take some $A \notin \mathcal{U}$, then

$$\mathcal{U}' = \{Y' \subseteq I : Y \setminus A \subseteq Y' \text{ for some } Y \in \mathcal{U}\}$$

this is a filter since

$$Y_1 \setminus A \subseteq Y_{1'} \text{ and } Y_2 \setminus A \subseteq Y_{2'} \Rightarrow (Y_1 \cap Y_2) \setminus A = (Y_1 \setminus A) \cap (Y_2 \setminus A) \subseteq Y_{1'} \cap Y_{2'}$$

and is obviously upwards closed. Now $\mathcal{U} \subseteq \mathcal{U}'$ since for every $Y \in \mathcal{U}$ we have $Y \setminus X \subseteq Y$ and so since $\mathcal{U}$ is an ultrafilter then $\mathcal{U} = \mathcal{U}'$. But note that $I \in \mathcal{U}$ so $I \setminus A \in \mathcal{U}'$ and so $I \setminus A \in \mathcal{U}$.

On the other hand assume that the second condition holds, then let F be a filter containing $\mathcal{U}$, then if F contains a subset $A \notin \mathcal{U}$ then $I \setminus A \in \mathcal{U}$ and so $I \setminus A \in F$. But then $A \cap (I \setminus A) = \varnothing \in F$ which contradicts the definition of a filter. □

Corollary 1.4.1.1: If $\mathcal{U}$ is an ultrafilter

$$A \cup B \in \mathcal{U} \Leftrightarrow A \in \mathcal{U} \vee B \in \mathcal{U}$$

Remark: An Ultrafilter has a very natural description as a finitely additive measure on I, who's only values are 0 and 1. The measure is defined by $\mu(A) = 1 \Leftrightarrow A \in I$.

In this context, if $p(x)$ holds on all $x \in A$ for some $A \in \mathcal{U}$, then we can think of this as $p(x)$ holding almost everywhere. It is through this lens that we will often think of ultrafilters, so keep this in mind as you read the rest of this section.

Definition 1.4.3: If $(\mathcal{M}_i)_{i \in I}$ are L-structures we can define $\prod_{i \in I} \mathcal{M}_i$ to be an L-structure with the natural pointwise interpretation of all the constants, relations, and functions.

This definition is not really satisfying from the point of view of model theory since it rarely preserves any structure. For example the product of two fields is not a field. However, we can take the quotient of the product by a maximal ideal to get a field, this is the approach we will try to mimic with model theory and ultrafilters.

Definition 1.4.4: Let I be a set. Let $(\mathcal{M}_i : i \in I)$ be a sequence of L-structures. Let $\mathcal{U}$ be an ultrafilter on I, the *ultraproduct*

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U}$$

is defined as follows.

On $\prod_{i \in I} |\mathcal{M}_i|$ we define the equivalence relation $\underset{\mathcal{U}}{\sim}$ by

$$(a_i) \underset{\mathcal{U}}{\sim} (b_i) \text{ if } \{i \in I : a_i = b_i\} \in \mathcal{U}$$

one can easily show that this is indeed an equivalence relation.

The universe of $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is just this infinite Cartesian product quotiented by this equivalence relation. The constants are interpreted as just the sequence of interpretations on each $\mathcal{M}_i$. Functions are interpreted pointwise as one would expect. Relations are interpreted as

$$R^{\prod_{i \in I} \mathcal{M}_i / \mathcal{U}}\left(\left[\left(a_i^1\right)\right]_{\underset{\mathcal{U}}{\sim}}, ..., \left[\left(a_i^k\right)\right]_{\underset{\mathcal{U}}{\sim}}\right) \text{ if } \left\{i \in I : \mathcal{M}_i \vDash R\left(a_i^1, ..., a_i^k\right)\right\} \in \mathcal{U}$$

Remark: One needs to check that the last two interpretations are well defined, but this is easy to do by the definition of an ultrafilter.

Remark: If $\mathcal{U}$ is the principal ultrafilter generated by $i_0 \in I$ then

$$\prod_{i \in I} \mathcal{M}_i \Big/ \mathcal{U} \simeq \mathcal{M}_{i_0}$$

Theorem 1.4.2 (Łoś's theorem): Let $\prod \mathcal{M}_i / \mathcal{U}$ be an ultraproduct, fix any formula $\varphi(x_1, ..., x_n)$ and $(a_i^1), ..., (a_i^n) \in \prod \mathcal{M}_i$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi([(a_i^1)], ..., [(a_i^n)]) \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi(a_i^1, ..., a_i^n)\} \in \mathcal{U}$$

Proof: The atomic case is covered by the definition of an ultraproduct.

We now induce on the complexity of φ,

- For $\varphi = \varphi_1 \land \varphi_2$ we have by definition

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2$$

now set

$$A = \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \quad B = \{i \in I : \mathcal{M}_i \vDash \varphi_2\}$$

then we know that for any A, B we have

$$A \in \mathcal{U}, B \in \mathcal{U} \Leftrightarrow A \cap B \in \mathcal{U}$$

now by inductive hypothesis we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_1 \text{ and} \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi_2 \Leftrightarrow A \in \mathcal{U} \text{ and } B \in \mathcal{U}$$

and so combined this gives us exactly what we want.

- For $\varphi = \neg\varphi_1$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \prod \mathcal{M}_i \Big/ \mathcal{U} \nvDash \varphi_1$$

but since $\mathcal{U}$ is an ultrafilter then by Proposition 1.4.1 we have that

$$\{i \in I : \mathcal{M}_i \vDash \varphi\} \in \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi\}^c \notin \mathcal{U} \Leftrightarrow \{i \in I : \mathcal{M}_i \vDash \varphi_1\} \notin \mathcal{U}$$

which is exactly what we want. This also gives us the disjunction case.

- For $\varphi = \exists\psi$ we have

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \varphi \Leftrightarrow \exists (a_i) \in \prod \mathcal{M}_i : \prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \psi([a_i])$$

but by inductive hypothesis this is equivalent to

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \in \mathcal{U}$$

and so we have

$$\{i \in I : \mathcal{M}_i \vDash \psi(a_i)\} \subseteq \{i \in I : \mathcal{M}_i \vDash \exists x\, \psi(x)\}$$

and thus the right set here is also in $\mathcal{U}$ which proves what we wanted to show. □

Corollary 1.4.2.1: If the $\mathcal{M}_i$ are all elementarily equivalent then

$$\mathrm{Th}\left(\prod \mathcal{M}_i \Big/ \mathcal{U}\right) = \mathrm{Th}(\mathcal{M}_i)$$

Definition 1.4.5: If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then $\prod \mathcal{M}_i / \mathcal{U}$ is called the *ultrapower* of $\mathcal{M}$.

Corollary 1.4.2.2: Let T be a set of sentences T has a model iff every finite subset of T has a model.

Proof: Assume that L is countable and T is countable and enumerate $T = \{\sigma_1, \sigma_2, ...\}$. Then set T_n to be the truncation of T, that is $T_n = \{\sigma_1, ..., \sigma_n\}$. By assumption we have the existence of some models $\mathcal{M}_n$ with $\mathcal{M}_n \vDash T_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$.

Set $\mathcal{M} = \prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$, then

$$\mathcal{M} \vDash \sigma \Leftrightarrow \{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma\} \in \mathcal{U}$$

Now for a fixed σ_i we have $\mathcal{M}_n \vDash \sigma_i$ if $n \geq i$ so

$$\{n \in \mathbb{N} : \mathcal{M}_n \vDash \sigma_i\} \in \mathcal{U}$$

because it is cofinite and a non-principal ultrafilter contains all cofinite sets. Thus

$$\prod \mathcal{M}_i \Big/ \mathcal{U} \vDash \sigma_i$$

The uncountable case is a bit more complicated, we start with defining

$$F = \{\Delta \subseteq T : \Delta \text{ is finite}\}.$$

Now set $X_\Delta = \{Y \in F : \Delta \subseteq Y\}$, then I claim that the set

$$D = \{Y \subseteq F : X_\Delta \subseteq Y \text{ for some } \Delta\}$$

is a filter. This is easy to see by just checking the definition. Now since it is a filter it is contained in some maximal ultrafilter $\mathcal{U}$. Now for each finite subset $\Delta \in F$ we have some model $\mathcal{M}_\Delta \vDash \Delta$ so we can consider $\mathcal{M} = \prod_{\Delta \in F} \mathcal{M}_\Delta / \mathcal{U}$. Now for a fixed $\sigma \in T$ we have that

$$\{\Delta \in F : \mathcal{M}_\Delta \vDash \sigma\} \supseteq X_{\{\sigma\}} \in \mathcal{U},$$

and so $\mathcal{M} \vDash \sigma$. $\square$

1.5. Types and Definable Sets

We will now develop more tools to use with models, first of these is the **type**, in short, a type is to formulas what a satisfiable theory is to sentences.

Definition 1.5.1: Let L be countable, and T a complete L-theory. Let $\mathcal{M} \vDash T$ then for $a \in |\mathcal{M}|$ we say that the *type* of a is

$$\mathrm{tp}^{\mathcal{M}}(a) = \{\varphi(x) : \mathcal{M} \vDash \varphi(a)\}.$$

If two elements a, b have the same type then we cannot distinguish a, b with first order formulas.

More generally, if $\overline{a}$ is a tuple of elements of $\mathcal{M}$ then the *type* of $\overline{a}$ is

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \{\varphi(x) : \mathcal{M} \vDash \varphi(\overline{a})\}.$$

We will also use the following notation

$$F_L(\overline{x}) = \{\text{formulas with variables } \overline{x}\}$$

if $\varphi(\overline{a}) \in F_L(\overline{x})$ and $\mathcal{M}$ is a model

$$\varphi(\mathcal{M}) = \{\overline{a} \in \mathcal{M} : \mathcal{M} \vDash \varphi(\overline{a})\}$$

Definition 1.5.2: $\varphi(\overline{x})$ is T-*consistent* if $T \vdash \exists \overline{x} \varphi(\overline{x})$ or equivalently $\varphi(\mathcal{M}) \neq \varnothing$.

Definition 1.5.3: A set of formulas $p(\overline{x}) \subseteq F_L(\overline{x})$ is T-*consistent* if for every finite subset $p_0(\overline{x}) \subseteq p(\overline{x})$ we have

$$T \vdash \exists \overline{x} \left(\bigwedge_{\varphi \in p_0} \varphi(\overline{x}) \right)$$

Definition 1.5.4: A *type in* T is a set of formulas $p(\overline{x})$ which is T-consistent, we call it a 1-type if $\overline{x} = x$ and an n-type if $\overline{x} = (x_1, ..., x_n)$

Definition 1.5.5: A type $p(\overline{x})$ is *complete* if for every formula $\varphi(\overline{x}) \in F_L(\overline{x})$ either $\varphi(\overline{x}) \in p$ or $\neg\varphi(\overline{x}) \in p$

Example: $\mathrm{tp}^{\mathcal{M}}(\overline{x})$ is always a complete type

Remark: If $\mathcal{M} \prec \mathcal{N}$, and $\overline{a} \in \mathcal{M}$ then $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{a})$.

Slightly generalizing the concept of a type we have the following

Definition 1.5.6: For a set of parameters $A \subseteq |\mathcal{M}|$ we define

$$T(A) = \mathrm{Th}_{L(A)}(\mathcal{M}),$$

that is all the true $L(A)$-sentences in $\mathcal{M}$.

A type *over* A is a type in $T(A)$.

We then have the generalization of the notation,

$$F_{L(A)}(\overline{x}) = \{\varphi(\overline{x}, \overline{a}) : \overline{a} \in A, \varphi(\overline{x}, \overline{y}) \in F_L(\overline{x}, \overline{y})\}$$

and

$$\mathrm{tp}^{\mathcal{M}}\left(\overline{b} \,/\, A\right) = \left\{\varphi(\overline{x}, \overline{a}) : \mathcal{M} \vDash \varphi\left(\overline{b}, \overline{a}\right)\right\}$$

as well as

$$S_n^T(A) = \{\text{all complete n-types in } T \text{ on } A\}$$

Definition 1.5.7: A type $p(\overline{x})$ is realized in a model $\mathcal{M}$ if there exists $\overline{a} \in \mathcal{M}$ with $p(\overline{x}) \subseteq \mathrm{tp}^{\mathcal{M}}(\overline{a})$.

Example: If $T = DLO_0$ and $\mathcal{M} = \mathbb{Q}$ then

$$p(x) = \left\{s < x, x < r : s < \sqrt{2} < r\right\}$$

is not realized in $\mathbb{Q}$.

Types have several basic properties that we will use quite often.

Proposition 1.5.1: If $p(\overline{x})$ is a type over $A \subseteq |\mathcal{M}|$ then there exists $\mathcal{M} \prec \mathcal{N}$ such that $p(\overline{x})$ is realized in $\mathcal{N}$.

Proof: Let $\overline{c}$ be new constants, define

$$T' = \{\varphi(\overline{c}) : \varphi(\overline{x}) \in p(\overline{x})\} \cup \mathrm{Th}_{L(M)}(M)$$

and model of T' will realize p because the interpretation of $\overline{c}$ will realize p.

Since $\mathrm{Th}_{L(M)}(M) \subseteq T'$ any model of T' will be an elementary extension of $\mathcal{M}$. It is thus enough to show that T' is consistent.

By assumption every finite subset of $p(\overline{x})$ will be consistent with $\mathrm{Th}_{L(M)}(M)$ and thus by compactness T' is consistent. $\square$

Corollary 1.5.1.1: Every type is a subset of a complete type since if p is realized by $\overline{b} \in \mathcal{N}$ then $p \subseteq \mathrm{tp}^{\mathcal{N}}\left(\overline{b} \,/\, A\right)$

We can also prove the above corollary in a different way, using Zorn's lemma, we will need some more notation.

Definition 1.5.8: A subset $F \subseteq \mathbb{B} \setminus \{0\}$, where $\mathbb{B}$ is a Boolean algebra, is a *filter* if
- If $a, b \in F$ then $a \cdot b \in F$.
- If $a \in F$ and $a \leq b$ then $b \in F$

An *ultrafilter* is a maximal filter with respect to inclusion.

Example: The principal ultrafilter $\mathcal{U}$ on $\mathbb{B}$ if $\mathcal{U} = \{a \in \mathbb{B} : a \geq a_0\}$ for some atom a_0.

Definition 1.5.9: If $\mathbb{B}$ is a Boolean algebra then $S(\mathbb{B})$ is the set of all ultrafilters over $\mathbb{B}$, we can give it a topology generated by

$$[a] = \{\mathcal{U} \in S(\mathbb{B}) : a \in \mathcal{U}\}$$

Proposition 1.5.2:
1. $\{[a] : a \in \mathbb{B}\}$ is indeed a basis of a topology.
2. $[a]^c = [-a]$
3. $[a + b] = [a] \cup [b]$
4. $[a \cdot b] = [a] \cap [b]$
5. The topology defined above is Hausdorff and compact.

Proof:
1. This will follow from 4.
2. For any ultrafilter $\mathcal{U}$ that does not contain a we must have $-a \in \mathcal{U}$ and so

$$\mathcal{U} \in [a] \Leftrightarrow \mathcal{U} \notin [-a]$$

3. Since $a, b \leq a + b$ then

$$(\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b]) \Rightarrow (a + b) \in \mathcal{U} \Rightarrow \mathcal{U} \in [a + b]$$

on the other hand $a + b \in \mathcal{U} \Rightarrow (a \in \mathcal{U}) \vee (b \in \mathcal{U})$ and so

$$\mathcal{U} \in [a + b] \Rightarrow (\mathcal{U} \in [a]) \vee (\mathcal{U} \in [b])$$

4. Since $a \cdot b \leq a, b$ then almost by definition

$$(a \in \mathcal{U}) \wedge (b \in \mathcal{U}) \Leftrightarrow a \cdot b \in \mathcal{U}$$

5. For any two distinct ultrafilters $\mathcal{U}, \mathcal{U}'$, then for some x we have $x \in \mathcal{U}$ and $x \notin \mathcal{U}'$. Then $\mathcal{U} \in [x], \mathcal{U}' \notin [x]$ as well as $\mathcal{U} \notin [-x], \mathcal{U}' \in [-x]$ and so the topology is Hausdorff. To show compactness let $\bigcup_i [a_i] = S(\mathbb{B})$, then $\{-a_i : i \in I\}$ cannot be a subset of any ultrafilter $\mathcal{U}$, for then

$$-a_i \in \mathcal{U}, \forall i \in I \Rightarrow a_i \notin \mathcal{U}, \forall i \in I \Rightarrow \mathcal{U} \notin [a_i], \forall i \in I \Rightarrow \mathcal{U} \notin \bigcup_i [a_i].$$

Thus, some finite subset of $-a_i$'s must have product zero since otherwise $\{-a_i : i \in I\}$ satisfies the finite intersection property and thus is contained in some ultrafilter. But then if $\left\{-a_{i_1}, ..., -a_{i_k}\right\}$ has zero product then any ultrafilter cannot contain all of them, thus any ultrafilter $\mathcal{U}$ has to contain some a_{i_j} and so $\bigcup_k \left[a_{i_k}\right] = S(\mathbb{B})$.

□

Theorem 1.5.3 (Stone's Theorem): For every Boolean algebra $\mathbb{B}$ there exists a set I with $\mathbb{B} \subseteq \mathcal{P}$

Proof: Set $I = S(\mathbb{B})$ and the map $a \mapsto [a]$ is clearly a homomorphism by the above proposition, to see it is 1 to 1 we use the proof for Hausdorffness above to see that $[a] \neq [b]$. □

Proposition 1.5.4: Let $\mathcal{U}$ be an ultrafilter, $\mathcal{U}$ is principal iff it is isolated in $S(\mathbb{B})$.

Proof: Assume that $\{\mathcal{U}\}$ is an open set, then $\{\mathcal{U}\} = [a]$ for some a. Now if a is not atomic then $0 < b < a$ for some b and so $[a] = [a \cdot b] \cup [a \cdot (-b)]$ but $[a \cdot b], [a \cdot (-b)]$ are both non-empty and not equal since they both contain the ultrafilters generated by the filter

$$\{Y \in \mathbb{B} : a \cdot b \leq Y\} \text{ and } \{Y \in \mathbb{B} : a \cdot (-b) \leq Y\}$$

this contradicts the fact that $[a]$ contains only one element. Thus a is an atom and so the principal ultrafilter of a is in $[a]$. Since $[a] = \{\mathcal{U}\}$ we have that U must be the principal ultrafilter of a.

On the other hand if $\mathcal{U}$ is principal then $\mathcal{U} \in [a]$ for some atom a but since its atomic anything in $[a]$ must be the principal ultrafilter of a. Thus $[a] = \{\mathcal{U}\}$ and so $\mathcal{U}$ is isolated. □

Definition 1.5.10: Let T be a complete theory and $\mathcal{M} \vDash T$ then

$$\operatorname{Def}(\mathcal{M}) = \{\varphi(\mathcal{M}) : \varphi \in F_L(x)\}$$

is a Boolean algebra of subsets of $\mathcal{M}$ called the algebra of *definable* subsets of $\mathcal{M}$.

Proposition 1.5.5: The map $\iota : F_L(\overline{x}) \to \mathrm{Def}(\mathcal{M})$ given by

$$\iota : \varphi \mapsto \varphi(\mathcal{M})$$

is a homomorphism.

Remark: $\ker(\iota) = \{\varphi : \varphi(\mathcal{M}) = \varnothing\}$ is the set of T-inconsistent formulas.

We have then by Isomorphism theorem for rings

$$F_L(\overline{x}) / \ker(\iota) \cong \mathrm{Def}(\mathcal{M})$$

We can also identify $S_n^T(\varnothing)$ with $S(F_L(\overline{x}))$ which makes it a compact set with basic open sets $[\varphi(\overline{x})] = \{p \in S_n^T(\varnothing) : \varphi(\overline{x}) \in p\}$.

Proposition 1.5.6: If L is countable then $S_n^T(\varnothing)$ is homeomorphic to a closed subset of the Cantor space.

Proof: To see this we will turn $S_n^T(\varnothing)$ into an infinite binary tree, first enumerate $F_L(\overline{x}) = \{\varphi_1, ...\}$ then for every type $p \in S_n^T(\varnothing)$ we have either $\varphi_1 \in p$ or $\neg\varphi_1 \in p$. This gives a splitting of $S_n^T(\varnothing)$ into two open subsets, we then split again on φ_2 and get 4 open subsets. Continuing this construction, we get that the complete types will be infinite branches in this tree, and it is well known that such an infinite binary tree is isomorphic to the Cantor space. □

Remark: This construction can also be done with L uncountable, we then get a homomorphism to $2^{|L|}$ seen as a product space.

The closed sets of $S_n^T(\varnothing)$ are of the form $[p(\overline{x})] = \{q \in S_n^T(\varnothing) : p \subseteq q\}$.

All of these also hold if we change $S_n^T(\varnothing)$ to $S_n^T(A)$

Definition 1.5.11: If $\mathcal{M}$ is a model of T and $\kappa \geq \aleph_0$ is an infinite cardinal, we say that $\mathcal{M}$ is *κ-saturated* if for every subset $A \subseteq |\mathcal{M}|$ of size less than κ every type in $S_n^T(A)$ is realized in $\mathcal{M}$.

$\mathcal{M}$ is *saturated* if $\mathcal{M}$ is $|\mathcal{M}|$-saturated.

Remark: $\{x \neq a : a \in \mathcal{M}\}$ is not realized in any model $\mathcal{M}$, so no model is κ-saturated for any $\kappa > |\mathcal{M}|$.

We will next show how to construct saturated models, to complete this we will need a lemma.

Definition 1.5.12: For a cardinal γ, $\mathrm{cf}(\gamma)$ is called the co-finality of γ and is the cardinality of the shortest unbounded sequence in γ.

Theorem 1.5.7 (König's theorem): For a cardinal γ, $\mathrm{cf}(2^\gamma) > \gamma$.

Lemma 1.5.8: If $(\mathcal{N}_\alpha)_{\alpha<\kappa}$ is an elementary chain, that is $\mathcal{N}_\alpha \prec \mathcal{N}_\beta$ for $\alpha < \beta$. Then if $\mathcal{N} = \bigcup_{\alpha=0}^{\kappa} \mathcal{N}_\alpha$ we have $\mathcal{N}_\alpha \prec \mathcal{N}$ for all α.

Proof: Let $\varphi(\overline{a})$ be a formula, we show that $\mathcal{N}_\alpha \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N}_\alpha$ for all alpha by induction. Since every $\mathcal{N}_\alpha$ is contained in $\mathcal{N}$ then this is true for all atomic formula φ. Now we induct on the structure of φ, for logical connectives this is trivial. Now assume that $\varphi = \exists x\, \psi(x, \overline{a})$, then certainly $\mathcal{N}_i \vDash \varphi \Rightarrow \mathcal{N} \vDash \varphi$, now if $\mathcal{N} \vDash \varphi(\overline{a})$ then there is some $j \geq i$ such that $b \in |\mathcal{N}_j|$ and so $\mathcal{N}_j \vDash \psi(b, \overline{a})$ so $\mathcal{N}_j \vDash \varphi(\overline{a})$ and so $\mathcal{N}_i \vDash \varphi(\overline{a})$. □

Theorem 1.5.9: For every κ, for every $\mathcal{M}$, there exists a model $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ and $\mathcal{N}$ is κ-saturated.

If κ is *weakly inaccessible*, that is $\lambda < \kappa \Rightarrow 2^\lambda \leq \kappa$ (note that such cardinals cannot be proved to exist in ZFC) then for every $\mathcal{M}$ with $|\mathcal{M}| \leq \kappa$ there exists $\mathcal{N}$ with $\mathcal{N} \succ \mathcal{M}$ saturated with size κ.

Proof: Assume that L is countable, then $S_n^T(A) \leq 2^{|A| + \aleph_0}$ by Proposition 1.5.6. Let $\mu = 2^\kappa$, note that $\mathrm{cf}(\mu) > \kappa$ by Theorem 1.5.7.

We will now construct a sequence of models $(\mathcal{M}_\alpha)_{\alpha<\mu}$ with $\mathcal{M}_0 = \mathcal{M}$ and at limit α we have $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$, we will assume that $|\mathcal{M}_\alpha| < \mu$. At successor steps $\alpha = \beta + 1$, we want to find $\mathcal{M}_\alpha$ with $\mathcal{M}_\beta \prec \mathcal{M}_\alpha$ such that for all $A \subseteq \mathcal{M}_\beta$ with $|A| < \kappa$, every type in $S_n^T(A)$ is realized in $\mathcal{M}_\alpha$. Now we know that for every single type $p(\overline{x})$ by Proposition 1.5.1 we can add a realization of that type, and then by Theorem 1.2.2 we can get that realization with size at most μ, so we just need to do induction again to add every type.

Let us count how many types we need to add, we know that for any fixed A we have $|S_n^T(A)| \leq 2^{\kappa+\aleph_0} = \mu$. Now for any cardinality β we have that the number of subsets A with $|A| = \beta$ is

$$\mu^\beta = (2^\kappa)^\beta = 2^{\kappa\times\beta} = 2^\kappa = \mu$$

so in total we have $\sum_{\lambda<\kappa} \mu^\lambda = \kappa\mu = \mu$ steps and so our final model $\mathcal{M}_{\alpha+1}$ is also of size at most μ which completes the induction. □

Example: There are strange consequences to this theorem, for example there are models of Piano Arithmetic that satisfy a statement encoding "PA is inconsistent".

We can see that the process of adding types is not very difficult, in model theory we have a saying about this: "Any fool can realize a type, but it takes a model theorist to omit one". We have not yet looked at omitting types, but the definition is exactly what you would expect.

Definition 1.5.13: For a complete theory T, a model $\mathcal{M} \vDash T$ and a type $p(\overline{x})$. We say that $\mathcal{M}$ *omits* $p(\overline{x})$ if it does not realize it, i.e. $p(\mathcal{M}) = \varnothing$.

Now the difficulty in omitting types is that some types can **never** be omitted.

Example: If c is a constant of a language L then the type of the interpretation of c can never be omitted.

But some types can be omitted

Example: The type of a transcendental number in ACF_p is distinct from that of an algebraic number, and can be omitted, for example in $\hat{\mathbb{Q}}$.

The first example here is an important one to keep in mind since all the properties of that type can be proved from the single formula $x = c$.

Definition 1.5.14: A type $p(\overline{x})$ is isolated if there exists a formula $\varphi(\overline{x}) \in p(\overline{x})$ such that for every $\psi(\overline{x}) \in p(\overline{x})$ we have

$$T \vdash (\varphi(\overline{x}) \Rightarrow \psi(\overline{x}))$$

Proposition 1.5.10: $p(\overline{x}) \in S_{n(A)}$ is isolated iff $\{p\}$ is open in $S_{n(A)}$.

Proof: Exercise □

This characterization is important due to the following fact.

Proposition 1.5.11: If $p(\overline{x})$ is isolated, then p cannot be omitted.

Proof: Let $\varphi(\overline{x})$ be the generating formula for p, then

$$\exists x \varphi(\overline{x})$$

is a true sentence in T and thus any witness of this sentence is a realization of the type. □

Now apriori we would not expect this converse to hold since it feels like being isolated is quite the strong condition, but in fact the converse does hold, which is shown in this theorem.

Theorem 1.5.12: If $p(\overline{x})$ is not isolated, then there exists $\mathcal{M} \vDash T$ which omits $p(\overline{x})$.

There are many proofs of this theorem but we will use one called **Henkin's construction**. This proof method is also the modern method for proving Theorem 1.2.3.

Proof: Let L be a countable language and let $\{c_n\}_{n\in\mathbb{N}}$ be a family of new constants not in L, enumerate all formulas in $L \cup \{c_n\}_{n\in\mathbb{N}}$ as φ_n. Let $f : \mathbb{N} \to \mathbb{N}$ be increasing such that $c_{f(n)}$ does not appear in $\varphi_0, ..., \varphi_n$.

We define the **Henkin axioms**

$$H_i = (\exists x \varphi_i(x)) \to \varphi_i\left(c_{f(i)}\right).$$

We now construct a sequence of sets of sentences $T_0 = T \subseteq T_1 \subseteq T_2 \subseteq ...$ such that

$$T_{2n+1} = T_{2n} \cup \{H_n\} \quad \text{and} \quad T_{2n+2} = T_{2n+1} \cup \left\{\neg\varphi_{n(c_n)}\right\} \text{ for some } \varphi_{n(\overline{x})} \in p(\overline{x})$$

Then taking the union of these sets we will get an axiomization of a consistent theory. We can then use Zorn's lemma to get a complete theory containing it and then if we set our universe to be the set of constants quotiented by the relation

$$c_i = c_j \text{ as elements if } \left(c_i = c_j\right) \text{ as a formula is in } T$$

Now a model satisfying this theory will not realize the type $p(\overline{x})$ since if it did then some constant would realize it which would contradict the fact that our theory contains $\neg\varphi(c_n)$ for every n.

All that is left to do is to check that at every odd step these sentences are indeed consistent and that at even steps we can pick specific φ_n to make the set of sentences consistent.

For the even steps assume that T_{2n+1} is consistent but for every $\psi(\overline{x}) \in p(\overline{x})$ we have that $T_{2n+1} \cup \{\neg\psi(c_n)\}$ is inconsistent. Then T_{2n+1} is T where we added some finitely many sentences, so we can write $T_{2n+1} = T \cup \left\{\psi_j(\overline{c}, c_n) : j < k\right\}$ for some k and ψ_j.

Now set

$$\varphi(\overline{y}, x) = \bigwedge_{j<k} \psi_j(\overline{y}, x)$$

then for every $\psi(\overline{x}) \in p(\overline{x})$ we have $T \cup \{\varphi(\overline{c}, c_n)\} \cup \{\neg\psi(c_n)\}$ is inconsistent so

$$T \vdash (\varphi(\overline{c}, c_n) \to \psi(c_n))$$

But now since the T does not contain c_n as a constant we can replace all instances of c_n with x and all instances of $\overline{c}$ with $\overline{y}$ in the proof and get that

$$T \vdash (\varphi(\overline{y}, x)) \to \psi(x))$$

but then this means that

$$T \vdash \forall \overline{y}(\varphi(\overline{y}, x) \to \psi(x))$$

but we have that

$$\begin{aligned}\forall\overline{y}(\varphi(\overline{y},x)\to\psi(x)) &= \forall\overline{y}(\neg\varphi(\overline{y},x)\vee\psi(x)) = \neg\exists\overline{y}(\varphi(\overline{y},x)\wedge\neg\psi(x)) \\ &= \neg(\exists\overline{y}\varphi(\overline{y},x)\wedge\neg\psi(x)) = (\exists\overline{y}(\varphi(\overline{y},x)))\to\psi(x)\end{aligned}$$

then $\exists\overline{y}(\varphi(\overline{y},x))$ implies every ψ in the type $p(\overline{x})$, but also $\exists\overline{y}(\varphi(\overline{y},x))$ is true in T_{2n+1} and thus is consistent with T and thus is in the type p. This contradicts our assumption that $p(\overline{x})$ is not isolated. □

We now have a powerful way to think about and use types in proofs.

Corollary 1.5.12.1: In a complete theory T,

$$p \text{ is isolated} \Leftrightarrow \text{every model of } T \text{ realizes } p$$

Now that we have the tools to omit types, we can use it to characterize the $\aleph_0$-categorical theories.

Theorem 1.5.13 (Ryll-Nardzweski): Let T be a complete theory over a countable language L, the following are equivalent.
- T is $\aleph_0$-categorical.
- $\forall n, S_n^T(\varnothing)$ is finite.

Proof: $(1 \Rightarrow 2)$. Suppose that $S_n^T(\varnothing)$ is infinite, we know that it is always a closed subset of the Cantor set. As an infinite compact space, $S_n^T(\varnothing)$ has a non isolated point, corresponding to a non isolated type p. By the omitting types theorem, there exists a model which omits p, since it is a type there is another model which realizes p, those two models then cannot be isomorphic. We can then make them both countable by Theorem 1.2.2 which completes this side of the proof.

$(2 \Rightarrow 1)$. We assume that $S_n^T(\varnothing)$ is finite. This implies that if $A \subseteq \mathcal{M} \vDash T$, with A being finite, then $S_n^T(A)$ is also finite since we can inject it into $S_{n+1}^T(\varnothing)$. So every type in $S_n^T(A)$ is isolated. Now let $\mathcal{M}, \mathcal{N} \vDash T$ be countable models, enumerate them as $\mathcal{M} = \{a_0, a_1, ...\}$ and $\mathcal{N} = \{b_0, b_1, ...\}$. We will now do a back and forth construction, at step n we have a partial isomorphism $f_n : A_n \to B_n$. Define the tuples $\overline{a} = (a_1, ..., a_n), \overline{b} = (b_1, ..., b_n)$ containing all elements of A_n and B_n respectively. From the fact that it is a partial isomorphism we will know that

$$\mathrm{tp}_n^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_n^{\mathcal{N}}\left(\overline{b}\right).$$

Now let us create the construct the maps by induction, at step 0 we pick some $a \in \mathcal{M}$ then $\mathrm{tp}_n^{\mathcal{M}}(a)$is isolated. Since it is isolated every model of T realizes it and so in particular there is an element $b \in \mathcal{N}$ that realizes the type and map a to it.

At the inductive even steps we will pick some $a_{n+1} \in \mathcal{M}$ and note that $\mathrm{tp}_n^{\mathcal{M}(A)}(a_{n+1})$ is again isolated so again there is some element $b_{n+1} \in \mathcal{N}$ such that $\mathrm{tp}_n^{\mathcal{N}(B)}(b_{n+1}) = \mathrm{tp}_n^{\mathcal{M}(A)}(a_{n+1})$ this type and so we can map a_{n+1} to b_{n+1}. At the odd steps we do the same thing as but pick $b \in \mathcal{N}$ first. □

Example: In ACF_p we have that the type of any irreducible polynomial is isolated while the type of the transcendental number is not isolated.

1.6. Automorphism groups

In algebra for some algebraic structure an important role is played by the automorphism groups of these structures. As model theory is a sort of algebra without algebra we will also use automorphism groups.

Definition 1.6.1: Let $\mathcal{M}$ be a countable structure of a countable language L. We define the automorphism group $\mathrm{Aut}(\mathcal{M})$ to be

$$\mathrm{Aut}(\mathcal{M}) := \{F : \mathcal{M} \to \mathcal{M} : F \text{ is an automorphism}\}$$

$\mathrm{Aut}(\mathcal{M})$ acts on $\mathcal{M}^n$ for all n, and is in fact a Polish topological group.

Proposition 1.6.1: $\mathrm{Aut}(\mathcal{M})$ is a Polish group, that is separable, infinite, and admits a complete metric.

Proof: Given $f \in \mathrm{Aut}(\mathcal{M})$, neighborhoods of f are

$$U^f_{a_1,\ldots,a_n} = \{g \in \mathrm{Aut}(\mathcal{M}) : g(a_1) = f(a_1), \ldots, g(a_n) = f(a_n)\}$$

Define the sets $[A, B]$ for finite tuples $A, B \subseteq \mathcal{M}$ by

$$[A, B] = \{f \in \mathrm{Aut}(\mathcal{M}) : f(A) = B\}$$

A complete metric can be defined as

$$d(f, g) := \exp_2(-\min\{n : f(n) \neq g(n) \text{ or } f^{-1}(n) \neq g^{-1}(n)\})$$

□

Theorem 1.6.2: $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical if and only if for all n, $\mathrm{Aut}(\mathcal{M})$ acts on $\mathcal{M}^n$ with finitely many orbits.

Proof: Exercise. □

1.7. Infinite-Ary-Logic and Scott Analysis

We now want to take a short look at different types of logic.

$\mathcal{L}_{\omega_1,\omega}$ is the extension of finite order logic over a countable language L, where in formulas we allow infinite countable $\bigvee, \bigwedge$.

More precisely,

1. The atomic formulas of $\mathcal{L}_{\omega_1,\omega}$ are the same as in first order logic.
2. If φ_k is a countable set of formulas then

$$\bigwedge_{k\in\omega} \varphi_k \text{ and } \bigvee_{k\in\omega} \varphi_k$$

are both in $\mathcal{L}_{\omega_1,\omega}$

3. If φ is in $\mathcal{L}_{\omega_1,\omega}$ then $\exists x(\varphi(x))$ and $\forall y(\varphi(y))$ are both in $\mathcal{L}_{\omega_1,\omega}$.

Now recall that, in ordinary logic, for a finite model $\mathcal{M}$, there exists a sentence σ with $\mathcal{M} \vDash \sigma$ and

$$(\mathcal{N} \vDash \sigma) \Rightarrow \mathcal{M} \cong \mathcal{N}$$

Our goal now is to generalize this using our new type of logic to the case of countable models.

Definition 1.7.1: Let $\mathcal{M}$ be a countable structure. Define $\underset{\alpha}{\equiv}$ on $\mathcal{M}^n$ for α an ordinal, n a natural number, by transfinite induction. For the base case

$$\overline{a} \underset{0}{\equiv} \overline{b} \quad \text{if} \quad \mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}\left(\overline{b}\right),$$

in the limit case

$$\overline{a} \underset{\gamma}{\equiv} \overline{b} \quad \text{if } \overline{a} \underset{\beta}{\equiv} \overline{b}, \forall \beta < \gamma,$$

and in the successor step

$$\begin{aligned} \overline{a} \underset{\alpha+1}{\equiv} \overline{b} \quad \text{if} \quad & \forall c \in \mathcal{M}, \exists d \in \mathcal{M}\ (\overline{a}, c) \underset{\alpha}{\equiv} \left(\overline{b}, d\right) \\ \text{and} \quad & \forall d \in \mathcal{M}, \exists c \in \mathcal{M}\ (\overline{a}, c) \underset{\alpha}{\equiv} \left(\overline{b}, d\right) \end{aligned}$$

We record some important properties of these relations

Proposition 1.7.1:
- $\underset{\alpha}{\equiv}$ is an equivalence relation on $\mathcal{M}^n$ for all $n \in \mathbb{N}$.
- If $\alpha < \beta$ and $\overline{a} \underset{\beta}{\equiv} \overline{b}$ then $\overline{a} \underset{\alpha}{\equiv} \overline{b}$.
- For every $\overline{a}$, there is an ordinal $\alpha < \omega_1$ such that

$$[\overline{a}]_{\alpha} = [\overline{a}]_{\beta} \quad \text{for all } \beta \geq \alpha$$

where $[\overline{a}]_{\alpha}$ is the equivalence class of $\overline{a}$ with respect to $\underset{\alpha}{\equiv}$.

Essentially we are saying that the equivalence classes of these relations form a decreasing sequence in alpha which stabilizes at some countable ordinal.

Proof: Exercise. □

In fact an even stronger property is true

Proposition 1.7.2: There exists $\alpha < \omega_1$, such that $\underset{\alpha}{\equiv}$ is the same equivalence relation as $\underset{\beta}{\equiv}$ for all $\beta \geq \alpha$.

Proof: For each n, $\underset{\alpha}{\equiv}$ forms a decreasing sequence of subsets of $\mathcal{M}^n \times \mathcal{M}^n$, which is a countable set, so it must stabilize. $\square$

This proposition motivates the following definition.

Definition 1.7.2: The *Scott height* (or *rank*) of a countable structure $\mathcal{M}$ is defined as

$$\mathrm{SH}(\mathcal{M}) = \min\left\{\alpha < \omega_1 : \underset{\alpha}{\equiv} \text{ is the same as } \underset{\alpha+1}{\equiv}\right\}$$

We now want to use these tools to work towards our characterizing sentence for countable structures. We now define an equivalence on models that mirrors Definition 1.7.1.

Definition 1.7.3: We define $\underset{\alpha}{\equiv}$ on countable L structures through transfinite induction. For the base case

$$\mathcal{M} \underset{0}{\equiv} \mathcal{N} \text{ if } \mathcal{M} \equiv \mathcal{N},$$

for the limit case

$$\mathcal{M} \underset{\gamma}{\equiv} \mathcal{N} \text{ if } \mathcal{M} \underset{\beta}{\equiv} \mathcal{N} \text{ for all } \beta < \gamma,$$

and for the successor step

$$\begin{aligned} \mathcal{M} \underset{\alpha+1}{\equiv} \mathcal{N} \text{ if } \quad & \forall a \in \mathcal{M}, \exists b \in \mathcal{N}, (\mathcal{M}, a) \underset{\alpha}{\equiv} (\mathcal{N}, b) \\ \text{and} \quad & \forall b \in \mathcal{N}, \exists a \in \mathcal{M}, (\mathcal{M}, a) \underset{\alpha}{\equiv} (\mathcal{N}, b) \end{aligned}$$

We can see that this definition in fact generalizes Definition 1.7.1.

Proposition 1.7.3: $\overline{a} \underset{\alpha}{\equiv} \overline{b}$ if and only if $(\mathcal{M}, \overline{a}) \underset{\alpha}{\equiv} \left(\mathcal{M}, \overline{b}\right)$.

Now with this definition we can start to construct some characterizing sentences.

Lemma 1.7.4: $\forall \alpha < \omega_1$, $\forall \overline{\alpha} \in m, \exists \varphi_\alpha^{\mathcal{M}, \overline{a}}(x) \in \mathcal{L}_{\omega_1, \omega}$, such that $\forall \mathcal{N}, \forall \overline{b} \in \mathcal{N}$,

$$(\mathcal{M}, \overline{a}) \underset{\alpha}{\equiv} \left(\mathcal{N}, \overline{b}\right) \Leftrightarrow \mathcal{N} \vDash \varphi_\alpha^{\mathcal{M}, \overline{a}}\left(\overline{\beta}\right)$$

Proof: We prove by induction on α, in the case of $\alpha = 0$

$$\varphi_0(\overline{x}) = \bigwedge_{\varphi \in \operatorname{tp}^{\mathcal{M}}(\overline{a})} \varphi(\overline{x}).$$

If α is a limit ordinal then

$$\varphi_{\alpha}^{\mathcal{M},\overline{a}}(\overline{x}) = \bigwedge_{\beta<\alpha} \varphi_{\beta}^{\mathcal{M},\overline{a}}.$$

Finally for $\alpha + 1$ we have

$$\varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(\overline{x}) = \left(\bigwedge_{b\in\mathcal{M}} \exists y \varphi_{\alpha}^{\mathcal{M},(\overline{a},b)}(\overline{x}, y)\right) \land \left(\forall y \bigwedge_{b\in\mathcal{M}} \varphi_{\alpha}^{\mathcal{M},(\overline{a},b)}(\overline{x}, y)\right)$$

□

Unfortunately, the sentences are not exactly what we want, they only guarantee isomorphic models under a fairly strong assumption.

Theorem 1.7.5 (Scott): Let $\mathcal{M}, \mathcal{N}$ be countable structures with

$$\mathrm{SH}(\mathcal{M}) = \mathrm{SH}(\mathcal{N}) = \alpha,$$

if $\mathcal{M} \underset{\alpha+\omega}{\equiv} \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.

Proof: Our proof will employ a back and fourth method, assume that at the step n we have $(\mathcal{M}, a_1, ..., a_n) \underset{\alpha+1}{\equiv} (\mathcal{N}, b_1, ..., b_n)$. Assume then that we are on an even step and want to add an element a_{n+1} to this equivalence, we leave this induction step as an exercise.

□

We also have a partial converse to this result.

Proposition 1.7.6: Suppose that $\mathrm{SH}(\mathcal{M}) = \alpha$ and $\mathcal{M} \underset{\alpha+\omega}{\equiv} \mathcal{N}$, then $\mathrm{SH}(\mathcal{N}) = \alpha$.

Proof: First we want to show that $\mathrm{SH}(\mathcal{N}) \leq \alpha$. Choose $\overline{a}, \overline{b} \in \mathcal{N}^n$ and suppose that $\overline{a} \underset{\alpha}{\equiv} \overline{b}$. We want to show that $\overline{a} \underset{\alpha+1}{\equiv} \overline{b}$ using $\mathcal{N} \underset{\alpha+\omega}{\equiv} \mathcal{M}$. Find $\overline{c}, \overline{d} \in \mathcal{M}^n$ such that

$$(\mathcal{M}, \overline{c}) \underset{\alpha+1}{\equiv} (\mathcal{N}, \overline{a}) \text{ and } \left(\mathcal{M}, \overline{d}\right) \underset{\alpha+1}{\equiv} \left(\mathcal{N}, \overline{b}\right)$$

then we also have

$$\left(\mathcal{N}, \overline{b}\right) \underset{\alpha+1}{\equiv} \left(\mathcal{M}, \overline{d}\right) \underset{\alpha+1}{\equiv} (\mathcal{M}, \overline{c}) \underset{\alpha+1}{\equiv} (\mathcal{N}, \overline{a})$$

and thus $\mathrm{SH}(\mathcal{N}) \leq \alpha$.

For the other inequality we just swap $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 1.7.6.1: Let $\mathcal{M}$ be a countable structure, there exists $\alpha < \omega_1$ such that for every countable structure $\mathcal{N}$

$$\mathcal{N} \simeq \mathcal{M} \Leftrightarrow \mathcal{N} \underset{\alpha}{\equiv} \mathcal{M}$$

However, with a bit of trickery, we can define a sentence which does uniquely classify our countable model.

Definition 1.7.4: Let $\mathcal{M}$ be an L structure, $\alpha = \mathrm{SH}(\mathcal{M})$. We define the *Scott Sentence* of $\mathcal{M}$ as

$$\phi = \varphi_\alpha^{\mathcal{M},\varnothing} \wedge \bigwedge_{n=0}^{\infty} \bigwedge_{\overline{a} \in \mathcal{M}} \left[\forall \overline{x} \left(\varphi_\alpha^{\mathcal{M},\varnothing}(\overline{x}) \to \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(\overline{x})\right)\right]$$

Theorem 1.7.7 (Scott Isomorphism Theorem): Let $\mathcal{M}$ be a countable structure for every countable structure $\mathcal{N}$,

$$\mathcal{N} \cong \mathcal{M} \Leftrightarrow \mathcal{N} \vDash \phi^{\mathcal{M}}$$

Proof: The forward direction is simple, if the two models are isomorphic $\mathcal{N}$ satisfies the sentence of $\mathcal{M}$ since they have the same sentences.

For the backwards direction we want to use back and forth, we will use induction and assume we have some tuple $\overline{a}$ and a partial isomorphism $f_n : \mathcal{M} \to \mathcal{N}$, in the sense that $(\mathcal{M}, \overline{a}) \underset{\alpha}{\equiv} \left(\mathcal{N}, f_{n(\overline{a})}\right)$.

For $n = 0$ we have $\mathcal{M} \underset{\alpha}{\equiv} \mathcal{N}$ since $\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\varnothing}$. Now assume that we have constructed the map for n, then we have $(\mathcal{M}, \overline{a}) \underset{\alpha}{\equiv} \left(\mathcal{N}, f_{n(\overline{a})}\right)$, then since $\mathcal{N} \vDash \varphi^{\mathcal{M}}$ then we get

$$\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\overline{a}}\left(f_{n(\overline{a})}\right) \Rightarrow \mathcal{N} \vDash \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}\left(f_{n(\overline{a})}\right)$$

but we know that

$$\mathcal{N} \vDash \varphi_\alpha^{\mathcal{M},\overline{a}}(f_n(\overline{a}))$$

so we must have

$$\mathcal{N} \vDash \varphi_{\alpha+1}^{\mathcal{M},\overline{a}}(f_n(\overline{a}))$$

and so

$$(\mathcal{M}, \overline{a}) \underset{\alpha+1}{\equiv} (\mathcal{N}, f_n(\overline{a})).$$

Now by Definition 1.7.1 we get that for any element in $a \in \mathcal{M}$ we can pick an element $b \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, a) \underset{\alpha}{\equiv} (\mathcal{N}, f_n(\overline{a}), b)$ and so we set f_{n+1} to be the extension of f_n with $f_{n+1}(a) = b$.

This describes how we do the odd steps, on even steps we just swap $\mathcal{N}$ and $\mathcal{M}$. □

1.8. Quantifier Elimination

Definition 1.8.1: A theory T has *quantifier elimination*, if for every formula $\varphi(\overline{x})$ there exists a quantifier free formula $\psi(\overline{x})$ such that

$$T \vdash \forall(\overline{x})(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

At face value this seems like a hopelessly strong property, but in fact we can make any theory have quantifier elimination if we expand our language. This is called *Skolemization*.

Definition 1.8.2: A theory T has *Skolem functions*, if for every formula $\varphi(\overline{x}, y)$ there exists a term $t_\varphi(\overline{x})$ such that

$$T \vdash \left[(\exists y(\varphi(\overline{x}, y))) \to \varphi\left(\overline{x}, t_\varphi(\overline{x})\right)\right]$$

Proposition 1.8.1: If T has Skolem functions then it has quantifier elimination.

Proof: We prove by induction on the complexity of a formula $\varphi(\overline{x})$, for atomic formulas this is trivial. For conjunctions, disjunctions and negations this is also trivial. Now if $\varphi(\overline{x}) = \exists y(\psi(\overline{x}, y))$ then through Skolem functions we get □

If T has Skolem functions and $\mathcal{M} \vDash T$ with $A \subseteq \mathcal{M}$, we can define $\mathrm{Sc}(A)$ to be the closure of A under all Skolem functions, sometimes called the Skolem hull of A.

Proposition 1.8.2: $\mathrm{Sc}(A) \prec \mathcal{M}$

Proof: Proof is trivial by Theorem 1.2.1. □

Let T be a theory in L, we can add enough Skolem functions as follows.

- We replace L with L' with new added function symbols.
- We replace T with $T' = T \cup \left\{\exists y(\varphi(\overline{x}, y)) \to \varphi\left(\overline{x}, f_\varphi(\overline{x})\right) : \varphi\right\}$
- We replace $\mathcal{M}$ with $\mathcal{M}'$ where we interpret the functions using the witnesses we know exist.

We now use induction, we set
- $L^{n+1} = (L^n)'$
- $T^{n+1} = (T^n)'$
- $\mathcal{M}^{n+1} = (\mathcal{M}^n)'$

then in the limit we have
- $L^s = \bigcup_{n<\omega} L^n$
- $T^s = \bigcup_{n<\omega} T^n$
- $M^s = \bigcup_{n<\omega} M^n$

Proposition 1.8.3:
- $\mathcal{M}^s \vDash T^s$, and T^s has Skolem function.
- T^s is a conservative extension, in the sense that

$$T \vdash \sigma \Leftrightarrow T^s \vdash \sigma$$

Proof: Exercise. □

Proposition 1.8.4: DLO_0 has quantifier elimination.

Proof: We induct on the logical structure of $\varphi(\overline{x})$, we show that there exists a quantifier free formula $\psi(\overline{x})$ such that

$$DLO_0 \vdash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})).$$

For atomic formulas this is trivial, for logical connectives this is also trivial, now assume that $\varphi = \exists x(\varphi'(\overline{x}))$. Since $\exists x(\alpha \vee \beta) \leftrightarrow \exists x(\alpha) \vee \exists x\beta$, we can assume that φ' is in Normal Form, that is

$$\varphi' \leftrightarrow \bigvee_i \exists x \bigwedge_j a^i_j$$

then WLOG $\varphi' = \alpha_1 \wedge ... \wedge \alpha_n$ with α_i atomic or a negation of an atomic formula.

Now write $c_1(\overline{y}), ..., c_m(\overline{y})$ to be the quantifier free formulas which describe a total order on $y_1, ..., y_n$ which possibly identifies some of them. Now for each $i \leq m$ if

$$\mathbb{Q} \vDash c_i\left(\overline{b}\right) \wedge c_i\left(\overline{b}'\right),$$

then there exists an automorphism of $(\mathbb{Q}, \leq)$ mapping $\overline{b}$ to $\overline{b}'$. For each $i \leq m$ let $\overline{b}_i$ such that $\mathbb{Q} \vDash c_i\left(\overline{b}_i\right)$, then consider the index set

$$I = \left\{i \leq m : \mathbb{Q} \vDash \exists \overline{x} \varphi'\left(x, \overline{b}_i\right)\right\}$$

then we have

$$\mathbb{Q} \vDash \left(\exists x \varphi(x, \overline{y}) \leftrightarrow \forall_{j \in I} c_j(\overline{y})\right)$$

because if $\overline{y}$ satisfies the left formula then it has some ordering and so we can use the automorphisms to map $\overline{y}$ to some $\overline{b}_i$ and then $i \in I$ and thus the right side also holds. Similarly we can go the other way.

Now since DLO_0 is complete we can lift the above sentence from $\mathbb{Q}$ to DLO_0 and get our result. □

What we see in this proof is that quantifier elimination is intimately related to the type structure for finite tuples. We can make this relation more precise.

Proposition 1.8.5: Let $p \in S_n(\varnothing)$, write p_0 for $\{\varphi \in p : \varphi \text{ is quantifier free}\}$.

A complete theory has quantifier elimination if and only if

$$\forall p \in S_n(\varnothing), \quad T \cup p_0 \vdash p$$

Proof: The forward direction is trivial, we just take any $\varphi \in p$ and use quantifier elimination to get an equivalent quantifier free version which must also lie in p and thus lie in p_0, then by equivalence we get the result.

For the backwards direction assume the condition above holds, then let φ be a formula, then $[\varphi] \subseteq S_n(\varnothing)$ be the corresponding open set. Then for every $p \in [\varphi]$ we have $T \cup p_0 \vdash \varphi$ so by compactness for some finite collection ψ_i^p of quantifier free formulas we have $T \cup \{\varphi_i^p : i \leq n\} \vdash \varphi$. Then set

$$\psi^p = \bigwedge_i \psi_i^p$$

and note that $[\psi^p] \subseteq [\varphi]$. Now since open sets of the form $[\psi^p]$ cover $[\varphi]$ which is compact, we can take a finite subcollection p_j such that $[\psi^{p_j}]$ cover $[\varphi]$, then $[\varphi] = \bigcup_j^k [\psi^{p_j}]$ and then

$$T \vdash \varphi \leftrightarrow \bigvee_{j=1}^k \psi^{p_j}$$

□

Proposition 1.8.6: ACF_p has quantifier elimination.

Proof: Let $p \in S_n(\varnothing)$, we need $T \cup p_0 \vdash p$. Choose a large algebraically closed field K and let $\overline{a}, \overline{b} \in K$ such that both realize p_0. We will show that there exists $\varphi \in \mathrm{Aut}(K)$ such that $\varphi(\overline{a}) = \overline{b}$, this will then imply that $\mathrm{tp}(\overline{a}) = \mathrm{tp}\left(\overline{b}\right)$ which proves what we want.

Recall that $\langle \overline{a} \rangle, \left\langle \overline{b} \right\rangle$ are the subrings generated by $\overline{a}$ and $\overline{b}$ respectively. Now map $a_i \mapsto b_i$, we want to extend this map to $\langle \overline{a} \rangle \to \left\langle \overline{b} \right\rangle$, recall that elements of $\langle \overline{a} \rangle$ are are of the form $P(\overline{a})$ where $P \in \mathbb{Z}[\overline{x}]$, which we can also write as $\tau_1(\overline{a}) - \tau_2(\overline{a})$ where τ_1, τ_2 are two terms. We now map

$$\tau_1(\overline{a}) - \tau_2(\overline{a}) \to \tau_1\left(\overline{b}\right) - \tau_2\left(\overline{b}\right)$$

one can easily check that this is a well defined map.

We now extend the isomorphism to the field of fractions for $\langle \overline{a} \rangle$ and $\left\langle \overline{b} \right\rangle$ in exactly the same way, by mapping

$$\frac{\tau_1(\overline{a})}{\tau_2(\overline{a})} \to \frac{\tau_2\left(\overline{b}\right)}{\tau_2\left(\overline{b}\right)}$$

and then once again we can extend to the algebraic closure of this field of fractions.

Finally we have a map of countable algebraically closed subfields $L \to L$, we can extend this map to all of K since K has a transcendental basis of L and so we can permute this transcendental basis whichever way we like to extend this map. $\square$

1.9. Algebraic Geometry

Now that we have quantifier elimination of ACF_p we can use that to very quickly prove the foundations of algebraic geometry

Theorem 1.9.1 (Lefchetz's principle): Let σ be a sentence in the language of fields. TFAE
1. σ is true in every algebraically closed field of characteristic 0.
2. σ is true in every algebraically closed field of characteristic p for all but finitely many p.
3. σ is true in every algebraically closed field of characteristic p for infinitely many p

Proof: Recall that for finite p,

$$ACF_p = ACF \cup \{\text{characteristic} = p\},$$

and

$$ACF_0 = ACF \cup \{\text{characteristic} \neq p : p \text{ prime}\}$$

Now (2) ⇒ (3) is trivial, so we prove (1) ⇒ (2) and (3) ⇒ (1).

For (1) ⇒ (2), assume that $ACF_0 \vdash \sigma$, then there is a finite subcollection of sentences $\text{characteristic} \neq p$ such that

$$ACF \cup \{\text{characteristic} \neq p_i : i \leq n\} \vdash \sigma$$

so we are done.

For (3) ⇒ (1), suppose that $ACF_0 \nvdash \sigma$ and $ACF_p \vdash \sigma$ for infinitely many p. Then by completeness $ACF_0 \nvdash \neg\sigma$ so by (1) ⇒ (2) there exists a prime p_0 such that for all prime numbers $p \geq p_0$ we have $ACF_p \vdash \neg\sigma$ and so we get a contradiction. □

There are some fun consequences of this theorem.

Theorem 1.9.2 (Ax): If $f : \mathbb{C}^N \to \mathbb{C}^N$ is a map where every coordinate is a polynomial, then if it is injective, then it is surjective.

Proof: Let $\hat{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. We claim that every injective polynomial map $f : \hat{\mathbb{F}}_p^N \to \hat{\mathbb{F}}_p^N$ is surjective. If we have this then by Theorem 1.9.1 we can transfer this result to $\mathbb{C}$.

Now to prove the claim first note that every polynomial has finitely many coefficients and that

$$\hat{\mathbb{F}}_p = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$$

has the property that every finite subset generates a finite field. Then WLOG we may assume that all the coefficients of f are in $\mathbb{F}_{p^m}$ for some fixed m. Then we get that f induced a map $f_{(n)} : \mathbb{F}_{p^n}^N \to \mathbb{F}_{p^n}^N$ for $n \geq m$. By assumption all $f_{(n)}$ are injective and since these are finite fields they must also be surjective. Hence since $f = \bigcup_{n=m}^{\infty} f_{(n)}$ we get that f is also surjective.

Now to transfer this result to $\mathbb{C}$, fix $d \in \mathbb{N}$ the degree of the polynomial and $N \in \mathbb{N}$ the number of variables. We write $g(\overline{x}, \overline{a})$ the degree d polynomial in $\overline{x}$ with coefficients $\overline{a}$. Then consider the sentence

$$\forall \overline{a}((\forall \overline{x} \forall \overline{y}(g(\overline{x}, \overline{a}) = g(\overline{y}, \overline{a}) \Rightarrow \overline{x} = \overline{y}) \Rightarrow \forall \overline{y} \exists \overline{x}(f(\overline{x}, \overline{c}) = \overline{y}))$$

this sentence encodes exactly the statement of the theorem for polynomials of degree $\leq d$. Hence by Theorem 1.9.1 since these sentences are true in $\hat{\mathbb{F}}_p$ then they are also true in $\mathbb{C}$. □

Coming back to quantifier elimination, we have an assortment of corollaries stemming from Proposition 1.8.6

Corollary 1.9.1: Let $K < L$ both be algebraically closed fields, if $F(\overline{x})$ is a system of polynomial equations and inequalities with coefficients from K with a solution in L, then the system also has a solution in K.

Proof: Let $\varphi(\overline{y}) = \exists \overline{x} F(\overline{x}, \overline{y})$ where $\overline{y}$ are the coefficients of the polynomials.

By quantifier elimination we have that $\varphi(\overline{y})$ is equivalent to a quantifier free formula $\psi(\overline{y})$. Then if for some choice of coefficients $L \vDash \psi(\overline{c})$ then $K \vDash \psi(\overline{c})$ and so we are done. □

Corollary 1.9.2 (Weak Hilbert Nullstellensatz): Let K be an algebraically closed field, $f_1, ..., f_n \in K[\overline{x}]$. f_i have a common zero in K^n if and only if $1 \notin (f_1, ..., f_n)$.

Proof: The forward direction is very easy, if they have a common zero then everything in the ideal has that same common zero.

On the other hand if 1 is not in the ideal, let I be a maximal ideal containing $(f_1, ..., f_n)$ then set

$$L = \widehat{K[\overline{x}]} \,/\, I$$

which is clearly an algebraically closed field containing K. Now in L there are common roots, they are the variables $x_1, ..., x_n$. Hence by the we get the desired result. □

We can now apply this to some basic algebraic geometry.

Definition 1.9.1:
1. If $S \subseteq K[\overline{x}]$ we set

$$V(S) = \{\overline{a} \in K^n : f(\overline{a}) = 0, \forall f \in S\}$$

2. If $Y \subseteq K^n$ we set

$$I(Y) = \{f \in K[\overline{x}] : f(\overline{a}), \forall \overline{a} \in Y\}$$

We call a subset V of K^n *Zariski-closed* if $V = V(S)$ for some $S \in K[\overline{x}]$.

An ideal is *radical* if it is closed under taking roots.

Proposition 1.9.3: For all $X, Y \subseteq K^n$
1. $I(Y)$ is a radical ideal
2. If X is Zariski-closed, then $X = V(I(X))$.
3. If $X \subsetneq Y$ and X, Y are Zariski-closed, then $I(Y) \subsetneq I(X)$.
4. The Zariski-closed sets form a topology, that is they are closed under finite unions and arbitrary intersections. In particular if X, Y are Zariski-closed then

$$X \cup Y = V(I(X) \cap I(Y))$$

and

$$X \cap Y = V(I(X) + I(Y)).$$

Proof: Exercise. □

Theorem 1.9.4 (Hilbert basis theorem): If K is a field, then $K[\overline{x}]$ is a Noetherian ring. That is, there is no infinite increasing chain of ideals. In particular, every ideal is finitely generated.

Corollary 1.9.4.1: If K is a field, then there is no infinite decreasing sequence of Zariski-closed sets.

Proof: We apply Hilbert's Basis theorem along with the third proposition. □

Definition 1.9.2: An ideal I in a ring is *prime* if

$$a \cdot b \in I \Rightarrow a \in I \text{ or } b \in I$$

Clearly every prime ideal is radical.

Theorem 1.9.5 (Primary decomposition): If $I \subseteq K[\overline{x}]$ is a radical ideal, then there are prime ideals $J_1, ..., J_n$ such that

$$I = J_1 \cap ... \cap J_n$$

We can now prove the strong form of Corollary 1.9.2

Theorem 1.9.6 (Hilbert Nullstellensatz strong form): Let K be algebraically closed, if $I \subsetneq J$ and both are radical in $K[\overline{x}]$, then

$$V(J) \subsetneq V(I)$$

Proof: Note that the non-strict inclusion is trivial, the hard part is to prove the non-strict inclusion. That is, we want to find a common root of I which is not a common root of J. Let $p \in J \setminus I$, we want want to find a point $\overline{b}$ which is a common root of I but $p\left(\overline{b}\right) \neq 0$.

We decompose $I = I_1 \cap ... I_n$ into prime ideals and let i be such that $p \notin I_i$.

By Hilbert basis theorem we have $I_i = (f_1, ..., f_n)$ so we want to find a root of $f_1, ..., f_n$ which is not a root of p. Let $R = \frac{K[\overline{x}]}{I_i}$ then R is an integral domain since I_i is prime, let R_0 be the field of fractions of R and $L = \widehat{R_0}$.

In L consider the system

$$\begin{cases} f_i = 0 : 1 \leq i \leq m \\ p \neq 0 \end{cases}$$

it has a solution in L and thus by it also has a solution in K and so we are done. □

Corollary 1.9.6.1: If I is a radical ideal then $I = I(V(I))$

Proof: Apply Theorem 1.9.6 to $J = I(V(I))$. □

Lemma 1.9.7: Let K be a field,

1. A subset of K^n is definable over K by an atomic formula if and only if it is of the form $V(p)$ for some $p \in K[x_1, ..., x_n]$.
2. A subset of K^n is definable over K by a quantifier free formula if and only if it is a Boolean combinations of Zariski closed sets.

Proof: (1) is straight forward, as is the forward direction for (2).

For the backward direction of (2), assume that X is a Boolean combination of Z_i for some Zariski-closed family Z_i, then by definition we have

$$Z_i = V(p_1^i) \cap ... \cap V(p_{n_i}^i$$

then immediately we have that X is a boolean combination of

$$V\left(p_j^i\right)$$

$\square$

Definition 1.9.3: A set in K^n is constructible if it satisfies (2) in Lemma 1.9.7.

Theorem 1.9.8 (Chevalley): Let K be algebraically closed, the images of constructible sets by polynomial maps are constructible.

Proof: Let $X \subseteq K^n$ be constructible, $p : K^n \to K^m$ be a polynomial map

$$p(X) = \{\overline{y} \in K^n : \exists \overline{x}(\overline{x} \in X \wedge \overline{y} = p(\overline{x}))\}$$

then since $p(\overline{x})$ is definable over K subset of K^m. Since the theory of K has quantifier elimination, $p(\overline{x})$ is definable by a quantifier free formula and thus is constructible. $\square$

1.10. Homogeneous Structures

Definition 1.10.1: $\mathcal{M}$ is *κ-homogeneous* if for every subset $A \subseteq \mathcal{M}$ with $|A| < \kappa$, every elementary embedding $f : A \to \mathcal{M}$ and every element $a \in \mathcal{M}$ there is an extension $g : A \cup \{a\} \to \mathcal{M}$ which is also an elementary embedding.

$\mathcal{M}$ is called *homogeneous* if it is $|\mathcal{M}|$-homogeneous. $\mathcal{M}$ is *strongly κ-homogeneous* if we have an extension $g : \mathcal{M} \to \mathcal{M}$ of f instead.

Remark: Sometimes strong homogeneity is called ultra homogeneity.

One might wonder why we do not similarly define $\mathcal{M}$ to be strongly homogeneous if it is strongly $|\mathcal{M}|$-homogeneous. This is explained by the following proposition.

Proposition 1.10.1: $\mathcal{M}$ is homogeneous if and only if it is strongly $|\mathcal{M}|$-homogeneous.

Proof: Exercise, quite simple using back and forth. $\square$

Proposition 1.10.2: If $\mathcal{M} \equiv \mathcal{N}$ are saturated and are of the same cardinality then $\mathcal{M} \cong \mathcal{N}$.

Proof: We prove, as expected, by back and forth. Set $\kappa = \|\mathcal{M}\| = \|\mathcal{N}|$ and numerate both models as

$$\mathcal{M} = \{a_\alpha : \alpha < \kappa\} \text{ and } \mathcal{N} = \{b_\alpha : \alpha < \kappa\}.$$

We will construct a partial map $f_\alpha : A_\alpha \to B_\alpha$ with $|f| \leq 2\,|\alpha|$ such that $f_\alpha \subseteq f_{\alpha+1}$ and $a_\alpha \in A_\alpha, b_\alpha \in B_\alpha$.

We start with the base case of $\alpha = 0$ where $f_\alpha = \varnothing$. For the limit case suppose that f_β is constructed for $\beta < \alpha$, we write

$$f'_\alpha = \bigcup_{\beta<\alpha} f_\beta$$

and we look at the type $p = \mathrm{tp}^{\mathcal{M}}(a_\alpha \,/\, A_\alpha)$. Notice that for any formula $\varphi \in p$ we can replace all the parameters in A_α with their image under f_α, so we can define $f'_\alpha(p)$ which is then a complete type over B_α and thus $f'_\alpha(p) \in S_n(B_\alpha)$. Then set b to be the realization of this type, then we define the extension $f''_\alpha(a_\alpha) = b$. We similarly do the same for the backwards direction, we take a type in $\mathcal{N}$ and map it back to $\mathcal{M}$. Then we set $f_\alpha = f''_\alpha$. □

Theorem 1.10.3: Suppose $\mathcal{M} \equiv \mathcal{N}$ are homogeneous of the same cardinality, then if $\mathcal{M}, \mathcal{N}$ realize the same complete n-types over the empty set for each n, then $\mathcal{M} \cong \mathcal{N}$.

Before we prove this we need a small lemma

Lemma 1.10.4: For any $A \subseteq \mathcal{M}$ there is some embedding $A \to \mathcal{N}$.

Proof: Induction on $|A|$.
1. If A is finite then since $\mathcal{M}$ and $\mathcal{N}$ realize the same types this is immediate.
2. If $|A| = \mu \geq \aleph_0$ then we can enumerate $A = \{a_\alpha : \alpha < \mu\}$ and so by a sub induction on α we construct $f(a_\alpha)$. Suppose that for some fixed α we have constructed $f(a_\beta)$ for $\beta < \alpha$. Then let $A_\alpha = \{a_\beta : \beta \leq \alpha\}$ then by our outer induction hypothesis there exists an embedding $g : A_\alpha \to \mathcal{N}$. Note that we are not done since the g could be incompatible with f, but notice that that $f \circ g^{-1}$ is a map $g(A_\alpha \setminus \{a_\alpha\}) \to \mathcal{N}$ then by homogeneity we can extend this to an elementary embedding $h : g(A_\alpha) \to \mathcal{N}$ and then we set $f(a_\alpha) = h(g(a_\alpha))$.

□

The intuitive explanation for this proof is that by induction we get a sequence of maps f_α and then by homogeneity we can arrange the images of the maps so that they sit on top of each other for increasing α which is enough to construct a limit map.

Now to prove the theorem.

Proof: We now use a back and forth argument to prove the theorem. We will not delve into the full details here but simply mention that when we want to add an element to the partial isomorphism f_α, we use the above lemma to get a new map g with an extended domain. But then to make that map compatible with the previous maps we can use homogeneity again to align the images so that the image of g sits on top of the images of f_α, then we use that as our extension. $\square$

Definition 1.10.2: A model $\mathcal{M}$ is called *κ-universal* if for every $\mathcal{N} \equiv \mathcal{M}$ with $\|\mathcal{N}\| \leq \kappa$ there exists an elementary embedding $f : \mathcal{N} \to \mathcal{M}$.

$\mathcal{M}$ is called *universal* if it is $|\mathcal{M}|$-universal.

$\mathcal{M}$ is $< \aleph_0$-universal if for every n, $\mathcal{M}$ realizes all types in $S_n^{\mathrm{Th}(\mathcal{M})}(\varnothing)$.

Theorem 1.10.5: If $\mathcal{M}$ is κ-saturated then $\mathcal{M}$ is κ-homogeneous and κ-universal.

Proof: For κ-homogeneity if we have $|A| < \kappa$ and an embedding $f : A \to \mathcal{M}$, then pick any $a \in \mathcal{M}$. We can take the type $p = \mathrm{tp}(a \,/\, A)$ and map it to $q = f(p)$ and define $f(a)$ to be the element that realizes this type.

For κ-universality we let $\mathcal{N} \equiv \mathcal{M}$, and $\|\mathcal{N}\| \leq \kappa$.
Then we enumerate $\mathcal{N} = \{a_\alpha : \alpha < \kappa\}$, and we construct $f(a_\alpha)$ by induction. We set $p = \mathrm{tp}\big(a_\alpha \,/\, \{a_\beta : \beta < \alpha\}\big)$ and then $q = f(p)$ and so we just set $f(a_\alpha)$ to be any element which realizes q. $\square$

Theorem 1.10.6: If $\mathcal{M}$ is κ-homogeneous and $< \aleph_0$-universal then $\mathcal{M}$ is κ-saturated.

Proof: Let $A \subseteq \mathcal{M}$ with $|A| < \kappa$, let $p \in S(A)$, we want to show that $p(M) \neq \varnothing$, we prove this by induction on $|A|$.

First assume that $|A|$ is finite, then let $\mathcal{N}$ be an extension $\mathcal{M} \prec \mathcal{N}$ which realizes the type p through some element b. Then consider the type $q = \mathrm{tp}^{\mathcal{N}}(A \cup b)$, by $< \aleph_0$ universality we get that $\mathcal{M}$ realizes q through some set A' and element b'. But then by homogeneity since A and A' have the same type we can map A', b' to inside $\mathcal{M}$ so that the image of A' is A. But then the image of b' realizes p so we are done.

Next we use induction, assume that the statement holds for all A' with $|A'| < \mu$, we want to show it holds for $|A| = \mu$. Enumerate $\{a_\alpha : \alpha < \mu\}$, then let p_0 be all the formulas in p that do not use any of the constants in A. Since $\mathcal{M}$ realizes p_0 let b be a witness p_0, let $\mathcal{N}$ again be an extension of $\mathcal{M}$ which realizes p with b' as a witness.

Now b and b' have the same type, so if we consider $\mathrm{tp}^{\mathcal{N}(a_0 \,/\, b')}$ we can replace b' by b in every formula and obtain a type over b in $\mathcal{M}$, by inductive hypothesis this type will be witnessed by an element a_0' in $\mathcal{M}$. We then repeat this by induction, assuming we found a_β' for $\beta < \alpha$, then we can consider the type $\mathrm{tp}^{\mathcal{N}}(a_\alpha \,/\, a_{<\alpha} b')$, we again replace $a_{<\alpha} b'$ in the parameters by $a_{<\alpha}' b$ and then we get the element a_α' in $\mathcal{M}$.

We thus obtain $A' = \{a_\alpha' : \alpha < \mu\}$ such that b satisfies the same formula over A' as b' satisfies over A. We then can use homogeneity to map A', b into $\mathcal{M}$ so that the image of A is A', then the image of b is a witness to type p. $\square$

1.11. Fraïssé Theory

Definition 1.11.1: Let $\mathcal{M}$ be a countable structure in a countable language L. The $\text{Age}(\mathcal{M})$ is the family of finitely generated submodels of $\mathcal{M}$. Alternatively $\text{Age}(\mathcal{M})$ is the set of Isomorphism classes of finitely generated L-models that can be embedded into $\mathcal{M}$.

Proposition 1.11.1:
1. Hereditary Property (HP) - If $A \in \text{Age}(\mathcal{M})$ and B embeds into A then $B \in \text{Age}(\mathcal{M})$.
2. Joint Embedding Property (JEP) - If $A, B \in \text{Age}(\mathcal{M})$ then there exists $C \in \text{Age}(\mathcal{M})$, such that A, B both embed into C.

Proof: Exercise. □

Proposition 1.11.2: Let K be a countable class of finitely generated L-structures, if K satisfies HP and JEP as above, then $K = \text{Age}(\mathcal{M})$ for some countable model $\mathcal{M}$.

Proof: Enumerate $K = \{B_1, B_2, ...\}$. By induction construct a sequence $A_n \in K$ such that $A_1 \subseteq A_2 \subseteq ...$ as follows. Start with $A_0 = B_0$, now given A_n we let $A_{n+1} \in K$ be an element such that A_n, B_n both embed into A_{n+1}. Now at the end we take $\mathcal{M} = \bigcup_{n=1}^{\infty} A_n$.

It is clear that $\text{Age}(\mathcal{M})$ is contained in K since every finitely generated submodel of $\mathcal{M}$ is a submodel of some A_n and thus is in K. On the other hand for each $B_i \in K$ then it embeds into A_{i+1} and thus into $\mathcal{M}$. □

Definition 1.11.2: A class K has the amalgamation property (AP) if for every $A, B, C \in K$ such that A embeds into both B and C, then there exists a $D \in K$ such that B embeds into D and C embeds into D. D is sometimes called an amalgam of B, C over A.

Definition 1.11.3: A countable structure $\mathcal{M}$ is called *ultrahomogeneous* if every partial isomorphism $A \to B$ between finitely generated substructures $A, B \subseteq \mathcal{M}$ extends to an automorphism $\mathcal{M} \to \mathcal{M}$.

Proposition 1.11.3: If $\mathcal{M}$ is ultrahomogeneous then $\text{Age}(\mathcal{M})$ satisfies AP.

Proof: Exercise. □

It turns out that the converse to this result is also true.

Theorem 1.11.4 (Fraïssé): If K satisfies HP, JEP, AP then there exists a unique ultrahomogeneous $\mathcal{M}$ such that $K = \operatorname{Age}(\mathcal{M})$. This is often denoted as $M = \lim K$, and called the Fraïssé limit of K.

Before we start with the proof we need to introduce a bit of theory.

Definition 1.11.4: A structure $\mathcal{M}$ is weakly homogeneous, if for all finitely generated $A, B \subseteq \mathcal{M}$ with $A \subseteq B$ and all embeddings $f : A \to \mathcal{M}$ we have an extension $g : B \to \mathcal{M}$.

Proposition 1.11.5: If $\mathcal{M}$ is ultrahomogeneous then $\mathcal{M}$ is weakly homogeneous.

Proof: Exercise. □

One can notice that the definition of weak homogeneity is ideal for extending back and forth maps, as is confirmed in the next proposition.

Lemma 1.11.6: For $\mathcal{M}, \mathcal{N}$ weakly homogeneous with $\operatorname{Age}(\mathcal{M}) = \operatorname{Age}(\mathcal{N})$, every isomorphism $f : A \to B$ between finitely generated substructures $A \subseteq \mathcal{M}$, $B \subseteq \mathcal{N}$, extends to a full isomorphism $g : \mathcal{M} \to \mathcal{N}$.

Proof: We use back and forth, as usual we will attempt to build a map $f_n : A_n \to B_n$, starting with $f : A \to B$. On even induction steps we try to extend the domain of f_n from A_n to A_{n+1}. Notice that A_{n+1} is in $\operatorname{Age}(\mathcal{M})$ and thus by assumption also in $\operatorname{Age}(\mathcal{N})$. Then by weak homogeneity we have that $f_n : A_n \to \mathcal{N}$ extends into $f_{n'} : A_{n+1} \to \mathcal{M}$, we then call $B_{n+1} = f_{n'}(A_{n+1})$ and $f_{n+1} = f_{n'}$.

On odd steps we do the same thing but swap $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 1.11.6.1: For $\mathcal{M}$ countable, weakly homogeneous is equivalent to ultrahomogeneous.

Proof (Fraïssé's Theorem): Uniqueness is easily shown since ultrahomogeneity implies weak homogeneity which allows us to apply Lemma 1.11.6 to the empty isomorphism.

For existence, by the corollary above, it is enough to find a weakly homogeneous countable model $\mathcal{M}$ such that $\operatorname{Age}(\mathcal{M}) = K$.

Enumerate $K = \{B_n : n \in \mathbb{N}\}$, and all pairs of embedding $f_n : B_{k_n} \to B_{\ell_n}$. We want to construct a sequence $A_n \subseteq A_{n+1}$ such that $A_n \in K$ for all n, with the additional property that if we have an embedding $B_{k_n} \to A_m$ for some $m \leq n$ then 2we also have an extension $B_{\ell_n} \to A_{n+1}$.

To guarantee this we can use AP. The union of A_n is then the desired structure. □

1.12. Monster Model

Let κ be a big cardinal (not too large, something of the order $2^{2^{\cdot^{\cdot^{\omega}}}}$). Ideally we would like a saturated model of size κ, but as we saw in Theorem 1.5.9 this is often quite difficult to achieve. Instead, in practice, we often use a κ-saturated model which is κ-strongly homogeneous.

Theorem 1.12.1 (Monster Model): For $\kappa \geq \aleph_0$, T complete and L countable, there exists a model $\mathfrak{C} \vDash T$ which is κ-saturated and κ-strongly homogeneous.

Before we prove this we will need a tiny lemma.

Lemma 1.12.2: For all $\mathcal{N} \vDash T$ there exists an elementary extension $\mathcal{N} \prec \mathcal{N}'$, such that

- For all $A \subseteq \mathcal{N}$ with $|A| < \kappa$ all of $S(A)$ are realized in $\mathcal{N}'$
- For all $f : A \to B$ elementary embedding between two $A, B \subseteq \mathcal{N}$ with $|A|, |B| < \|\mathcal{N}\|$, f can be extended to $f' : A' \to B'$ also an elementary embedding with $A \cup \mathcal{N} \subseteq A'$ and $B \cup \mathcal{N} \subseteq B'$.

Proof: Let $\mu = \|\mathcal{N}\|$, then for the first condition we simply pick $\mathcal{N}'$ which is μ^+-saturated through Theorem 1.5.9.

Now assume that we have an embedding $f : A \to B$, since $\mathcal{N}'$ is μ-saturated we can, by a simple inductive argument, construct an extension $g : \mathcal{N} \to \mathcal{N}'$. Now the issue here is that $g(\mathcal{N})$ might contain $\mathcal{N}$ which is required by the lemma.

To fix this set $\mathcal{N}_0 = g(\mathcal{N})$, and construct $h : \mathcal{N}_0 \cup \mathcal{N} \to \mathcal{N}'$ extending g^{-1}. We then can set $A' = h(\mathcal{N}_0 \cup \mathcal{N})$ and $B' = \mathcal{N}_0 \cup \mathcal{N}$. Then $h^{-1} : A' \to B'$ is an extension of f as desired. □

Proof (of Theorem 1.12.1): We will construct an elementary chain $\mathcal{N}_\alpha$ with $\alpha < \kappa^+$, $\mathcal{N}_\alpha \vDash T$.

- $\mathcal{N}_0$ can be arbitrary
- For limit α we will have $\mathcal{N}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{N}_\gamma$
- For $\alpha + 1$ we will have $\mathcal{N}_{\alpha+1}$ be the extension of $\mathcal{N}_\alpha$ provided by Lemma 1.12.2.

We then set $\mathfrak{C} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$, note that since κ^+ is regular then $\mathrm{cf}(\kappa^+) = \kappa^+$.

Now we check κ-saturation, if $A \subseteq \mathcal{M}$, $|A| < \kappa$, $p \in S(A)$, we know that $A \subseteq \mathcal{N}_\alpha$ for some α by regularity of κ. So we also know that p is realized in $\mathcal{N}_{\alpha+1}$ and thus since this is an elementary chain p is realized through the same element in $\mathfrak{C}$.

Next we check κ-homogeneity, if $A, B \in \mathfrak{C}$, $|A|, |B| < \kappa$, $f : A \to B$, there is α such that $A, B \subseteq \mathcal{N}_\alpha$ again by regularity. We then fix $f = f_\alpha$ use Lemma 1.12.2 to extend f_α to $f_{\alpha+1}$, we repeat this for successor steps and for limit steps we union. We can continue doing this for all α to extend f to an automorphism $g : \mathfrak{C} \to \mathfrak{C}$. □

Remark: Since $\mathfrak{C}$ has so many automorphisms, it is often useful to consider, for any subset $A \subseteq \mathfrak{C}$ with $|A| \subseteq \kappa$, the group $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ consisting of automorphisms of $\mathfrak{C}$ that fix A. This group naturally acts on $\mathfrak{C}^n$.

One can easily show that for any two tuples $\overline{x}, \overline{y}$, they are in the same orbit of $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ if and only if $\mathrm{tp}(\overline{x} \,/\, A) = \mathrm{tp}(\overline{y} \,/\, A)$. So orbits of $\mathrm{Aut}(\mathfrak{C} \,/\, A)$ are equivalent to realizations of $S(A)$.

1.13. Indiscernibles

Definition 1.13.1: Let $(I, \leq)$ be a linear order, a set $A = \{a_i : i \subseteq I\} \subseteq \mathcal{M}$ is called *order-indiscernible* if for all formulas $\varphi(x_1, ..., x_n)$ and linear suborders $\forall i_i < ... < i_n, j_1 < ... < j_n \in I$ we have

$$\mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right) \Leftrightarrow \mathcal{M} \vDash \varphi\left(a_{j_1}, ..., a_{j_n}\right)$$

Example: DLO_0 with $A = \mathcal{M} = I = \mathbb{Q}$ is order indiscernible.

If $K > L$ are algebraically closed fields with K transcendental over L, then a transcendental basis of K over L is also an example.

Any basis of an infinite dimensional vector space.

Theorem 1.13.1 (Ehrenfeucht–Mostowski): Let T be a theory with infinite models, I arbitrary, there exists a model $\mathcal{M} \vDash T$ with an order-indiscernible sequence $(a_i : i \in I)$ of infinite cardinality.

Proof: Let $(c_i : i \in I)$ be new constants and set $L' = L \cup \{c_i : i \in I\}$. We want to show that the augmented theory

$$T' := T \cup \left\{\varphi\left(c_{i_1}, ..., c_{i_n}\right) \leftrightarrow \varphi\left(c_{j_1}, ..., c_{j_n}\right)\right\} : \varphi \in F_L, i_1 < ... < i_n, j_1 < ... < j_n\}$$

is consistent.

We use compactness, let $S \subseteq F_L$ be a finite subset of formulas, then we define the theory

$$T'' = T \cup \left\{\varphi\left(c_{i_1}, ..., c_{i_n}\right) \leftrightarrow \varphi\left(c_{j_1}, ..., c_{j_n}\right)\right\} : \varphi \in S, i_1 < ... < i_n, j_1 < ... < j_n\}$$

and we will construct a model of T''.

Lemma 1.13.2: If $A \subseteq \mathbb{N}$, $\{a_n : n \in A\} \subseteq \mathcal{M}$, and φ a formula. There exists $B \subseteq A$ infinite such that for any sequences $i_1 < ... < i_n, j_1 < ... < j_n \in B$ we have

$$\mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right) \Leftrightarrow \mathcal{M} \vDash \varphi\left(a_{j_1}, ..., a_{j_n}\right)$$

Proof: Ramsey's theorem says that if we color ordered n-tuples of A infinite, then there exists an infinite subset $B \subseteq A$ such that all ordered tuples of B have the same color.

We define the coloring

$$C\left(a_{i_1}, ..., a_{i_n}\right) = \begin{cases} 1 \text{ if } \mathcal{M} \vDash \varphi\left(a_{i_1}, ..., a_{i_n}\right) \\ 0 \text{ otherwise} \end{cases},$$

then the infinite set given by Ramsey's theorem is exactly the subset B we want.□

Applying the Lemma for each formula in S we get that any model satisfies T'' and so T'' is consistent and so by compactness so is T'. □

Corollary 1.13.2.1: Let T be a theory with infinite models, then there is a model $\mathcal{M} \vDash T$ with

$$\mathrm{Aut}(\mathbb{Q}, <) < \mathrm{Aut}(\mathcal{M})$$

where the second $<$ means subgroup.

Proof: Consider the Skolemization T^S of T, use Theorem 1.13.1 to get a model $\mathcal{M}^S \vDash T^S$ with an order-indiscernible sequence $\left(a_q : q \in \mathbb{Q}\right)$.

Set $\mathcal{N}^S$ be the smallest submodel containing $\left(a_q : q \in \mathbb{Q}\right)$ then $\mathcal{N}^S \prec \mathcal{M}^S$ since with Skolemization we have quantifier elimination. Let $\mathcal{N}$ be a reduct of $\mathcal{N}^S$ to L.

If $\varphi \in \mathrm{Aut}(\mathbb{Q}, <)$ we want to define $\overline{\varphi} \in \mathrm{Aut}(\mathcal{N})$. Since everything in $\mathrm{Aut}(\mathcal{N})$ has the form of a Skolem term in the elements a_q we can define

$$\overline{\varphi}\left(a_q\right) = a_{\varphi(q)}$$

on the generating elements, and

$$\overline{\varphi}\left(t\left(a_{i_1}, ..., a_{i_n}\right)\right) = t\left(a_{\varphi(i_1)}, ..., a_{\varphi(i_n)}\right)$$

on terms.

The fact that $\overline{\varphi}$ is well defined and an automorphism will follow from the fact that $\left(a_q : q \in \mathbb{Q}\right)$ is order-indiscernible.

One can then easily check that the map $\varphi \mapsto \overline{\varphi}$ is an embedding

$$\mathrm{Aut}(\mathbb{Q}, <) \to \mathrm{Aut}(\mathcal{N}),$$

which gives us the subgroup relation. □

Definition 1.13.2: A model $\mathcal{M}$ is an *Ehrenfeucht–Mostowski model* if there exists an infinite order-indiscernible sequence $(a_i : i \in I) \subseteq \mathcal{M}$ such that $\mathcal{M}$ is generated by $(a_i : i \in I)$. This generating sequence is called a *spine* of $\mathcal{M}$.

Theorem 1.13.3: Let $\mathcal{M}$ be an Ehrenfeucht-Mostowski model with theory $T = \operatorname{Th}(\mathcal{M})$ over a countable language L.

- The number of types in $S_n^T(\varnothing)$ realized in $\mathcal{M}$ is countable.
- If I is well-ordered, then for all $A \subseteq \mathcal{M}$ the number of types in $S_n^T(A)$ realized in $\mathcal{M}$ is at most $|A| + \aleph_0$.

Proof: For simplicity we restrict to $n = 1$, $\forall a \in \mathcal{M}$ we can write a as a term

$$a = t\left(a_{i_1}, ..., a_{i_n}\right), \quad i_1 < ... < i_n$$

note that if a and a' are assigned the same term (but possibly with different input sequence) then the assumption that a_i are order-indiscernible gives us that $\operatorname{tp}^{\mathcal{M}}(a) = \operatorname{tp}^{\mathcal{M}}(a')$. Thus we have a well-defined surjective map

$$\text{terms} \to \text{types realized}$$

so since our language is countable the number of terms is countable and hence we get the first statement of the theorem.

For the second statement, we again assume $n = 1$, and we notice that for any term $t\left(a_{i_1}, ..., a_{i_n}\right)$ if we we have a, a' both equal to this term for different sequences of inputs, then they have the same type over A only if we have an automorphism taking the arguments of a to the arguments of a' while fixing A.

We will for now assume that $A \subseteq (a_i : i \in I)$ and deal with this later, then such an automorphism exists if the arguments a_{i_k} have the same relative positions. Proof is complicated and I got confused, will finish later. □

Definition 1.13.3: For T a complete theory, L-countable and κ a cardinal.

- T is called *κ-stable* if for any model $\mathcal{M} \vDash T$ and all subsets $A \subseteq \mathcal{M}$, $|A| < \kappa$, we have $|S_1(A)| \leq \kappa$.
- T is stable if it is κ-stable for some κ.
- T is *totally transcendental* (TT) if it is κ-stable for *all* κ.

Example: DLO_o is not $\aleph_0$-stable since $S_1(\mathbb{Q}) \cong \mathbb{R}$ which has larger cardinality than $\aleph_0$.