Question 1

Statement

Evaluate the following line integrals
- $\int_C y\,\mathrm{d}s$ where $C : x = t^2, y = 2t, 0 \le t \le 3$.
- $\int_C xy^4\,\mathrm{d}s$ where C is the right half of the circle $x^2 + y^2 = 16$.
- $\int_C (x^2y + \sin x)\,\mathrm{d}y$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2).

Solution

- First we need to calculate the length element $\mathrm{d}s$. We use the formula

  $$\mathrm{d}s = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\mathrm{d}t = \sqrt{(2t)^2 + (2)^2}\,\mathrm{d}t = 2\sqrt{1+t^2}\,\mathrm{d}t$$

  we then can compute the integral

  $$\int_C y\,\mathrm{d}s = \int_0^3 2t\left(2\sqrt{1+t^2}\right)\mathrm{d}t = \int_1^{10} 2\sqrt{u}\,\mathrm{d}u = \left[\frac{4}{3}u^{\frac{2}{3}}\right]_1^{10} = \frac{4}{3}\left(10\sqrt{10} - 1\right)$$

- First thing we want to do is to parametrize the circle, this is almost always done with polar coordinates with r fixed, for us this will be $x = 4\cos(t), y = 4\sin(t)$. Now the whole circle is given by $0 \le t \le 2\pi$, but since we only want the right part we will instead want $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. With this parametrization we get that

  $$\mathrm{d}s = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\mathrm{d}t = \sqrt{(4\cos t)^2 + (4\sin t)^2}\,\mathrm{d}t = 4\,\mathrm{d}t$$

  and so we can compute

  $$\int_C xy^4\,\mathrm{d}s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos(t)(4\sin(t))^4 4\,\mathrm{d}t = 4^6\int_{-1}^{1} u^4\,\mathrm{d}u = \frac{2\cdot 4^6}{5}$$

- This time our parametrization is easy since we can simply use $x = t, y = t^2$. Note that our integration now is with respect to $\mathrm{d}y$ which we compute to be

$$\mathrm{d}y = \frac{\partial y}{\partial t}\,\mathrm{d}t = 2t\,\mathrm{d}t$$

This then gives us

$$\begin{aligned}\int_0^{\pi} \big((t)^2(t^2) + \sin t\big)2t\,\mathrm{d}t &= 2\int_0^{\pi} t^5 + t\sin t\,\mathrm{d}t = \frac{1}{3}\pi^6 + 2\int_0^{\pi} t\sin t\,\mathrm{d}t \\ &= \frac{1}{3}\pi^6 + 2[t(-\cos t)]_0^{\pi} - 2\int_0^{\pi}(-\cos t)\,\mathrm{d}t \\ &= \frac{1}{3}\pi^6 + 2\pi\end{aligned}$$

Question 2

Statement

Evaluate the following line integrals

- $\int_C y^2 z\,\mathrm{d}s$, where C is the line segment from $(3, 0, 2)$ to $(1, 2, 5)$.
- $\int_C xye^{yz}\,\mathrm{d}y$, where $C : x = t, y = t^2, z = t^3, 0 \le t \le 1$.
- $\int_C z\,\mathrm{d}x + xy\,\mathrm{d}y + y^2\,\mathrm{d}z$, where $C : x = \sin(t), y = \cos(t), z = \tan(t), -\frac{\pi}{4} \le t \le \frac{\pi}{4}$

Solution

- Our first job is to parametrize the line segment, for line segments there is a nice formula and it is

$$(3, 0, 2)(1 - t) + (1, 2, 5)t = (3 - 2t, 2t, 2 + 3t)$$

with $0 \le t \le 1$. This then gives us

$$\mathrm{d}s = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\,\mathrm{d}t = \sqrt{(-2)^2 + (2)^2 + (3)^2}\,\mathrm{d}t = \sqrt{17}\,\mathrm{d}t$$

and so we can compute

$$\sqrt{17}\int_0^1 (2t)^2(2+3t)\,\mathrm{d}t = \sqrt{17}\int_0^1 8t^2+12t^3\,\mathrm{d}t = \sqrt{17}\left[\frac{8}{3}t^3+3t^4\right]_0^1$$
$$= \sqrt{17}\left[\frac{8}{3}+3\right] = \frac{17\sqrt{17}}{3}$$

- Here our parametrization is given and the integral is with respect to $\mathrm{d}y$ and so we calculate

$$\mathrm{d}y = \frac{\partial y}{\partial t}\,\mathrm{d}t = 2t\,\mathrm{d}t$$

  which then lets us compute

$$\int_0^1 t^3e^{t^5}2t\,\mathrm{d}t = \frac{2}{5}\int_0^1 e^u\,\mathrm{d}u = \frac{2}{5}(e-1)$$

- Again the parametrization is given and so we simply compute

$$\mathrm{d}x = \cos(t)\,\mathrm{d}t, \quad \mathrm{d}y = -\sin(t)\,\mathrm{d}t, \quad \mathrm{d}z = \sec^2(t)\,\mathrm{d}t$$

  this then gives us

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\tan(t)\cos(t) - \sin^2(t)\cos(t) + \cos^2(t)\sec^2(t)\right)\mathrm{d}t$$

  and so after simplifying we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sin(t) - \sin^2(t)\cos(t) + 1\right)\mathrm{d}t$$

  we split this integral up into 3 parts and we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\,\mathrm{d}t = \frac{\pi}{2} \qquad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(t)\,\mathrm{d}t = 0 \qquad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2(t)\cos(t)\,\mathrm{d}t = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u^2\,\mathrm{d}u = \frac{1}{3\sqrt{2}}$$

  giving us

$$\frac{\pi}{2} + \frac{1}{3\sqrt{2}}$$

Question 3

Statement

Evaluate the line integral $\int_C F \cdot \mathrm{d}r$ (there were some typos here in the original questions).

- $F(x, y) = xy^2\mathbf{i} - x^2\mathbf{j}$ where C is the curve $r(t) = t^3\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 1$.
- $F(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ where C is the curve $r(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}, 0 \le t \le 1$.
- $F(x, y, z) = xz\mathbf{i} + z^3\mathbf{j} + y\mathbf{k}$ where C is the curve $r(t) = e^t\mathbf{i} - e^{2t}\mathbf{j} + e^{-t}\mathbf{k}, -1 \le t \le 1$.

Solution

- First we will compute the velocity of C,

$$\frac{\partial r}{\partial t} = (3t^2, 2t)$$

and we also have

$$F(x(t), y(t)) = (t^5, -t^6)$$

and so we can compute

$$\int_C F \cdot \mathrm{d}r = \int_0^1 (3t^2, 2t) \cdot (t^5, -t^6)\,\mathrm{d}t = \int_0^1 (3t^7 - 2t^7)\,\mathrm{d}t = \frac{1}{8}$$

- Again we first compute the velocity of C,

$$\frac{\partial r}{\partial t} = (3t^2, -2t, 1)$$

as well as

$$F(x(t), y(t), z(t)) = \sin(t^3)\mathbf{i} + \cos(t^2)\mathbf{j} + t^4\mathbf{k}$$

giving us

$$\begin{aligned}\int_0^1 \sin(t^3)3t^2 - \cos(t^2)2t + t^4\,\mathrm{d}t &= \int_0^1 \sin(u)\,\mathrm{d}u - \int_0^1 \cos(u)\,\mathrm{d}u + \int_0^1 t^4\,\mathrm{d}t \\ &= 1 - \cos(1) - \sin(1) + \frac{1}{5} \\ &= \frac{6}{5} - \cos(1) - \sin(1)\end{aligned}$$

- Finally we one agian compute the velocity of C,

$$\frac{\partial r}{\partial t} = (e^t, 2e^{2t}, -e^{-t})$$

and also get

$$F(x(t), y(t), z(t)) = 1\mathbf{i} + e^{-3t}\mathbf{j} + e^{2t}\mathbf{k}$$

giving us

$$\int_{-1}^{1} e^t + 2e^{-t} - e^t \, \mathrm{d}t = \int_{-1}^{1} 2e^{-t} \, \mathrm{d}t = 2\big(e^1 - e^{-1}\big)$$