1. Basic Definitions and Concepts

1.1. Models and Languages

Definition 1.1.1: A *model* or *structure* is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\right)$$

where
- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark 1.1.2: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.1.3: A *language* (or *signature*) is a tuple

$$L = \left(\left(\underline{f_i}\right)_{i\in I'}, \left(\underline{R_j}\right)_{j\in J'}, \left(\underline{c_k}\right)_{k\in K'}\right)$$

where now the f_i are function *symbols* with arity $a'_i \in \mathbb{N}$, each R_j are relation *symbols* with arity $a'_j \in \mathbb{N}$, and c_j are constant *symbols*.

A model $\mathcal{M}$ is an L-structure if

$$I = I', J = J', K = K', a_i = a'_i, a_j = a'_j$$

If $\mathcal{M}$ is an L-structure then the *interpretations* of the symbols of the language are defined as

$$\underline{f_i}^{\mathcal{M}} = f_i, \underline{R_j}^{\mathcal{M}} = R_j, \underline{c_k}^{\mathcal{M}} = c_k$$

Remark 1.1.4: For a model $\mathcal{M}$ we will sometimes denote $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe.

We have defined the symbols of L but how do we speak it? We will need the following

- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: x, y, z, v
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Remark 1.1.5: We will often use $\overline{a}$ to denote the ordered collection $(a_1, ..., a_n)$ where n will be clear from context.

Definition 1.1.6: *L-terms* are defined inductively as follows
- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, ..., \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, ..., \tau_n)$$

is a term.

An L-term is said to be *constant* if it does not contain any variables.

Definition 1.1.7: If $\mathcal{M}$ is an L-structure and τ is a constant L-term then the *interpretation* of τ, $\tau^{\mathcal{M}}$, is defined equivalently
- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in M$

Example: $L = (+, \cdot, 0, 1)$ then $M = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0

Definition 1.1.8: An *L-formula* is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

  $$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

  $$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called *atomic* L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x(x = 1),$$
$$(\exists x(x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Now this is all first order logic, but one might wonder, what makes it "first"? This comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in |\mathcal{M}|$, in *second* order logic we can quantify over subsets $S \subseteq |\mathcal{M}|$ like all relations for example. We can also see this as $S \in \mathcal{P}(|\mathcal{M}|)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(|\mathcal{M}|))$, and so on.

In this course, however, we will only be looking at first order logic.

Definition 1.1.9: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are *bound* in φ', and we say that φ is the *range* of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is *free* if it is not bound in φ.

An L-*sentence* is an L-formula with no free variable.

Definition 1.1.10: Let φ be a formula containing x, $\varphi({}^{\tau}/_{x})$ will denote the formula obtained by replacing every free occurrence of x by τ.

Now one would expect that substitution should never change the meaning of a logical statement, but in fact, this is not quite right. Consider the case $\varphi = \forall y(y = x)$, the substitution $\varphi\left(\frac{y}{x}\right)$ is changes the meaning of the statement from "all y are equal to x" to "all y are equal to themselves". We want to avoid this outcome, which we can formalize as follows.

Definition 1.1.11: A substitution $\varphi({}^{\tau}/_{x})$ is called *correct* if no free variable of τ becomes bound in $\varphi({}^{\tau}/_{x})$

Definition 1.1.12: If $A \subseteq |\mathcal{M}|$ and $\mathcal{M}$ is an L-structure then $L(A)$ is the language

$$L \cup \{a : a \in A\}$$

We extend our definition of interpretation of terms to terms of $L(|\mathcal{M}|)$ by setting $\underline{a}^{\mathcal{M}} = a$

Definition 1.1.13: Let $\mathcal{M}$ be an L-structure and σ an $L(|\mathcal{M}|)$-sentence. We say that σ is true in $\mathcal{M}$, and write $\mathcal{M} \vDash \sigma$ if

- If σ is of the form $\tau_1 = \tau_2$ then $M \vDash \sigma$ if and only if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ (note that while this may look circular, the first equality is in the space of *terms* while the second is in the universe $|\mathcal{M}|$)
- If σ is of the form $\underline{R}_j(\tau_1, ..., \tau_n)$, then $\mathcal{M} \vDash \sigma$ if and only if $\left(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}\right) \in R_j$
- If σ is of the form $\sigma_1 \wedge \sigma_2$ then $\mathcal{M} \vDash \sigma_1 \wedge \sigma_2$ if $\mathcal{M} \vDash \sigma_1$ and $\mathcal{M} \vDash \sigma_2$. A similar definition follows for the other logical connectives.
- If σ is of the form $\exists x\, \varphi$ then $\mathcal{M} \vDash \varphi$ if there exists $a \in |\mathcal{M}|$ with $\mathcal{M} \vDash \varphi(\underline{a}/_x)$. Similarly for $\forall x\, \varphi$.

Definition 1.1.14: Let $\mathcal{M}$ be a model. The *theory* of $\mathcal{M}$ is defined also

$$\mathrm{Th}(\mathcal{M}) = \{\sigma \text{ is an } L\text{-sentence} : \mathcal{M} \vDash \sigma\}$$

We say that two L-structures, $\mathcal{M}$ and $\mathcal{N}$, are elementary equivalent, and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

We write that $\mathcal{M} \subseteq \mathcal{N}$ to mean that $\mathcal{M}$ is a substructure of $\mathcal{N}$, meaning that

$$|\mathcal{M}| \subseteq |\mathcal{N}|, \underline{f_i}^{\mathcal{M}} \subseteq \underline{f_i}^{\mathcal{N}}, \underline{R_j}^{\mathcal{M}} = \underline{R_j}^{\mathcal{N}} \cap |\mathcal{M}|^{a_j}, \text{ and } \underline{c_k}^{\mathcal{M}} = \underline{c_k}^{\mathcal{N}}$$

We write $\mathcal{M} \equiv \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there is a bijection g with

$$g\left(\underline{c_k}^{\mathcal{M}}\right) = \underline{c_k}^{\mathcal{N}}$$
$$(a_1, ..., a_n) \in \underline{R_j}^{\mathcal{M}} \Leftrightarrow (f(a_1), ..., f(a_n)) \in \underline{R_j}^{\mathcal{N}}$$
$$f\left(\underline{f_i}^{\mathcal{M}}(a_1, ..., a_n)\right) = \underline{f_i}^{\mathcal{N}}(a_1, ..., a_n)$$

We write $\mathcal{M} \prec (\preccurlyeq)\ \mathcal{N}$ to mean $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is true if $|M| \subseteq |N|$ and for every formula $\varphi(\overline{x})$ and for every $\overline{a} \subseteq M$ we have

$$M \vDash \varphi(\overline{a}) \Leftrightarrow N \vDash \varphi(\overline{a})$$

Theorem 1.1.15 (Tarski-Vaught test): Suppose $\mathcal{M}$ is an L-structure, $A \subseteq |\mathcal{M}|$, then A is the universe of an elementary substructure iff the following condition holds, called the Tarski-Vaught test

> For every formula $\varphi(x, \overline{y})$ in L and every $\overline{a} \subseteq A$, if $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ then there exists $b \in A$ such that $M \vDash \varphi(b, \overline{a})$

Proof: First the $\Leftarrow$ direction, assume that the T-V test holds, then we need to show that A is a substructure. First we use $\varphi = (x = c)$ to show that A contains all constants of $\mathcal{M}$, then $\varphi = (x = \varphi_i(\overline{a}))$ for $\overline{a} \subseteq A$, and we define the interpretation of $\underline{R_j}$ to be exactly $\underline{R_j}^{\mathcal{M}} \cap A^{a_j}$ to make it a substructure.

Now A being a substructure is equivalent to

$$A \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{M} \vDash \varphi(\overline{a})$$

for all $\overline{a} \subseteq A$ and φ being an *atomic* formula. So now we only need to prove this is true for the other formula types.

- The connective types are immediate.
- Let us assume $\varphi(\overline{x}) = \exists y\, \psi(y, \overline{x})$. Then $M \vDash \varphi(\overline{a})$ iff $\mathcal{M} \vDash \exists y\, \psi(y, \overline{a})$ iff there exists $b \in A$ with $\mathcal{M} \vDash \psi(b, \overline{a})$. But by definition this last form is equivalent to $A \vDash \exists y\, \psi(y, \overline{a})$

□

Theorem 1.1.16 (Lowenheim-Skolem downwards Theorem): Let L be countable, for any L-structure $\mathcal{M}$ and every $A \subseteq |\mathcal{M}|$, there exists an elementary substructure $N \prec M$with $A \subseteq |\mathcal{N}|$

$$\|\mathcal{N}\| = |A| + |L| + \aleph_0$$

Proof: Set $\kappa = |A| + |L| + \aleph_0$, by transfinite induction on κ we will define a sequence a_α for $\alpha < \kappa$ of elements in $\mathcal{M}$, where at each step α we will try to satisfy a formula $\varphi_a(x) \in L(A \cup a_{<\alpha})$, we will then set $|\mathcal{N}| = \{a_\alpha : \alpha < \kappa\}$. □