General ideas: Mathematical structures from a higher perspective. First developed by Tarski, then demonstrated use in algebra, grew to be its own field. We will study introductory Model Theory, end with Morley's categoricity theorem.

1. Basic Definitions and Concepts

Definition 1.1: A **model** or **structure** is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i\in I}, (R_j)_{j\in J}, (c_k)_{k\in K}\right)$$

where
- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark 1.2: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.3: A **language** (or **signature**) is a tuple

$$L = \left(\left(\underline{f}_i\right)_{i \in I'}, \left(\underline{R}_j\right)_{j \in J'}, \left(\underline{c}_k\right)_{k \in K'}\right)$$

where now the f_i are function **symbols** with arity $a_{i'} \in \mathbb{N}$, each R_j are relation **symbols** with arity $a_{j'} \in \mathbb{N}$, and c_j are constant **symbols**.

A model $\mathcal{M}$ is an L-structure if

$$I = I', J = J', K = K', a_i = a_{i'}, a_j = a_{j'}$$

If $\mathcal{M}$ is an L-structure then the **interpretations** of the symbols of the language are defined as

$$(f_{i'})^{\mathcal{M}} = f_i, (R_{j'})^{\mathcal{M}} = R_j, (c_{k'})^{\mathcal{M}} = c_k$$

Remark 1.4: For a model $\mathcal{M}$ we will sometimes denote $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe.

We have defined the symbols of L but how do we speak it? We will need the following
- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: x, y, z, v
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Definition 1.5: L-terms are defined inductively as follows
- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, ..., \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, ..., \tau_n)$$

is a term.

An L-term is said to be constant if it does not contain any variables.

Definition 1.6: If M is an L-structure and τ is a constant L-term then the **interpretation** of τ, $\tau^{\mathcal{M}}$, is defined equivalently
- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in M$

Example: $L = (+, \cdot, 0, 1)$ then $M = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0

Definition 1.7: An L-**formula** is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

  $$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

  $$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called **atomic** L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x(x = 1),$$
$$(\exists x(x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Now this is all first order logic, but one might wonder, what makes it “first”, this comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in M$, in *second* order logic we can quantify over subsets $S \subseteq M$ like all relations for example.We can also see this as $S \in \mathcal{P}(M)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(M))$.

In this course, however, we will only be looking at first order logic.

Definition 1.8: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are **bound** in φ', and we say that φ is the **range** of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is **free** if it is not bound in φ.

An L-**sentence** is an L-formula with no free variable.