1. Basic Definitions and Concepts

1.1. Models and Languages

Definition 1.1.1: A *model* or *structure* is a tuple

$$\mathcal{M} = \left(M, (f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\right)$$

where
- M is a set called the universe
- f_i are functions $f : M^{a_i} \to M$
- R_j are relations $R_j \subseteq M^{a_j}$
- c_k are constants $c_k \in M$.

Remark 1.1.2: Sometimes constants can be seen as 0-ary functions.

Example: Consider the model $(\mathbb{C}, +, \cdot, \exp)$, consisting of the universe $\mathbb{C}$ with the 3 functions $+, \cdot, \exp$. Note that we will often write out the functions inside the brackets as above, it will be clear if an object is a function,relation or constant from context.

Example: Another model would be $(\mathbb{R}, +, \cdot, <)$, consisting of the universe $\mathbb{R}$ with the 2 functions $+, \cdot$ and the 2-ary relation $<$.

Example: $(\mathbb{Z}_4, +_4, 0)$, here 0 is a constant.

Example: An important example is $(V, \in)$ where V is any set which sort of encodes set theory (though there are several issues with this).

We can see already that models can encode many objects that we study in math, and there are many many more such encodings.

All of this is very semantic encoding of a mathematical structure, but we will also be concerned with the syntactical encoding.

Definition 1.1.3: A *language* (or *signature*) is a tuple

$$L = \left(\left(\underline{f_i} \right)_{i \in I'}, \left(\underline{R_j} \right)_{j \in J'}, \left(\underline{c_k} \right)_{k \in K'} \right)$$

where now the f_i are function *symbols* with arity $a_i' \in \mathbb{N}$, each R_j are relation *symbols* with arity $a_j' \in \mathbb{N}$, and c_j are constant *symbols*.

A model $\mathcal{M}$ is an L-structure if

$$I = I', J = J', K = K', a_i = a_i', a_j = a_j'$$

If $\mathcal{M}$ is an L-structure then the *interpretations* of the symbols of the language are defined as

$$\underline{f_i}^{\mathcal{M}} = f_i, \underline{R_j}^{\mathcal{M}} = R_j, \underline{c_k}^{\mathcal{M}} = c_k$$

Remark 1.1.4: For a model $\mathcal{M}$ we will sometimes denote $|\mathcal{M}|$ to refer to the universe of a model and $\|\mathcal{M}\|$ to denote the cardinality of said universe.

We have defined the symbols of L, but how do we speak it? We will need the following

- Logical symbols, these will consist of
  - Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$
- Auxiliary symbols: Parentheses, Commas
- Variables: $x, y, z, v, ...$
- Equivalency Symbol: $=$

As with any language we will build up our language first with nouns and then with phrases.

Remark 1.1.5: We will often use $\overline{a}$ to denote the ordered collection $(a_1, ..., a_n)$ where n will be clear from context.

Definition 1.1.6: *L-terms* are defined inductively as follows
- Any constant symbol is an L-term
- Any variable symbol is an L-term
- If $\tau_1, ..., \tau_n$ are L-terms f_i is a function with arity n then

$$f_i(\tau_1, ..., \tau_n)$$

is a term.

An L-term is said to be *constant* if it does not contain any variables.

Definition 1.1.7: If $\mathcal{M}$ is an L-structure and τ is a constant L-term then the *interpretation* of τ, $\tau^{\mathcal{M}}$, is defined equivalently
- If $\tau = c_k$ then $\tau^{\mathcal{M}} = c_k^{\mathcal{M}}$
- If $\tau = f_i(\tau_1, ..., \tau_n)$ then $\tau^{\mathcal{M}} = f_i^{\mathcal{M}}(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}) \in |\mathcal{M}|$

Example: $L = (+, \cdot, 0, 1)$ then $\mathcal{M} = (\mathbb{N}, +, \cdot, 0, 1)$ is an L-structure in which the L-term

$$\tau = 1 + 1 + 1$$

has the interpretation 3.

However, in the L-structure $(\mathbb{Z}_3, +_3, \cdot_3, 0, 1)$ the interpretation is instead 0

Definition 1.1.8: An *L-formula* is also defined inductively
- If τ_1, τ_2 are L terms then $\tau_1 = \tau_2$ is an L-formula
- If $\tau_1, ..., \tau_n$ are L-terms then $R_j(\tau_1, ..., \tau_n)$ is a formula if R_j is an n-ary relation.
- If φ_1, φ_2 are L-formulas, then

  $$\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \neg\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_1 \Leftrightarrow \varphi_2$$

  are all L-formulas.
- If φ is an L-formula, x is a variable, then

  $$\forall x\, \varphi, \exists x\, \varphi$$

  are both L-formulas.

The first 2 of these are called *atomic* L-formula.

Example: The following are all formulas,

$$1 = 1 + 1, x = 1, 0 = 1, 1 = 1, (1 = 1) \wedge \neg(0 = 1), \forall x(x = 1),$$
$$(\exists x(x = 1)) \Rightarrow (\forall x \forall y\, x = y), \forall x \forall x\, 1 = 1$$

Now this is all first order logic, but one might wonder, what makes it "first"? This comes from what things we can quantify over. In first order logic we can only quantify over elements $x \in |\mathcal{M}|$, in *second* order logic we can quantify over subsets $S \subseteq |\mathcal{M}|$ like all relations for example. We can also see this as $S \in \mathcal{P}(|\mathcal{M}|)$. Third order logic would then be quantification over $S \in \mathcal{P}(\mathcal{P}(|\mathcal{M}|))$, and so on.

In this course, however, we will only be looking at first order logic.

Definition 1.1.9: If φ is an L-formula then in the formulas

$$\varphi' = \forall x\, \varphi \text{ or } \varphi' = \exists x\, \varphi$$

we say that all occurrences of x are *bound* in φ', and we say that φ is the *range* of $\forall x$ or $\exists x$ respectively.

An occurrence of a variable x in a formula φ is *free* if it is not bound in φ.

An L-*sentence* is an L-formula with no free variables.

Definition 1.1.10: Let φ be a formula containing x (which we will follow denote as $\varphi(x)$), $\varphi({}^{\tau}/_{x})$ will denote the formula obtained by replacing every free occurrence of x by τ.

Now one would expect that substitution should never change the meaning of a logical statement, but in fact, this is not quite right. Consider the case $\varphi = \forall y(y = x)$, the substitution $\varphi({}^{y}/_{x})$ is changes the meaning of the statement from "all y are equal to x" to "all y are equal to themselves". We want to avoid this outcome, which we can formalize as follows.

Definition 1.1.11: A substitution $\varphi({}^{\tau}/_{x})$ is called *correct* if no free variable of τ becomes bound in $\varphi({}^{\tau}/_{x})$

Definition 1.1.12: If $A \subseteq |\mathcal{M}|$ and $\mathcal{M}$ is an L-structure then $L(A)$ is the language

$$L \cup \{a : a \in A\}$$

We extend our definition of interpretation of terms to terms of $L(|\mathcal{M}|)$ by setting $\underline{a}^{\mathcal{M}} = a$

Definition 1.1.13: Let $\mathcal{M}$ be an L-structure and σ an $L(|\mathcal{M}|)$-sentence. We say that σ is true in $\mathcal{M}$, and write $\mathcal{M} \vDash \sigma$ if

- If σ is of the form $\tau_1 = \tau_2$ then $M \vDash \sigma$ if and only if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ (note that while this may look circular, the first equality is in the space of *terms* while the second is in the universe $|\mathcal{M}|$)
- If σ is of the form $\underline{R}_j(\tau_1, ..., \tau_n)$, then $\mathcal{M} \vDash \sigma$ if and only if $\left(\tau_1^{\mathcal{M}}, ..., \tau_n^{\mathcal{M}}\right) \in R_j$
- If σ is of the form $\sigma_1 \wedge \sigma_2$ then $\mathcal{M} \vDash \sigma_1 \wedge \sigma_2$ if $\mathcal{M} \vDash \sigma_1$ and $\mathcal{M} \vDash \sigma_2$. A similar definition follows for the other logical connectives.
- If σ is of the form $\exists x \, \varphi$ then $\mathcal{M} \vDash \sigma$ if there exists $a \in |\mathcal{M}|$ with $\mathcal{M} \vDash \varphi(\underline{a}/_x)$. Similarly for $\forall x \, \varphi$.

1.2. Model equivalences

Definition 1.2.1: Let $\mathcal{M}$ be a model. The *theory* of $\mathcal{M}$ is defined also

$$\mathrm{Th}(\mathcal{M}) = \{\sigma \text{ is an } L\text{-sentence} : \mathcal{M} \vDash \sigma\}$$

We say that two L-structures, $\mathcal{M}$ and $\mathcal{N}$, are elementarily equivalent, and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

We write that $\mathcal{M} \subseteq \mathcal{N}$ to mean that $\mathcal{M}$ is a substructure of $\mathcal{N}$, meaning that

$$|\mathcal{M}| \subseteq |\mathcal{N}|, \underline{f_i}^{\mathcal{M}} \subseteq \underline{f_i}^{\mathcal{N}}, \underline{R_j}^{\mathcal{M}} = \underline{R_j}^{\mathcal{N}} \cap |\mathcal{M}|^{a_j}, \text{ and } \underline{c_k}^{\mathcal{M}} = \underline{c_k}^{\mathcal{N}}$$

We write $\mathcal{M} \simeq \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there is a bijection g with

$$g\left(\underline{c_k}^{\mathcal{M}}\right) = \underline{c_k}^{\mathcal{N}}$$
$$(a_1, ..., a_n) \in \underline{R_j}^{\mathcal{M}} \Leftrightarrow (g(a_1), ..., f(a_n)) \in \underline{R_j}^{\mathcal{N}}$$
$$g\left(\underline{f_i}^{\mathcal{M}}(a_1, ..., a_n)\right) = \underline{f_i}^{\mathcal{N}}(a_1, ..., a_n)$$

We write $\mathcal{M} \prec (\preccurlyeq) \, \mathcal{N}$ to mean $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is true if $\mathcal{M} \subseteq \mathcal{N}$ and for every formula $\varphi(\overline{x})$ and for every $\overline{a} \subseteq |\mathcal{M}|$ we have

$$\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \varphi(\overline{a})$$

Theorem 1.2.2 (Tarski-Vaught test): Suppose $\mathcal{M}$ is an L-structure, $A \subseteq |\mathcal{M}|$, then A is the universe of an elementary substructure iff the following condition holds, called the Tarski-Vaught test

> For every formula $\varphi(x, \overline{y})$ in L and every $\overline{a} \subseteq A$, if $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ then there exists $b \in A$ such that $\mathcal{M} \vDash \varphi(b, \overline{a})$

Proof: First the $\Leftarrow$ direction, assume that the T-V test holds, then we need to show that A is a substructure. First we use $\varphi = (x = c)$ to show that A contains all constants of $\mathcal{M}$, then $\varphi = (x = \varphi_i(\overline{a}))$ for $\overline{a} \subseteq A$, and we define the interpretation of $\underline{R_j}$ to be exactly $\underline{R_j}^{\mathcal{M}} \cap A^{a_j}$ to make it a substructure.

Now A being a substructure is equivalent to

$$A \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{M} \vDash \varphi(\overline{a})$$

for all $\overline{a} \subseteq A$ and φ being an *atomic* formula. So now we only need to prove this is true for the other formula types.

- The connective types are immediate.
- Let us assume $\varphi(\overline{x}) = \exists y\, \psi(y, \overline{x})$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ iff $\mathcal{M} \vDash \exists y\, \psi(y, \overline{a})$ iff there exists $b \in A$ with $\mathcal{M} \vDash \psi(b, \overline{a})$. But by definition this last form is equivalent to $A \vDash \exists y\, \psi(y, \overline{a})$

Assume, on the other hand, that A is the universe of an elementary substructure $\mathcal{A}$, then we need to prove the T-V test holds, assume then that for some formula $\varphi(x, \overline{y})$ in L and some $\overline{a} \subseteq A$ we have $\mathcal{M} \vDash \exists x\, \varphi(x, \overline{a})$ and so since it is an elementary substructure we also have that $\mathcal{A} \vDash \exists x\, \varphi(x, \overline{a})$ and so we must have some $x \in A$ such that $\varphi(x, \overline{a})$ holds. $\square$

Theorem 1.2.3 (Lowenheim-Skolem downwards Theorem): Let L be countable, for any L-structure $\mathcal{M}$ and every $A \subseteq |\mathcal{M}|$, there exists an elementary substructure $\mathcal{N} \prec \mathcal{M}$ with $A \subseteq |\mathcal{N}|$

$$\|\mathcal{N}\| = |A| + |L| + \aleph_0$$

Proof: Using induction we will define a sequence A_n of subsets of $\mathcal{M}$, where at each step n we will try to satisfy all statements in $L(A_{n-1})$, we will then set $|\mathcal{N}| = \bigcup_n A_n$.

First we set $A_0 = A$, then at step $n > 0$, we will consider all formulas in $L(A_{n-1})$ (there are $|\kappa \times \mathbb{Z}| = |\kappa|$ many of them) and for each formula $\varphi(\overline{x})$ we will pick some collection of elements $\overline{a} \subseteq |\mathcal{M}|$ such that $\mathcal{M} \vDash \varphi(\overline{a})$, then we will add $\overline{a}$ to A_{n-1}, adding these elements for each formula gives us A_n. $\square$

Remark 1.2.4 (Skolem's Paradox): Let $ZFC^* \subseteq ZFC$ be a finite substructure which proves cantor's theorem. Let $V \vDash ZFC^*$. By the previous theorem we can find a countable $\mathcal{M} \prec V$ for which $\mathcal{M} \vDash ZFC^*$ and $\mathcal{M} \vDash$ "exists an uncountable set".

Definition 1.2.5: In FOL we have the concept of a *proof system*, consisting of two parts. *Axioms*, and *proofs* which is a finite sequence of L-formulas such that every step is either an axiom of follows from the previous steps using an inference rule.

Example: An example proof system has the following 4 types of axioms.
- All instances of propositional tautologies are axioms.
- $[\forall x\, \varphi \to \psi] \to [\varphi \to \forall \psi]$ as long as x is not free in φ.
- $\forall x \to \varphi({}^t/_s)$ where t is any L-term where the substitution is correct.
- $x = x$,
  $x = y \to t(..., x, ...) = t(..., y, ...)$ for any L-term,
  $x = y \to (\varphi(..., x, ...) \to \varphi(..., y, ...))$

And the following inference rules.
- If φ and $\varphi \to \psi$ then ψ.
- If φ then $\forall x\, \varphi$.

We will use the notation $\Gamma \vdash \varphi$ to mean "Γ proves φ" and define it as the existence of a proof whose final step is φ and every step is either an axiom or an element of Γ or follows from a previous step or by an inference in φ.

Definition 1.2.6: We say that Γ is consistent if there exists φ such that $\Gamma \nvdash \varphi$.

By a famous theorem of Gödel that we will not prove in this class we can actually not care about any proof system details.

Theorem 1.2.7 (Gödel's completeness theorem): Let Γ be a set of sentences in L then Γ is consistent if and only if Γ has a model.

We will not prove this theorem in this class but we will use an important corollary of it.

Corollary 1.2.7.1 *(Compactness Theorem)*: Let Γ be a set of L-sentences, Γ has a model if and only if every finite subset of Γ has a model.

Proof: The $\Rightarrow$ direction is immediate, the hard part is the $\Leftarrow$ direction. By Gödel's completeness theorem, we can replace "having a model" with "is consistent".

We now prove this by contrapositive, assume that Γ is inconsistent, then we have $\Gamma \vdash \exists x\,(x = x) \wedge (\neg(x = x))$, now this proof consists of finitely many steps and thus can only use finitely many statements in Γ, let Γ_0 be that subset of statements. Since we can prove a contradiction using Γ_0 it must also be inconsistent, thus one of the finite subsets of Γ is inconsistent. □

As an example use we have the following theorem.

Theorem 1.2.8 (Lowenheim-Skolem upwards Theorem): If $\mathcal{M}$ is an infinite L-structure where L is countably infinity then $\forall k > \|\mathcal{M}\|$ there exists a model $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| = k$

Proof: Let us consider the language $L' = L(\mathcal{M}) \cup \{c_\alpha : \alpha < \kappa\}$ where c_α are new constants. Now set

$$\Gamma = \mathrm{Th}(\mathcal{M}) \cup \left\{c_\alpha \neq c_\beta : \alpha \neq \beta < \kappa\right\}$$

We want to show now that Γ is consistent, to see this we use compactness and take an arbitrary finite subset Γ_0. Let $\alpha_1, ..., \alpha_n$ be such that

$$\Gamma_0 \subseteq \mathrm{Th}(\mathcal{M}) \cup \left\{c_{\alpha_i} \neq c_{\alpha_j} : i \neq j\right\}$$

choose then any $a_1, ..., a_n \in |\mathcal{M}|$ which are distinct and interpret c_{α_i} as a_i to get a model of Γ_0, hence Γ_0 is consistent.

Now we have by Gödel's completeness theorem that there exists a model $\mathcal{N}$ such that $\mathcal{N} \vDash \Gamma$ then by construction we have $\mathcal{M} \prec \mathcal{N}$ and $\|\mathcal{N}\| \geq \kappa$ and so by downwards theorem we can now decrease the cardinality until we reach κ. □

Definition 1.2.9: A *theory* is a set Γ of sentences such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.

A theory T is *complete* if for every sentence φ either $\varphi \in T$ or $\neg\varphi \in T$.

Remark 1.2.10:

- For any model $\mathcal{M}$ the theory $\mathrm{Th}(\mathcal{M})$ is complete.
- For any theory T which is complete, there exists a model $\mathcal{M}$ with $T = \mathrm{Th}(\mathcal{M})$.

Corollary 1.2.10.1: If $\mathcal{M}$ is infinite then there exists $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$.

Proof: We simply pick some $\kappa > \|\mathcal{M}\|$ and then use the upwards theorem to get a model $\mathcal{N}$ with $\mathcal{M} \prec \mathcal{N}$ with $\|\mathcal{N}\| = \kappa$, now there can't exist a bijection between the two since they have different cardinalities. □

1.3. Categoricity

Definition 1.3.1: Let κ be an infinite cardinal, a theory T is κ-categorical if it has infinitely many models but exactly one model (up to isomorphism) of size κ.

Proposition 1.3.2: If T is κ-categorical, then T is complete.

Proof: Suppose that T is not complete, let σ be such that $\sigma \notin T$ and $\neg\sigma \notin T$, then let $T_1 = T \cup \{\sigma\}$ and $T_2 = T \cup \{\neg\sigma\}$. Both are consistent but are not isomorphic, this contradicts the fact that there is only one model of this size. □

Example: Consider the language $L = (<)$, a dense linear order (DLO_0) is the theory generated by the additional axioms: $<$ is total, dense and has no endpoints.

- Total means $\forall x\, \forall y(x = y \vee x < y \vee y < x)$
- Dense means $\forall x\, \forall y(x < y \Rightarrow \exists z x < z < y)$
- No endpoints means $\neg(\exists z\, \forall x(x \neq z \Rightarrow x < z))$ for the max endpoint and similarly for the min endpoint.

Examples of such a structure include $\mathbb{Q}$, $\mathbb{R}$ and many others.

It turns out, however, that the only countable such structure is $\mathbb{Q}$, up to isomorphism.

Proposition 1.3.3 (Cantor): DLO_0 is $\aleph_0$-categorical.

Proof: Let $(A, <)$ and $(B, <)$ be two countable models of DLO_0, we enumerate them $A = \{a_0, a_1, ...\}$ and $B = \{b_0, b_1, ...\}$.

We now use the "back and forth" method, which essentially incrementally pairs up elements of A with elements of B, and in the limit this will give us a bijection which will be our isomorphism.

More formally we will construct a sequence $\varphi_n : A_n \to B_n$ where $A_n \subseteq A, B_n \subseteq B$ where each φ_n is monotone increasing, and so that at step $2n$ we have $a_n \in A_n$ and $b_n \in B_n$.

For the base case we take a_0 and pair it with anything in B, lets say b_{20}, now we look at the smallest (in the sense of the enumeration) element b_i in B (in this case b_0), and try to map it to something in A. Now b_i will be somehow related to b_{20}, we can now use the density and the lack of endpoints to always find an element in A that has the same relations as b_i so we can always find a proper pairing.

□

Corollary 1.3.3.1: $DLO_0 = \mathrm{Th}(\mathbb{Q}, <)$, and so is complete.

Example: ACF_p is the theory generated by the axioms of an algebraically closed field of characteristic p.

The key question for any theory is, "is this theory complete?". We want to use our previous method and show that ACF_p is categorical for some cardinal, but it turns out that it is not α_0-categorical. To see this we note that $\hat{\mathbb{Q}}, \widehat{\mathbb{Q}[x]}, \widehat{\mathbb{Q}[x, y]}, \ldots$ are all non-isomorphic algebraically closed fields, where x, y are transcendental and $\widehat{}$ denotes algebraic closure.

Proposition 1.3.4: ACF_p is κ-categorical for every uncountable κ.

Proof: If $K, L \vDash ACF_p$ of size κ. The transcendental degree, the size of a field's transcendental basis, will also be equal to κ, then any bijection between transcendental bases will extend to an isomorphism between K and L. □

Corollary 1.3.4.1: ACF_p is complete.

We now want to discuss how to check that two models are elementarily equivalent.

Definition 1.3.5: Given a formula φ its *quantifier depth* qd is defined by induction,
- If φ is atomic $\mathrm{qd}(\varphi) = 0$.
- If φ is a logical connective like $\varphi_1 \vee \varphi_2$ then $\mathrm{qd}(\varphi) = \max(\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2))$
- If φ is a quantifier $\exists x\, \varphi'$ then $\mathrm{qd}(\varphi) = \mathrm{qd}(\varphi') + 1$, similarly for $\forall$.

We write $\mathcal{M} \underset{n}{\equiv} \mathcal{N}$ to mean "$\mathcal{M}$ is equivalent to $\mathcal{N}$ up to order n" if for every sentence σ of quantifier depth less than n we have $\mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{N} \vDash \varphi$.

We now define a tool for proving such partial equivalences.

Definition 1.3.6 (Ehreufeucht-Fraisse Games): Let L be finite relational, $\Gamma(\mathcal{M}, \mathcal{N})$ is a two player game where player I is called the spoiler and player II is called the prover. Together they will construct a function $f : A \to B$ where $A \subseteq |\mathcal{M}|$ and $B \subseteq |\mathcal{N}|$.

Player I plays first and either plays an element of $m \in |\mathcal{M}|$, challenging player II to put m in the domain of f, or they play an element $n \in |\mathcal{N}|$ challenging player II to put it in the range of f. Player II then plays the corresponding pairing for whatever player I played. Then player II starts again and they continue forever. Player II wins if the resulting f is an isomorphism of the induced structures on A and B.

Theorem 1.3.7: TFAE

- $\mathcal{M} \equiv \mathcal{N}$
- The prover has a winning strategy in $\Gamma(\mathcal{M}, \mathcal{N})$.