Question 1

Statement

Consider a physical system in which a rocket's total energy is constant, given by

$$E = \frac{1}{2}mv^2e^{kmv}$$

where m is the mass of the rocket, v is its velocity, and k is a constant equal to $4\mathrm{s\,kg^{-1}\,m^{-1}}$.

Assume that at a certain snapshot in time, $t = 0$, we measure the mass of the rocket to be 1kg, its velocity to be $1\mathrm{m\,s^{-1}}$ and its acceleration to be $\frac{\partial v}{\partial t}|_{t=0} = 5\mathrm{m\,s^{-2}}$. Determine the rate at which it is losing mass, in other words, what is $\frac{\partial m}{\partial t}|_{t=0}$?

(a) $-4\mathrm{kg\,s^{-1}}$ (b) $-5\mathrm{kg\,s^{-1}}$ (c) $-6\mathrm{kg\,s^{-1}}$ (d) $-7\mathrm{kg\,s^{-1}}$ (e) None of the above

Solution

This is an implicit differentiation question since we are solving for the change of one variable when some implicit equation of them holds. Since E is constant we have that

$$\begin{aligned}
0 = \frac{\partial E}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{2}mv^2e^{kmv} \\
&= \frac{1}{2}v^2e^{kmv}\frac{\partial m}{\partial t} + mve^{kmv}\frac{\partial v}{\partial t} + \frac{1}{2}mv^2e^{kmv}(kv)\frac{\partial m}{\partial t} + \frac{1}{2}mv^2e^{kmv}(km)\frac{\partial v}{\partial t} \\
&= (1 + mkv)\frac{1}{2}v^2e^{kmv}\frac{\partial m}{\partial t} + (2 + mkv)\frac{1}{2}mve^{kmv}\frac{\partial v}{\partial t}
\end{aligned}$$

We now have the values at $t = 0$ of all variables here except the one we want to know. We thus get

$$\begin{aligned}
0 &= (1 + 4)\frac{1}{2}e^4\frac{\partial m}{\partial t}\mathrm{m^2\,s^{-2}} + (2 + 4)\frac{1}{2}e^4(5)\mathrm{kg\,m^2\,s^{-3}} \\
&= 5\frac{\partial m}{\partial t} + 30\mathrm{kg\,s^{-1}}
\end{aligned}$$

giving us $\frac{\partial m}{\partial t} = -6\mathrm{kg\,s^{-1}}$

Question 2

Statement

Consider the region $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$. For which of the following functions is the point (0.8, 0.6) an absolute maximum over R?

(a) $f(x, y) = x^3 - x + y$ (b) $f(x, y) = (4x + 3y - 1)^2$ (c) $f(x, y) = e^{xy}$ (d) $f(x, y) = 4x + 3y$ (e) none of the above

Solution

This one was the hardest question for the most students, the key is that the the point is on the boundary of R and so we can immediately use Lagrange multipliers to eliminate some options.

The gradient of the constraint function is easily seen to be $(2x, 2y) = (1.6, 1.2)$ and so we need to compute the gradients of the options, these are

$$\begin{aligned}
\nabla(x^3 - x + y) &= \left(3(0.8)^2 - 1, 1\right) = (1.92, 1) \\
\nabla\left((4x + 3y - 1)^2\right) &= 2(4 \cdot (0.8) + 3 \cdot (0.6) - 1)(4, 3) = (32, 24) \\
\nabla(e^{xy}) &= e^{0.8 \cdot 0.6}(0.6, 0.8) \\
\nabla(4x + 3y) &= (4, 3)
\end{aligned}$$

We can quickly see that both options (a) and (c) cannot be correct since the gradient is not colinear with $(1.6, 1.2)$. Options (b) and (d) are both on the table. We now explicitly use Lagrange multipliers on those two options. Let us first try for (b), this gives us

$$\lambda(2x, 2y) = 2(4x + 3y - 1)(4, 3)$$

which then gives us

$$\lambda x = 16x + 12y - 4, \quad \lambda y = 12x + 9y - 3$$

we can multiply the left equation by 3 and the right equation by 4 and then subtract to get

$$3\lambda x - 4\lambda y = 0$$

and so (x, y) must lie on the line $y = \frac{3}{4}x$. Since it must also be on the circle $x^2 + y^2 = 1$ and so that leaves us with the points $(-0.8, -0.6)$ as well as $(0.8, 0.6)$. We notice that the value at $(-0.8, -0.6)$ is 36 which is higher than the value at $(0.8, 0.6)$ which is 16 so $(0.8, 0.6)$ this is not true for (b).

Next we have (d), we get

$$\lambda(2x, 2y) = (4, 3)$$

and so we again have that (x, y) lies on the line $y = \frac{3}{4}x$. We then get that the points are again $(-0.8, -0.6)$ and $(0.8, 0.6)$.

But now the value at $(-0.8, -0.6)$ is -5 and the value at $(0.8, 0.6)$ is 5 and so it could be a maximum. All that remains to check is that there are no other critical points in the interior of R. But this is immediate from the fact that the gradient $(4, 3)$ is never zero and so we get that (d) is the right answer.

Question 3

Statement

Consider the triangle T with corners $(0, 0), (0, 2), (4, 0)$ and constant density function 1. Which point is its center of mass?

(a) $\left(1, \frac{2}{3}\right)$ (b) $\left(\frac{4}{3}, \frac{2}{3}\right)$ (c) $\left(\frac{4}{3}, \frac{1}{2}\right)$ (d) $\left(1, \frac{1}{2}\right)$ (e) None of the above

Solution

First let us compute the mass of the triangle, we can easily find that the region enclosed by the triangle is given by $T = \left\{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 2 - \frac{x}{2}\right\}$

$$\int_0^4 \int_0^{2-\frac{x}{2}} 1 \,\mathrm{d}y \,\mathrm{d}x = \int_0^4 \left(2 - \frac{x}{2}\right) \mathrm{d}x = \left[2x - \frac{x^2}{4}\right]_0^4 = 8 - 4 = 4$$

Next we want to calculate all the moments of the triangle T

$$\int_0^4 \int_0^{2-\frac{x}{2}} x \,\mathrm{d}y \,\mathrm{d}x = \int_0^4 x\left(2 - \frac{x}{2}\right) \mathrm{d}x = \left[x^2 - \frac{x^3}{6}\right]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}$$

and also

$$\begin{aligned}\int_0^4 \int_0^{2-\frac{x}{2}} y \,\mathrm{d}y \,\mathrm{d}x &= \int_0^4 \frac{1}{2}\left(2 - \frac{x}{2}\right)^2 \mathrm{d}x = \int_0^4 2 - x + \frac{x^2}{8} \,\mathrm{d}x \\ &= \left[2x - \frac{x^2}{2} + \frac{x^3}{24}\right]_0^4 = 8 - 8 + \frac{8}{3} = \frac{8}{3}\end{aligned}$$

So now we divide to get that the center of mass is $\left(\frac{4}{3}, \frac{2}{3}\right)$.

Question 4

Statement

In class we found the following estimate $\pi \leq \iint_D e^{x^2+y^2} \leq e\pi$ where D is the region $D = \{(x,y) \mid x^2+y^2 \leq 1\}$. Compute the exact value of this integral. The answer is

(a) $\pi(e-1)$ (b) $2e$ (c) $\frac{\pi e}{2}$ (d) 2π (e) None of the above

Solution

We use polar coordinates to compute this integral

$$\int_0^{2\pi}\int_0^1 e^{r^2} r \,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}e^u \,\mathrm{d}u\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}(e-1)\,\mathrm{d}u\,\mathrm{d}\theta = 2\pi\frac{1}{2}(e-1) = \pi(e-1)$$

and so a is the right answer.

Question 5

Statement

Find the dimensions of a rectangular tile with maximal area subject to the constraint that the total perimeter of the tile is 64cm. Please show all your work. (Hint: set this question up as a Lagrange multiper problem).

Solution

Let us consider a rectangular tile with side lengths x cm and y cm. It is clear that the area is given by xy and the perimeter is given by $2x+2y$. We thus want to maximize xy subject to the constraint that $2x+2y = 64\text{cm}$. To that end we find the gradient of this constraint, it is clearly given by $(2x, 2y)$. The gradient of the function we are trying to optimize is (y, x) we then want

$$(y, x) = \lambda(2x, 2y)$$

for some λ. We also have the equation $2x+2y = 64\text{cm}$ and so we can deduce

$$32\text{cm} = y + x = \lambda(2x+2y) = \lambda 64\text{cm}$$

and so we must have $\lambda = \frac{1}{2}$ and so we must have $x = y$. Under the constraint $2x+2y = 64\text{cm}$ we get $x = y = 16\text{cm}$ which has maximimal area 256cm^2

Question 6

Statement

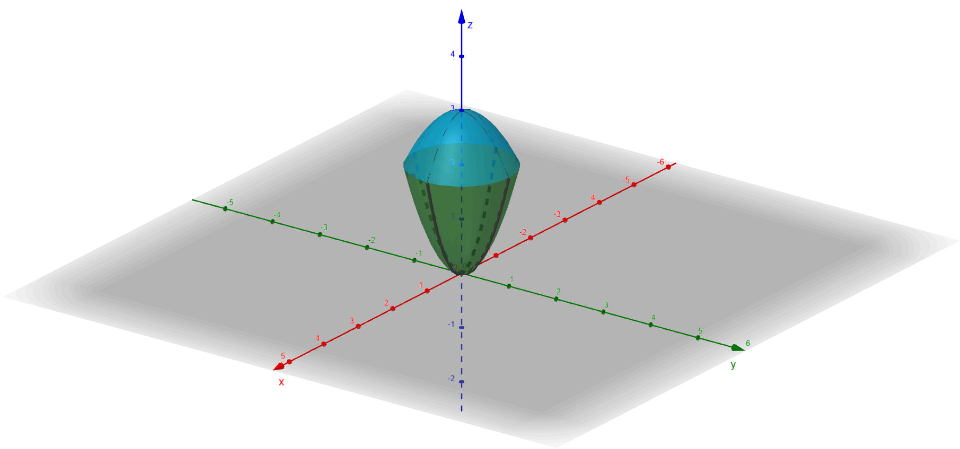
Let E be the region between the surfaces $S_1 = \{(x, y, z) \in \mathbb{R}^3, z = 3 - x^2 - y^2\}$ and $S_2 = \{(x, y, z) \in \mathbb{R}^3, z = 2x^2 + 2y^2\}$. Sketch the region E and compute its volume. Please show all your work.

Solution

First we need to sketch the surface, we notice immediately that S_1 is a downward pointing paraboloid and S_2 is an upward pointing paraboloid, and that both are centered on the z axis. Their intersection is given by the set of points where

$$3 - x^2 - y^2 = 2x^2 + 2y^2 \Longrightarrow 1 = x^2 + y^2$$

and so is a circle. We can thus try and sketch this region. We should get something like this



From this sketch we can easily see that to get the volume we want to integrate the difference between the heights of the two paraboloids over the circle $x^2 + y^2 \leq 1$. The top paraboloid in polar coordinates is given by $z = 3 - r^2$ and the bottom one is given by $2r^2$. We thus get that the difference is $3 - 3r^2$. Thus our volume is

$$V = \int_0^{2\pi} \int_0^1 (3 - 3r^2) r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \frac{3}{2} - \frac{3}{4} \, \mathrm{d}\theta = \frac{3}{2}\pi$$