

# CONFORMALLY INDUCED MEAN CURVATURE FLOW

Jacob Reznikov  
Master of Science

Mathematics and Statistics  
McGill University  
Montreal, Quebec, Canada

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# Abstract

This thesis aims to loosely cover the subject of geometric flows, and more specifically a variant of the mean curvature flow. The subject of geometric flows uses results in PDE theory theory, more specifically parabolic PDEs, to gain new insight about Riemannian geometry.

One of the biggest problems often tackled by mean curvature flows is the Isoperimetric problem. The Isoperimetric problem asks us to classify the spaces that minimize perimeter for a given volume (hence the name), the first use of mean curvature flows to attack this problem was due to Huisken in 1984 [1] who proved it in case of boundaries of convex domains in Euclidean, and since then there have been attempts to push his methods further. More recently, in 2013, Guan and Li [2] constructed a new normalized flow which allows them to prove the inequality in the case of boundaries of star-shaped domains in Space forms. Shortly after, in 2018, Guan, Li and Wang pushed this flow even further which allowed them to prove the result in a certain class of warped product spaces [3].

This thesis continues the effort to use such flows, together with my collaborator Joshua Flynn we pushed the flow even further, allowing us to prove the result for far larger class of domains in an even wider class of spaces.

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# Contribution

This thesis and each chapter within was written solely by myself, with occasional assistance from colleagues regarding phrasing. The body of the thesis is largely taken from the paper co-authored by myself and Joshua Flynn.

# Chapter 1

## Introduction

### 1.1 Background on the Isoperimetric Problem

The Isoperimetric Problem asks a seemingly simple question:

Among all regions occupying a given volume which has the least perimeter?

In the simple case of 2 and 3 dimensions we all know that the answer is a circle and a sphere respectively, and when we extend it to  $n$  dimensions in Euclidean space, the answer remains the  $n - 1$  dimensional hypersphere. However, if we try to generalize this question to a larger domain the answer very quickly becomes intractable.

One of the ways to answer such a question is with the method of geometric flows, we will start with a shape which will likely not be optimal, and then we will morph it over time to make it optimal.

### 1.2 Concepts and notation

This document assumes general knowledge of differential and Riemannian geometry, see [4] and [5] for great introductions, respectively.

For the rest of this document we will use the following notation,  $N$  is an  $n + 1$  dimensional Riemannian manifold with metric  $\bar{g}$  within which we have a compact domain  $\Omega$  with boundary  $\partial\Omega = M$  such that  $F : M \rightarrow N$  is an embedding making  $M$  a Riemannian hypersurface. We then set  $g := F^*\bar{g}$  to be the induced metric on  $M$ . We will in general identify  $M$  with its image  $F(M)$  and use the two interchangeably. We will call  $\mathfrak{X}(N)$  the set of surfaces that can be defined as above.

The Riemannian metric  $\bar{g}$  defines with it a Riemannian volume form which we will call  $dV$ , this form can be restricted to  $\Omega$  to give and allow us to define

$$\text{Volume}(\Omega) = \int_{\Omega} dV,$$

we will often write  $\text{Volume}(M)$  as our  $\Omega$  can be determined uniquely an orientation on  $M$  (more on that later). Similarly the metric  $g$  defines a volume form on  $M$  which we will call  $dS$ , using which we define

$$\text{Area}(M) = \int_M dS.$$

We can now define the Isoperimetric profile of  $N$  to be the function

$$I(v) := \inf\{\text{Area}(M) : M \in \mathfrak{X}(N) \text{ and Volume}(M) = v\},$$

The Isoperimetric Problem now asks us to

1. Show there exist elements  $M \in \mathfrak{X}(N)$  that attain the infimum above.
2. Characterize these elements.

We will now start to build up the concepts that allow us to solve this problem.

### **1.2.1 Extrinsic Riemannian geometry**

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