Comprehensive Explanation of the Equations and Their Interrelations

Below is a cohesive explanation that connects each of the listed equations and provides context for how they are used in optical systems, remote sensing, and image formation. The text is structured so that each equation builds on fundamental concepts—such as numerical aperture (NA), f-number, magnification, ground sampling distance (GSD), resolution metrics, and field of view—while highlighting the relationships among them.

1. Numerical Aperture (Eq045)

$$\mathrm{NA} = n_k\,|\sin U_k| \;\approx\; n_k\,|u_k|$$

This equation defines the **numerical aperture** (NA) of an optical system. In optical engineering, NA measures how much light an optical element (e.g., a lens) can gather from a given object space or how widely it can project the light into the image space. Here:

- n_k is the refractive index of the medium (often air, for which $n_k \approx 1$).
- U_k is the half-angle of the maximum cone of light that can enter or exit the system.
- u_k is a small-angle approximation for $\sin U_k$, valid when U_k is small (in radians).

When angles are small ($\sin U_k \approx U_k$), the expression $\mathrm{NA} \approx n_k\,u_k$ becomes a convenient simplification. The NA is critical in determining the resolution of an optical system—higher NA indicates a better ability to distinguish finer spatial details.

2. f-Number and Its Relation to NA (Eq046)

$$f/\# \;\equiv\; \frac{f_E}{D_{\mathrm{EP}}} \;\approx\; \frac{1}{2\,\mathrm{NA}}$$

This equation relates the **f-number** $\big(f/\#\big)$ of an optical system to its numerical aperture. The f-number is defined as the ratio of the effective focal length (f_E) to the entrance pupil diameter (D_{EP}). In simpler terms, it is a measure of how "fast" or "slow" a lens is—a lower f-number means a larger entrance pupil relative to the focal length, allowing more light into the system.

The approximate relationship $f/\# \approx 1/(2\,\mathrm{NA})$ highlights that a larger NA corresponds to a smaller (faster) f-number. In remote-sensing or photographic contexts, a large-aperture lens (small f-number) can collect more light and can often achieve better resolution (up to practical limits such as diffraction and sensor characteristics).

3. Working f-Number (Eq047)

$$f/\#_w \;\approx\; (1-m)\,f/\#$$

The **working f-number** ($f/\#_w$) is an adjusted version of the system's f-number when considering **magnification** (m). Magnification refers to the ratio of the image size to the object size. In many remote-sensing or imaging systems, magnification is small (e.g., a camera imaging the Earth's surface from orbit). The relationship here shows that as magnification (m) increases, the effective or working f-number increases, slightly changing how we describe the light-gathering ability of the system in practice.

4. Pupil Magnification (Eq048)

$$m_{\text{PUPIL}} \;=\; \frac{\bar{\omega}}{\bar{\omega}'}$$

Pupil magnification (m_{PUPIL}) is the ratio of the entrance pupil diameter to the exit pupil diameter or, equivalently, the ratio of angles subtended by the pupil in object space ($\bar{\omega}$) versus image space ($\bar{\omega}'$). This is relevant for understanding how the aperture stop (entrance or exit pupil) is imaged within the optical system. By examining pupil magnification, an optical designer can ensure that light is collected and relayed efficiently through the lens system.

5. Ground Sample Distance, Panchromatic (Eq049)

$$\text{GSD}_p \;=\; \frac{p_p \times h_{\text{alt}}}{f}$$

The **Ground Sample Distance (GSD)** indicates the size of one pixel on the ground. For the **panchromatic** channel, denoted here by GSD_p, we see that:

- p_p is the pixel pitch (i.e., the physical size of one pixel in the panchromatic detector),
- h_{alt} is the altitude (distance from the imaging system to the ground),
- f is the focal length of the optical system.

In remote-sensing, GSD is a fundamental parameter describing the spatial resolution on the ground. A smaller GSD means finer spatial details can be resolved in the final imagery.

6. Nyquist Frequency (Eq050)

$$\nu_N \;=\; \frac{1}{2\,p_p}$$

The **Nyquist frequency** ν_N is a critical concept in sampling theory. For a detector with pixel pitch p_p, ν_N sets the highest spatial frequency that can be correctly sampled without aliasing. In units of cycles per millimeter (or cycles per meter), this tells us that the sensor can reliably capture details up to $\frac{1}{2\,p_p}$. If the optical system or scene contains higher spatial frequencies, aliasing will occur in the sampled image.

7. Diffraction Cutoff Frequency (Eq051)

$$\nu_c \;=\; \frac{D}{\lambda\, f}$$

The **diffraction cutoff frequency** ν_c is the theoretical upper limit on the spatial frequencies that can pass through an aperture of diameter D. This limit arises from the physical wave nature of light. Here:

- λ is the wavelength of the light,
- f is the focal length,
- D is the aperture diameter.

If ν_c is lower than the sensor's Nyquist frequency, the system is said to be **diffraction-limited** in that diffraction sets the maximum spatial frequency that can be resolved, rather than the sampling of the sensor or other aberrations.

8. Ground Spot Size (Eq052)

$$\text{GSS} = \frac{\lambda\, h}{D}$$

The **Ground Spot Size (GSS)** is an estimate of how large the diffraction-limited spot is once projected onto the ground. For a system at altitude h with an aperture of diameter D:

- λ is the wavelength of light,
- h is the distance (altitude),
- D is the system aperture.

As D grows larger, the spot on the ground becomes smaller, improving resolution. Conversely, higher altitude h or longer wavelength λ will increase the ground spot size.

9. Image Quality Factor (Eq053)

$$Q_{\text{img}} = \frac{\lambda\, f_{\text{number}}}{p_{\text{ms}}}$$

This **image quality factor** (Q_{img}) serves as a dimensionless measure of how well the optical resolution (in terms of diffraction) matches the sampling resolution. Here:

- λ is the wavelength,
- f_{number} is the lens's f-number,
- p_{ms} is the pixel pitch for the **multispectral** sensor.

A larger Q_{img} can imply that the optical resolution is well-matched or oversampled by the sensor, whereas a smaller Q_{img} could indicate that the sensor is undersampling, or that the optical system is not fully leveraging the pixel resolution.

10. Swath Width (Eq054)

$$\text{Swath} = 2\, h_{\text{alt}}\, \tan(\text{HFOV})$$

In remote sensing, the **Swath Width** is the width of the ground area captured in a single pass or single image. By knowing the **Horizontal Field of View (HFOV)** and the altitude h_{alt}, one can calculate how wide the imaged footprint is on the ground. This is crucial for mission planning: a larger HFOV or altitude increases the swath, reducing the number of passes needed to cover a given area.

11. Ground Sample Distance, Multispectral (Eq055)

$$\text{GSD}_{ms} = \frac{p_{\text{ms}}\, h_{\text{alt}}}{f}$$

This equation mirrors Eq049 but uses p_{ms}, the pixel pitch of the **multispectral** sensor. Multi-spectral detectors often have different pixel sizes from panchromatic ones (sometimes larger, to improve signal-to-noise in each spectral band). Hence, the GSD in the multispectral channels can differ from the panchromatic GSD.

12. Resolved Ground Detail, Panchromatic (Eq056)

$$r_p \;=\; 2\,\mathrm{GSD}_p$$

The term r_p can be interpreted as a diameter or spacing on the ground that just spans two pixels (often taken as a measure of the smallest distinguishable feature in a simple approximation). By setting $r_p = 2\,\mathrm{GSD}_p$, we adhere to the Nyquist sampling criterion that at least two samples (pixels) are needed to resolve a feature unambiguously.

13. Resolved Ground Detail, Multispectral (Eq057)

$$r_{ms} \;=\; 2\,\mathrm{GSD}_{ms}$$

Analogous to the panchromatic channel, for the **multispectral** sensor, the resolved ground detail r_{ms} is about twice the multispectral GSD. If the multispectral detector has a coarser pixel pitch than the panchromatic detector, it will yield a larger ground sample distance and thus larger minimal resolvable detail on the ground.

Putting It All Together

These equations collectively describe key concepts in optical imaging and remote sensing:

1. **Optical Throughput & Aperture**
   NA and $f/\#$ define how much light the lens gathers and the theoretical resolution limit (Eqs045, 046). The working f-number (Eq047) refines these concepts when magnification is considered.

2. **Pupil Imaging & Magnification**
   Understanding pupil magnification (Eq048) helps in lens design, ensuring the system's aperture is efficiently used.

3. **Sensor Sampling & Spatial Resolution**
   GSD (Eqs049, 055) gives the ground footprint per pixel for different spectral bands (panchromatic vs. multispectral). Nyquist frequency (Eq050) clarifies the maximum spatial frequency that can be sampled properly, while the diffraction cutoff (Eq051) sets the physical limit based on the aperture.

   Together, ν_N and ν_c indicate whether a system is sensor-limited or diffraction-limited.

4. **On-Ground Spot Size & Image Quality**
   Ground spot size (Eq052) reveals how diffraction at altitude translates to a minimum spot. Q_{img} (Eq053) measures the match between optical resolution and pixel sampling in multispectral imaging.

5. **Coverage & Swath**
   Swath width (Eq054) is essential for mission planning in remote sensing—wider swath covers more ground per pass, often trading off with resolution or required altitude.

6. **Effective Resolution Metrics**
   r_p and r_{ms} (Eqs056, 057) reflect practical ground resolutions in panchromatic and multi-spectral images. They show how GSD translates to a resolvable feature size, adhering to sampling theory.

Overall, these equations underscore the interplay between **optical design parameters** (f-number, aperture diameter, focal length), **sampling parameters** (pixel pitch, Nyquist frequency), and **operational parameters** (altitude, field of view). By applying each formula appropriately, one can design or evaluate a remote-sensing system to achieve desired resolution and coverage. This holistic view helps engineers and scientists optimize systems for everything from high-altitude atmospheric imaging to satellite-based Earth observation, balancing image quality, coverage (swath), and the practicality of sensor design.