$$\underset{x,\ st\ \mathbb{1}'x=1}{\operatorname{argmin}} \ x'\Sigma x = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}'\Sigma^{-1}\mathbb{1}}$$ → minimum variance + without risk-free.

$$\underset{x'\mathbb{1}=1}{\operatorname{argmin}} \ x'\mu - x'\Sigma x = \frac{\Sigma^{-1}\mu}{\mathbb{1}\Sigma^{-1}\mu}$$

$$\underset{x,\ st\ \mathbb{1}'x=1,\ x'\mu=e}{\operatorname{argmax}} \ x'\mu - \frac{\gamma}{2}x'\Sigma x = \Sigma^{-1}\cdot\frac{(Ae-B)\mathbb{1}+(C-De)\mu}{AC-BD}$$

où, $A = \mathbb{1}'\Sigma^{-1}\mu$, $C = \mu'\Sigma^{-1}\mathbb{1}$

$B = \mu'\Sigma^{-1}\mu$, $D = \mathbb{1}'\Sigma^{-1}\mathbb{1}$

→ without risk free + return fixé

$$\underset{x}{\operatorname{argmin}} \ x'\Sigma x \ . \ (\text{st} \ \ x'\mu=e) = e\,\frac{\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\mu}$$

→ with risk-free + return fixé

$$\underset{x}{\operatorname{argmax}} \ x'\mu - \frac{\gamma}{2}x'\Sigma x = \frac{1}{\gamma}\Sigma^{-1}\mu$$

→ with risk-free

$$\frac{1}{\gamma} = \frac{e}{\mu'\Sigma^{-1}\mu}$$

portfolio weights return: $x'\mu + (1 - x'\mathbb{1})R_f$ → risk-free and return

$$= x'(\mu - R_f\mathbb{1}) + R_f$$

tangence portfolio: $\mathbb{1}'x_{tg} = 1 \Rightarrow \boxed{\gamma_{tg} = \mathbb{1}'\Sigma^{-1}\mu}$

d'où $x_{tg} = \dfrac{\Sigma^{-1}\mu}{\mathbb{1}'\,\Sigma^{-1}\mu} = \underset{x,\ \text{s.t } x'\mathbb{1}=1}{\operatorname{argmin}}\left[x'\mu - x'\Sigma x\right]$

Cas général : $\alpha\, \underbrace{x_{tg}}_{\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix}} + (1-\alpha)\, \underbrace{\mathbb{R}_f}_{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ R_f \end{pmatrix}} = x(\alpha)$

$$\mathbb{E}(x(\alpha)) = \alpha\,\mathbb{E}(x_{tg}) + (1-\alpha)\,\mathbb{R}_f$$

$$= \alpha\left[\mathbb{E}(t_g) - \mathbb{R}_f\right] + R_f$$

$$\sigma(x(\alpha)) = \alpha\,\sigma(x_{tg}) = \alpha\, x_{tg}'\,\Sigma\, x_{tg}$$

$$\Rightarrow \boxed{\mathbb{E}(x(\alpha)) = \frac{\mathbb{E}(t_g) - \mathbb{R}_f}{x_{tg}'\,\Sigma\, x_{tg}} \cdot \sigma(x(\alpha)) + R_f}$$

Relation entre α et γ ?

à α fixé on choisit le portefeuille risqué, $\alpha\, x_{tg}$

ce qui correspond au choix de $\frac{1}{\gamma}\,\Sigma^{-1}\mu$

$$\Rightarrow \alpha\, x_{tg} = \frac{1}{\gamma}\,\Sigma^{-1}\mu \Rightarrow \alpha\,\frac{\Sigma^{-1}\mu}{\mathbb{1}\,\Sigma^{-1}\mu} = \frac{1}{\gamma}\,\Sigma^{-1}\mu$$

$$\Rightarrow \boxed{\gamma = \frac{\mathbb{1}\,\Sigma^{-1}\mu}{\alpha}}$$