ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/3845146

Approximating NURBS curves by arc splines

Conference Paper · February 2000

DOI: 10.1109/GMAP.2000.838249 · Source: IEEE Xplore

CITATIONS
5

READS
481

1 author:

Xunnian Yang
Zhejiang University

54 PUBLICATIONS **524** CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Project Matrix weighted NURBS and matrix weighted rational subdivision View project

All content following this page was uploaded by Xunnian Yang on 23 June 2017.

The user has requested enhancement of the downloaded file.

Approximating NURBS Curves by Arc Splines

Xunnian Yang
Department of Applied Mathematics, Zhejiang University
Yuquan, Hangzhou 310027, People's Republic of China
Email: yxn@math.zju.edu.cn

Abstract

It is desirable to approximate a smooth curve by arc spline with fewest segments within prescribed tolerance. In this paper, we present an efficient algorithm for fitting planar smooth curve by arc spline. The main idea is that we construct the optimal arc spline by optimizing the interpolating biarc curve. The scheme consists of three steps: (1)sampling the curve based on consecutive tangents deviation; (2)construct the interpolating arc spline; (3)reduce the arc number to the minimum within prescribed tolerance. The algorithm can control the approximating error efficiently and results in the fewest number of arc segments.

1. Introduction

Arc spline is a kind of geometric curve made of circular arcs and straight line segments. Arc splines are frequently used as the tool path of CNC machinery[1,2]. In programming the tool path of CNC machinery, the number of arc segments should be decreased in order to reduce the number of instructions and tool motions. Approximating general smooth curves, especially NURBS[3,4] curves by arc splines play an important role for geometric processing. Research in this area has been active in recent years[5-10].

Marciniak and Putz[11] proposed the formula for fitting a spiral by arc spline with minimum segments and the optimal arc spline can be found by solving nonlinear equations. Qiu, et al[8] used minimax approximation and numerical techniques to construct optimal circular arc interpolation for NC tool path generation. Ong et al[7] construct optimal fitting arc curve by minimizing the bounding area of B-spline curve and the arc spline curve, and nonlinear optimization method has to be used to find the solution. Meek and Walton[5,6] studied the problem of fitting quadratic NURBS curve by arc spline by employing divide and conquer strategy. Ahn et al[9] construct the optimal approximation of quadratic Bezier curves based on accurate error measurement and solving cubic equations. Chuang, et al[10] construct one-sided approximate arc spline for B-spline curve based on convex hull property of B-spline curve. Yeung and Walton[2] fit smooth arc spline to the smooth curve by fitting the dense sampled data from the continuous curves. Wallner[16] gave multiresolution analysis of arc spline based on wavelets decomposition technique.

In this paper we present a practical approach to approximate smooth curve by optimal arc spline within the prescribed tolerance. We test our algorithm by approximating NURBS curves. The method consists of following main steps:

1. Sample and interpolate the NURBS curve by arc spline with tolerance τ_1. In fact, tolerance τ_1 are always smaller than 10^{-6} and can be ignored.
2. Merging consecutive arc segments into optimal arc spline with fewest segments within tolerance τ_2. The total deviation is $\tau_1 + \tau_2$.

The method owns several advantages over existing arc approximating method. Firstly, it can approximate any type of smooth parametric curves by arc spline. Secondly, constructing the interpolating arc spline by sampling the original curve are efficient and the error are controllable. At the end, the optimal approximating arc spline can be obtained in an efficient way and the segment number is much smaller compared with existing methods.

2. Preliminaries

To fit the parametric curve by arc spline, we first construct the biarc spline interpolating the sampled points and matching the tangents at the points. A biarc curve between two sampled points consists of two smoothly connected arc segments that interpolate the two end points and the end tangents. If we sample the parametric curve with enough points and density, the fitting error from biarc curve to smooth parametric curve can be estimated from the following theorems[12]:

Theorem 1. Let C_A and C_B be the arcs that interpolate points A and B and unit tangent vector at A or at B, respectively. Any biarc that joins point A to the distinct point B and matches given tangent vectors at the two points lies between the bounding circular arcs C_A and C_B.

Theorem 2. If a given spiral of positive increasing curvature is repeatedly divided into segments such that the lengths of the segments approach zero, each segment will eventually be enclosed by the bounding circular arcs derived from that segment.

Theorem 3. As the length of a spiral approaches zero, the distance from the spiral to the approximating biarc approaches $\frac{1}{13.5}$ of the distance between the bounding circular arcs.

Proof, see ref[12].

Suppose P_A and P_B are two consecutive points associated with tangent vectors T_A and T_B on the curve. Let α and β be the angles from T_A to P_AP_B and from P_AP_B to T_B, respectively, then the maximum distance between the two circular arcs is $\frac{\|P_B - P_A\|}{2}\left|\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}\right|$.

For biarc construction, we cite the formula from Su and Liu[13]. The biarc can be constructed in a local coordinate system with origin at P_A and x coordinate paralleling T_A (see Fig. 1). The conversion of points or vectors in one coordinate system to another can be achieved by direct transformation. Let O_1 and O_2 be the centers of two arc segments with radii r_1 and r_2, respectively. The contact point is denoted as P_C and the unit tangent vector at P_C is $U = \begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}$, where θ is an angle given ahead. Then the unit normal vector N_1 at point P_A is $N_1 = \begin{pmatrix}0 & 1\end{pmatrix}$ and the normal vector at point P_C is $V = \begin{pmatrix}-\sin\theta & \cos\theta\end{pmatrix}$. The rest free variables for the two arcs can be computed as follows:

$$
\begin{aligned}
l &= \|P_B - P_A\| \\
r_1 &= \frac{l}{2\sin((\alpha+\beta)/2)} * \frac{\sin((\beta-\alpha+\theta)/2)}{\sin(\theta/2)} \\
r_2 &= \frac{l}{2\sin((\alpha+\beta)/2)} * \frac{\sin((2\alpha-\theta)/2)}{\sin((\alpha+\beta-\theta)/2)} \\
P_c &= P_A + r_1(N_1 - V) \qquad (1) \\
O_1 &= P_A + r_1 N_1 \\
O_2 &= P_c + r_2 V
\end{aligned}
$$

If slope angles α and β have the same sign, we can then construct a C_shaped biarc panel, otherwise construct a S_shaped biarc panel. For free variable θ, we choose $\theta = \alpha$ for C_shaped biarc panel and $\theta = (3\alpha - \beta)/2$ for S_shaped biarc panel. This choice can keep the monotony of the curve curvature which play much important role for the global fairness of the curve.

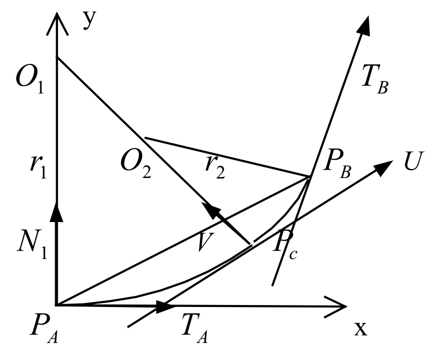


Figure 1. Biarc construction

Theorem 4. Let the curvature of a cubic spline that interpolates points P_1 and P_2 and matches the tangents at these two points be k_1 and k_2. If $k_1 < k_2$, then $k_1 < \frac{1}{r_1} < \frac{1}{r_2} < k_2$. If $k_1 > k_2$ then the inequality should be turned over.

Proof. See ref[13].

From theorem 4 we can conclude that the curvature of the interpolate biarc curve can still keep the monotone property of spirals if the segment tends to zero.

3. Fitting NURBS curves by arc spline

A NURBS curve with order k can always be represented as:

$$P(t) = \frac{\sum_{i=0}^{n} P_i w_i N_{i,k}(t)}{\sum_{i=0}^{n} w_i N_{i,k}(t)} \qquad (2)$$

where P_i, i=0, 1, ⋯, n are the control points and w_i, i=0, 1, 2, ⋯, n are the weights for the curve. The blending function $N_{i,k}(t)$ can be defined recursively,

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t)$$

and $N_{i,0}(t) = \begin{cases} 1 & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$ (3)

where t_i are the so-called knots forming a knot vector $\{t_0, t_1, \cdots, t_{n+k+1}\}$.

To approximate a NURBS curve, we can first divide the NURBS curve into a series of rational Bezier curve by knot insertion algorithm[14,15], and then subdivide the Bezier curve for arc spline interpolation within prescribed tolerance. A rational Bezier curve can be represented as $P(t) = \frac{\sum_{i=0}^{n} P_i w_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)}$, where $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are the Bernstein blending functions, and w_i are the weights. Let $P(t) = \frac{P_w(t)}{w(t)}$, then $P'(t) = \frac{P'_w(t) - P(t)w'(t)}{w(t)}$.

To approximate P(t) by arc spline, we sample the curve at t_i=i/N, i=0,1,⋯,N and compute the derivative at t_i simultaneously. Suppose the sampled point at t_i is P_i and the unit vector at t_i is T_i, then the distance between two bounding circular arcs interpolating P_{i-1} and P_i can be estimated based on theorem 3. If the fitting error is less than the prescribed tolerance τ_1, accept the subdivision, else sample another new point at $(t_{i-1}+t_i)/2$ and test the two subintervals respectively. This procedure can guarantee that the fitting arc spline is smooth and within prescribed tolerance. Generally, this error can be reduced to 10^{-6} to 10^{-8} and can always be ignored. Even more, since we subdivide the curve so many times that the curvature of the arc spline is monotone when the curvature of the parametric curve does.

4. Data optimization for arc spline

In this section, we'll derive explicit formula for optimizing the arc spiral consisting of arc segments with monotone curvature within the tolerance. The arc spiral is part of a chain of enclosed or enclosing circles and each arc segment except the first and last one is the part between two osculating points on the circle. We reduce the arc number by expanding the arc with biggest curvature to its limit within given tolerance for spiral with decreasing curvature, and shrinking the arc with smallest curvature for spiral with increasing curvature. When an arc is expanded or shrank, tangent continuity condition should be preserved. We implement this strategy by optimizing every three consecutive arc segments into two smoothly connected approximate arc segments within the permitted tolerance step by step. For each step, the tolerance should be recomputed for global constraints. This procedure continues until there are no two arcs can replace the current three arcs within the permitted tolerance. Then the first arc is the optimal approximate arc and we continue the optimization procedure from the second one. We obtain the optimal approximate arc spline to the arc spiral when the last three arcs are optimized into a biarc panel. The detail for optimization procedure is given below.

4.1. Arc optimization for arc spiral with decreasing curvature

Suppose O_0, O_1 and O_2 be three smooth connected arc segments with radii r_0, r_1, and r_2, respectively, and the radii satisfy $r_0 < r_1 < r_2$. The arc segments start from P_0 and end at P_3, two contact points for the curve are P_1 and P_2. To determine two new smoothly connected arc segments from the three arcs with controllable deviation, we construct the new arcs by expanding arc O_0 and shrinking arc O_2 with P_0 and P_3

fixed (see Fig2). Let $V_0 = \frac{O_0 - P_0}{\left\| O_0 - P_0 \right\|}$ and $V_2 = \frac{O_2 - P_3}{\left\| O_2 - P_3 \right\|}$, then the centers for two new arc segments can be obtained as: $O_a = O_0 + tV_0$ and $O_b = O_2 - uV_2$, where t and u are two variables that can be decided by continuity condition and tolerance constraint. The radii for arc O_a and O_b are $r_a = r_0+t$ and $r_b = r_2-u$, respectively.

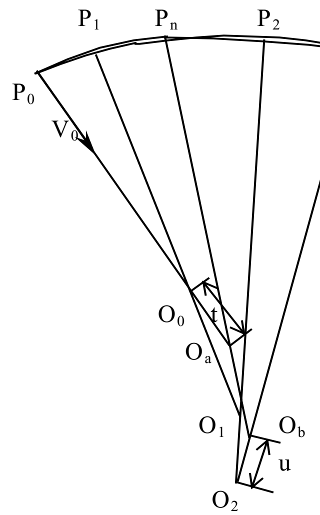


Figure 2. Optimize triarc segment by biarc

To guarantee arc segments O_a and O_b are tangent continuous at contact point, the radii for the two arcs should satisfy the following condition:

$$r_a + \left\| O_a - O_b \right\| = r_b \tag{4}$$

by expressing O_a, O_b, r_a and r_b as the function of t and u, we obtain the following formula:

$$\left\| O_2 - O_0 - uV_2 - tV_0 \right\| = r_2 - r_0 - u - t \tag{5}$$

Let $D = O_2 - O_0$ and $t_0 = r_2 - r_0$, then we can derive the following formula

$$u = \frac{X_0 + X_1 t}{Y_0 + Y_1 t} \tag{6}$$

where $X_0 = t_0^2 - D^2$, $X_1 = 2(DV_0 - t_0)$, $Y_0 = 2(t_0 - DV_2)$ and $Y_1 = 2(V_0 V_2 - 1)$. When t or u is chosen the two new arcs can then be obtained. Let $V_n = \frac{O_a - O_b}{\left\| O_a - O_b \right\|}$, the contacting point for arc O_a and O_b is $P_n = O_a + r_a V_n$.

To guarantee that arc O_a is within permitted tolerance ε of the initial curve, we expect that the distance from point P_1 to arc O_a is no more than ε (see Fig. 3). Let $V_b = \frac{P_1 - O_0}{\left\| P_1 - O_0 \right\|}$, the nearest point on arc O_a to P_1 is P, then $P = O_0 + (r_0 + d)V_b$. Because $O_0 = O_a - tV_0$, then $P - O_a = -tV_0 + (r_0 + d)V_b$.

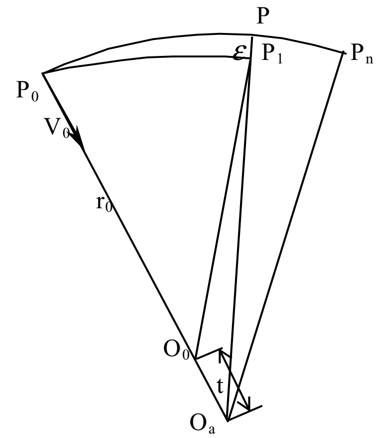


Figure 3. Tolerance constraint for arc expansion

Since the length of vector $P - O_a$ is r_a, then

$$r_a^2 = \left[-tV_0 + (r_0 + d)V_b \right]^2 \le \left[-tV_0 + (r_0 + \varepsilon)V_b \right]^2 \tag{7}$$

and because $r_a = r_0 + t$, we can conclude that

$$t \le t_b = \frac{2r_0\varepsilon + \varepsilon^2}{2\left[r_0 + (r_0 + \varepsilon)V_0 V_b \right]} \tag{8}$$

To guarantee the validity of the new arc segments, when arc O_0 is expanded, another constraint is that the circle can not exceed arc O_2(see Fig. 4):

$$\left\| O_a - O_2 \right\| + r_a \le r_2 \tag{9}$$

from this inequality we can conclude that

$$t \le t_c = \frac{(r_2 - r_0)^2 - (O_0 - O_2)^2}{2(r_2 - r_0) + (O_0 - O_2)V_0} \tag{10}$$

The maximum value for t can now be chosen as t=min(t_b, t_c). This also indicates that the new arc curve is still arc spiral.

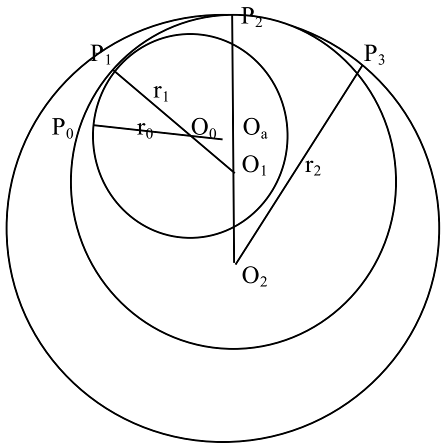


Figure 4. Expansion for arc O_0 is within arc O_2

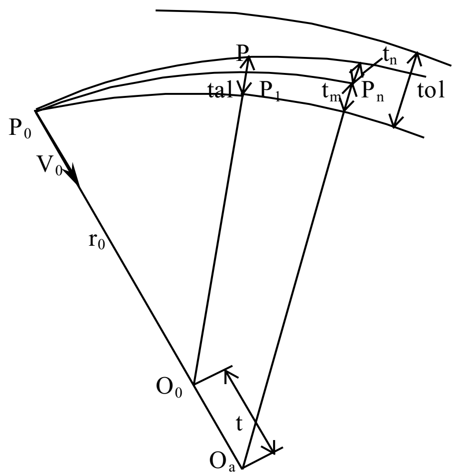


Figure 5. Compute the active tolerance

Let d_a be the distance from point P_n to arc O_1, $d_a = \|P_n - O_1\| - r_1$, and the distance from point P_2 to arc O_b is $d_b = \|P_2 - O_b\| - r_b$. Since arc O_a may be an arc by expanding a former one, then d_a should be no more than the active bounding tolerance tal, i.e., $d_a \le tal$. Because we construct arc O_b by shrinking arc O_2, then d_b should satisfy the inequality $d_b \le tbl$ to achieve the tolerance constraint. If both of these inequalities hold simultaneously, we can replace arcs O_0, O_1 and O_2 by arcs O_a and O_b, else set arc O_0 aside and choose arcs O_1, O_2 and another arc segment from the spline for further optimization.

When the optimization procedure starts, we choose the active tolerance tal and tbl both as tol, else the active tolerance should be recomputed according to the geometry of the fitting biarcs.

The distance from point P_1 to the expanded arc O_a can be computed as $t_l = r_a - \|P_1 - O_a\|$, then the permitted expansion magnitude at point P_1 is tal-t_l (see Fig. 5). Because lines O_0P_1 and O_aP_n are approximately parallel, then arc O_a should expand no more than $t_e = scl * (tal - t_l)$ at point P_n, where scl is a scaling coefficient and can be estimated as $scl = \frac{\|P_0 - P_n\|}{\|P_0 - P_1\|}$. Since point P_n lies outside of arc O_1, the distance from point P_n to arc O_1 is $t_m = \|P_n - O_1\| - r_1$, and the expansion magnitude for arc O_a at point P_n should not exceed $t_f = tol - t_m$ too. The permitted tolerance at point P_n is tal=min(t_e, t_f). It is difficult to estimate tbl accurately, we provide a well working empirical formula here. Let the arc next to arc O_2 be arc O_3 and the last end point of arc O_3 is P_4. To guarantee that point P_2 is within the current permitted tolerance tbl for latter optimization steps when $d_b > 0$, tbl for point P_3 can be estimated as $(tbl - d_b)/scl + d_a$, where $scl = \frac{\|P_2 - P_4\|}{\|P_2 - P_3\|} * 0.65$. When d_b is equal to zero, i.e., P_2 is on arc O_b, tbl can be kept unchanged.

4.2. Arc optimization for arc spiral with increasing curvature

Suppose O_0, O_1 and O_2 are three smoothly connected arc segments with increasing curvature, i.e., $r_0 > r_1 > r_2$. The derivation for approximate arcs is similar as arc spiral with decreasing curvature. We can assume that the arc spiral starts at P_0 and ends at P_3, and the two contact points are P_1 and P_2. The two approximate arcs O_a and O_b are shrank and expanded from arc O_0 and O_2, respectively. As in case 4.1, let V_0 be the unit vector pointing from P_0 to O_0 and V_2 be the unit vector pointing from P_3 to O_2. The centers for two new arcs can be expressed as $O_a = O_0 - uV_0$ and $O_b = O_2 + tV_2$. The radii for the two arcs are $r_a = r_0 - u$ and $r_b = r_2 + t$. From the smooth contacting condition we can deduce

$$r_b + \left\| O_b - O_a \right\| = r_a \tag{11}$$

Let $D = O_2 - O_0$ and $t_0 = r_0 - r_2$, from this equality we can deduce the function of t as:

$$t = \frac{X_0 + X_1 u}{Y_0 + Y_1 u} \tag{12}$$

where $X_0 = D^2 - t_0^2$, $X_1 = 2\left(DV_0 + t_0\right)$, $Y_0 = -2\left(t_0 + DV_2\right)$ and $Y_1 = 2\left(1 - V_0 V_2\right)$.

The value of u can be chosen according to tolerance constraint. Let $V_n = \frac{O_b - O_a}{\left\| O_b - O_a \right\|}$, the contacting point for two new approximate arcs is $P_n = O_a + r_a V_n$. Similar as case 3.1, the shrink magnitude for arc O_0 at point P_1 should no more than permitted tolerance ε. Let $V_b = \frac{P_1 - O_0}{\left\| P_1 - O_0 \right\|}$, the tolerance constraint for variable u is

$$u \le u_b = \frac{2r_0\varepsilon + \varepsilon^2}{2\left[r_0\left(1 + V_0 V_b\right) + \varepsilon\right]} \tag{13}$$

When arc O_0 is shrank, to preserve the continuity property of new arc segments, arc O_2 should still lies inside the shrank arc, i.e. $\left\| O_2 - O_a \right\| \le r_a - r_2$. From this inequality we can deduce the following bound for u:

$$u \le u_c = \frac{t_0^2 - D^2}{2\left(t_0 + DV_0\right)} \tag{14}$$

the value of u can be chosen $u = \min(u_b, u_c)$.

Let d_a be the distance from point P_n to arc O_1, $d_a = \left\| P_n - O_1 \right\| - r_1$ and the distance from point P_2 to arc O_b is $d_b = \left| \left\| P_2 - O_b \right\| - r_b \right|$. Since arc O_1 maybe the expanded arc, then d_a should satisfy the inequality $d_a \le 0$ as well as the bounding tolerance $d_a \ge (-tal)$. Similarly as case 4.1, d_b should satisfy the inequality $d_b < tbl$ for keeping point P_2 within tolerance. If all these inequalities hold simultaneously, we can replace arcs O_0, O_1 and O_2 by arcs O_a and O_b, else set arc O_0 aside and choose three arcs from the spline for further optimization.

The distance from point P_1 to the shrank arc O_a can be computed as $t_l = \left\| P_1 - O_a \right\| - r_a$, then the permitted shrink magnitude at point P_1 is tal-t_l. Let $scl = \frac{\left\| P_0 - P_n \right\|}{\left\| P_0 - P_1 \right\|}$, then arc O_a should shrink no more than $t_e = scl * \left(tal - t_l\right)$ at point P_n. The distance from point P_n to arc O_1 is $t_m = \left\| P_n - O_1 \right\| - r_1$ the shrink magnitude for arc O_a at point P_n should not exceed $t_f = tol - t_m$ too. The permitted maximum tolerance at point P_n is tal=min(t_e, t_f). Active tolerance tbl for point P_2 can be similarly as case 4.1.

4.3. The algorithm

For every smooth parametric curve, the approximate arc spline are always composed of a series of arc spirals with monotone curvature. Arc spirals with decreasing or increasing curvature can then be optimized by the former formulas. When three chosen arcs are replaced by two optimized arc segments within the tolerance, the radii for the spiral are still monotone. In fact, we can optimize a smooth arc curve in a uniform way. The main procedure for arc spline optimization is given below:

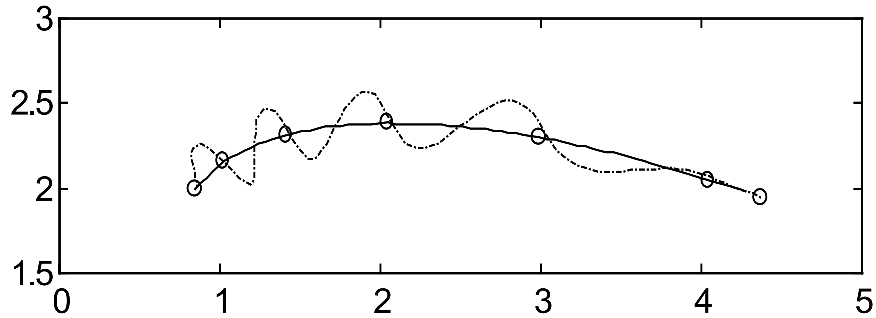
Algorithm Optimization-Procedure-for-Arc-Spline

input: a smooth circular arc spline and the tolerance *tol*

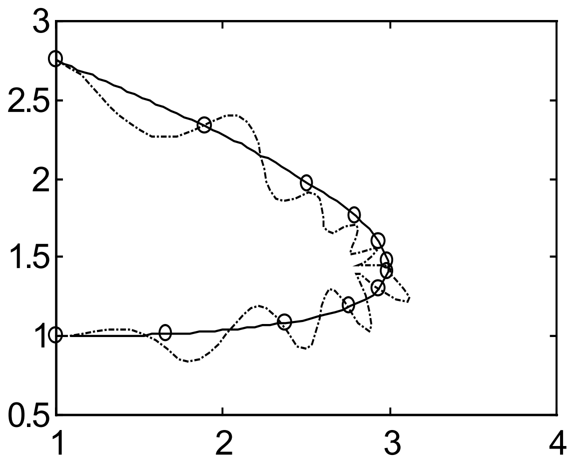
output: an optimal new smooth arc spline within *tol*

```
choose first two arcs from the spline as arc1 and arc2;
set tal = tol;
while (next arc != NULL)
{
   arc0 = arc1;
   arc1 = arc2;
   arc2 = next arc;
   if (arc radii are monotone)
   {
         Compute two optimized arcs arc_a and arc_b;
         if (both arc_a and arc_b are within the tolerance)
         {
               delete arc0;
               arc1 = arc_a;
```

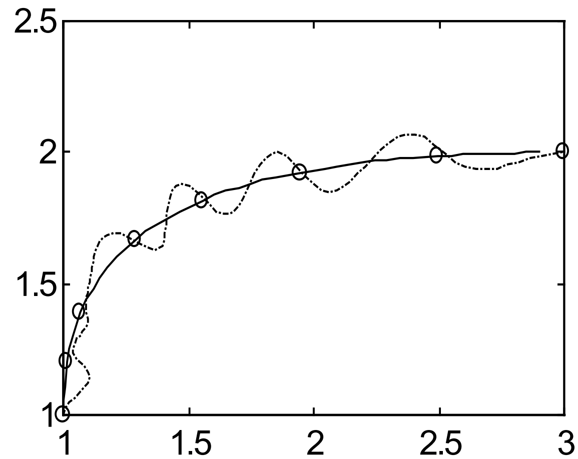
```
        arc2 = arc_b;
        compute the tolerance tal;
}       else tal = tbl = tol;
                }       else  tal = tbl = tol;
        }       /* end of while */
        output the final arc spline.
```



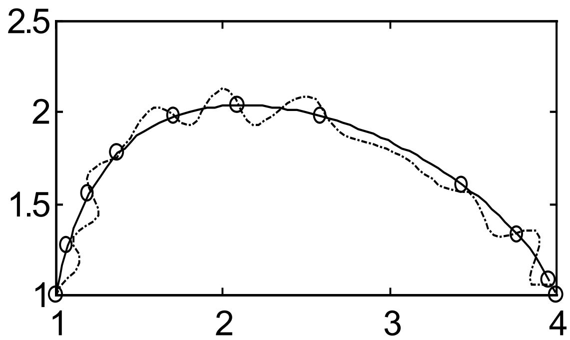
Example 1 (tol=0.001)



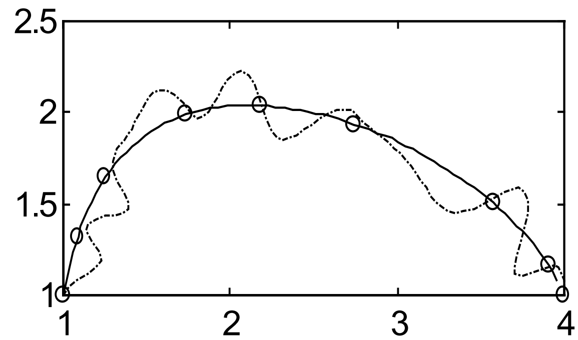
Example 2 (tol=0.001)



Example 3 (tol=0.0005)



Example 4a (tol=0.0005)



Example 4b (tol=0.001)

5. Examples

We have applied our algorithm on a number of parametric curves and obtain satisfying approximating results. We cite a few examples here to show the efficiency of our algorithm.

In Example 1 we approximate a rational quadratic Bezier curves with control points (0.85,1.99), (1.5, 2.9), (4.37,1.95) and weights 1, 0.8, 1, respectively. By sampling 51 points on the curve and interpolating the points with biarc spline consists of 100 arcs. An optimized G^1 arc spline consists of only 6 arcs is obtained with tolerance 0.001. The fitting error of this and following examples magnified 200 times is plotted in dashed curve. For Example 2, the data is the same as example 3 of Ahn et al[9], we optimize the arc spline interpolating 101 points on the curve. There are only 11 arcs are left in the final arc spline while Ahn's example consists of 15 segments. For example 3, we use the control points as (1,1), (1,2), (3,2) and weights 1, 1.2, 1. We obtain the optimal approximating arc spline consisting of 7 arc segments by first sampling the Bezier curve with 50 points and optimizing the interpolating arc spline within tolerance 0.0005. In example 4, we construct the optimal arc spline approximating a cubic rational Bezier curve with control points (1,1), (1.3,2.5), (3.5,2.2), (4,1) and weights 1, 1.2, 0.8, 1. In example 4a, we obtain optimal approximate arc spline with 10 segments within tolerance 0.0005 and in example 4b the approximate arc spline consists of just 8 arcs within tolerance 0.001.

6. Conclusions and discussion

In this paper we present a very practical algorithm for approximating smooth parametric curve by optimal G^1 arc spline within the given tolerance. The final arc spline interpolates the curve ends and matches the end tangents. We illustrate the algorithm by approximating NURBS curves with arc splines.

The crux for approximating parametric curve by arc spline is how to estimate the fitting error accurately and robustly. We employ here the theory of Meek and Walton[12] and an interpolating biarc spline within arbitrary prescribed accuracy can be obtained by sampling the parametric curve with enough density. The optimization procedure can be applied on the fitting arc spline, explicit formula have been derived to construct the optimal approximate arc spline.

If there are some arc segments with large radii or there are some straight lines in the fitting arc spline, the optimization process may be influenced by the numerical errors or more care should be taken. Approximating parametric curve by optimal one-sided arc spline can be deduced with a few modification of the current procedure.

Acknowledgements

The author owns thanks to Prof. Wang Guozhao for his support and help. This work is supported by the NNSF(19971079), Foundation of State Key Basic Research 973 Development Programming Item(G1998030600) and the Natural Science Foundation of Zhejiang Province. The author also acknowledges K. C. Wong Foundation for financial support of this project.

References

[1] K. M. Bolton, "Biarc curves", *Computer Aided Design*, 1975, 7, pp. 89-92.

[2] M. Yeung, and D. J. Walton, "Curve fitting with arc splines for NC tool path generation", *Computer Aided Design*, 1994, 26(11), pp. 845-849.

[3] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design, a practical guide.2*. New York: Academic Press, 1990.

[4] L. Piegl, "On NURBS: A survey", *IEEE CG&A*, 1991, 11(1), pp. 55-71.

[5] D. S. Meek, and D. J. Walton, "Approximating quadratic NURBS curves by arc splines", *Computer Aided Design*, 1993, 25(6), pp. 371-376.

[6] D. S. Meek, and D. J. Walton, "Approximating of quadratic Bezier curves by arc splines", *Journal of Computational and Applied Mathematics*, 1994, 54, pp. 107-120.

[7] C.L. Ong, S. Wong, H.T. Loh, and X.G. Hong, "An optimization approach for biarc curve fitting of B-spline curves", *Computer Aided Design*, 1996,

28(12) pp. 951-959.

[8] H. Qiu, K. Cheng, and Y. Li, “Optimal circular arc interpolation for NC tool path generation in curve contour manufacturing”, *Computer Aided Design*, 1997, 29(11), pp. 751-760.

[9] Y. J. Ahn, H. O. Kim, and K.-Y. Lee, “G^1 arc spline approximation of quadratic Bezier curves”, *Computer Aided Design*, 1998, 30(8), pp. 615-620.

[10] S.-H. Chuang, and C. Z. Kao, “One-sided arc approximation of B-spline curves for interference-free offsetting”, *Computer Aided Design*, 1999, 31, pp. 111-118.

[11] K. Marciniak, and B. Putz, “Approximation of spirals by piecewise curves of fewest circular arc segments”, *Computer Aided Design*, 1984, 16(2), pp. 87-90.

[12] D. S. Meek, and D. J. Walton, “Approximating smooth planar curves by arc splines”, *Journal of Computational and Applied Mathematics*, 1995, 59, pp. 221-231.

[13] B. Q. Su, and D. Y. Liu, *Computational Geometry*, Academic Press, USA(1989).

[14] W. Boehm, “Inserting new knots into B-spline curves”, *Computer Aided Design*, 1980, 12(4), pp. 199-201.

[15] W. Boehm, “Generation of the Bezier points of B-spline curves and surfaces”, *Computer Aided Design*, 1981, 13, pp. 365-366.

[16] J. Wallner, “Generalized multiresolution analysis for arc splines”, in: Dahlen, M., Lyche, T. And Schumaker, L. L., eds., *Mathematical Methods for Curves and Surfaces II*, Vanderbilt University Press, pp. 537-544.

View publication stats