Numerical linear algebra course

Midterm, Fall 2023

Variant 1

Theoretical tasks

1. **(1 pts)**

   (a) What is the name of the transformation that is represented as the following matrix in the 2D case?

   $$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

   ☐ Giver rotation
   ☐ Given rotation
   ☐ Givens rotation
   ☐ Gineve rotation
   ☐ No specific name for such transformation

   (b) Construct such unitary transformation that zeros out the second coordinate of vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

2. **(2 pts)** What is the complexity of matrix by vector product in the general case, where a matrix is of size $m \times n$? Can it be improved? Why?

3. **(4 pts)** What is the invertible matrix? Assume matrix $I_n + AB$ is invertible, where $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$ and I_n denotes the identity matrix of size n. Is matrix $I_m + BA$ invertible or not? Provide proof of your claim.

4. **(3 pts)** Proof that $(XY)^\dagger = Y^\dagger X^\dagger$.

5. **(2 pts)** Show that for any k there exists matrix A: $\text{rank}(A) = k$ such that $\|A\|_F = \sqrt{\text{rank}(A)}\|A\|_2$

Practical tasks

1. **(3 pts)** Assume matrix A has eigendecomposition $A = U\Lambda U^*$. Derive the eigendecomposition of a block matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and explain why does it exist?

2. **(4 pts)** Assume you are given a matrix $A = \begin{bmatrix} -4 & 6 \\ 9 & -1 \end{bmatrix}$ and you run the power method. Will the power method converge? If it will converge, comment on what is a convergence speed and what is the stationary point. If it will not converge, please explain why.

3. **(2 pts)** Does LU decomposition exist for matrix $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$? If it exists, compute it. If it does not exist, compute PLU decomposition.

4. **(2 pts)** Calculate SVD of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{pmatrix}$.

5. **(4 pts)** Find the matrix inverse to A:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$