

Scientific Computing

Lecture 4

Integral Transforms and Integral Equations

Nikolay Koshev & Maxim Fedorov

October 18, 2023

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Structure of the lecture

Part 1. Convolution

Part 2. Fourier Transform

Part 3. Radon Transform

Part 4. Wavelet Transform

Part 5. Integral equations classification

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Part 1: Convolution

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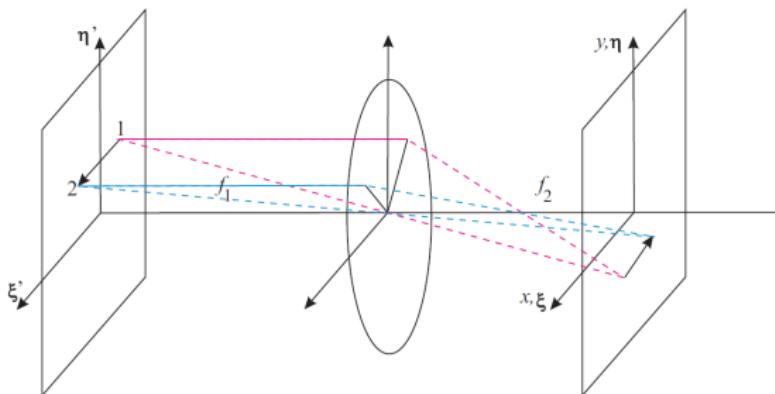
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The distorted images reconstruction

- ▶ Smeared images
- ▶ Defocused images
- ▶ Electronic microscopy
- ▶ The convolution
- ▶ The equation

Smeared images



Each point of the top image becomes an interval at the bottom image.

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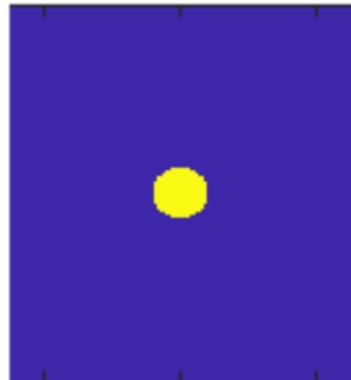
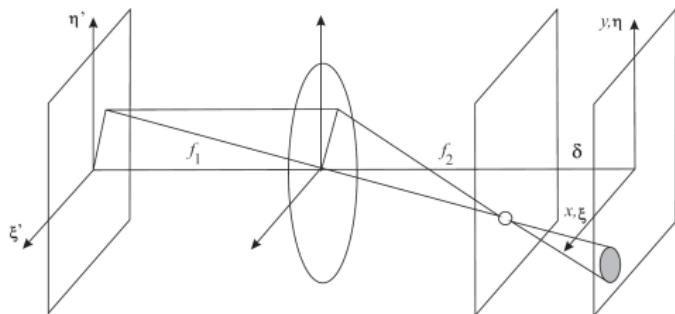
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Defocused images



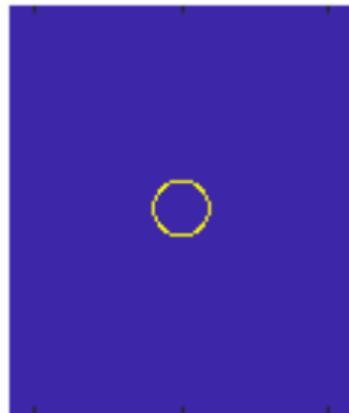
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Defocused images



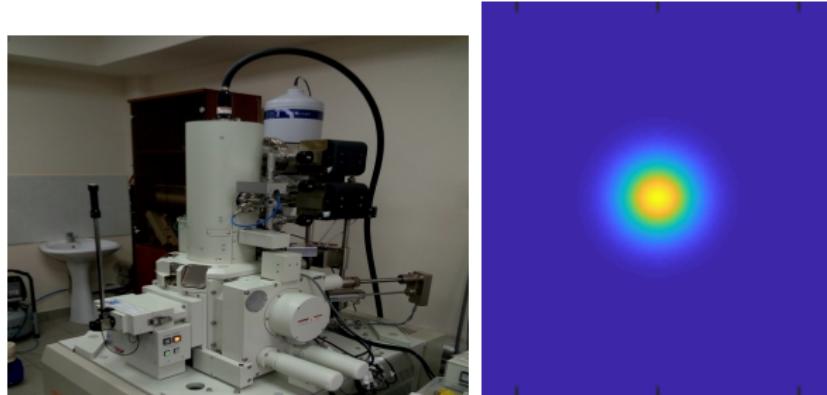
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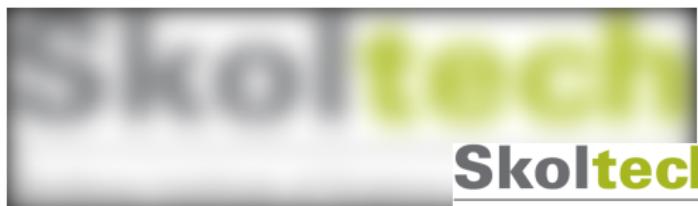
Defocused images



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The convolution

- ▶ Let the functions $k(\mathbf{x})$ and $f(\mathbf{x})$ are both defined for $\mathbf{x} \in \mathbb{R}^n$.
- ▶ The function $g(\mathbf{x})$ is called convolution of k and f if:

$$g(\mathbf{x}) = k * f \equiv \int_{\mathbb{R}^n} k(\mathbf{x} - \xi) f(\xi) d\xi.$$

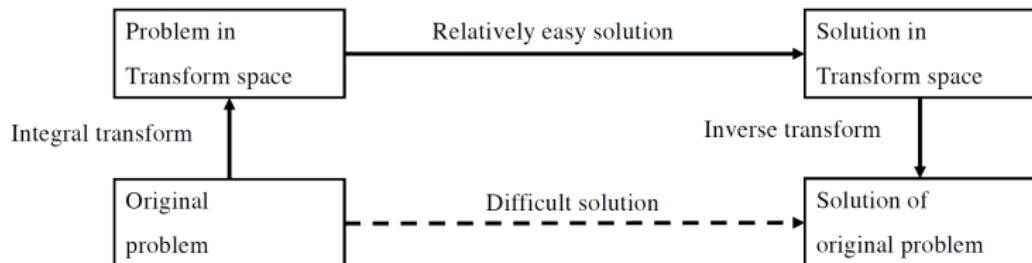
- ▶ The function $k(\mathbf{x})$ is the convolution kernel.
- ▶ The convolution means than each value $f(\mathbf{x})$ becomes some spot defined with $k(\mathbf{x})$. Thus, the function $k(\mathbf{x})$ is also sometimes called the **Point spread function**
- ▶ In the problems of image processing, the kernel k is often being defined by hardware function.
- ▶ If $\hat{\cdot}$ denotes the Fourier transform of \cdot , then:

$$\hat{k * f} = \hat{k} \hat{f}.$$

Integral transforms

$$g(\mathbf{x}) = \int_{\Omega} K(\mathbf{x}, \xi) f(\xi) d\xi, \quad \mathbf{x}, \xi \in \Omega \subset \mathbb{R}^n$$

- ▶ $K(\mathbf{x}, \xi)$ - the Kernel;
- ▶ Mapping a function $f(\mathbf{x})$ in \mathbf{x} -space into another function $g(\xi)$ in ξ -space
- ▶ Fourier, Wavelet, Z-transform, Laplace, Hilbert, Radon, etc



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Part 2: Fourier Transform

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Fourier Transform



1768-1830

Fourier Transform

- ▶ Each function defined in spatial or time domain (**real** domain) by some function $h(t)$, can be presented also in **frequency** or **reciprocal** domain with a function $H(f)$ of frequency f .
- ▶ One goes back and forth between these two representations by means of the Fourier transform,

$$H(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt; \quad h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df;$$

Or, using angular frequency $\omega = 2\pi f$:

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt; \quad h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$$

Fourier Transform: examples

- ▶ FT of Dirac δ -function: $\int_{-\infty}^{\infty} \delta(t) e^{-2\pi i f t} dt = 1$:

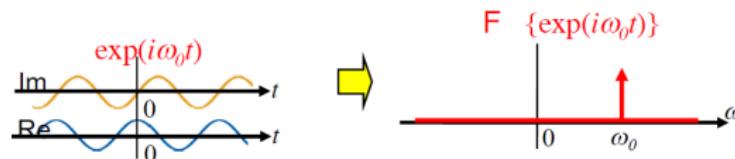


- ▶ FT of $\cos(\omega_0 t)$:

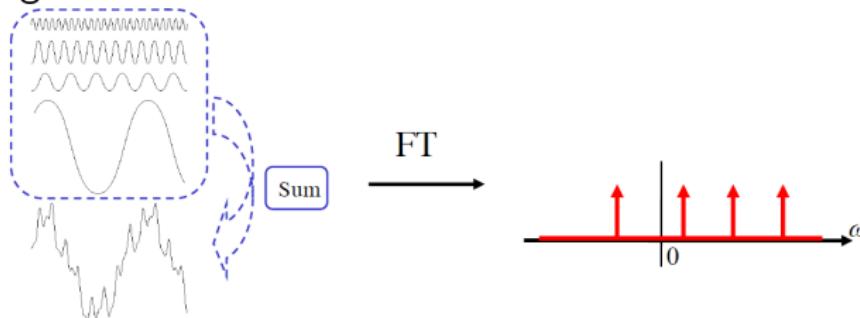


Fourier Transform: examples

- ▶ FT of $e^{\pi i \omega_0 t} = \sqrt{2\pi} \delta(\omega - \omega_0)$:



- ▶ The idea of FT: periodic functions could be represented as a weighted sum of sines and cosines:



Multidimensional Fourier Transform

Let $f(\mathbf{x}) \in L_2(\mathbb{R}^n)$, $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$. Then

$$F(\omega) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\omega \cdot \mathbf{x}} d\mathbf{x}, \quad \omega \in \mathbb{R}^n$$

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} F(\omega) e^{i\omega \cdot \mathbf{x}} d\omega.$$

- ▶ Correspondence between symmetries in the two domains:

$h(t)$ is real $\Rightarrow H(-f) = H^*(f);$

$h(t)$ is imaginary $\Rightarrow H(-f) = -H^*(f);$

$h(t)$ is even $\Rightarrow H(-f) = H(f);$

$h(t)$ is odd $\Rightarrow H(-f) = -H(f);$

$h(t)$ is real and even $\Rightarrow H(-f)$ is real and even;

$h(t)$ is real and odd $\Rightarrow H(-f)$ is imaginary and odd;

$h(t)$ is imaginary and even $\Rightarrow H(-f)$ is imaginary and even;

$h(t)$ is imaginary and odd $\Rightarrow H(-f)$ is real and odd;

Fourier Transform: properties

► Scaling and shifting:

$h(t) \Leftrightarrow H(f)$: Fourier Transform

$h(at) \Leftrightarrow \frac{1}{|a|} H\left(\frac{f}{a}\right)$: time scaling

$\frac{1}{|b|} h\left(\frac{t}{b}\right) \Leftrightarrow H(bf)$: frequency scaling

$h(t - t_0) \Leftrightarrow H(f)e^{2\pi ift_0}$: time shifting

$h(t)e^{-2\pi if_0 t} \Leftrightarrow H(f - f_0)$: frequency shifting

► Convolution theorem:

$$g * h \equiv \int_{-\infty}^{\infty} g(x)h(x - \xi)d\xi \Leftrightarrow G(f)H(f). \quad (1)$$

Cross-correlation of two functions:

$$g \star h = \int_{-\infty}^{\infty} g(x + \xi)h(\xi)d\xi$$

- ▶ Correlation theorem; Autocorrelation, Wiener-Khinchin theorem::

$$g \star h \Leftrightarrow G(f)H^*(f), \quad \Rightarrow \quad g \star g = |G(f)|^2$$

- ▶ Parseval's theorem:

$$\|h(t)\|_{L^2} = \|H(f)\|_{L^2} \Rightarrow \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$

Fourier Transform: properties

- ▶ Derivatives:

$$\mathcal{F}[h'(t)] = -i\omega H(f); \quad \mathcal{F}[h^{(n)}] = (-i\omega)^n H(f).$$

- ▶ Application: D'Alambert equation

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 y(x, t)}{\partial t^2}, \quad y(x, 0) = f(x)$$

- ▶ After application of FT to both sides of this equation:

$$(-i\omega)^2 Y(\omega, t) = \frac{1}{\nu^2} \frac{\partial^2 Y(\omega, t)}{\partial t^2}$$
$$F(\omega) = Y(\omega, 0) = \mathcal{F}[f(x)]$$

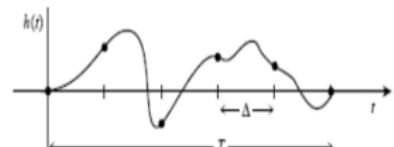
Sampling theorem

- ▶ Suppose function $h(t)$ is sampled at evenly spaced intervals in time:
$$h_n = h(n\Delta), n = \dots, -2, -1, 0, 1, 2, \dots, \quad \frac{1}{\Delta}$$
 - sampling rate
- ▶ For any sampling interval Δ , there is a special frequency f_c , called **Nyquist critical frequency**, given by $f_c = 1/2\Delta$
 - ex: critical sampling of a sine wave of Nyquist critical frequency is two sample points per cycle.
- ▶ A function f is "bandwidth limited" if its Fourier transform is 0 outside of a finite interval $[-L, L]$;
- ▶ Sampling Theorem: If a continuous function $h(t)$, sampled at an interval Δ , is bandwidth limited to frequency smaller than f_c , i.e., $H(f) = 0$ for all $|f| > f_c$, then the function $h(t)$ is completely determined by its samples h_n .

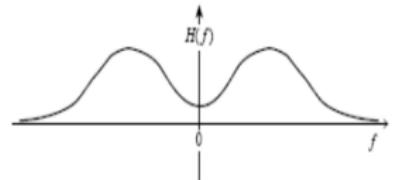
$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin(2\pi f_c(t - n\Delta))}{\pi(t - n\Delta)}.$$

Sampling theorem

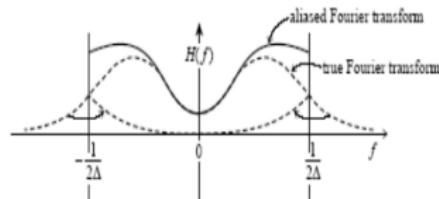
- ▶ For bandwidth limited signals, such as e.g. music in concert hall, sampling theorem tells us that the entire information content of the signal can be recorded by sampling rate Δ^{-1} equal to twice the maximum frequency pass by the amplifier
- ▶ For the function that is not bandwidth limited to less than the Nyquist critical frequency, frequency component that lies outside of the frequency range, $-f_c < f < f_c$ is spuriously moved into that range (aliasing).



(a)



(b)



(c)

Discrete Fourier Transform (DFT)

- ▶ Suppose we have N consecutive sampled values

$$h_k \equiv h(t_k), \quad t_k = \Delta k, \quad k = 0, 1, 2, \dots, N - 1,$$

where Δ is the sampling interval, and assume N is even.

- ▶ With N numbers of input, we can produce no more than N independent number of outputs. Therefore, we seek estimates only at the discrete values;

$$f_n \equiv \frac{n}{N\Delta}, \quad n = -\frac{N}{2}, \dots, \frac{N}{2}$$

- ▶ Then, discrete Fourier Transform (DFT) is

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n} \equiv \Delta H_n$$

- ▶ DFT maps N complex numbers (the h_k 's) into N complex numbers (the H_n 's).
- ▶ Trapezoidal approximation of the integral

Discrete Fourier Transform (DFT)

$$H_{N-n} = \sum_{k=0}^{N-1} h_k e^{2\pi k(N-n)/N} = \sum_{k=0}^{N-1} h_k e^{2\pi k(-n)/N} e^{2\pi i k N / N} = H_{-n}$$

- ▶ It's periodic in n , with period N : $H_{-n} = H_{N-n}$.
- ▶ With this conversion, one lets the n in H_n vary from 0 to $N-1$. Then n and k vary exactly over the same range.
- ▶ With this convention
 - zero frequency: $n = 0$
 - positive frequencies, $0 < f < f_c \Rightarrow 1 \leq n \leq N/2 - 1$
 - negative frequencies, $-f_c < f < 0 \Rightarrow N/2 + 1 \leq n \leq N - 1$
 - the value $n = N/2 \Rightarrow$ both $f = f_c$ and $f = -f_c$
- ▶ The DFT has symmetry properties almost exactly the same as the continuous Fourier transform.

Discrete Fourier Transform (DFT)

- ▶ The **Discrete Inverse Fourier Transform** (DIFT) is:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i kn/N}$$

- ▶ Parseval's theorem:

$$\sum_{k=0}^{N-1} |h_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2$$

Fast Fourier Transform (FFT)

- ▶ DFT appears to be an $O(N^2)$ process.
- ▶ Danielson and Lanczos; DFT of length N can be rewritten as the sum of two DFT of length $N/2$.

$$\begin{aligned} H_k = \sum_{j=0}^{N-1} e^{2\pi i j k / N} h_j &= \sum_{j=0}^{N/2-1} e^{2\pi i (2j)n / N} h_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i (2j+1)k / N} = \\ &\sum_{j=0}^{N/2-1} e^{2\pi i j k / (N/2)} h_{2j} + e^{2\pi i k / N} \sum_{k=0}^{N/2-1} e^{2\pi i j k / (N/2)} h_{2k+1} = \\ H_k^0 + e^{2\pi i k / N} H_k^1 &= H_k^0 + W^k H_k^1, \quad W \equiv e^{2\pi i / N}. \quad (2) \end{aligned}$$

- ▶ Such reduction may be applied at H_k^0 .
- ▶ For $N = 2^R$ the reduction can be applied multiple times until we subdivide the data into the R transforms of length 1.
- ▶ For every pattern of $\log_2 N$ number of 0's and 1's, there is one-point transformation that is just one of the input number h_n .

Main Applications of FFT

- ▶ Signal & Image Analysis (e.g. finding periodicities, feature extraction, correlation analysis etc)
- ▶ Signal & Image Processing (e.g. compression, de-noising etc)
- ▶ Convolution operations (mind the convolution theorem and $O(N \log N)$ scaling)
- ▶ Numerical Solution of Integral Equations (discretization in the reciprocal domain)
- ▶ Treatment of long-range potential problem in N-body simulations (electrostatic and gravitational interactions)

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Part 3: Radon Transform

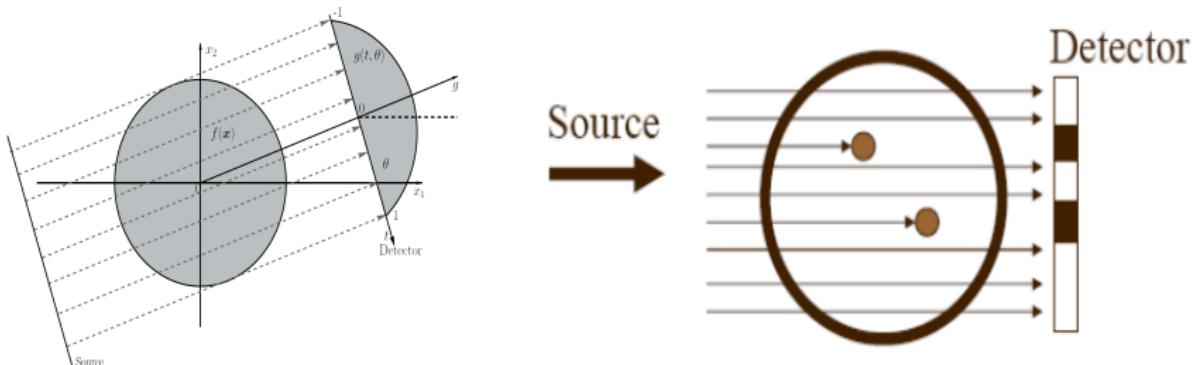
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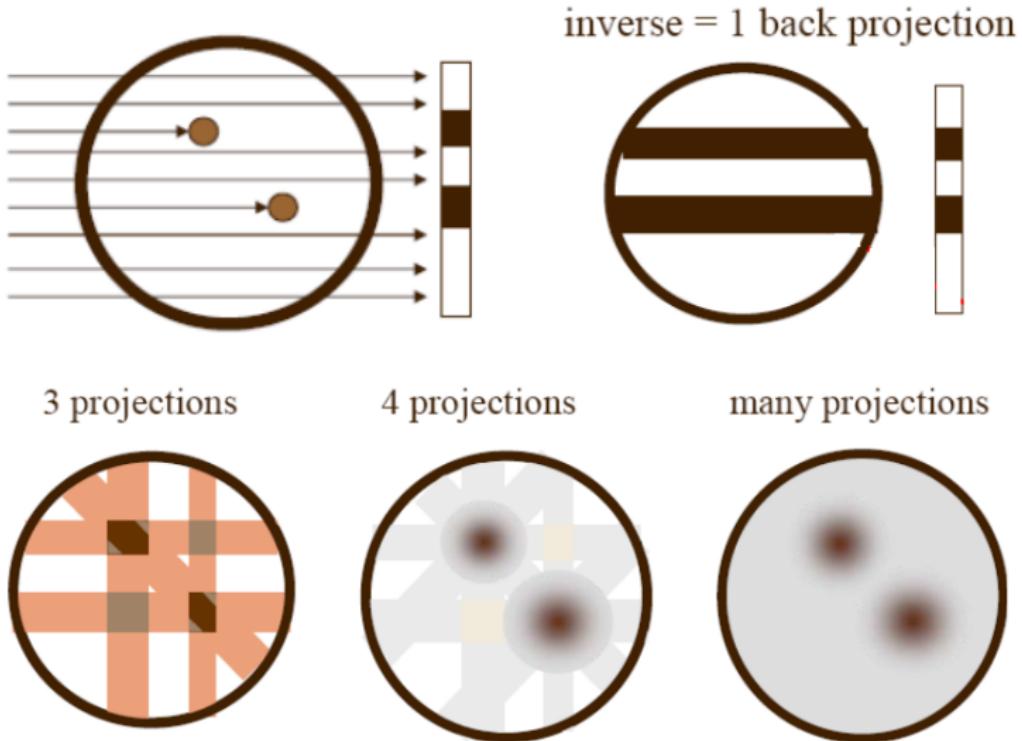
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The X-ray tomography



- ▶ Measurements: attenuation of X-rays from many different angles
- ▶ Possibility of reconstruction the inner structure of the object of study in a series of cross sections or planes
- ▶ Combination of X-ray pictures of many planes (cross sections) allows the reconstruction of 3D structures
- ▶ The reconstruction also can be considered as a convolution

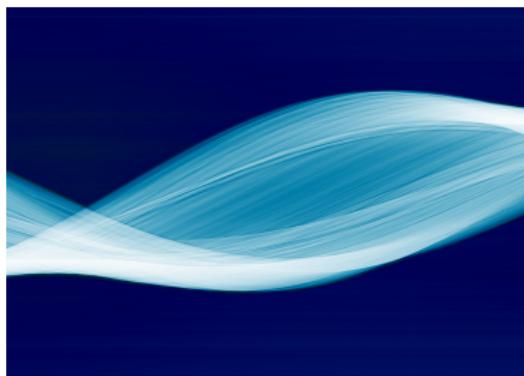
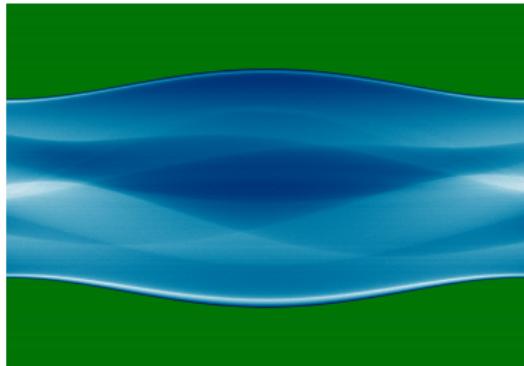
The X-ray tomography



The X-ray tomography



Source



Sinogram

Johann Karl August Radon

- ▶ Born in Decin (Austrian monarchy, now North Bohemia, CZ) in 1887
- ▶ Austrian mathematician living in Vienna
- ▶ Discover the transform and its inversion in 1917 as pure theoretical result
- ▶ No practical applications during his life
- ▶ Died in 1956 in Vienna

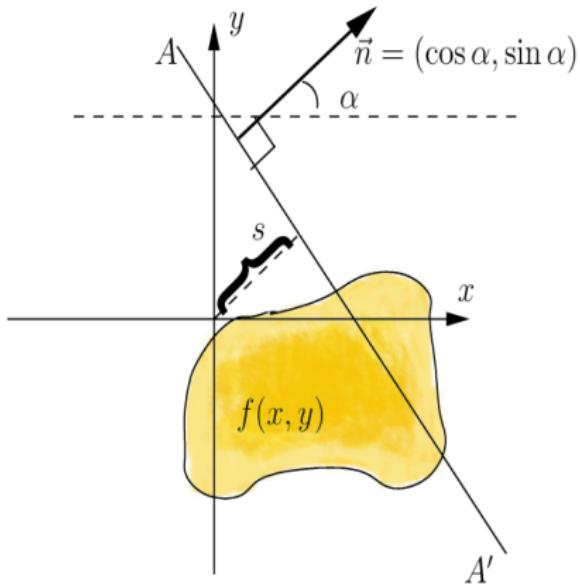


J. Radon

Actual applications of inverse Radon Transform

- ▶ CT - Computer Tomography
- ▶ MRI - Magnetic Resonance Imaging
- ▶ PET - Positron Emission Tomography
- ▶ SPECT - Single Photon Emission Computer Tomography

- ▶ Input space coordinates x, y
- ▶ Input function $f(x, y)$
- ▶ Output space coordinates α, s
- ▶ Output function $F(\alpha, s)$



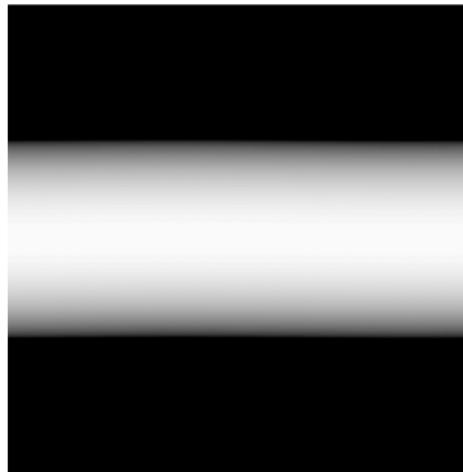
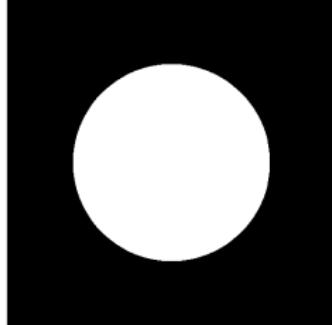
► Radon Transform

$$F(\alpha, s) = \int_{-\infty}^{\infty} f(t \cdot \sin\alpha + s \cdot \cos\alpha, -t \cdot \cos\alpha + s \cdot \sin\alpha) dt$$

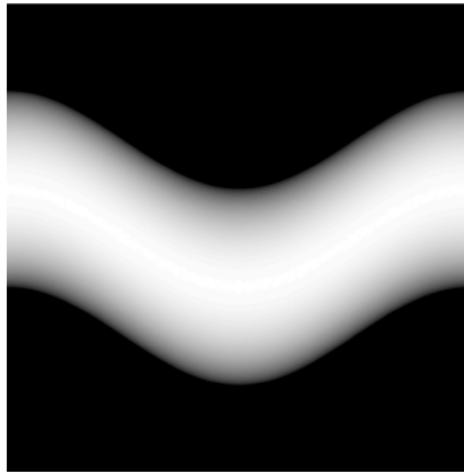
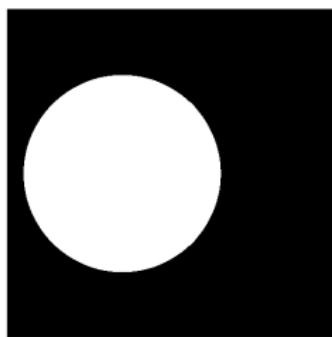
► Inverse Radon Transform

$$f(x, y) = \int_0^{2\pi} F(\alpha, x \cdot \cos\alpha + y \cdot \sin\alpha) d\alpha.$$

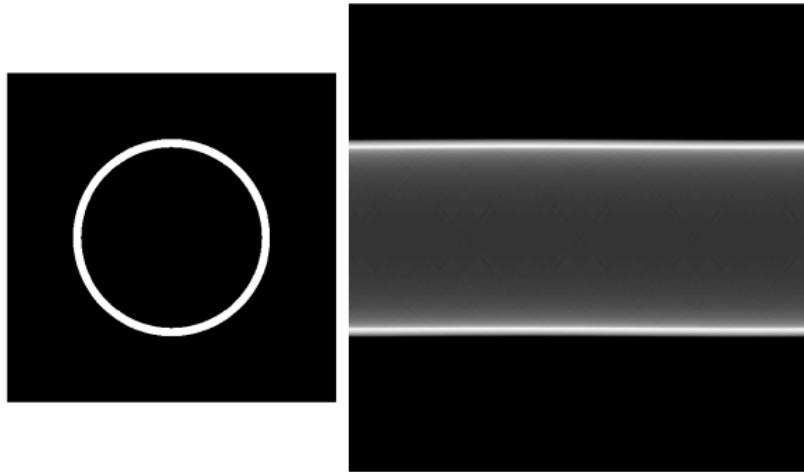
Radon Transform example



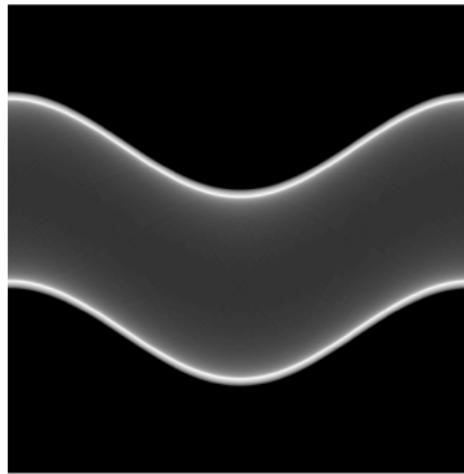
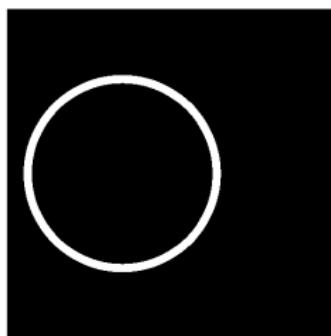
Radon Transform example



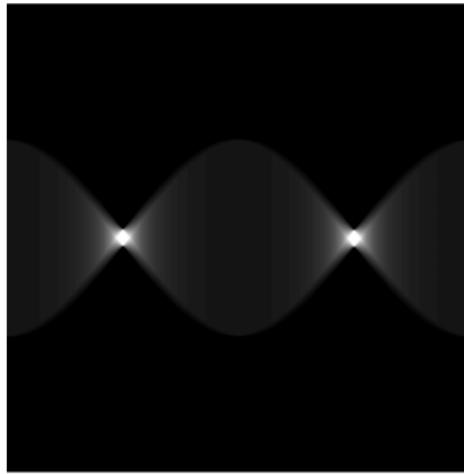
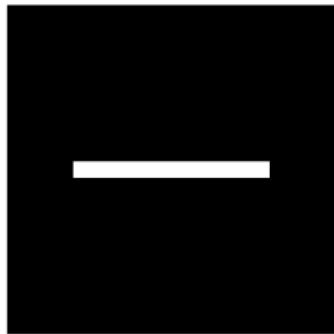
Radon Transform example



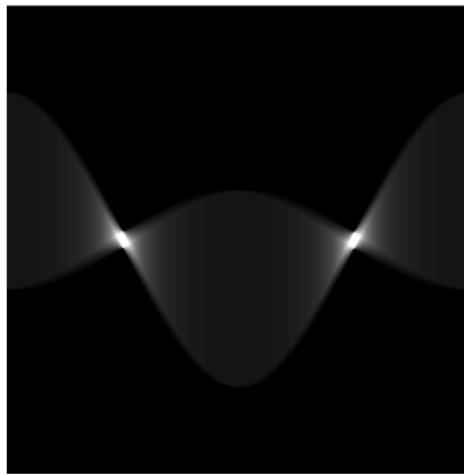
Radon Transform example



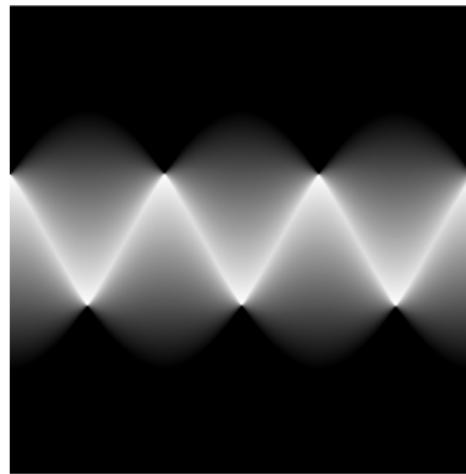
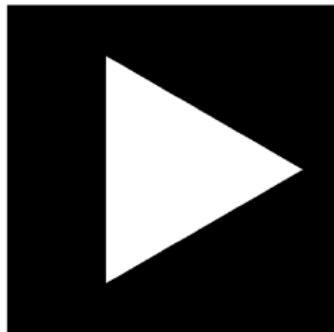
Radon Transform example



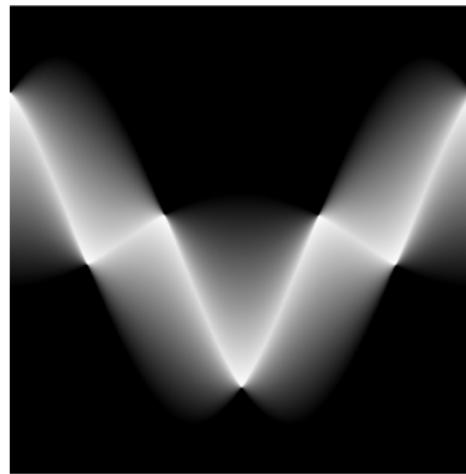
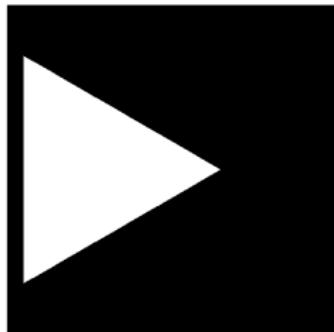
Radon Transform example



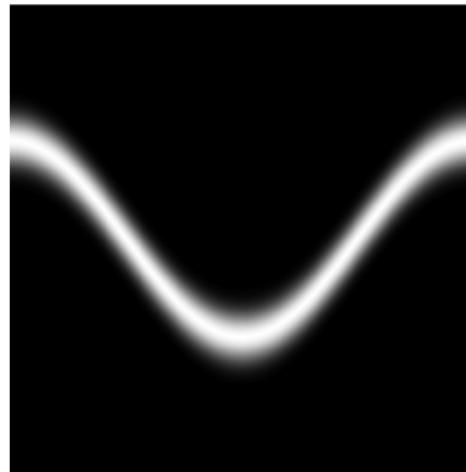
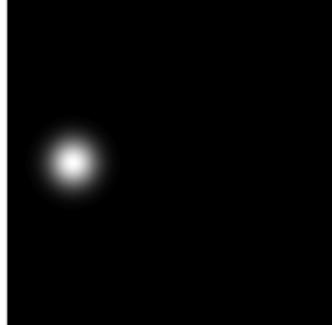
Radon Transform example



Radon Transform example



Radon Transform example



Natural transform as result of measurement:

- ▶ Gamma ray decay from local density map
- ▶ Extinction from local concentration map
- ▶ Total radioactivity from local concentration map
- ▶ Total echo from local nuclei concentration map
- ▶ 3D reality is investigated via 2D slices

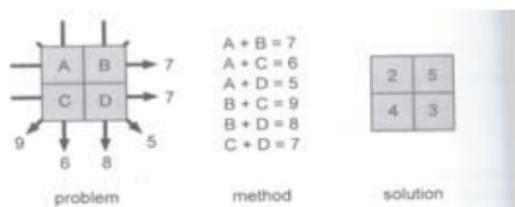
Artificial realization:

- ▶ Noise - RT - noise - IRT simulations
- ▶ Image decryption as a fun
- ▶ TSR invariant recognition of objects

- ▶ Image of any $f + g$ is $F + G$
- ▶ Image of cf is cF for any real c
- ▶ Rotation of f causes translation of F in α
- ▶ Scaling of f in (x, y) causes scaling of F in s
- ▶ Image of a point (2D Dirac function) is sine wave line
- ▶ Image of n points is a set of n sine wave lines

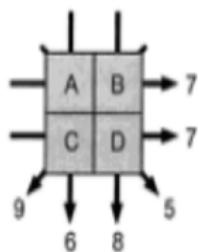
ART or Algebraic Reconstruction Technique

- ▶ Consider the basic 2×2 image known only by projection values:



- ▶ Given are sums, we have to reconstruct values of pixels A, B, C and D

ART or Algebraic Reconstruction Technique



problem

$$\begin{aligned}A + B &= 7 \\A + C &= 6 \\A + D &= 5 \\B + C &= 9 \\B + D &= 8 \\C + D &= 7\end{aligned}$$



method

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 5 \\ 9 \\ 8 \\ 7 \end{pmatrix}$$

solution

$$K \cdot x = b$$

Over-determined non-square matrix K

Multiply by K^T first

Invert square matrix

$$K^T K \cdot x = K^T b$$

$$x = [K^T K]^{-1} K^T b$$

Larger problems must be solved iteratively using standard methods for solving large matrix operation problems.

Fourier Slice Theorem

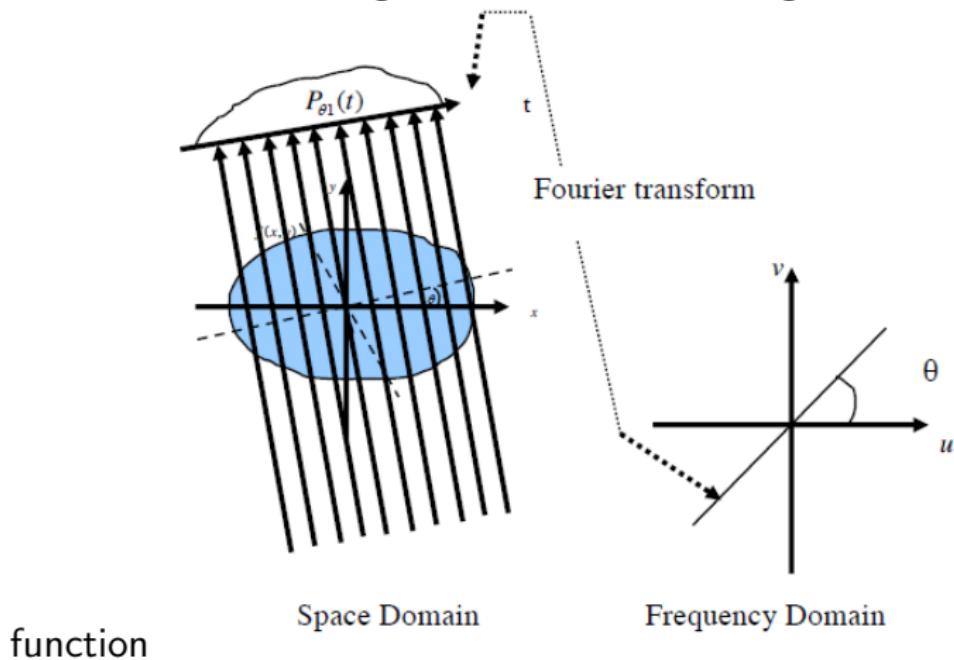
- ▶ Denote the projection at angle $\alpha = P_\alpha(s)$
- ▶ $S_\alpha(\omega) = \hat{P}_\alpha(\omega)$ - 1D Fourier image of the projection.
- ▶ $\hat{f}(\omega, \nu)$ - the Fourier image of the feature (object) function

The Fourier Slice Theorem:

$$\hat{f}(\omega, \nu) = S_{\alpha=\nu}(\omega).$$

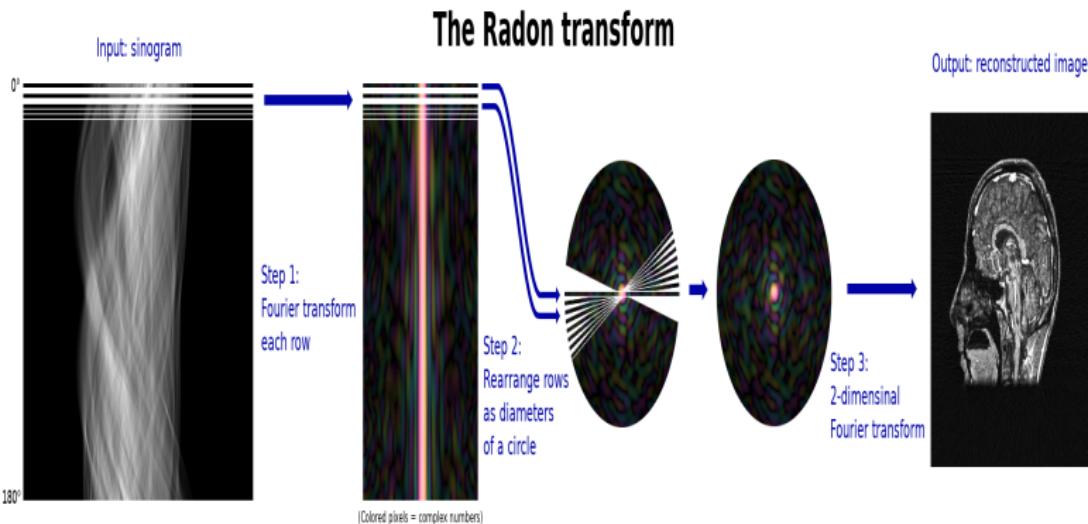
Fourier Slice Theorem

The FST maps the 1D Fourier image of the projection at angle α into the radial line with angle α in 2D Fourier image of the feature



Fourier Slice Theorem

The FST maps the 1D Fourier image of the projection at angle α into the radial line with angle α in 2D Fourier image of the feature function



The Convolution and the Radon transform

Let $g = Rf$, and $b = Bg \equiv BRf$. The operator B is called **the Backprojection operator**:

$$\int_{\mathbb{R}_+} Rf(s, \alpha) g(s, \alpha) ds d\alpha = \int_{\mathbb{R}^2} f(x) Bg(x) dx.$$

- ▶ The FST-based convolution:

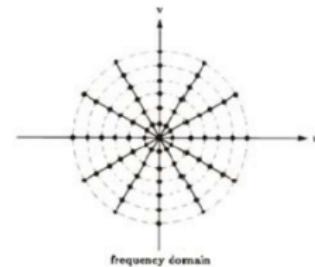
$$b = BRg = g * \left(\frac{1}{|x|} \right)$$

- ▶ The Log-Polar backprojection:

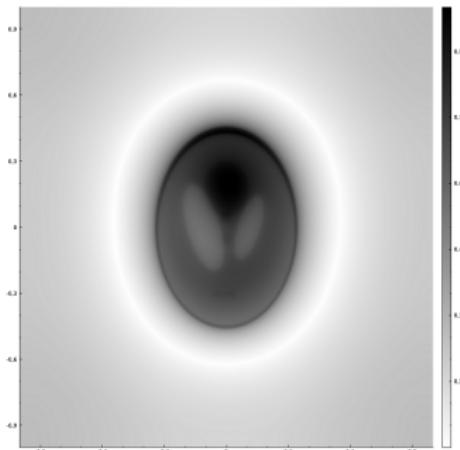
$$[Bg]_L(s, \alpha) = [g]_L * [K]_L(s, \alpha), \quad \text{where } [K]_L(s, \alpha) = \delta(1 - e^s \cos \alpha).$$

Filtered Backprojection

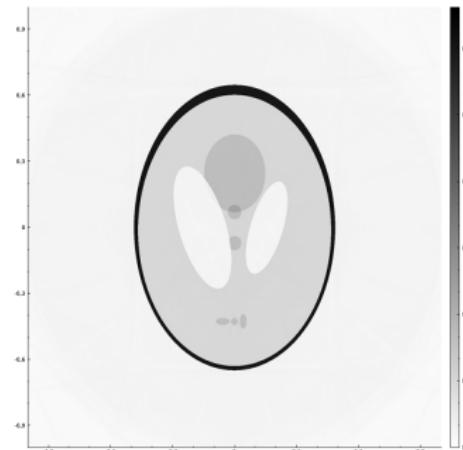
- ▶ We have to account for the non-uniform sampling.
- ▶ The ideal filter for this is the pie shaped wedges shown with Fig. (a).
- ▶ Unfiltered processing would simply use response (b)
- ▶ To emulate the response (a), people have used the filter response (c).
 - ▶ Let us have K projections over 180 degrees. Width of the wedge will be $2\pi|\omega|$ (a).
 - ▶ The ramp filter in (c) corresponds to that width.



Filtered Backprojection



Unfiltered



Filtered

Radon Transform

- ▶ 2D FFT of original
- ▶ Resampling to polar coordinates (2D interpolation)
- ▶ Line-by-line inverse 1D FFT brings result

Inverse Radon Transform

- ▶ Line-by-line 1D FFT of original
- ▶ Resampling to rectangular coordinates (2D interpolation)
- ▶ 2D LF filtering in frequency domain
- ▶ Inverse 2D FFT brings result

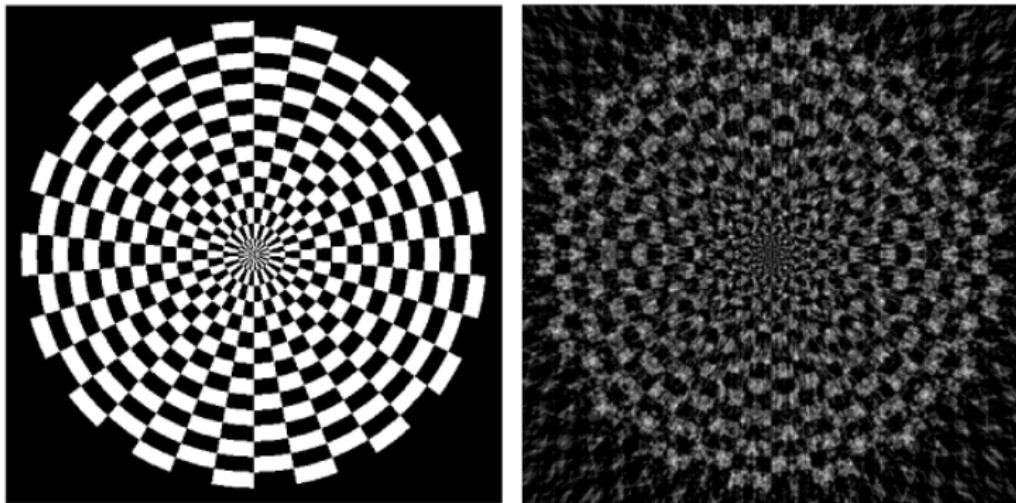
FBP

- ▶ Computationally cheap
- ▶ Clinically usually 500 projections per slice
- ▶ problematic for noisy projections

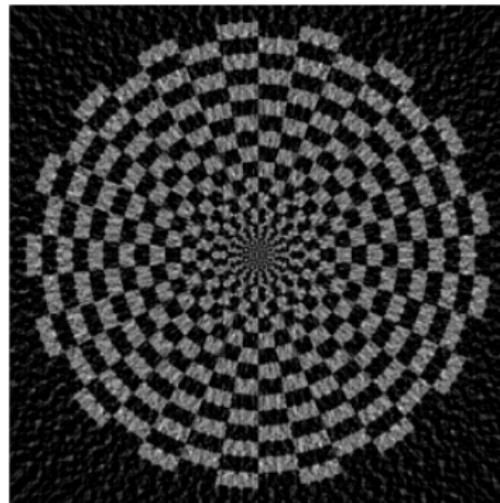
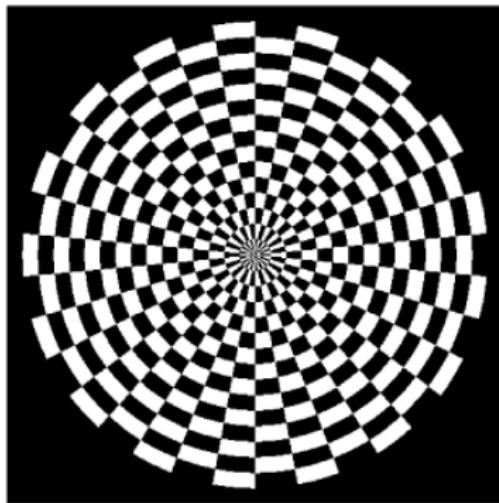
ART

- ▶ Still slow
- ▶ Better quality for fewer projections
- ▶ better quality for non-uniform project
- ▶ "guided" reconstruct (initial guess!)

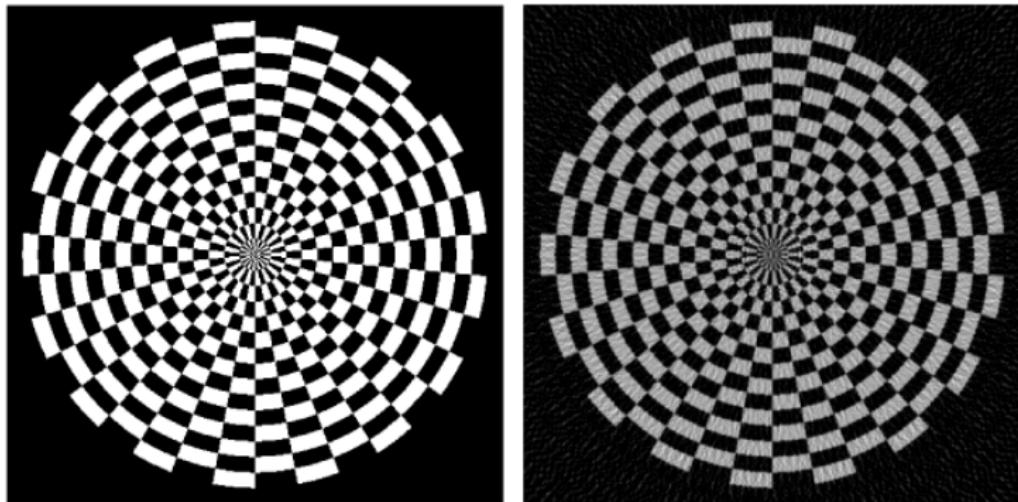
Reconstruction from 32 angles



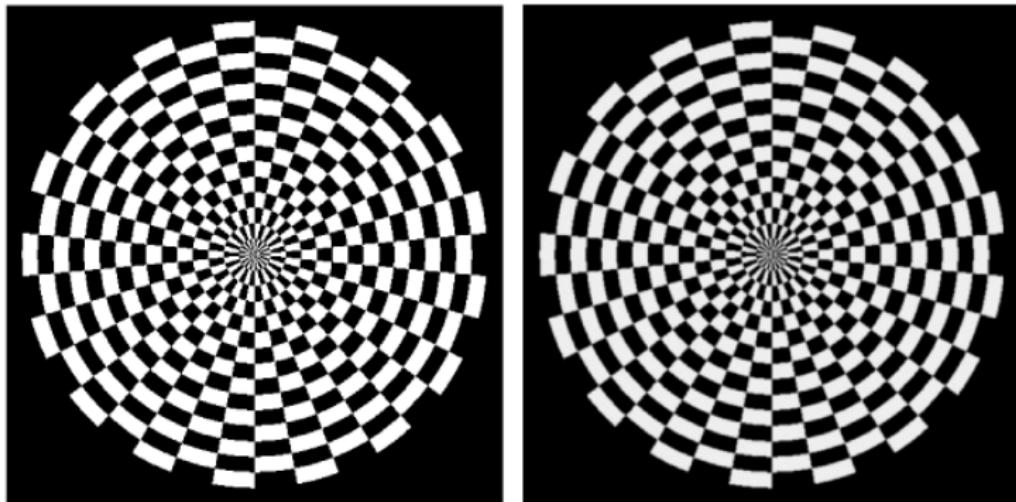
Reconstruction from 64 angles



Reconstruction from 128 angles

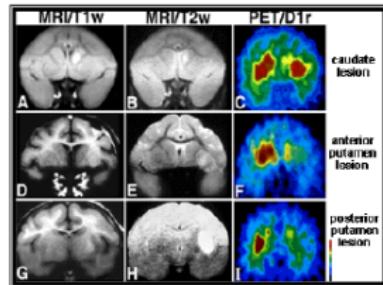


Reconstruction from 512 angles

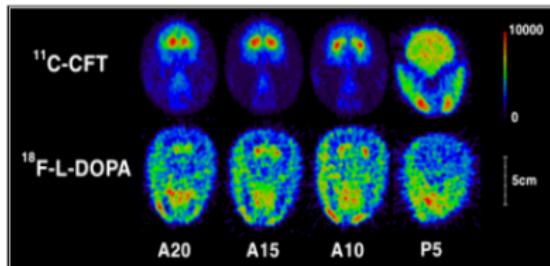


Applications & Types of Tomography

Medical Applications	Type of Tomography
Full body scan	X-ray
Respiratory, digestive systems, brain scanning	PET Positron Emission Tomography
Respiratory, digestive systems.	Radio-isotopes
Mammography	Ultrasound
Whole Body	Magnetic Resonance (MRI, NMR)



MRI and PET showing lesions in the brain.



PET scan on the brain showing Parkinson's Disease

Scientific Computing

Lecture 5

Part 4: Wavelet Transform

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October 18, 2023

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Or: What's **wrong** with Fourier?

- ▶ By using Fourier Transform , we loose the time information : WHEN did a particular event take place ?
- ▶ FT can not locate drift, trends, abrupt changes, beginning and ends of events, etc.
- ▶ Calculating use complex numbers.

Time and Space definition

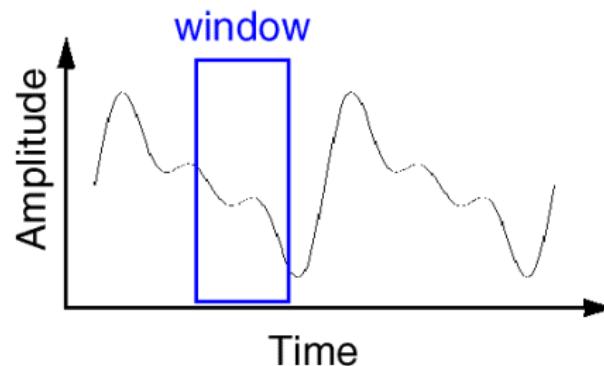
- ▶ Time - for one dimension waves we start point shifting from source to end in time scale .
- ▶ Space - for image point shifting is two dimensional .
- ▶ Here they are synonyms .

$$\psi_k(t) = \delta_k(t) = \begin{cases} 1, & k == t \\ 0, & k \neq t \end{cases}$$

Can exactly show the time of appearance but have not information about frequency and shape of signal.

Short Time Fourier Analysis

In order to analyze small section of a signal, Denis Gabor (1946), developed a technique, based on the FT and using windowing : STFT



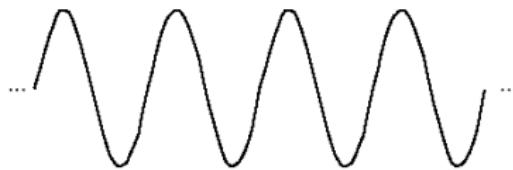
- ▶ A compromise between time-based and frequency-based views of a signal.
- ▶ both time and frequency are represented in limited precision.
- ▶ The precision is determined by the size of the window.
- ▶ Once you choose a particular size for the time window - it will be the same for all frequencies.

What's **wrong** with Gabor?

- ▶ Many signals require a more flexible approach - so we can vary the window size to determine more accurately either time or frequency.

What is Wavelet Analysis?

- ▶ And what is a wavelet...?



Sine Wave



Wavelet (db10)

- ▶ A wavelet is a waveform of effectively limited duration that has an average value of zero.
- ▶ Short time localized waves with zero integral value.
- ▶ Possibility of time shifting.
- ▶ Flexibility.

The Continuous Wavelet Transform

For easier understanding, consider the 1D wavelets in a time domain.

- ▶ $x(t)$ - time-evolving signal.
- ▶ The Mother Wavelet: $\Psi(t)$.
- ▶ Daughter Wavelets: $\Psi\left(\frac{t-b}{a}\right)$ - shifted and scaled versions of the MW.
- ▶ Continuous Wavelet Transform (CWT):

$$X(a, b) = \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} dt x(t) \Psi^*\left(\frac{t-b}{a}\right)$$

- ▶ Continuous Inverse Wavelet Transform:

$$x(t) = \frac{1}{C_\Psi} \iint_{\mathbb{R}^2} X(a, b) \frac{1}{|a|^{1/2}} \tilde{\Psi}\left(\frac{t-b}{a}\right) db \frac{da}{a^2}$$

- ▶ C_Ψ - the admission constant; $\tilde{\Psi}$ - dual wavelet.

The Continuous Wavelet Transform (CWT)

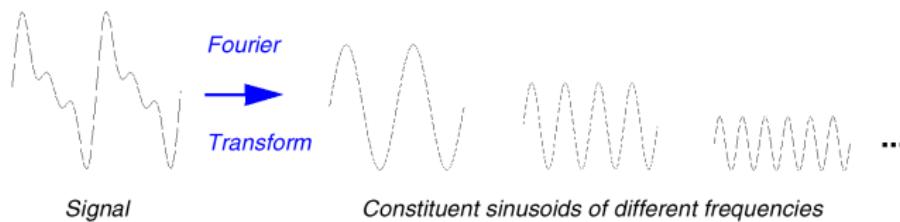
- ▶ A mathematical representation of the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

- ▶ Meaning: the sum over all time of the signal $f(t)$ multiplied by a complex exponential, and the result is the Fourier coefficients $F(\omega)$

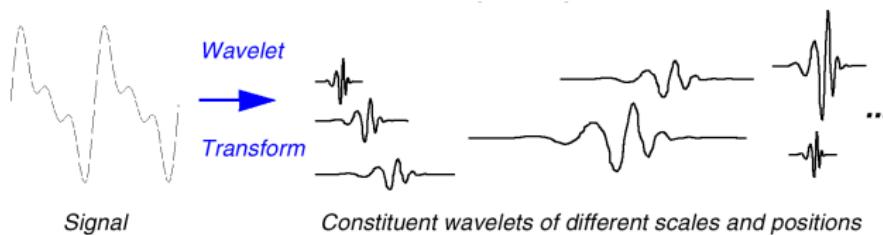
Wavelet Transform

- ▶ Those coefficients, when multiplied by a sinusoid of appropriate frequency ω yield the constituent sinusoidal component of the original signal:



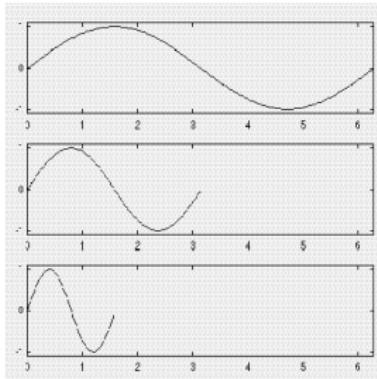
Wavelet Transform

- ▶ And the result of the CWT are Wavelet coefficients .
- ▶ Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelet of the original signal:



Scaling

- ▶ Wavelet analysis produces a time-scale view of the signal.
- ▶ Scaling means stretching or compressing of the signal.
- ▶ scale factor (a) for sine waves:

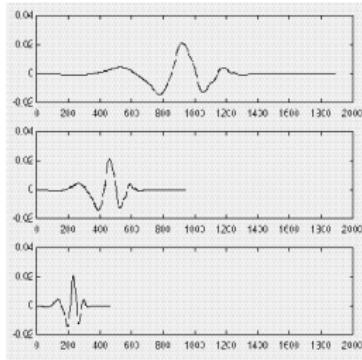


$$f(t) = \sin(t); a = 1$$

$$f(t) = \sin(2t); a = \frac{1}{2}$$

$$f(t) = \sin(4t); a = \frac{1}{4}$$

- ▶ Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t); a = 1$$

$$f(t) = \Psi(2t); a = \frac{1}{2}$$

$$f(t) = \Psi(4t); a = \frac{1}{4}$$

The Wavelet function

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}} \Psi\left(\frac{x - b}{a}\right),$$

here

- ▶ b - shift coefficient
- ▶ a - scale coefficient

2D function:

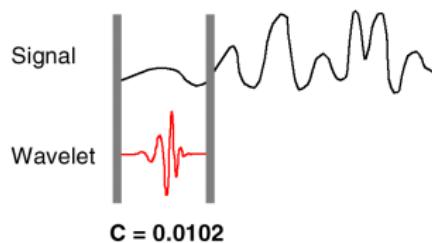
$$\Psi_{a,b_x,b_y}(x, y) = \frac{1}{|a|} \Psi\left(\frac{x - b_x}{a}, \frac{y - b_y}{a}\right),$$

Continuous Wavelet Transform

- ▶ Reminder: The CWT Is the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function

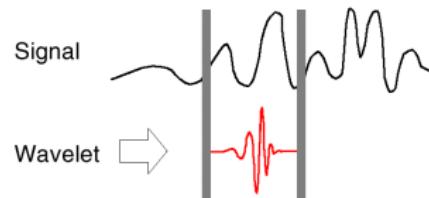
Step 1: Take a Wavelet and compare it to a section at the start of the original signal.

Step 2: Calculate a number, C , that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.

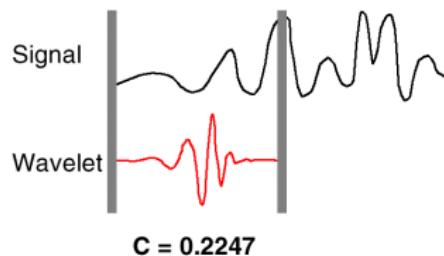


Continuous Wavelet Transform

Step 3: Shift the wavelet to the right and repeat steps 1-2 until you've covered the whole signal

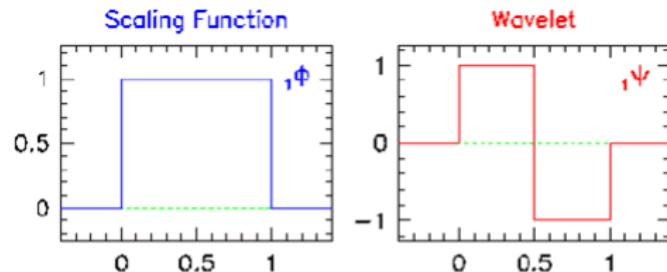


Step 4: Scale (stretch) the wavelet and repeat steps 1-3

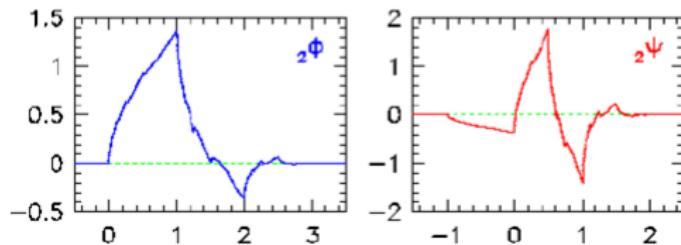


Wavelet functions examples

► Haar function



► Daubechies function



Scientific Computing

Lecture 5

Part 5: Integral Equations

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The integral equations

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

Here A is an integral operator, which can be expressed as follows:

$$Ay(\mathbf{x}) = \lambda \int_{\Omega} K(\mathbf{x}, \xi) y(\xi) d\xi$$

- ▶ Ω is some domain (may represent the whole space).
- ▶ $K(\mathbf{x}, \xi)$ is a Kernel.
- ▶ $y(\mathbf{x})$ is unknown function to be found.
- ▶ $f(\mathbf{x})$ is a right-hand side.

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

- ▶ **The integral equation of the first kind:** $g(\mathbf{x}) \equiv 0$.

$$Ay(\mathbf{x}) = f(\mathbf{x})$$

These equations are commonly ill-posed.

- ▶ **The integral equation of the second kind:**

$$y(\mathbf{x}) - \lambda Ay(\mathbf{x}) = f(\mathbf{x}).$$

These equations are commonly well-posed.

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

- ▶ Homogeneous equation: $f(\mathbf{x}) \equiv 0$.
Homogeneous IE with $g(\mathbf{x}) \equiv 1$ are eigenvalues problems.
- ▶ Heterogeneous equation: $f(\mathbf{x}) \neq 0$.

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

Here A is an integral operator, which can be expressed as follows:

$$Ay(\mathbf{x}) = \lambda \int_{\Omega} K(\mathbf{x}, \xi) y(\xi) d\xi$$

- ▶ **Volterra IE:** the domain Ω is changeable domain. The 1D example:

$$y(x) - \int_0^x K(x, s) y(s) ds = f(x)$$

- ▶ **Fredholm IE:** the domain Ω is a constant domain.

Simple integration

Consider the integration problem:

$$f'(x) = g(x), 0 \leq x \leq 1$$

In order to calculate antiderivative, we have to integrate the latter equation with Volterra differential operator:

$$f(x) = f(0) + \int_0^x g(\xi) d\xi.$$

or

$$f(x) = f(1) - \int_x^1 g(\xi) d\xi.$$

Thank you for your attention!