Numerical linear algebra course

Midterm, Fall 2023

Variant 2

Theoretical tasks

1. **(1 pts)**
   (a) What is the name of the transformation that is represented as the following matrix in the 2D case?
   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2vv^\top,$$
   where $||v|| = 1$.
   - ☐ Housholder reflection
   - ☐ Householder reflection
   - ☐ Householder projection
   - ☐ Householder rotation
   - ☐ No specific name for such transformation

   (b) Find such v that this transformation makes vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ collinear to e_1.
2. **(2 pts)** What is the complexity of the straightforward matrix by matrix product in the case of square matrices of size n? Can it be improved? Why? If it can be improved, describe the algorithm idea and provide the resulting complexity.
3. **(4 pts)** Proof the following equality $(I + uv^\top)^{-1} = I - \frac{uv^\top}{1+v^\top u}$. Why $1 + v^\top u \neq 0$?
4. **(3 pts)** Proof that $XX^\dagger X = X$.
5. **(2 pts)** Show that $\frac{\|A\|_2^2}{\|A\|_F^2} \leq 1$ for any matrix A. In which case does equality hold?

Practical tasks

1. **(3 pts)** Assume matrix A has singular value decomposition $A = U\Sigma V^*$. Derive the singular value decomposition of a block matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$.
2. **(4 pts)** Assume you are given a matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and you run the power method. Does the power method converge? If it converges, comment on what is a convergence speed and what is the stationary point. If it will not converge, please explain why.
3. **(2 pts)** Does QR decomposition exist for matrix $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \end{bmatrix}$? Why? If it exists, compute it.
4. **(2 pts)** Calculate SVD of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$.

5. **(4 pts)** Compute determinant of matrix A:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$