Solutions NLA Midterm 2023 for Variant 1

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Theoretical Task 1

a) Identifying the Transformation

Given matrix:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

This matrix represents a **Givens rotation** in 2D, a fundamental tool in numerical linear algebra for introducing zeros into vectors or matrices.

b) Constructing a Unitary Transformation

Task: Construct a unitary transformation that zeros out the second coordinate of a 2D vector (x_1, x_2).

Solution: The required transformation is a Givens rotation. The angle α is determined by the elements of the vector:

$$\alpha = \begin{cases} \arctan\left(\frac{-x_2}{x_1}\right) & \text{if } x_1 \neq 0 \end{cases}$$

The Givens rotation matrix then becomes:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

This matrix will rotate the vector (x_1, x_2) to a new position where its second component is zeroed out.

Theoretical Task 2

Question: What is the complexity of matrix by vector product in the general case where a matrix is of size $m \times n$? Can it be improved? Why?

Solution:

1. *Complexity Analysis:*

   - A matrix-vector multiplication where the matrix is of size $m \times n$ and the vector is of size n involves computing each element of the resulting m-dimensional vector.

- For each element in the resulting vector, you perform n multiplications and $n-1$ additions.
- Thus, for m elements, the total operations are $m \times (n+n-1)$, which simplifies to $2mn - m$.
- Therefore, the complexity is $O(mn)$, which represents a linear operation count in terms of the number of elements in the matrix.

2. *Improvement Possibilities:*

- The complexity of $O(mn)$ is generally considered efficient for matrix-vector multiplication, especially for dense matrices.
- However, if the matrix has a special structure (like being sparse, diagonal, triangular, etc.), the complexity can be reduced by exploiting these properties. For example, in a sparse matrix, where most elements are zero, you only need to perform operations for non-zero elements, significantly reducing the computational load.
- Another approach to improve efficiency, especially for large matrices, is to use optimized libraries and hardware acceleration (like GPUs). These optimizations often involve parallel processing and other hardware-specific enhancements.

Theoretical Task 3

Question: Given $I_n + AB$ is invertible with $A \in R^{n\times m}$ and $B \in R^{m\times n}$, is $I_m + BA$ also invertible?

Solution:

1. *Given Condition:* It's given that I_n+AB is invertible, thus $\det(I_n+AB) \neq 0$.

2. *Block Matrix Formulation:* Consider the block matrix

$$M = \begin{pmatrix} I_n & A \\ -B & I_m \end{pmatrix}$$

3. *Determinant of the Block Matrix:* Using the determinant formula for block matrices:

$$\det(M) = \det(I_n)\det(I_m + BA) = \det(I_n + AB)$$

This shows $\det(I_m+BA) = \det(I_n+AB)$. (Weinstein-Aronszajn identity)

4. *Conclusion:* Since $\det(I_n + AB) \neq 0$, it implies $\det(I_m + BA) \neq 0$, hence $I_m + BA$ is also invertible.

Theoretical Task 5

Question: Show that for any k there exists a matrix A: $\text{rank}(A) = k$ such that $\|A\|_F = \text{rank}(A)\|A\|_2$.

Solution:

1. *Understanding the Norms:* The Frobenius norm ($\|A\|_F$) is the square root of the sum of the absolute squares of its elements, and the spectral norm ($\|A\|_2$) is the largest singular value of A.

2. *Constructing the Matrix:* Construct a diagonal matrix A of size $k \times k$ with equal singular values (e.g., all 1s) on its diagonal. Thus, $\text{rank}(A) = k$.

3. *Calculating the Norms:* The Frobenius norm of A is $\|A\|_F = \sqrt{k}$, and the spectral norm is $\|A\|_2 = 1$.

4. *Establishing the Relationship:* Since all singular values are 1, $\|A\|_F = \sqrt{k} = \text{rank}(A)\|A\|_2$.

5. Or it can be a matrix with all σ's on its diagonal or just a matrix with one element on the top-left corner.

Practical Task 2

Question: Given $A = \begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$, will the power method converge? If so, comment on the convergence speed and the stationary point.

Solution:

1. *Eigenvalues Calculation:* The eigenvalues of A are calculated to be -10 and 5.

2. *Convergence Analysis:* Since there is a dominant eigenvalue (-10), the power method will converge.

3. *Convergence Speed:* The convergence speed is relatively fast, as indicated by the ratio of the absolute values of the dominant to sub-dominant eigenvalue, which is 2.

4. *Stationary Point:* The stationary point of the power method will be the eigenvector corresponding to the eigenvalue -10.

Practical Task 3

Question: Compute the LU decomposition for the matrix $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

Solution:

1. The matrix A is singular with rank 1 and the first leading principal minor is non-zero, hence it admits an LU factorization.

2. The LU decomposition of A is:

   - Lower Triangular Matrix L:

   $$\begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}$$

   - Upper Triangular Matrix U:

   $$\begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}$$

Practical Task 4

Question: Calculate SVD of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{pmatrix}$.

Solution:

1. *Calculate AA^* (Matrix A times Conjugate Transpose of A):*

$$AA^* = \begin{pmatrix} 5 & 9 & 4 \\ 9 & 18 & 9 \\ 4 & 9 & 5 \end{pmatrix}$$

2. *Find Eigenvalues of AA^*:* The eigenvalues are 27 and 1.

3. *Calculate Singular Values (Σ):* The singular values are the square roots of the eigenvalues of A^*A, giving:

$$\Sigma = \begin{pmatrix} 3\sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

4. *Compute Eigenvectors of AA^* (Matrix U):*

$$U = \begin{pmatrix} \frac{\sqrt{6}}{6} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

5. *Compute V:*

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Practical Task 5

Question: Find the matrix inverse to A

Solution: We will use the Sherman-Morrison formula, where $A = I + uv^T$ and both u and v are vectors of all ones. The inverse of matrix A using the Sherman-Morrison formula is:

$$A^{-1} = I - \frac{uv^T}{1 + v^T u} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$$