# Numerical linear algebra course

Midterm, Fall 2023

Variant 3

## Theoretical tasks

1. **(1 pts)**

   (a) What is the name of the following expression for the hermitian matrix $A$?

   $$\frac{x^* A x}{x^* x}$$

   ☐ Raleigh quotient
   ☐ Rayleygh quotient
   ☐ Rayleigh quotient
   ☐ Railigh qutient
   ☐ No specific name for such expression

   (b) Find such $v$ that maximizes this expression for matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix}$.

2. **(3 pts)** What is the complexity of the straightforward Schur decomposition computing for a matrix of size $n$? Can it be improved? Why? If it can be improved, describe the algorithm idea and provide the resulting complexity.

3. **(3 pts)** Proof the following equality $(I + uv^\top)^{-1} = I - \frac{uv^\top}{1+v^\top u}$. Why $1 + v^\top u \neq 0$?

4. **(3 pts)** If you have a matrix $A = UV^\top$, where $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{m \times k}$, $k < m < n$. Derive the expression for $A^\dagger$.

5. **(3 pts)** Show that $\frac{\|AB\|_F}{\|A\|_2 \|B\|_F} \leq 1$ for any matrices $A$ and $B$. In which case does equality hold?

## Practical tasks

1. **(3 pts)** Assume matrix $A$ has eigendecomposition $A = U\Lambda U^*$. Derive the eigendecomposition of a block matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and explain why does it exist?

2. **(3 pts)** Assume you are given a matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and you run the power method. Does the power method converge? If it converges, comment on what is a convergence speed and what is the stationary point. If it will not converge, please explain why.

3. **(2 pts)** Does SVD decomposition exist for matrix $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \end{bmatrix}$ ? Why? If it exists, compute it.

4. **(3 pts)** Compute $\text{cond}_\infty(A)$ for $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

5. **(3 pts)** Compute determinant of matrix $A$:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & -5 \end{pmatrix}$$