Solutions NLA Midterm 2023 for Variant 3

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Numerical Linear Algebra Midterm Solutions

Theoretical Task 1

(a) The expression given is:

$$\frac{x^* A x}{x^* x}$$

This expression is known as the **Rayleigh quotient**.

(b) The eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix}$ are:

- Eigenvalues:
  - $\lambda_1 = \frac{5-\sqrt{5}}{2}$
  - $\lambda_2 = \frac{5+\sqrt{5}}{2}$
- Eigenvectors:
  - For λ_1: $v_1 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}$
  - For λ_2: $v_2 = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

The vector v that maximizes the Rayleigh Quotient for matrix A is $v_1 = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$.

Theoretical Task 2

Question: What is the complexity of the straightforward Schur decomposition computing for a matrix of size n? Can it be improved? Why? If it can be improved, describe the algorithm idea and provide the resulting complexity.

Solution

1. The straightforward Schur decomposition of a matrix of size $n \times n$ typically has a computational complexity of $O(n^3)$. This complexity arises from the process of reducing the matrix to upper triangular form using Householder transformations or Givens rotations.

2. While the complexity can potentially be improved in specific cases, such as for sparse matrices, the general complexity for dense matrices remains $O(n^3)$.

3. To improve efficiency, one can consider using divide-and-conquer algorithms or algorithms that exploit parallelism. These methods can reduce the constant factor in the complexity but do not change the overall $O(n^3)$ complexity in the general case.

Theoretical Task 3

We are to prove that for any vectors u and v such that $1+v^T u \neq 0$, the following identity holds:

$$(I + uv^T)^{-1} = I - \frac{uv^T}{1 + v^T u}.$$

Proof:

Let $A = uv^T$ and $\alpha = 1 + v^T u$. The matrix $I + A$ is invertible as $\alpha \neq 0$. We shall show that $(I + A)(I - \frac{A}{\alpha}) = I$:

$$\begin{aligned}
(I + A)(I - \frac{A}{\alpha}) &= I - \frac{A}{\alpha} + A - \frac{A^2}{\alpha} \\
&= I - \frac{A}{\alpha} + A - \frac{(\alpha - 1)A}{\alpha} \\
&= I.
\end{aligned}$$

The third equality follows from $A^2 = (uv^T)(uv^T) = u(v^T u)v^T = (\alpha - 1)uv^T = (\alpha - 1)A$, hence $\frac{A^2}{\alpha-1} = A$. Therefore, $(I - \frac{A}{\alpha})$ is the inverse of $(I + A)$, and the identity is proven.

To show that $1 + v^T u \neq 0$ we can use Matrix determinant lemma.

N.B. The formula that needed to be proved is a special case of the Sherman–Morrison formula.

Theoretical Task 5

Question: Show that $\|AB\|_F \leq \|A\|_2 \|B\|_F$ for any matrices A and B, and identify in which case does equality hold.

Solution

1. The Frobenius norm of a matrix M, denoted as $\|M\|_F$, is the square root of the sum of the absolute squares of its elements. The spectral norm (or 2-norm) of a matrix A, denoted as $\|A\|_2$, is the largest singular value of A.

2. The inequality $\|AB\|_F \leq \|A\|_2\|B\|_F$ can be proven using the properties of the spectral norm and the Frobenius norm. The Frobenius norm is submultiplicative, and the Frobenius norm is bigger or equal then the spectral norm.

3. Equality holds if and only if B is a rank-1 matrix and A is a scalar multiple of a unitary matrix that aligns with the singular vector of B corresponding to its singular value.

Practical Task 2

Given matrix A:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix},$$

we are asked to determine whether the power method converges when applied to A, and if it does, to discuss the convergence speed and the stationary point.

Solution:

The eigenvalues of A are calculated to be $2 + 2i$ and $2 - 2i$. For the power method to converge, there must be a dominant eigenvalue, which is not the case here as both eigenvalues have the same magnitude.

Conclusion:

The power method will not converge for matrix A as it lacks a dominant eigenvalue. The method relies on a dominant eigenvalue to ensure convergence to the corresponding eigenvector. With multiple eigenvalues of equal magnitude, the power method will not reliably converge to a single eigenvector.

Practical Task 3

Question: Compute the SVD (Singular Value Decomposition) for the matrix

$$A = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix}$$

Solution:

The Singular Value Decomposition of the matrix A is given by $A = U\Sigma V^T$, where U is the matrix of left singular vectors, Σ is the diagonal matrix of singular values, and V^T is the transpose of the matrix of right singular vectors.

$$U = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix},$$
$$\Sigma = \begin{pmatrix} \sqrt{26} & 0 \\ 0 & 0 \end{pmatrix},$$
$$V^T = \begin{pmatrix} -\frac{3\sqrt{13}}{13} & \frac{2\sqrt{13}}{13} \\ \frac{2\sqrt{13}}{13} & \frac{3\sqrt{13}}{13} \end{pmatrix}.$$

Practical Task 4

Question: Compute $\text{cond}_\infty(A)$ for $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

Solution:

The condition number $\text{cond}_\infty(A)$ of a matrix A, with respect to the infinity norm, is defined as:

$$\text{cond}_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$$

where $\|A\|_\infty$ is the maximum absolute row sum of A, and $\|A^{-1}\|_\infty$ is the maximum absolute row sum of the inverse of A.

For the given matrix A, the infinity norm $\|A\|_\infty$ is the maximum absolute row sum, which is 5. The infinity norm of the inverse of A, $\|A^{-1}\|_\infty$, is 0.8. Thus, the condition number is calculated as:

$$\text{cond}_\infty(A) = 5 \cdot 0.8 = 4$$

Practical Task 5

Solution: We will use the Matrix Determinant Lemma to compute the determinant of $A+uv^T$, where A is the identity matrix with the last diagonal element being -6, and u and v are vectors with all ones.

The Matrix Determinant Lemma states that:

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)$$

where A is an invertible matrix, and u and v are column vectors. In our case, A is the identity matrix with the last diagonal element being -6, and u and v are vectors with all ones. Applying the lemma:

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)$$

Given that A is almost an identity matrix, its determinant is simply the product of its diagonal elements, which is -6. The term $v^T A^{-1} u$ adds up all elements of A^{-1} multiplied by the corresponding elements of u and v, which are all ones.

Therefore, the determinant is computed as:

$$\det(A + uv^T) = -29$$