Solutions NLA Midterm 2023 for Variant 2

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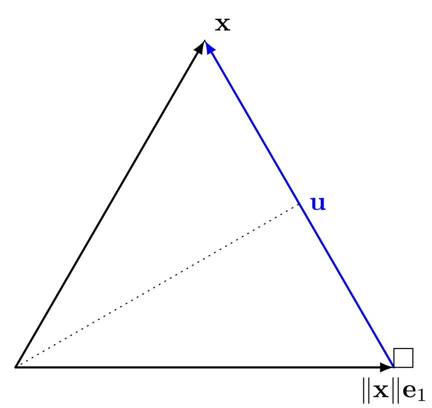
Theoretical Task 1

a) Identifying the Transformation

$\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T$

This transformation represents a **Householder reflection**.



b) Finding v vector

Given a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the goal is to find a vector $\mathbf{v}$ such that the Householder transformation $\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T$ will make $\mathbf{x}$ collinear with $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The Householder vector $\mathbf{v}$ is computed as follows:

$$\begin{aligned}
\|\mathbf{x}\| &= \sqrt{x_1^2 + x_2^2}, \\
\mathbf{u} &= \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1, \\
\mathbf{v} &= \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{(x_1 - \sqrt{x_1^2 + x_2^2})^2 + x_2^2}} \begin{pmatrix} x_1 - \sqrt{x_1^2 + x_2^2} \\ x_2 \end{pmatrix}.
\end{aligned}$$

Theoretical Task 2

Complexity of Matrix Multiplication: The straightforward computation of matrix multiplication for square matrices of size n has a complexity of $O(n^3)$.

Improved Complexity: This complexity can be improved. Strassen's algorithm, for instance, reduces it to $O(n^{2.8074})$. The best-known methods, as of the last update, have reduced the complexity to approximately $O(n^{2.3728596})$. These algorithms utilize advanced mathematical techniques, including divide-and-conquer strategies, to decrease the number of multiplication operations required.

Theoretical Task 3

We are to prove that for any vectors u and v such that $1+v^T u \neq 0$, the following identity holds:

$$(I + uv^T)^{-1} = I - \frac{uv^T}{1 + v^T u}.$$

Proof:

Let $A = uv^T$ and $\alpha = 1 + v^T u$. The matrix $I + A$ is invertible as $\alpha \neq 0$. We shall show that $(I + A)(I - \frac{A}{\alpha}) = I$:

$$\begin{aligned}
(I + A)(I - \frac{A}{\alpha}) &= I - \frac{A}{\alpha} + A - \frac{A^2}{\alpha} \\
&= I - \frac{A}{\alpha} + A - \frac{(\alpha - 1)A}{\alpha} \\
&= I.
\end{aligned}$$

The third equality follows from $A^2 = (uv^T)(uv^T) = u(v^T u)v^T = (\alpha - 1)uv^T = (\alpha - 1)A$, hence $\frac{A^2}{\alpha - 1} = A$. Therefore, $(I - \frac{A}{\alpha})$ is the inverse of $(I + A)$, and the identity is proven.

To show that $1 + v^T u \neq 0$ we can use Matrix determinant lemma.

N.B. The formula that needed to be proved is a special case of the Sherman–Morrison formula.

Theoretical Task 4

We aim to prove that for any matrix X, the following identity holds: $XX^+X = X$, where X^+ denotes the Moore-Penrose pseudoinverse of X.

Proof:

Given the Singular Value Decomposition (SVD) of X as $X = U\Sigma V^*$, where U and V are unitary matrices, and Σ is a diagonal matrix of singular values, the pseudoinverse X^+ is defined as $X^+ = V\Sigma^+U^*$. Then we have:

$$
\begin{aligned}
XX^+X &= (U\Sigma V^*)(V\Sigma^+U^*)(U\Sigma V^*) \\
&= U\Sigma(V^*V)\Sigma^+(U^*U)\Sigma V^* \\
&= U\Sigma\Sigma^+\Sigma V^* \\
&= U\Sigma V^* \\
&= X.
\end{aligned}
$$

The product $\Sigma\Sigma^+\Sigma$ simplifies to Σ, as Σ^+ is constructed to be the pseudoinverse of Σ, ensuring that non-zero singular values are reciprocated. Therefore, $XX^+X = X$, proving the identity.

Theoretical Task 5

We are to show that for any matrix A, the following inequality holds: $\|A\|_2^2 \leq \|A\|_F^2$, and identify when equality holds.

Proof:

The spectral norm $\|A\|_2$ is the largest singular value of A, denoted $\sigma_{\max}(A)$. The Frobenius norm $\|A\|_F$ is the square root of the sum of the squares of the singular values of A. Let the singular values of A be $\sigma_1, \sigma_2, \ldots, \sigma_r$. Thus, we have:

$$
\begin{aligned}
\|A\|_2^2 &= \sigma_{\max}^2, \\
\|A\|_F^2 &= \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2.
\end{aligned}
$$

Since $\|A\|_2$ is the largest singular value, $\|A\|_2^2 = \sigma_{\max}^2 \leq \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2 = \|A\|_F^2$. Equality holds if and only if all singular values except the largest one are zero, i.e., when A is a rank-1 matrix (or the zero matrix).

Practical Task 1

Question: Assume matrix A has singular value decomposition $A = U\Sigma V^*$. Derive the singular value decomposition of the block matrix $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ and explain why it exists.

Solution: Given the singular value decomposition $A = U\Sigma V$, we know that $A = V\Sigma^* U^*$. The block matrix can be represented as:

$$
\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & U\Sigma V^* \\ V\Sigma^* U^* & 0 \end{pmatrix}
$$

We can decompose this block matrix as:

$$
\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{pmatrix} \begin{pmatrix} 0 & V^* \\ U^* & 0 \end{pmatrix}
$$

This decomposition is valid as the matrices U and U^* are unitary, and the block matrix structure allows for this form of decomposition. The product of these matrices yields the original block matrix, establishing its singular value decomposition.

Practical Task 2

Given matrix A:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix},$$

we are asked to determine whether the power method converges when applied to A, and if it does, to discuss the convergence speed and the stationary point.
Solution:sub

The eigenvalues of A are calculated to be $2 + 2i$ and $2 - 2i$. For the power method to converge, there must be a dominant eigenvalue, which is not the case here as both eigenvalues have the same magnitude.

Conclusion:

The power method will not converge for matrix A as it lacks a dominant eigenvalue. The method relies on a dominant eigenvalue to ensure convergence to the corresponding eigenvector. With multiple eigenvalues of equal magnitude, the power method will not reliably converge to a single eigenvector.

Practical Task 3

Given the outer product of two vectors as matrix D:

$$D = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix},$$

we perform QR decomposition.

Solution:

The columns of D are linearly dependent, so the second column in the QR decomposition is manually chosen to be orthogonal to the first. The resulting QR decomposition is:

$$Q = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$R = \begin{pmatrix} 3\sqrt{2} & -2\sqrt{2} \\ 0 & 0 \end{pmatrix}.$$

Conclusion:

The QR decomposition of matrix D is obtained with Q being an orthogonal matrix and R being an upper triangular matrix, reflecting the linear dependency of the columns of D.

Practical Task 4

The Singular Value Decomposition (SVD) of the matrix A is:

$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{5}}{5} & 0 & -\frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix}.$$

Practical Task 5

Using the Matrix Determinant Lemma:

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)$$

Given $A = I$ (the identity matrix) and vectors u and v as vectors with all ones, the product $v^T u$ simplifies to n, where n is the dimension of A. Thus, we have:

$$\det(I + uv^T) = (1 + n) \det(I)$$

Since $\det(I)$ is 1, the determinant of $I + uv^T$ is $1 + n = 6$.