Connection and Curvature

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Problem 1.1.

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Problem 3.1. *Prove the second Bianchi identity in the natural frame:*

$$R_{ijkl,h} + R_{ijlh,k} + Rijhk, l = 0$$

Problem 3.2. *Suppose the Riemann curvature tensor R for a Riemannian manifold (M^m, g) satisfy the following:*

$$R(X,Y,Z,W) = \frac{1}{m-1}\{S(Y,Z)g(X,W) - S(Y,W)g(X,Z)\}$$

, where S is the Ricci tensor, and if $m \geq 3$, (M^m, g) has constant curvature.

Problem 3.3. *Suppose (M^3, g) is a three-dimensional Riemannian manifold. At any $p \in M^3$, take the coordinate system $\{x^i\}$, s.t. at p we have $g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = 0, i \neq j$. Prove that for $i, j, k \neq$ at p, the following holds:*

$$R_{ij} = \frac{1}{g_k k} R_{ikjk}$$

$$R_{ii} = \frac{1}{g_{jj}} R_{ijij} + \frac{1}{g_{kk}} R_{ikjk}$$

$$R_{ijij} - g_{ii} R_{jj} - g_{jj} R_{ii} + \frac{1}{2} \rho g_{ii} g_{jj}$$

Problem 3.4. *Suppose (M^m, g) is a connected Einstein manifold and $m \geq 3$,*

(i) *if $m = 3$, then (M^m, g) has constant curvature.*

(ii) *if the scalar curvature ρ of (M^m, g) does not vanish, then there is no parallel vector field on (M^m, g).*

Problem 3.5. *Suppose $M^2 \subset \mathbb{R}^3$ is an immersed surface, and has the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^3$, prove that the sectional curvature on M^2 is the Gauss curvature.*

Problem 3.6. *Calculate the sectional curvature, Ricci curvature and scalar curvature of the sphere*

$$S^m(r) = \left\{ x \in \mathbb{R}^{m+1} | \sum_i (x^i)^2 = r^2 (\text{constant}) \right\}$$

. The Riemannian metric on $S^m(r)$ is induced by the Euclidean metric on $\mathbb{R}^{m+1}$.

Proof. Let's elaborate on the Riemann curvature tensor. □

4 Harmonic Forms

Problem 4.1. *Suppose $\alpha \in A^1(M)$, then*

$$\int_M (\delta \alpha) \eta = 0$$

.

Proof. ∎

Problem 4.2. *Suppose M is compact smooth Riemannian manifold without boundary, $h, f \in C^2(M)$. Prove the following Green formula:*

$$\int_M (h\Delta f - f\Delta h)\eta = 0$$

.

Proof. Recall that $\Delta = -(\mathrm{d}\delta + \delta\mathrm{d})$. $h\Delta$.

According to theorem 3.4.4, we have $(\mathrm{d}\alpha, \beta) = (\alpha, \delta\beta)$.

we have

$$\int_D (h\Delta f)\eta = -\int_D \langle \mathrm{d}h, \mathrm{d}f\rangle\eta$$

∎

Problem 4.3. *Take the geodesic polar coordinates on (M, g):*

$$g = (\mathrm{d}r)^2 + g_{ij}(r,\theta)\mathrm{d}\theta^i\mathrm{d}\theta^j, 1 \leq i, j, k \leq m-1$$

. Suppose $f = f(r)$ is independent from $\{\theta^i\}$, prove that

$$\Delta f = f'' + \frac{m-1}{r}f' + f'\frac{\partial}{\partial r}\log\sqrt{G}$$

, where $G = \det(g_{ij})$ and $'$ denotes ordinary differentiation with respect to r.

Proof. ∎