Topological Spaces and Continuous Functions

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1 Basis for a Topology

2 The Subspace Topology

Problem 2.1. *A map $f : X \to Y$ is said to be an* ***open map*** *if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to X$ are open maps.*

Solution: ◀

Problem 2.2. *Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.*

Proof. We would like to show that there exist a homeomorphism between these two spaces. That their open subsets correspond or contain each other.

Let's recall the definition of dictionary order topology. It is a kind of order topology. There is no largest or smallest element in $\mathbb{R} \times \mathbb{R}$ so the only elements in the basis is of the type form (a, b). Write $a = (x_a, y_a), b = (x_b, y_b)$, we have $x_b \geq x_a$ an d if $x_b = x_a$ then $y_b > y_a$.

And recall the definition of the product topology. For a finite index set, the basis of the product consists of products of basis, which are further reduced to $\{x\} \times U$ where U is in the basis of $\mathbb{R}$, which should be an open interval (y, y'). Such kinds of open set is contained in the basis of the dictionary order topology, as it's just $((x, y), (x, y'))$. With this kind of open sets we are able to constitute $x \times (-\infty, y), x \times (y, +\infty), x \times (-\infty, +\infty)$ and then easily other kind of open sets in the basis of the dictionary order topology.

Are there any geometric intuitions behind the scenes?

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