Segway - inverse pendulum

and what math tools you need to simulate it

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Motivation

Segway

- A Segway PT - short for Segway Personal Transporter - is a two-wheeled, self-balancing personal transporter.

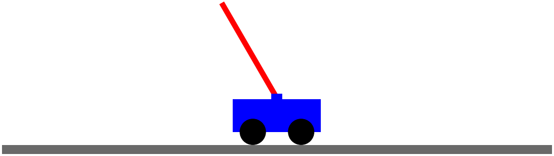
Source: http://de-de.segway.com/

- How does it balance itself?

Motivation

Goals

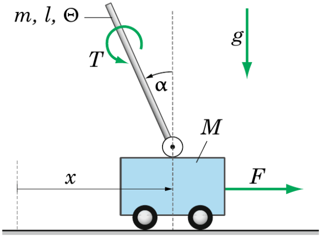
- We want to understand the function of the Segway by simulating the simplified model of an inverse pendulum on a moving platform:



- Throughout the presentation we will pay particular attention to the **math tools** needed for the derivation and simulation of the model.

Model

- The model, as depicted in *Das Mathe-Praxis-Buch*, J. Härterich, A. Rooch, 2014, Springer Verlag:



Source: Das Mathe-Praxis-Buch, p. 4, Abb. 1.1

- With M mass of the cart, m mass of the pendulum, l length of the pendulum, α angular displacement of the pendulum, g gravitional force, F force applied on the cart, x position of the cart, T disturbance (e.g. wind, pushing), Θ moment of inertia.

Equations of motion

Derivation

- To obtain equations of motion for such a system we use **Lagrangian mechanics**.
- For our inverse pendulum we obtain:

$$\ddot{x} = -\frac{ml}{M+m}\dot{\alpha}^2 \sin(\alpha) + \frac{ml}{M+m}\ddot{\alpha}\cos(\alpha) + \frac{1}{M+m}F$$
$$\ddot{\alpha} = \frac{3}{4l}\ddot{x}\cos(\alpha) + \frac{3g}{4l}\sin(\alpha) + \frac{3}{4ml^2}T$$

- To reduce writing overhead we simplify:

$$\ddot{x} = c_0\dot{\alpha}^2 \sin(\alpha) + c_1\ddot{\alpha}\cos(\alpha) + c_2 F$$
$$\ddot{\alpha} = c_3\ddot{x}\cos(\alpha) + c_4\sin(\alpha) + c_5 T$$

- We obtain a **system of coupled nonlinear ordinary differential equations of second order**.
  - **ordinary differential equation** - equation contains an entity u and its ordinary derivatives e.g. $\ddot{u}$, $\frac{\partial u}{\partial x}$
  - **nonlinear** - equation contains nonlinear expressions e.g. $\dot{u}^2$, $\sin(u)$
  - **second order** - highest occuring derivative
  - **system** - more than one equation
  - **coupled** - equation regarding $\ddot{u}$ contains entities, that are not u or one of its derivatives
- Such an equation is in general not solveable by analytical means only.

Equations of motion

Partly decoupling

- By inserting equation 1 in equation 2 and vice versa we obain "less" coupled equations:

$$\ddot{x} = \big(c_0\dot{\alpha}^2\sin(\alpha) + c_1c_4\sin(\alpha)\cos(\alpha) + c_2F + c_1c_5T\big)/\big(1 - c_1c_3\cos^2(\alpha)\big)$$
$$\ddot{\alpha} = \big(c_0c_3\dot{\alpha}^2\sin(\alpha)\cos(\alpha) + c_4\sin(\alpha) + c_2c_3\cos(\alpha)F + c_5T\big)/\big(1 - c_1c_3\cos^2(\alpha)\big)$$

- Now our system is more convenient but still highly **nonlinear**.

Linearization

- To simplify our model, we want to remove any nonlinear term. This procedure is called **linearization**.
- It relies heavily on approximation and therefor resulting systems only have a limited validity.
- We use **Taylor's theorem**, more correct **Taylor polynomials of first order**, to approximate the terms:

$$\dot{\alpha}^2 \approx 0$$
$$\sin(\alpha) \approx \alpha$$
$$\cos(\alpha) \approx 1$$

Equations of motion

Linearized

- After replacing nonlinear parts with linear approximations we obtain:

$$\ddot{x} = \big(c_1 c_4 \alpha + c_2 F + c_1 c_5 T\big) / \big(1 - c_1 c_3\big)$$
$$\ddot{\alpha} = \big(c_4 \alpha + c_2 c_3 F + c_5 T\big) / \big(1 - c_1 c_3\big)$$

- It is necessary to realize how small the margin is in which the approximation - and therefor the linear model - holds.
- Now our system is linear but still of **second order**.
- There are at least two reasons to **reduce the order** of the system of differential equations:
  - We are interested in the position x of the cart as well as the angular displacement of the pendulum α on it. So far the equations describe only the change of the velocity $\ddot{x}$ of the cart and the change of the angular velocity $\ddot{\alpha}$ of the pendulum.
  - We will later be able to apply iterative methods to solve the system numerically e.g. the **classical Runge-Kutta method**.

System of differential equations

Reducing order

- The entities we want to observe are $\vec{x} = \left(x, \dot{x}, \alpha, \dot{\alpha}\right)^T$. The time **derivative** is $\dot{\vec{x}} = \left(\dot{x}, \ddot{x}, \dot{\alpha}, \ddot{\alpha}\right)^T$.
- By appending the system with the equations

$$\dot{x} = \dot{x}, \quad \text{and} \quad \dot{\alpha} = \dot{\alpha}$$

  we obtain

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{\alpha} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{c_1 c_4}{1 - c_1 c_3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{c_4}{1 - c_1 c_3} & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \alpha \\ \dot{\alpha} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{c_2}{1 - c_1 c_3} \\ 0 \\ \frac{c_2 c_3}{1 - c_1 c_3} \end{pmatrix} F + \begin{pmatrix} 0 \\ \frac{c_1 c_5}{1 - c_1 c_3} \\ 0 \\ \frac{c_5}{1 - c_1 c_3} \end{pmatrix} T$$

  Keep in mind that F and T are **scalars** in this **one-dimensional** model problem.
- In **matrix notation**

$$\dot{\vec{x}} = A\vec{x} + \vec{b}_1 F + \vec{b}_2 T$$

  it is more visible that we now have a **system of linear differential equations of first order**.

System of differential equations

Analytical solution

- When omitting F and T we restrict external influence T or a force F upon the cart. We then have the opportunity to **solve** the system

$$\dot{\vec{x}} = A\vec{x}$$

  by means of **analytical methods**.
- The solutions to the system will be a **linear combination** of

$$\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i,$$

  where λ_i are the **eigenvalues** and $\vec{v}_i$ are the **eigenvectors** of the systems matrix A.
- When A is **diagonalizable**, creating the solutions via linear combination is trivial. Unfortunately our A is **not diagonalizable**:
  - When the eigenvalues of A are **complex** we get real solutions via the **real** and the **imaginary part** of the complex solutions.
  - When the eigenvalues of A are real, but their **algebraic multiplicity** differs from their **geometric multiplicity** we have to find **generalized eigenvectors** to append our solutions with.

Eigenvalues and Eigenvectors

Eigenvalues

- Eigenvalues are the roots of the **characteristic polynomial** of A, which is computed by the **determinant**:

$$\det(A - \lambda I) = \mathcal{P}(\lambda).$$

- While the determinant of matrices up to the size 3×3 can easily be obtained via the rule of Sarrus, for higher-dimensional matrices we use **Laplace's formula**.
- For (artificial) concrete values $M = 0.5\text{kg}$, $m = 1.75\text{kg}$, $l = 0.5\text{m}$ and $g = 10\text{m}/\text{s}^2$ we get $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = -6,\ \lambda_4 = 6.$

Eigenvectors

- Eigenvectors are the solutions to the **homogeneous system of linear equations**

$$(A - \lambda_i I)\vec{v}_i = 0$$

- For λ_3 and λ_4 we find the related $\vec{v}_i$. For $\lambda_{1,2}$ we additionally have to find a generalized eigenvector $\vec{w}$.

System of differential equations

Analytical solution

- The solution has the form:

$$\begin{aligned}\vec{x}(t) &= d_1\vec{v}_1 e^{\lambda_{1,2}t} + d_2(t\vec{v}_1 + \vec{w})e^{\lambda_{1,2}t} + d_3\vec{v}_3 e^{\lambda_3 t} + d_4\vec{v}_4 e^{\lambda_4 t} \\ &= \begin{pmatrix} d_1 - 7c_4e^{-6t} + 7d_3e^{6t} + d_2(1+t) \\ d_2 + 42c_4e^{-6t} + 42d_3e^{6t} \\ -18d_4e^{-6t} + 18d_3e^{6t} \\ 108d_4e^{-6t} + 108d_3e6t \end{pmatrix}.\end{aligned}$$

- Due to the natur of the **e-function** it is obvious that $e^{\lambda t}$ will be **unconstrained** as long as $\lambda > 0$ and tend to $+\infty$, for growing t.
- It is necessary to find a way to **control the eigenvalues** of A.

Control

Control matrix

- We use a **linear control approach** $F = -K\vec{x}$ with a matrix K, that is effectively a vector - in our one-dimensional case - and get

$$\begin{aligned}\dot{\vec{x}} &= A\vec{x} + \vec{b}_1 F = A\vec{x} - \vec{b}_1 K\vec{x} \\ &= (A - \vec{b}_1 K)\vec{x} = A'\vec{x}.\end{aligned}$$

- By correct choice of K we can control the eigenvalues of A'. This procedure is called **pole placement**. In MATLAB this method is provided in a function called place().
- If you are interested in how this method works have a look at *Robust Pole Assignment in Linear State Feedback*, J. Kautsky, N.K. Nichols, and P. Van Dooren, International Journal of Control, 41 (1985), pp. 1129-1155.

Simulation

Iterative method

- There are many **explicit** and **implicit** iterative methods to solve systems of linear ordinary differential equations of first order - and therefor single linear differential equations of first order as well. The simplest is the **Euler method**.
- We use the **classical Runge-Kutta method**, due to its good properties in **stability** and **stiffness**:

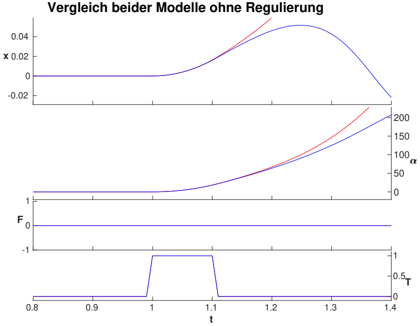
$$\begin{aligned}
\vec{k}_1 &= \vec{f}(t_n, \vec{y}_n), \\
\vec{k}_2 &= \vec{f}(t_n + \frac{1}{2}h, \vec{y}_n + \frac{1}{2}h\vec{k}_1), \\
\vec{k}_3 &= \vec{f}(t_n + \frac{1}{2}h, \vec{y}_n + \frac{1}{2}h\vec{k}_2), \\
\vec{k}_4 &= \vec{f}(t_n + h, \vec{y}_n + h\vec{k}_3), \\
\vec{y}_{n+1} &= \vec{y}_n + \frac{1}{6}h(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4).
\end{aligned}$$

- Figuratively speaking, we obtain the new **descent direction** as a weighted average of four different descent directions.
- The properties of such a method need to be **proven mathematically**.

Solution

Compare linear and nonlinear case

- We compare the numerical solutions of the linear (red) and the nonlinear (blue) case without control:

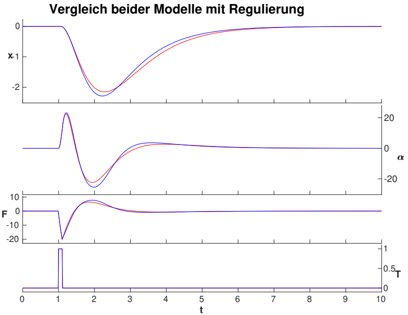


- We observe two characteristics:
  - Up until a value for α, there is barely a difference between the linear and nonlinear cases - as we expected when we linearized the system.
  - The solution for the linear case tends to infinity - as we expected due to the positive eigenvalue of A.

Solution

Compare linear and nonlinear case

- We compare the numerical solutions of the linear (red) and the nonlinear (blue) case with control:

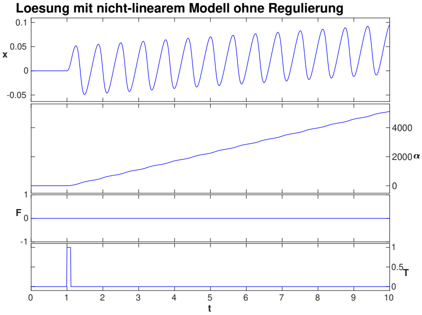


- We observe only slight differences between the two cases that minimize over time, due to the control we applied.

Solution

Nonlinear case

- We review the numerical solution of the nonlinear (blue) case without control:



- We observe that the momentum T we applied to the system reflects in the overall movement of the cart over time.

Sources

- *Das Mathe-Praxis-Buch*, J. Härterich, A. Rooch, 2014, Springer Verlag
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- *The dynamics of a Mobile Inverted Pendulum (MIP)*, S. Ostovari, N. Morozovsky, T. Bewley, 2012, UCSD