AI Assignment # 3

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1) Given:

$P(M = Air \cup M = Train) = 0.80$
$P(P = Business \mid M = Air) = 0.20$
$P(P = Leisure \mid M = Air) = 0.30$
$P(P = Leisure \mid M = Train) = 0.40$
$P(M = Car \cap S = High) = 0.25$
$P(M = Bus \cap S = Low) = 0.015$
$P(P = Business \mid M = Bus) = 0.350$
$P(M = Air \mid S = High) = 0.60$
$P(M = Train \mid P = Business)$
$= P(M = Train \mid P = Leisure) = 0.50$

Where → M is Mode of transport (Air, Bus, Car, Train)
P → Purpose (Business, Leisure)
S → Stress Level (High, Low)

a) (i) Direct Sampling: $P(x) = count(x)/N$: $O(N)$

The strengths of Direct Sampling are:

* It is easy to implement.
* It is well-suited for simple probabilistic distributions
* It gives unbiased estimates

The weaknesses are:

* It is less efficient for rare event.
  For eg: $P(m = Bus \cap S = Low) = 0.015$

* Large sample sizes may be required for accurate estimates

(ii) Rejection Sampling : Accept sample with probability

$P(\text{Accept}) = \min(1, q(x)/M\,p(x))$

where M is proposal distribution constant efficiency

$$M = \frac{\text{Area under target}}{\text{Area under proposal}}$$

The strengths of Rejection Sampling are:

* It is good for conditional probabilities
* It can handle complex distributions

The weaknesses are:

* It may be insufficient if the rejection rate is high
* It may waste samples for constraints like "80% prefer air or train"

(iii) Gibbs Sampling : For each iteration

$P(M_{i+1} \mid P_i, S_i) \cdot P(P_{i+1} \mid M_{i+1}, S_i) \cdot P(S_{i+1} \mid P_{i+1}, M_{i+1})$

The strengths of Gibbs sampling are:

* It is efficient for high-dimensional data having conditional probabilities.
* It is effective for estimating joint distributions.

The weaknesses are:

* It requires specification of conditional probabilites
* It is more complex to implement.

In the context of the given travel dataset, direct sampling is the most suitable approach, as the available information enables direct calculation of the relevant probabilities. This makes it straightforward to determine the joint probability distribution.

b) No. of trains = 30

$N = 100$ (total sample size)

$P(P = \text{Leisure} \mid M = \text{train}) = 0.400$

Expected no. of people $= 30 \times 0.400 = 12$

Using Bayes Theorem:

$P(M = \text{Train} \cap P = \text{Leisure})$

$= P(P = \text{Leisure} \mid M = \text{Train}) \times P(M = \text{Train})$

$= 0.400 \times \frac{30}{100} = 0.12$

$= 12\%$

c) Using Bayes Rule for air and business:

$P(M = \text{Air} \cap P = \text{Business}) = P(M = \text{Air}) \cdot P(P = \text{Business} \mid M = \text{Air})$

$= 0.80 \times 0.20$

$= 0.16$

d) The effects of increasing sample size are:
(i) Accuracy Improvement: Accuracy improvement can be explained through the application of the law of large numbers, which states that as the sample size increases, estimates tend to converge more closely to the true value. This convergence occurs because larger samples effectively reduce random variation, leading to more reliable and precise outcomes.

(ii) Precision Improvement: This is marked here by narrower confidence intervals, providing more reliable estimates of rare events, while also reducing the standard error, thereby enhancing the overall accuracy of the results.

(iii) For this dataset, there are better estimates of rare events, such as bus travel (1.5%), along with improved reliability in stress-related measurements. Additionally, the approach provides more precise conditional probability estimates, while balancing the trade-off between computational cost and efficiency.

2) (a) Initialization:

B → Person reads book

C → Person participates in book clubs

J → Person accesses academic journals

Then,

1) $P(B \cup J) = 0.91$
2) $P(J|B) = 0.400$

   $P(\neg J|B) = 0.600$
3) $P(C|B) = 0.320$
4) $P(J \cap \neg B) = 0.227$
5) $P(\neg B \cap \neg J) = 0.090$
6) $P(J|\neg B) = 0.716$
7) $P(C \cap J) = 0.088$
8) $P(C \cup J) = 0.631$
9) $P(J|C) = 0.4$
10) $P(J) = 0.5$
11) $P(C|\neg B) = 0.0044$

The list of priors is

1) $P(J) = 0.50$
2) $P(B) = 0.683$
3) $P(C) = 0.22$

d(a) To verify the correctness of the propositions, we assess whether they satisfy the axioms of probability:

i) $P(X) \geq 0$ for any event X
$P(X)$ satisfied for all 11 statements

ii) Sum of probabilities of all possible outcomes is 1

From statements ① and ⑤:
$P(B \cup J) = 0.91$, $P(\neg B \cap \neg J) = 0.090$

Then, $P(B \cup J) + \neg P(B \cup J) = 1$

$(\because \quad = P(\neg A \cap \neg B)$

$\therefore$ Axiom is satisfied by all the statements

iii) Law of mutually exclusivity:
$P(X \cup Y) = P(X) + P(Y)$ if events X and Y are independent

Take events B and J.
We would establish that
$(B \cup J)$ and $(\neg B \cap \neg J)$ can be mutually exclusive

Now, $P(B \cap J) = P(J|B) \times P(B)$
$= 0.4 \times P(B)$

$P(B \cup J) = P(B) + P(J) - P(B \cap J)$
$0.91 = 0.5 + P(B) - (0.4 \times P(B))$
$0.41 = 0.6 \times P(B)$
$P(B) = 0.6833$

Then,
$P(B \cup J) = 0.91$
$P(\neg B \cap \neg J) = 0.09$
$P(B \cup J)$ ~~and~~ $+ P(\neg B \cap \neg J) = 1$
They are mutually exclusive events

Similarly,

$$P(B\cap \neg J)+P(\neg B\cap J)+P(B\cap J)$$
$$= P(\neg J|B)\times P(B)+0.227+0.4\times P(B)$$
$$= 0.6\times P(B)+0.227+0.4\times P(B)$$
$$= 1.0\times P(B)+0.227$$
$$= P(B)+0.227$$
$$= 0.6833+0.227$$
$$= 0.910$$
$$= P(B\cup J)$$

$\therefore$ Axiom is satisfied for all the statements

(C) $P(B)=0.6833$

$P(J|B)=0.400$

Then, $P(B\cap J)=P(J|B)\times P(B)$

$= 0.4\times 0.6833$

$= 0.2732$

Independence of J and C

$$P(C\cap J)=P(C)\cdot P(J)$$
$$P(C)=P(C\cap B)+P(C\cap \neg B)$$
$$P(C\cap B)=P(C|B)\cdot P(B)$$
$$=0.320\times 0.6833=0.2186$$
$$P(C\cap \neg B)=P(C|\neg B)\cdot P(\neg B)=0.0014$$
$$=0.0044\times 0.317=$$

Then,

$$P(C)=0.2186+0.0014=0.220$$

So, $P(C\cap J)=0.220\times 0.5=0.110$

$\neq 0.088$

Therefore, (B and J), (C and J) are non-independent

(c) Using the following observations, we will now populate the Joint Probability Distribution Table.

B	C	J	$P(C \cap B \cap J)$
F	F	F	0.0896
F	F	T	0.2268
F	T	F	0.00039
F	T	T	0.0099
T	F	F	~~0.2188~~ 0.2788
T	F	T	0.1862
T	T	F	0.131
T	T	T	0.087

(d) checking conditional Independence for all random variables

(i) $P(C \wedge B \mid J) = \dfrac{P(C \wedge B \wedge J)}{P(J)}$

$= \dfrac{0.087}{0.5} = 0.174$

$P(C \mid J) = \dfrac{P(J \mid C) \cdot P(C)}{P(J)} = \dfrac{(0.4) \times (0.22)}{0.5}$

$P(B \mid J) = \dfrac{P(J \mid B) \cdot P(B)}{P(J)} = \dfrac{(0.4) \times (0.68)}{0.5}$

$\Rightarrow P(B \mid J) \times P(C \mid J) = 0.096$

$\rightarrow P(B \wedge C \mid J) \neq P(B \mid J) \cdot P(C \mid J)$

$\therefore$ They are not conditionally Independent with respect to J.

(ii) $P(B \wedge J \mid C) = \dfrac{P(B \wedge J \wedge C)}{P(C)} = \dfrac{0.087}{0.22}$

$= 0.3954$

$P(B \mid C) = \dfrac{P(C \mid B) \cdot P(B)}{P(C)} = \dfrac{0.32 \times 0.683}{0.22}$

$P(J \mid C) = 0.4$ (defined in part-(a))

$\Rightarrow P(B \wedge J \mid C) = P(B \mid C) \times P(J \mid C)$

$\therefore$ B and J are conditionally independent with respect to C

(iii) $P(J \wedge C \mid B) = \dfrac{P(B \wedge J \wedge C)}{P(B)} = \dfrac{0.087}{0.683}$

$= 0.128$

$P(J \mid B) \cdot P(C \mid B) = \cancel{0.4} \; 0.4 \times 0.32$

$= 0.128$

$\Rightarrow P(J \wedge C \mid B) = P(J \mid B) \times P(C \mid B)$

$\therefore$ J and C are conditionally independent with respect to B.

3) Take Random variables:
A: Adversarial Perturbations
B: Backdoor attacks

P(B): Probability adversial attack is a backdoor attack
P(A): Probability adversial attack is an adversial perturbation
P(M): Probability of missclassification

Using Bayes Rule :-

$$P(A|M) = \frac{P(M|A) \cdot P(A)}{P(M)}$$

$$P(M) = P(M|A) \cdot P(A) + P(M|B) \cdot P(B)$$

a) Given the random variables from above, for the Bayesian inference model

b) Prior probabilities: P(A), P(M), P(B) are defined.
Likelihood:
P(M/A) → probability of observing missclassification given that it was caused by adversarial perturbation
P(M/B) → probability of observing missclassification given that it was caused by backdoor attack

Posterior probability:
P(A|M) → posterior probability of adversial perturbation causing the missclassification

$P(B|M) \rightarrow$ posterior probability of backdoor attack causing the miss classification

(c) Initially A and B were considered independent events, the posterior probabilities of A given the missclassification was observed as

$$P(A|M) = \frac{P(M|A) \times P(A)}{P(M)}$$

$$P(M) = P(M|A) \cdot P(A) + P(M|B) \cdot P(B)$$

However, after the report about increasing prevalance of back door attacks, the information about likelihood of B changes. This implies that prior probablity of B, $P(B)$ increases. Thus, the posterior probability of A, $P(A|M)$ decreases.