Analytical Mechanics (Take Home Exam)

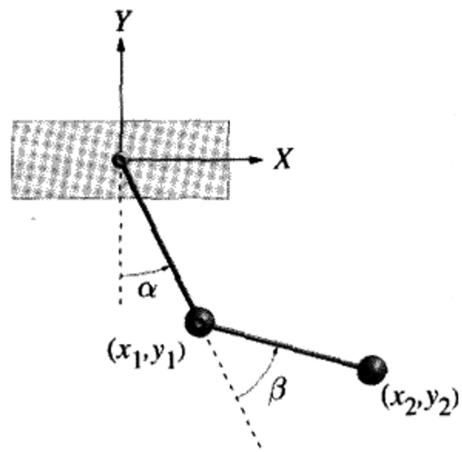
Aya Khaled Muhammed

January 2025

Contents

Take Home Exam Problem Statement

Simulate the chaotic behavior of the following double pendulum system.



Origin is at top pivot. Masses are both equal to m, lengths are both l.

Deriving Hamilton's Equations

Step 1: Describing the kinematics of the system

$$x_1 = l\cos\alpha \qquad\qquad x_2 = l\cos\alpha + l\cos(\alpha+\beta)$$
$$y_1 = -l\cos\alpha \qquad\qquad y_2 = -l\cos\alpha - l\cos(\alpha+\beta)$$

Step 2: Choosing the generalized coordinates

$$q_1 = \alpha \qquad\qquad\qquad q_2 = \beta$$

Step 3: Writing the Lagrangian Function

The Kinetic energy of the system is given by:

$$T = \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m(\dot{x_2}^2 + \dot{y_2}^2)$$

The potential energy of the system is given by:

$$V = mgy_1 + mgy_2$$

Therefore, the Lagrangian of the system is given by:

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2 + \dot{x_2}^2 + \dot{y_2}^2) - mgy_1 - mgy_2\end{aligned}$$

Step 4: Non-dimensionalization

By dividing the Lagrangian by mgl and changing the time scale:

$$t \to t\sqrt{\frac{g}{l}}$$

We get the following non-dimensional Lagrangian function:

$$\mathcal{L}(q_i, \dot{q_i}) = \cos(q_1 + q_2) + 2\cos(q_1) + \left(\dot{q_1}\dot{q_2} + \dot{q_1}^2\right)\cos(q_2) + \frac{1}{2}\dot{q_2}^2 + \dot{q_1}\dot{q_2} + \frac{3}{2}\dot{q_1}^2$$

Step 5: Obtaining the non-dimensional generalized momenta

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{q_1}} = (3 + 2\cos(q_2))\,\dot{q_1} + (1 + \cos(q_2))\,\dot{q_2} \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{q_1}} = (1 + \cos(q_2))\,\dot{q_1} + \dot{q_2} \end{aligned}$$

Step 6: Solving for the rates of change of the generalized coordinates

$$\begin{aligned} \dot{q_1} &= \frac{p_2\cos(q_2) + p_2 - p_1}{\cos(q_2)^2 - 2} \\ \dot{q_2} &= \frac{(p_1 - 2p_2)\cos(q_2) + p_1 - 3p_2}{\cos(q_2)^2 - 2} \end{aligned}$$

Step 7: Writing the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum_i p_i\dot{q_i} - \mathcal{L} \\ &= p_1\dot{q_1} + p_2\dot{q_2} - \mathcal{L} \\ &= \frac{(1 + \sin^2(q_2))(2\cos(q_1 + q_2) + 4\cos(q_1)) + 2(p_1p_2 - p_2^2)\cos(q_2) - 3p_2^2 + 2p_1p_2 - p_1^2}{2\cos(q_2)^2 - 4} \end{aligned}$$

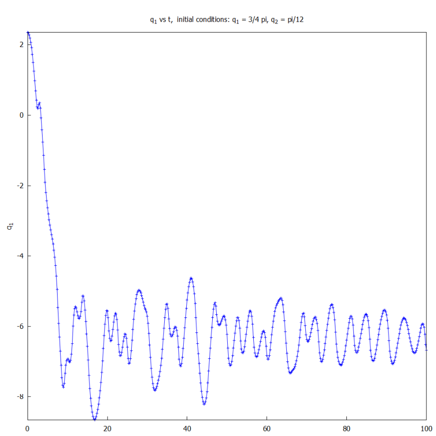
Step 8: Writing Hamilton's Equations

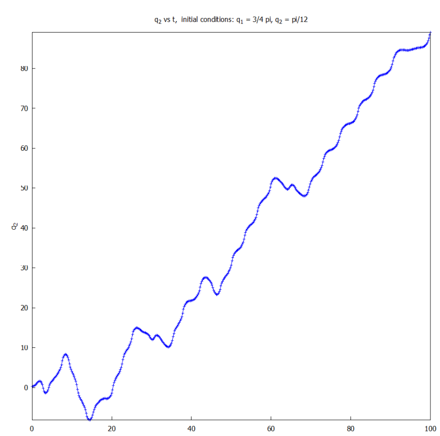
$$
\begin{aligned}
\dot{p_1} &= -\frac{\partial \mathcal{H}}{\partial q_1} \\
&= -\sin(q_1 + q_2) - 2\sin(q_1) \\
\dot{p_2} &= -\frac{\partial \mathcal{H}}{\partial q_2} \\
&= -\sin(q_1 + q_2) - \frac{(p_2^2 - p_1 p_2)\sin^3(q_2) + \left(3p_1p_2 - 3p_2^2 + (2p_1p_2 - p_1^2 - 3p_2^2)\cos(q_2)\right)\sin(q_2)}{\left(\sin^2(q_2) + 1\right)^2} \\
\dot{q_1} &= \frac{\partial \mathcal{H}}{\partial p_1} \\
&= \frac{p_2\cos(q_2) + p_2 - p_1}{\cos(q_2)^2 - 2} \\
\dot{q_2} &= \frac{\partial \mathcal{H}}{\partial p_2} \\
&= \frac{(p_1 - 2p_2)\cos(q_2) + p_1 - 3p_2}{\cos(q_2)^2 - 2}
\end{aligned}
$$

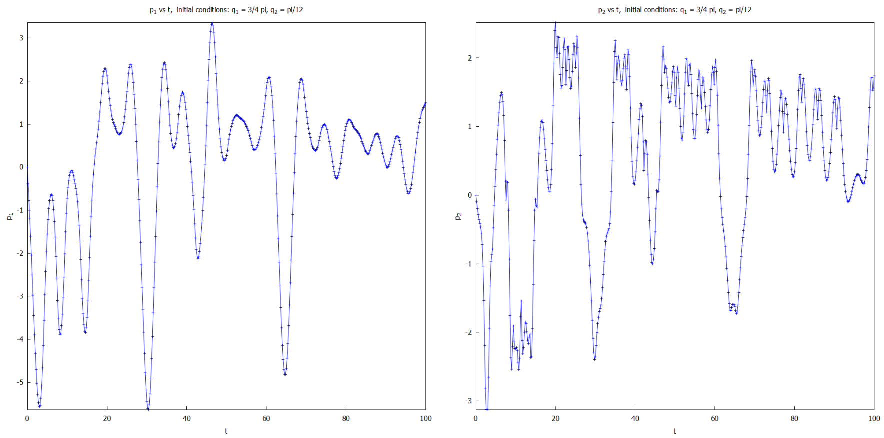
Numerically Solving Hamilton's Equations

Initial Conditions: $\alpha = \frac{3\pi}{4},\ \beta = \frac{\pi}{12}$

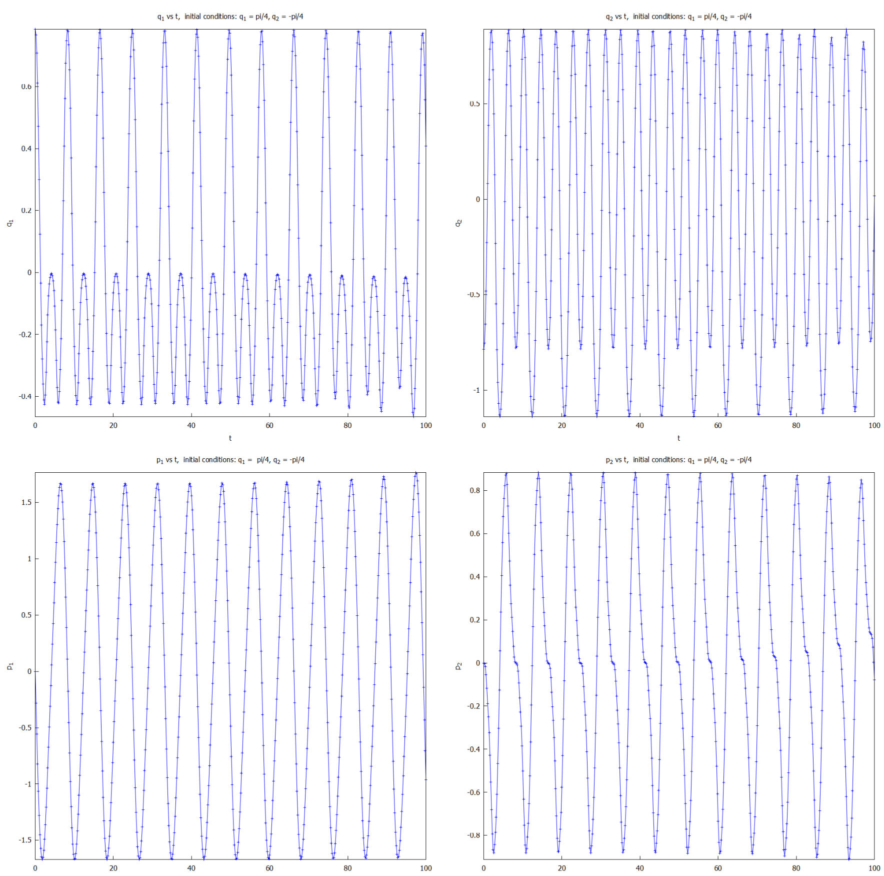
The system behavior is chaotic as shown:







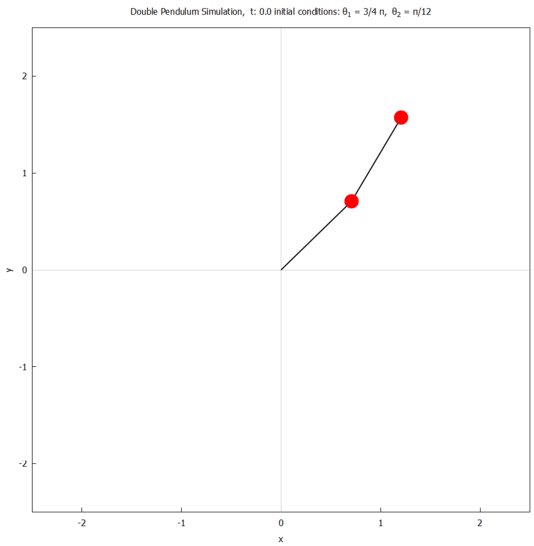
Initial Conditions: $\alpha = \frac{\pi}{4},\ \beta = -\frac{\pi}{4}$



Animating The Double Pendulum System

Initial Conditions: $\alpha = \frac{3\pi}{4},\ \beta = \frac{\pi}{12}$

Click the image to start the animation:



Initial Conditions: $\alpha = \frac{\pi}{4},\ \beta = -\frac{\pi}{4}$

Click the image to start the animation: