Insect Outbreak:
Searching for Patterns in Heterogeneous Environments

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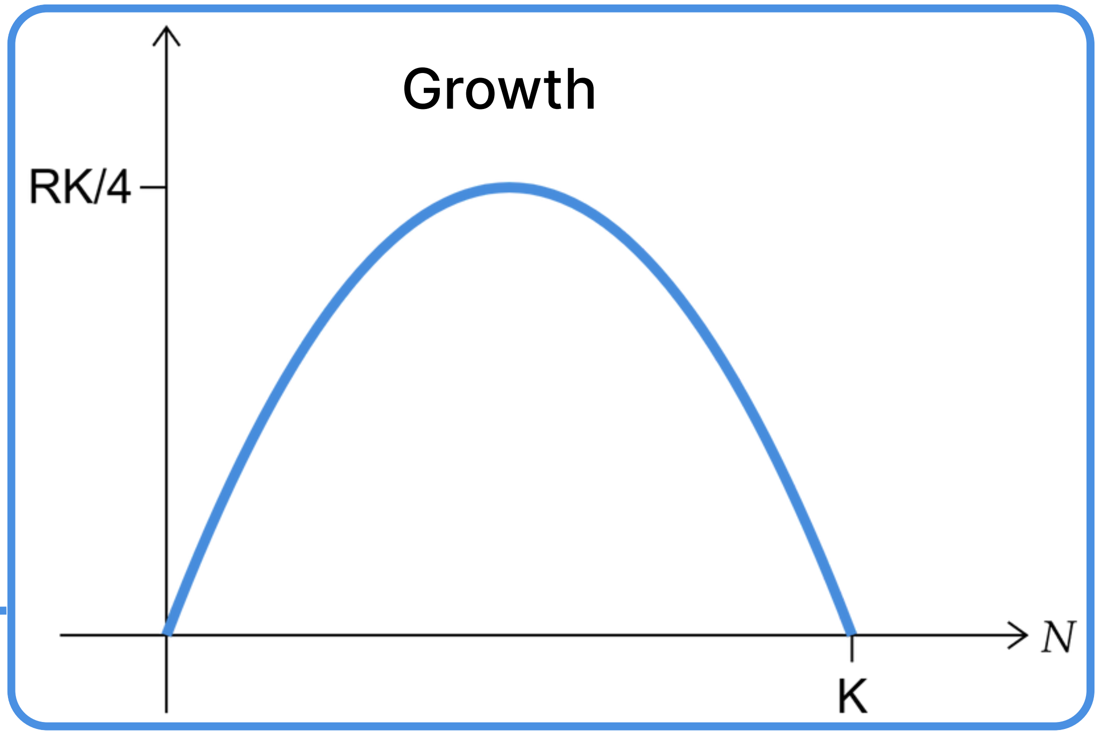
Explaining an Outbreak

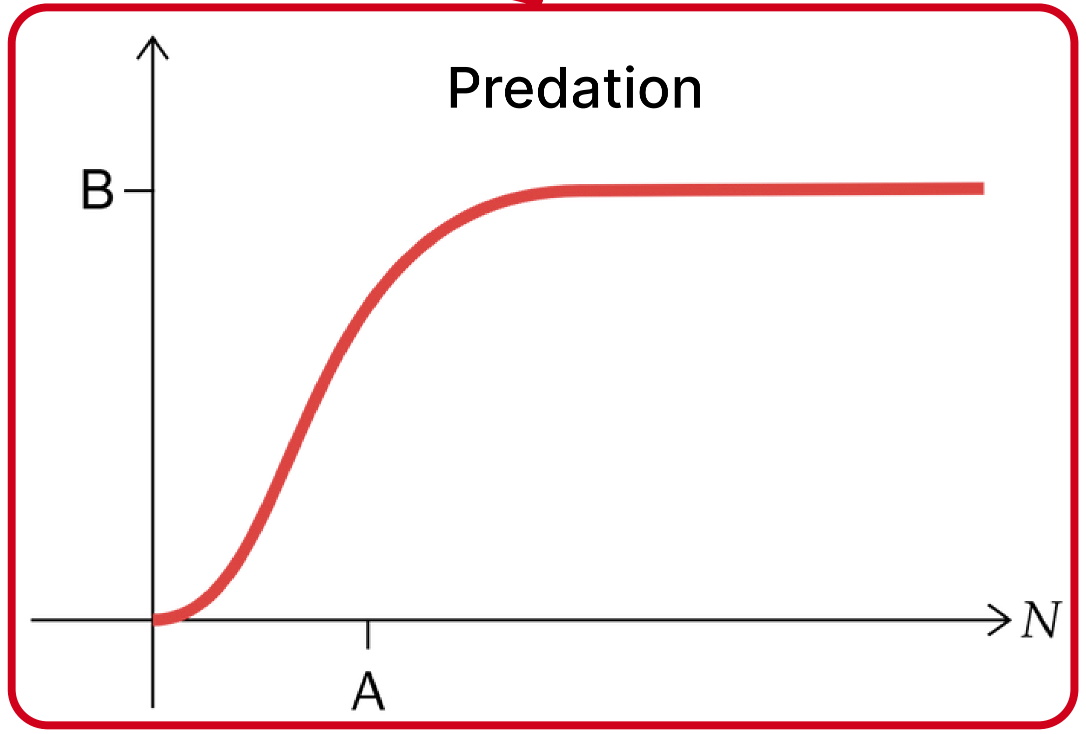
Can an insect outbreak be thought of as a phase transition?

- Gradual change of parameters over time (i.e. growth of the forest)
- Sudden jump from low (i.e. refuge) population to high (i.e. outbreak) population.

Ludwig et al. (1978) proposed a simple population dynamics model to explain this phenomenon in Canadian spruce budworm. This model assumed spatially homogeneous parameters.

$$\frac{dN}{dt} = \boxed{RN\left(1 - \frac{N}{K}\right)} - \boxed{\frac{BN^2}{A^2 + N^2}}$$





Dimensionless Formulation

To simplify our analysis, we can reduce the number of parameters by non-dimensionalizing the equation:

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{k}\right) - \frac{u^2}{1 + u^2}$$

Where

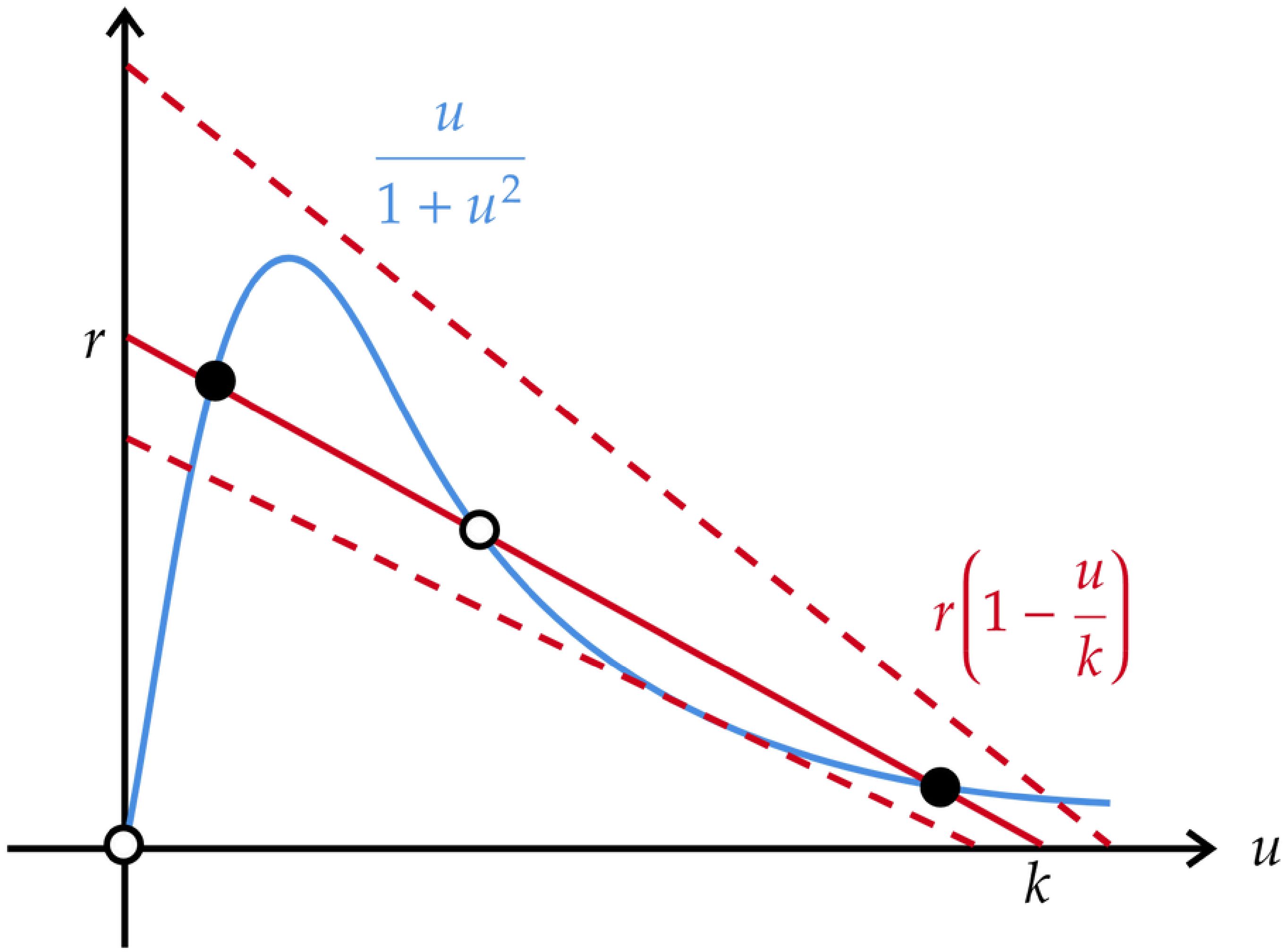
$u = \frac{N}{A}$ → Rescaled Population

$\tau = \frac{BT}{A}$ → Rescaled Time

$r = \frac{AR}{B}$ → Rescaled Growth Rate

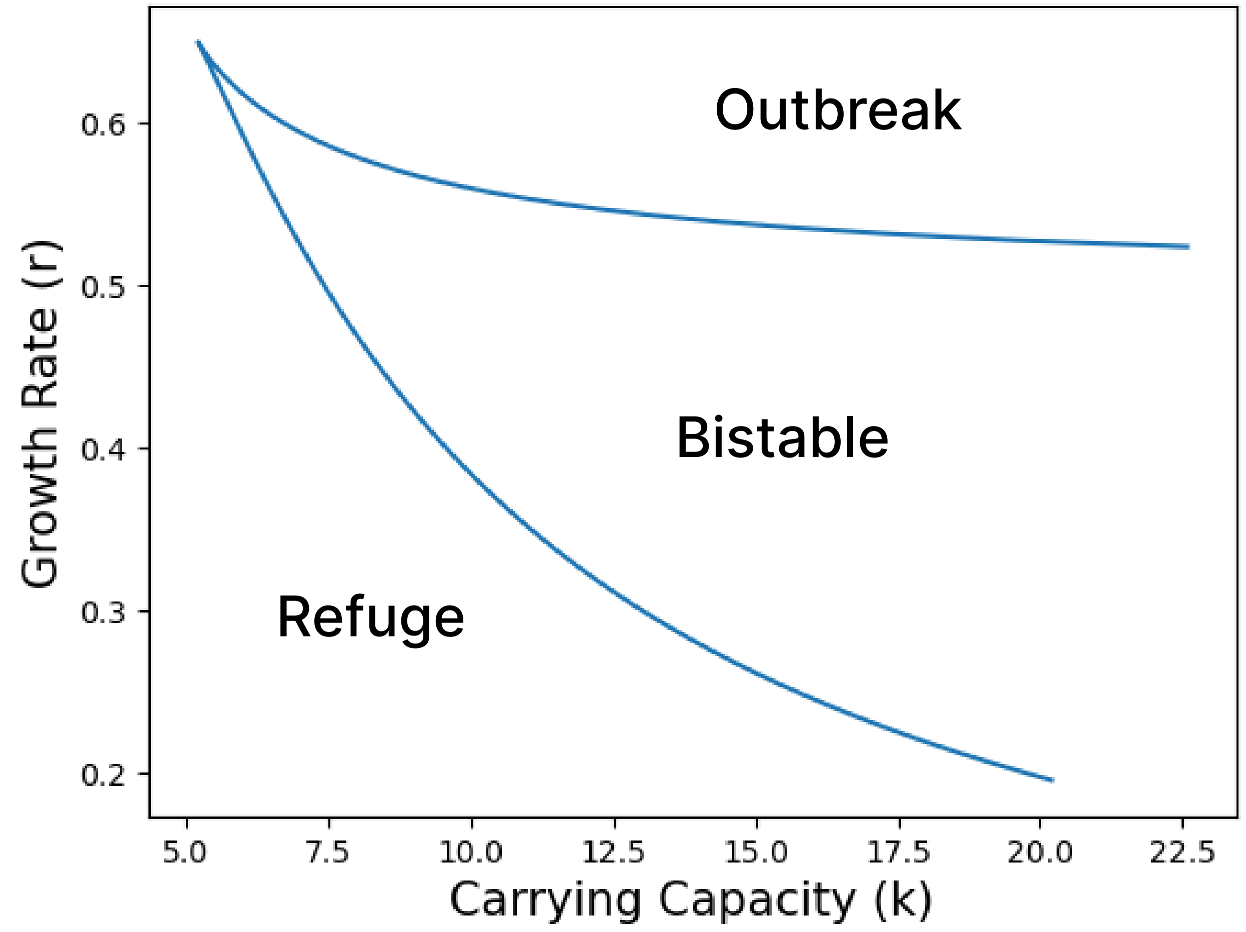
$k = \frac{K}{A}$ → Rescaled Carrying Capacity

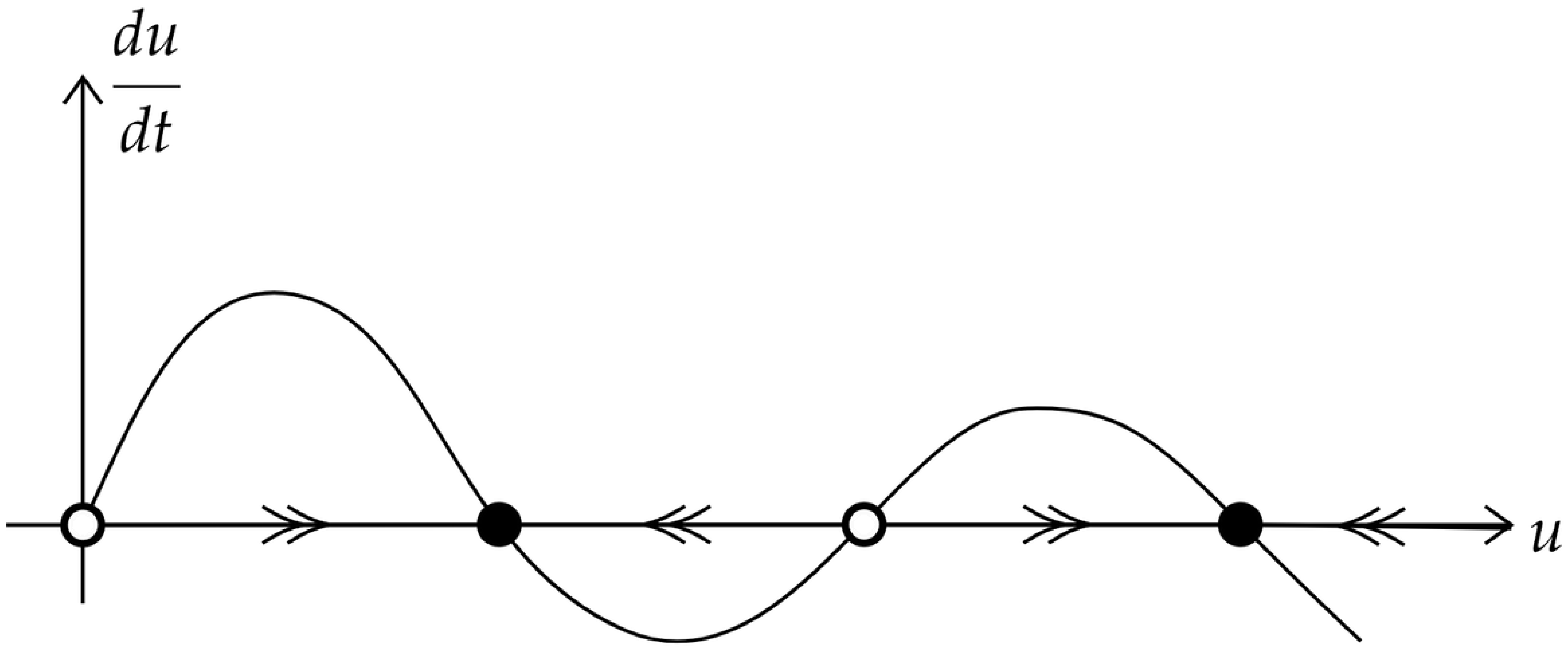
Fixed Point Analysis



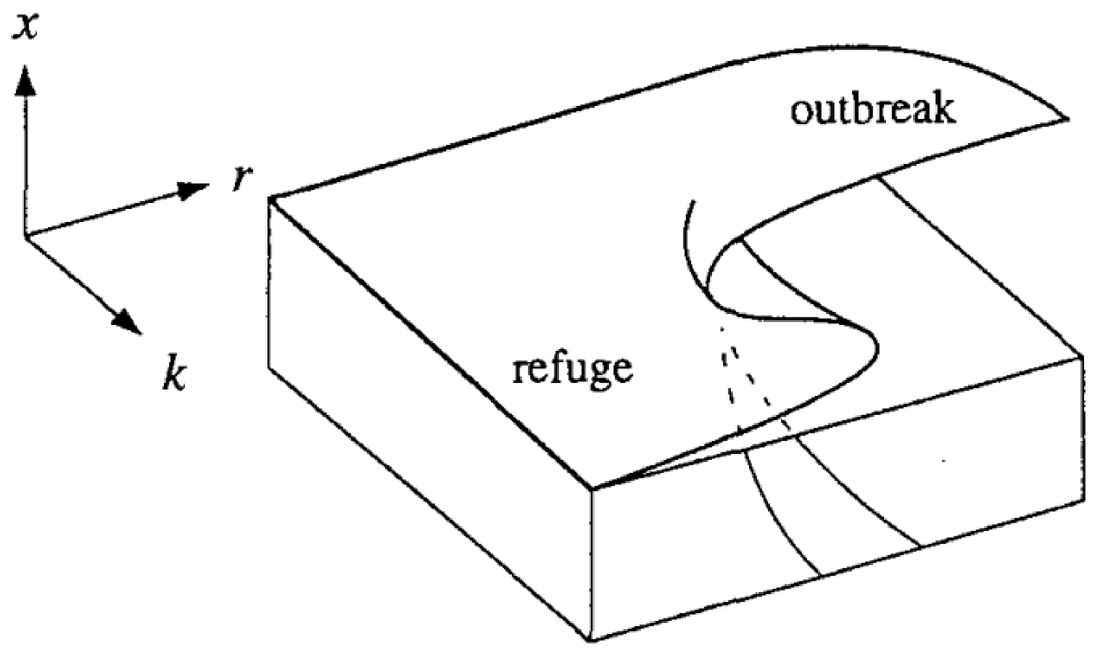
For long-term behaviour, we look for the **fixed points** (i.e. du/dt = 0). We find that we can get upto 3 fixed points, depending on the values of *r* and *k*.

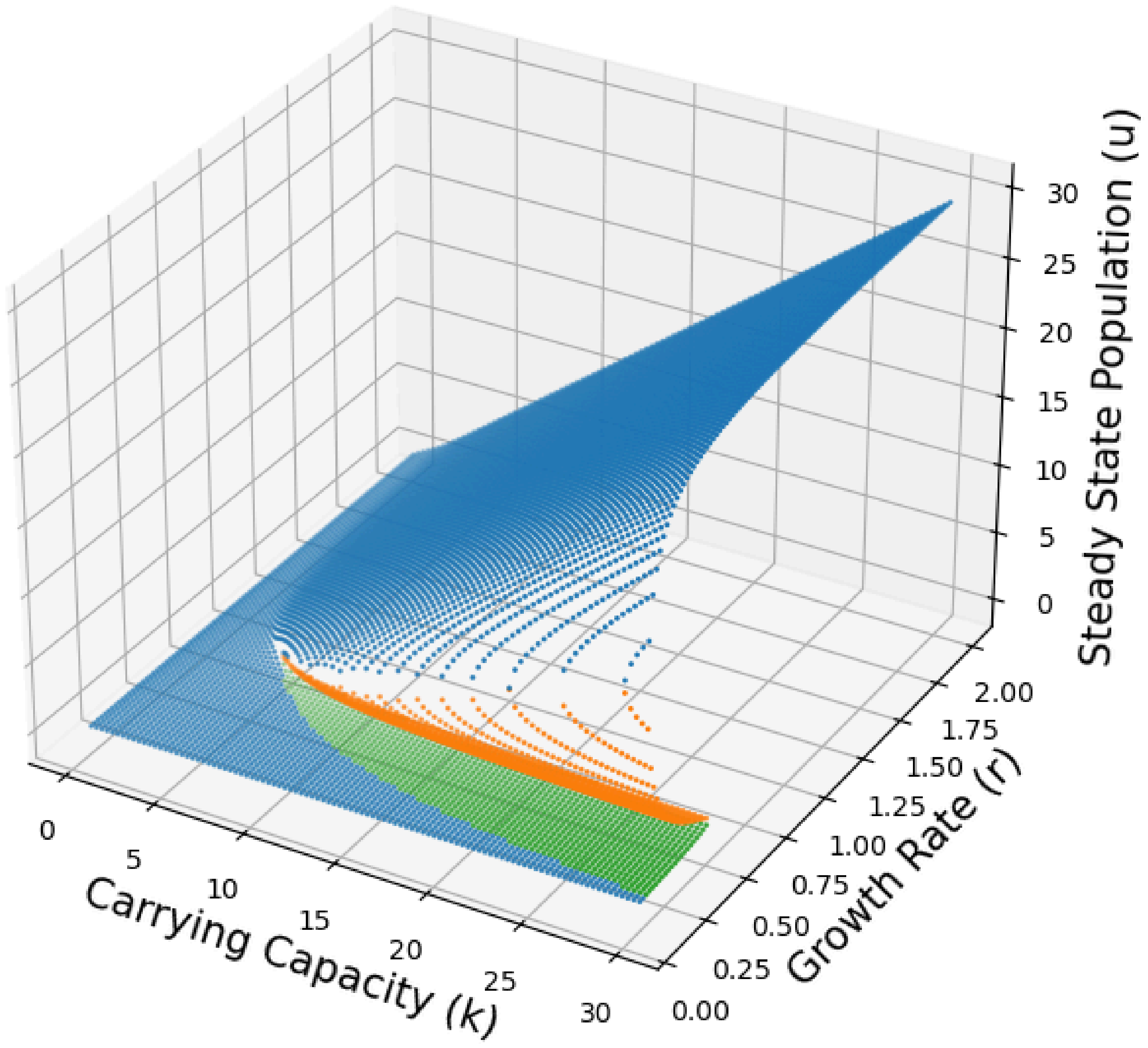
$$\frac{du}{d\tau} = 0 \Longrightarrow r\left(1 - \frac{u}{k}\right) = \frac{u}{1+u^2}$$

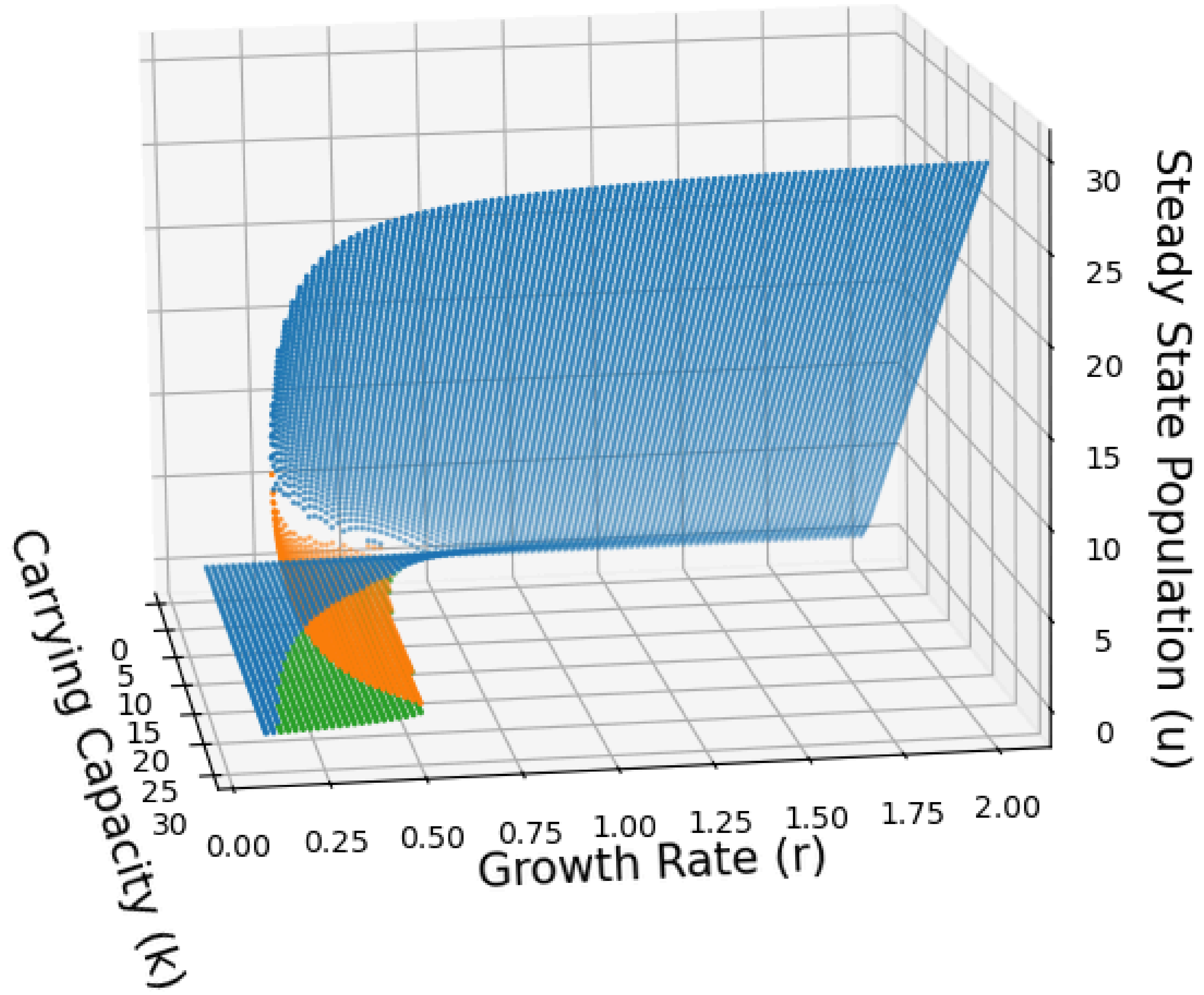




Cusp Catastrophe Curve





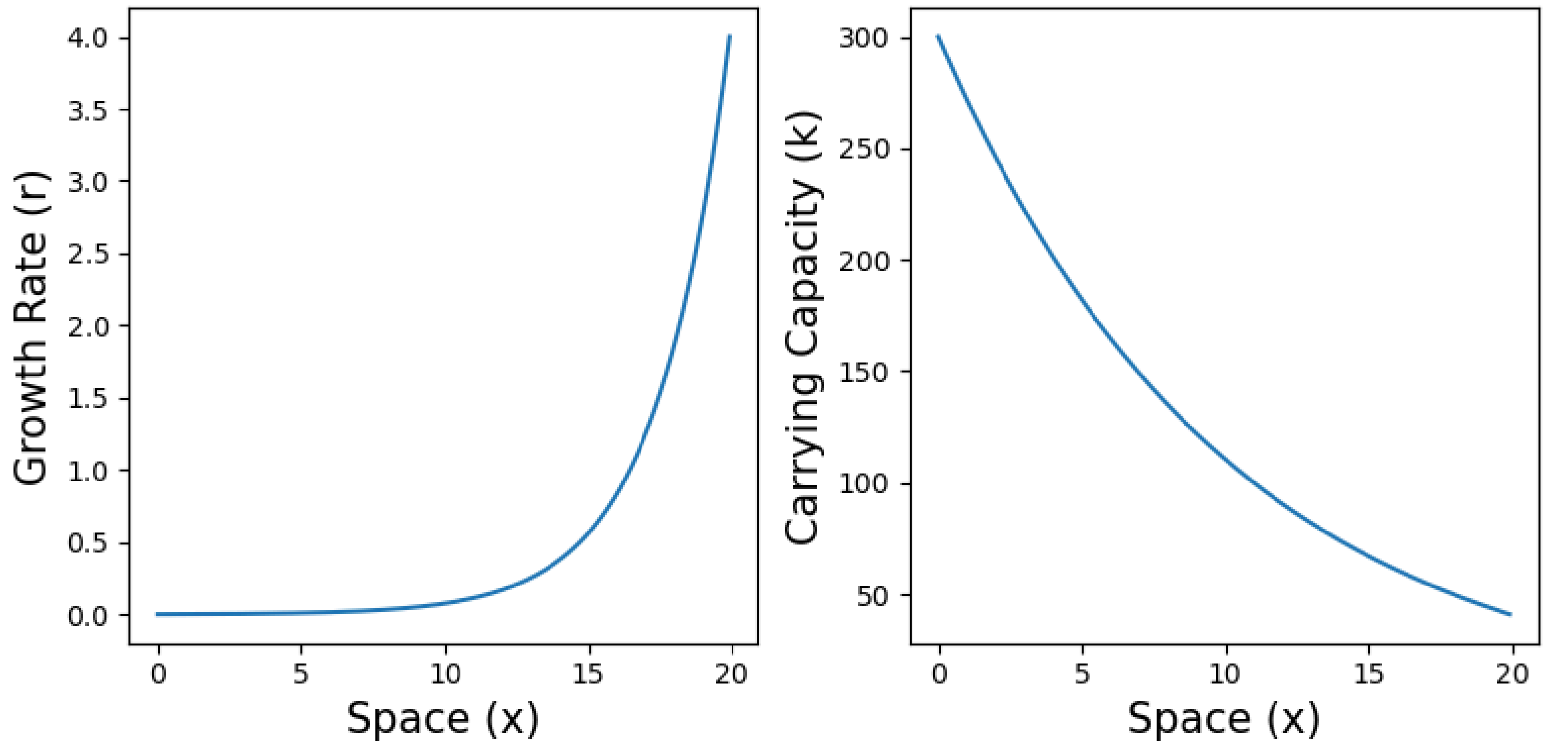


Introducing Spatial Heterogeneity

Natural environments are rarely homogenous. Heterogeneities might exist due to random events or external forces.

Spatial gradients in resources might influence how populations grow and even lead to patterns.

- Sunlight gradient at the surface of the ocean
- Nutrient gradient around a food source
- Chemical gradient inside a developing embryo



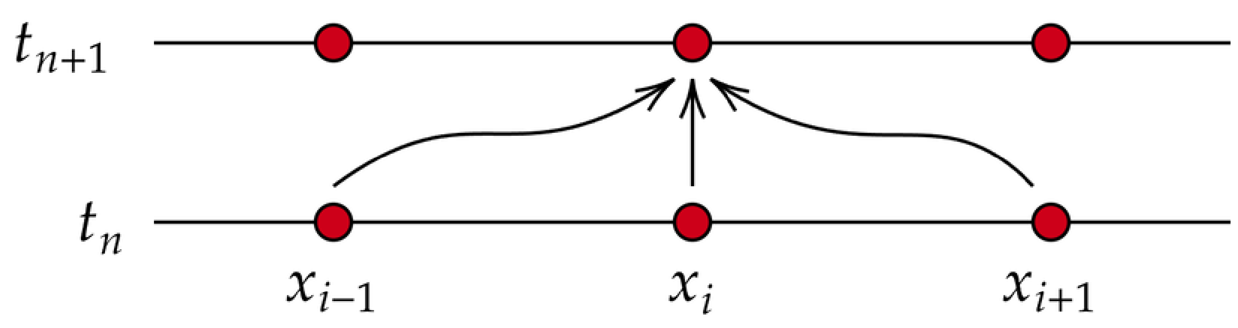
$$\frac{\partial u}{\partial \tau} = \underbrace{D\frac{\partial^2 u}{\partial x^2}}_{\text{Diffusion}} + \underbrace{r(x)u\left(1 - \frac{u}{k(x)}\right) - \frac{u^2}{1+u^2}}_{\text{Reaction}}$$

$$r(x) = r_0 \exp(-r_1 x)$$

$$k(x) = k_0 \exp(-k_1 x)$$

Numerically Solving RDEs

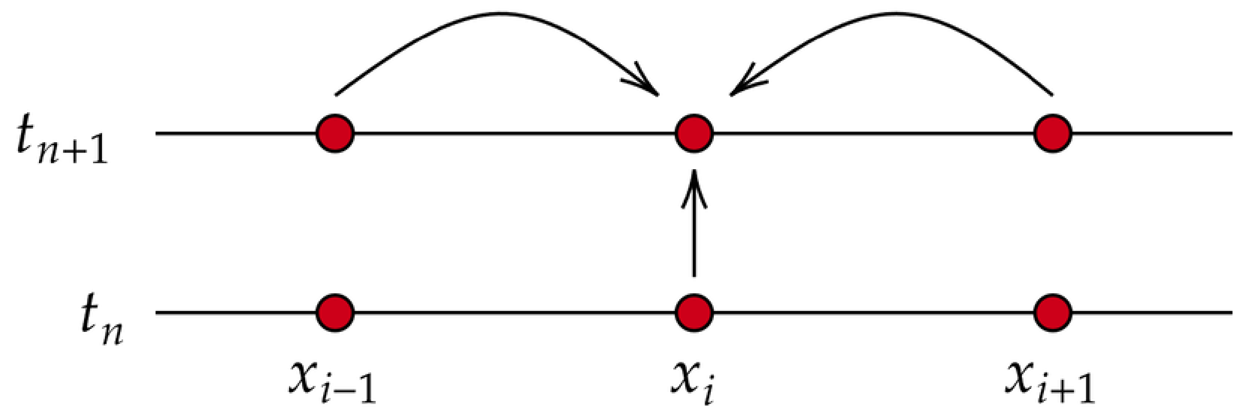
Explicit Scheme



Pros:
- Easy to implement
- Faster to compute

Cons:
- Has limited stability
- Boundary effects lag by 1 step

Implicit Scheme



Pros:
- Much more stable than explicit
- No lag at boundary

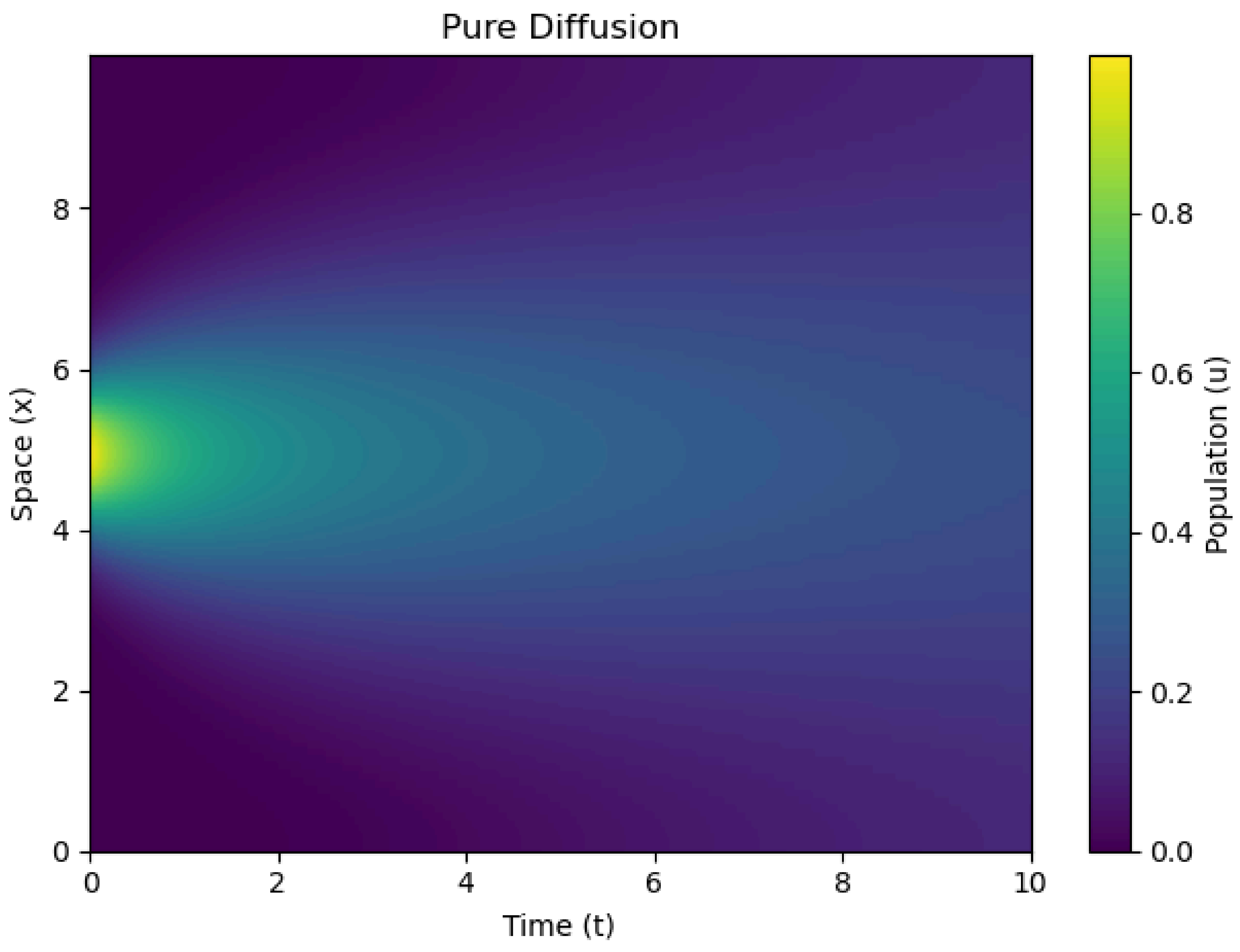
Cons:
- Difficult to implement (especially for nonlinear equations)
- Takes longer to compute

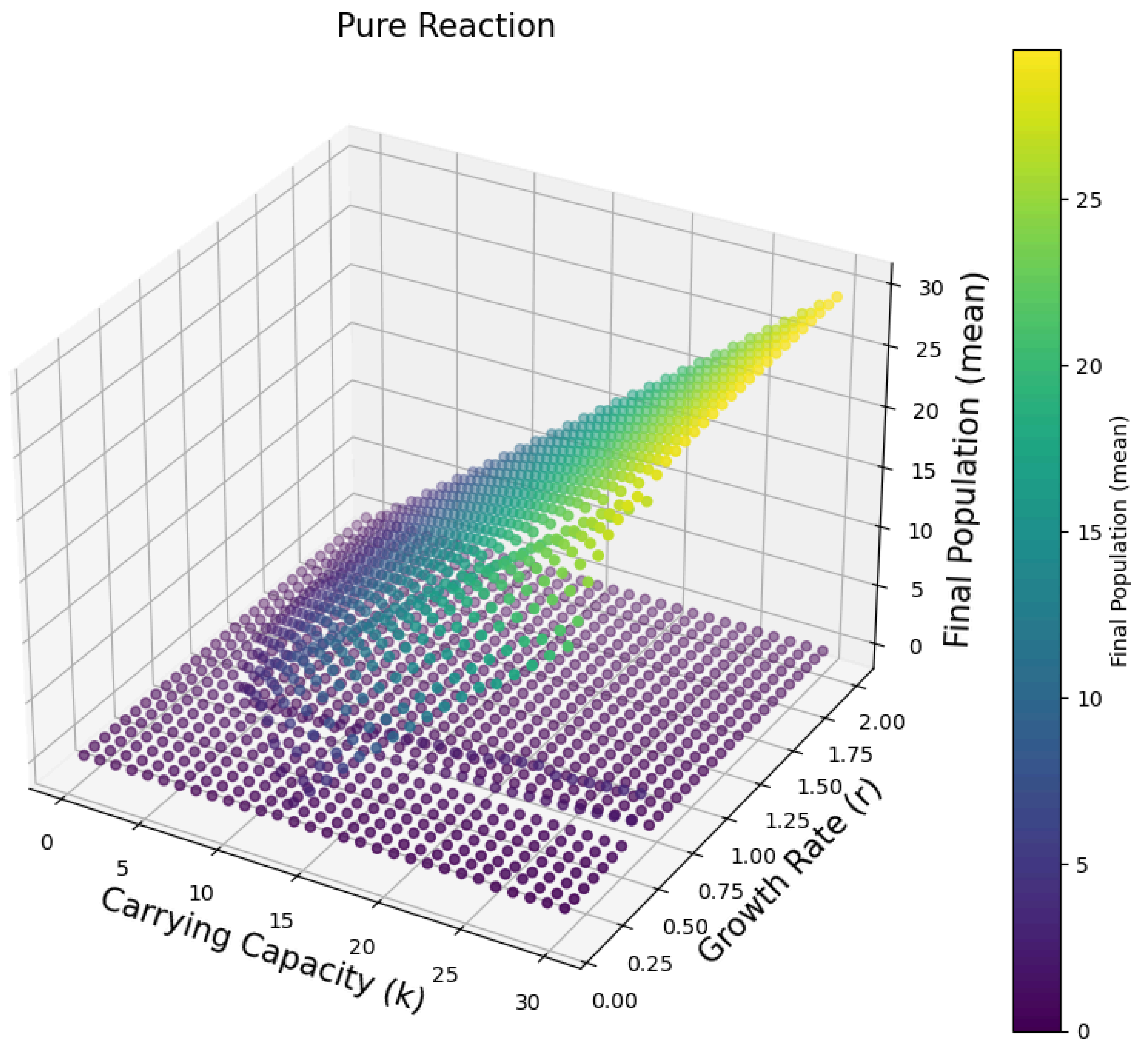
$$\frac{\partial u}{\partial \tau} = D\frac{\partial^2 u}{\partial x^2} + f(u, x)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta\tau} = D\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f\left(u_i^n, x_i\right)$$

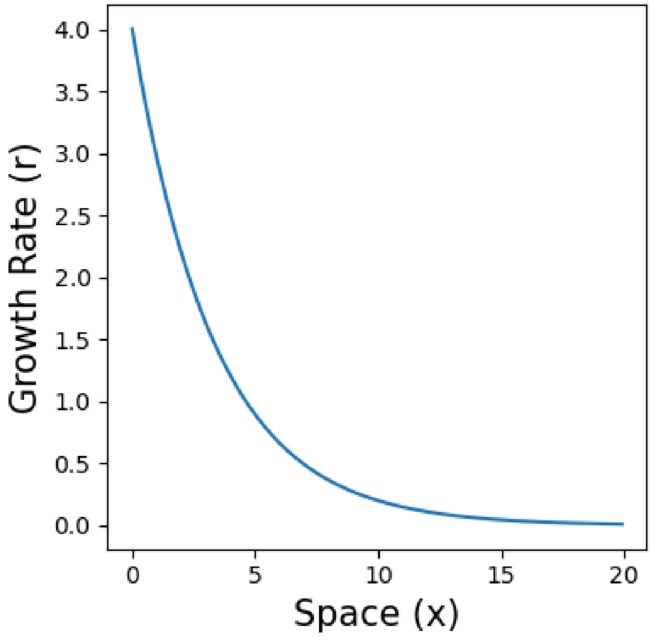
$$\frac{u_i^{n+1} - u_i^n}{\Delta\tau} = D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + f\left(u_i^{n+1}, x_i\right)$$

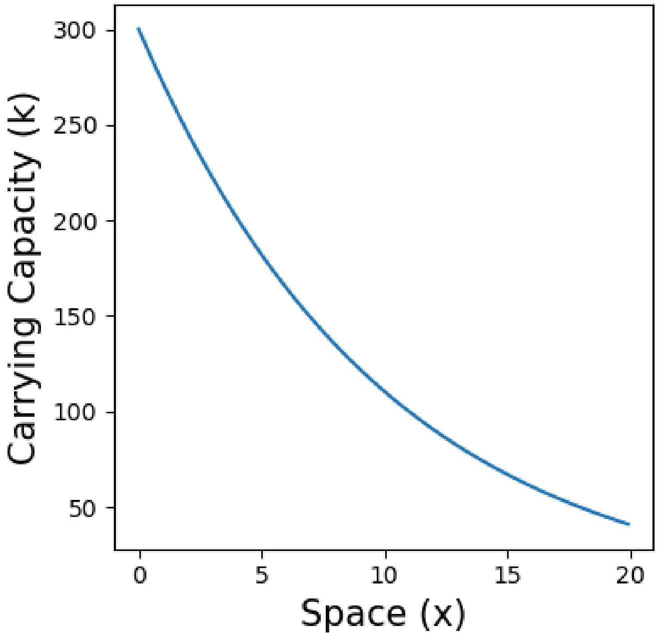
Verifying the Method





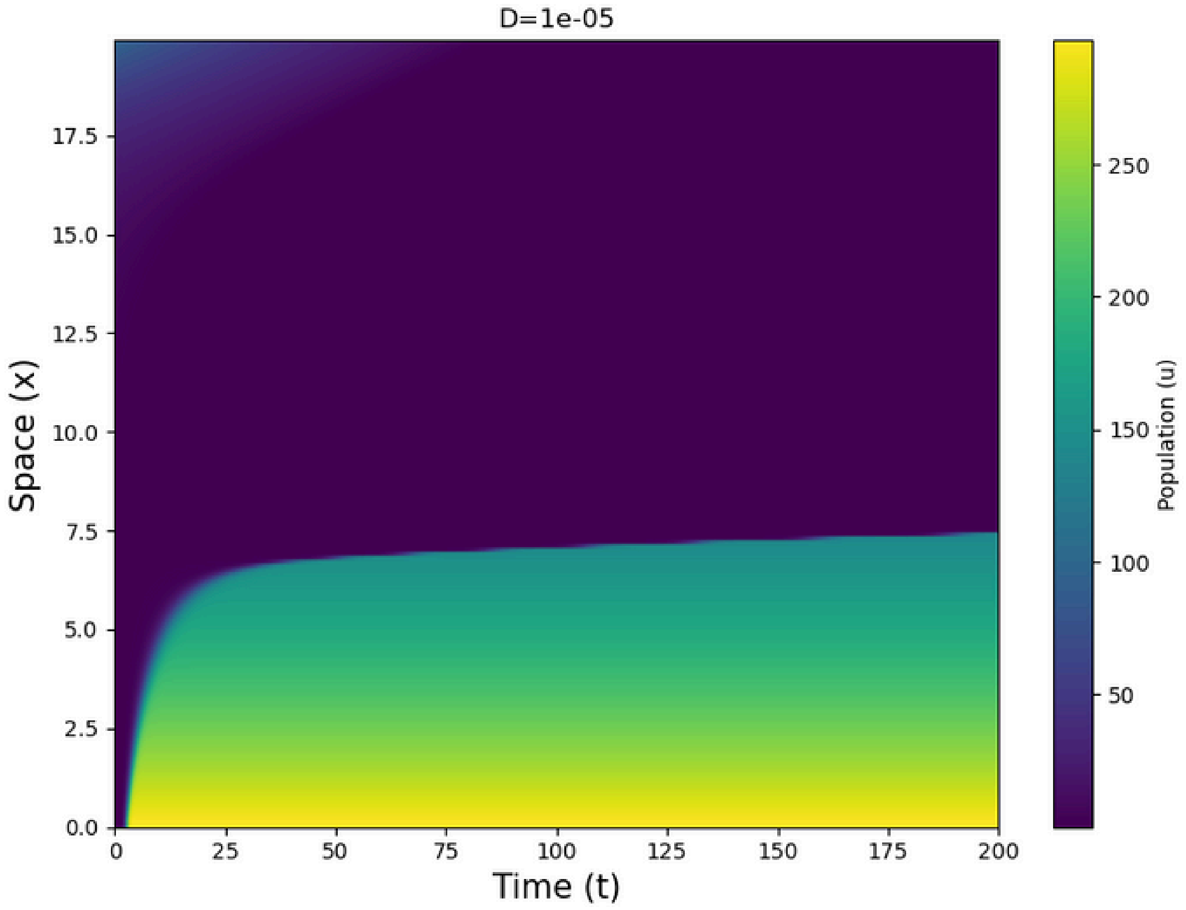
Parallel Gradients

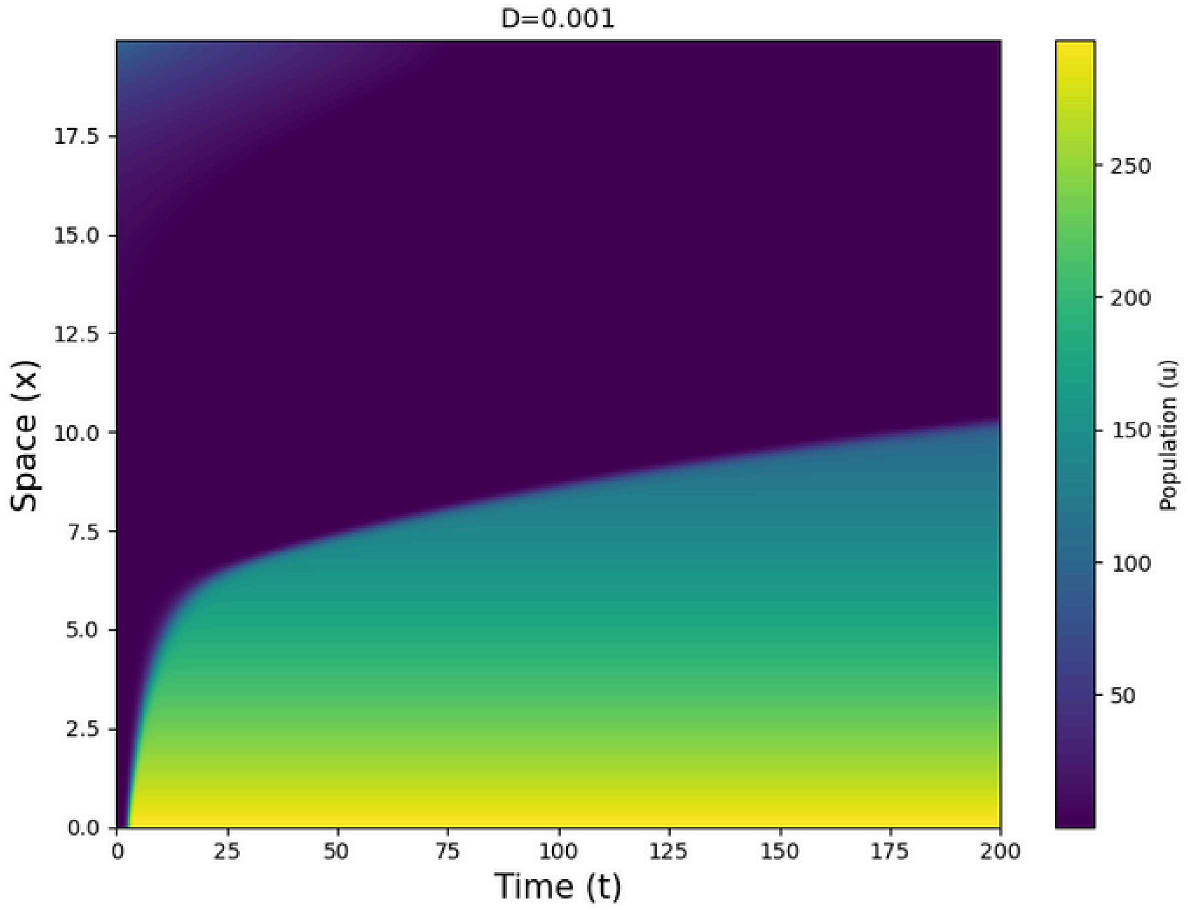


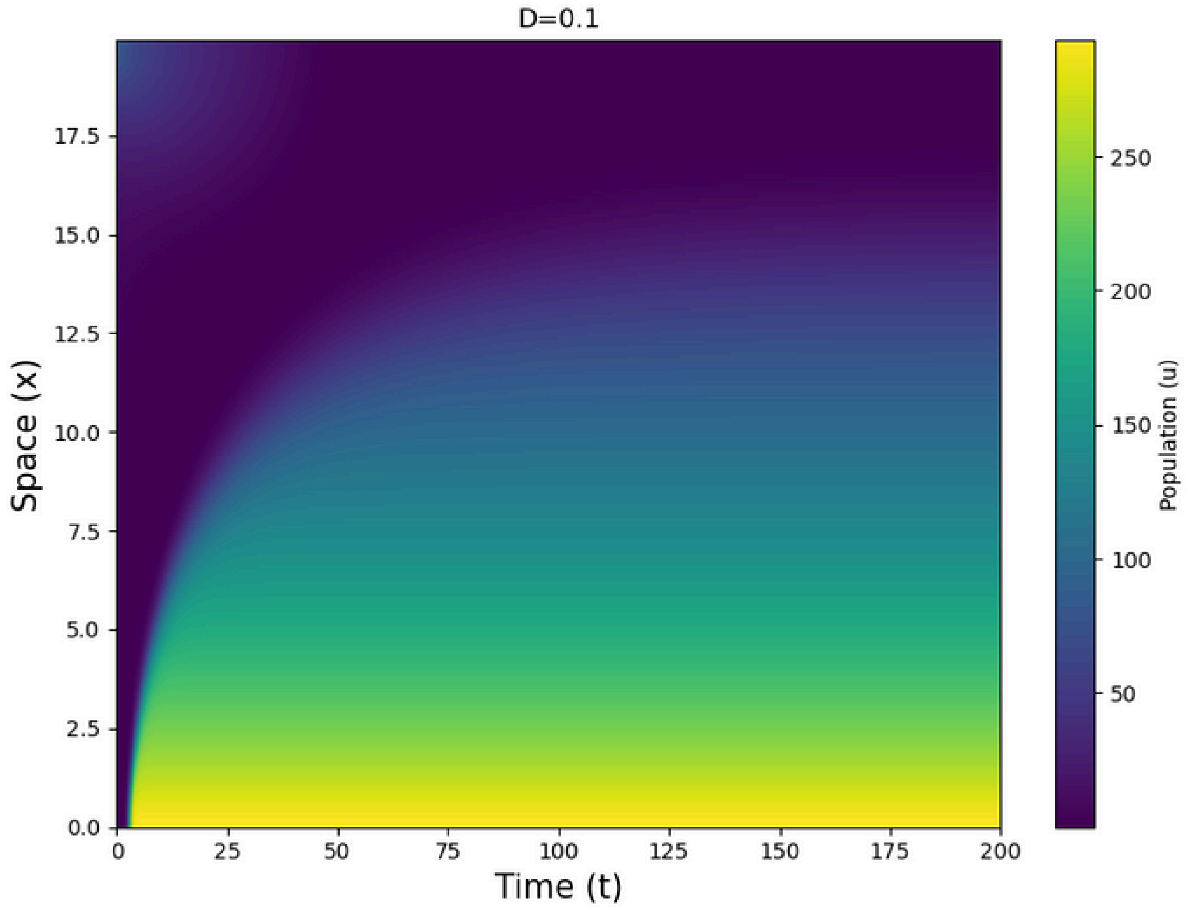


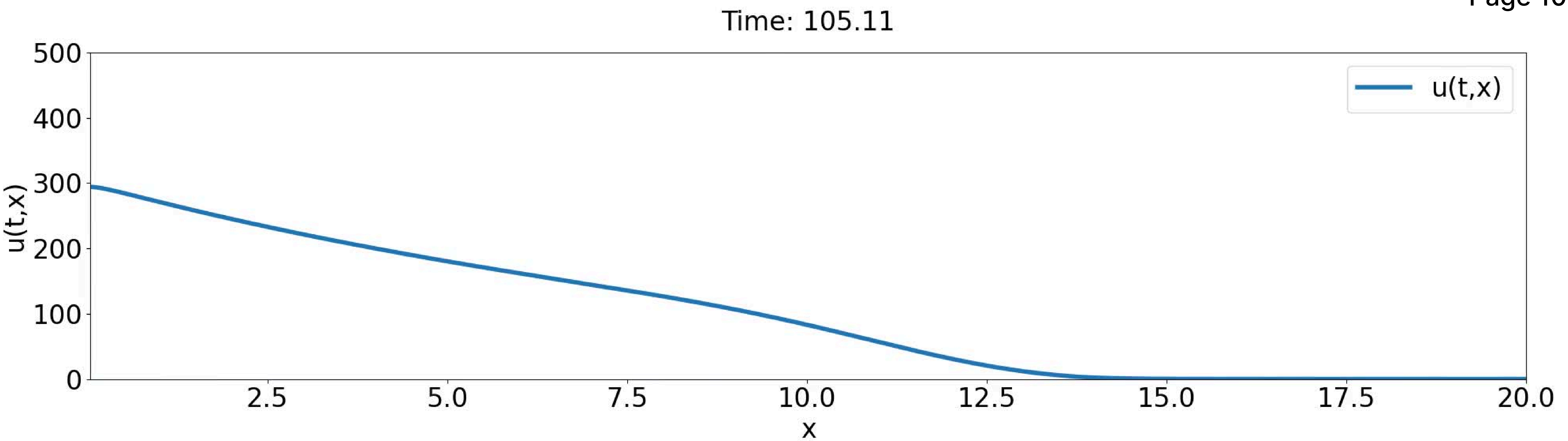
As the diffusion is strengthened, the population shows a travelling wave in space which asymptotically appears to approximate the carrying capacity gradient.

The speed of the wave is not constant, but it slows down as it approaches the boundary.









This wave-like behavior appears to be similar to the KPP-Fisher equation, which is also a RDE with a logistic growth reaction term, but no predation term. This is likely because the predation term tends towards a constant at sufficiently large populations, essentially reducing our equation to a rescaled and shifted KPP-Fisher equation.

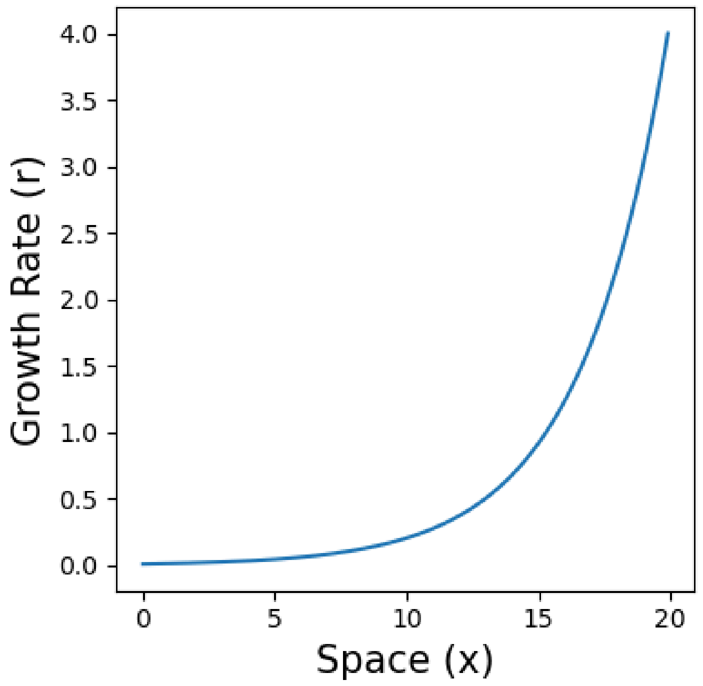
KPP-Fisher Equation

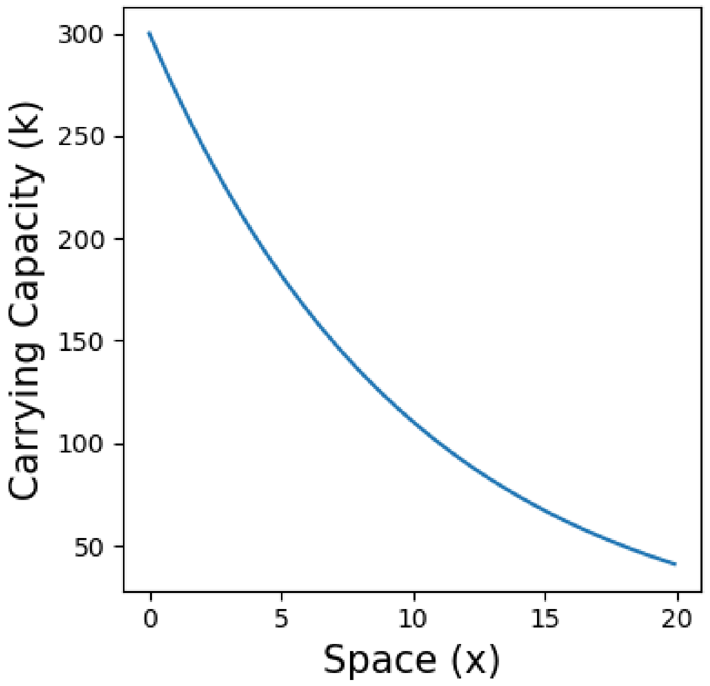
$$\frac{\partial u}{\partial \tau} = D\frac{\partial^2 u}{\partial x^2} + ru(1-u)$$

Insect Outbreak Equation

$$\frac{\partial u}{\partial \tau} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1-\frac{u}{k}\right) - \frac{u^2}{1+u^2}$$

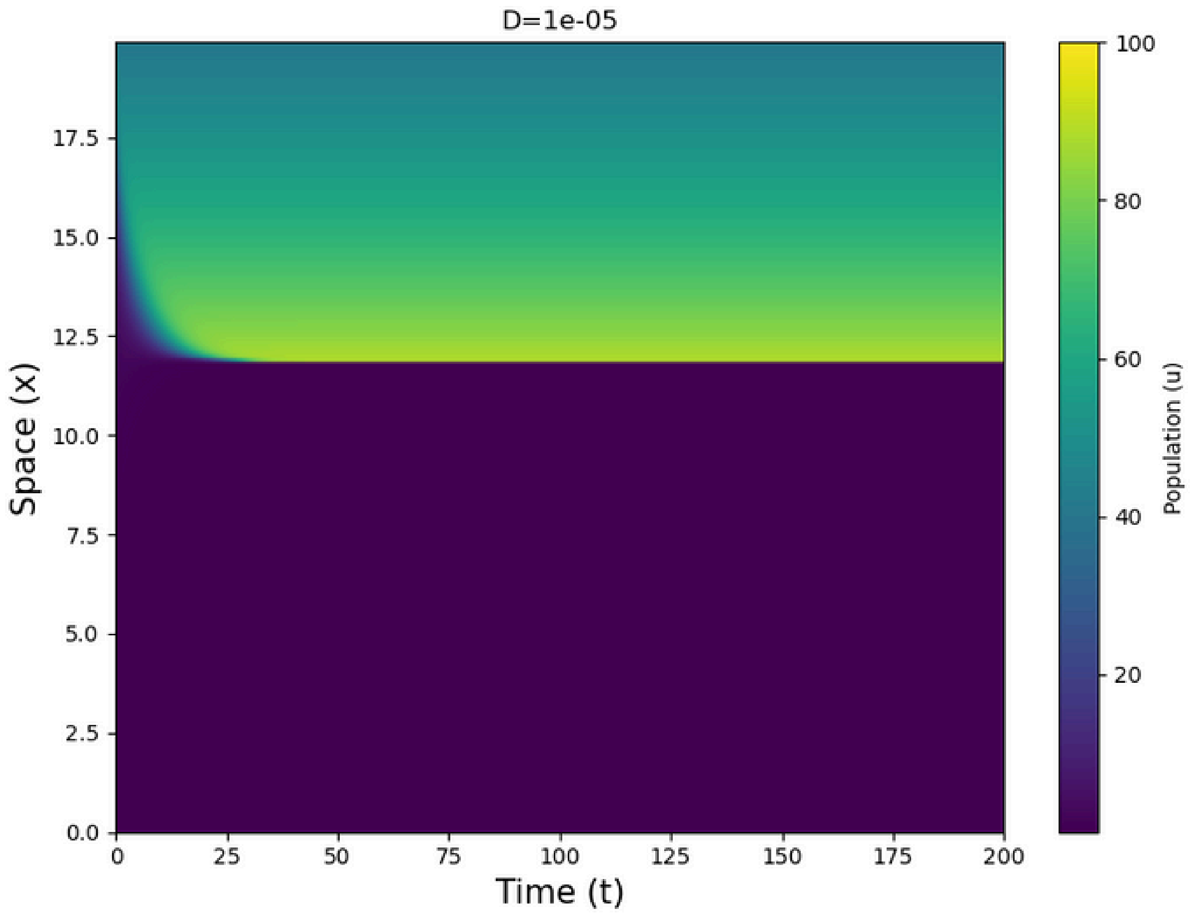
Anti-Parallel Gradients

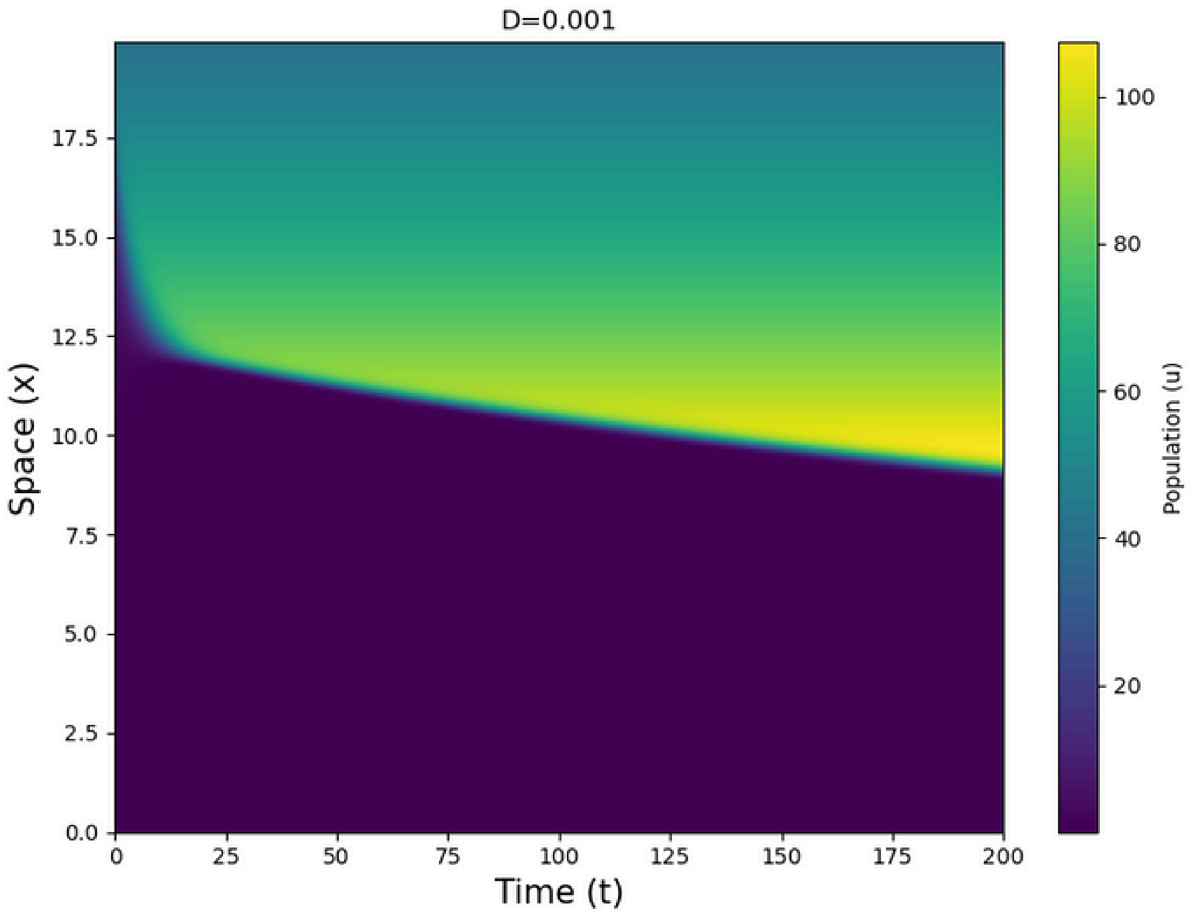


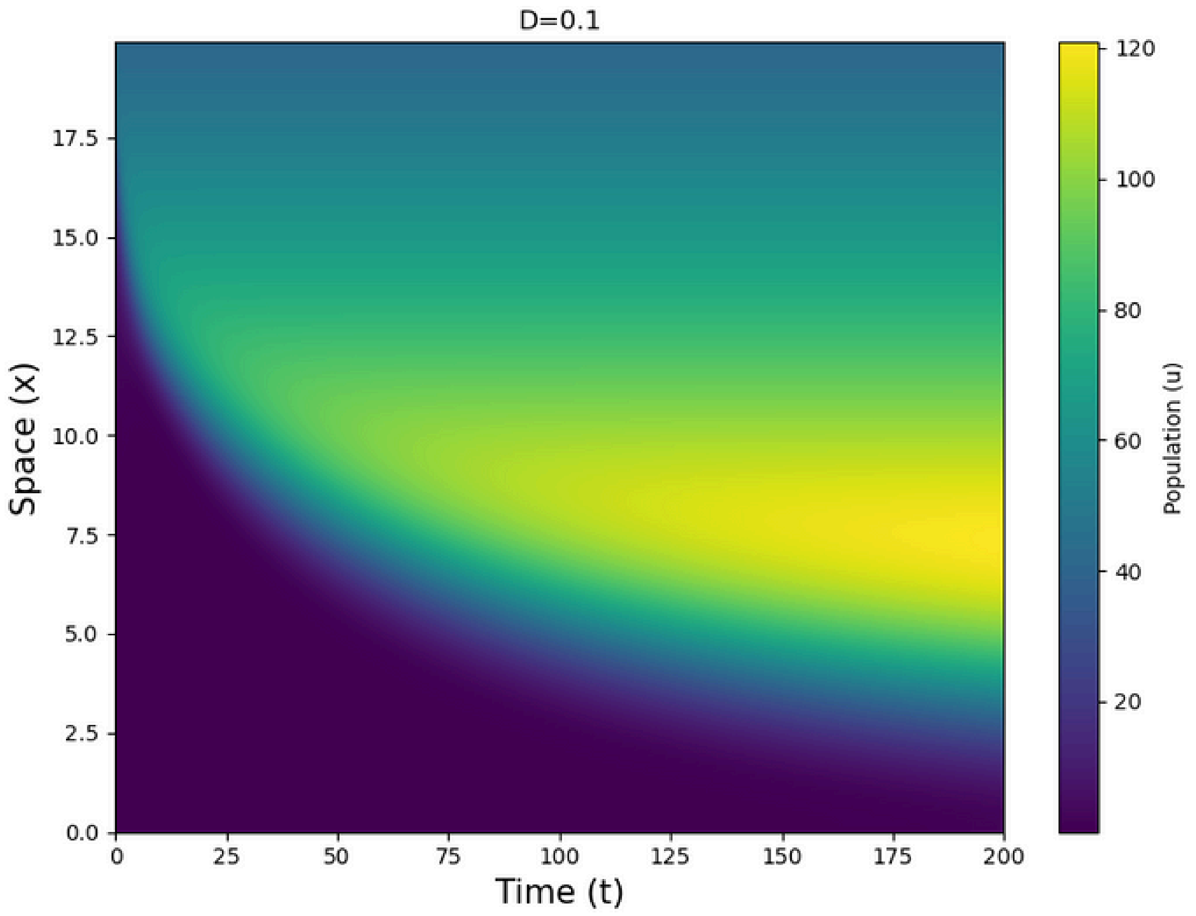


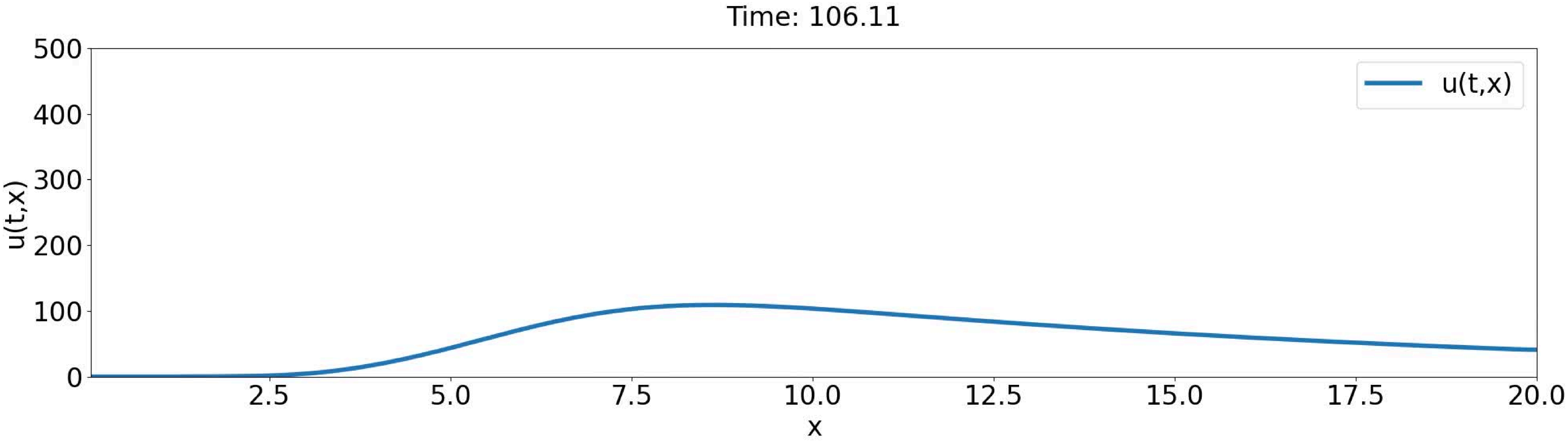
The population seems to tend towards the carrying capacity gradient again, but this time it begins from the end and the wave is led by a peak rather than a dip.

This wave also slows down as it approaches the boundary.








Time: 106.11
500
400
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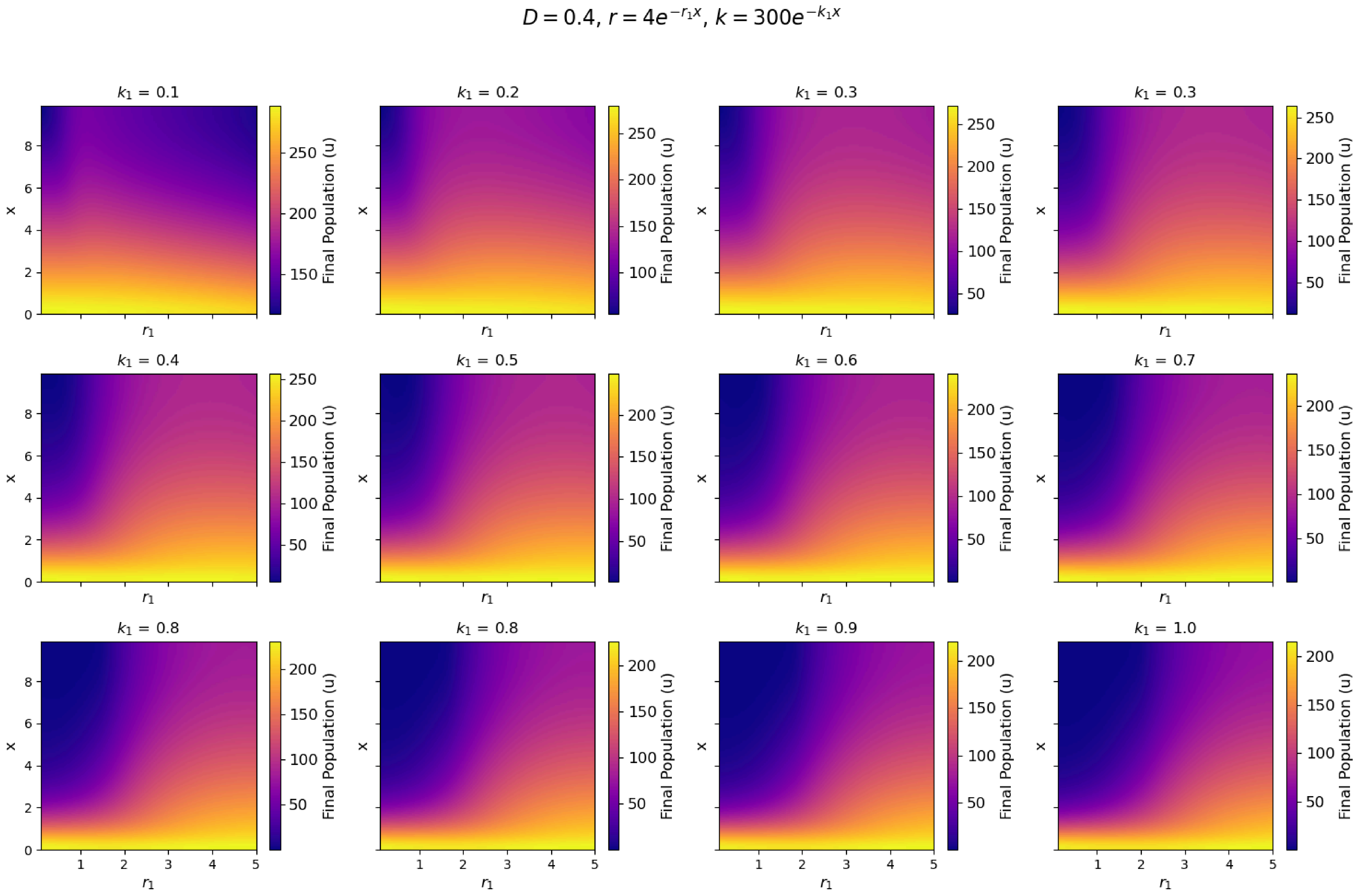
200
100
0
u(t,x)
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
x
u(t,x)

Parameter Dependence

There doesn't seem to be much qualitative change in behaviour for different gradients.

Changing the growth rate gradient affects the speed of the waves.

Changing the carrying capacity affects the asymptotic spatial distribution.



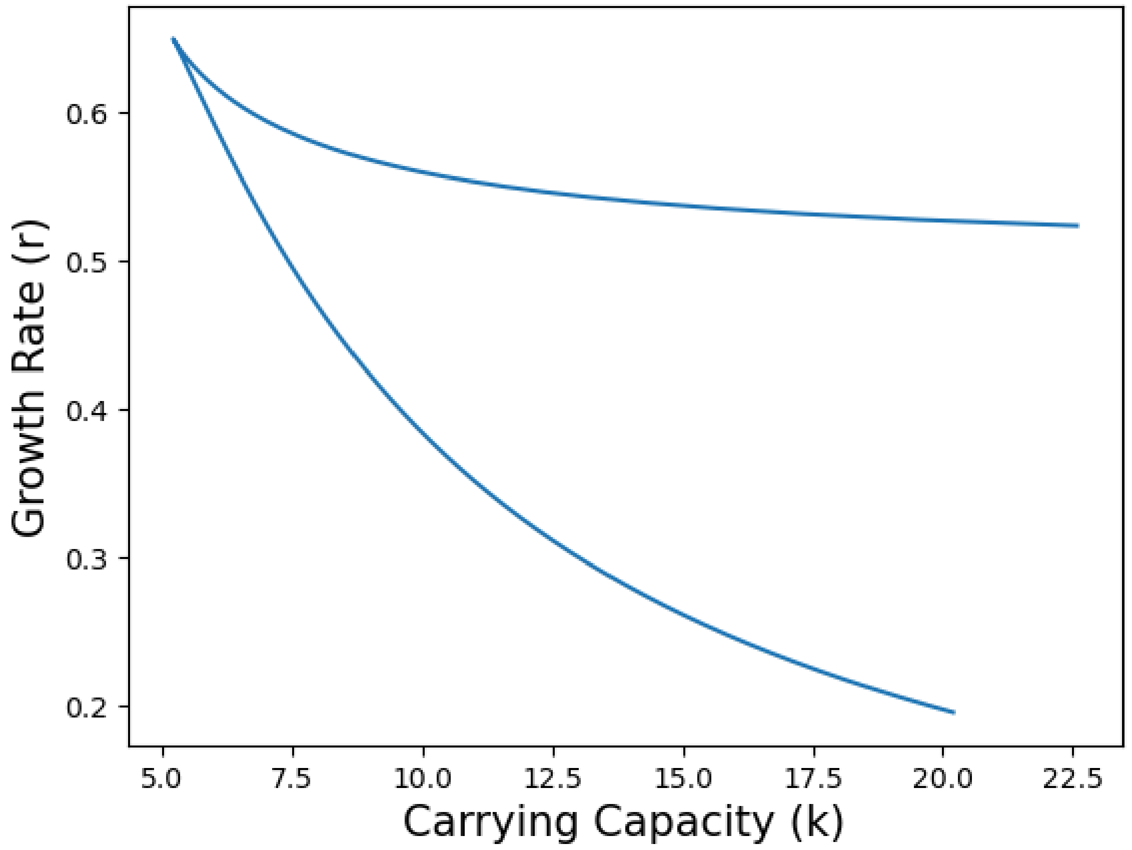
Work in Progress

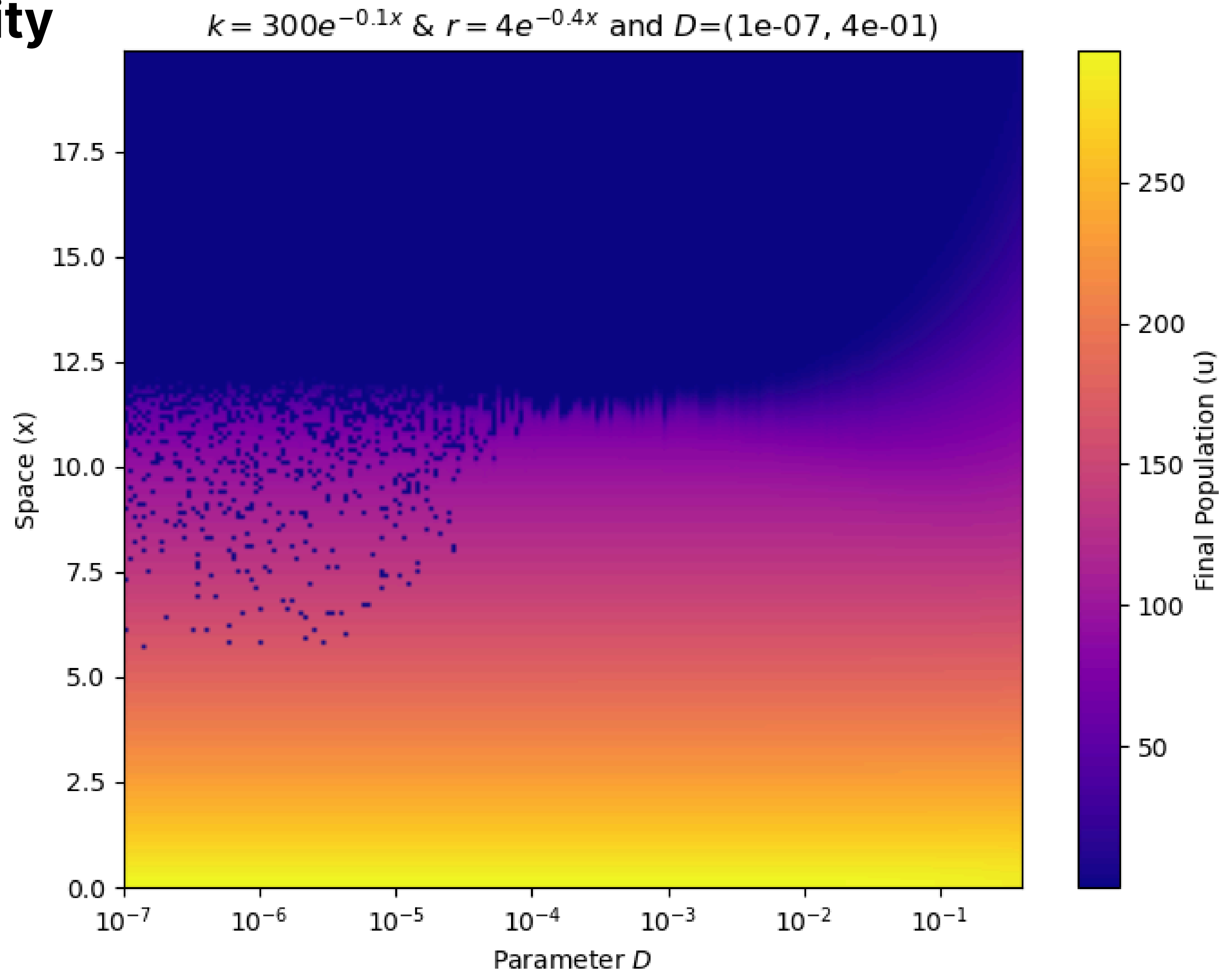
I am currently exploring several directions with this project:

01

Effect of Diffusion and Collapse of Bistability

As diffusion is increased, the transition from refuge to outbreak appears to smoothen, effectively eliminating the phase-transition behaviour.





02

Linear Stability Analysis

Considering a small perturbation close to the stable fixed points u_0, we can analytically search for instabilities in the RDE:

$$u = u_0 + \varepsilon,\ \frac{\partial u_0}{\partial \tau} = 0 \implies \frac{\partial \varepsilon}{\partial \tau} = D\frac{\partial^2 \varepsilon}{\partial x^2} + \left[r\left(1 - \frac{2u_0}{k}\right) - \frac{2u_0}{\left(1 + u_0^2\right)^2} \right]\varepsilon$$

Assuming a Fourier series solution:

$$\varepsilon = \sum_q \tilde{\varepsilon} e^{iqx + \lambda\tau} \implies \lambda = \left[-q^2 D + r\left(1 - \frac{2u_0}{k}\right) - \frac{2u_0}{\left(1 + u_0^2\right)^2} \right]$$

03

Adding Variables (Turing Patterns)

Constructing a 2-variable reaction-diffusion system to look for spatial patterns like spots, stripes and waves.

Spatial gradients can often allow for multiple coexisting patterns in space, and can be used to explain phenomena such as cell differentiation.

Budworm

$$\frac{\partial u}{\partial \tau} = D_u \frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{k}\right) - \frac{u^2 v}{1+u^2}$$

Bird (Predator)

$$\frac{\partial v}{\partial \tau} = D_v \frac{\partial^2 v}{\partial x^2} + g\left(\frac{u^2 v}{1+u^2}\right) - \delta v$$

Foliage (Prey)

$$k = \alpha s, \quad \frac{\partial s}{\partial t} = D_s \frac{\partial^2 s}{\partial x^2} + r_s s\left(1 - \frac{s}{k_s}\right)$$

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