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Analysis of a Planar Simple Pendulum Using a Video Tracker

Lab Report 1

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Aim

- Determine the amplitude (A) and time period (T) of a planar undamped simple pendulum using $A\sin(2\pi t/T+\phi)$ as a model.
- Identify the damping coefficient (γ), Q-factor (Q) and the damped sine function in the form $A\sin(2\pi t/T+\phi)e^{-\gamma t/2}$ of a planar damped simple pendulum.
- Find the distribution and standard deviation of the time periods of multiple oscillations of a planar damped simple pendulum.
- Plot the x-t, y-t and x-y graphs for the motion of an elliptical undamped simple pendulum and compare the time periods of each.
- Determine the equation of the ellipse from the equations along the x and y axes - $x = A_x\cos(\omega t)$ and $y = A_y\cos(\omega t-\phi)$.

Experimental Setup

List of Apparatus:

- Retort Lab Stand$\times$2
- Split Cork
- Thread
- Metallic Bob
- Neon Sticker

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- Measuring Tape
- Protractor
- Weighing Scale
- Camera Phone

Frame Rate of Camera Phone = 60 fps
Least Count of Tape Measure = 0.1 cm
Least Count of Weighing Scale = 0.1 g
Least Count of Protractor = 1°

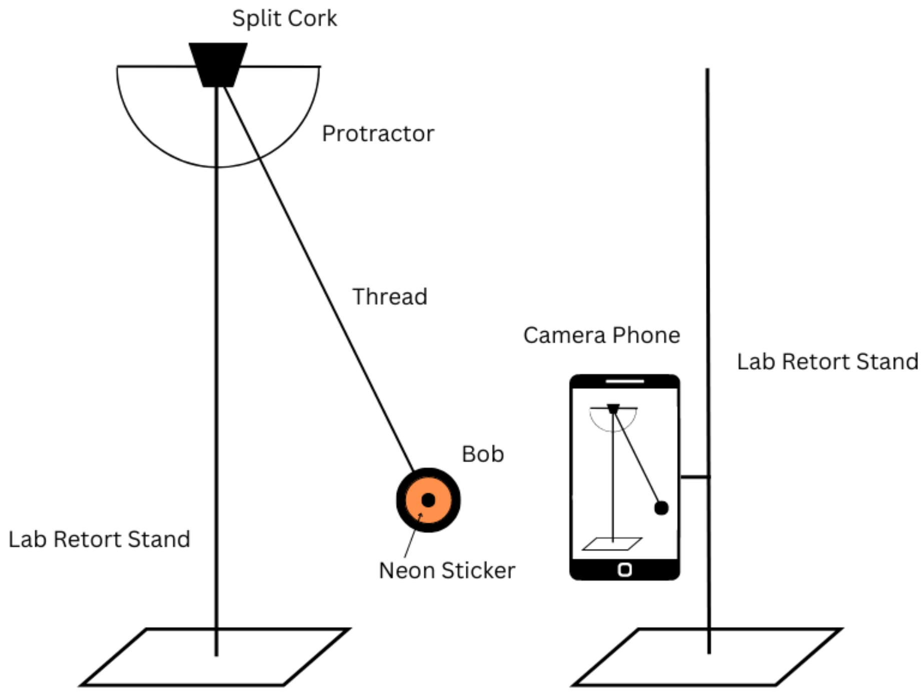

Figure 1: Experimental setup for a simple pendulum with video tracking

Theoretical Background

Planar Pendulum

The motion of an undamped simple pendulum can be described by the differential equation $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$. However, for small amplitudes we can approximate $\sin\theta \approx \theta$, hence $\ddot{\theta} + \frac{g}{l}\theta = 0$. We can guess that the solution of this second-order, linear, homogeneous, ordinary differential equation is a sinusoidal wave of the form:

$$x = A_0 \sin(\omega t + \phi) \tag{1}$$

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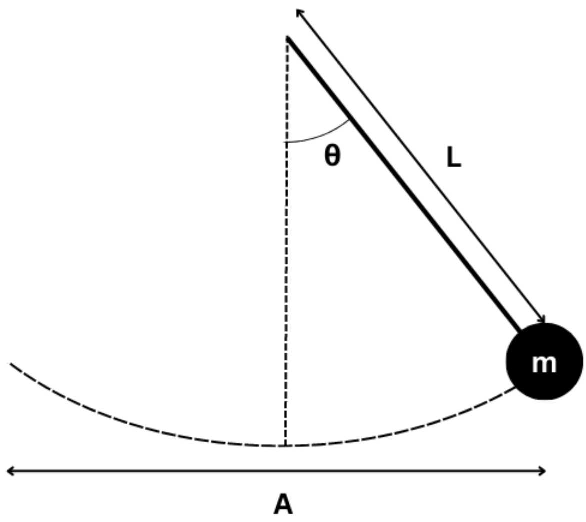

Figure 2: Schematic diagram of a planar simple pendulum

Where A is the amplitude, ϕ is the phase difference, $\omega = \sqrt{g/l}$ and the time period $T = 2\pi/\omega = 2\pi\sqrt{l/g}$. This can be verified by taking the double derivative of equation (1):

$$x = A_0 \sin(\omega t + \phi)$$

$$\implies \dot{x} = \omega A_0 \cos(\omega t + \phi)$$

$$\implies \ddot{x} = -\omega^2 A_0 \sin(\omega t + \phi)$$

$$\implies \ddot{x} = -\omega^2 x$$

$$\implies \ddot{x} = -\frac{g}{l} x$$

Given this equation, we should be able to determine the time period and amplitude of our undamped planar pendulum by fitting a curve with equation (1) on our data. Over a longer time period, we will also observe damping which can be accounted for by multiplying our equation with an exponential damping factor:

$$x = A_0 \sin(\omega t + \phi) e^{-\gamma t/2} \tag{2}$$

Where γ is the damping factor. We may abbreviate $-\gamma/2$ as d for the sake of simplicity. To find this value from our data, we can use the function of amplitude $A_m = A_0 e^{-\gamma t/2}$ and take its natural log to get a straight line:

$$\ln(A_m) = -\frac{\gamma t}{2} + \ln(A_0) \tag{3}$$

Whose slope should be $-\gamma/2$ or d. Additionally, the natural frequency $\omega_0 = 2\pi/T$ can be found from the time period and subsequently the Q-factor of the pendulum can be calculated by $Q = \omega_0/\gamma$.

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Elliptical Pendulum

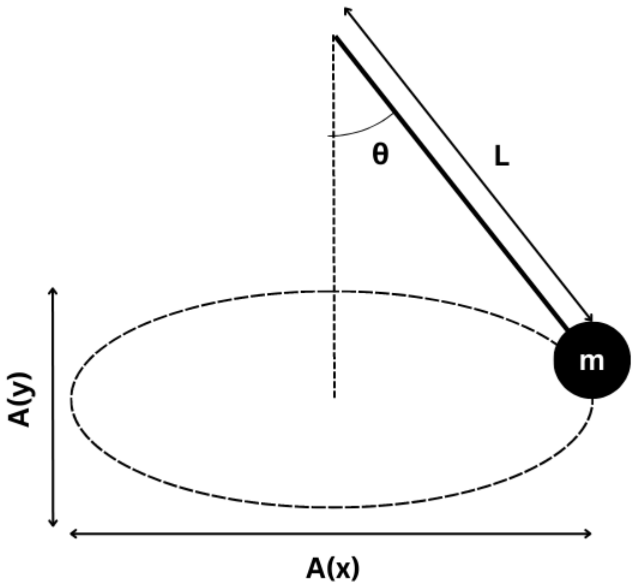

Figure 3: Schematic diagram of a elliptical simple pendulum

The motion of an elliptical pendulum forms a cone in 3 dimensions, where the bob traces an ellipse which is a section of the cone. Within the plane of the ellipse, assuming that the time period along the x and y axes are the same (thus tracing the same ellipse over and over) the equations of motion are:

$$x = A\cos(\omega t) \tag{4}$$

$$y = B\cos(\omega t - \phi) \tag{5}$$

If there is a difference in the time period of the x and y oscillations, the ellipse will begin to precess along its axis, causing the shape and orientation of the ellipse to change over time.

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Procedure

Step 1: Measure the mass of the bob with the weighing scale. Make a small black dot in the center of the neon sticker and stick it on the side of the bob that faces the camera.

[Precaution: Keep the bob in the center of the scale to ensure equal distribution of weight. Let the bob come to rest and untwist before applying the sticker so that it is always facing the camera.]

Step 2: Measure the length of the pendulum with the tape measure by lining the tape vertically with the thread. Take readings at the top and the bottom of the bob and average these values to determine the length till the center of mass of the bob (Figure 2).

$$L = \frac{L_1 + L_2}{2} \tag{6}$$

[Precaution: Ensure that the thread is taut and vertically suspended while measuring. Use a flat surface to mark the length from the top and bottom of the bob.]

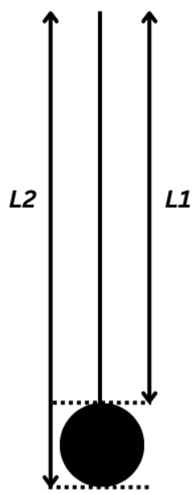

Figure 4: Determining the length of the pendulum to the center of mass of the bob

Step 3: Place a meter scale in the same plane as the motion of the pendulum and set-up the camera phone such that the bob and scale are clearly visible and in focus.

[Precaution: The phone should be vertical and centered about the axis of the pendulum so that there is minimal shift correction required to adjust the sine waves laterally.]

Step 4: Hold the bob lightly and bring it up until the thread aligns with the desired amplitude of release on the protractor. Release the bob quickly and allow for 2 oscillations before starting to record the oscillations (to allow the pendulum to stabilize). Take a video for at least 100 oscillations (typically 3-5 min).

[Precaution: Keep the thread taut before release and ensure that the bob does not bounce or swing elliptically after release. The string should not be twisted so that the sticker does not go out of frame during the video.]

Step 5: Upload the video to the Tracker app and calibrate the scale with the help of the meter scale in frame. Use the auto-track function to generate the x, y, and t data of the pendulum's motion.

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[Precaution: The frame rate of the video should preferably be 60 fps to get a smoother distribution of data and to avoid any frame skipping while tracking. While calibrating, use the longest length along the meter stick visible in frame to minimize visual error.]

Step 6: To record the motion of an elliptical pendulum, apply the sticker on the bottom of the bob and set up the camera below the pendulum, facing upwards. Repeat steps 2-5 for an elliptical pendulum.

[Precaution: Place the camera such that the point of suspension is directly above it so that the axes can be easily aligned. If placing a meter scale in the same plane as the bob proves difficult, the diameter of the bob can be used to calibrate the tracker.]

Observations

Planar Pendulum

Constant Parameters:
- Length of Pendulum $= 40.0 \pm 0.1$ cm
- Mass of Bob $= 32.0 \pm 0.1$ g
- Frame rate of Camera = 60 fps

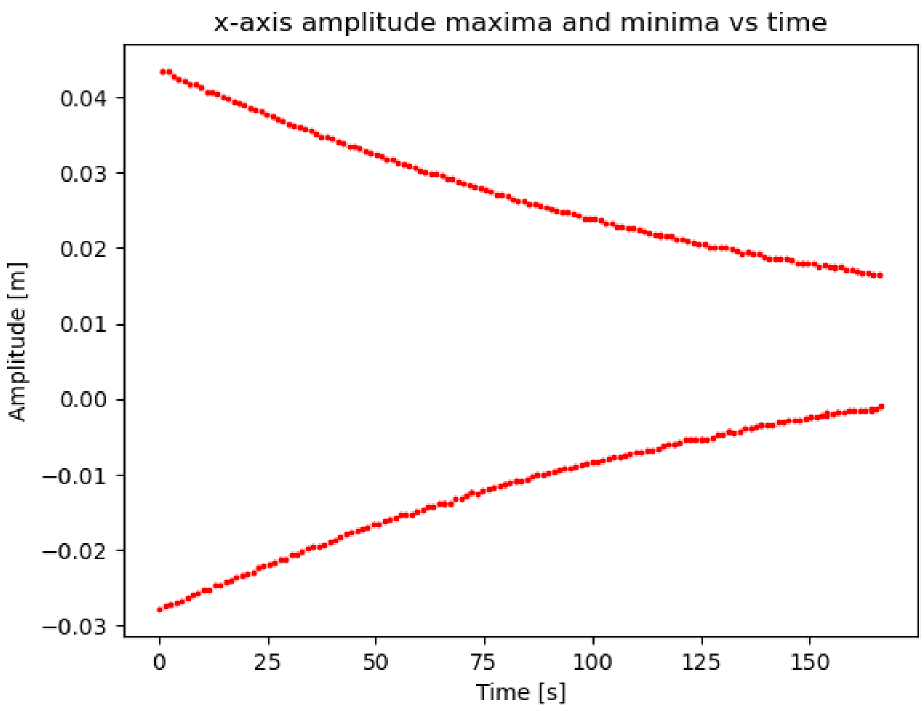

Figure 5: The data for the x-t motion of a damped planar pendulum shows reducing amplitude over a long time.

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To determine the trend of the data, we must observe only a small number of oscillations. Here is the scatter plot for only 10 oscillations:

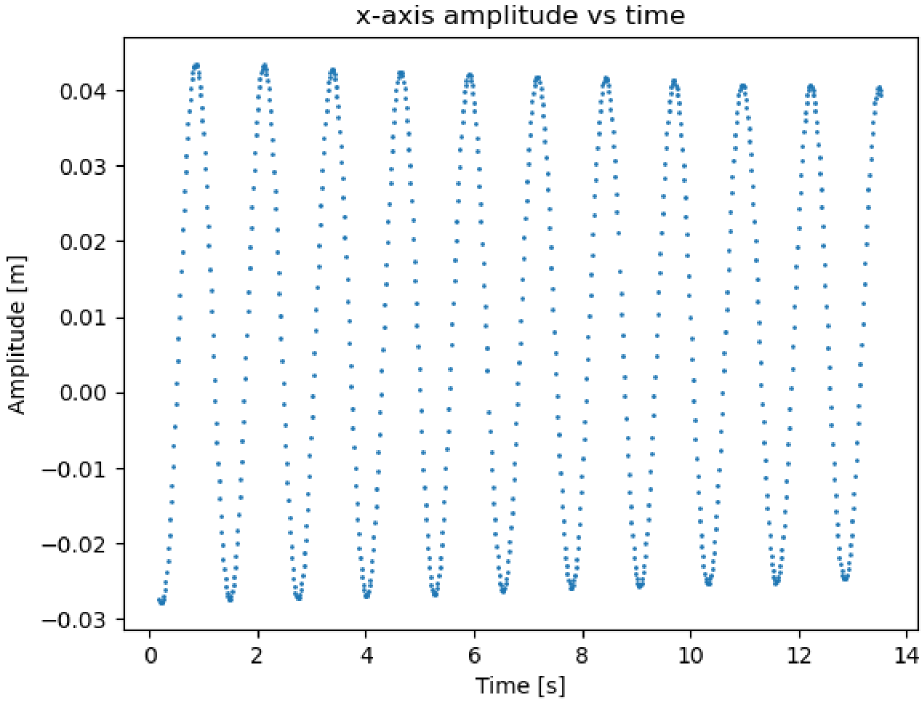

Figure 6: The trend is clearly periodic and sinusoidal

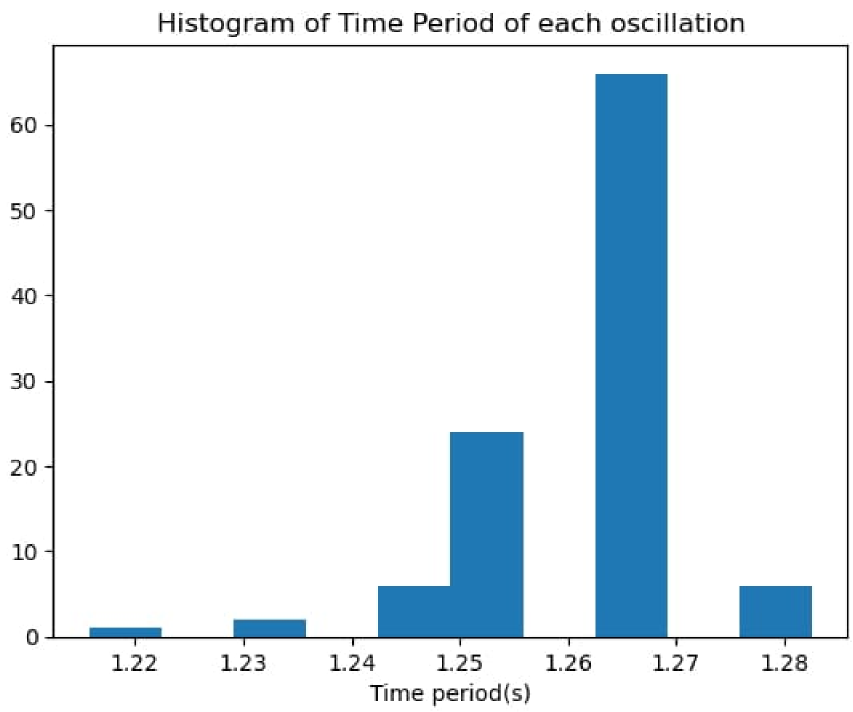

Figure 7: The time period shows minimal distribution about the mean and negligible change over long time intervals

Mean time period = 1.261 s
Standard Deviation of time period = 0.0108 s

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The time taken for 131 oscillations is 166 s, giving an average time period of:

$$T_{avg} = \frac{T_n}{N} = \frac{166}{131} = 1.267s$$

Elliptical Pendulum

Constant Parameters:
Length of Pendulum = 40.0 ± 0.1 cm
Mass of Bob = 32.0 ± 0.1 g
Frame rate of Camera = 60 fps

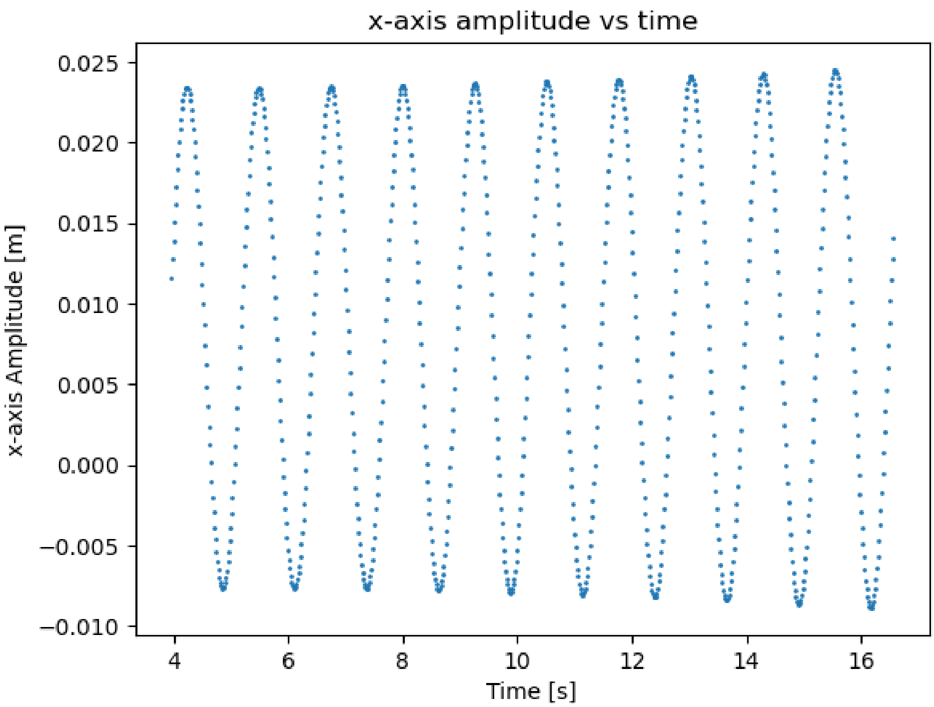

Figure 8: The x amplitude seems to be increasing over time. This is due to the greater effect of precession acting against the effect of damping.

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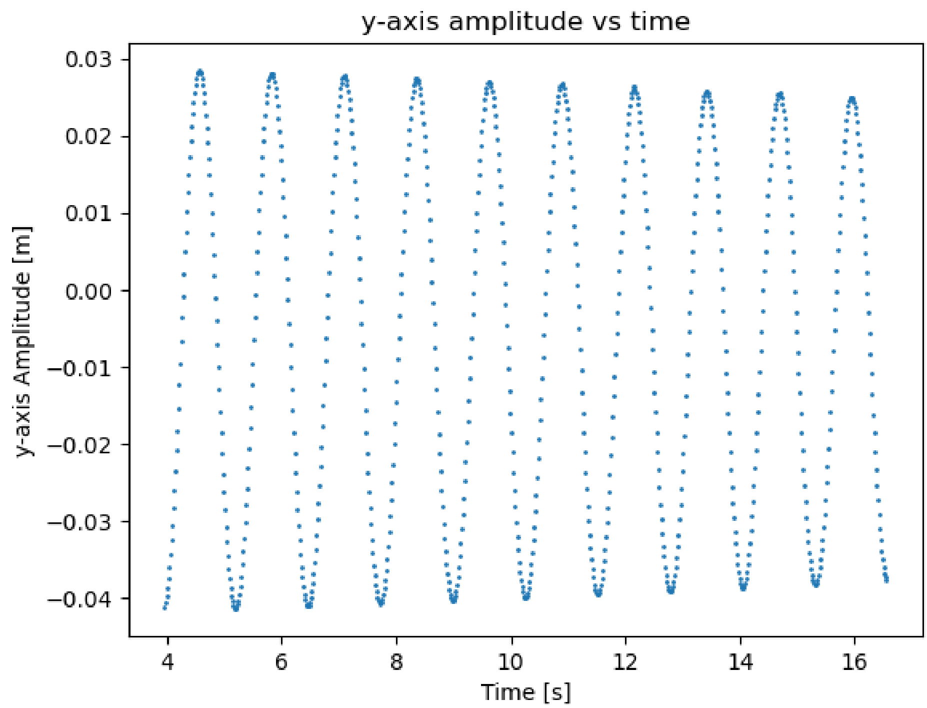

Figure 9: The damping of the y is relatively accelerated due to additional effect of precession along with damping.

The x-y plot of an elliptical pendulum shows the ellipse traced by the bob over a few oscillations:

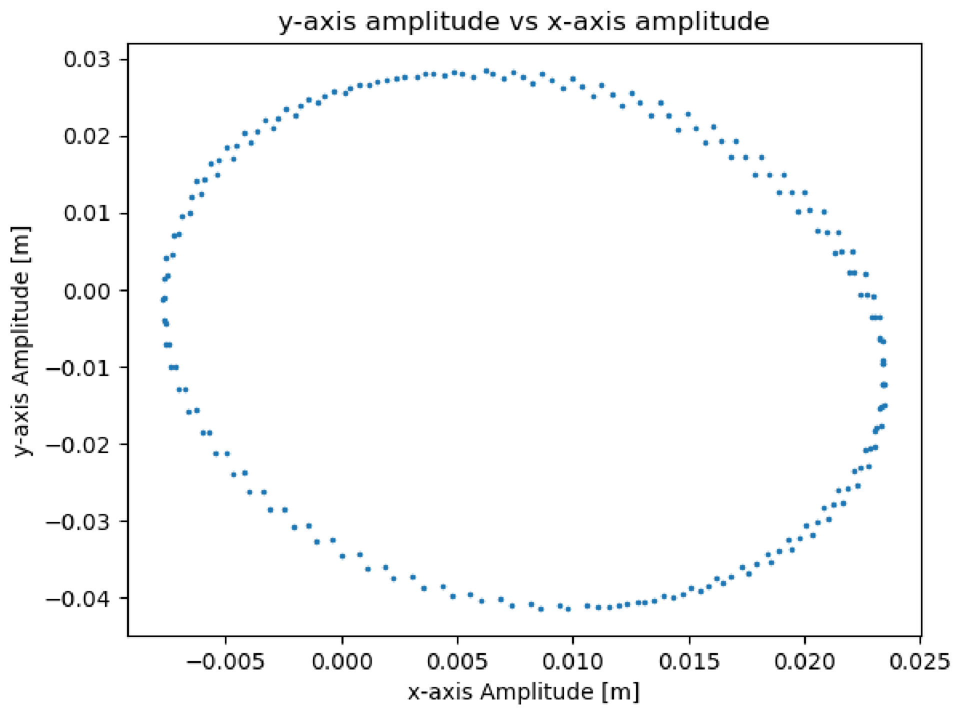

Figure 10: The precession of the pendulum due to the difference in time period along the x and y axis causes a slight shift in the ellipse after each oscillation.

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Analysis

Planar Pendulum

Using Equation (1) as a model, we fit the data in Figure (6) with a damped sine function [note the addition of a constant k to account for lateral shift error]:

$$x = A_0 \sin\left(\frac{2\pi t}{T} + \phi\right) + k \tag{7}$$

Where,

$A_0 = 0.0369m$
$T = 1.262s$
$\phi = -2.778$
$k = 0.0077m$

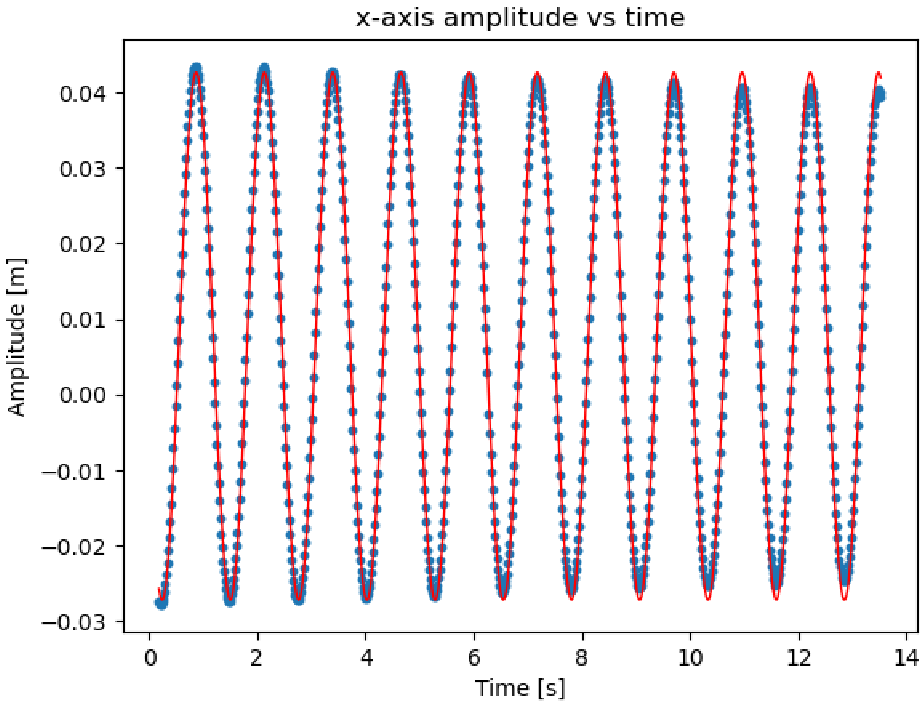

Figure 11: The damping is minimal, yet noticeable for 10 oscillations. Nonetheless, it can be estimated fairly accurately with an ordinary sine wave.

The time period of the fitted graph differs from the average time period by 0.005 s, which can be considered negligible.

The maxima were isolated using a python program that found all x-values that were greater than their immediate neighbours and put them into an array (See Appendix 1). Using the function of amplitude $A_m = A_0 e^{-\gamma t/2}$, we found its natural log to get a straight line:

$$\ln(A_m) = -\frac{\gamma t}{2} + \ln(A_0) \tag{8}$$

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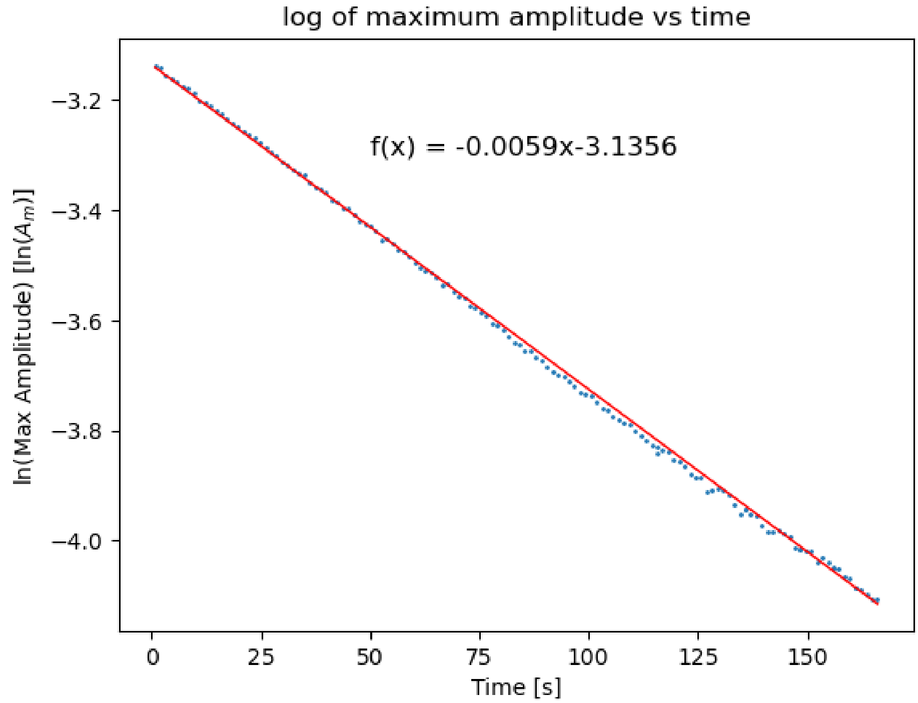

Figure 12: The log of maximum amplitude shows a linear trend against time, with a slope of -0.0059.

Figure 6 shows that a damped sine wave can better model the data for amplitude along the x-axis versus time. The equation is:

$$x = A_0 \sin\left(\frac{2\pi t}{T} + \phi\right)e^{dt} + k$$

From the fitted function we can find the damping factor and natural frequency of the pendulum:

$$\gamma = -2d = 0.0162$$

$$\omega_0 = \frac{2\pi}{T} = 4.98317s^{-1}$$

From this, the Q-factor of the pendulum can be found:

$$Q = \frac{\omega_0}{\gamma} = 307.60309$$

The damping factor found from the slope of Figure 8 is $\gamma = -2d = 0.0118$, which is within 5% of the estimated value from the fitted curve. This is an acceptable margin of error since the order of magnitude of both values is the same.

However, since the frequency of a damped oscillator is actually $\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4} = 4.98314s^{-1}$, we find that the true value of time period of the damped oscillator is $T_1 = 2\pi/\omega_1 = 1.260888s$. The deviation of time period is of the order $10^{-6}s$. Therefore the effect of damping on time period can be considered negligible.

The standard deviation observed in T is of the order 10^{-2}. Since it is clear that this error cannot originate due to damping, it is possible that it comes from the method of determining

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the time period of each cycle.

To find the time period of each individual oscillation, we wrote a program that finds the interval between every maxima. However, since the tracker data is discontinuous, there may be a slight difference in the intervals which could explain the error and deviation.

Elliptical Pendulum

Assuming the time period for the oscillations on the x-axis and y-axis are the same, we can derive the equation of the ellipse:

$$x = Acos\omega t,\ y = Bcos(\omega t + \phi)$$

$$\implies \frac{x}{A} = cos(\omega t)$$

$$\implies sin(\omega t) = \sqrt{1 - \frac{x^2}{A^2}}$$

$$\implies \frac{y}{B} = cos(\omega t)cos\phi - sin(\omega t)sin\phi$$

$$\implies \frac{y}{B} = \frac{x}{A}cos\phi - \sqrt{1 - \frac{x^2}{A^2}}sin\phi$$

$$\implies \frac{y}{B} - \frac{x}{A}cos\phi = -\sqrt{1 - \frac{x^2}{A^2}}sin\phi$$

$$\implies \frac{y^2}{B^2} + \frac{x^2}{A^2}cos^2\phi - 2\frac{yx}{BA}cos\phi = (1 - \frac{x^2}{A^2})sin^2\phi$$

$$\implies \frac{y^2}{B^2} + \frac{x^2}{A^2}cos^2\phi - 2\frac{yx}{BA}cos\phi = sin^2\phi - sin^2\phi\frac{x^2}{A^2}$$

$$\implies \frac{y^2}{B^2} + \frac{x^2}{A^2}cos^2\phi + sin^2\phi\frac{x^2}{A^2} - 2\frac{yx}{BA}cos\phi = sin^2\phi$$

$$\implies \frac{y^2}{B^2} - 2\frac{yx}{BA}cos\phi + \frac{x^2}{A^2}(cos^2\phi + sin^2\phi) = sin^2\phi$$

$$\implies \boxed{\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2\frac{yx}{BA}cos\phi = sin^2\phi}$$

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Using the equations (4) and (5) as a model, we can fit the data of figure (8) and (9), respectively:

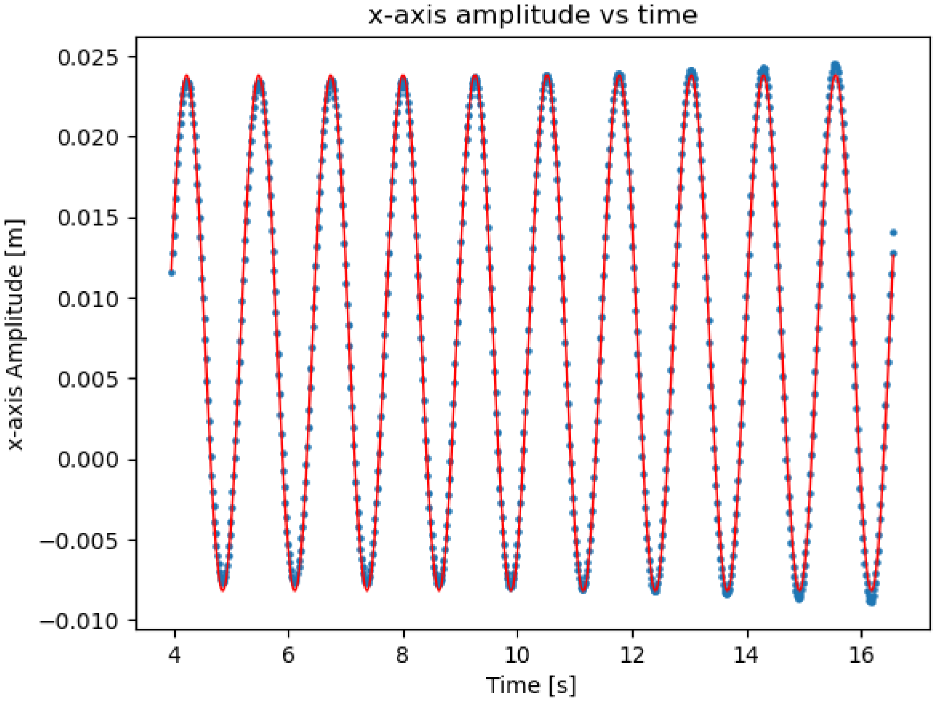

Figure 13: The x-t graph is modelled by a sine wave

Where,
$A = 0.016m$
$T = 1.258s$
$p = -0.7$
$k = 0.0078$

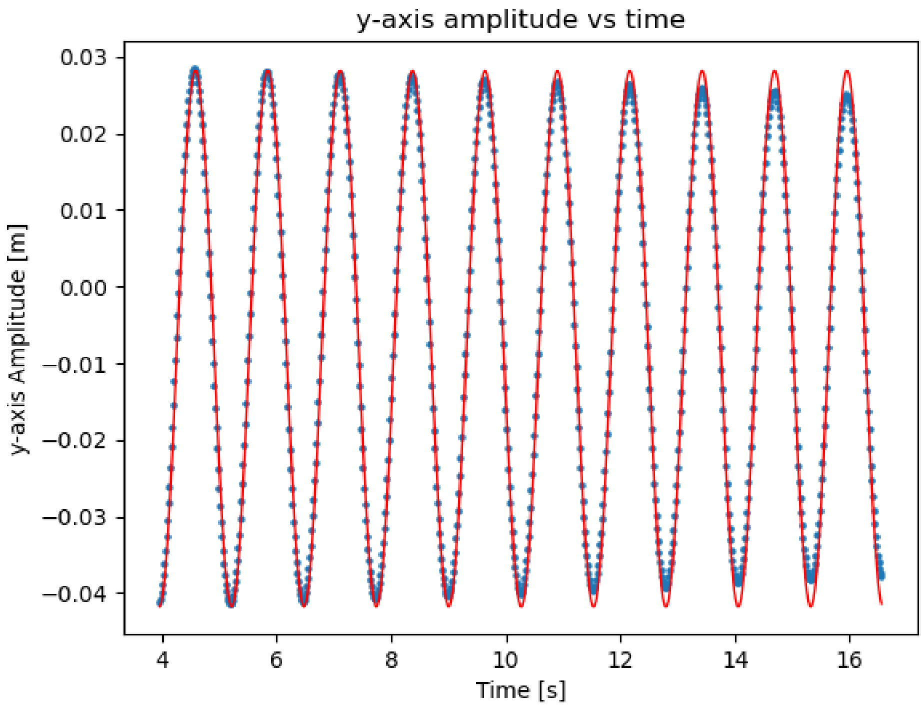

Figure 14: The y-t graph is modelled by a sine wave

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Where,
$A = 0.035$
$T = 1.263$
$p = -2.4$
$k = -0.0068$

Error Analysis

Though it is difficult to determine the relative error in any of the parameters without a deeper understanding of the tracking software, the potential sources of error are:

- The calibration meter stick with least count 0.001 m
- The frame rate of the camera is 60 fps, which means that the data is discontinuous for 1/60th second intervals.
- The estimated values to fit the function were visually fit onto the data and have a limited degree of accuracy.
- The tracker has three variables to determine the extent to which the frames may differ to continue tracking the same point.
    1. Evolve - The maximum amount the template frame is allowed to change to be considered the same point. We used 20% which is the default value.
    2. Tether - The maximum distance the the point is allowed to move between frames to be considered the same point. Our tether value was 5%.
    3. Automark - The minimum match score required for automatic tracking. The automark score was 4 for our experiment.

Results

Planar Pendulum

Using equation (1) for the motion of an undamped planar pendulum, we estimated the value of amplitude (A_0) and time period (T) for the first few oscillations:

$$\boxed{A_0 = 0.0369m}$$

$$\boxed{T = 1.267s}$$

$$\boxed{g = \frac{4\pi^2 L}{T^2} = 9.94m/s^2}$$

The damping coefficient (γ) and the Q-factor are:

$$\boxed{\gamma = 0.0162}$$

$$\boxed{Q = 307.60309}$$

The effect of damping on time period was found to be in the order of 10^-6 and is negligible.

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The mean time period $\boxed{T_m = 1.26088s}$

The standard deviation of time period is $\boxed{\sigma = 0.01080s}$

These fluctuations are likely due to the discontinuous nature of tracking data rather than damping since they are significantly larger than the change in T caused by damping.

Elliptical Pendulum

The formula of an ellipse from the equations of motion of an elliptical pendulum is:

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2\frac{yx}{BA}cos\phi = sin^2\phi \tag{9}$$

The difference in time period along the x-axis and y-axis is $\boxed{0.005s}$, which leads to the gradual precession of the ellipse over time.

Discussion

To reduce the potential for error, we must increase the smoothness of the data by using:

- A higher FPS camera to capture more data points per second.
- A higher shutter speed camera to get a higher resolution of the image, allowing the tracker to follow the point more closely.
- A dark room with a single stable source of light illuminating the pendulum to avoid any sudden lighting changes and disturbances in the video.
- A curvefit algorithm or program that can estimate the constants in the damped sine function model to a greater number of decimal points.

Appendix

The tracker data for the planar pendulum can be found at - Planar Pendulum Video Tracking Data.txt. The code for the graphs can be found at - Simple Pendulum Video Tracking.ipynb.

The maxima data was isolated by the following code:

```
mx = [] %string of all maxima
mn = [] %string of all minima
t1 = [] %string of timestamps of each maxima
t2 = [] %string of timestamps of each minima

for i in range(1, 9999):

        if(x_val[i-1] > x_val[i] < x_val[i + 1] and x_val[i]<0):
        %If x_val < its neighbours and is negative, it is a minima

            mn.append(x_val[i])
            t2.append(time[i])
```

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```
    elif(x_val[i-1] < x_val[i] > x_val[i + 1] and x_val[i]>0):
    %If x_val > its neighbours and is positivie, it is a maxima

        mx.append(x_val[i])
        t1.append(time[i])
```

This code generates 4 lists - mx, mn, t1 and t2 for the maxima, minima and their respective timestamps. These lists were then converted into numpy arrays before being used to plot the $\ln(A_m)$ vs t graph.