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Kater's Pendulum
Lab Report 3

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Aim

- To study the oscillations of a compound pendulum.
- To find the local value of acceleration due to gravity using a Kater's pendulum.

Theoretical Background

A compound pendulum is an extended object whose mass is distributed throughout its body and is suspended from a pivot point on the body. The time period of such a pendulum is:

$$T = 2\pi\sqrt{\frac{I_l}{mgl}}$$

Where I_l is the moment of inertia of the body about the axis of rotation (at l), g is acceleration due to gravity, m is the total mass of the body, and l is the distance between the pivot point and the centre of mass.

The parallel axis theorem tells us that the moment of inertia of a body about an axis parallel to the axis passing through the centre of mass (I_{CM}) is given by $I_l = I_{CM} + ml^2$. The moment of inertia about the center of mass can also written as mk^2 where k is the radius of gyration of the object. Hence, the time period becomes:

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}} \tag{1}$$

We can see that in the above equation, all point at a distance of l and k^2/l from the center of mass will have the same time period. On a one-dimensional rod, this will correspond to four points: l, $-l$, k^2/l and $-k^2/l$. Note that as l approaches the center of mass, k^2/l will blow up and therefore no longer lie on the rod.

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For a Kater's pendulum suspended from two different knife edges (figure 1), we can get the following relations for the time periods using equation 1:

$$\frac{T_1^2}{4\pi^2} = \frac{k^2 + l_1^2}{gl_1}$$

$$\frac{T_2^2}{4\pi^2} = \frac{k^2 + l_2^2}{gl_2}$$

Combining the above equations to eliminate k^2, we find,

$$\frac{1}{g} = \frac{1}{8\pi^2}\left(\frac{T_1^2 + T_2^2}{l_1 + l_2} + \frac{T_1^2 - T_2^2}{l_1 - l_2}\right) \tag{2}$$

Thus we are able to find g independent of the mass distribution of the pendulum. However, this formula still requires us to know the exact position of the center of mass to find $l_1 - l_2$. This can be circumvented by choosing a configuration such that $T_1 = T_2 = T$ which would leave us with the following relation:

$$g = 4\pi^2\left(\frac{l_1 + l_2}{T^2}\right) \tag{3}$$

We know that a point where this equation holds true can be found since there are two distinct distances which have the same time period T.

We can approximate equation 2 with the following substitutions: $T = (T_1 + T_2)/2$, $\Delta T = (T_1 - T_2)$ and $L = l_1 + l_2$. If we ignore the higher powers of ΔT, g can be approximated as:

$$g \approx \frac{4\pi^2 L}{T^2}\left(1 + \left(\frac{\Delta T}{T}\frac{L}{L - 2l_A}\right)\right) \tag{4}$$

Experimental Setup

- A reversible Kater's Pendulum
- A wall mount
- A spirit level
- A measuring tape
- A pair of pliers
- A wooden and metal weight of the same shape
- A stopwatch
- A heavy wedge with which to determine the centre of mass of the pendulum rod

Least Count of Stopwatch = 0.01 s
Least Count of Measuring Tape = 0.1 cm
Human Reaction Time Error = 0.05 s [see Appendix]

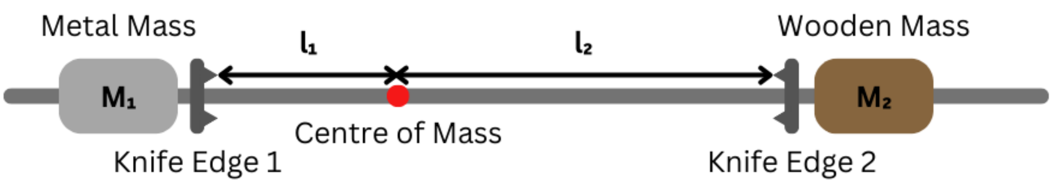

Figure 1: Experimental setup of the Kater's Pendulum
(source: made on https://www.canva.com)

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Note: T_1 is the time period that corresponds to the pendulum being suspended from knife edge 1 with a distance of l_1 from the centre of mass, and likewise for T_2.

Procedure

Part A

1. Using just the rod of the Kater's pendulum as a compound pendulum, find and mark its centre of mass using the wedge.

2. Use the spirit level to check if the wall mount is levelled and adjust it accordingly.

3. Attach a single knife edge at a distance l from the centre of mass and tighten it using a pair of pliers so that it does not slip.

4. Suspend the pendulum from the knife edge using the wall mount and allow it to oscillate with small amplitude. Measure the time period of 10 oscillations.

5. Change the position of the knife edge and measure the new l. Repeat the measurement of the time period for 10 oscillations.

6. Repeat these measurements while moving the knife edge from one end of the rod to the other. Plot a graph for T vs l and show that there are four points which correspond to the same time period.

Part B

1. Attach both knife edges on either side of the centre of the rod and then attach the metal and wooden masses on either end of the rod. Measure the distance between the two knife edges $(l_1 + l_2)$.

2. Take a measurement for the time period of 10 oscillations with each knife edge for a given configuration and divide each reading by 10 to get T_1 and T_2.

3. Change the position of the masses with respect to the knife edges and repeat the measurements for T_1 and T_2.

4. Keep track of when the sign of $T_1 - T_2$ changes and try to find the point where $T_1 = T_2$. Once the difference between T_1 and T_2 is sufficiently small, repeat the measurement for 100 oscillations in both orientations to get a more accurate reading for T_1 and T_2.

5. Find the centre of mass using the wedge for the configuration in which $T_1 \approx T_2$ and use it to find l_1 and l_2.

6. Use these readings to calculate the value of g using equation 3.

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Observations

Part A

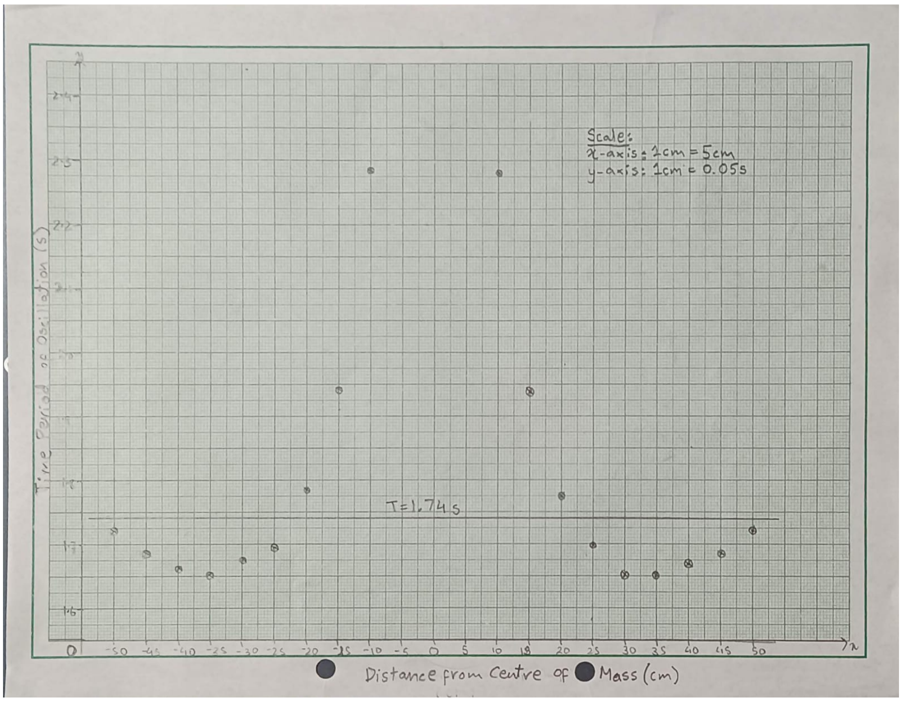

Figure 2: Above a certain distance from the center of mass ($\sim 20\ cm$)
we can find 4 points which correspond to the same time period (raw data in Appendix)

Part B

We kept mass 1 stationary at a distance of $2.7 \pm 0.1\ cm$ and adjusted mass 2 to find the point at which $10T_1 - 10T_2$ was minimised.

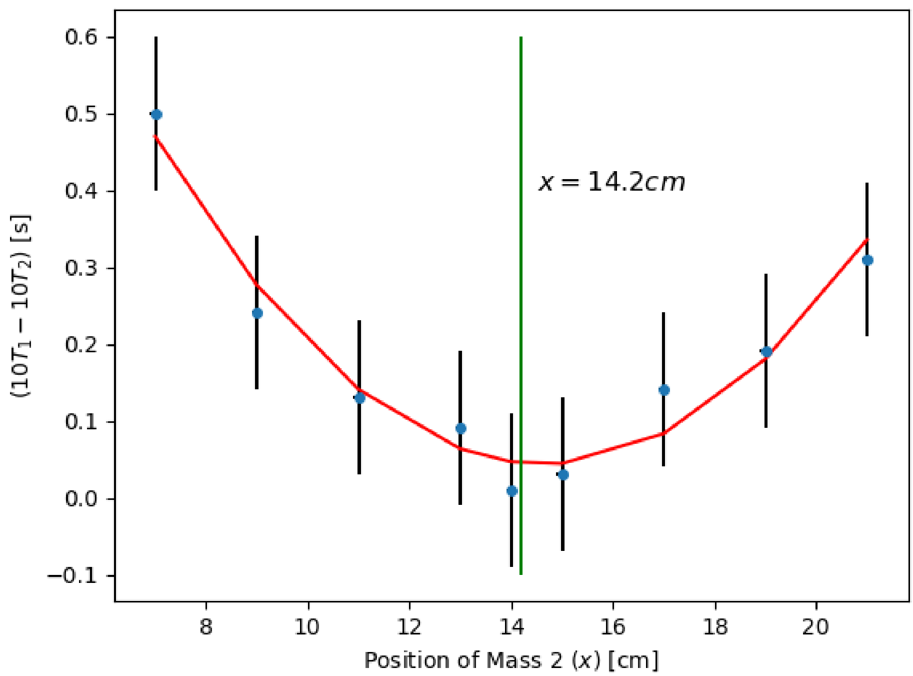

Figure 3: The experimental mimina of $10T_1 - 10T_2$ was found at
$x = 14.2 \pm 0.1\ cm$ at which point $T_1 \approx T_2$ (raw data in Appendix)

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The minima point of $10T_1 - 10T_2$ was found experimentally where M_1 was $2.7 \pm 0.1\ cm$ from knife edge 1 and M_2 was $14.2 \pm 0.1\ cm$ from knife edge 2.

$l_1 + l_2 = 70.9 \pm 0.1\ cm$

$l_1 = 35.4 \pm 0.1\ cm$

$l_2 = 35.5 \pm 0.1\ cm$

For 100 oscillations, we found the time period of T_1 and T_2 to be (Note: the error is $0.1/100 = 0.001\ s$ since the human error is 0.05 s for both the start and stop of 100 oscillations, which is then divided by 100):

$T_1 = 1.737 \pm 0.001\ s$

$T_2 = 1.734 \pm 0.001\ s$

Analysis

Part A

We expect the data in figure 2 to conform to equation 1. We can do this by curve-fitting the data with a function of the form:

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

We attempted to curve-fit this plot by assuming a value of $k = 34.4\ cm$. This gives us the following plot:

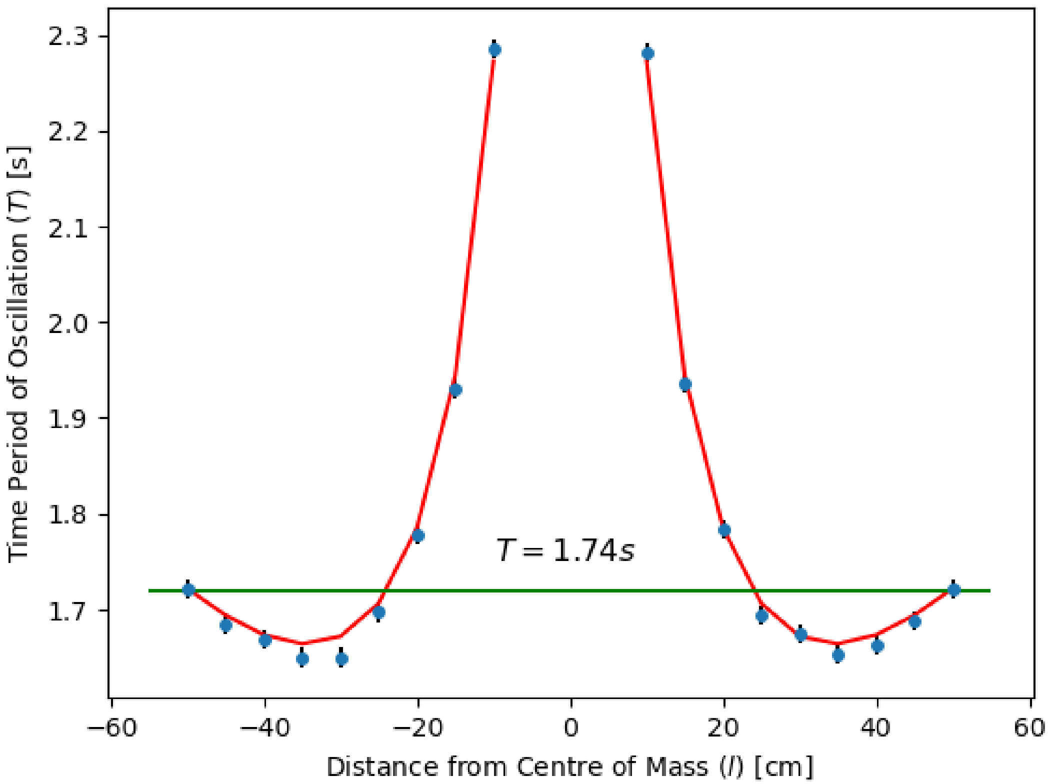

Figure 4: Below a time period of 1.74 s we found that there are 4 possible suspension points that give the same time period

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Clearly our data fits our model equation. We can rearrange equation 1 to find the suspension lengths l that corresponds to a given time period T:

$$l^2 - \frac{T^2 g}{4\pi^2} l + k^2 = 0$$

Solving this quadratic we find:

$$l = \frac{T^2 g}{8\pi^2} \pm \sqrt{\frac{T^4 g^2}{64\pi^4} - k^2}$$

If we take the product of these two solutions we get:

$$l^+ \cdot l^- = \left(\frac{T^2 g}{8\pi^2}\right)^2 - \left(\frac{T^4 g^2}{64\pi^4} - k^2\right)$$

$$\implies l^+ \cdot l^- = \frac{T^4 g^2}{64\pi^4} - \frac{T^4 g^2}{64\pi^4} + k^2 = k^2$$

Hence we find,

$$l^+ = k^2 / l^-$$

This gives us the roots of the equation as l and k^2/l for a given time period T. But since we have negative distances as well, we get another pair of solutions: $-l$ and $-k^2/l$. Therefore, the four suspension points that give us the same time period are l, $-l$, k^2/l, and $-k^2/l$.

We can verify our value of k by finding the four intersection points for $T = 1.74\ s$:

$$l_1 = 22.4 \pm 0.1\ cm$$
$$l_2 = 52.8 \pm 0.1\ cm$$
$$l_3 = -22.4 \pm 0.1\ cm$$
$$l_4 = -52.8 \pm 0.1\ cm$$

Since we know $k^2 = l_1 \cdot l_2$, we can find the radius of gyration is $k = 34.4 \pm 0.1\ cm$.

Part B

From figure 3 it is clear that there is a minimum value of $10T_1 - 10T_2$ which lies close to zero. Experimentally, we found this point when M_1 was $2.7 \pm 0.1\ cm$ from knife edge 1 and M_2 was $14.2 \pm 0.1\ cm$ from knife edge 2.

In this configuration, $l_1 + l_2 = 70.9 \pm 0.1\ cm$, $T_1 = 1.737 \pm 0.001\ s$, and $T_2 = 1.734 \pm 0.001\ s$. Using equation 3 we can find the value of g with this data.

$$g = 8\pi^2 \left(\frac{l_1 + l_2}{T_1^2 + T_2^2}\right) = 9.29 \pm 0.02\ m/s^2$$

Error Analysis

Part A

The error in k depends on the length measurement least count with the following relation:

$$\Delta k = \left(\frac{1}{2} \cdot \frac{\Delta l^+}{l^+} + \frac{1}{2} \cdot \frac{\Delta l^-}{l^-}\right) k$$

$$\implies \Delta k = \left(\frac{1}{2} \cdot \frac{0.1}{22.4} + \frac{1}{2} \cdot \frac{0.1}{52.8}\right) 34.4 = 0.1\ cm$$

Hence, $k = 34.4 \pm 0.1\ cm$.

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Part B

The error in g can be found using the general formula for the propagation of error in a multi-variable function:

$$\delta f = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \delta x_i^2}$$

Here our f is g and the dependent variables are l_1, l_2, T_1 and T_2 with their respective least count errors. We know that g is given by:

$$g = 8\pi^2 \left(\frac{l_1 + l_2}{T_1^2 + T_2^2}\right)$$

Hence,

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial l_1}\right)^2 \delta l_1^2 + \left(\frac{\partial g}{\partial l_2}\right)^2 \delta l_2^2 + \left(\frac{\partial g}{\partial T_1}\right)^2 \delta T_1^2 + \left(\frac{\partial g}{\partial T_2}\right)^2 \delta T_2^2}$$

$$\delta g = \sqrt{\left(\frac{8\pi^2}{T_1^2 + T_2^2}\right)^2 0.1^2 + \left(\frac{8\pi^2}{T_1^2 + T_2^2}\right)^2 0.1^2 + \left(-\frac{16\pi^2 T_1(l_1 + l_2)}{(T_1^2 + T_2^2)^2}\right)^2 0.001^2 + \left(-\frac{16\pi^2 T_2(l_1 + l_2)}{(T_1^2 + T_2^2)^2}\right)^2 0.001^2}$$

$$\implies \delta g = \sqrt{(13.1)^2\, 0.1^2 + (13.1)^2\, 0.1^2 + (-535.0)^2\, 0.001^2 + (-535.9)^2\, 0.001^2} = 2.00\, cm/s^2 = 0.02 m/s^2$$

Therefore the value of $g = 9.29 \pm 0.02 m/s^2$.

Results

Part A

For a compound pendulum such as a uniform massive rod, there are four suspension point which give the same time period. These points correspond to l, $-l$, k^2/l, and $-k^2/l$.

The radius of gyration for the rod we used is $k = 34.4 \pm 0.1\ cm$

Part B

The value of gravitational acceleration is $g = 9.29 \pm 0.02\ m/s^2$

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Appendix

In Lab 1 we conducted an experiment in which we tested our personal human reaction time error by using a stopwatch to time the presence of a coloured square that appeared on a screen for a fixed duration. We then plotted a histogram of our timings (figure 4) and found the standard deviation of our readings.

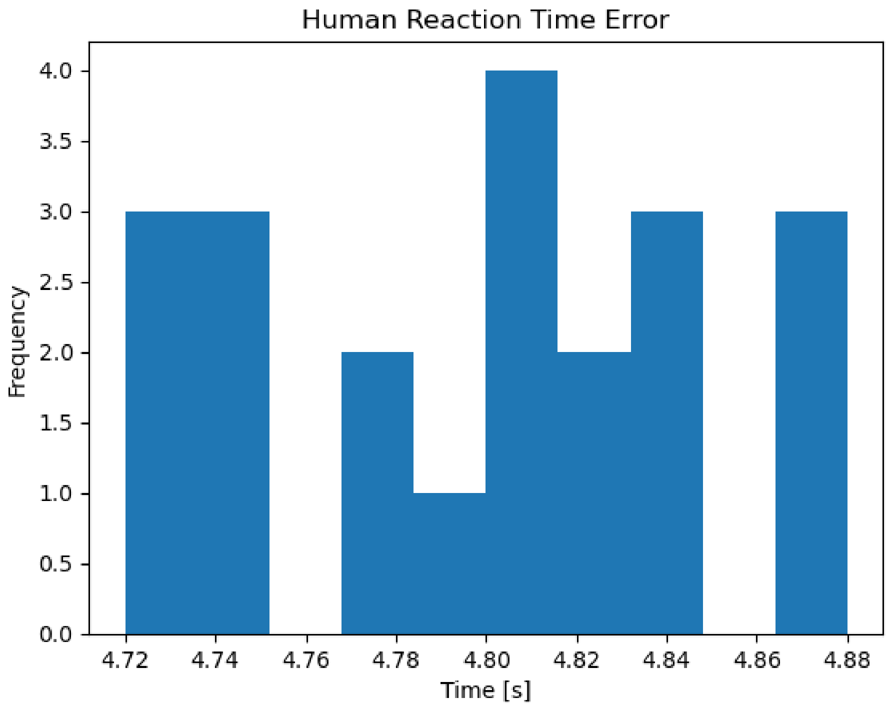

Figure 5: The standard deviation of these readings corresponds to the error we expect in an average time reading taken by us

Mean = 4.8 s
Standard Deviation = 0.05 s

We use the standard deviation to assume our human reaction time error is ~ 0.05 s at a single point of measurement (i.e. a start or stop). For a complete reading we double this error since there is error present at both the start and the stop.

Part A

l $[cm]$	$10T$ $[s]$	T $[s]$
50.0	17.2	1.72
45.0	16.8	1.68
40.0	16.6	1.66
35.0	16.5	1.65
30.0	16.7	1.67
25.0	16.9	1.69
20.0	17.8	1.78
15.0	19.3	1.93

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l [cm]	$10T$ [s]	T [s]
10.0	22.8	2.28
-10.0	22.8	2.28
-15.0	19.3	1.93
-20.0	17.7	1.77
-25.0	16.9	1.69
-30.0	16.5	1.65
-35.0	16.5	1.65
-40.0	16.6	1.66
-45.0	16.8	1.68
-50.0	17.2	1.72

Table 1: Time period of a compound pendulum for various distances of suspension point from the centre of mass

Part B

Position of M_2 [cm]	$10T_1 - 10T_2$ [s]
7.0	0.50
9.0	0.24
11.0	0.13
13.0	0.09
14.0	0.01
15.0	-0.03
17.0	-0.14
19.0	-0.19
21.0	-0.31

Table 2: The time period difference for different positions of wooden mass M_2