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Equipotential Surfaces
Lab Report 4

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Aim

- To study the properties of electric fields in two-dimensions by mapping equipotential curves for various systems.

Theoretical Background

Maxwell's description of the divergence and curl of a static electric field says that

$$\nabla \cdot E = \rho/\epsilon_o \tag{1}$$

$$\nabla \times E = 0 \tag{2}$$

From Stoke's Theorem, we find that $\nabla \times E = 0$ implies $\oint E \cdot dl = 0$. Since the loop integral goes to zero irrespective of path, we can define a quantity V equivalent to the following line integral,

$$V(r) = -\int_O^r E \cdot dl$$

Where O is a standard reference point (which does not coincide with r) and V is the potential at point r. The potential difference between two points is therefore:

$$V(b) - V(a) = -\int_a^b E \cdot dl$$

The fundamental theorem of gradients states that

$$V(b) - V(a) = \int_a^b (\nabla V) \cdot dl$$

Comparing this with our expression for potential difference, we can conclude:

$$E = -\nabla V \tag{3}$$

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This implies that the electric field always points in the direction in which the potential increases most rapidly (i.e. the gradient of V). It therefore follows that in the direction perpendicular to the electric field there is no change in potential. The surface on which the potential is constant is known as an *equipotential surface* and it lies locally perpendicular to the electric field at all points. Putting this back into equation 1, we find Poisson's equation:

$$\nabla^2 V = -\rho/\epsilon_o \tag{4}$$

If there is no net charge in the space being considered, then $\rho = 0$ and Poisson's equations reduces to Laplace's equation.

$$\nabla^2 V = 0 \tag{5}$$

Parallel-Bar System

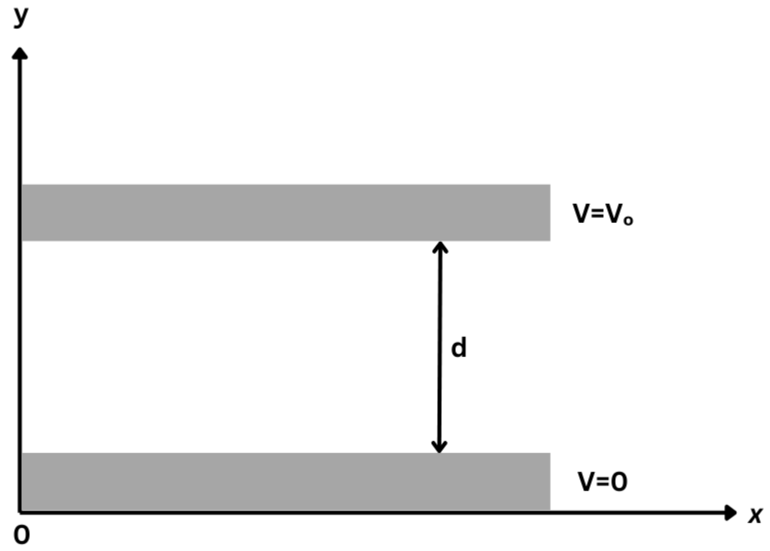

Figure 1: Schematic diagram of a parallel bar system
Source: made on www.canva.com

Consider the two-dimensional system represented in figure 1 consisting of two bars, one of which is grounded and kept at the origin while the other is placed at a distance d along the y-axis and at a potential of V_o. Since there is no variation of potential along the x-axis, the Laplace equation for the region between the bars (assuming the bars are infinitely long), reduces to:

$$\frac{d^2V}{dy^2} = 0$$

This can be integrated twice to get the equation for potential,

$$V = ky + C$$

Using the boundary condition that $V = 0$ when $y = 0$, we find $C = 0$. Hence, $V(y) = ky$. Using the second boundary condition, at distance d, the potential $V_o = kd$. Dividing this with the general equation, we derive the following relation:

$$V(y) = V_o \left(\frac{y}{d}\right) \tag{6}$$

From equation 6 we would expect to see a constant $V(y)$ for a given value of y. This implies that the equiotential surfaces between the two bars will simply be straight lines parallel to the x-axis (i.e. where y is constant).

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Concentric Cylinder System

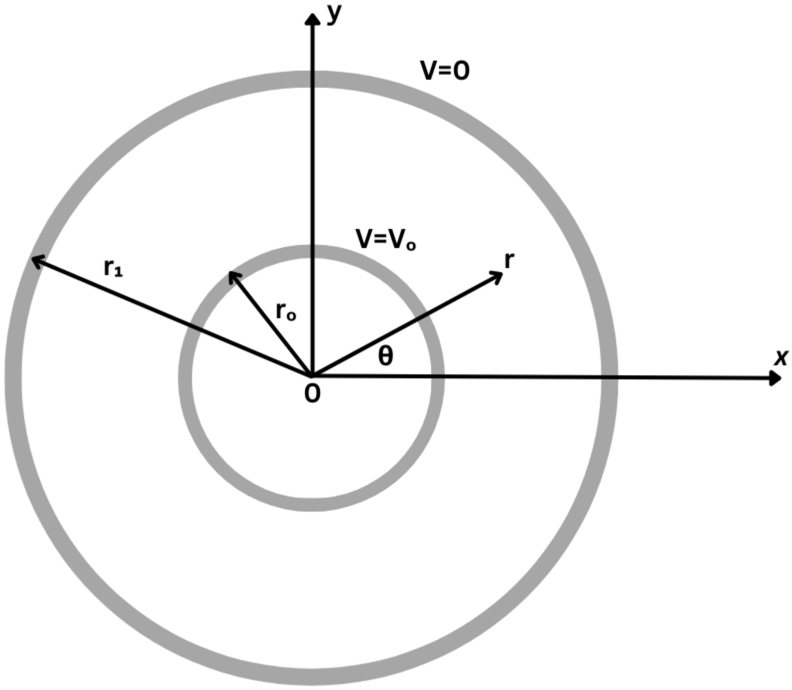

Figure 2: Schematic diagram of a concentric cylinder system
Source: made on www.canva.com

Consider the two-dimensional system in figure 2 consisting of two concentric cylinders of radius r_o and r_1, respectively, where $r_o < r_1$. The inner cylinder is maintained at a potential V_o while the outer cylinder is grounded. If we consider the Laplace equation for the region between the cylinders in cylinderical coordinates, we can ignore the θ term due to symmetry. This leaves us with following equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

Integrating this twice gives us the following expression for potential:

$$V = a \ln r + b$$

Applying the boundary conditions, at $r = r_1$ the potential $V_1 = a \ln r_1 + b$. Hence, $V = a \ln(r/r_1) + V_1$. The other boundary condition gives us $V_o = a \ln r_o + b$ at $r = r_o$. Hence, $a = (V_1 - V_o)/\ln(r_1/r_o)$. Putting this into $V = a \ln(r/r_o) + V_o$, we get:

$$V = \frac{(V_1 - V_o)}{\ln\left(\frac{r_1}{r_o}\right)} \ln\left(\frac{r}{r_o}\right) + V_o$$

Note that in our case where $V_1 = 0$, the equation reduces to:

$$V(r) = \frac{V_o}{\ln\left(\frac{r_o}{r_1}\right)} \ln\left(\frac{r}{r_o}\right) + V_o \tag{7}$$

Equation 7 tells us that we should expect to see a constant $V(r)$ for a given value of r. This corresponds to concentric circular equipotential surfaces for $r_o < r < r_1$.

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Experimental Setup

- An acrylic tray with a graph sheet underneath it
- An electrolytic medium (tap water)
- An AC power supply
- A set of metallic cuboid and cylindrical electrodes
- A digital multi-meter set to measure voltage
- A pointed probe attached to a stand
- A set of connecting cords

Least Count of Probe Reading = 0.3 cm

Least Count of Voltmeter = 0.01 V

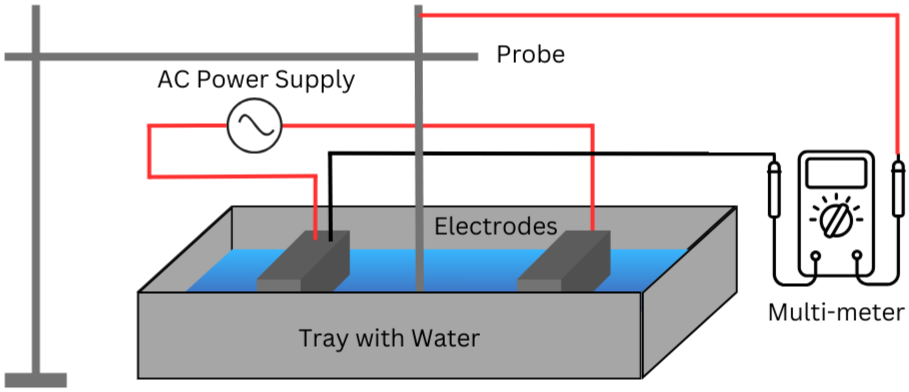

Figure 3: Experimental setup to measure equipotential surfaces
Source: made on www.canva.com

Procedure

1. Place two long bar electrodes in the tray symmetrically about the origin, parallel to the x-axis.
2. Connect an AC power supply to both electrodes and fill the tray with tap water until the bars are half submerged.
   [Precaution: Use an AC voltage supply since DC voltage may cause electrolysis and erode the aluminium electrodes. Do not submerge the electrodes completely in the water.]
3. Attach the ground of the multi-meter to one of the electrodes and the positive end to the pointed probe.
4. Measure the potential at an arbitrary point between the two electrodes using the probe and note down the position on the graph sheet.
5. Carefully move the probe and find the points at which the potential is the same.
6. Repeat this process for at least 3 more potentials between the electrodes. Plot these equipotential surfaces and find their shape.
7. Compare with the theoretically expected result in the appropriate limit.
8. Repeat the above steps with a concentric cylinder system and then an arbitrary system of your choice.

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Observations

Part A

Voltage Supplied (V_o) = 10.08 ± 0.01 V
Position of Bar 1 (d_1) = 4.0 ± 0.1 cm
Position of Bar 2 (d_2) = -4.0 ± 0.1 cm

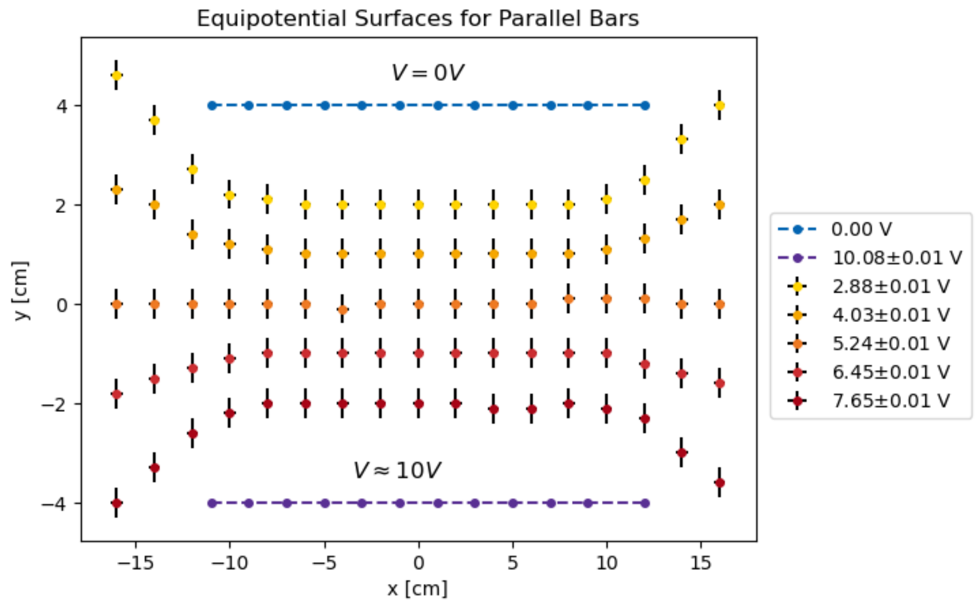

Figure 4: The equipotential curves for a parallel bar system appear to be parallel to the x-axis until the edges where they begin to diverge

Part B

Voltage Supplied (V_o) = 10.08 ± 0.01 V
Radius of Inner Cylinder (r_o) = 2.0 ± 0.1 cm
Radius of Outer Cylinder (r_1) = 8.0 ± 0.1 cm

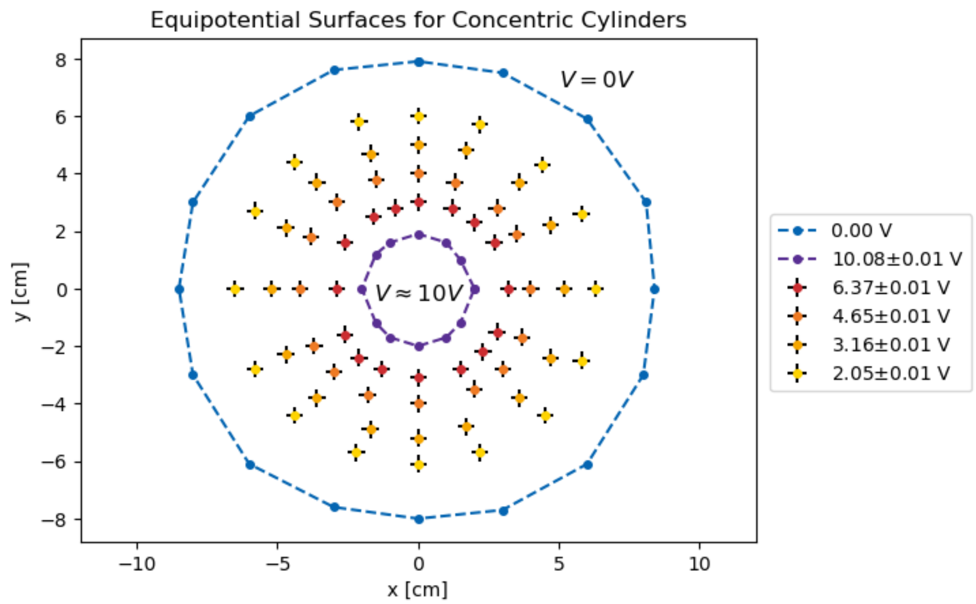

Figure 5: The equipotential curves for a concentric cylinder system appear to be radially symmetric

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Part C

For this part of the experiment, we chose a system consisting of a long bar which was grounded and two cylinders placed symmetrically about the y-axis and both are maintained at a potential V_o.

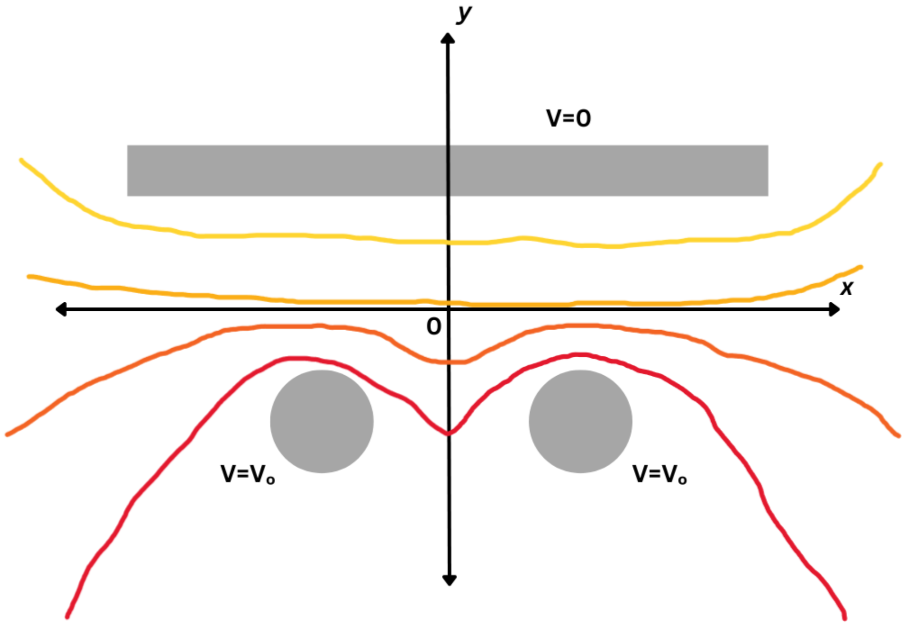

Figure 6: Schematic diagram of the bar and two cylinder system with the expected shape of the equipotential curves
Source: made on www.canva.com

Voltage supplied (V_o) = 10.16 ± 0.01 V

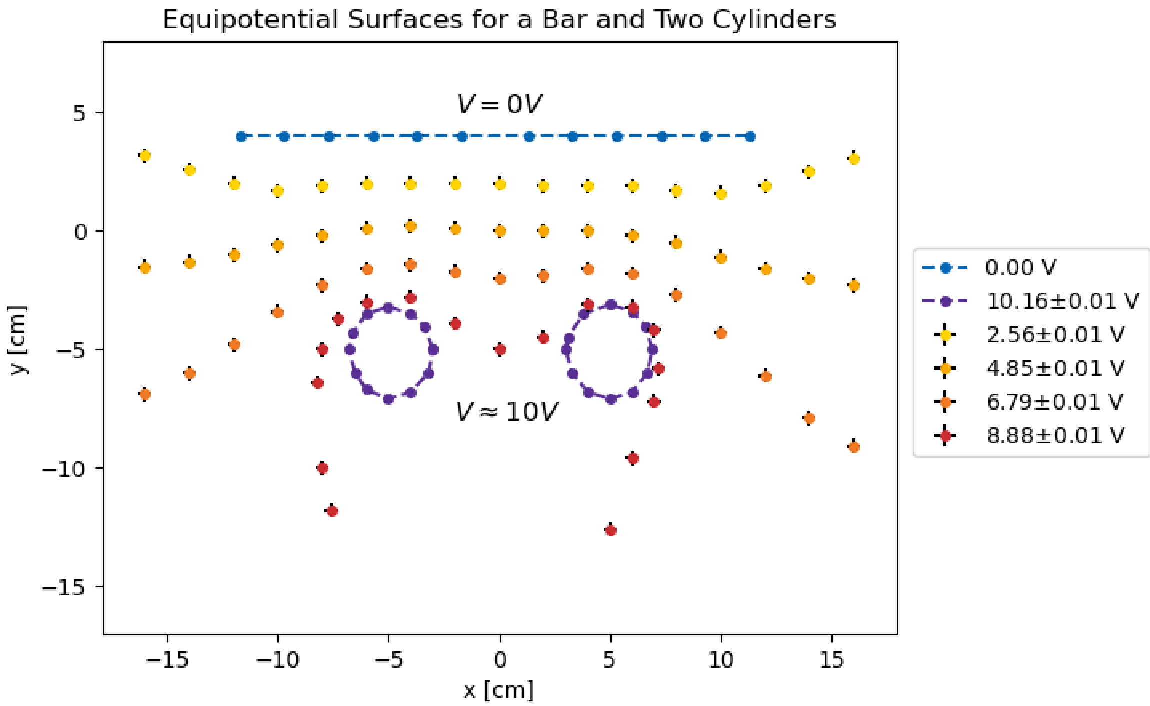

Figure 7: The equipotential curves for our system roughly coincides with the expected equipotential curves in figure 6

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Analysis

Part A

In figure 4 we can clearly see that at a certain distance from the edge of the bars, the equipotential curves are straight lines parallel to the x-axis. This is in accordance with equation 6 which predicts that for infinitely long bars (which can be considered the case at a sufficient distance from the ends), the potential is constant for a given value of y. Close to the ends of the bar the equipotential curves begin to diverge since there are lines of force which emerge from the ends and meet the other electrode, thereby changing the Laplace equation for this region. But within a certain limit, we can assume:

$$V(y) = V_o\left(\frac{y}{d}\right)$$

However, in our case the origin is between the bars so we have different boundary conditions. At $y = d_1$, $V = 0$ and at $y = d_2$, $V = V_o$. Applying these conditions to $V = ky + C$ we get:

$$V(y) = \frac{V_o y}{(d_2 - d_1)} - \frac{V_o d_1}{(d_2 - d_1)} \tag{8}$$

We can verify this equation by plotting a graph of potential against y (at $x = 0$ in the graph below):

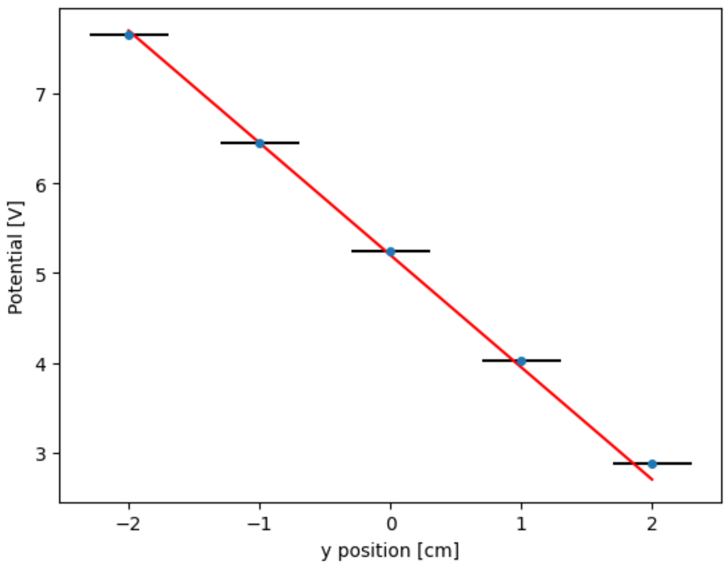

Figure 8: The potential varies linearly with the y-position of the equipotential surface

The slope of the above graph is -1.24 V/cm and the intercept is 5.02 V. The expected value of $V_o/(d_2 - d_1) = -1.26$ V/cm and the intercept is $-V_o d_1/(d_2 - d_1) = 5.04$ V.

Part B

In figure 5 we can see that the equipotential surfaces are radially symmetric in accordance with equation 7 which predicts that for a given radius r the potential will be constant.

$$V(r) = \frac{V_o}{\ln\left(\frac{r_o}{r_1}\right)} \ln\left(\frac{r}{r_o}\right) + V_o$$

We can verify this equation by plotting a graph of potential against $\ln(r/r_o)$:

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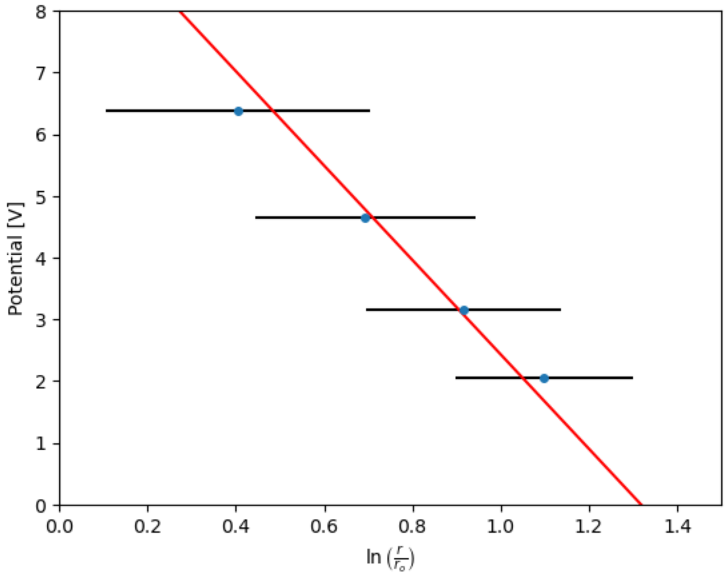

Figure 9: The relationship between potential and $\ln(r/r_o)$ appears to be linear and assuming a y-intercept at $V_o = 10.16$ V we get a slope of $m = -7.65$ V

Using the known value of y-intercept of 10.16 ± 0.01 V, we find the slope is $m = -7.65$ V. The expected slope from equation 7 is $V_o/\ln(r_o/r_1) = -7.27$ V.

Part C

The expected shape of the equipotential surfaces for the bar and two cylinder system was predicted based on our understanding of how the field lines would connect the electrodes. We noticed that the region close to the top of the cylinders is particularly sensitive to potential change since every surface below them must also curve over the cylinders to lie between the bar and cylinders. This means that the gradient of the potential is not uniform, implying a changing electric field in the region between the bar and cylinders. Further analysis would require solving the Laplace equation for this system which would prove a challenging task given its asymmetric geometry.

Discussion and Error Analysis

Part A

The error in the expected value of slope can be found from the least count errors of the potential and position:

$$\Delta m = |m|\sqrt{\left(\frac{\Delta V_o}{V_o}\right)^2 + \left(\frac{\Delta(d_2 - d_1)}{|d_2 - d_1|}\right)^2}$$

$$\Delta m = 1.26\sqrt{\left(\frac{0.01}{10.08}\right)^2 + \left(\frac{0.2}{8}\right)^2} = 0.02\ V/cm$$

Hence, the experimentally found slope $m = -1.24$ lies within the acceptable margin of error since the slope is given by $V_o/(d_2 - d_1) = -1.26 \pm 0.02$ V/cm. The percentage error of slope is 1.6%.

The expected intercept is $-V_o d_1/(d_2 - d_1) = 5.04$ with error originating from length and potential measurement least counts.

$$\Delta m = |m|\sqrt{\left(\frac{\Delta V_o}{V_o}\right)^2 + \left(\frac{\Delta d_1}{d_1}\right)^2 + \left(\frac{\Delta(d_2 - d_1)}{|d_2 - d_1|}\right)^2}$$

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$$\Delta m = 1.26\sqrt{\left(\frac{0.01}{10.08}\right)^2 + \left(\frac{0.1}{4}\right)^2 + \left(\frac{0.2}{8}\right)^2} = 0.04\ V$$

Hence, the experimentally found intercept $c = 5.02$ V lies within the acceptable margin of error since the slope is given by $-V_o d_1/(d_2 - d_1) = 5.04 \pm 0.04$ V. The percentage error of intercept is 0.4%.

Part B

The error in this value comes from the error in V_o, r_o and r_1. The propogation of error in $\ln(r/r_o)$ and $\Delta \ln(r/r_o)/\ln(r/r_o) \approx \Delta r/r + \Delta r_o/r_o$:

$$\Delta m = |m|\sqrt{\left(\frac{\Delta V_o}{V_o}\right)^2 + \left(\frac{\Delta r_0}{r_o}\right)^2 + \left(\frac{\Delta r_1}{r_1}\right)^2}$$

$$\Delta m = 7.27\sqrt{\left(\frac{0.01}{10.08}\right)^2 + \left(\frac{0.1}{2}\right)^2 + \left(\frac{0.1}{8}\right)^2} = 0.37\ V$$

Hence, the expected slope is $V_o/\ln(r_o/r_1) = -7.27 \pm 0.37$ V. The experimentally found slope of $m = -7.65$ V therefore lies within the margin of error. The percentage error of slope is 5.2%.

Results

Part A

In a parallel bar system the equipotential surfaces (at a certain distance from the ends of the bars) follow the following equation:

$$V(y) = \frac{V_o y}{(d_2 - d_1)} - \frac{V_o d_1}{(d_2 - d_1)}$$

Where $\frac{V_o y}{(d_2 - d_1)} = -1.26 \pm 0.02$ V/cm and $\frac{V_o d_1}{(d_2 - d_1)} = -5.04 \pm 0.04$ V.

Part B

In a concentric cylinder system the equipotential surfaces follow the following equation:

$$V(r) = \frac{V_o}{\ln\left(\frac{r_o}{r_1}\right)} \ln\left(\frac{r}{r_o}\right) + V_o$$

Where $V_o/\ln(r_o/r_1) = -7.27 \pm 0.37$ V and $V_o = 10.16 \pm 0.01$ V.

Part C

The equipotential surfaces for our system are non-uniformly distributed between the bar and cylinders such that the potential gradient (which is proportional to the electric field) is much higher close to the potential source cylinders. Further analysis would require solving the Laplace equation for this system which would prove a challenging task given its asymmetric geometry.

Appendix

The raw data for the experiment can be found in the following drive file
$https : //drive.google.com/file/d/1MBhxsXRKR_PtkxQzaEKWx{-}sJdtPnw95T/view?usp = sharing$