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Coupled Pendulum

Lab Report 5

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Date of Experiment: April 5th, 2024
Date of Submission: April 15th, 2024

Aim

- Determine the spring constant of different soft massive springs using both static and dynamic methods.
- Find the normal modes of a coupled pendulum system.
- Setup beats in a coupled pendulum system and use the Fourier transform to show that it is the sum of normal mode frequencies.
- Show that arbitrary oscillations are a linear combination of normal modes using the Fourier transform to analyse position data.

Theoretical Background

A coupled pendulum system consists of two simple pendulums whose bobs are connected with a spring. Such a system has two degrees of freedom. If we assume the angular displacements of each pendulum are small, we can parameterise the system in terms of the horizontal displacements of the bobs - x_1 and x_2 (figure 1). We can use Newton's second law ($F = ma$) to set up the equations of motion. Assuming $m_1 = m_2$ and $l_1 = l_2$, the equations of motion are:

$$\frac{d^2x_1}{dt^2} = -g\frac{x_1}{l} + \frac{k}{m}(x_2 - x_1)$$

$$\frac{d^2x_2}{dt^2} = -g\frac{x_2}{l} + \frac{k}{m}(x_1 - x_2)$$

Where $l_1 = l_2 = l$, $m_1 = m_2 = m$, g is gravitational acceleration and k is the spring constant of the spring attached between the two bobs. Note that these equations are coupled and therefore cannot be solved independently.

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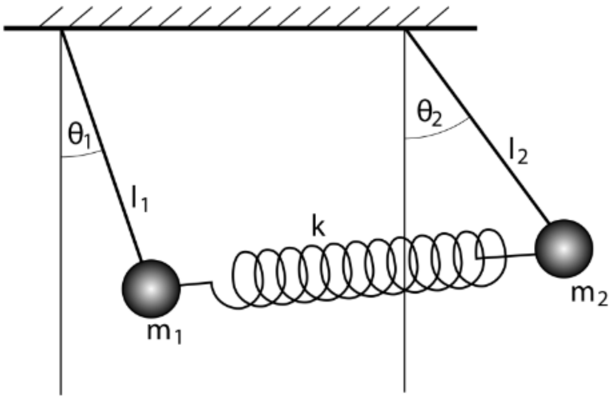

Figure 1: Schematic diagram of a coupled pendulum system
Source: Lab 3 Coupled Pendulum Handout

In order to uncouple the system, we can take the sum and difference of these equations:

$$\frac{d^2(x_2 + x_1)}{dt^2} = -\frac{g}{l}(x_2 + x_1)$$

$$\frac{d^2(x_2 - x_1)}{dt^2} = -\left(\frac{g}{l} + \frac{2k}{m}\right)(x_2 - x_1)$$

Let $X = x_2 + x_1$ and $Y = x_2 - x_1$:

$$\frac{d^2X}{dt^2} = -\omega_1^2 X \tag{1}$$

$$\frac{d^2Y}{dt^2} = -\omega_2^2 Y \tag{2}$$

Where $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} - \frac{2k}{m}}$. These equations are uncoupled and can be solved as independent simple harmonic oscillators. The solutions for equation 1 and 2, respectively, are:

$$X = a_1 \cos(\omega_1 t + \delta_1)$$

$$Y = a_2 \cos(\omega_2 t + \delta_2)$$

Since $x_2 = (X + Y)/2$ and $x_1 = (X - Y)/2$, we can combine these solutions to find the horizontal positions of each bob:

$$x_1 = b_1 \cos(\omega_1 t + \delta_1) + b_2 \cos(\omega_2 t + \delta_2) \tag{3}$$

$$x_2 = b_1 \cos(\omega_1 t + \delta_1) - b_2 \cos(\omega_2 t + \delta_2) \tag{4}$$

Where b_1, b_2, δ_1 and δ_2 are constants that are determined by the initial conditions. For any arbitrary system, the positions of x_1 and x_2 are described by the superposition of two waves with frequencies ω_1 and ω_2.

If we consider a special case of initial conditions where $x_1 = x_2$ and $\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0$, then $Y = 0$ and the system is entirely described by ω_1 and both the pendulums oscillate in phase. If $x_1 = -x_2$ and $\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0$, then $X = 0$ and the system is entirely described by ω_2 and both the pendulums oscillate with a phase difference of π. These are known as the normal modes of the system and any arbitrary motion is simply the superposition of these two states.

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The phenomenon of beats is observed when $x_1 = A$, $x_2 = 0$ and $\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0$. The equations of motion can be simplified to the following:

$$x_1 = A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right) \tag{5}$$

$$x_2 = A\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_1 - \omega_2}{2}t\right) \tag{6}$$

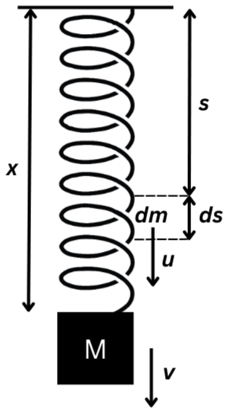

Figure 2: Schematic diagram of a soft massive spring
Source: made on www.canva.com

Since the springs we use have a non-negligible mass, we must account for the mass of the spring itself. The kinetic energy of the spring is given by:

$$K_s = \int \frac{1}{2}u^2 dm$$

Where dm is an infinitesimal mass element of the spring at position s, and u is the velocity of that mass element. If we assume the spring stretches homogeneously, then $dm = (m/x)ds$ and the velocity is $u = sv/x$ where m is the mass of the spring, x is the total length of the spring at time t and v is the velocity of the suspended mass. The integral therefore becomes:

$$K_s = \frac{1}{2}\frac{m}{x^3}v^2 \int_0^x s^2 ds$$

$$\implies K_s = \frac{1}{2}\frac{m}{3}v^2$$

The gravitational potential energy of the spring can similarly be found using:

$$U_s = \int gsdm$$

$$\implies U_s = \frac{mg}{x}\int_0^x sds$$

$$\implies U_s = \frac{1}{2}mgx$$

Hence, the total energy of the spring will be:

$$E = U_M + K_M + U_s + K_s + U_k$$

$$\implies E = \frac{1}{2}Mv^2 + Mgx + \frac{1}{2}\frac{m}{3}v^2 + \frac{1}{2}mgx + \frac{1}{2}kx^2$$

$$\implies E = \frac{1}{2}\left(M + \frac{m}{3}\right)v^2 + \frac{1}{2}\left(M + \frac{m}{2}\right)gx + \frac{1}{2}kx^2$$

Where M is the mass of the suspended weight. Differentiating with respect to time, we get the equation of motion for the spring:

$$\left(M + \frac{m}{3}\right)a = -kx - \left(M + \frac{m}{2}\right)g$$

$$\implies \left(M + \frac{m}{3}\right)a = -kx_{eff}$$

At equilibrium, the spring constant is:

$$k = -\left(M + \frac{m}{2}\right)\frac{g}{x_o} \tag{7}$$

And while oscillating, the frequency of oscillation is:

$$\omega = \sqrt{\frac{k}{\left(M + \frac{m}{3}\right)}} \tag{8}$$

Experimental Setup

- Two simple pendulums with equal length and mass
- Styrofoam sheet or any white background
- Neon stickers
- Springs of different spring constants
- Stands with clamps
- Stopwatch
- Weights
- Weighing scale
- Measuring Tape
- Spirit level
- Calibration stick
- Camera and tripod
- Tracker software

Least count of weighing scale = 0.1 g
Least count of measuring tape = 0.1 cm

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Procedure

Spring Constant

1. Use a weighing scale to measure the mass of a spring.
2. Suspend the spring vertically from a stand and measure its unstretched length with a measuring tape.
3. Attach a mass to the suspended spring and allow it to come to rest. Measure the extension of the spring at equilibrium for the given mass.
4. Give the mass a small vertical displacement and let the spring to oscillate. Measure the time period of 10 oscillations multiple times. Divide these values by 10 and average them to find the time period of each oscillation.
5. Repeat steps 3 and 4 for multiple different masses.
6. Plot the extensions vs the effective mass times gravitational acceleration (g = 9.8 m/s^2) i.e. the force to determine the spring constant. Compare this value with the spring constant found by plotting the angular frequency squared (ω^2) vs the reciprocal of the effective mass.
7. Repeat this analysis for multiple different springs and find their spring constants via both static and dynamic methods.

Coupled Pendulum

1. Attach each end of the spring to the bobs of the simple pendulums. Ensure that at rest both the bobs are at the same level. Place the styrofoam sheet behind the bobs to allow the camera to easily capture them.
2. Set up a camera and tripod to record the movement of the bobs. Keep a calibration stick in the frame after measuring its length to calibrate the data in the video. Stick a neon sticker on each bob and mark a dot in the center of each sticker as a tracking reference.
3. First give the bobs an equal displacement in the same direction to set them in phase with each other. This is the first normal mode of oscillation. Record a 30 second video of this motion.
4. Second, give the bobs an equal displacement in opposite directions to set them π radians out of phase. This is the second normal mode of oscillation. Record another 30 second video of this motion.
5. Finally, keep one bob stationary and give the other bob a push. This induces beats where the energy starts in one bob and is gradually transferred to the other bob until it is stationary. Record another video of this motion.
6. Using the spring with the highest spring constant (and therefore most distinct normal modes), set the bobs into arbitrary motion and record their motion.
7. Upload the videos onto the Tracker software and track the motion of the dots on the neon stickers to determine the horizontal motion of the bobs. Use a Fast Fourier Transform (FFT) to find the frequency content of the motion and show that any motion of the bobs is a superposition of the two normal mode frequencies.

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Observations

Spring Constant

Least count of measuring tape = 0.1 cm
Least count of weighing scale = 0.1 g

Spring	Mass of Spring m_s [g]	Unextended Length x_0 [cm]
A	15.9	21.8
B	9.6	19.6
C	20.3	20.2
D	23.9	19.7

Table 1: The mass and unextended length of each spring

Mass [g]	x_a [cm]	x_b [cm]	x_c [cm]	x_d [cm]
10.2	33.1	34.2	23.9	23.4
7.1	29.4	29.6	22.7	22.2
5.0	27.1	26.6	21.9	21.4
3.1	24.9	23.9	21.2	20.6
2.1	23.9	22.4	20.9	20.3

Table 2: The extended equilibrium length of each spring when different masses were suspended from them

For the dynamic method, 3 readings were taken for each mass and the mean value was used to determine the time period of the oscillations.

Mass [g]	T_a [s]	T_b [s]	T_c [s]	T_d [s]
10.2	0.846	0.887	0.521	0.568
7.1	0.763	0.796	0.485	0.508
5.0	0.692	0.705	0.446	0.482
3.1	0.641	0.626	0.407	0.450
2.1	0.598	0.577	0.383	0.429

Table 3: The mean time period of oscillations of each spring for different masses

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Coupled Pendulum

Tracker was used to track the motion of the bobs, labelled mass A and mass B in the graphs. The amplitude of oscillation has been normalised and mass A has been plotted between 0 and 1 while mass B has been plotted between -1 and 0. This is to allow easy comparison between the waveforms.

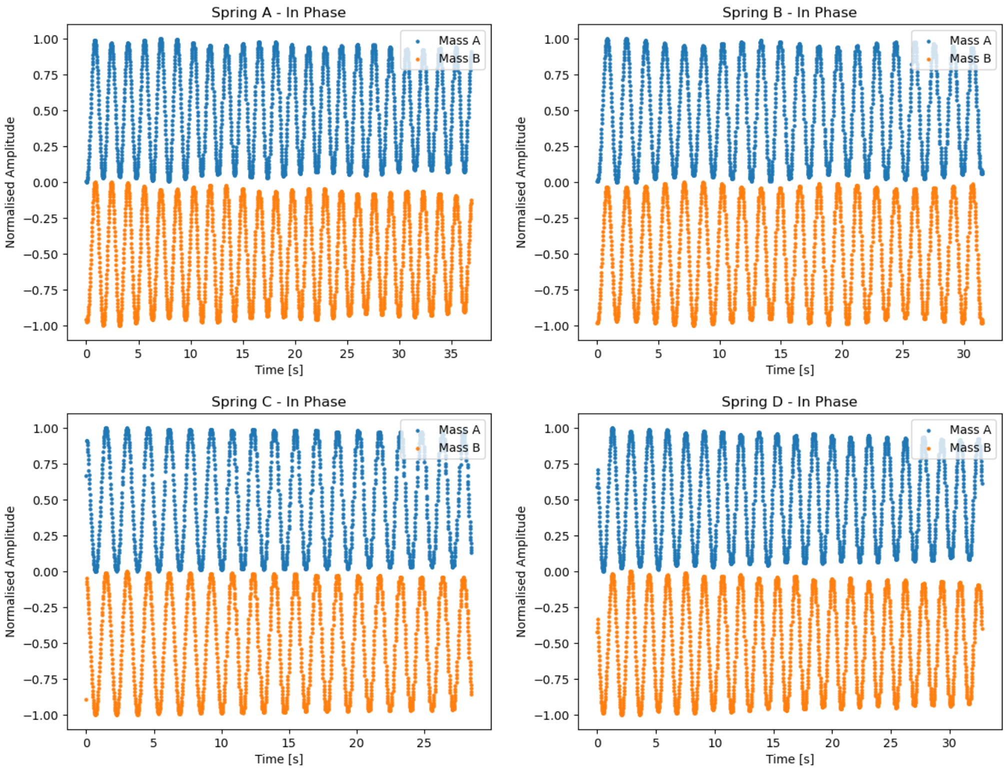

Figure 3: In phase oscillations (normal mode 1) for each spring

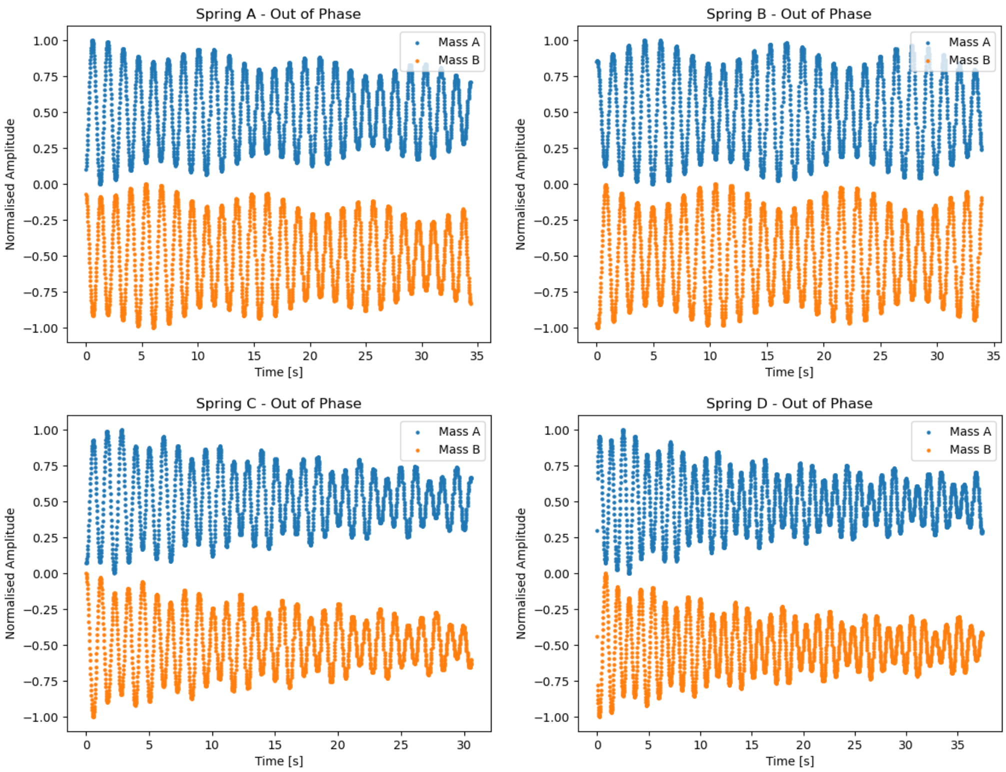

Figure 4: Out of phase oscillations (normal mode 2) for each spring

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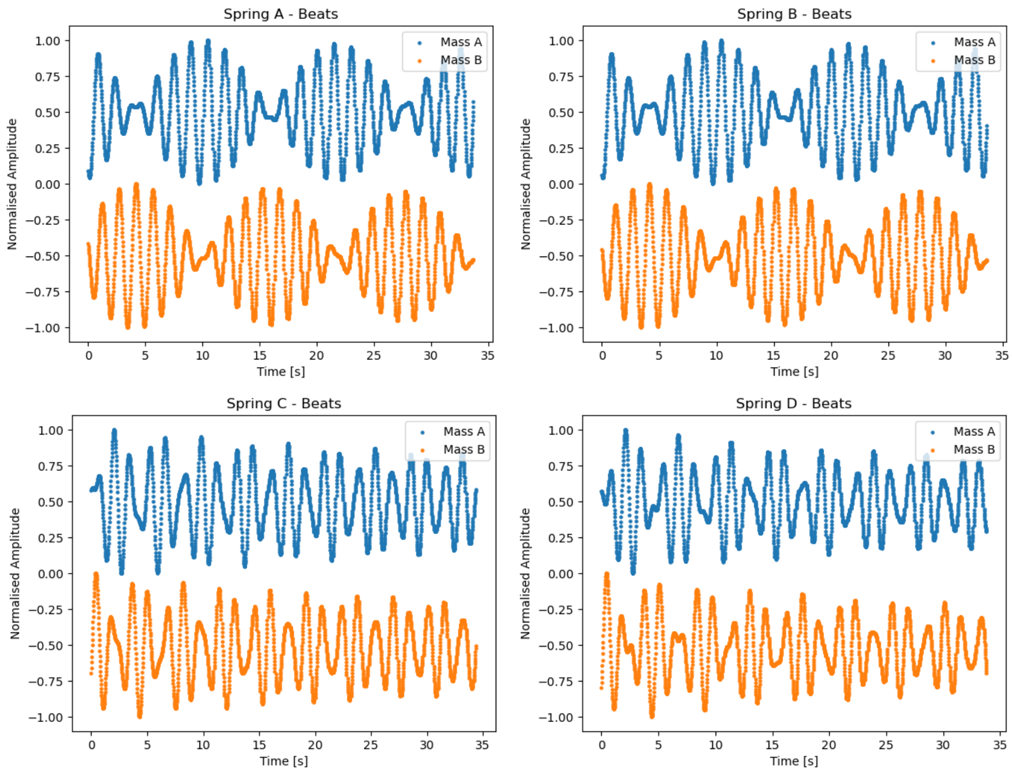

Figure 5: Beat oscillations for each spring

Spring C was observed to have the highest spring constant and was therefore used to demonstrate arbitrary oscillations.

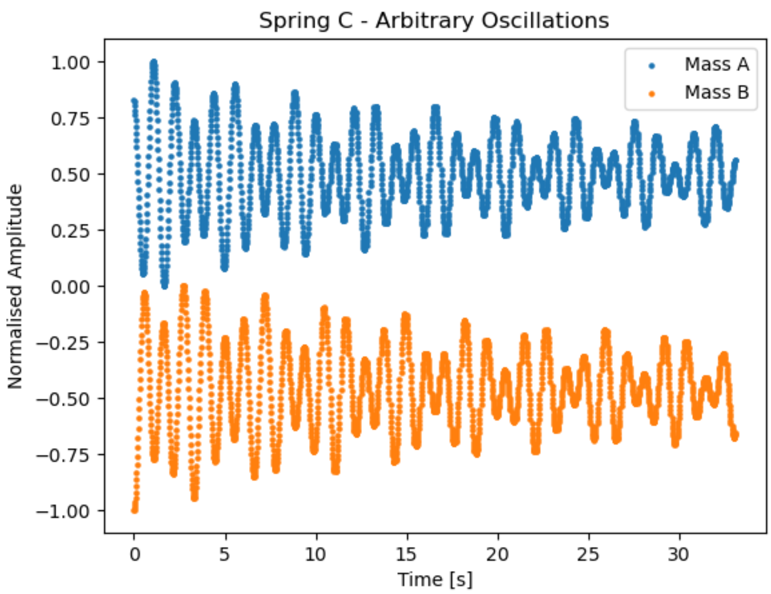

Figure 6: Arbitrary oscillations in spring C

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Data and Error Analysis

Spring Constant

Using equation 7, if we plot the force ($F = (M + m_s/2)g$) vs the extension (x_o), the slope should give us the spring constant of the spring.

$$k = -\frac{F}{x_o} = -\left(M + \frac{m}{2}\right)\frac{g}{x_o}$$

We can used the data from table 1 and 2 to determine the equilibrium extension of the spring and the effective mass. We took g = 9.8 m/s^2.

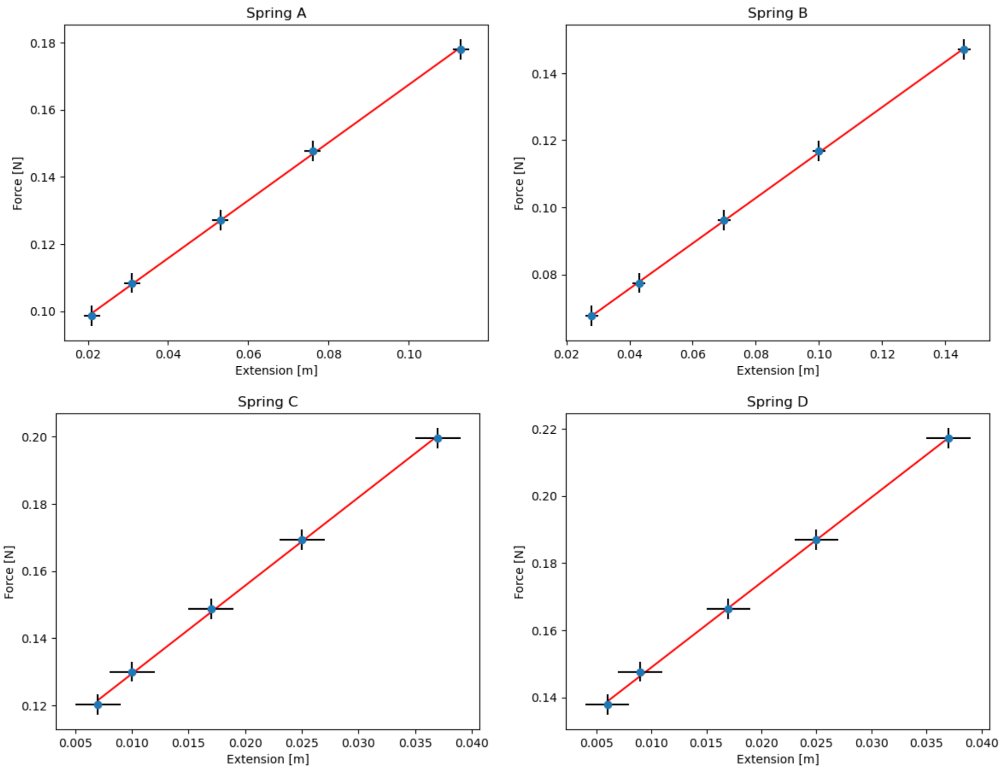

Figure 7: The force on the spring-mass system varies linearly with respect to the extension of the spring. The slope of this graph is used to find the spring constant of each spring.

From the slope of the above graphs, we found that spring constants for the springs:

Spring A (k_a) = 0.862 N/m
Spring B (k_b) = 0.676 N/m
Spring C (k_c) = 2.626 N/m
Spring D (k_d) = 2.533 N/m

The error in the measurement of the mass and extension of the spring contribute to the error in k. We can account for this by considering the error in quadrature:

$$\Delta k = k\sqrt{\left(\frac{\Delta(M + m/2)}{(M + m/2)}\right)^2 + \left(\frac{\Delta x_o}{x_o}\right)^2}$$

Where $\Delta M + m/2 = 0.15$ g and $\Delta x_o = 0.1$ cm. The above formula was used to plot the errorbars in the graphs. A linear regression analysis with a larger data set would be required to determine the exact error in the experimentally found values of spring constant.

The time period of the oscillations of the springs were used to find their angular frequency ($\omega = 2\pi/T$) whose square was plotted against the reciprocal of the mass to find the spring constant. The dynamic analysis is a result of equation 8:

$$\omega^2 = \frac{k}{(M + \frac{m}{3})}$$

The data from table 1 and 3 were used to find the angular frequency and effective mass of each spring-mass system.

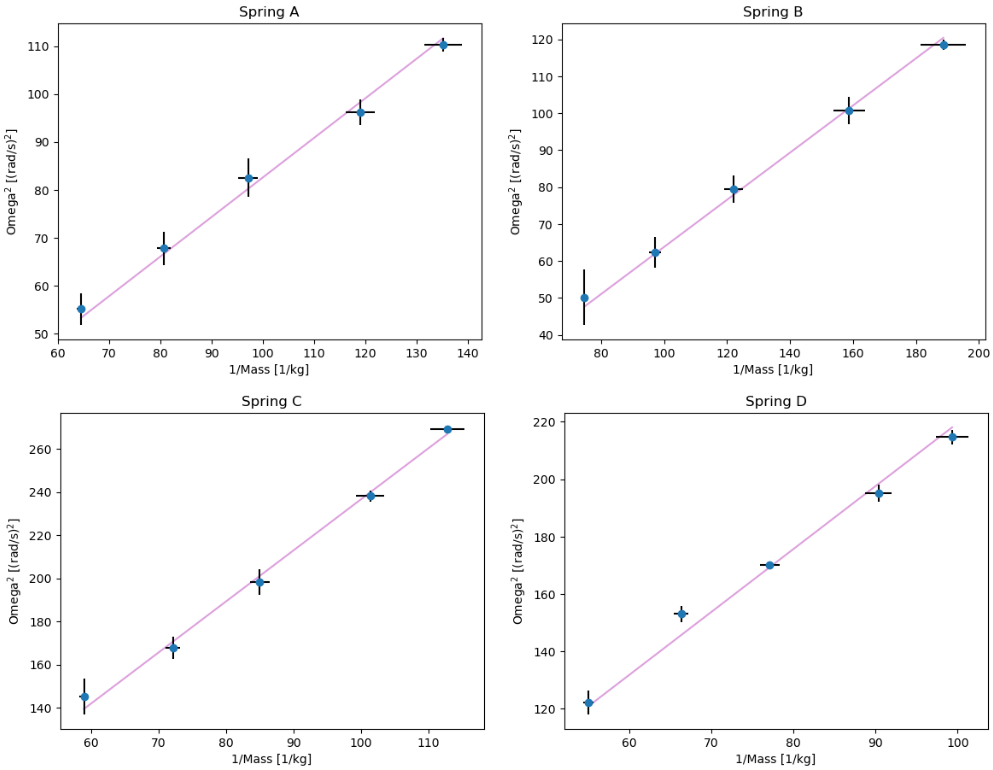

Figure 8: The square of the angular frequency of the spring-mass system varies linearly with respect to the reciprocal of the effective mass. The slope of this graph is used to find the spring constant of each spring.

From the slope of the above graphs, we found that spring constants for the springs:

Spring A (k_a) = 0.826 N/m
Spring B (k_b) = 0.638 N/m
Spring C (k_c) = 2.366 N/m
Spring D (k_d) = 2.195 N/m

The error in the measurement of the time period and mass of the spring-mass system contribute to the error in k. Since the time period measurement is subject to human error, each measurement was performed three times and the mean values were used. The standard deviation of each measurement was used as ΔT.

$$\Delta T = \sqrt{\frac{\sum(T_i - \mu)^2}{3}}$$

The error in the mass was simply $\Delta m = 0.1$ g. A linear regression analysis with a larger data set would be required to determine the exact error in the experimentally found values of spring constant.

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Coupled Pendulum

By performing a Fast Fourier Transform (FFT) of the tracker data in figures 3-6, we were able to determine the frequency content of each oscillation. The frequency spectrum analysis is shown in the following plots:

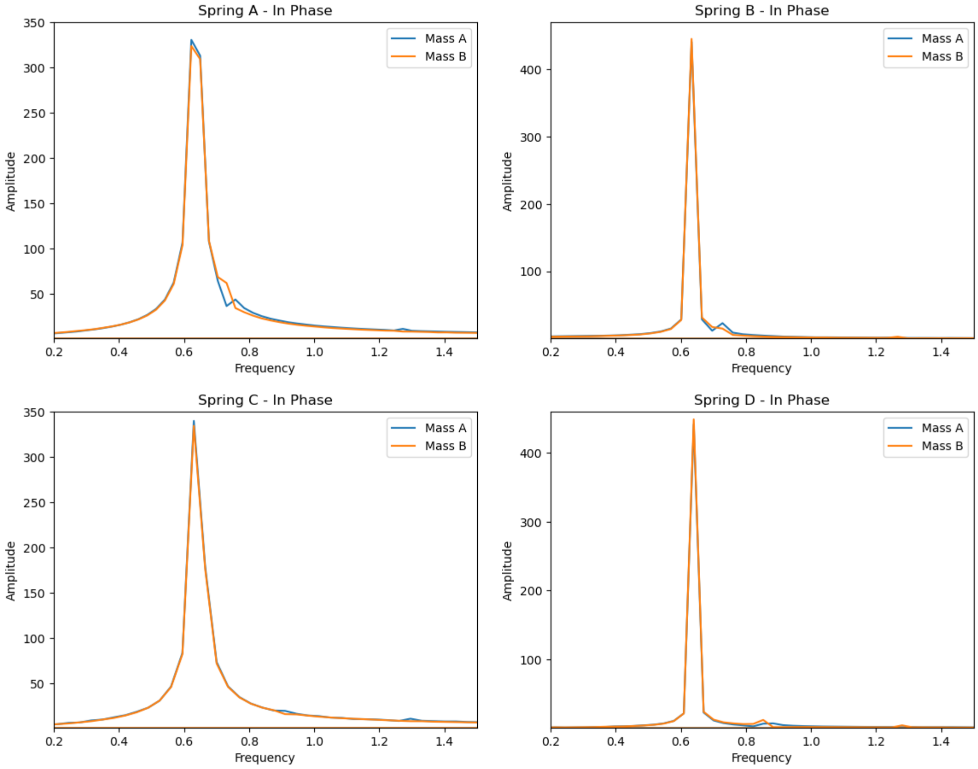

Figure 9: FFT of the in-phase oscillations (normal mode 1) for each spring

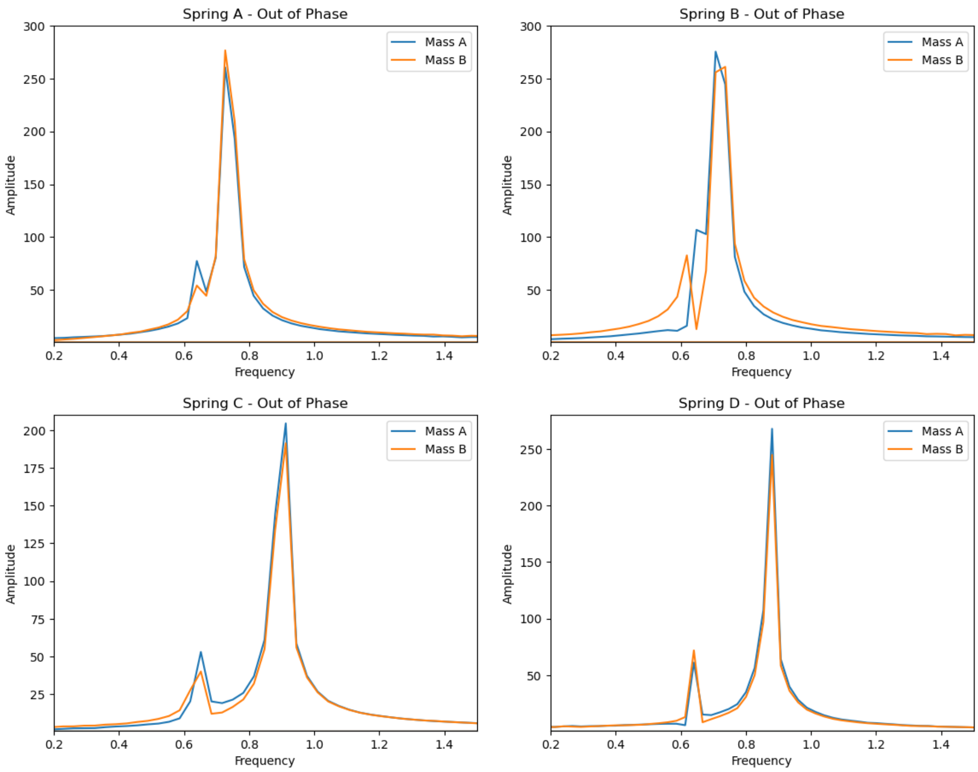

Figure 10: FFT of the out of phase oscillations (normal mode 2) for each spring

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The normal mode frequencies are characterised by a single distinct peak since the normal mode is defined by a single frequency. The frequency analysis shows a smaller peak in some of the data. This is because there may have been a small contribution from the second normal mode due to a slight phase difference between the two pendulums. However, for the purpose of determining the normal mode frequencies, these smaller peaks can be ignored. The experimentally determinded normal mode frequencies are:

Spring A f_1 (In-Phase) = 0.622 Hz
Spring A f_2 (Out-of-Phase) = 0.727 Hz

Spring B f_1 (In-Phase) = 0.633 Hz
Spring B f_2 (Out-of-Phase) = 0.707 Hz

Spring C f_1 (In-Phase) = 0.630 Hz
Spring C f_2 (Out-of-Phase) = 0.913 Hz

Spring D f_1 (In-Phase) = 0.639 Hz
Spring D f_2 (Out-of-Phase) = 0.880 Hz

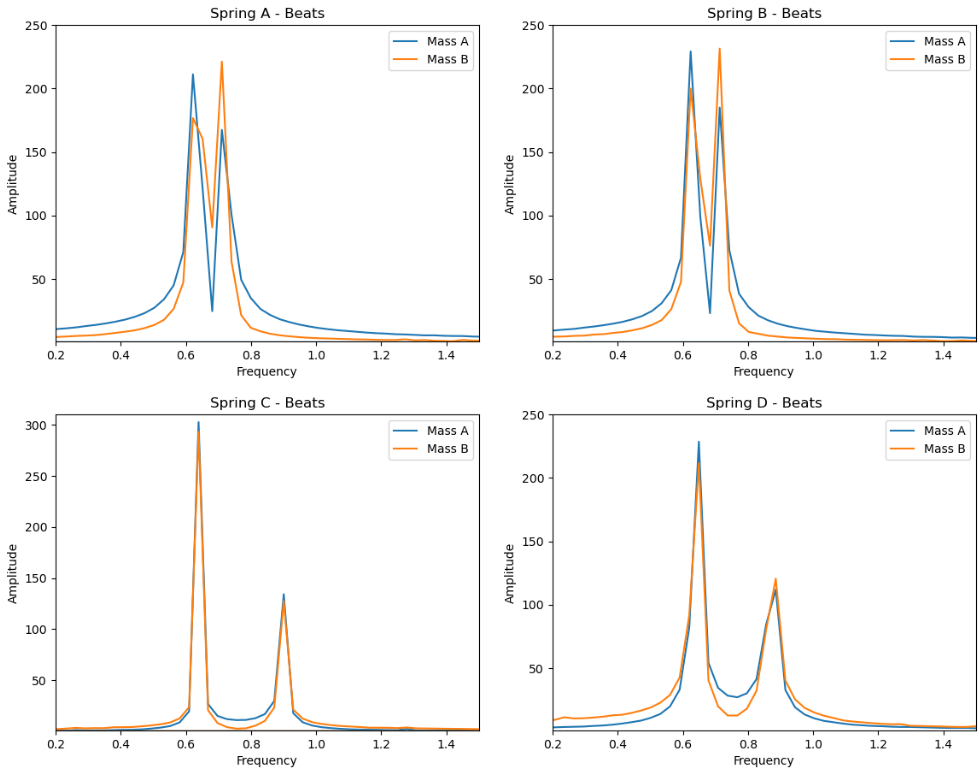

Figure 11: FFT of the beat oscillations for each spring

Beats are a result of the sum of both normal modes which is evident from the double peaks present in the above graphs. Spring A and B have lower spring constants, and hence the normal mode frequencies are closer together since $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} - \frac{2k}{m}}$. Due to the proximity of the normal modes, the peaks are less distinct and appear to overlap substantially. However, the peak values are the same as the individual frequencies determined from the out of phase and in phase analysis. Furthermore, the waveform of the beats in figure 5 match equations 5 and 6:

$$x_1 = A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right), x_2 = A\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_1 - \omega_2}{2}t\right)$$

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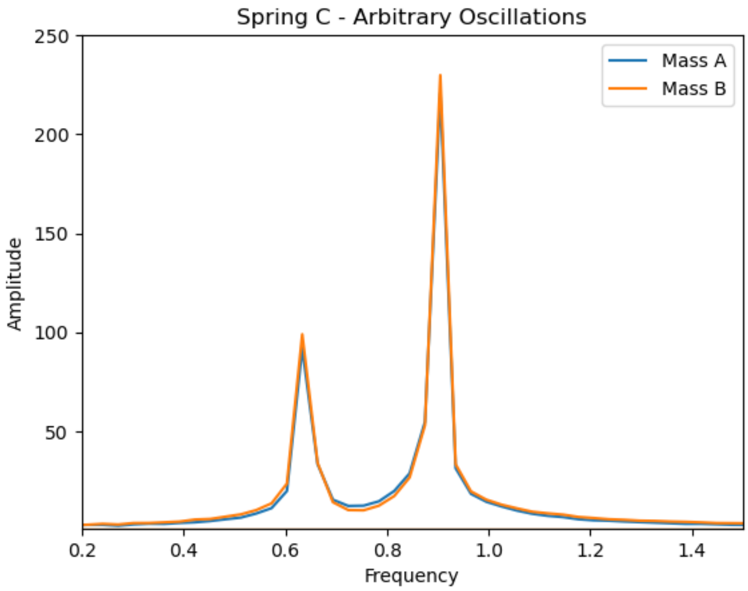

Figure 12: FFT of an arbitrary oscillation of spring C

The FFT of arbitrary oscillations with spring C show that any random set of initial conditions will lead to an oscillation that is a superposition of the two normal mode frequencies. Spring C was used for this part of the experiment since its normal mode frequencies were far apart due to the high spring constant. This is clear from the distinct peaks in figure 12.

Results

The spring constant of each spring found by studying its equilibrium state is:

Spring A (k_a) = 0.862 N/m
Spring B (k_b) = 0.676 N/m
Spring C (k_c) = 2.626 N/m
Spring D (k_d) = 2.533 N/m

The spring constant of each spring found by studying its oscillations is:

Spring A (k_a) = 0.826 N/m
Spring B (k_b) = 0.638 N/m
Spring C (k_c) = 2.366 N/m
Spring D (k_d) = 2.195 N/m

The normal mode frequencies of each coupled pendulum system found by taking the FFT of in-phase and out-of-phase oscillations are:

Spring A f_1 (In-Phase) = 0.622 Hz
Spring A f_2 (Out-of-Phase) = 0.727 Hz

Spring B f_1 (In-Phase) = 0.633 Hz
Spring B f_2 (Out-of-Phase) = 0.707 Hz

Spring C f_1 (In-Phase) = 0.630 Hz
Spring C f_2 (Out-of-Phase) = 0.913 Hz

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Spring D f_1 (In-Phase) = 0.639 Hz
Spring D f_2 (Out-of-Phase) = 0.880 Hz

Beats and arbitrary oscillations in a coupled pendulum system are just a superposition of the two normal mode frequencies with varying amplitudes. The normal modes are characterised by a single frequency and the amplitude of the second frequency disappears.

Discussion

- The normal mode FFT graphs have smaller peaks present at the second normal mode. This occurs due to a small contribution from the other frequency since the pendulums were set off with a slight phase difference. This can be eliminated by precisely measuring the horizontal displacement of both bobs before setting them in motion to ensure they are precisely in-phase or π radians out-of-phase.
- The tracker software identifies the position of the bob at every frame of the video. Hence, the granular discontinuity of the data can be controlled by controlling the frame rate of the camera. If a camera capable of capturing a higher number of frames per second (fps) is used, then the data will be more continuous, allowing for smoother tracking and cleaner Fourier analysis.
- The measurement of the time period of oscillations with a stop watch are subject to human error, but this error can be reduced by increasing the number of oscillations counted since the error only exists at the start and the end of the readings. Hence, if we were to count 100 oscillations instead of 10 oscillations, our error would reduce by a factor of 10.

References

- *Compound Pendula* (2021) Lab 3 Handout
- Wikipedia contributors (2024) *"Effective mass (spring–mass system)."* Wikipedia, The Free Encyclopedia.
- French, A. P. (1971). *Vibrations and Waves.* http://ci.nii.ac.jp/ncid/BA10414750