Noise Fundamentals

Lab Report 5

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Abstract

In this experiment we used the Noise Fundamentals apparatus to measure the Johnson noise produced by a resistor and the dependence of mean square voltage on resistance, bandwidth and temperature. The device used a series of amplifiers and frequency filters to boost the signal produced by the resistor. By selecting different resistances and bandwidths, their effect on the magnitude of noise could be measured. The temperature dependence was determined by immersing the resistor probe in liquid nitrogen and recording the temperature and mean square voltage fluctuation.

Introduction

In 1926, while studying electronic circuit noise at Bell Labs, John Johnson discovered an irreducible low-level noise in resistors that was proportional to temperature. Harry Nyquist who was also a theorist at Bell Labs provided a theoretical explanation for this effect, showing that it was a result of the random thermal motion of stationary charge carriers. This phenomenon is now called "Johnson noise" or "Johnson-Nyquist noise".

This type of noise arises from the fundamental properties of electric conductors rather than the design or materials of the circuit. Johnson noise is directly influenced by resistance, bandwidth, and temperature, and can be described by equations involving fundamental constants. In this experiment we aim to:

- Measure the Johnson noise produced by a resistor and show that it fluctuates randomly.
- Determine the dependence of the mean square voltage on resistance, bandwidth and temperature.
- Estimate the value of the Boltzmann constant by measuring Johnson noise and using Nyquist's equation.

Theoretical Background

Harry Nyquist used a thought experiment in his 1928 paper "Thermal Agitation of Electric Charge in Conductors" to derive the expression for the mean square voltage produced by a resistor due to Johnson noise. His thought experiment consists of two resistors (represented by ideal resistors in series with a noisy voltage source) connected to each other by a long loss-less transmission line of length L (figure 1). The signal produced by both resistors propagates at velocity v through the circuit, so if the impedance of the transmission line is made to be equal to $R_1 = R_2 = R$, then the signal from the resistors is entirely absorbed by the opposite resistor.

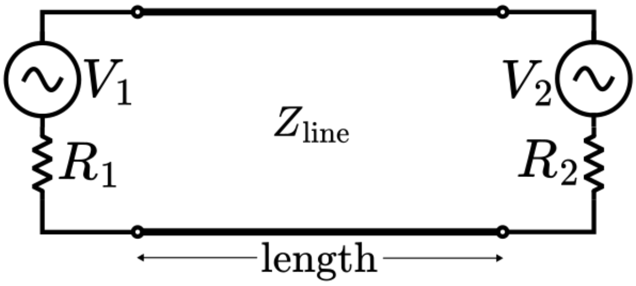

Figure 1: Circuit diagram for Nyquist's though experiment
Source: Nyquist's derivation of ideal resistor noise, Johnson–Nyquist noise, Wikipedia

Nyquist then imagined shorting both ends of the line, thereby trapping in-flight energy on the line. This energy would now be reflected within the line, forming a summation of standing waves. For a range of frequencies Δf, the number of oscillation modes are $n = 2L\Delta f/v$ since $\lambda = v/f = 2L/n$. According to the equipartition theorem of classical thermodynamics, the average energy contained within each radiation mode is equal to $k_B T$:

$$\langle E \rangle = \frac{\int_0^\infty E e^{(-E/k_B T)} dE}{\int_0^\infty e^{(-E/k_B T)} dE} = k_B T$$

Where k_B is the Boltzmann constant and T is temperature. Hence, the average energy in the line is $2k_B T L \Delta f/v$, so each resistor on average contributes $k_B T L \Delta f/v$. Since the time spent in transit is L/v, the average power supplied by each resistor is:

$$\langle P \rangle = k_B T \Delta f$$

The resistances are equal, so the current in the circuit is $I = V/2R$. Hence the power will be $P = I^2 R = V^2/4R$. Putting this into the above equation we find Nyquist's equation:

$$\langle V^2 \rangle = 4 k_B R T \Delta f \tag{1}$$

However, in order to measure Johnson noise, the signal must be passed through a high-gain amplifier to be able to detect it. The amplification circuit generates its own noise which can be measured in terms of the mean square voltage fluctuation $\langle V_N^2 \rangle$. Since the signal for both sources is combined, the measured voltage is:

$$\langle V_{out}^2 \rangle = G^2 \langle [V_J + V_N]^2 \rangle$$

Where V_{out} is the measured voltage, G is the gain supplied by the amplifier, V_J is the Johnson noise voltage (which we are interested in measuring) and V_N is the amplifier noise voltage. However, V_J and V_N are completely independent of each other so $\langle (V_J + V_N)^2 \rangle = \langle V_J^2 \rangle + \langle V_N^2 \rangle$. Hence,

$$\langle V_{out}^2 \rangle = G^2 \langle V_J^2 \rangle + G^2 \langle V_N^2 \rangle \tag{2}$$

If we were to plot $\langle V_{out}^2 \rangle$ against different values of $\langle V_J^2 \rangle$ (by varying resistance, say), the intercept would give us $\langle V_N^2 \rangle$ and we would subsequently be able to isolate the value of $\langle V_J^2 \rangle$. This value of $\langle V_J^2 \rangle$ obeys equation 1 and can used used to analyse the relationship between Johnson noise and resistance, bandwidth and temperature.

Experimental Setup

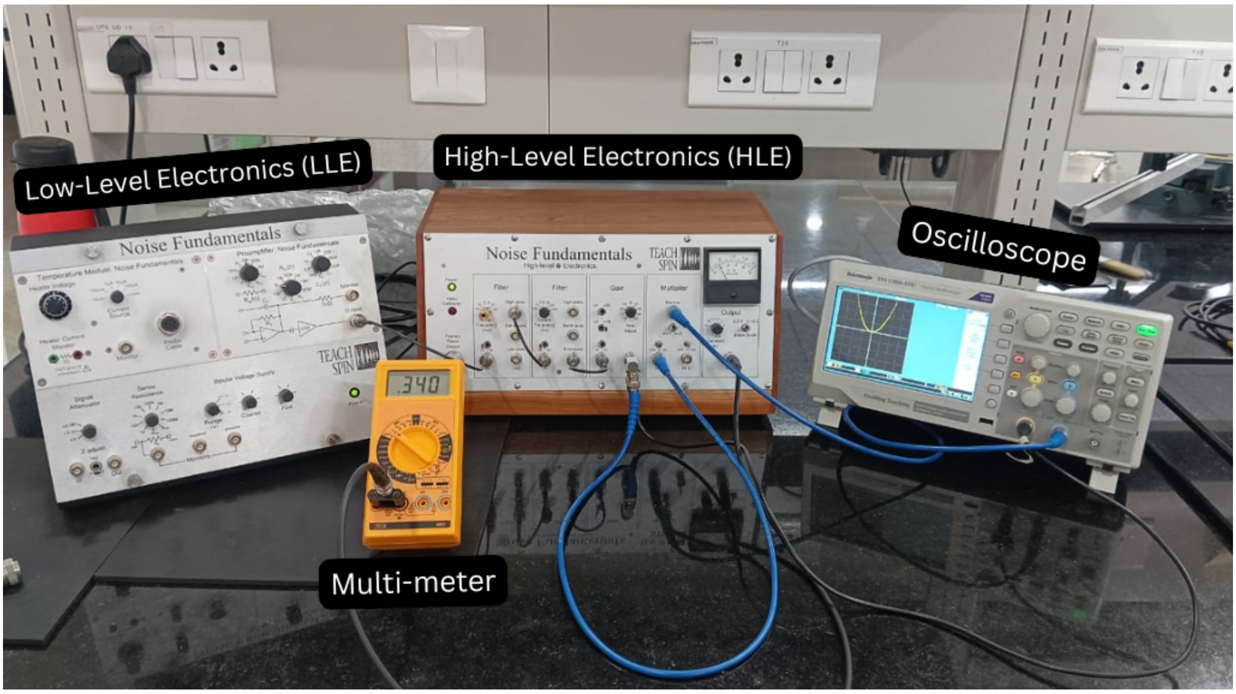

Figure 2: Experimental apparatus for studying Johnson noise

This experiment used the TeachSpin Noise Fundamentals NF1-A apparatus to study Johnson noise. The setup consists of a low-level electronics (LLE) box, high-level electronics (HLE) box and a variable-temperature probe mounted in a dewar (not pictured above). An oscilloscope and multi-meter were used to display the output of the instruments.

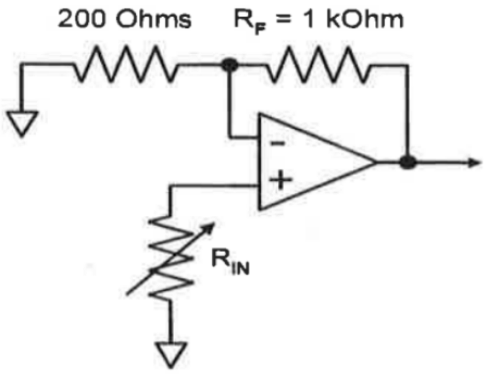

Figure 3: Schematic diagram for the pre-amplifier circuit
Source: Noise Fundamentals NF1-A Instructors Manual, TeachSpin Inc.

The low-level electronics box contains a resistance selector R_{in} and a pre-amplification circuit whose resistance is denoted by R_f (figure 3). The box is lined with a Faraday cage in order to isolate the components from external interference. The temperature module of the LLE contains a heater dial and a connector for the temperature probe. The high-level electronics box consists of a low-pass filter, high-pass filter, variable gain module and a signal multiplier. The output of the multiplier is displayed after taking the time-average of the signal. The range and interval over which the average can be controlled. The rest of the components in the instrument are irrelevant for our experiment.

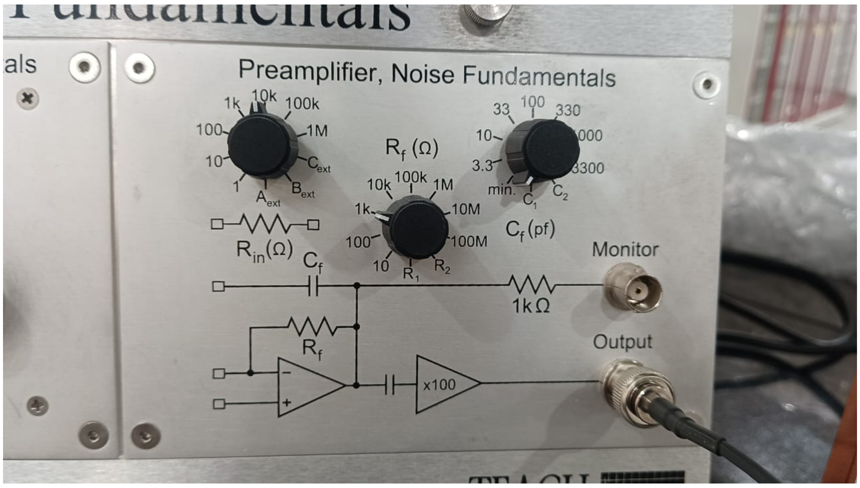

Figure 4: LLE settings to measure the dependence of resistance, bandwidth and temperature on Johnson noise mean square voltage

For the purpose of our experiment, we set $R_f = 1k\Omega$ which produces a gain of 600 from the pre-amplifier. The pre-amp output on the LLE is fed to the first frequency filter on the HLE via a BNC connector cable. The first filter is used as a high-pass filter and the second one is used as a low-pass filter by making the connections shown in figure 4. The low-pass filter output is fed through another amplifier that supplies variable gain. The toggle switches are set to AC-coupling for all components.

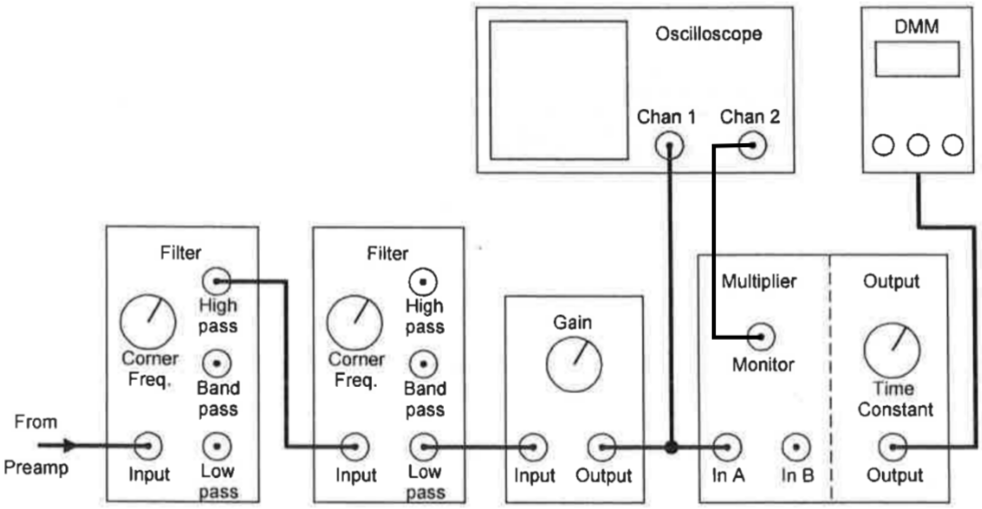

Figure 5: Schematic diagram for the HLE connections to measure Johnson noise
Source: Noise Fundamentals NF1-A Instructors Manual, TeachSpin Inc.

Finally, the output of the amplifier is split and sent to both the oscilloscope's channel 1 and the multiplier input. The multiplier squares the signal when set to $A \times A$ and its monitor signal is sent to channel 2 of the oscilloscope. The time-average is taken over 1s and displayed on the 0-2V scale. The xy-mode of the oscilloscope plots the output against its squared average, forming a parabola. Finally, the time-averaged output is sent to a multi-meter which displays the mean-square voltage of the amplified signal when it is set to a voltmeter with range 0-2V.

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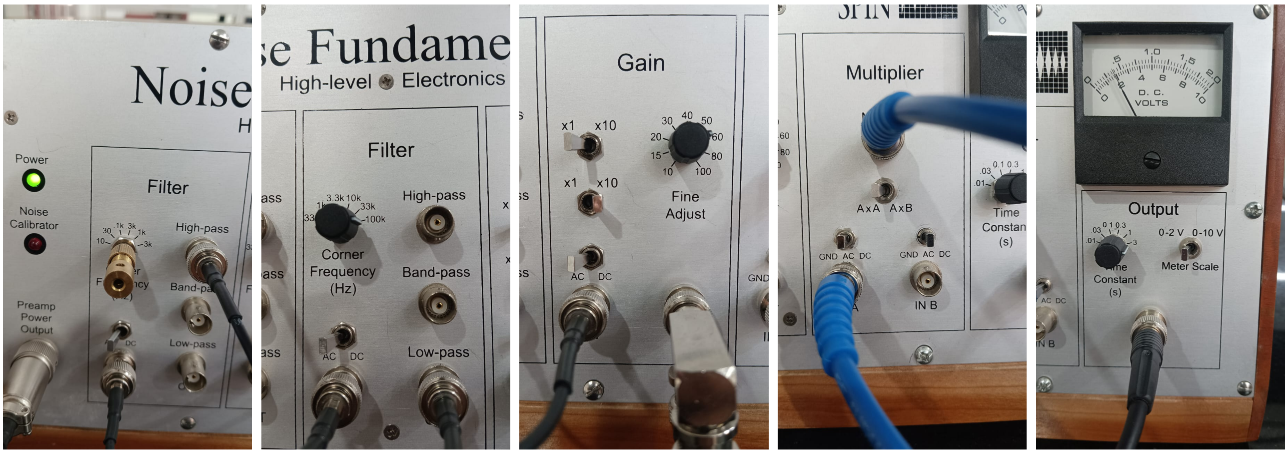


Figure 6: Close-up of HLE settings (from left to right) for the high-pass filter, low-pass filter, gain, multiplier and output.

List of instruments and materials required:

- TeachSpin NF1-A Noise Fundamentals apparatus (including LLE, HLE and power supply adapter)
- Tektronix TBS 1102B-EDU Digital Oscilloscope
- Metro-Q MTQ 888 Digital Multimeter $\times$ 2
- BNC connector cables
- Variable-temperature probe
- Dewar and mount
- Liquid nitrogen

Least count of voltmeter = 0.001 V

Procedure

Resistance Dependence

1. Make the connections detailed in the section above. Select the following settings on the HLE, LLE and multimeter:
   Pre-amp Resistance (R_f) = 1 kΩ
   High-Pass (HP) Filter = 0.1 kHz
   Low-Pass (LP) Filter = 100 kHz
   Voltmeter Range = 0-2 V

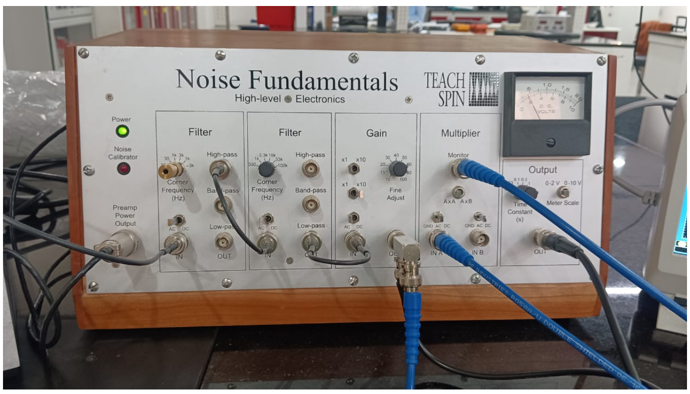


Figure 7: HLE connections to measure the dependence of resistance, bandwidth and temperature on Johnson noise mean square voltage

2. Switch on the power supply for the HLE and oscilloscope after completing the connections. Select the lowest input resistor (1Ω) and adjust the variable gain until the voltmeter and inbuilt dial display a value between 0V and 2V (*Precaution:* maintain all readings just below 1V to avoid distortion of the signal due to non-linearities which are introduced when amplifying outside of the -1V to 1V range).

3. Once an appropriate gain has been selected, the oscilloscope should display a parabolic curve. This verifies that the multiplier is performing its task. Record the gain, resistance and meter voltage reading for increasing values of resistance. Adjust the gain to keep the readings within the appropriate range (*Precaution:* if the meter voltage suddenly shoots past the 2V limit, quickly reduce the gain to avoid damaging the components).

4. Plot the meter voltage against the resistance to determine the amplifier noise and isolate $\langle V_J^2 \rangle$. Use this to find the relationship between $\langle V_J^2 \rangle$ and R and use the slope to determine the value of the Boltzmann constant.

Bandwidth Dependence

1. Using the same settings as the previous part, fix the input resistance to 10kΩ and systematically vary the bandwidth to observe the dependence of mean square voltage on bandwidth. Change the gain until an appropriate meter reading is reached.

2. Record the true bandwidth (from the Noise Fundamentals handout), $|f_2 - f_1|$, gain and meter voltage. Plot the bandwidth and $|f_2 - f_1|$ against $\langle V_J^2 \rangle$ and use the slope to determine their relationship. Also use the slope to estimate the value of the Boltzmann constant.

Temperature Dependence

1. Switch off the instruments and open the LLE box to rewire the R_A, R_B and R_C resistors to connect to the external temperature probe. Connect the probe cable to the LLE and plug in a second voltmeter to the temperature module for monitoring the temperature. The voltage to temperature conversion is given by a nearly linear relationship provided in the instrument manual.

2. Select the following settings:
   Pre-amp Resistance (R_f) = 1 kΩ
   Input Resistance (R_{in}) = R_B = 10 kΩ
   High-Pass (HP) Filter = 1 kHz
   Low-Pass (LP) Filter = 10 kHz
   Voltmeter Range = 0-2 V

3. Fill the dewar about halfway with liquid nitrogen and mount it such that the temperature probe is entirely immersed in the liquid. Monitor the temperature until it reaches its minimum value (it should be close to 77K, the boiling point of liquid nitrogen, which corresponds to around 990mV according to the calibration curve).

4. Turn on the heater voltage to maximum and as the nitrogen begins to evaporate, observe the temperature of the probe drop. At regular intervals of temperature voltage record the meter voltage and gain. Calculate $\langle V_J^2 \rangle$ and the temperature from the calibration curve. Plot them against each other and find their relationship.

Observations and Analysis

Upon passing the Johnson noise through the multiplier and plotting it against the output, we receive a parabola on the oscilloscope. This parabola shows how the mean square voltage fluctuation is symmetric and has a mean value of zero.

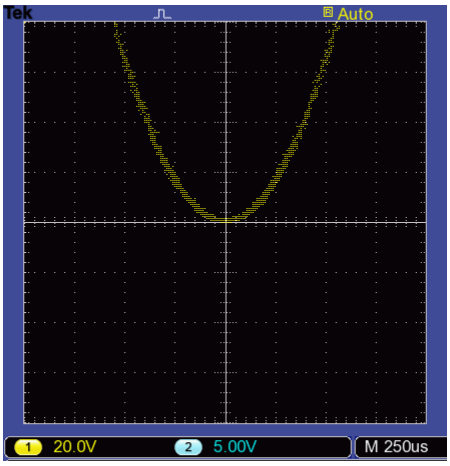


Figure 8: It was noticed that the thickness of the parabola was proportional to the intensity of the meter voltage and it was most stable when close to 1V

Resistance Dependence

In order to study the dependence of resistance on the mean square voltage fluctuation produced by Johnson noise, we first have to remove the noise produced by the amplifier from the signal. According to the instructor's manual for the apparatus, the meter reading displayed has the form:

$$\langle V_{meter} \rangle = \frac{(G_1 G_2)^2}{10 \text{ V}} \langle V_J^2 + V_N^2 \rangle \tag{3}$$

This was used to convert the meter readings to $\langle V_J^2 + V_N^2 \rangle$ which was plotted against resistance to determine the amplifier noise contribution.

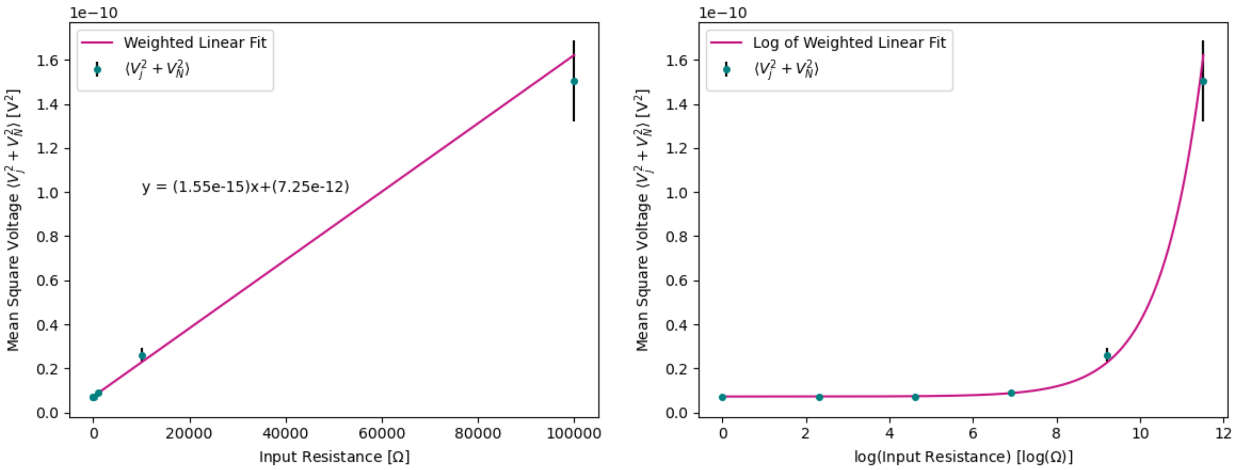


Figure 9: $\langle V_J^2 + V_N^2 \rangle$ vs R shows a linear trend whose intercept is equal to $\langle V_N^2 \rangle$ since it is independent of R. The log plot shows the data spread more clearly.

Since $\langle V_J^2 + V_N^2 \rangle = \langle V_J^2 \rangle + \langle V_N^2 \rangle$ and V_N is independent of R while $V_J = 0$ at $R = 0$, the y-intercept of the linear graph (figure 8) is the value of V_N. The error in V_N was determined from the covariance matrix of the weighted linear fit from the curve-fit function of the `scipy.optimize` package.

$$V_N = (7.249 \pm 0.005) \times 10^{-12} \text{ V}$$

This value was subtracted from all the data points to remove the effect of amplifier noise, effectively isolating the Johnson noise of the resistor. $\langle V_J^2 \rangle$ was then plotted against resistance to determine its relationship and estimate the value of the Boltzmann constant.

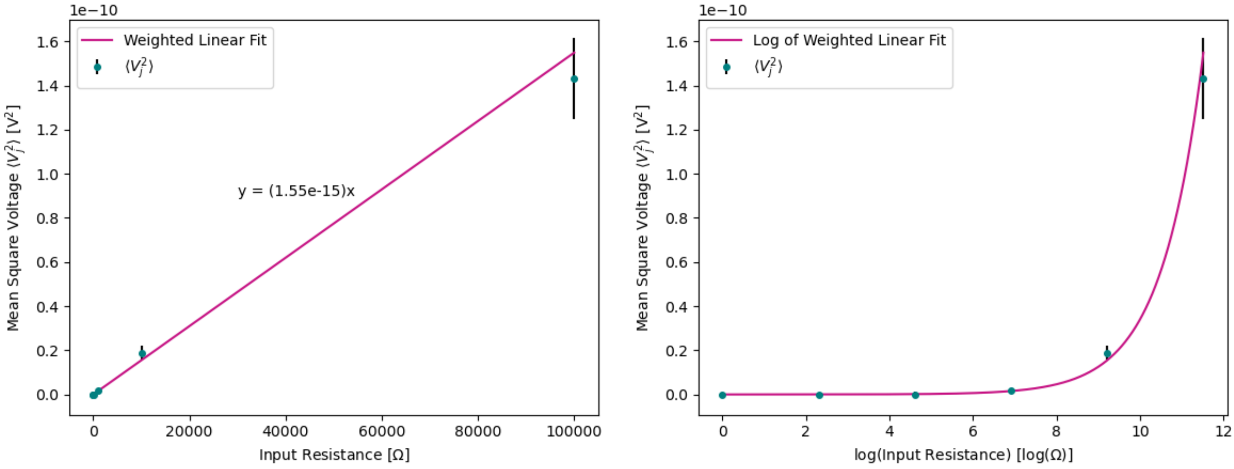


Figure 10: Since the removal of V_N was linear, the slope of the trend does not change

The Nyquist equation tells us that $\langle V_J^2 \rangle \propto R$, so the slope of the graph can be used to calculate the value of the Boltzmann constant:

$$k_B = \left(\frac{\langle V_J^2 \rangle}{R} \right) \frac{1}{4T\Delta f}$$

Where $T = 300$K (room temperature) and the bandwidth used was $\Delta f = 110961$Hz according to the frequency bandwidth table provided in the handout. The error in the slope was determined from the covariance matrix of the weighted linear fit from the curve-fit function of the `scipy.optimize` package.

$$k_B = (1.163 \pm 0.001) \times 10^{-23} \text{ J/K}$$

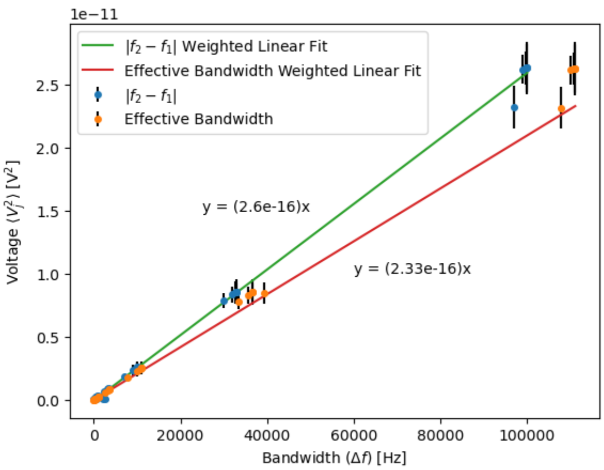
Bandwidth Dependence

The dependence of Johnson noise on bandwidth was tested by keeping the resistance constant at $R_{in} = 10\text{k}\Omega$ and changing the bandwidth. Since the bandwidth can be varied by changing either the low-pass filter frequency (f_1) or high-pass frequency (f_2), all possible combinations of f_1 and f_2 were taken.

f_1 \ f_2	0.33 kHz	1 kHz	3.3 kHz	10 kHz	33 kHz	100 kHz
10 Hz	355	1,100	3,654	11,096	36,643	111,061
30 Hz	333	1,077	3,632	11,074	36,620	111,039
100 Hz	258	1,000	3,554	10,996	36,543	110,961
300 Hz	105	784	3,332	10,774	36,321	110,739
1000 Hz	9	278	2,576	9,997	35,543	109,961
3000 Hz	0.4	28	1,051	7,839	33,324	107,740

Table 1: The effective bandwidth in Hertz (Hz) for various values of f_1 and f_2

The meter voltage was converted to $\langle V_J^2 \rangle$ in the same way as the previous part. Both the effective bandwidth and $|f_2 - f_1|$ were plotted against $\langle V_J^2 \rangle$ to show the difference in their behavior.



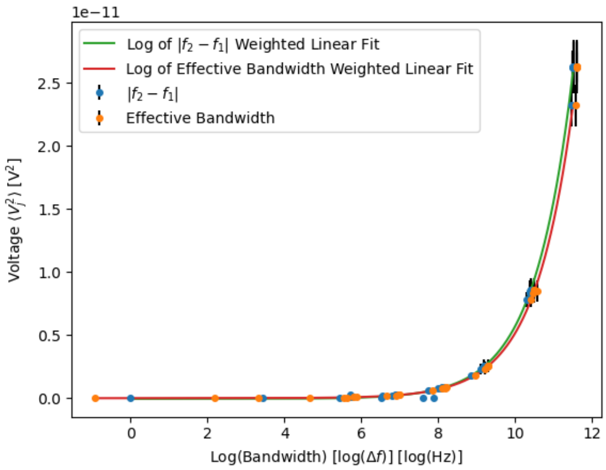


Figure 11: The effective bandwidth increases faster than $|f_2 - f_1|$, causing the slope to reduce.

From both these slopes, we can find the value of the Boltzmann constant:

$$k_B = \left(\frac{\langle V_J^2 \rangle}{\Delta f} \right) \frac{1}{4RT}$$

Where $R = 10\text{k}\Omega$ and $T = 300\text{K}$. From $|f_2 - f_1|$, we find:

$$k_{B1} = (2.16 \pm 0.02) \times 10^{-23} \text{ J/K}$$

From the effective bandwidth, we find:

$$k_{B2} = (1.94 \pm 0.01) \times 10^{-23} \text{ J/K}$$

Clearly, the effective bandwidth gives us a value that is closer to the true value of k_B, indicating that $|f_2 - f_1|$ is not an accurate measure of bandwidth but merely an approximation.

Temperature Dependence

Finally, we determined the effect of temperature on Johnson noise by cooling the resistor to 77K by immersing it in liquid nitrogen and recording how $\langle V_J^2 \rangle$ changes as the resistor is gradually warmed past room temperature. In order to convert the probe voltage to temperature, the following calibration curve was used:

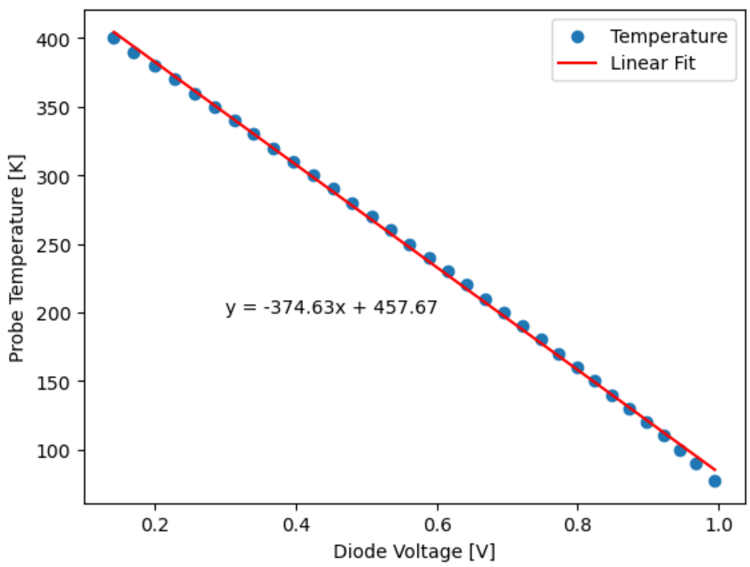


Figure 12: The temperature calibration curve provided in the instructor's manual for the diode voltage in the temperature probe.

Using the equation of the above graph, we could convert the diode voltage into temperature:

$$T = -374.63V + 457.67$$

Using the same procedure as the last two parts, we isolated $\langle V_J^2 \rangle$ and plotted it against T to find their relationship and estimate the value of the Boltzmann constant.

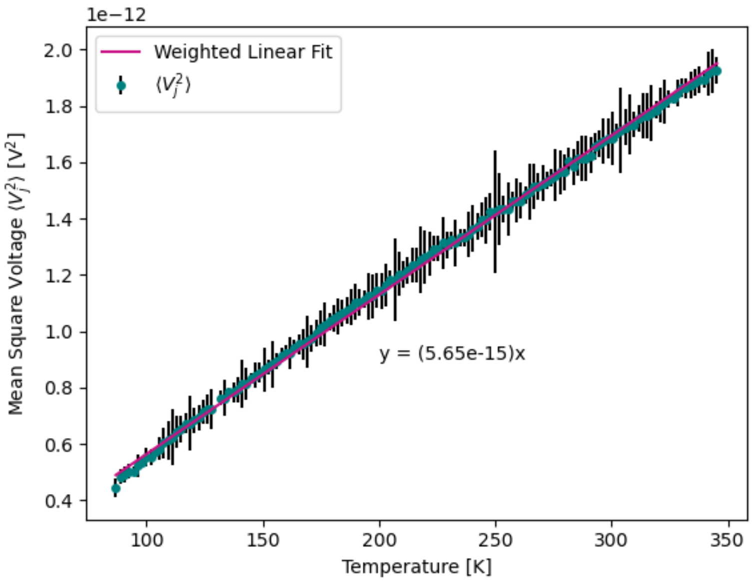


Figure 13: The mean square voltage varies linearly with temperature as expected by the Nyquist equation.

The slope of this graph allows us to estimate the value of the Boltzmann constant.

$$k_B = \left(\frac{\langle V_J^2 \rangle}{T} \right) \frac{1}{4R\Delta f}$$

Where $R = R_B = 10\text{k}\Omega$ and $\Delta f = 9997\text{Hz}$, which is the effective bandwidth when $f_1 = 1\text{kHz}$ and $f_2 = 10\text{kHz}$. This gives us the following value of Boltzmann constant:

$$k_B = (1.412 \pm 0.005) \times 10^{-23} \text{ J/K}$$

This is by far the most accurate value of k_B that we have calculated, most likely due to the larger and more consistent spread of data collected.

Error Analysis and Discussion

The error in k_B in all cases originates from the error in the slope of the respective graph since the other quantities are fixed. The error in the slope was found by taking the square-root of the diagonal elements of the covariance matrix generated from by weighted linear fit from the curve-fit function of the `scipy.optimize` package. This gives us the standard deviation error in the slope. Note that the weighted linear fit accounts for the error in individual data points by weighting them according to their errorbars, giving higher weight to data points with smaller error (and therefore greater precision). By quadrature:

$$\frac{\Delta k_B}{k_B} = \frac{\Delta \text{slope}}{\text{slope}}$$

For the resistance dependence, the error in k_B is:

$$\Delta k_B = 0.001 \times 10^{-23} \text{ J/K}$$

For the bandwidth dependence (considering effective bandwidth), the error in k_B is:

$$\Delta k_B = 0.01 \times 10^{-23} \text{ J/K}$$

For the temperature dependence, the error in k_B is:

$$\Delta k_B = 0.005 \times 10^{-23} \text{ J/K}$$

The known value for $k_B = 1.381 \times 10^{-23}$J/K, so according to the percentage error formula:

$$\text{Percentage Error in } k_B = \left(1 - \frac{k_B(\text{measured})}{k_B(\text{real})}\right) \times 100$$

Hence, the percentage error for the Boltzmann constant values determined from resistance, bandwidth and temperature are 15.6%, 40.8% and 2.3%, respectively. Hence, the best value was determined from the temperature.

The noise measurements require the amplification of the signal by linear amplifiers that boost the signal without distorting it. However, this linear operation only occurs over a finite range of output voltages and therefore the gain of the signal must be adjusted to keep the voltage below this threshold. For the HLE amplifiers, the threshold is -10V to 10V, which after being divided by 10V by the meter (see equation 3), gives us a range of -1V to 1V. We have attempted to keep the voltage below 1V for most readings, but since the gain settings were discrete, it was impossible to avoid taking certain readings slightly above 1V. This may have introduced some error in the data due to signal distortion.

We also noticed that higher resistances tend to deviate more from the expected trend of the data. It is unclear why this might be occurring but in order to get better values of k_B, we would recommend using smaller resistances and avoiding large ones.

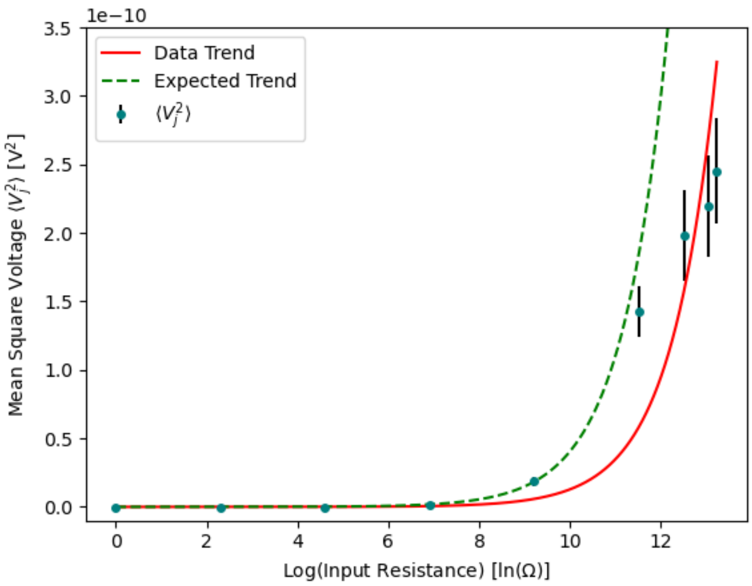


Figure 14: Resistances beyond 10kΩ show significant deviation from the expected trend from the true value of the Boltzmann constant.

Results

We measured Johnson noise and showed that it is a randomly fluctuating voltage with a mean value of zero by plotting a parabola of the mean square voltage against the output of the noise fundamentals apparatus. Furthermore, we verified the linear relationship between $\langle V_J^2 \rangle$ and resistance, bandwidth and temperature. This confirms the Nyquist equation:

$$\langle V_J^2 \rangle = 4k_B RT\Delta f$$

The value of the Boltzmann constant determined from resistance, bandwidth and temperature are:

	Boltzmann Constant [10^{-23} J/K]
Resistance	1.163 ± 0.001
Bandwidth	1.94 ± 0.01
Temperature	1.412 ± 0.005

Table 2: The experimentally determined values of the Boltzmann constant from Johnson noise

The true value of $k_B = 1.381 \times 10^{-23}$J/K. The percentage error for the Boltzmann constant values determined from resistance, bandwidth and temperature are 15.6%, 40.8% and 2.3%, respectively.

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