CALCULUS - PRACTICAL 1.

1. $$\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$$

$$= \lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} \times \frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$$

$$= \lim_{x\to a}\frac{(a+2x-3x).(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x).(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{1}{3}\lim_{x\to a}\frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(a-x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{1}{3}\frac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$$

$$= \frac{1}{3}\times\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$= \frac{1}{3}\times\frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$= \frac{1}{3}\times\frac{\cancel{4}^{\,2}\sqrt{\cancel{a}}}{\cancel{2}\sqrt{3\cancel{a}}}$$

$$= \frac{2}{3\sqrt{3}}$$

2. $\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$= \lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$= \lim_{y\to 0} \frac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$

$= \lim_{y\to 0} \frac{\cancel{y}}{\cancel{y}\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$

$= \frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a})}$

$= \frac{1}{\sqrt{a}\,(\sqrt{a}+\sqrt{a})}$

$= \frac{1}{\sqrt{a}\,(2\sqrt{a})}$

$= \frac{1}{2a}$

4.
Ans.

3. $\lim_{x\to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \frac{\pi}{6}$, where $h \to 0$

$= \therefore \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{6}\right) . \sqrt{3}\sin\left(h+\frac{\pi}{6}\right)}{\pi - 6\left(h+\frac{\pi}{6}\right)}$

using $\cos(A+B)$
$= \cos A \cos B - \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \lim_{h\to 0} \frac{\cos h \cos\frac{\pi}{6} - \sin h \sin\frac{\pi}{6} - \sqrt{3}\sin h \cos\frac{\pi}{6} + \cos h \sin\frac{\pi}{6}}{\pi - \cancel{6}\left(\frac{h+\pi}{\cancel{6}}\right)}$

$= \lim_{h\to 0} \frac{\cos h . \frac{\sqrt{3}}{2} - \sin h \frac{1}{2} - \sqrt{3}\left(\sin h\frac{\sqrt{3}}{2} + \cos h \frac{1}{2}\right)}{\cancel{\pi} - 6h \cancel{-\pi}}$

$= \lim_{h\to 0} \frac{\cos\frac{\sqrt{3}}{2}h - \sin\frac{h}{2} - \sin\frac{3h}{2} - \cos\frac{\sqrt{3}}{2}h}{-6h}$

$= \lim_{h\to 0} \frac{-\sin\frac{4h}{2}}{+6h} = \lim_{h\to 0} \frac{\sin 4h}{3 \cdot 2h}$

$= \frac{1}{3}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{3}\times 1 = \frac{1}{3}$

4. $\lim_{x \to \infty} \left[\dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

By rationalizing numerator & denominator both

$$= \lim_{x\to\infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}\right] \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \left[\frac{(x^2+5-x^2+3)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{(x^2+3-x^2-1)\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)}\right]$$

$$= \lim_{x\to\infty} \frac{\left(\overset{4}{\cancel{8}}\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{\left(\cancel{2}\sqrt{x^2+5}+\sqrt{x^2-3}\right)}$$

$$= 4 \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\ 1+\frac{5}{x^2}} + \sqrt{x^2\ 1-\frac{3}{x^2}}}$$

After applying limit,

we get $= 4$

4.
Ans.

5. $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \pi/2$
$\quad = \dfrac{\cos x}{\pi - 2x}$, for $\dfrac{\pi}{2} < x < \pi$ $\Big\}$ at $x = \pi/2$

$$f(\pi/2) = \frac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos 2(\pi/2)}} \qquad \therefore f(\pi/2) = 0$$

f at $x = \pi/2$ define,

$$= \lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\pi/2^+} \frac{\cos x}{\pi - 2x}$$

By substituting method, $x - \dfrac{\pi}{2} = h$

$x = h + \dfrac{\pi}{2}$ when $h \to 0$

$$= \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - 2\left(h+\frac{\pi}{2}\right)}, = \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$$

using $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \lim_{h\to 0} \frac{\cos h \cdot \cos\frac{\pi}{2} - \sin h \cdot \sin\frac{\pi}{2}}{-2h}$$

$$= \lim_{h\to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$= \lim_{h\to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h} = \frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \frac{1}{2}$$

b. $\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^-} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$

Using $\sin 2x = 2\sin x \cdot \cos x$

$= \lim_{x\to\pi/2^-} \dfrac{2\sin x \cdot \cos x}{\sqrt{2}\sin x}$

$= \lim_{x\to\pi/2^-} \dfrac{2\cos x}{\sqrt{2}}$

$\dfrac{2}{\sqrt{2}} \lim_{x\to\pi/2^-} \cos$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ f is not continuous at $x = \pi/2$.

5. ii. $f(x) = \dfrac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x < 6$

$= \dfrac{x^2-9}{x+3}$ $\quad 6 \le x < 9$

at $x=3$ and $x=6$

at $x = 3$

$f(3) = \dfrac{x^2-9}{x-3} = 0$

f at $x=3$ define

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is defined at $x=3$

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (x+3) = 6$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \dfrac{x^2-9}{x-3} = \dfrac{(x-3)(x+3)}{(x-3)}$

$\therefore$ L.H.L = R.H.L

f is continuous at $x=3$, for $x=6$

$f(6) = \dfrac{x^2-9}{x+3} = \dfrac{36-9}{6+3} = 3$

2. $\lim_{x\to6^+} \dfrac{x^2-9}{x+3}$

$= \lim_{x\to6^+} \dfrac{(x-3)(x+3)}{(x+3)}$

$\lim_{x\to6^+} (x-3) = 6.3 = 3$

$\lim_{x\to6^-} x+3 = 3+6 = 9$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ Function is not continuous

$$f(x) = \frac{1-\cos 4x}{x^2} \quad x \neq 0 \Big\} \text{ at } x = 0$$
$$= k$$

Ans - f is continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k$$

$$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$$

$$2(2)^2 = k$$

$$k = 8$$

4 $f(x) = (\sec^2 x)^{\cot^2 x} \quad x \neq 0 \Big\}$ at $x = 0$

$$= k \quad x = 0$$

Ans. $f(x) = (\sec^2 x)^{\cot^2 x}$

$$\lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$$ we know that $\lim_{x\to 0} (1+f(x))^{\frac{1}{f(x)}} = e$

$$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

$$k = e$$

$$f(x) = \frac{\sqrt{3}-\tan x}{\pi - 3x} \quad x \neq \pi/3 \Big\} \text{ at } x = \frac{\pi}{3}$$
$$= k \quad x = \pi/3$$

$$x - \frac{\pi}{3} = h$$

$x = h + \frac{\pi}{3}$ where $h \to 0$

$$f\left(\frac{\pi}{3}+h\right) = \frac{\sqrt{3} - \tan(\pi/3+h)}{\pi - 3\left(\frac{\pi}{3}+h\right)}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan(\pi/3+h)}{\pi - 3(\pi/3+h)}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - \dfrac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\tan h}}{\pi - \pi - 3h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3}\cdot\tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3}\cdot\sqrt{3}\tan h) - (\sqrt{3} + \tan h)}{1 - \tan\pi/3\cdot\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\frac{(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h)}{1-\sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1-\sqrt{3}\tan h)}$$

$$= \frac{4}{3} \frac{1}{(1-\sqrt{3}(0))}$$

4.
Ans.

$$= \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}$$

2. $f(x) = \left.\begin{matrix} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ = 9 & x = 0 \end{matrix}\right\}$ at $x=0$

$$f(x) = \frac{1-\cos 3x}{x\tan x}$$

$$\lim_{x\to 0} = \frac{2\sin^2 3/2\, x}{x\tan x}$$

$$\lim_{x\to 0} = \frac{\frac{2\sin^2 \frac{3x}{2}}{x^2} \times x^2}{\frac{x \cdot \tan x}{x^2} \times x^2}$$

$$= 2 \lim_{x\to 0} \frac{\left(\frac{3}{2}\right)^2}{1} = 2 \times \frac{9}{4 \times 2} = \frac{9}{2}$$

$\lim_{x\to 0} f(x) = \frac{9}{2}$ $\quad 9 = f(0)$

$\therefore f$ is not continuous at $x=0$

Redefine function

$$f(x) = \left\{\begin{matrix} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ \frac{9}{2} & x = 0 \end{matrix}\right\}$$

Now $\lim_{x\to 0} f(x) = f(0)$

$\therefore f$ has removable discontinuity at $x = 0$.

7. $f(x) = \frac{(e^{3x}-1)\sin x^\circ}{x^2}$ $x \neq 0$ $\Big\}$ at $x=0$
$= \frac{\pi}{6}$ $x = 0$

$\lim_{x\to 0} \frac{e^{3x}-1}{x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$\lim_{x\to 0} 3.\frac{e^{3x}-1}{3x} \lim_{x\to 0} \sin\left(\frac{\pi x}{180}\right)$

$\lim_{x\to 0} 3.\frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$3 \lim_{x\to 0} \frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

4 Ans.

$3 \log e \frac{\pi}{180} = \frac{\pi}{60} = f(0)$

$\therefore f$ is continuous at $x = 0$

8. $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x = 0$

is continuous at $x = 0$

Given f is continuous at $x = 0$

$\lim_{x\to 0} f(x) = f(0)$

$= \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$

$= \lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$= \lim_{x\to 0} \frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$

$= \log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$

$= \log e + 2 \lim_{x\to 0} \left(\frac{\sin x/2}{x}\right)^2$

multiply with 2 in Num & Denominator

$= 1 + 2 \times \frac{1}{4}\ \frac{3}{2} = f(0)$

$$f(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \quad x \neq \pi/2$$

$f(x)$ is continuous at $x = \pi/2$

$$\lim_{x\to\pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$= \lim_{x\to\pi/2} \frac{2 - 1 + \sin x}{\cos^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x\to\pi/2} \frac{1 \cancel{+\sin x}}{(1-\sin x)(\cancel{1+\sin x})(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x\to\pi/2} \frac{1}{(1-\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

$$f(\pi/2) = \frac{1}{4\sqrt{2}}$$

Ans.

Practical 2 Derivative

Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable

$\cot x$

$f(x) = \cot x$

$$Df(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\cot x - \cot a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x-a}$$

$$= \lim_{x\to a} \frac{\tan a - \tan x}{(x-a)\tan x \tan a}$$

put $x - a = h$

$x = a+h$, as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{\tan(\cancel{a}-a-h) \cdot 1 + \tan a \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} -\frac{\tan h}{h} \times \frac{1+\tan a\tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a} = \frac{-\sec^2 a}{\tan^2 a} = \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

f is differentiable $\forall a \in \mathbb{R}$.

Cosec x

$f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sin a - \sin x}{(x-a) \sin a \sin x}$$

put $x - a = h$

$x = a + h$, as $x \to a$, $h \to 0$

$$Df(x) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a) \sin a \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$$

$$= \lim_{h \to 0} -\frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h)}$$

Ans.

Sec x

$f(x) = \sec x$ 18

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a) \cos a \cos x}$$

Put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\frac{a+a+h}{2} \sin\frac{a-a-h}{2}}{h \times \cos a \cos(a+h)}$$

Q2. If $f(x) = 4x+1, \quad x \le 2$ at $x=2$, then
$= x^2+5 \quad x>0$

Find function is differentiable or not

Ans LHD

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4 \times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

Ans. $Df(2^-) = 4$

RHD: $$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 2+2 = 4$$

$Df(2^+) = 4$

RHD = LHD

f is differentiable at $x=2$

If $(f(x)) = 4x+7 \quad x<3$
$= x^2+3x+1 \quad x \ge 3$ at $x=3$

find f is differentiable or not

Ans: RHD $$Df(3^+) = \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3 \times 3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)(x-3)}{(x-3)} = 3+6 = 9$$

$Df(3^+) = 9$

L.H.D $= Df(3^-)$

$$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x \to 3} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$$

$Df(3^+) = 4$

∴ R.H.D ≠ L.H.D

∴ f is not differentiable at $x = 3$

Qu. If $f(x) = 8x-5, \quad x \le 2$

$= 3x^2-4x+7, \quad x > 2$ at $x=2$

find f is differentiable or not

Ans. $f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

R.H.D

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2-4x+7-11}{x-2}$$

$$\lim_{x \to 2^+} \frac{3x(x-2)+2(x-2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x+2\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 3 \times 2 + 2 = 8$

$Df(2^+) = 8$

L.H.D

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x-5-11}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x-16}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 8$

$Df(2^-) = 8$

LHD = RHD

∴ f is differentiable at $x = 3$.

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4.

Ans.

Application of Derivative.

Q1 Find the intervals in which function is increasing or decreasing.

a. $f(x) = x^3 - 5x - 11$

Solution: f is increasing iff $f'(x) > 0$

$\because f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

$\therefore 3x^2 - 5 > 0$

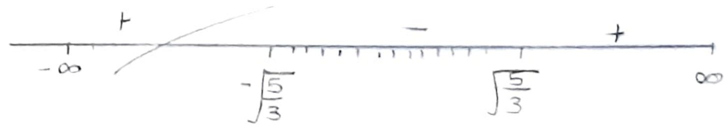
$\therefore x = \pm\sqrt{\frac{5}{3}}$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

Now f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$x = \pm\sqrt{\frac{5}{3}}$



$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

b. $f(x) = x^2 - 4x$

Solution:

f is increasing iff $f'(x) > 0$

$\because f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\therefore 2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$\therefore x = 2$

$\therefore x \in (2, \infty)$

Now f is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x = 2$

$\therefore x \in (-\infty, 2)$

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Ans

c. $f(x) = 2x^3 + x^2 - 20x + 4$

Solution: f is increasing iff $f'(x) > 0$

$\because f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 6x^2 + 12x - 10x - 20 > 0$

$\therefore 6x(x+2) - 10(x+2) > 0$

$\therefore (x+2)(6x-10) > 0$

$\therefore x = -2, \frac{5}{3}$

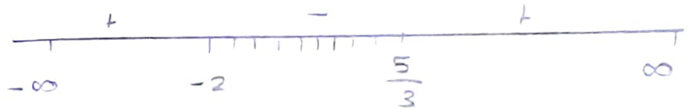
$\therefore x \in (-\infty, -2) \cup (5/3, \infty)$

Now f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$\therefore (x+2)(6x-10) < 0$

$x = -2, 5/3$



$\therefore x \in \left(-2, \frac{5}{3}\right)$

d. $f(x) = x^3 - 27x + 5$

Solution: f is increasing iff $f'(x) > 0$

$f(x) = x^3 - 27x + 5$

$\therefore f(x) = x^3 - 27x + 5$

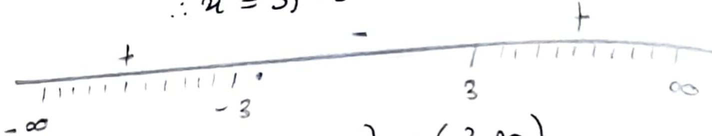
$\therefore f'(x) = 3x^2 - 27 > 0$

$\therefore 3x^2 - 27 > 0$

$\therefore 3(x^2 - 9) > 0$

$\therefore x^2 - 9 > 0$

$\therefore x = 3, -3$



$\therefore x \in (-\infty, -3) \cup (3, \infty)$

Now f is decreasing iff

$f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore 3(x^2 - 9) < 0$

$x^2 - 9 < 0$

$x = 3, -3$

$\therefore x \in (-3, 3)$

4
Ans

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e. $f(x) = 69 - 24x - 9x^2 + 2x^3$

Solution: f is increasing iff $f'(x) > 0$

$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore -24 - 18x + 6x^2 > 0$

$-6(-4 - 3x + x^2) > 0$

$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$(x-4)(x+1) > 0$

$x = 4, -1$

$x \in (-\infty, -1) \cup (4, \infty)$

Now f is decreasing iff $f'(x) < 0$

$\therefore -24 - 18x + 6x^2 < 0$

$6(-4 - 3x + x^2) < 0$

$(x-4)(x+1) < 0$

$x = 4, -1$

$x \in (-1, 4)$

Find the intervals in which function is concave upwards and concave downwards.

$y = 3x^2 - 2x^3$

Solution: $f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

f is concave upwards iff $f''(x) > 0$

$6 - 12x > 0$

$6(1 - 2x) > 0$

$(1 - 2x) > 0$

$-(2x - 1) > 0$

$-\infty \qquad \frac{1}{2} \qquad \infty$

$x \in \left(-\infty, \frac{1}{2}\right)$

f is concave downwards iff $f''(x) < 0$

$6(1 - 2x) < 0$

$-2(2x - 1) < 0$

$x \in \left(\frac{1}{2}, \infty\right)$

$y = f(x)$

$f(x) = x^4 - 6x^3 + 12x^2 - 5x + 2$

$f'(x) = 4x^3 - 18x^2 + 24x - 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upwards iff

$f''(x) > 0$

$12x^2 - 36x + 24 > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 > 0$

$(x - 2)(x - 1) > 0$

$x = 2$

f is concave downwards

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c) $y = x^3 - 27x + 5$

solution: $y = f(x)$

$\therefore f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

$\therefore f$ is concave upwards iff

$f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x = 0$

$-$ $+$

$-\infty$ 0 ∞

$\therefore f$ is concave downward iff

$f''(x) < 0$

$\therefore \cancel{6x < 0}$

$\therefore x < 0$

$\therefore x = 0$

$+$

$-\infty$ 0 ∞

$x \in (-\infty, 0)$

Ans

$y = 69 - 24x - 9x^2 + 2x^3$

Solution: $y = f(x)$

$f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore f''(x) = -18 + 12x$

$\therefore f$ is concave upward iff

$f''(x) > 0$

$\therefore -18 + 12x > 0$

$\therefore 6(2x - 3) > 0$

$2x - 3 > 0$

$x = 3/2$

$-\infty$ $\frac{3}{2}$ ∞

$x \in \left(+\frac{3}{2}, \infty\right)$

$\therefore f$ is concave downwards iff

$f''(x) < 0$

$\therefore -18 + 12x < 0$

$\therefore 6(2x - 3) < 0$

$\therefore x = 3/2$

$-$ $+$

$-\infty$ $\frac{3}{2}$ ∞

6. $y = 2x^3 + x^2 - 20x + 4$

$\therefore y = f(x)$

$f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore f''(x) = 12x + 2$

$\therefore$ f is concave upwards iff $f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 6x + 1 > 0$

$\therefore x = -1/6$

$-\infty \quad -1/6 \quad \infty$

$x \in \left(-\frac{1}{6}, \infty\right)$

$\therefore$ f is concave downward iff

$f''(x) < 0$

$12x + 2 < 0$

$2(6x + 1) < 0$

$6x + 1 < 0$

$x = -1/6$

$6x + 1 < 0$

$x + \frac{1}{6} < 0$

$x < -\frac{1}{6}$

$x \in \left(-\frac{1}{6}, -\infty\right)$

18/12/17

PRACTICAL No. 4.

Find maximum and minimum value of the following function.

$f(x) = x^2 + \frac{16}{x^2}$

$\therefore f'(x) = 2x - \frac{32}{x^3}$

For maxima / minimum

$f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$\therefore 2x = \frac{32}{x^3}$

$\therefore x^4 = 16$

$x = \pm 2.$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = f''(-2) = 2 + \frac{96}{(\pm 2)^4} = 2 + \frac{96}{16} = 8 > 0$

$\therefore f(x)$ is minimum at $x = \pm 2$

$\therefore f(2) = 8$ is the minimum value.

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2. $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = 15x^4 + 15x^2$

for maxima / minimum

$f'(x) = 15x^4 - 15x^2 = 0$

$\therefore x^4 - x^2 = 0$

$\therefore x^2(x^2 - 1) = 0$

$\therefore x = 0, -1, 1$

$f''(x) = 60x^3 - 30x$

$f''(0) = 0$

$f''(-1) = -60 + 30 = -30 < 0$

An.

$f''(1) = 60 - 30 = 30 > 0$

$f(x)$ is minimum 1 & maximum at -1

$f(-1) = 3 + 5 - 3 = 5$

$f(1) = 3 - 5 + 3 = 1$

$f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

For maxima / minima

$f'(x) = 0$

$3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$

$f''(2) = 6 > 0$

$\therefore f(x)$ is maxima at $x = 0$ and minima at $x = 2$

$f(0) = 1$

$f(2) = -3$

iv. $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

for maximal/ minima

$f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

$f''(x) = 12x - 6$

$f''(-1) = -12 - 6 = -18 < 0$

An $f(2) = 24 - 6 = 18 > 0$

$f(x)$ is maximum at $x = -1$ & minimum at $x = 2$

$\therefore f(-1) = 8$

$f(2) = -19$

Q2. Find the root of the following equation by newton's method.

i. $f(x) = x^3 - 3x^2 - 55x + 9.5 \quad (x_0 = 0)$

$\therefore f'(x) = 3x^2 - 6x - 55$

$x_1 = 0$

$f(x_0) = 9.5$

$f'(x_0) = -55$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{9.5}{-55}$

$x_1 = 0.1727$

$f(x_1) = -0.0828$

$f'(x_1) = -55.9467$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.1727 - \frac{-0.0828}{-55.9467}$

$x_2 = 0.1712$

$f(x_2) = 0.0011$

$f'(x_2) = -55.9393$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.1712 - \frac{0.0011}{-55.9393}$

$\therefore x_3 = 0.1712$ = root of eqn.

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ii. $f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

$f(2) = -9$

$f(3) = 6$

$\therefore 6$ is closer to 0 on the number line.

$x_0 = 3$

$f(x_0) = 6$

$f'(x_0) = 23$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 2.7391$

$f(x_1) = 0.5942$

$f'(x_1) = 18.508$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5942 / 18.508$

$x_2 = 2.707$

$f(x_2) = 0.0085$

$f'(x_2) = 17.9835$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.707 - \frac{0.0085}{17.9835}$

$x_3 = 2.7065$

$f(x_3) = -0.0005$

$f'(x_3) = 17.9755$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$= 2.7065 - \frac{-0.0005}{17.9757}$

$x_3 = 2.7065$

$\therefore 2.7065$ is the root of the given equation

iii. $f(x) = x^3 - 1.8x^2 - 10x + 17$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = 1 - 1.8 - 10 + 17 = 6.2$

$f(2) = -2.2$

$\therefore -2.2$ is closer to 0 on the number line

$\therefore x_0 = 2$

$f(x_0) = -2.2$

$f'(x_0) = -5.2$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 2 - \frac{-2.2}{-5.2}$

$x_1 = 1.5769$

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$f(x_1) = (1.5769)^3 - 1.8(1.5769)^2 - 10(1.5769) + 17$

$= 0.6762$

$f'(x_1) = -8.217.$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5769 - \frac{0.6762}{-8.217}$

$x_2 = 1.6592$

$\therefore f(x_2) = 0.0204$

$f'(x_2) = -7.7143$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 1.6592 - \frac{0.0205}{-7.7143}$

$x_3 = 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$f(x_3) = 0$

$\therefore$ 1.6618 is the root of the function

Practical 5

Solve the following integration

$\int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}} dx$

$= \int \frac{1}{\sqrt{(x+1)^2 - 2^2}} dx$

$I = \ln|x+1+\sqrt{x^2+2x-3}|$

$I = \int (4e^{3x}+1)\, dx$

$= 4\int e^{3x} dx + \int 1\, dx$

$I = \frac{4e^{3x}}{3} + x + c$

$I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$= 2\int x^2 dx - 3\int \sin x\, dx + 5\int x^{1/2} dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{5\times 2}{3} x^{3/2} + c$

$I = \frac{2x^3}{3} + 3\cos x + \frac{10}{3} x\sqrt{x} + c$

iv. $I = \int \frac{x^3 + 3x + 5}{\sqrt{x}} \, dx$

Put $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \quad \therefore \frac{dx}{\sqrt{x}} \; 2\,dt$

$I = \int \frac{(\sqrt{x})^6 + 3(\sqrt{x})^2 + 4}{\sqrt{x}} \, dx$

$=$ ~~Put $\sqrt{x} = t$~~

~~$\therefore \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \quad \therefore \frac{dx}{\sqrt{x}}$~~ $2\,dt$

~~$I = \int$~~

$= 2 \int \frac{t^6 + 3t^2 + 4 \, dt}{\sqrt{x}}$

$= 2 \int t^6 + 3t^2 + 4 \; dt$

$= 2 \int \left[\frac{t^7}{7} + \frac{3t^3}{3} + 4x \right] + C$

$= 2 \left[\frac{x^{7/2}}{7} + x^{3/2} + 4x^{1/2} \right] + C.$

v. $I = \int t^7 \sin(2t^4) \, dt$

$= \int t^4 \sin(2t^4) \cdot t^3 \, dt$

put $t^4 = x$

$4t^3 = \frac{dx}{dt}$

$\therefore t^3 \, dt = \frac{1}{4} dx$

$\therefore I = \frac{1}{4} \int x \cdot \sin(2x) \, dx$

$\therefore I = \frac{1}{4} \left[x \int \sin 2x \, dx - \int \left(\frac{d}{dx} x \int \sin 2x \, dx \right) dx \right]$

$= \frac{1}{4} \left[-\frac{x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx \right]$

$= \frac{1}{16} \sin 2x - \frac{x \cdot \cos 2x}{8} + C$

$I = \frac{1}{16} \sin 2t^4 - \frac{t^4 \cdot \cos 2t^5}{8} + C$

vi. $I = \int \sqrt{x}\,(x^2-1)\,dx$

$I = \int x^2\sqrt{x}\,dx - \int \sqrt{x}\,dx$

$= \int x^{5/2}\,dx - \int x^{1/2}\,dx$

$= \dfrac{x^{7/2}}{7/2} - \dfrac{x^{3/2}}{3/2} + C$

$= \dfrac{2}{7}x^3\sqrt{x} - \dfrac{2}{3}x\sqrt{x} + C$

vii. $I = \int \dfrac{1}{x^3}\sin\left(\dfrac{1}{x^2}\right)dx$

$\dfrac{1}{x^2} = t$

$\therefore \dfrac{-2}{x^3} = \dfrac{dt}{dx}$

$\therefore \dfrac{dx}{x^3} = -\dfrac{1}{2}\,dt$

$I = -\dfrac{1}{2}\int \sin t\,dt$

$= -\dfrac{1}{2}\times -\cos t + C$

$= \dfrac{\cos t}{2} + C$

$I = \dfrac{\cos(1/x^2)}{2} + C$

$I = \int \dfrac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$

Put $\sin x = t$

$\therefore \cos x\,dx = dt$

$\therefore I = \int \dfrac{1}{t^{2/3}}\,dt$

$= \int t^{-2/3}\,dt$

$= \dfrac{t^{-2/3+1}}{-2/3+1} + C$

$= 3t^{1/3} + C$

$I = 3\sqrt[3]{\sin x} + C$

ix. $I = \int e^{\cos^2 x} \cdot \sin 2x \, dx$

put $\cos^2 x = t$

$\therefore -2\cos x \sin x = \frac{dt}{dx}$

$\therefore \sin 2x \, dx = -dt$

$\therefore I = -\int e^t \, dt$

$= -e^t + C$

$\therefore I = -e^{\cos^2 x} + C$

x. $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

A. $x^3 - 3x^2 + 1 = t$

$\therefore 3x^2 - 6x = \frac{dt}{dx}$

$\therefore (x^2 - 2x) \, dx = \frac{dt}{3}$

$\therefore I = \frac{1}{3} \int \frac{1}{t} \, dt$

$= \frac{1}{3} \log |t| + C$

$I = \frac{1}{3} \log |x^3 - 3x^2 + 1| + C$

Application of Integration & Numerical Integration

Find the length of the following curve

$n = t \sin t,\ y = 1 - \cos t \quad t \in (0, 2\pi)$

arc length $= \int_0^{4} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^3 dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\ dt$

$= \int_0^{2\pi} \sqrt{2 - \cos t}\ dt$

$= \int_0^{2\pi} \sqrt{2}\ |\sin t/2|\ dt$

$= \cancel{\int_0^{2\pi}} 2 \sin \frac{t}{2}\ dt$

$= \left[-4 \cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$

$= \left[-4 \cos \pi\right] - (-4 \cos 0) = 4 + 4 = 8$

2. $y = \sqrt{4-x^2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \times (-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$I = \int_{-2}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}\, dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\int_{-2}^{2} \frac{1}{\sqrt{2^2-x^2}}\, dx$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 2\left[\pi/2 - \left(-\frac{\pi}{2}\right)\right]$$

$$L = 2\pi$$

$y = x^{3/2}$ in $[0,4]$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$L = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{3/2} \times \frac{1}{9}\right]_0^4$$

$$= \frac{-1}{27}\left[(4+0)^{3/2} - (4+36)^{3/2}\right]$$

$$L = \frac{1}{27}\left(40^{3/2} - 8\right) \text{ unit}$$

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4. $x = 3\sin t,\ y = 3\cos t,\quad t \in (0, 2\pi)$

$$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$$

$$l = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dx}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} 3\sqrt{x}\, dt$$

$$= 3\int_0^{2\pi} dt$$

$$= 3[x]_0^2\, dt$$

$$= 3(2\pi - 0)$$

$\therefore L = 6\pi$ units.

5. $x = \frac{1}{6} y^3 + \frac{1}{2y}$ on $y \in (1, 2)$

$$\therefore \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$$

$$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \frac{(y^4 - 1)}{4y^2}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4 - 1) + 4y^2 \cdot 1}{4y}}\, dy$$

$$= \int_1^2 \cancel{\frac{y^5 + 1}{(2y^2)^2}}\, dy$$

$$= \int_1^2 \frac{y^4 + 1}{2y^2}\, dy$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^2\, dy$$

$$\int_0^2 e^{x^2}dx = \frac{h}{3}\left[\cancel{(y_0+y_4)} + 4(y_1+y_3) + 2(y_2\right.$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right]$$

$$L = \frac{17}{12} \quad \text{units.}$$

$$\int_0^2 e^{x^2}\,dx \text{ with } n=4$$

$$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

x	0	0.5	1	1.5	2
y	1	1.284	2.718	9.4875	54.5982

$$\int_0^2 e^{x^2}\,dx = \frac{2}{3}\left[(y_0+y_4) + 4(y_1+y_3) + y_2\right]$$

$$= \frac{0.5}{3}\left[(1+54.5982) + 4(1.285+9.4877) + 2e\right]$$

$$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 5.436\right]$$

$$\int_0^2 e^{x^2}\,dx = 17.3535$$

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ii $\int_0^4 x^2\,dx \quad n=4$

$l = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

$\int_0^4 x^2\,dx = \frac{l}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$

$= \frac{1}{3}\left[0+16+4(1+9)+2\times4\right]$

$= \frac{1}{3}\left[16+4(10)+8\right]$

$= \frac{1}{3}\left[\right.$

$= \frac{64}{3}$

$\int_0^4 x^2\,dx = 21.3333$

iii $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n=6$

$l = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$

x	0	$\pi/18$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y	0	0.4167	0.5848	0.7071	0.8017	0.8752	0.9306

$\int_0^{\pi/3} \sqrt{\sin x}\,dx = \frac{l}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$

$= \frac{\pi/18}{3}\left[0.4167+0.9306+2(\ldots)(0.4167+0.7071+0.8752)+2(0.5848+0.8017)\right]$

$= \frac{\pi}{54}\left[1.3473+4(1.999)+2(1.3865)\right]$

$= \frac{\pi}{54}\left[1.3473+7.996+2.773\right]$

$= \frac{\pi}{54} \times 12.1163$

$\int_0^{\pi/3} \sqrt{\sin x}\,dx = 0.7049$

AK
08/10/2020

PRACTICAL 7

Q. Solve the following differential eqn

1. $x\frac{dy}{dx} + y = e^x$

$x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$P(x) = 1/x \quad Q(x) \; \frac{e^x}{x}$

$If = e^{\int P(x)dx}$

$= e^{\int 1/x dx}$

$= e^{\ln x}$

$If = e$

$y(If) = \int Q(x)(If)dx + C$

$yx = \int \frac{e^x}{x} \cdot x \, dx + C$

$= \int e^x dx + C$

$xy = e^x + C$

$e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + \frac{2e^x y}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$

$P(x) = 2 \quad Q(x) = e^{-x}$

$If = e^{\int P(x)dx}$

$IF = e^{\int 2dx}$

$= e^{2x}$

$y(If) = \int Q(x)(If)dx + c$

$= \int e^{-x} e^{2x} dx + c$

$= \int e^x dx + c$

$y \cdot e^{2x} = e^x + c$

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Soln :- $\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$f(x) = 2/x \qquad Q(x) = \frac{\cos x}{x^2}$

$IF = e^{\int p(x)\,dx}$

$= e^{\int 2/x\,dx}$

$= e^{2\ln x}$

$= e^{\ln x^2}$

$IF = x^2$

$y(IF) = \int Q(x)\,(IF)\,dx + C$

$y \cdot x^2 = \int \frac{\cos x}{x^2} \cdot x^2\,dx + C$

$= \int \cos x + C$

$x^2 y = \sin x + C.$

$\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

Ans $\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x \qquad Q(x) = \sin x / x^3$

$IF = e^{\int p(x)\,dx}$

$= e^{\int 3/x\,dx}$

$= e^{3\ln x}$

$= e^{\ln x^3}$

$IF = x^3$

$y(IF) = \int Q(x)\,dx + C$

$x^3 y = \int \frac{\sin x}{x^3}\, x^3\,dx + C$

$= \int \sin x\,dx + C$

$x^3 y = -\cos x + C$

Q2

v. $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$p(x) = 2 \quad Q(x) = 2x/e^{2x} = 2x\,e^{-2x}$

$If = e^{\int (Px)\,dx}$

$= e^{\int 2\,dx}$

$= e^{2x}$

$y(If) = \int Q(x)(If)\,dx + C$

$= \int 2x\,e^{-2x}\,e^{2x}\,dx + C$

$= \int 2x\,dx + C$

$ye^{2x} = x^2 + C$

vi. $\sec^2 x \cdot \tan y\,dx + \sec^2 y \tan x\,dy = 0$

$\sec^2 x \cdot \tan y\,dx = -\sec^2 y \tan x\,dy$

$\frac{\sec^2 x\,dx}{\tan x} = \frac{-\sec^2}{\sin y}\,dx$

$\therefore \int \frac{\sec^2 dx}{\tan x} = \int -\frac{\sec^2 y\,dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + C$

$\therefore \log|\tan x - \tan y| = c$

$\therefore \tan x \cdot \tan y = e^c$

vii. $\frac{dy}{dx} = \sin^2(x - y + 1)$

Put $x - y + 1 = v$

Diff both Sides

$x - y + 1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore 1 - \frac{dv}{dx} = \frac{dy}{dx}$

$\therefore 1 - \frac{dv}{dx} = \sin^2 v$

$\frac{dv}{dx} = 1 - \sin^2 v$

$\frac{dv}{dx} = \cos^2 v$

$\frac{dv}{\cos^2 v} = dx$

$\int \sec^2 v\,dv = \int dx$

$\therefore \tan v = x + c$

$\tan(x + y - 1) = x + 1$

8. $\frac{dy}{dx} = \frac{2x+3y-1}{6x+4y+6}$

put $(2x+3y) = v$

$2 + \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\frac{(v-1)}{(v+2)}$

$\therefore \frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\therefore \frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$= \frac{3v+3}{v+2}$

$= 3\left(\frac{v+1}{v+2}\right)$

$= \int\left(\frac{v+2}{v+1}\right) dv = 3\int dx$

$\int \frac{v+2}{v+1} dv + \int \frac{1}{v+1} dv = 3x + C$

$v + \log|v+1| = 3x + C$

$\therefore 2x+3y+\log|2x+3y+1| = 3x + C$

$3y = x - \log|2x+3y+1| + C$

Topic : Euler's Method

1. $\frac{dy}{dx} = y + e^x - 2 \quad y(0) = 2 \quad h = 0.5$ find $y(2) = ?$

Ans. $f(x) = y + e^x - 2 \quad x_0 = 0 \quad y(0) = 2 \quad h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		
5				

$\therefore y(2) = 9.8215$

2. $\frac{dy}{dx} = 1 + y^2, \; y(0) = 1 \quad h = 0.2$ find $y(1) = ?$

$x_0 = 0, \; y_0 = 0 \quad h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.411	0.9234
4	0.8	0.9234	1.8821	1.2939
5	0.1	1.2939		

$y(1) = 1.2939$

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23. $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0)=1$ $h=0.2$ find $f(1)$

$x_0 = 0,$ $y(0)=1$ $h=0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
		1	0	1
0	0	1	0.4472	1.0844
1	0.2	1.0844	0.6059	1.2105
2	0.4	1.2105	0.7040	1.3513
3	0.6	1.3513	0.7696	1.5051
4	0.8	1.5051		
5	1			

$\therefore y(1) = 1.5051$

24. $\frac{dy}{dx} = 3x^2+1$ $y(1)=2$ find $y(2)$ $h=0.5$
$h = 0.25$

$y_0 = 2$ $x_0 = 1$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
			4	4
0	1	2	7.75	7.875
1	1.5	4		
2	2	7.875		

$y(2) = 7.875$

$y_0 = 2$ $x_0 = 1$ $h = 0.25$

2)

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	59.1569	19.3360
3	1.75	19.3360	1122.6426	299.9960
4	2	299.9960		

$y(2) = 299.9960$

25. $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1)=1$ $h=0.2$

$x_0 = 1$ $y_0 = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
			3	3.6
0	1	1		
1	1.2	3.6		

$y(1.2) = 3.6$

Practical 9

Limits and Partial order derivaties

Q1. Evaluate the following limit

1) $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

$= \lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

Apply limit.

$= \frac{(-4)^3-3(-4)+(-1)^2-1}{(4)(-1)+5} = \frac{-64+12+1-1}{4+5} = \frac{-52}{9}$

2) $\lim_{(x,y)\to(4,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

$= \lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2-y^2-4x)}{x+3y}$ # Apply limit

$= \frac{(0+1)(2)^2+(0)^2-4(2)}{2+3(0)} = \frac{1(4+0-8)}{2}$

$= \frac{4-8}{2} = \frac{4}{2} = -2$

3) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3x^2y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3x^2y^2}$

Apply limit

$= \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)} = \frac{1-1}{1-1} = \frac{0}{0}$ Limit does not exist

find f_x, f_y for each of the foll. of

$f(x,y) = xy\,e^{x^2+y^2}$

$f_x = y\,(1 \cdot e^{x^2+y^2})$

$= y\,e^{x^2+y^2} + 2x \cdot y e^{x^2+y^2}$

$f_y = x\,(1 \cdot e^{x^2+y^2}) + xy(e^{x^2+y^2} \cdot 2y)$

$= x \cdot x^{x^2+y^2} + 2xy^2 e^{x^2+y^2}$

$f_x = y\,e^{x^2+y^2} + 2x^2 y e^{x^2+y^2}$

$\therefore f_y = xe^{x^2+y^2} + 2xy^2 e^{x^2+y^2}$

ii) $f(x,y) = e^x \cos y$

$f_x = \cancel{\cos y e}$

$f_y = e^x - \sin y$

$f_y = -\sin y\, e^x$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = y^2 3x^2 - 3y^2 x + 0 + 0$

$= 3x^2y^2 - 6xy$

$f_y = x^3 2y - 3x^2 + 3y^2$

$= 2x^2y - 3x^2 + 3y$

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Q. Usind def find values of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2}{1+y^2}$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

According to given $(a,b) - (0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h - 0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore f_x = 2,\ f_y = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h - 0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$\lim_{h\to 0} \frac{0-0}{h} = 0$$

$f_x = 2 \quad f_y = 0$

find all second order partial derivatives and verify whether $f_{xy} = f_{yx}$

$$f(x,y) = \frac{y^2 - xy}{x^2}$$

$$f_{xx} = \frac{d^2f}{dx^2} \quad f_{yy} = \frac{d^2f}{dy^2}$$

Applying $\frac{u}{v}$ rule

$$f(x) = \frac{x^2(0-y) - (y^2 - xy)2x}{x^4}$$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^4}$$

$$\therefore f_x = \frac{x^2y - 2xy^2}{x^4}$$

$$f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - 4x^5y + 8x^4y^2}{x^8}$$

$$= \frac{-2x^5y + 6x^4y^2}{x^8} \quad f_{xx} = \frac{6y^2 - 2xy}{x^4}$$

$$dy = \frac{1}{x^2}(2y - x) \quad \therefore f_y = \left(\frac{2y - x}{x^2}\right)$$

$$\therefore f_{yy} = \frac{1}{x^2} 2 = \frac{2}{x^2}$$

$$f_{xy} = \frac{2y - x}{x^2}$$

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Q. $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{d}{dx} x^3 + 3xy^2 - \log(x^2+1)$

$f_x = 3x^2 + 6xy^2 \frac{-2x}{x^2+1}$

$f_y = \frac{d}{dy} x^3 + 3x^2y^2 - \log(x^2+1)$

$= 0 + 6x^2y - 0$

$f_y = 6x^2y$

$f_{xx} = \frac{d}{dx} f_x$

$= \frac{d}{dx}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$f_x = 6x + 6y^2 \frac{-4x - 2x^2 + 2}{(x^2+1)^2}$

$f_{yy} = \frac{d}{dy} f_y$

$= \frac{d}{dy} 6x^2y$

$f_{yy} = 6x^2$

$f_{xy} = \frac{6x}{dy}$

$= \frac{d}{dy}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$f_{xy} = 12xy$

$f_{yx} = \frac{d}{dx} f_y$

$= \frac{d}{dx} 6x^2y$

$f_{xy} = 12xy$

$f(x,y) = \sin(xy) + e^{x+y}$

$f(x,y) = \sin(xy) + e^x . e^y$

$f_x = \frac{d}{dx}(\sin(xy) + e^x e^y)$

$f_x = y\cos(xy) + e^x e^y$

$f_y = \frac{d}{dy}(\sin(xy) + e^x . e^y)$

$= (x\cos(xy)) + e^x . e^y)$

$f_{xx} = \frac{d}{dx} f_x = \frac{d}{dx} y\cos(xy) + e^x e^y$

$f_{xx} = -y \sin(xy) + e$

$f_{yy} = \frac{d}{dy} f_y$

$= \frac{d}{dy} (x\cos(xy) + e^x . e^y)$

$f_{yy} = -x^2 \sin^2(xy) + e^x . e^y$

25. 1. $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$
2. $f(x,y) = 1-x+y\sin x$ at $(\frac{\pi}{2}, 0)$
3. $f(x.y) = \log x + \log y$ at $(1,1)$

Ans

1. $F(x,y) = \sqrt{x^2+y^2}$ $(a,b) = (1,1)$

$\therefore f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$ $f(1,1) = \sqrt{1+1} = \sqrt{2}$

$f_x = \frac{x}{\sqrt{x^2+y^2}}$ $f_y = \frac{2y}{2\sqrt{x^2+y^2}}$

$f_x(1,1) = \frac{1}{\sqrt{2}}$ $f_y = \frac{y}{\sqrt{x^2+y^2}}$

$\therefore f_y(1,1) = \frac{1}{\sqrt{2}}$

$L(x,y)$ $F(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} \quad \frac{+x+y-2}{\sqrt{2}}$

$(x,y) = \frac{x+y}{\sqrt{2}}$

$f(x,y) = 1-x+y\sin x$

$f(\pi/2, 0) = 1 - \frac{\pi}{2} + 0 + \sin \pi/2$

$F(\pi/2, 0) = \frac{2-\pi}{2}$

$f_x = -1 + y\cos x$ $f_y = \sin x$

$f_x(\pi/2, 0) = -1 + 0 \times \cos\pi/2$ $f_y(\pi/2, 0) = \sin\pi/2$

$f_x(\pi/2, 0) = -1$ $f_y(\pi/2, 0) = 1$

$L(x,y) = f(\pi/2, 0) + f_x(\pi/2, 0)(x-\pi/2) + f_y(\pi/2,0)(y-0)$

$L(x,y) = \frac{2-\pi}{2} + (-1)(x-\pi/2) + (y)$

$1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y$

$(x,y) = 1-x+y$

3) $f(x,y) = \log x + \log y$

$F(1,1) = \log 1 + \log 1$

$F(1,1) = 0$

$f_x = \frac{1}{x}$ $f_y = \frac{1}{y}$

$f_y(1,1) = 1$

$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 + 1(x-1) + 1(y-1)$

$L(x,y) = x + y - 2$

Practical - 10

Q1. Find the directional derivative of the following at given points and in direction of given vectors

1. $f(x,y) = x+2y-3 \quad a=(1,-1) \quad u = 3i-j$

Here $u = 3i-j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit Vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1\,2-3 =$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = 1+\frac{3}{\sqrt{10}}) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1+\frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

$= \lim_{h\to 0} \frac{-4+h/\sqrt{10}+4}{h}$

$D_u f(a) = \frac{1}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

$= \lim_{h\to 0} \quad 4+$

$f(x) = y^2-4x+1 \quad a=(3,4) \quad u = i+5j$

Here $u = i+5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2+(5)^2} = \sqrt{26}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5$

$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5h}{\sqrt{21}}\right)$

$f(a+hu) = 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$$= \frac{25h^2}{26} + \frac{40h - 4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}}}{h}$$

$$\frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$$

iii. $2x + 3y \quad a = (1,2), \; u = (3i + 4j)$

Here $u = 3i + 4j$ is not a unit vector

$$|u| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Unit Vector along v is $\frac{u}{|u|} = \frac{1}{5}(3,4)$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$f(a) = f(1,2) = 2(1) + 3(2) = 8$$

$$f(a+hu) = f\left(1 + \frac{3h}{5}, 2 + \frac{4}{5}\right)$$

$$f(a+hu) = 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= \frac{18h}{5} + 8$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$$= \frac{18}{5}$$

Q2. find gradient vector for following function at given points

i. $f(x,y) = x^y + y^x \quad a = (1,1)$

$$f_x = y \cdot x^{y-1} \; y^x \log y$$

$$f_y = x^y \log x + xy^{x-1}$$

$$f(x,y) = (f_x, f_y)$$

$$= (yx^{y-1} + y^2 \log y, \; x^y \log x + xy^{x-1})$$

$$f(1,1) = (1+0, 1+0)$$

$$= (1,1)$$

ii. $f(x,y) = (\tan^{-1} x) \cdot y^2 \quad a = (1,-1)$

$$f_x = \frac{1}{1+x^2} \cdot y^2$$

$$f_y = 2y \tan^{-1} x$$

$$f(x,y) = (f_x, f_y)$$

$$= \left(\frac{y^2}{1+x^2}, \; 2y \tan^{-1} x\right)$$

$$f(1,-1) = \left(\tfrac{1}{2},\ \tan^{-1}(1)(-2)\right)$$
$$= \left(\tfrac{1}{2},\ \tfrac{\pi}{4}(-2)\right)$$
$$= \left(\tfrac{1}{2},\ -\tfrac{\pi}{2}\right)$$

iii. $F(x,y,z) = xyz - e^{x+y+z}$ $\quad a = (1,-1,0)$

$F_x = yz - e^{x+y+z}$
$F_y = xz - e^{x+y+z}$
$F_z = xy - e^{x+y+z}$

$F(x,y,z) = F_x, F_y, F_z$
$= yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z}$

$F(1,-1,0) = \left((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}, (1)(-1)e^{0}\right)$
$= (0 - e^{0}, 0 - e^{0}, -1\, e^{0})$
$= (-1, -1, -2)$

Q3. Find the eqn of tangent and normal to each of the following using curves at given points.

i. $x^2\cos y + e^{xy} = 2$ at $(1,0)$

$F_x = \cos y\, 2x + e^{xy}\, y$
$F_y = x^2(-\sin y) + e^{xy}\cdot x$
$(x_0, y_0) = (1,0) \quad \therefore x_0 = 1$

eqn of tangent $\quad F_x(x - x_0) + F_y(y - y_0) = 0$

$F_x(x_0, y_0) = \cos 0\, 2(1) + e^{0}\cdot 0$
$= 1(2) + 0$
$= 2$

$F_y(x_0, y_0) = (1)^2(-\sin 0) + e^{0}\cdot 1$
$= 0 + 1$
$= 1$

$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0 \rightarrow$ It is required eqn of tangent

eqn of normal
$= ax + by + c = 0$
$= bx + ay + d = 0$

$1(x) + 2(y) + d = 0$
$x + 2y + d = 0$
$= 1 + 2(0) + d = 0$ $\quad$ at $(1,0)$
$d + 1 = 0$
$d = -1$

$x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$

$F_x = 2x + 0 - 2 + 0 + 0 = 2x - 2$
$F_y = 0 + 2y - 0 + 3 + 0 = 2y + 3$
$(x_0, y_0) = (2,-2) \quad \therefore x_0 = 2,\ y_0 = -2$

$F_x(x_0, y_0) = 2(2) - 2 = 2$
$F_y(x_0, y_0) = 2(-2) + 3 = -1$

eqn of tangent

$F_x(x - x_0) + F_y(y - y_0) = 0$
$2(x-2) + (-1(y+2)) = 0$
$2x - 2 - y - 2 = 0$
$2x - y - 4 = 0 \Rightarrow$ Required eqn of tangent

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eqn of normal

$= ax + by + c = 0$

$bx + ay + d = 0$

$= -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-2 + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$-6 + d = 0$

$\therefore d = 6$

Q4. find the eqn of tangent and normal line to each of the following surface.

i. $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$

$fx = 2x - 0 + 0 + z$

$fx = 2x + z$

$fy = 0 - 2z + 3 + 0$

$= -2y + x$

$(x_0, y_0, z_0) = (2, 1, 0) \therefore x_0 = 2, y_0 = 1, z_0 = 0$

$fx(x_0, y_0, z_0) = 2(2) + 0 = 4$

$fy(x_0, y_0, z_0) = 2(0) + 3 = 3$

$fz(x_0, y_0, z_0) = -2(1) + 2 = 0$

eqn of tangent

$fx(x_0 - x_0) + fy(y_0 - y_0) + fz(z_0 - z_0) = 0$

$= 4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$= 4x - 8 + 3y - 3 = 0$

$4x + 3y - 11 = 0$ ⇒ Required eqn of tangent.

eqn of normal at $(4, 3, -11)$

$\frac{x - x_0}{fx} = \frac{y - y_0}{fy} = \frac{z - z_0}{fz}$ 80

$= \frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 11}{0}$

$3xyz - x - y + z = -4$ at $(1, -1, 2)$

$3xyz - x - y + z + 4 = 0$ at $(1, -1, 2)$

$fx = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

$fy = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

$fz = 3xy - 0 - 0 + 1 + 0$

$= 3xy + 1$

$(x_0, y_0, z_0) = (1, -1, 2) \therefore x_0 = 1, y_0 = -1, z_0 = 2$

$fx(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$fy(x_0, y_0, z_0) = 3(1)(2) - 1 = 5$

$fz(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

eqn of tangent

$-7(x - 1) + 5(y + 1) - 2(z - 2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ ⇒ This is the required eqn of tangent

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eq of normal at $(-7,5,-2)$

$$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

25. Find the local maxima and minima for following

1. $f(x,y) = 3x^2+y^2-3xy+6x-4y$

$f_x = 6x+0-3y+6-0$

$= 6x-3y+6$

$f_y = 0+2y-3x+0-4$

$= 2y-3x-4$

$F_x = 0$

$\cancel{2y-3x-4=0}$

$\cancel{2y-3x=4}$

$6x-3y+6=0$

$3(2x-y+2)=0$

$2x-y+2=0$

$2x-y=-2 \rightarrow (1)$

$F_y = 0$

$2y-3x-4=0$

$2y-3x=4 \rightarrow (2)$

Multiply eqn 1 with 2

$4x-2y=-4$

$2y-3x=4$

$x=0$

substitute value of x in eqn (1)

$2(0)-y=-2$

$-y=-2$

$\therefore y=2$

Critical points are $(0,2)$

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r>0$

$= rt-s^2$

$= 6(2)-(-3)^2$

$= 12-9$

$= 3>0$

$\therefore f$ has maximum at $(0,2)$

$3x^2+y^2-3xy+6x-4y$ at $(0,2)$

$3(0)^2+(2)^2-3(0)(2)+6(0)-4(2)$

$0+4-0+0-8$

$=-4$

18.

ii. $f(x,y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy$
$f_y = 3x^2 - 2y$

$f_x = 0$

$8x^3 + 6xy = 0$
$2x(4x^2 + 3y) = 0$
$4x^2 + 3y = 0 \rightarrow (1)$

$f_y = 0$

$3x^2 - 2y = 0 \rightarrow (2)$

Multiply eqⁿ (1) with 3
(2) with 4

$12x^2 + 9y = 0$
$-12x^2 - 8y = 0$

$1y = 0 \quad y = 0$

substitute value of y in eqn (1)

$4x^2 + 3(0) = 0$
$4x^2 = 0$
$x = 0$

Critical point is $(0,0)$
$r = f_{xx} = 24x^2 + 6x$
$t = f_{yy} = 0 - 2 = -2$
$s = f_{xy} = 6x = 0 \quad 6x = 6(0) = 0$

at $(0,0)$

$= 24(0) + 6(0) = 0$

$\therefore r = 0$

$rt - s^2 = 0(-2) - (s)^2$

$= 0 - 0 = 0$

$r = 0$ and $rt - s^2 = 0$

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$f(x,y)$ at $(0,0)$

$2(0)^4 + 3(0)^2(0) - (0)$

$0 + 0 - 0$

$= 0$

iii. $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0 \therefore 2x + 2 = 0$
$x = -1$

$f_y = 0$

$-2y + 8 = 0$
$y = \frac{-8}{-2} = 4$
$\therefore y = 4$

$\therefore$ Critical point is $(-1, 4)$

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$r > 0$

$rt - s^2 = 2(-2) - (0)^2$
$= -4 - 0$
$-4 < 0$

$f(x,y)$ at $(-1,4)$

$(-1)^2 + (4)^2 + 2(-1) + 8(4) - 70$

$= 1 + 16 - 2 + 32 - 70$

$= 17 + 30 - 70$

$= 37 - 70 \quad = -33$

AV
05/02/2022