Quiz 1 Solutions
AMTL100: CALCULUS

1. Let $f(x) = 3x - 2\sin x + 8$. Note that $f(0) = 8 > 0$ and $f(-\pi) = 3(-\pi) + 8 < 0$. Since f is continuous therefore by intermediate value theorem $\exists x \in \mathbb{R}$ such that $f(x) = 0$.
   Now $f'(x) = 3 - 2\cos x$. Since $-1 \leq \cos x \leq 1 \implies f'(x) > 0 \implies f$ is strictly increasing. Therefore f has exactly one real solution.

2. We have $f(x) = x^4 - 6x^2 + 4$, so

$$\begin{aligned} f'(x) =& 4x^3 - 12x \\ =& 4x(x^2 - 3) \end{aligned}$$

   Therefore the critical points are $x = 0, \ \sqrt{3}$ and $-\sqrt{3}$.
   Now $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$ and $f''(0) = -12 < 0$ which implies f has a local maximum at $x = 0$.
   Since $f''(\sqrt{3}) = f''(-\sqrt{3}) = 24 > 0$ therefore f has a local minimum at $x = \pm\sqrt{3}$.
   Note that

$$\begin{aligned} f''(x) =& 12(x-1)(x+1) \\ =& \begin{cases} > 0 \ if \ x < -1 \ or \ x > 1 \\ < 0 \ if \ -1 < x < 1 \end{cases} \end{aligned}$$

Therefore at $x = \pm 1$ concavity of f changes i.e. $x = \pm 1$ are the points of inflection.

3. We know that the Taylor's polynomial of order 3 of the function f is given by

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Now,

$$\begin{aligned} f(x) =& x^3 + 3x^2 - 2x + 1 \\ f'(x) =& 3x^2 + 6x - 2 \\ f''(x) =& 6x + 6 \\ f'''(x) =& 6 \\ f^{(k)}(x) =& 0 \quad \forall k \geq 4. \end{aligned}$$

So $f(1) = 3,\ f'(1) = 7,\ f''(1) = 12,\ and\ f'''(1) = 6.$

$$\implies P_3(x) = 3 + 7(x-1) + \frac{12}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$\implies P_3(x) = 3 + 7(x-1) + 6(x-1)^2 + (x-1)^3$$

And the remainder term is given by

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-1)^4 = 0.$$