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HTML Content

Power series method

Example ①: Solve $y'' = y$

We assume that the solution $y(x)$ can be written as a power series in x, i.e.,

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

Then by the term-by-term differentiation,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

& $$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y(x)$ is a soln. to $y'' = y$ iff

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n} = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n\right]x^n = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} - a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad \text{for all } n = 0, 1, 2, \ldots}$$

$\longrightarrow$ recurrence relation.

Put $n=0$: $a_2 = \frac{a_0}{2\times 1} = \frac{a_0}{2!}$

Put $n=2$: $a_4 = \frac{a_2}{4\times 3} = \frac{a_0}{4\times 3\times 2\times 1} = \frac{a_0}{4!}$.

Inductively, $\boxed{a_{2n} = \frac{a_0}{(2n)!}}$ for $n = 0, 1, 2, \ldots$

Putting $n=1$, we get

$$a_3 = \frac{a_1}{3\times 2} = \frac{a_1}{3!}$$

Put $n=3$: $$a_5 = \frac{a_3}{5\times 4} = \frac{a_1}{5!}$$

$$\therefore \boxed{a_{2n+1} = \frac{a_1}{(2n+1)!} \text{ for } n=1,2,3,\ldots}$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{a_1}{(2n+1)!} x^{2n+1}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

Note that

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$= \frac{e^{x} + e^{-x}}{2}$$

and $y_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$= \frac{e^{x} - e^{-x}}{2}$$

$$\therefore \quad y(x) = a_0\left(\frac{e^{x}+e^{-x}}{2}\right) + a_1\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$= c_1 e^{x} + c_2 e^{-x},$$

where $c_1 = \frac{a_0+a_1}{2}$

$c_2 = \frac{a_0-a_1}{2}$.

Example ② Solve $y'' + y = 0$
by using power series method.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

The $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$\therefore \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$

$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$ for $n = 0, 1, 2, \ldots$

$\Rightarrow a_{2n} = \frac{(-1)^n a_0}{(2n)!}$ for $n = 0, 1, 2, \ldots$

$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$

$$\therefore \quad y = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1}$$

$$= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}_{\substack{\shortparallel \\ 1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\cdots \\ \shortparallel \\ \cos x}} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\substack{\shortparallel \\ \sin x}}$$

$$\therefore \quad y = a_0 \cos x + a_1 \sin x \, .$$

When is the power series applicable?

Theorem: Consider the linear ODE of the form $y'' + p(x)y' + q(x)y = r(x)$.

Suppose $p(x)$, $q(x)$ and $r(x)$ can be expressed as power series in x (or about $x = a$). Then the solution $y(x)$ can be written as a power series

$$\sum_{n=0}^{\infty} a_n x^n \quad \left(\text{or } \sum_{n=0}^{\infty} a_n (x-a)^n \right)$$

Remark (i) The above theorem is valid for any nth order linear ODE also.

(ii) Generally, we use this method when $p(x)$, $q(x)$, $r(x)$ are polynomials.

Example ③: Solve $y'' + xy = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\therefore \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Coeff. of x^0 : $\quad 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

Coeff. of x^{n+1} $(n \geq 0)$: $\quad (n+3)(n+2) a_{n+3} + a_n = 0$

$$\Rightarrow \quad \boxed{a_{n+3} = \frac{-a_n}{(n+3)(n+2)}} \text{ for } n = 0, 1, 2, \ldots$$

Since $a_2 = 0$, we get $a_5 = 0 = a_8 = a_{11} = \cdots$

$$\therefore \quad \boxed{a_{2+3k} = 0 \quad \text{for } k = 0, 1, 2, \ldots}$$

Putting $n=0$: $a_3 = \frac{-a_0}{3\times 2} = \frac{-a_0}{6}$

Put $n=3$: $a_6 = \frac{-a_3}{6\times 5} = \frac{a_0}{30\times 6} = \frac{a_0}{180}$,

Put $n=6$: $a_9 = \frac{-a_0}{9\times 8\times 180}$, ...

Put $n=1$: $a_4 = \frac{-a_1}{12}$

$n=4$: $a_7 = \frac{-a_4}{7\times 6} = \frac{a_1}{42\times 12}$,

We can find as many terms as we want.

$$y = a_0\left[1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - + \cdots\right] + a_1\left[x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - + \cdots\right]$$

Exercise: Solve the following using the power series method:

① $y'' - xy' + y = 0$

② $y'' - y' = 0$

③ $(2x^2 - 3x + 1)y'' + 2xy' - 2y = 0$

Remark: For ①, note that $y = x$ is a solution. Try to find a second linearly indep. soln. using the reduction of order method.

Ref: Chapter 5.1 in Kreyszig.