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Contents

Sum and Direct Sum

1 Sum of Subspaces

Suppose W_1 and W_2 are subspaces of a vector space V. Then the sum $W_1 + W_2$ is defined as

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Theorem 1. *$W_1 + W_2$ is a subspace of V.*

Proof. Clearly, $0 = 0 + 0 \in W_1 + W_2$.
Let $u, v \in W_1 + W_2$ and $\alpha, \beta \in \mathbb{F}$, then $u = w_1 + w_2,\ w_1 \in W_1,\ w_2 \in W_2$, and $v = w_1' + w_2',\ w_1' \in W_1,\ w_2' \in W_2$.

$$\therefore\ \alpha u + \beta v = (\underbrace{\alpha w_1 + \beta w_1'}_{\in W_1}) + (\underbrace{\alpha w_2 + \beta w_2'}_{\in W_2}) \in W_1 + W_2.$$

□

Theorem 2. $W_1 + W_2 = \operatorname{span}(W_1 \cup W_2)$.

Proof. $W_1 \subseteq W_1 + W_2,\ W_2 \subseteq W_1 + W_2$

$$\begin{aligned} \therefore \quad & W_1 \cup W_2 \subseteq W_1 + W_2 \\ \Rightarrow \quad & \operatorname{span}(W_1 \cup W_2) \subseteq W_1 + W_2 \end{aligned}$$

(because $\operatorname{span}(W_1 \cup W_2)$ is the smallest subspace containing $W_1 \cup W_2$).
Conversely, if $w_1 + w_2 \in W_1 + W_2$,
then $w_1 + w_2 = 1.w_1 + 1.w_2 \in \operatorname{span}(W_1 \cup W_2)$

$$\Rightarrow \quad W_1 + W_2 \subseteq \operatorname{span}(W_1 \cup W_2)$$

Hence, $W_1 + W_2 = \operatorname{span}(W_1 \cup W_2)$. □

Theorem 3. *Let V be a finite dimensional vector space and W_1, W_2 be subspaces of V. Then*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Proof. Let $\beta_1 = \{v_1, v_2, \ldots, v_k\}$ be a basis for $W_1 \cap W_2$ (If $W1 \cap W_2 = \{0\}$, then $\beta_1 = \phi$) Since $W_1 \cap W_2$ is contained in W_1, and $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent subset of W_1, there exists a basis

$$\beta_2 = \{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m\}$$

for W_1.
Similarly, there is a basis

$$\beta_3 = \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_l\}$$

for W_2.
Claim: $\beta = \{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_l\}$ is a basis for $W_1 + W_2$.

- β is linearly independent:

  Let

$$\begin{aligned} & a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_mu_m + c_1w_1 + \cdots + c_lw_l = 0 \\ \Rightarrow \quad & a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_mu_m = -c_1w_1 - \cdots - c_lw_l. \qquad (1) \end{aligned}$$

The L.H.S. $\in W_1$ and the R.H.S $\in W_2$.

$$\begin{aligned} \therefore \quad & -(c_1w_1 + \cdots + c_lw_l) \in W_1 \cap W_2 \\ \Rightarrow \quad & -(c_1w_1 + \cdots + c_lw_l) = d_1v_1 + \cdots + d_kv_k \\ \Rightarrow \; & d_1v_1 + \cdots + d_kv_k + c_1w_1 + \cdots + c_lw_l = 0 \end{aligned}$$

Since $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ is linearly independent,

$$d_1 = 0 = \cdots = d_k, \quad c_1 = 0 = \cdots = c_l.$$

$\therefore$ from 1, $a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_mu_m = 0$

$$\implies \quad a_1 = \cdots = a_k = 0 = b_1 = \cdots = b_m$$

(since $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ is linearly independent). Thus, β is linearly independent.

- $\mathrm{span}(\beta) = W_1 + W_2$.

Let $w_1' + w_2' \in W_1 + W_2$.
Then

$$\begin{aligned} w_1' &= \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_m u_m \\ w_2' &= \gamma_1 v_1 + \cdots + \gamma_k v_k + \delta_1 w_1 + \cdots + \delta_l w_l \\ \Rightarrow w_1' + w_2' &= (\alpha_1 + \gamma_1)\, v_1 + \cdots + (\alpha_k + \gamma_k)\, v_k \\ &\quad + \beta_1 u_1 + \cdots + \beta_m u_m + \delta_1 w_1 + \cdots \delta_l w_l \\ &\quad \in \mathrm{span}(\beta) \end{aligned}$$

$\therefore \beta$ is a basis for $W_1 + W_2$.

$$\begin{aligned} \Rightarrow \dim\left(W_1 + W_2\right) &= k + m + l = (k+m) + (k+l) - k \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2). \end{aligned}$$

□

2 Direct Sum

Definition 1. ***(Direct Sum)*** *The sum $(W_1 + W_2)$ is called a direct sum if $W_1 \cap W_2 = \{0\}$.*
Notation: $W_1 \oplus W_2$ denotes the direct sum.

Note that: $\dim\left(W_1 \oplus W_2\right) = \dim\left(W_1\right) + \dim\left(W_2\right)$.

Theorem 4. *$V = W_1 \oplus W_2$ iff every vector $v \in V$ can be written uniquely as the sum of vectors in W_1 & W_2.*

Proof. $\Longrightarrow$ suppose $v = w_1 + w_2 = w_1' + w_2{}'$, where $w_1, w_1' \in W_1$; $w_2, w_2' \in W_2$.

$$\begin{aligned} &\Rightarrow \quad w_1 - w_1' = w_2' - w_2 \in W_1 \cap W_2 = \{0\} \\ &\Rightarrow w_1 = w_1' \quad \& \quad w_2 = w_2'. \end{aligned}$$

$\Longleftarrow$ Suppose any $v \in V$ can be written uniquely as $v = w_1 + w_2$, where $w_1 \in W_1$, $w_2 \in W_2$.

To show: $V = W_1 \oplus W_2$
For this, we need to show $W_1 \cap W_2 = \{0\}$.

Let $w \in W_1 \cap W_2$.
Then $w = w + 0 \in W_1 + W_2$
also, $w = 0 + w \in W_1 + W_2$, by uniqueness, $w = 0$. $\square$

Remark 1. *If W_1, W_2, W_3 are three subspaces, we can consider $W_1+W_2+W_3$ which is also a subspace.*

Example 1. Is $\dim(W_1 + W_2 + W_3) = \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$?

Solution This is not correct. Take $V = \mathbb{R}^2$, $W_1 = x$-axis, $W_2 = y$-axis, $W_3 = \{y = x \text{ line}\}$. Then $W_1 + W_2 + W_3 = \mathbb{R}^2$ and $W_i \cap W_j = \{0\}$. $\therefore$ L.H.S. $=2$ but R.H.S.$=3$. $\blacksquare$

Row Space, Column Space and Null Space of a Matrix

3 Row Space and Column Space

Let $A \in M_{m\times n}(\mathbb{F})$. There are m rows $R_1, R_2, \ldots, R_m$ and n columns $C_1, C_2, \ldots, C_n$ of A. Each row of A can be thought of as an element of $\mathbb{F}^n$ and each column of A can be thought of as an element of $\mathbb{F}^m$.

Definition 2. *The row space of A is the space spanned by the rows $R_1, R_2, \ldots, R_m$. So, the row space is a subspace of $\mathbb{F}^n$.*

Definition 3. *The column space of A is the space spanned by the columns $C_1, C_2, \ldots, C_n$. So, the column space is a subspace of $\mathbb{F}^m$.*

Example 2. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}$,
then Row space $(A) = \text{span}\,\{(1,2,3),(2,-1,4)\} \subseteq \mathbb{R}^3$, and
the Column space $(A) = \text{span}\,\{(1,2),(2,-1),(3,4)\} = \mathbb{R}^2$.

Definition 4. *The row rank and the column rank of A are the dimensions of the row space and the column space, respectively. i.e.*
row rank $(A) = \dim($row space $(A))$ and column rank $(A) = \dim($column space $(A))$.

Remark 2.
1. $0 \le$ *row rank* $(A) \le \min\{m,n\}$.
   (Since row space $(A) \subseteq \mathbb{F}^n \implies$ row rank $(A) \le n$, also, since row space $(A) =$ span $\{R_1, R_2, \ldots, R_m\} \implies$ row rank $(A) \le m$.)

2. $0 \le$ *column rank* $(A) \le \min\{m,n\}$.

Remark 3. *For* $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$,

$$\begin{aligned} AX &= x_1C_1 + x_2C_2 + \cdots + x_nC_n \\ &\in \text{span}\{C_1, C_2, \ldots, C_n\} \end{aligned}$$

$\therefore$ *Column space* $(A) = \{AX : X \in \mathbb{F}^n\}$.

Remark 4. *Observe that Row space* (A) = *Column space* (A^t), *and Column space* (A) = *Row space* (A^t).

Definition 5. ***(Rank of a matrix)*** *The rank of a matrix is the number of nonzero rows in the RRE form of the matrix.*

Question: How to find row rank (A) and a basis for row space (A)?

Note that if A is row equivalent to B, then row space (A) = row space (B). $\therefore$ If R is the RRE form of A, then row space (A) = row space (R). Also, the nonzero rows of an RRE matrix are linearly independent; therefore, the nonzero rows of R form a basis for the row space (A).

$$\begin{aligned} \therefore \text{row rank}(A) &= \text{the number of nonzero rows in the RRE form of } A \\ &= \text{Rank}(A). \end{aligned}$$

Remark 5. *We will prove later that: Row rank* (A) = *Column rank* (A).

4 Null Space

Definition 6. *The null space of* A *is defined as*

$$\begin{aligned} \textit{Null space } (A) &= \left\{X = (x_1, x_2, \ldots, x_n)^t : AX = 0\right\} \\ &= \textit{the solution space of the homogeneous system } AX = 0; \\ &\subseteq \mathbb{F}^n. \end{aligned}$$

$$\therefore \ \dim(\text{null space } (A)) = \dim(\text{solution space of } AX = 0).$$

If r is the number of nonzero rows in the RRE form of A and n is the number of columns of A, then $\dim(\text{null space }(A)) = n - r$.

To find a basis for the null space (A), we identify the $n-r$ free variables and then find solutions by putting one of the free variables equal to 1 and the rest 0.

Example 3. Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & -1 & 4 & 1 \\ 4 & 1 & 5 & 1 \\ 2 & -3 & 3 & 1 \end{pmatrix}$. Find a basis for the null space (A).

Solution

$$A \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 2R_1}]{} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -7 & 1 & 1 \\ 0 & -7 & 1 & 1 \\ 0 & -7 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -7 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 \to -\frac{1}{7}R_2]{} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 & \frac{9}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the above matrix is the RRE form of A; free variables are x_3 and x_4. Putting

$x_3 = \lambda$ and $x_4 = \mu$, we get

$$x_1 = -\frac{9}{7}\lambda - \frac{2}{7}\mu,$$
$$x_2 = \frac{1}{7}\lambda + \frac{1}{7}\mu.$$

$$\therefore \text{ Null space } (A) = \left\{ \left(-\frac{9}{7}\lambda - \frac{2}{7}\mu,\ \frac{1}{7}\lambda + \frac{1}{7}\mu,\ \lambda,\ \mu \right);\ \lambda, \mu \in \mathbb{R} \right\}.$$

Since,

$$\left(-\frac{9}{7}\lambda - \frac{2}{7}\mu,\ \frac{1}{7}\lambda + \frac{1}{7}\mu,\ \lambda,\ \mu \right) = \lambda \left(-\frac{9}{7}, \frac{1}{7}, 1, 0 \right) + \mu \left(-\frac{2}{7}, \frac{1}{7}, 0, 1 \right);$$

$\therefore$ A basis for the null space (A) is

$$\beta = \left\{ \left(-\frac{9}{7}, \frac{1}{7}, 1, 0 \right), \left(-\frac{2}{7}, \frac{1}{7}, 0, 1 \right) \right\}.$$

■