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1 Ordinary Differential Equations

Definition 1. *An equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation, i.e.*

$$F(x, y, y', \dots, y^{(n)}) = 0,$$

where x is an independent variable, $y = y(x)$ is a function of x, $y^{(k)}$ is the k^{th} derivative of y.

Definition 2. *The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.*

Definition 3. *(Solution of an ODE) A solution to an ODE is a function defined on some interval and satisfies the ODE. (For n^{th} order ODE, a solution must be at least n-times differentiable.)*

Question 1. *Does every ODE have a solution?*

Solution No, $\left(\frac{dy}{dx}\right)^2 + 1 = 0$ has no real solutions. ■

Question 2. *If an ODE has a solution, is it unique?*

Solution No, $\frac{dy}{dx} = 0$ has $y = c$ as a solution for every $c \in \mathbb{R}$. ■

Definition 4. ***Initial Value Problem (IVP):*** *An IVP is an ODE: $y^{(n)} = f(x, y, y', \dots, y^{n-1})$ together with initial conditions $y(x_0) = y_0,\ y'(x_0) = y_1,\ y^{n-1}(x_0) = y_{n-1}$.*

So, the first order IVP is $y' = f(x, y),\ y(x_0) = y_0$. It is natural to ask about the existence and uniqueness of solution of an IVP.

2 Methods of Solving First Order ODEs

2.1 Separable ODEs

Suppose the ODE can be written as $g(y)\frac{dy}{dx} = h(x)$, i.e.

$$g(y)dy = h(x)dx$$

Integrating both sides, we get

$$\int g(y)dy = \int h(x)dx + c.$$

Sometimes the ODE can be converted into a separable ODE by substitution, for example, $u = \frac{y}{x}$.

Example 1. Solve the differential equation $\frac{dy}{dx} = \frac{x+2y}{2x+y}$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{x+2y}{2x+y} \\ &= \frac{1+2\frac{y}{x}}{2+\frac{y}{x}} \end{aligned}$$

Put $u = \frac{y}{x}$, i.e. $y = ux$

$$\begin{aligned} \implies \frac{dy}{dx} &= u + x\frac{du}{dx} \\ \therefore u + x\frac{du}{dx} &= \frac{1+2u}{2+u} \\ \implies x\frac{du}{dx} &= \frac{1-u^2}{2+u} \\ \implies \int \frac{2+u}{1-u^2}du &= \int \frac{1}{x}dx \end{aligned}$$

■

2.2 Linear First Order ODEs

Definition 5. *A first order linear ordinary differential equation is an equation that can be expressed in the form* $\frac{dy}{dx} + p(x)y = g(x)$. *It is called homogeneous if* $g(x) = 0$*; otherwise, it is non-homogeneous.*

To solve this type of differential equation, we multiply the ODE by an integrating factor $u(x) = e^{\int p(x)dx}$, i.e.

$$\begin{aligned} e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}.p(x)y &= u(x)g(x) \\ \implies \frac{d}{dx}[u(x)y] &= u(x)g(x) \\ \implies u(x)y &= \int u(x)g(x)dx + c \\ \implies y &= \frac{1}{u(x)}\left[\int u(x)g(x)dx + c\right] \end{aligned}$$

First order linear ODEs always have solutions provided $p(x)$ and $g(x)$ are continuous on an interval I. In that case, the solution must exist on the interval I.

2.3 Exact ODEs

Consider first order ODE of the form $M(x, y)dx + N(x, y)dy = 0$. This is called an exact ODE if $M(x, y)dx + N(x, y)dy = d(u(x, y))$ for some $u(x, y)$. Then we have $d(u(x, y)) = 0$, which gives $u(x, y) = c$ as solutions for any constant c.

Example 2. Consider the differential equation $ydx + xdy = 0$. Since $ydx+xdy = d(xy)$, the above equation is exact and $xy = c$ gives the solutions.

Necessary Condition for Exactness:
By the chain rule,

$$\begin{aligned} \frac{d}{dx}[u(x,y)] &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \\ \implies d(u(x,y)) &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \end{aligned}$$

$\therefore$ if $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$, then the ODE $Mdx + Ndy = 0$ is exact. In this case,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \ \& \ \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

So,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Theorem 1. *Assume that $M(x, y)$ and $N(x, y)$ have continuous partial derivatives. Then $Mdx + Ndy = 0$ is an exact ODE if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.*

Proof. We have already proved the forward direction. Now, assume that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We want to find $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x,y) \ \& \ \frac{\partial u}{\partial y} = N(x,y).$$

Integrating $\frac{\partial u}{\partial x} = M(x, y)$ with respect to x, we get

$$u(x,y) = \int M(x,y)dx + h(y),$$

where $h(y)$ is any function of y. Differentiating the above equation w.r.t. y, we get

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + h'(y) \\ &= \int \frac{\partial M}{\partial y}dx + h'(y) \end{aligned}$$

we want: $\frac{\partial u}{\partial y} = N(x, y)$. So,

$$\int \frac{\partial M}{\partial y} dx + h'(y) = N(x, y)$$
$$\Longleftrightarrow h'(y) = N(x, y) - \int \frac{\partial M}{\partial y} dx$$

If we find $h(y)$ satisfying the above, then the ODE is exact. We can find such $h(y)$ provided $\frac{\partial}{\partial x}(R.H.S) = 0$, i.e.

$$\frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\int \frac{\partial M}{\partial y} dx\right) = 0$$
$$\text{i.e.} \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0,$$

which is true by assumption. □

Example 3. Solve $2xy\, dx + (x^2 + y^2)dy = 0$.

Solution Here, $M = 2xy$; $N = x^2 + y^2$.

$$\therefore \frac{\partial M}{\partial y} = 2x \ \& \ \frac{\partial N}{\partial x} = 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. So, $u(x, y) = c$ will be a solution, where $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$.

$$\therefore \frac{\partial u}{\partial x} = 2xy \implies u(x, y) = x^2 y + h(y)$$
$$\because \frac{\partial u}{\partial y} = N \implies x^2 + h'(y) = x^2 + y^2$$
$$\implies h'(y) = y^2$$

So, we can take $h(y) = \frac{y^3}{3}$.

$$\therefore u(x, y) = x^2 y + \frac{y^3}{3}.$$

Therefore, the general solution is $u(x, y) = c$, i.e. $x^2 y + \frac{y^3}{3} = c$. ■

If $Mdx + Ndy = 0$ is not exact, can we still solve it in some special cases?

Example 4. Consider $ydx - xdy = 0$. Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$, $\therefore$ the equation is not exact. Can we multiply the given equation by some function so that it becomes exact? Multiplying by $\frac{1}{x^2}$, we get

$$\frac{ydx - xdy}{x^2} = 0 \implies d\left(\frac{y}{x}\right) = 0$$

$\therefore \frac{y}{x} = c$ is the general solution.

2.4 Integrating Factors for Non-Exact ODEs

Suppose $Mdx + Ndy = 0$ is not exact, but we can find some function $u(x, y)$ such that $uMdx + uNdy = 0$ is exact. We must have

$$\begin{aligned} &\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN) \\ \iff &\frac{\partial u}{\partial y}M + uM_y = \frac{\partial u}{\partial x}N + uN_x \\ \iff &\frac{\partial u}{\partial y}M - \frac{\partial u}{\partial x}N = u(N_x - M_y) \end{aligned}$$

Suppose u is a function of x only. Then $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = u'(x)$. Then

$$\begin{aligned} &-u'(x)N = u(x)(N_x - M_y) \\ \iff &u'(x) = u(x)\left(\frac{M_y - N_x}{N}\right) \end{aligned}$$

If $\frac{M_y - N_x}{N}$ is a function of x only, then we can find

$$u(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

Similarly, if $\frac{M_y - N_x}{M}$ is a function of y only, then

$$u(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right)$$

is an integrating factor.

2.5 Summary

- First check if $M_y - N_x = 0$ or not.
- If $M_y - N_x = 0$ then the equation is exact.
- If $M_y - N_x \neq 0$ then check if either
  1. $\frac{M_y - N_x}{N}$ is a function of x only, then $u(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right)$ is an integrating factor.
  2. $\frac{M_y - N_x}{M}$ is a function of y only, then $u(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right)$ is an integrating factor.

Example 5. Consider $x \sin y \; dx + (x + 1)\cos y \; dy = 0$. Try to find an integrating factor to make it exact and then solve it.

Solution Here, $M = x \sin y$ & $N = (x+1)\cos y$, so

$$M_y = x \cos y \ \& \ N_x = \cos y,$$

$\because\ M_y - N_x \neq 0$, the equation is not exact. Observe that

$$\frac{M_y - N_x}{N} = \frac{x-1}{x+1},$$

is a function of x only. So,

$$u(x) = \exp\left(\int \frac{x-1}{x+1} dx\right) = \frac{e^x}{(x+1)^2},$$

is an integrating factor.

$$\therefore\ x \sin y\ u(x)\ dx + (x+1)\cos y\ u(x)\ dy = 0$$

is an exact differential equation. So, the solution will be given by

$$\begin{aligned}
g(x,y) &= \int x \sin y\ u(x)\ dx + h(y) \\
&= \sin y \int \frac{xe^x}{(x+1)^2} dx + h(y) \\
&= \sin y \frac{e^x}{x+1} + h(y) \\
\because\ \frac{\partial g}{\partial y} &= (x+1)\cos y\ u(x) \\
\implies \frac{e^x \cos y}{x+1} + h'(y) &= \frac{e^x \cos y}{x+1} \\
\implies h'(y) &= 0.
\end{aligned}$$

So, we can take $h(y) = c$.

$\therefore\ \frac{e^x \sin y}{x+1} = c$ is the general solution of the ODE. ■