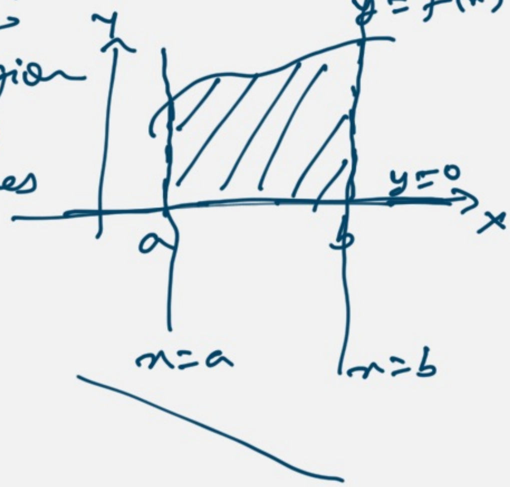
Applications of Definite Integrals

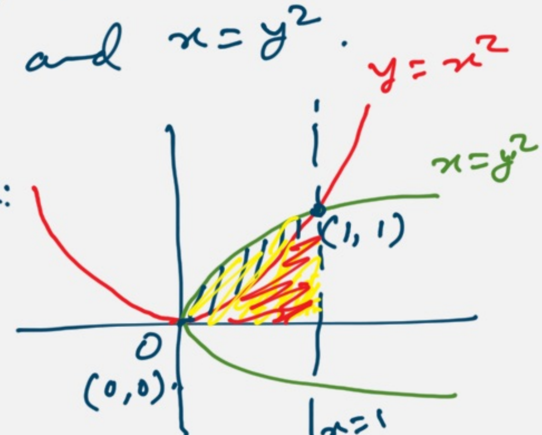
① Calculating area:

The area of the region bounded by the curves $y = f(x)$, and the lines $x = a$, $x = b$, $y = 0$ is given by

$$\int_a^b f(x)\,dx .$$



Example: Find the area bounded by the curves $y = x^2$ and $x = y^2$.



Solution:

Pts. of intersection:

$y = x^2 = (y^2)^2$

$\Rightarrow y = 0$ or $y = 1$

When $y = 0$, $x = 0$

When $y = 1$, $x = 1$

The region is bounded from above by $x = y^2$ and from below by $y = x^2$

∴ Required area

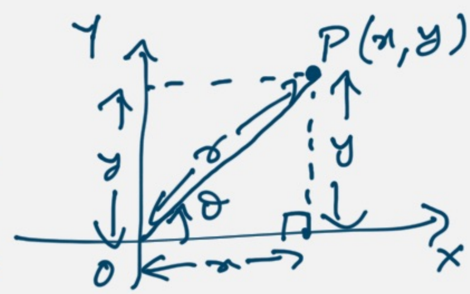
$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$

$$= \frac{2}{3}x^{3/2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Polar coordinates:

Let $P \equiv (x, y)$ in the Cartesian coordinates.



If r = length of OP

& θ = angle of OP with the +ve x-axis, then

$$x = r\cos\theta \;;\; y = r\sin\theta$$

Also, $r = \sqrt{x^2+y^2}$

$$\tan\theta = \frac{y}{x} \text{ if } x \neq 0.$$

Symmetry in polar coordinates

Suppose a curve is defined by
$r = f(\theta)$ in polar coordinates.

1. If $f(-\theta) = f(\theta)$, then the graph is symmetric about the x-axis.
2. If $f(\pi - \theta) = f(\theta)$, then the graph is symmetric about the y-axis.
3. If $f(\pi + \theta) = f(\theta)$, then the graph is symmetric about the origin.

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Examples:

1. (Leminiscate):
Consider $r^2 = \cos 2\theta$.
We see that the graph must be symmetric about the x-axis, y-axis and the origin.
Hence it is enough to trace the curve in the first quadrant.
Since $\cos 2\theta = r^2 \geq 0$, the domain of θ in the first quadrant is $[0, \frac{\pi}{4}]$.
For $\theta = 0$, $r = 1$ is the max. possible
For $\theta = \frac{\pi}{4}$, $r = 0$ is the min possible

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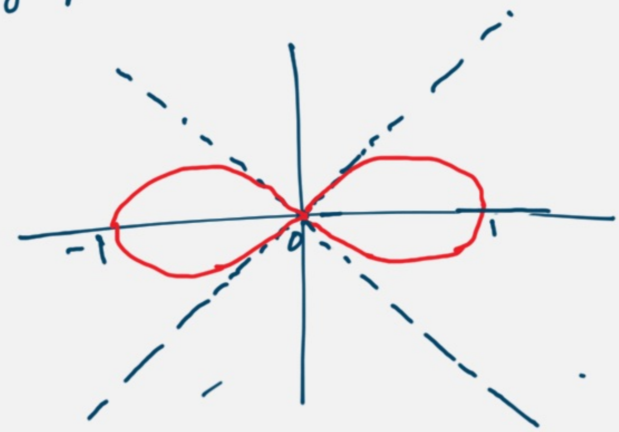
The graph looks like below:



② (Cardioid): $r = 1 - \cos\theta$

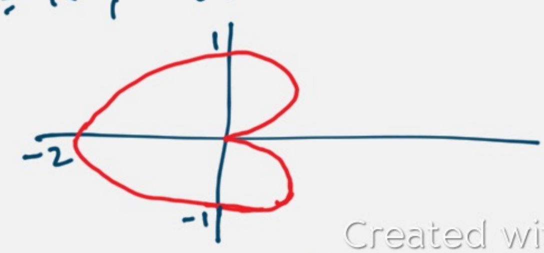
$(r, \theta) \in$ graph $\Leftrightarrow (r, -\theta) \in$ graph.

$\therefore$ The graph is symmetric about the x-axis.

So, it is enough to trace the curve for $0 \le \theta \le \pi$.

For $\theta = 0$, $r = 0$ is the minimum

For $\theta = \pi$, $r = 2$ is the maximum



Area in polar coordinates:

Consider the region bounded by the rays $\theta = \theta_1$ and $\theta = \theta_2$, and the curve $r = f(\theta)$.

The area is given by

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (f(\theta))^2\, d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta .$$

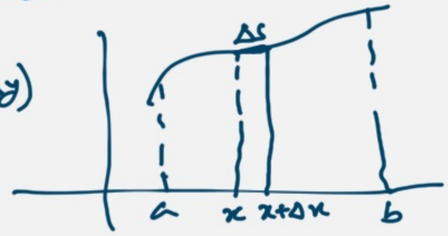
Problem: Find the area of the region enclosed by the cardioid $r = 2(1-\cos\theta)$

Solution:

$$\begin{aligned}
\text{Area} &= 2 \times \frac{1}{2}\int_0^{\pi} r^2\, d\theta \\
&= \int_0^{\pi} 4(1+\cos^2\theta - 2\cos\theta)\, d\theta \\
&= 4\int_0^{\pi}\left[1 + \frac{1+\cos 2\theta}{2} - 2\cos\theta\right] d\theta \\
&= 4\left[\pi + \frac{\pi}{2} + 0 + 0\right] \\
&= 6\pi
\end{aligned}$$

Arc Length

Consider a curve defined by $y = f(x)$ between $x = a$ and $x = b$.



$\Delta s \simeq$ distance between (x, y) and $(x+\Delta x, y+\Delta y)$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$\therefore$ The arc length is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$