Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Char. poly. of A, $p(x) = \det(xI - A)$

$$= \det \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

$$= x^2 + 1$$

Since $p(x)$ has no real roots, there are no real eigenvalues and eigenvectors.

However, if we allow complex eigenvectors, then $\lambda = \pm i$ (the complex roots of x^2+1) are complex eigenvalues, and we can calculate complex eigenvectors as follows:

For $\lambda = i$:

$$\lambda I - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$i z_1 + z_2 = 0 \iff z_2 = -i z_1$$

So, $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector for $\lambda = i$.

For $\lambda = -i$: $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-i z_1 + z_2 = 0 \iff z_2 = i z_1$$

So, $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector for eigenvalue $-i$.

Remark: If $\alpha \pm i\beta$ are eigenvalues of a real matrix A, then if $X \in M_{n\times 1}(\mathbb{C})$ such that $AX = (\alpha + i\beta)X$

(i.e. X is an eigenvector of A corresponding to eigenvalue $\alpha + i\beta$)

Then $A\overline{X} = (\alpha - i\beta)\overline{X}$

So, eigenvectors for conjugate pair of eigenvalues can be found by taking the complex conjugate of eigenvectors.

Diagonalizability

A matrix $A \in M_{n\times n}(\mathbb{R})$ (or $M_{n\times n}(\mathbb{C})$) is said to be diagonalizable if A is similar to a diagonal matrix.

i.e. A is diagonalizable if there exists an invertible matrix P s.t.

$\bar{P}^{1}AP = D$, where D is a diagonal matrix.

. Of course, every diagonal matrix is diagonalizable.

Suppose A is diagonalizable.

Then $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

$\Rightarrow \quad P^{-1}APe_i = De_i = \lambda_i e_i \quad ; \; e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to i\text{th}$

$\Rightarrow \quad A(Pe_i) = P(\lambda_i e_i) = \lambda_i (Pe_i)$

$\Rightarrow \quad Pe_i$ is an eigenvector of A with eigenvalue λ_i

(Note that $Pe_i \neq 0$ because if $Pe_i = 0 \Rightarrow P^{-1}Pe_i = 0 \Rightarrow e_i = 0$, which is not true)

So, we get eigenvectors $Pe_1, Pe_2, \cdots, Pe_n$ of the matrix A.

Also, $Pe_1, Pe_2, \cdots, Pe_n$ are linearly independent (because they are columns of invertible matrix P)

$\therefore \; \{Pe_1, Pe_2, \cdots, Pe_n\}$ is a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

So, if A is diagonalizable then we can find a basis consisting of eigenvectors of A.

Created with Doceri

Conversely, suppose $\{x_1, x_2, \cdots, x_n\}$ is a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A ie, $x_1, x_2, \cdots, x_n$ are n linearly indep. eigenvectors of A.

$\therefore \quad A x_i = \lambda_i x_i$ for $i = 1, 2, \cdots, n$.

(Note that $\lambda_1, \lambda_2, \cdots, \lambda_n$ need not be distinct)

Let P be the matrix whose columns are $x_1, x_2, \cdots, x_n$

Claim : $P^{-1} A P = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

Pf : We'll show : $AP = PD$

$$AP e_i = A(Pe_i) = A x_i$$
$$= \lambda_i x_i$$

Also, $PD e_i = P(\lambda_i e_i) = \lambda_i P e_i = \lambda_i x_i$

$\therefore \quad AP e_i = PD e_i \quad \forall i = 1, 2, \cdots, n.$

$\Rightarrow \quad AP = PD$

$\Rightarrow \quad P^{-1} A P = D$.

So, we got the following result.

Theorem: A is diagonalizable if and only if A has n linearly indep. eigenvectors.

Examples: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Is A diagonalizable?

What are the eigenvectors of A?

$p(x) = \det(xI - A) = \det\begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix}$

$= x^2$

$\Rightarrow$ $\lambda = 0$ is the only eigenvalue of A.

$\lambda I - A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of A, then

$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow$ $y = 0$

$\therefore$ $\left\{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \neq 0\right\}$ is the set of eigenvectors.

$\Rightarrow$ We cannot find two lin. indep. eigenvectors of A.

$\therefore$ A is not diagonalizable.

Prop: If λ_1 and λ_2 are two distinct eigenvalues of A, then the corresponding eigenvectors are lin. indep.

Pf: Let $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $x_1 \neq 0, x_2 \neq 0$, $\lambda_1 \neq \lambda_2$

To show: $\{x_1, x_2\}$ is lin. indep.

Assume $c_1 x_1 + c_2 x_2 = 0$ — (i)

$\Rightarrow A(c_1 x_1 + c_2 x_2) = A\,0 = 0$

$\Rightarrow c_1 A x_1 + c_2 A x_2 = 0$

$\Rightarrow c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$ — (ii)

$\lambda_2 \times$(i) — (ii) $\Rightarrow c_1(\lambda_2 - \lambda_1) x_1 = 0$

$\Rightarrow c_1(\lambda_2 - \lambda_1) = 0$ $(\because x_1 \neq 0)$

$\Rightarrow c_1 = 0$ $(\because \lambda_2 - \lambda_1 \neq 0)$

From (i), $c_2 x_2 = 0 \Rightarrow c_2 = 0$ $(\because x_2 \neq 0)$

Exercise: If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of with eigenvectors $x_1, x_2, \ldots, x_k$, then $\{x_1, x_2, \ldots, x_k\}$ is lin. indep.