Theorem: Convergent sequences are bounded.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and let $L = \lim_{n\to\infty} a_n$.

To show: $\exists M$ s.t. $|a_n| \le M \;\forall n$.

Since $\lim_{n\to\infty} a_n = L$, $\exists N \in \mathbb{N}$ (corresponding to $\varepsilon = 1$) s.t. for all $n > N$,

$$|a_n - L| < 1 \Rightarrow |a_n| < |L| + 1$$

Take $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |L|+1\}$

Then $|a_n| \le M \quad \forall n \in \mathbb{N}$.

Supremum and infimum:

For any subset $A \subseteq \mathbb{R}$, a number M is an upper bound of A if

$$a \le M \quad \forall a \in A.$$

Similarly, a number l is a lower bound of A if $l \le a \quad \forall a \in A$.

The supremum of A is the least upper bound of A, denoted by $\sup(A)$

The infimum of A is the greatest lower bound of A, denoted by $\inf(A)$.

e.g. $A = (0,1)$

$\sup(A) = 1$; $\inf(A) = 0$

This example shows that supremum and infimum of a set need not belong to that set.

$B = [0,1]$

$\sup(B) = 1$; $\inf(B) = 0$

If A is not bound above, then $\sup(A) = \infty$

If A is not bounded below, $\inf(A) = -\infty$.

$\inf(A) \le a \le \sup(A) \quad \forall a \in A.$

Theorem: Every bounded monotone sequence is convergent.

- For a seq. $\{a_n\}$ which is nondecreasing and bounded above, $\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \{a_n\}$
- For a seq. $\{a_n\}$ which is nonincreasing and bounded below, $\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \{a_n\}$.

<u>Example</u>:

① Let $x_1 = \sqrt{2}$; $x_{n+1} = \sqrt{2 + x_n}$ $\forall n \geq 1$

Does the seq. $\{x_n\}$ converge?

Is it does, find $\lim\limits_{n\to\infty} x_n$.

<u>Soln</u>: $x_n > 0$ $\forall n$.

Since $x_1 = \sqrt{2} < 2$, and

$x_n < 2 \Rightarrow x_{n+1} = \sqrt{2 + x_n} < \sqrt{2+2} = 2$

$\therefore$ By induction, $x_n < 2$ $\forall n$.

$\therefore$ $\{x_n\}$ is bounded.

<u>Claim</u>: $\{x_n\}$ is an increasing seq.

i.e. $x_n < x_{n+1}$ $\forall n$.

<u>For $n = 1$</u>: $x_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = x_2$

Assume $x_k < x_{k+1}$ for some k

Then $x_k + 2 < x_{k+1} + 2$

$\Rightarrow \sqrt{x_k + 2} < \sqrt{x_{k+1} + 2}$

$\Rightarrow x_{k+1} < x_{k+2}$.

By induction $x_n < x_{n+1}$ $\forall n \in \mathbb{N}$.

$\therefore$ By the prev. thm., $\lim_{n\to\infty} x_n$ exists.

Let $L = \lim_{n\to\infty} x_n$

Then since $x_{n+1} = \sqrt{2+x_n}$, we get

$$L = \sqrt{2+L} \Rightarrow L^2 = 2+L$$

$$\Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow (L-2)(L+1) = 0$$

$$\Rightarrow L = 2 \quad (\because L = -1 \text{ is not possible as } x_n > 0 \ \forall n)$$

Q. $x_1 = \sqrt{2}$, $x_2 = \sqrt{2+\sqrt{2}}$,

$x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, ..., $x_n = \sqrt{2+\sqrt{2+\sqrt{\cdots+2}}}$