AMTL 100 (CALCULUS)
Mock Quiz Solution

1. Find the following limits (with justification).

   (a) $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$ [2]

   (b) $\lim_{x\to \infty} \frac{2x^2+5}{3x^2-1}$ [2]

   (c) $\lim_{x\to \infty} \frac{\sin(x^2+1)}{x^2}$ [2]

   Solution: (a) For any $x \neq \pm 1$,

$$\frac{\sin(x^2-1)}{x-1} = \left(\frac{\sin(x^2-1)}{x^2-1}\right)(x+1).$$

As $x \to 1$, $(x^2-1) \to 0$, so using $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, we get

$$\lim_{x\to 1} \frac{\sin(x^2-1)}{x^2-1} = 1.$$

Also, $\lim_{x\to 1}(x+1) = 2$. Therefore, by the product rule of limits,

$$\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = 2.$$

(b)

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2-1} = \lim_{x\to\infty} \frac{2+5/x^2}{3-1/x^2} = \frac{2+0}{3-0} = \frac{2}{3}.$$

(c)

$$\frac{-1}{x^2} \leq \frac{\sin(x^2+1)}{x^2} \leq \frac{1}{x^2}.$$

Using the Sandwich Theorem, we get

$$\lim_{x\to\infty} \frac{\sin(x^2+1)}{x^2} = 0.$$

2. Use the formal definition of limit to prove that the function $f(x) = x^2 + 1$ is continuous at $c = 1$. [4]

   Solution: Since $f(1) = 2$, we need to prove that $\lim_{x \to 1} f(x) = 2$.
   Let $\epsilon > 0$ we given. We need to find $\delta > 0$ such $|f(x) - 2| < \epsilon$ whenever $|x - 1| < \delta$.
   Now, $|f(x) - 2| = |x^2 - 1| = |x - 1||x + 1| \leq |x - 1|(|x - 1| + 2)$.
   So, if $|x - 1| < \delta$, $|f(x) - 2| \leq \delta(\delta + 2)$. If we choose $\delta > 0$ satisfying $\delta(\delta + 2) = \epsilon$, then we are done.
   So, we need $\delta^2 + 2\delta - \epsilon = 0$, which is equivalent to $\delta = -1 \pm \sqrt{1 + \epsilon}$.
   Since we need $\delta > 0$, we take $\delta = -1 + \sqrt{1 + \epsilon}$.

   Another way: If we choose $\delta \leq 1$, then $|x-1| < \delta \leq 1 \implies |x+1| \leq |x-1|+2 < 3$.
   Thus, $|f(x) - 2| < 3|x - 1| < 3\delta$. So, we can take $\delta = \min\{1, \epsilon/3\}$.