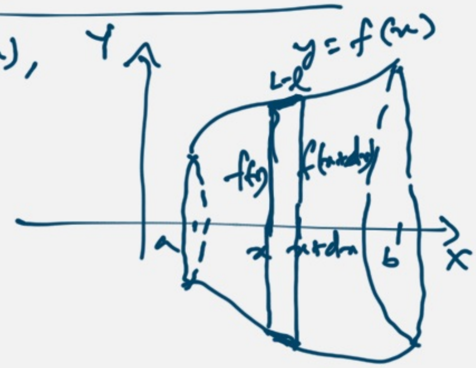
Surface area of solid of revolution

Suppose the curve $y=f(x)$, $a \le x \le b$, is rotated about the x-axis.



Surface area $= 2\pi\left(\frac{r_1+r_2}{2}\right) l$ $\longrightarrow$

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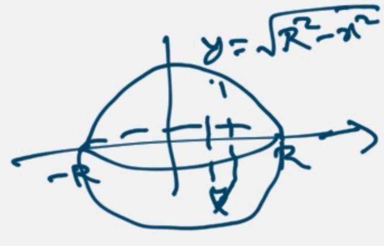
Surface area,

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= 2\pi \int_a^b y\,ds$$

Example: Find the surface area of sphere of radius R.



$$f(x) = \sqrt{R^2-x^2}$$

$$f'(x) = \frac{1}{2\sqrt{R^2-x^2}}\cdot(-2x)$$

$$= \frac{-x}{\sqrt{R^2-x^2}}$$

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$$1+(f'(x))^2 = 1+\frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}$$

$$\therefore \quad S = 2\pi \int_{-R}^{R} \cancel{\sqrt{R^2-x^2}} \cdot \frac{R}{\cancel{\sqrt{R^2-x^2}}}\,dx$$

$$= 2\pi R \cdot \int_{-R}^{R} dx = 2\pi R\,(2R) = 4\pi R^2$$

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Problem: Find the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis.

Soln: $y = \sqrt[3]{x} \Rightarrow x = y^3$, $1 \le y \le 2$

$f(y) = y^3$

$$S = 2\pi \int_1^2 y^3 \sqrt{1+(3y^2)^2}\,dy$$

$$= 2\pi \int_1^2 y^3 \sqrt{1+9y^4}\,dy$$

Put $u = 1+9y^4$

$du = 36y^3\,dy$

$$= 2\pi \int_{10}^{145} \frac{1}{36}\sqrt{u}\,du = \frac{\pi}{18}\,\frac{2}{3}\,u^{3/2}\Big|_{10}^{145}$$

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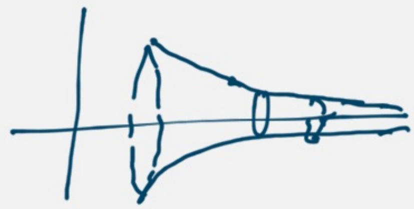
Problem: Find the surface area and volume of the solid generated by infinite curve

$$y = \frac{1}{x}, \quad 1 \le x < \infty.$$



Soln:

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$> 2\pi \int_1^\infty \frac{1}{x}\,dx = 2\pi \ln x \Big|_1^\infty = \infty$$

$\therefore$ It has infinite surface area.

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Volume, $V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$

$$= \pi \left(\frac{-1}{x}\right)\Big|_1^\infty = \pi$$

This is an example of a solid with finite volume but infinite surface area.

This is sometime described as the "painter's paradox": we can fill a can of infinite surface area with finite amount of paint.

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Improper integrals

Let $f(x)$ be a function which is bounded on $[a, b]$ for every $b > a$. Also, suppose that f is Riemann integrable on $[a, b]$ for every $b > a$.

Then $\int_a^b f(x)\,dx$ exists for every $b > a$.

We define the improper integral

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

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If the above limit exists, we say that the improper integral $\int_a^\infty f(x)\,dx$ converges.

If the limit does not exist, we say $\int_a^\infty f(x)\,dx$ diverges.

Example: ① $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1$$

$\therefore \int_1^\infty \frac{1}{x^2}\,dx$ converges.

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(2) $\int_1^\infty \frac{1}{x}\,dx$ diverges because

$\int_1^b \frac{1}{x}\,dx = \ln b \to \infty$ as $b \to \infty$.

(3) Show that $\int_1^\infty \frac{1}{x^p}\,dx$ converges

if and only if $p > 1$.

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(4) $\int_1^\infty \frac{1}{1+x^2}\,dx$

$\int_1^b \frac{1}{1+x^2}\,dx = \tan^{-1} b - \frac{\pi}{4}$

$\to \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ as $b \to \infty$

$\therefore \int_1^\infty \frac{1}{1+x^2}\,dx$ converges

& $\int_1^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

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(5) Does $\int_1^\infty \frac{1}{1+x^3}\,dx$ converge?

$0 \le \frac{1}{1+x^3} < \frac{1}{x^3}$

and $\int_1^\infty \frac{1}{x^3}\,dx$ converges.

$\therefore \int_1^\infty \frac{1}{1+x^3}\,dx$ converges.

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Comparison Test:

If $0 \le f(x) \le g(x)$ for $a \le x < \infty$,

then $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

Also, if $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

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