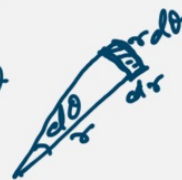## Double integral in polar coordinates

In polar coords. $dA = r\,dr\,d\theta$

$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\,r\,dr\,d\theta$$

Example: Area of a circle of radius R
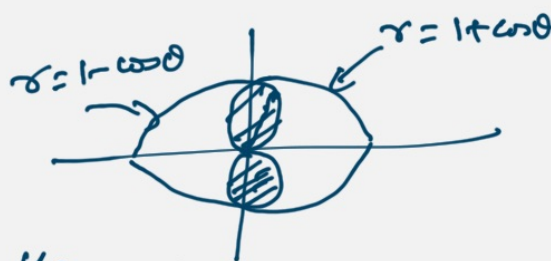
$$\iint dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r\,dr\,d\theta$$

$$= \int_{0}^{2\pi} \frac{R^2}{2}\,d\theta = \frac{R^2}{2}\times 2\pi = \pi R^2.$$

Example: Find the area common to the cardioids: $r = 1+\cos\theta$ and $r = 1-\cos\theta$.

$r = 1-\cos\theta$ $r = 1+\cos\theta$

Area = 4 × area in the first quadrant

$$= 4 \int_{\theta=0}^{\pi/2}\int_{r=0}^{1-\cos\theta} r\,dr\,d\theta = 4\int_{0}^{\pi/2}\frac{(1-\cos\theta)^2}{2}\,d\theta$$

$$= 2 \int_{0}^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

Example: $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} \cdot r\, dr\, d\theta = 2\pi \times \left(\frac{-e^{-r^2}}{2}\right)\Big|_0^{\infty}$$

$$= \pi$$

$$\Rightarrow I = \sqrt{\pi} \cdot$$

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Triple integrals:

$$dV = dx\, dy\, dz$$

$$\iiint f(x,y,z)\, dV$$

Example: Find the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Soln: $\iiint dx\, dy\, dz = \iint \left(\int_{x^2+3y^2}^{8-x^2-y^2} dz \right) dx\, dy$

$$= \iint (8 - 2x^2 - 4y^2)\, dx\, dy$$

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Ω is the projection of the intersecting surface on the xy-plane, ie.,
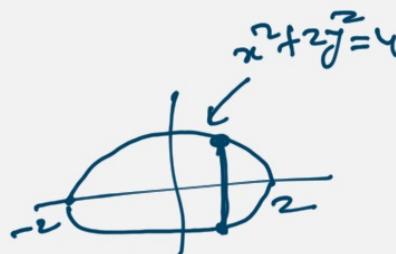
$$8 - x^2 - y^2 = x^2 + 3y^2$$

$\Rightarrow$ $$x^2 + 2y^2 = 4 .$$

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2) \, dy \, dx$$

$x^2 + 2y^2 = 4$

$$= 2 \int_{-2}^{2} \left(8y - 2x^2 y - \frac{4}{3}y^3 \right) \Big|_{0}^{\sqrt{\frac{4-x^2}{2}}} dx$$

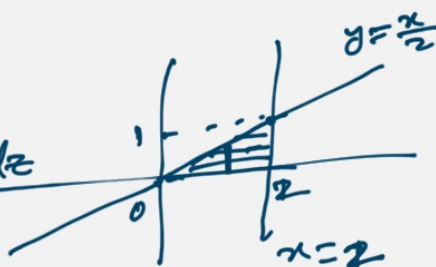$$= 8\sqrt{2}\pi \qquad (\text{Verify!})$$

Example: Evaluate $I = \int_{z=0}^{4} \int_{y=0}^{1} \int_{x=2y}^{2} \frac{z \cos(x^2)}{\sqrt{z}} \, dx \, dy \, dz$

Interchanging $dx \, dy$ with $dy \, dx$,

$$I = \int_{z=0}^{4} \int_{x=0}^{2} \int_{y=0}^{x/2} \frac{z \cos(x^2)}{\sqrt{z}} \, dy \, dx \, dz$$

$y = \frac{x}{2}$

$x = 2$

$$= \int_{z=0}^{4} \int_{x=0}^{2} \frac{z \cos(x^2)}{\sqrt{z}} \cdot \frac{x}{2} \, dx \, dz$$

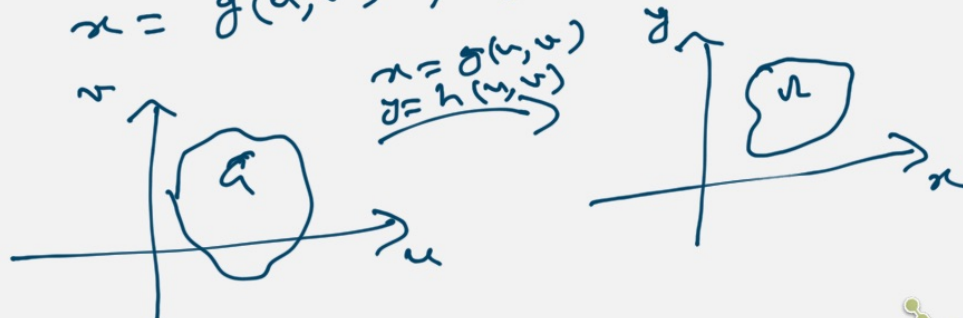$$= \int_{z=0}^{4} \frac{1}{2\sqrt{z}} \sin(x^2) \Big|_{0}^{2} \, dz = \sin(4) \int_{z=0}^{4} \frac{1}{2\sqrt{z}} \, dz$$

$$= 2\sin(4)$$

Change of variable formula:

Suppose a domain G in the uv-plane
is transformed onto a domain Ω
in the xy-plane by transformations:
$$x = g(u,v) , \quad y = h(u,v)$$

$$\iint_{\Omega} f(x,y)\, dx\, dy = \iint f(g(u,v), h(u,v))$$
$$|J(u,v)|\, du\, dv ,$$

where $J(u,v)$ is the Jacobian of
the transformation given by
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

determinant.

Example: Evaluate the integral

$$I = \int_{y=0}^{4} \int_{x=y/2}^{1+y/2} \left(\frac{2x-y}{2}\right) dx\, dy \qquad x = y/2$$

Let $u = \dfrac{2x-y}{2}$

$v = y/2$

i.e. $y = 2v, \quad x = u+v$

$v = 2$

$u = 0 \qquad v = 0$

$(2,4) \qquad (3,4) \qquad y = 4$

$(0,0) \qquad (1,0) \qquad y = 0$

$x = 1 + y/2$

$$\therefore \quad I = \int_{v=0}^{2} \int_{u=0}^{1} u\, |J(u,v)|\, du\, dv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$
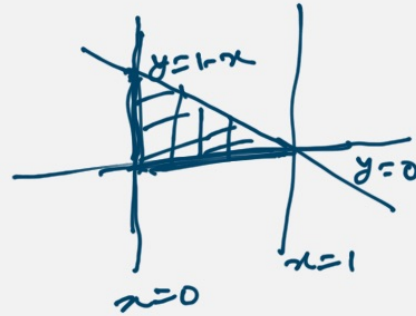
$$\therefore \quad I = \int_{v=0}^{2} \int_{u=0}^{1} 2u\, du\, dv = \int_{v=0}^{2} 1\, dv = 2.$$

Example: $I = \int_{x=0}^{1} \int_{y=0}^{1-x} \sqrt{x+y} \, (y-2x)^2 \, dy \, dx$
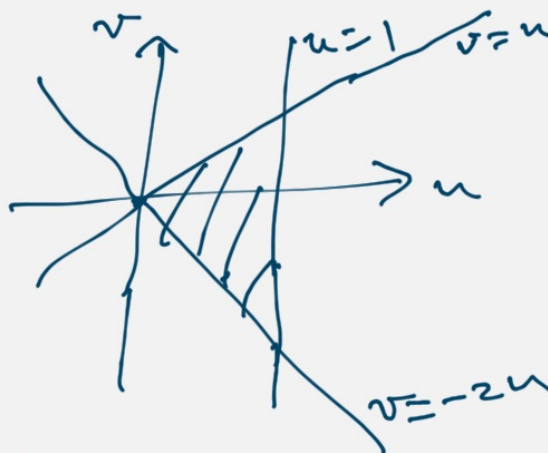
Let $u = x+y$
$\quad v = y - 2x$

i.e. $\quad x = \dfrac{u-v}{3}$

$\qquad y = \dfrac{2u+v}{3}$

$J(u,v) = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = \dfrac{1}{3}$.

When $y=0$, $\boxed{v = -2v}$

When $x=0$, $\boxed{v = u}$

When $y = 1-x$,
$\dfrac{2u+v}{3} = 1 - \dfrac{u-v}{3}$

$\boxed{u = 1}$.

$I = \int_{u=0}^{1} \int_{v=-2u}^{u} \sqrt{u} \, v^2 \cdot \dfrac{1}{3} \, dv \, du$

$= \int_{u=0}^{1} \dfrac{\sqrt{u}}{3} \cdot \left(\dfrac{v^3}{3}\right) \Big|_{v=-2u}^{u} \, du = \cdots$