Limits and continuity of a real-valued function of two variables

Defn: If a real-valued function $f(x,y)$ is defined in a neighborhood of (a,b), except possibly at (a,b), then we say

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \ (\in \mathbb{R}) \text{ if}$$

given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$$

$$\Rightarrow |f(x,y) - L| < \varepsilon.$$

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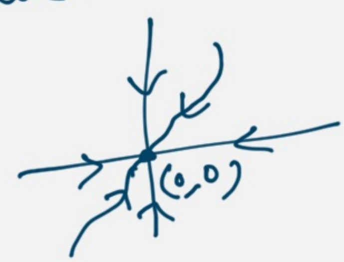
Examples:

① $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$, $(x,y) \neq (0,0)$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If it exists what is the value.

First, we try to find the limit (if it exists) along x-axis or y-axis.



Along x-axis: $y=0$

$f(x,0) = \dfrac{0}{x^2} = 0$ for all $x \neq 0$

$\therefore \lim_{x\to 0} f(x,0) = 0.$

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So, if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then it must be zero.

$$|f(x,y) - 0| = \left|\frac{4xy^2}{x^2+y^2}\right|$$

$$= \frac{4|x|y^2}{x^2+y^2}$$

$$\leq 4|x|$$

$$\leq 4\sqrt{x^2+y^2} < 4\delta$$

if $\sqrt{x^2+y^2} < \delta$, then

If we choose $4\delta = \varepsilon$ i.e. $\delta = \varepsilon/4$,

$|f(x,y) - 0| < \varepsilon$ if $0 < \sqrt{x^2+y^2} < \delta$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(2) $f(x,y) = \dfrac{xy}{x^2+y^2}$, $(x,y) \neq (0,0)$

Along x-axis:

$f(x,0) = 0 \Rightarrow \lim_{x\to 0} f(x,0) = 0$

Along the line $y = x$:

$f(x,x) = \dfrac{x^2}{2x^2} = \dfrac{1}{2} \quad \forall x \neq 0$

$\lim_{x\to 0} f(x,x) = \dfrac{1}{2}$

Since, the limit, if it exists, has to be unique, along every direction, we conclude that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

If $\lim_{x\to 0} f(x, mx)$ depends on m, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

③ $f(x,y) = \frac{2x^2 y}{x^4+y^2}$, $(x,y) \neq (0,0)$.

Along the line $y = mx$:

$$f(x, mx) = \frac{2x^2 \cdot mx}{x^4 + m^2x^2}$$

$$= \frac{2mx}{x^2+m^2}$$

$$\to 0 \text{ if } x \to 0.$$

Along $y = x^2$:

$$f(x, x^2) = \frac{2x^4}{x^4+x^4} = 1 \quad \forall x \neq 0$$

$$\neq 0$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(4) $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & \text{if } xy \neq 0 \\ 0, & xy = 0. \end{cases}$

$$\begin{aligned} |f(x,y)| &= \left| x\sin\left(\tfrac{1}{y}\right) + y\sin\left(\tfrac{1}{x}\right)\right| \\ &\leq |x|\left|\sin\left(\tfrac{1}{y}\right)\right| + |y|\left|\sin\left(\tfrac{1}{x}\right)\right| \\ &\leq |x| + |y| \\ &\leq 2\sqrt{x^2+y^2} \\ &< 2\delta = \varepsilon \end{aligned}$$

(By AM-G.M ineq. $\frac{|x|+|y|}{2} \leq \sqrt{|x||y|} \leq \sqrt{x^2+y^2}$)

If $\delta = \frac{\varepsilon}{2}$, then $|f(x,y) - 0| < \varepsilon$ if $0 < \sqrt{x^2+y^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Sometimes using polar coordinates helps in finding the limit.

e.g. $f(x,y) = \dfrac{x^3}{x^2+y^2}, (x,y) \neq (0,0)$

$x = r\cos\theta,\ y = r\sin\theta$

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta$$

$\therefore |f(x,y)| \leq r \to 0$ as $(x,y) \to (0,0)$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Continuity:

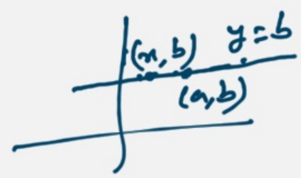
Defn: We say $f(x,y)$ is continuous at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

e.g.
$$f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$
is continuous at $(0,0)$.

Partial Derivatives:

Defn: The partial derivative of $f(x,y)$ with respect to x at (a,b) is defined as

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$



Similarly,

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Example :

Let $f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & , (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0). \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$$

Note that $f(x,y)$ is discontinuous at $(0,0)$.