Taylor's Theorem

The Taylor's polynomial of order n of a function $f(x)$ about $x=a$ is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

$f(x) = P_n(x) + R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

for some c between a and x.

Error estimate:

Suppose $|f^{(n+1)}(x)| \le M$ in the interval $(a-r, a+r)$.

Then for any $x \in (a-r, a+r)$,

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

$$\le \frac{M}{(n+1)!} r^{n+1}$$

Example: $f(x) = e^x$, $a = 0$.

$$f'(x) = e^x, \quad f''(x) = e^x, \cdots$$

$$f(0) = 1 = f'(0) = f''(0) = \cdots$$

$$\therefore P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$ for some c between 0 & x.

$$= \frac{e^c}{(n+1)!}x^{n+1}$$

If $x \in (-1, 1)$, then $c \in (-1, 1)$

$$\Rightarrow e^c < e^1 = e$$

$$\therefore |R_n(x)| < \frac{e}{(n+1)!}|x|^{n+1} < \frac{e}{(n+1)!}$$

Suppose we want to approximate $f(x) = e^x$ by $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ the error for $x \in (-1, 1)$ so that is no more than 10^{-5}. What should be n?

We choose n such that

$$\frac{e}{(n+1)!} < 10^{-5} \quad \text{i.e.} \quad (n+1)! > e \times 10^{5} \simeq 2.71 \times 10^{5}$$

Since $9! > 3 \times 10^{5}$, so if $n \geq 8$

then the $|\text{error}| < 10^{-5}$.

Example: $f(x) = \sin x$, $a = 0$

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
&\vdots
\end{aligned}$$

Taylor's polynomial of order $(2n+1)$

$$0 + 1.x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

What is the Taylor's polynomial of order 4 for the function $f(x) = \sin x$ about $a = 0$?

$$\sin x \simeq 0 + 1.x + \frac{0.}{2!} x^2 + \frac{(-1)}{3!} x^3 + \frac{0}{4!} x^4$$

$$= x - \frac{x^3}{3!}$$

Example: For what values of x can be replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 3×10^{-4} ?

Soln: Take $n = 4$.

Then $P_4(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$.

$$|\text{Error}| = |R_4(x)| = \left| \frac{f^{(5)}(c)}{5!} x^5 \right|$$

$$= \frac{|\cos c|}{5!} |x|^5 \le \frac{1}{5!} |x|^5$$

We want $|\text{error}| < 3 \times 10^{-4}$.

So we choose x such that

$$\frac{1}{5!}|x|^5 < 3 \times 10^{-4}$$

i.e. $|x|^5 < 360 \times 10^{-4}$

i.e. $|x| < \sqrt[5]{36 \times 10^{-3}} \simeq 0.514$

Remark: You can take $n=3$ also.

In that case,

$$|R_3(x)| = \left|\frac{f^{(4)}(c)}{4!} x^4\right| = \frac{|\sin c|}{24}|x|^4 \le \frac{|x|^4}{24}$$

$$\frac{x^4}{24} < 3 \times 10^{-4} \quad \text{i.e.} \quad |x| < \sqrt[4]{72 \times 10^{-4}} \approx 0.292$$

Example

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\vdots$

$$\cos x \simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{2n!}$$

Example

$$f(x) = \ln(1+x) \quad ; \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad ; \quad f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \quad ; \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad ; \quad f'''(0) = 2$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n} \quad ; \quad n = 1, 2, 3, \ldots$$

$$\ln(1+x) \approx 0 + 1 \cdot x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots + (-1)^{n+1}\frac{(n-1)!}{n!}x^n + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots (-1)^{n+1}\frac{x^n}{n} + \cdots$$

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Example

$$f(x) = \frac{1}{1-x} \quad ; \quad f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \quad ; \quad f'(0) = 1$$

$$f''(x) = \; \ldots$$

$$\frac{1}{1-x} \simeq 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$