Let $T : V \to W$ be a linear transf. and B_1, B_2 be two ordered bases for V & B_1', B_2' be two ordered bases for W.

Then we have two matrices

$$[T]_{B_1}^{B_1'} \quad \text{and} \quad [T]_{B_2}^{B_2'}.$$

Q. How are they related?

We know that for any $v \in V$,

$$[T(v)]_{B_1'} = [T]_{B_1}^{B_1'} [v]_{B_1}$$

and

$$[T(v)]_{B_2'} = [T]_{B_2}^{B_2'} [v]_{B_2}.$$

Let P be the change of bases matrix from B_1 to B_2. and let Q be the change of bases matrix from B_1' to B_2'.

Then $[v]_{B_2} = P[v]_{B_1} \quad \forall v \in V$

and $[w]_{B_2'} = Q[w]_{B_1'} \quad \forall w \in W$.

Taking $w = T(v)$, we get

$$[T(v)]_{B_2'} = Q[T(v)]_{B_1'}$$

$$\Rightarrow [T]_{B_2}^{B_2'} [v]_{B_2} = Q [T]_{B_1}^{B_1'} [v]_{B_1}$$

$$\Rightarrow \quad [T]_{B_2}^{B_2'} P [v]_{B_1} = Q [T]_{B_1}^{B_1'} [v]_{B_1} \quad \forall v \in V$$

$$\Rightarrow \quad [T]_{B_2}^{B_2'} P = Q [T]_{B_1}^{B_1'}$$

$$\Rightarrow \quad \boxed{[T]_{B_1}^{B_1'} = Q^{-1} [T]_{B_2}^{B_2'} P}$$

$$\text{or} \quad \boxed{[T]_{B_2}^{B_2'} = Q [T]_{B_1}^{B_1'} P^{-1}}$$

Defn: A <u>linear operator</u> is a linear transformation from a vector space V to itself.

Let $T : V \to V$ be a linear operator and let B be an ordered basis for V. Then $[T]_B^B$ will simply be denoted by $[T]_B$.

Now suppose B_1 & B_2 be two ordered bases for V and P is the change of basis matrix from B_1 to B_2.

Then $Q = P$.

$\therefore$ $\boxed{[T]_{B_1} = P^{-1} [T]_{B_2} P}$

Defn: (Similar matrices)
Two square matrices A and B of the same size say $n \times n$ are said to be similar if there exists an invertible matrix P s.t. $\boxed{B = P^{-1} A P}$

$\therefore$ $[T]_{B_1}$ & $[T]_{B_2}$ are similar matrices.

Note that similar matrices have the same determinant and same trace.

Pf: Let $B = P^{-1} A P$

Then
$$\det(B) = \det(P^{-1} A P) = \det(A P P^{-1}) = \det(A)$$

Also,
$$\operatorname{trace}(B) = \operatorname{trace}(P^{-1}(AP)) = \operatorname{trace}((AP)P^{-1}) = \operatorname{trace}(A)$$

$$\left(\begin{array}{c} \operatorname{trace}(CD) = \operatorname{trace}(DC) \\ \det(CD) = \det(DC) \end{array} \right)$$