## Root Test:

Let $\lim_{n\to\infty} |a_n|^{1/n} = \rho$.

(i) If $\rho < 1$, then $\sum_{n=1}^{\infty} |a_n|$ converges

(ii) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

(iii) If $\rho = 1$, the test fails.

**Example:** $a_n = \begin{cases} n/2^n & \text{if } n \text{ is odd} \\ 1/2^n & \text{if } n \text{ is even} \end{cases}$

$$|a_n|^{1/n} = \begin{cases} \frac{n^{1/n}}{2} & \text{if } n \text{ is odd} \\ 1/2 & \text{if } n \text{ is even} \end{cases}$$

Since $\lim_{n\to\infty} n^{1/n} = 1$, $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1$

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$\therefore$ The series $\sum_{n=1}^{\infty} a_n$ converges.

Is ratio test applicable?

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1/2^{n+1}}{n/2^n} & \text{if } n \text{ is odd} \\ \dfrac{(n+1)/2^{n+1}}{1/2^n} & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2n} & \text{if } n \text{ is odd} \\ \frac{n+1}{2} & \text{if } n \text{ is even} \end{cases}$$

$\to$ 0 for odd subseq.

$\to$ $\infty$ for even subseq.

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$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

So, ratio test is not applicable directly.

Absolute convergence

We say a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

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Theorem: Absolute convergence implies convergence.

Proof: Since $-|a_n| \leq a_n \leq |a_n|$,

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

If $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} 2|a_n|$ converges.

$\therefore$ By the comparison test, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges.

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Since $a_n = (a_n + |a_n|) - |a_n|$,

$\sum_{n=1}^{\infty} a_n$ converges.

Remark: The converse is not true. A series may be convergent but not absolutely convergent. We'll see such examples later.

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Example: Consider the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$.

Since $0 \leq \left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$, by the comparison test $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent (as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent).

$\therefore \sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is convergent.

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**Alternating series:**

A series $\sum_{n=1}^{\infty} a_n$ is called an alternating series if the terms are alternating between positive and negative signs.

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n \geq 0 \ \forall n$ is an alternating series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

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**Theorem:** Let $\boxed{a_n \geq 0}$, $\{a_n\}$ is $\boxed{\text{non-increasing}}$ and $\boxed{\lim_{n\to\infty} a_n = 0}$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

**Example:** ① $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Here $a_n = \frac{1}{n} > 0$, $\{a_n\}$ is decr.

and $\lim_{n\to\infty} a_n = 0$

$\therefore$ By the above theorem, it converges.

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② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges

However, the above two series are not absolutely convergent as $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge.

Defn: If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say the series $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u>.

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Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{|\sin(n)|}{n}$

This is an alternating series.

But $a_n = \frac{|\sin(n)|}{n}$ is not a decreasing sequence.

$\therefore$ We cannot apply the alternating series test (the previous theorem) for convergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{|\sin(n)|}{n}$

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