Recall of matrices :

A matrix is a rectangular array of (real) numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

has m rows and n columns.

A is of size $m \times n$.

$a_{ij} \in \mathbb{R}$ for $1 \le i \le m$, $1 \le j \le n$.

$M_{m\times n}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with entries from $\mathbb{R}$.

Operations on matrices

(1) Scalar multiplication :

For $A \in M_{m\times n}(\mathbb{R})$ and $\alpha \in \mathbb{R}$, αA is defined by multiplication each entry of A by α, i.e.,

$$(\alpha A)_{ij} = \alpha A_{ij}$$

e.g. $3 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix}$.

② **Addition:**

For two matrices A and B of the same size $A+B$ is defined as

$$(A+B)_{ij} = A_{ij} + B_{ij} .$$

③ **Multiplication:**

For $A \in M_{m\times n}(\mathbb{R})$ & $B \in M_{n\times k}(\mathbb{R})$, AB is defined as follows:

$$(AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}_{2\times 3}$$

$$B = \begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \end{pmatrix}_{3\times 4}$$

$$AB = \begin{pmatrix} 8 & 4 & 8 & 0 \\ 5 & 1 & 7 & -3 \end{pmatrix}_{2\times 4}$$

BA is not defined.

- Square matrix:

No. of rows = no. of columns.

$m \times m$

Identity matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n \times n}$$

2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3×3 " " $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- Determinant of a square matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(A) = ad - bc.$$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$\det A = a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Invertibility of a square matrix :

A square matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be invertible if there exists $B \in M_{n\times n}(\mathbb{R})$ such that

$$AB = I = BA$$

In this case B is called the inverse of A and denoted by A^{-1}.

Property of determinant:

$$\det(AB) = (\det A)(\det B)$$

Fact : $A \in M_{n\times n}(\mathbb{R})$ is invertible if and only if $\det(A) \neq 0$.

(If A is invertible, $\exists\, B$ s.t.

$$AB = I$$

$$\det(AB) = \det(I) = 1$$

$$\Rightarrow (\det A)(\det B) = 1$$

$$\Rightarrow \det A \neq 0 .)$$

$\Rightarrow$

System of linear equations

Example: 2 linear eqns in 2 unknowns

$$\left.\begin{aligned} 2x + y &= 5 \\ x - y &= 1 \end{aligned}\right\} \text{ 2 eqns in 2 unknowns } x \,\&\, y.$$

This can be written in matrix form as

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

or

$$x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$