Taylor's Theorem

Suppose f is a function which is continuously differentiable n-times and the $(n+1)^{th}$ derivative $f^{(n+1)}$ exists.

Then $f(x) = P_n(x) + R_n(x)$,

where
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

and
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$
where c is between a and x.

$P_n(x)$ is a polynomial of degree at most n.

$R_n(x)$ is called the remainder term.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the Taylor's polynomial of degree $\leq n$.

$R_n(x) = f(x) - P_n(x)$ gives the error when $f(x)$ is approximated by $P_n(x)$.

For $n=1$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$R_1(x) = \frac{f''(c)}{2!}(x-a)^2,$$

for some c between a & x.

Proof of Taylor's thm (for $n=1$)

$$f(x) = P_1(x) + R_1(x)$$

Let $\varphi(y) = a_0 + a_1(y-a) + a_2(y-a)^2$

Then $\varphi(a) = a_0$, $\varphi'(a) = a_1$

$$\varphi''(a) = 2a_2$$

We want: $\varphi(a) = f(a)$; $\varphi'(a) = f'(a)$

i.e. $a_0 = f(a)$; $a_1 = f'(a)$

Also, we want $f(x) = \varphi(x)$

$$= f(a) + f'(a)(x-a) + a_2(x-a)^2,$$

Let $F(y) = \varphi(y) - f(y)$

Then $F(a) = \varphi(a) - f(a) = 0$

$$F'(a) = \varphi'(a) - f'(a) = 0$$

$$F(x) = \varphi(x) - f(x) = 0$$

Applying Rolle's thm., $\exists\, c_1 \in (a, x)$ such that $F'(c_1) = 0$

Now, $F'(a) = 0 = F'(c_1)$

Applying Rolle's thm. again,

$\exists\, c \in (a, c_1)$ such that

$$F''(c) = 0$$

$$\Rightarrow \varphi''(c) - f''(c) = 0$$

$$\Rightarrow 2a_2 - f''(c) = 0 \Rightarrow a_2 = \frac{f''(c)}{2}$$

$$\therefore f(x) = \varphi(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

for some $c \in (a, x)$

Theorem (Generalization of Rolle's thm.)

Suppose F is n-times differentiable.

and $F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0$.

Also, assume that $F(b) = 0$.

Then $\exists\, c \in (a, b)$ s.t. $F^{(n)}(c) = 0$

Proof: $F(a) = 0 = F(b)$

By Rolle's thm., $\exists\, c_1$ s.t. $F'(c_1) = 0$

Now, $F'(a) = 0 = F'(c_1)$

$\Rightarrow \exists\, c_2 \in (a, c_1)$ s.t. $F''(c_2) = 0$

$\therefore \exists\, c_n \in (a, c_{n-1})$ s.t. $F^{(n)}(c_n) = 0$

Example: $f(x) = e^x$; $a = 0$

$P_n(x) = ?$, $R_n(x) = ?$

$f(0) = e^0 = 1$, $f'(x) = e^x, f''(x) = e^x, \dots$

$f^{(k)}(0) = e^0 = 1$

$$P_n(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

$$= \frac{e^c}{(n+1)!} x^{n+1}$$, where c is some no. between 0 & x.

$$|R_n(x)| = \frac{e^c}{(n+1)!} |x|^{n+1}$$

Q: What is the maximum error when we approximate the function $f(x) = e^x$ by the Taylor's polynomial of degree 10 in the interval $[-1, 1]$.

Soln: $f(x) = P_{10}(x) + R_{10}(x)$

$$R_{10}(x) = \frac{e^c}{11!} x^{11}, \text{ where } c \in (-1, 1)$$

$$|R_{10}(x)| = \frac{e^c}{11!} |x|^{11} \le \frac{e}{11!}$$

for $x \in [-1, 1]$.