Lagrange multiplier method with two constraints

Max./min $f(x,y,z)$

subject to $g(x,y,z)=0$ & $h(x,y,z)=0$. $\Big\} \to$ 2 eqns.

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h \text{ — 3 eqns}$$

5 unknowns x, y, z, λ, μ.

We solve these 5 eqns to find the critical points.

Example: Let C be the intersection of the surfaces $x^2+4y^2+4z^2=4$ and $x+y+z=0$. Using the Lagrange multiplier method, determine the points that are nearest and farthest from the origin.

Soln: We need to maximize/minimize

$$f(x,y,z) = x^2+y^2+z^2$$

subject to $g(x,y,z) = x^2+4y^2+4z^2-4=0$

$h(x,y,z) = x+y+z=0$.

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$\Rightarrow \quad (2x, 2y, 2z) = \lambda(2x, 8y, 8z) + \mu(1,1,1)$

$\Rightarrow \quad 2x = 2\lambda x + \mu \quad \text{—(i)}$

$\quad\quad 2y = 8\lambda y + \mu \quad \text{—(ii)}$

$\quad\quad 2z = 8\lambda z + \mu \quad \text{—(iii)}$

(ii) − (iii) $\Rightarrow \quad 2(y-z) = 8\lambda(y-z)$

$\Rightarrow$ either $y = z$ or $\lambda = \frac{1}{4}$

$\boxed{\text{If } \lambda = \frac{1}{4}}$, (i) $\Rightarrow \quad 2x = \frac{x}{2} + \mu$

$\Rightarrow \quad \frac{3x}{2} = \mu$

(ii) $\Rightarrow \quad 2y = 2y + \mu \Rightarrow \boxed{\mu = 0}$

$\therefore \frac{3x}{2} = 0 \Rightarrow \boxed{x = 0}$.

$x^2 + 4y^2 + 4z^2 = 4 \quad \text{—(iv)}$

$x + y + z = 0 \quad \text{—(v)}$

$x = 0 \Rightarrow \quad 4y^2 + 4z^2 = 4$

$\quad\quad y + z = 0$

$\Rightarrow \quad z = -y \; ; \quad y^2 + (-y)^2 = 1$

$\Rightarrow y^2 = \frac{1}{2}$

$\Rightarrow y = \pm\frac{1}{\sqrt{2}}$

$\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) ; \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

If $\boxed{y = z}$: (v) $\Rightarrow \quad x = -2y$

(iv) $\Rightarrow \quad x^2 + 8y^2 = 4$

$\Rightarrow (-2y)^2 + 8y^2 = 4$

$\Rightarrow y^2 = 1/3$

$\therefore \quad y = \pm \frac{1}{\sqrt{3}}, \quad z = y, \quad x = -2y$

Critical pts in this case: $\left(\frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; $\left(\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$f\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 = f\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$f\left(\frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = f\left(\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = 2$

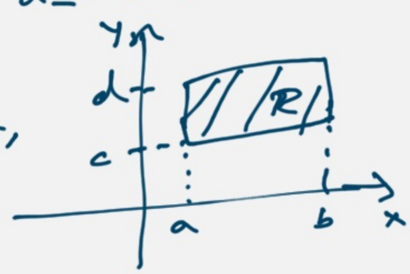
Nearest pts are $\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ & $\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ at distance $= 1$

Farthest pts. are $\left(\frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ & $\left(\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ at a distance $= \sqrt{2}$.

Double integrals

Let R be the rectangle $[a,b] \times [c,d]$

i.e. $R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\}$

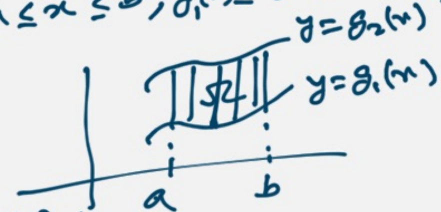


If $f(x,y)$ is a bounded real valued fn. defined on R, we can define

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\,dx \right) dy$$

For a general bounded region Ω, we say Ω is "y-regular" if

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

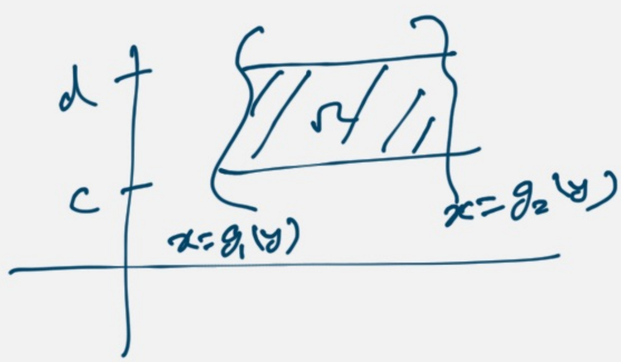


$$\iint_\Omega f(x,y)\,dx\,dy = \int_{x=a}^{b} \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

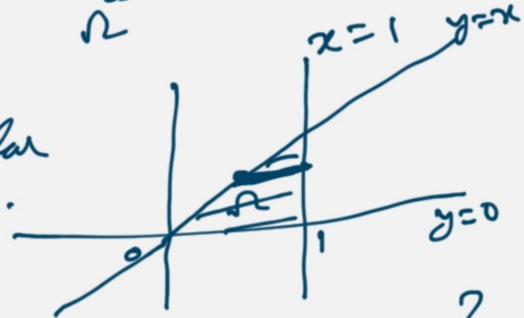
Similarly, Ω is "x-regular" if

$$\Omega = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ g_1(y) \le x \le g_2(y)\}$$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx \right) dy$$



Example: Let Ω be the triangular region bounded by $y=0$, $x=1$ and $y=x$. Evaluate the integral $\iint_{\Omega} (x+y+xy)\,dx\,dy$.



Solution:

Ω is both x-regular as well as y-regular.

y-regular: $\Omega = \{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ 0\le y\le x\}$

x-regular: $\Omega = \{(x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ y\le x\le 1\}$

$$\iint_{\Omega} (x+y+xy)\,dx\,dy$$

$$= \int_0^1 \left(\int_0^x (x+y+xy)\,dy \right) dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} + x\frac{y^2}{2}\right)\Big|_{y=0}^{x} dx$$

$$= \int_0^1 \left(x^2 + \frac{x^2}{2} + \frac{x^3}{2} - 0\right) dx$$

$$= \left(\frac{3}{2}\frac{x^3}{3} + \frac{x^4}{8}\right)\Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$

Also, $\iint_{\Omega} (x+y+xy)\,dx\,dy$

$$= \int_0^1 \left(\int_y^1 (x+y+xy)\,dx \right) dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + xy + \frac{x^2}{2}y \right)\Big|_{x=y}^{1} dy$$

$$= \int_0^1 \left(\frac{1}{2} + y + \frac{y}{2} - \frac{y^2}{2} - y^2 - \frac{y^3}{2} \right) dy$$

$$= \frac{1}{2}y + \frac{3}{2}\frac{y^2}{2} - \frac{3}{2}\frac{y^3}{3} - \frac{y^4}{8} \Big|_0^1$$

$$= \cancel{\frac{1}{2}} + \frac{3}{4} - \cancel{\frac{1}{2}} - \frac{1}{8} = \frac{5}{8}.$$

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Sometimes changing the order of integration makes it easier to evaluate.

e.g. Evaluate $I = \iint_{\Omega} \frac{\sin x}{x}\,dx\,dy$

$$\Omega = \{ (x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ y \le x \le 1 \}$$

$I = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\,dx \right) dy \rightarrow$ We get stuck!

Changing the order of integration:

$$I = \int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy \right) dx = \int_0^1 \sin x\,dx = 1 - \cos 1$$

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Example:
Evaluate $I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$,
where $a>0$, $b>0$.

Soln: Note that
$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\,dy$$

$$\therefore\ I = \int_0^\infty \left(\int_a^b e^{-xy}\,dy \right) dx$$
$$= \int_a^b \left(\int_0^\infty e^{-xy}\,dx \right) dy$$

$$= \int_a^b \left(-\frac{e^{-xy}}{y} \Big|_{x=0}^{x=\infty} \right) dy$$
$$= \int_a^b \frac{1}{y}\,dy = \ln b - \ln a = \ln\left(\frac{b}{a}\right).$$