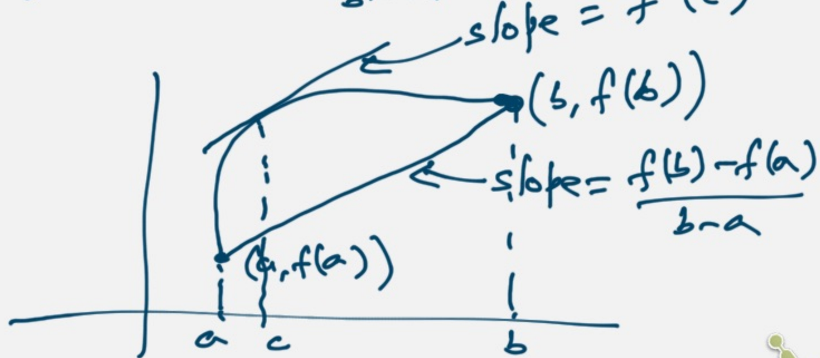
Mean Value Theorem:

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Proof:

Let $g(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$

and let $h(x) = f(x) - g(x)$

Then h is continuous on $[a, b]$ and h is differentiable on (a, b).

Also, $h(a) = f(a) - g(a) = 0$

$h(b) = f(b) - g(b) = 0$

$\therefore h(a) = h(b)$

By the Rolle's thm, $\exists\, c \in (a, b)$ s.t. $h'(c) = 0$

$\Rightarrow f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}.$

Corollary 1: Suppose $f'(x) = 0$ for all $x \in (a,b)$. Then f must be constant in the interval (a,b).

Proof: Let $x_1 < x_2$ be any two points in (a,b).

Then by the mean value theorem,

$\exists\, c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) = 0$. $\therefore$ $f(x_2) = f(x_1)$.

$\Rightarrow f$ is a constant.

Corollary 2 (i) If $f'(x) > 0$ on an interval (a,b), then f is increasing on (a,b)

(ii) If $f' < 0$ on (a,b), then f is decreasing on (a,b).

Proof: (i) Let $x_1 < x_2$ in (a,b).

To show: $f(x_1) < f(x_2)$.

By the MVT, $\exists\, c \in (x_1, x_2)$ s.t.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

$\Rightarrow f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_2) > f(x_1)$