Local maxima and minima of functions of two variables:

(a,b) is a point of local minimum if $f(a,b) \le f(x,y)$ for all (x,y) in some disk containing (a,b) and $(x,y) \in \text{Domain}(f)$.

Similarly, we define local maximum.

Necessary condition:

If (a,b) is an interior point of the domain of f and the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b), then

f has local min/max. $\Rightarrow \frac{\partial f}{\partial x}(a,b) = 0$ & $\frac{\partial f}{\partial y}(a,b) = 0$.

Critical points: An interior point (a,b) is a critical point of f if either $\frac{\partial f}{\partial x}(a,b) = 0 = \frac{\partial f}{\partial y}(a,b)$ or one of the partial derivatives does not exist at (a,b).

Theorem: If (a,b) is an interior point of the domain of f, then it is a point of local min. or local max. only if it is a critical point.

Sufficient condition:

Let $f_x = \frac{\partial f}{\partial x}(a,b) = 0$ & $\underbrace{\frac{\partial f}{\partial y}}_{}\overset{= f_y}{}(a,b) = 0$.

Let $A = f_{xx}(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b)$

$B = f_{xy}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$

$C = f_{yy}(a,b) = \frac{\partial^2 f}{\partial y^2}(a,b)$

(i) If $AC - B^2 < 0$, then f has a saddle point at (a,b) (i.e. neither local max. nor local min.).

(ii) If $AC - B^2 > 0$ and $A > 0$, then (a,b) is a point of local min.

(iii) If $AC - B^2 > 0$ and $A < 0$, then (a,b) is a point of local max.

(iv) If $AC - B^2 = 0$, then the test is inconclusive.

Second derivative test:

$AC-B^2<0$	saddle point
$AC-B^2>0, A>0$	local minimum
$AC-B^2>0, A<0$	local maximum
$AC-B^2=0$	No conclusion

Examples:

(i) $f(x,y) = x^2+y^2-4y+9$

$\frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = 2y-4$

$\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0 \Rightarrow x=0; y=2$

$\therefore$ $(0,2)$ is the only critical point.

$f_{xx} = 2$; $f_{xy}=0$; $f_{yy}=2$

$\therefore$ $A=2$; $B=0$; $C=2$

$AC-B^2 = 4>0$; $A>0$

$\Rightarrow (0,2)$ is a point of local min.

In this example, it is possible to see that $(0,2)$ is a point of global minimum:

$$f(x,y) = x^2 + y^2 - 4y + 9$$
$$= x^2 + (y-2)^2 + 5$$
$$\geq 5$$

and $f(0,2) = 5$

$\therefore$ $(0,2)$ is a point of global min.

② $f(x,y) = y^2 - x^2$

$f_x = -2x$; $f_y = 2y$

$f_x = 0$; $f_y = 0$ $\Rightarrow$ $(x,y) = (0,0)$.

$f(0,0) = 0$

$f_{xx} = -2$; $f_{xy} = 0$; $f_{yy} = 2$

$A = -2$; $B = 0$, $C = 2$

$AC - B^2 = -4 < 0$

$\Rightarrow$ $(0,0)$ is a saddle point.

(3) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2$

$f_y = x - 2y - 2$

$f_x = 0 = f_y \Rightarrow \begin{array}{l} y - 2x = 2 \\ x - 2y = 2 \,]\times 2 \end{array}$

$\Rightarrow \begin{array}{l} -3y = 6 \Rightarrow y = -2 \\ x = 2 + 2(-2) = -2 \end{array}$

$\therefore$ $(-2,-2)$ is the only critical pt.

$f_{xx} = -2$; $f_{xy} = 1$; $f_{yy} = -2$

$AC - B^2 = (-2)(-2) - 1^2 = 3 > 0$

$A = -2 < 0$ $\Rightarrow$ local max.

(4) $f(x,y) = x^2 + y^2 - 2xy$

$f_x = 2x - 2y$

$f_y = 2y - 2x$

$f_x = 0 = f_y \Rightarrow y = x$.

$\therefore$ Every (x,x) is a critical point.

$f_{xx} = 2$; $f_{xy} = -2$; $f_{yy} = 2$

$AC - B^2 = (2)(2) - (-2)^2 = 0$

$\therefore$ The second derivative test fails.

However, $f(x,y) = (x-y)^2 \geq 0 \ \forall (x,y)$

$f(x,x) = 0$

$\therefore$ Every point on the line $y=x$ is a point of local minimum (in fact, also global min.)

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(5) $f(x,y) = x^3 + 3xy + y^3$

$f_x = 3x^2 + 3y$

$f_y = 3x + 3y^2$

$f_x = 0 \Rightarrow x^2 + y = 0 \Rightarrow y = -x^2$

$f_y = 0 \Rightarrow x + y^2 = 0 \Rightarrow x = -y^2$

$y = -x^2 = -(-y^2)^2 = -y^4$

$\Rightarrow y(1+y^3) = 0 \Rightarrow y = 0$ or $y^3 = -1$

$\Rightarrow y = 0$ or $y = -1$

When $y=0$, $x=0$

When $y=-1$, $x = -(-1)^2 = -1$

$\therefore (0,0)$ & $(-1,-1)$ are the critical points

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Now, $f_{xx} = 6x$

$f_{xy} = 3$

$f_{yy} = 6y$

At $(0,0)$: $A = f_{xx}(0,0) = 0$

$B = f_{xy}(0,0) = 3$

$C = f_{yy}(0,0) = 0$

$AC - B^2 = -9 < 0$

$\Rightarrow (0,0)$ is a saddle point.

At $(-1,-1)$: $A = -6$; $B = 3$, $C = -6$

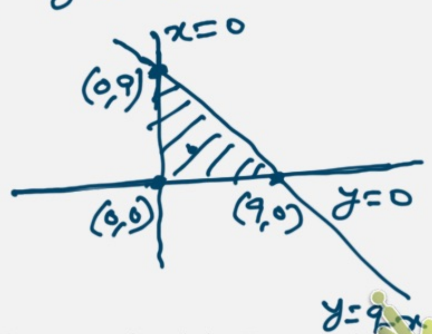
$AC - B^2 = 36 - 9 > 0$; $A < 0$

$\therefore (-1,-1)$ is a local. max.

Global maxima and minima

Example: Find the global maxima & minima of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.

Soln: First we find the critical points lying in the interior.



$f_x = 2 - 2x$

$f_y = 2 - 2y$

$f_x = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow y = 1$

$(1,1)$ is the only critical point. $f(1,1) = 4$

Then we look at the boundary of the domain.

- On $y = 0$ $g(x) = f(x,y) = f(x,0) = 2 + 2x - x^2$, $0 \le x \le 9$

$g'(x) = 2 - 2x = 0$ iff $x = 1$

$f(1,0) = g(1) = 3$

$f(0,0) = g(0) = 2$

$f(9,0) = g(9) = 2 + 2\times 9 - 9^2 = -61$

- On $\underline{x = 0}$: $f(0,y) = 2 + 2y - y^2$, $0 \le y \le 9$

$f(0,1) = 3$

$f(0,0) = 2$

$f(0,9) = -61$

• On $\underline{y = 9 - x}$:

$$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$$
$$= 2 + 2x + 18 - 2x - x^2 - x^2 + 18x - 81$$
$$= -2x^2 + 18x - 61 = h(x)$$

$h'(x) = -4x + 18 = 0 \iff x = \frac{9}{2}$

$$f\left(\frac{9}{2}, \frac{9}{2}\right) = -2\left(\frac{9}{2}\right)^2 + 18\left(\frac{9}{2}\right) - 61$$
$$= -\frac{41}{2}.$$

$f(9,0) = -61$, $f(0,9) = -61$

So, ~~the~~ possible values for global min/max are $4, 3, 2, -61, -\frac{41}{2}$

$\therefore$ The max. value is 4 at $(1,1)$

& the min. value is -61 attained at $(9,0)$ & $(0,9)$.