How to calculate $\int_a^b f(x)\,dx$?

Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ be any partition of $[a, b]$.

We define $\|P\| = \max_{1 \le i \le n} |x_i - x_{i-1}|$,

i.e., $\|P\|$ is the maximum of the lengths of subintervals in the partition.

If we divide $[a, b]$ into n equal subintervals, then $\|P\| = \frac{b-a}{n}$
$\to 0$ as $n \to \infty$

Thm: Let for each $n \in \mathbb{N}$, P_n be a partition of $[a, b]$ such that $\|P_n\| \to 0$ as $n \to \infty$.

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$$

$$= \lim_{n\to\infty} S(P_n, f),$$

where $S(P_n, f) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$,

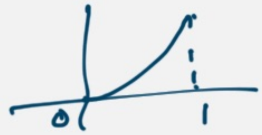
for some $\xi_i \in [x_{i-1}, x_i]$

Example

① Evaluate $\int_0^1 x^2\,dx$.

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$

Then $\|P_n\| = \frac{1}{n} \to 0$ as $n \to \infty$.



$$L(P_n, f) = f(0)\cdot\frac{1}{n} + f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + \dots + f\left(\frac{n-1}{n}\right)\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left[0 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n-1}{n}\right)^2\right]$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + \dots + (n-1)^2\right]$$

$$= \frac{1}{n^3}\cdot\frac{(n-1)n\cdot(2n-1)}{6} \to \frac{2}{6} = \frac{1}{3}$$

$$\therefore \boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$

② Evaluate $\int_1^2 \frac{1}{x}\,dx$.

$f(x) = \frac{1}{x}$

Let's try $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots, 1+\frac{n-1}{n}, 2\}$

$$U(P_n, f) = f(1)\cdot\frac{1}{n} + f\left(1+\frac{1}{n}\right)\cdot\frac{1}{n} + \dots + f\left(1+\frac{n-1}{n}\right)\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left[1 + \frac{1}{1+\frac{1}{n}} + \dots + \frac{1}{1+\frac{n-1}{n}}\right]$$

$$= \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots \frac{1}{2n-1}$$

We don't know how to evaluate $\lim_{n\to\infty} U(P_n, f)$.

Let's try partition P_n consisting of points in geometric progression, i.e.,

$P_n = \{1, r, r^2, \dots, r^{n-1}, r^n = 2\}$

ie. $r = 2^{1/n}$.

Then $\|P_n\| = \max\{r-1, r^2-r, \dots, r^n - r^{n-1}\}$

$= r^n - r^{n-1} = r^{n-1}(r-1)$

$= 2^{1-\frac{1}{n}}(2^{1/n}-1) \to 0$ as $n\to\infty$

$U(P_n, f) = f(1).(r-1) + f(r).(r^2-r) + f(r^2)(r^3-r^2)$
$+ \dots + f(r^{n-1})(r^n - r^{n-1})$
$= (r-1)\left[1 + \frac{1}{r}.r + \frac{1}{r^2}.r^2 + \dots + \frac{1}{r^{n-1}}.r^{n-1}\right]$

$\Rightarrow U(P_n, f) = (r-1)[1+1+\dots+1]$

$= n(r-1) = \dfrac{2^{1/n}-1}{1/n}$

$\to \ln 2$ (Use L'Hôpital's rule)

as $n\to\infty$.

$\therefore \boxed{\int_1^2 \frac{1}{x}\,dx = \ln 2}$

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Properties of definite integrals:

① $\displaystyle\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$

② $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

③ If $f(x) \le g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

④ $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

⑤ $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$