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HTML Content

Picard's Iteration Method

This method is used to find successive approximations to the unknown solution of a first order initial value problem.

Consider the IVP :

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \quad \text{— (1)}$$

This is equivalent to the following integral equation :

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad \text{— (2)}$$

(1) $\Rightarrow$ (2) by integrating.

(2) $\Rightarrow$ $y(x_0) = y_0 + \int_{x_0}^{x_0} f(t,y(t))dt = y_0$

Also, diff. (2), we get

$$\frac{dy}{dx} = 0 + f(x, y(x))$$

Since $y(t)$ is unknown, we can use that in the integral to evaluate.

So, we use $y(t) = y_0$ in the integral to get a function $y_1(x)$.

i.e. $$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

Then we use $y(t) = y_1(t)$ in the integral to get the next approx.

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

In general,

$$\boxed{y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\,dt}$$

$y_1(x)$, $y_2(x)$, ..., $y_n(x)$, ... are called the Picard's iterations.

Note that all $y_n(x)$ satisfy the initial condition $y_n(x_0) = y_0$.
But none of them may satisfy the ODE.

However, under some conditions on $f(x,y)$, $y_n(x)$ converges to the unique solution $y(x)$ to the IVP.

<u>Example</u>: $\frac{dy}{dx} = x+y$; $y(0)=0$

Note that this can be solved easily to get $y(x) = e^x - x - 1$.
Let's find the Picard's iterations.
Here, $f(x,y) = x+y$ & $x_0 = 0, y_0 = 0$.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$
$$= 0 + \int_0^x f(t,0)\,dt$$

$$\therefore \quad y_1(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

$$= 0 + \int_0^x (t + y_1(t))\,dt$$

$$= \int_0^x \left(t + \frac{t^2}{2}\right)dt = \frac{x^2}{2} + \frac{x^3}{6}$$

$$y_3(x) = 0 + \int_0^x f(t, y_2(t))\,dt$$

$$= \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)dt$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

By induction,

$$y_n(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}, \quad n \geq 1$$

$$= \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!}\right] - 1 - x$$

$\to e^x - 1 - x$, which is the solution.

In practice, we can use $y_n(x)$ for large enough n to approximate the solution $y(x)$ to the given IVP (when we are not able find the actual solution).

Existence and uniqueness of first order IVP

Some examples:

① $|y'| + |y| = 0$; $y(0) = 1$ has no solutions.

(because $|y'| + |y| = 0 \Rightarrow |y| = 0$
$\Rightarrow y \equiv 0$
$\Rightarrow y(0) = 0 \neq 1$)

② $y' = 2x$; $y(0) = 1$
has a unique soln. $y = x^2 + 1$

③ $xy' = y-1$; $y(0)=1$

$y = 1+cx$ is a solution for every $c \in \mathbb{R}$.

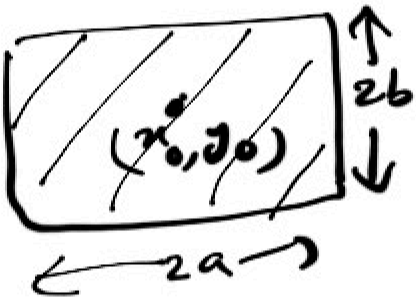
Hence, an IVP may have no soln., a unique or more than one soln.

<u>Theorem 1</u> (Sufficient conditions for existence of solns.)

Consider the IVP: $\frac{dy}{dx} = f(x,y)$; $y(x_0)=y_0$

Suppose the function $f(x,y)$ is continuous on a rectangle

$R = \{ (x,y) : |x-x_0| \le a, \ |y-y_0| \le b \}$



Assume $|f(x,y)| \le K$ $\forall (x,y) \in R$ for some constant K.

Then the IVP has at least one solution $y(x)$. Also, the solution must be defined for $x \in (x_0 - \alpha, x_0 + \alpha)$, where $\alpha = \min\{a, \frac{b}{K}\}$

<u>Remark</u>: ① The proof is nontrivial and we will not discuss it.

② The α that we get in the theorem need not be the maximum possible for a given IVP.

<u>e.g.</u> Consider $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$.

Let's find the maximum possible α given by Theorem 1.

Here, $f(x, y) = 1 + y^2$ is continuous everywhere. So, we can take a and b to be as large as we want.

If $R = \{(x,y) : |x| \le a, |y| \le b\}$,

then $|f(x,y)| = 1+y^2 \le 1+b^2$

$\forall (x,y) \in R$.

So, $K = 1+b^2$

Then $\frac{b}{K} = \frac{b}{1+b^2} \le \frac{1}{2}$

$\left(\because 1+b^2-2b = (b-1)^2 \ge 0 \right.$
$\left. \Rightarrow 1+b^2 \ge 2b \right)$

$\therefore \alpha = \min\{a, \frac{b}{K}) \le \frac{b}{K} \le \frac{1}{2}$

So, the max. value α that we get using existence thm. is $\alpha = \frac{1}{2}$.

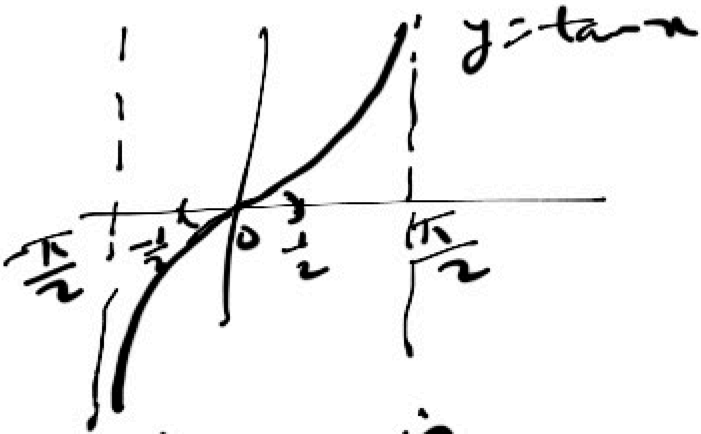
$\therefore$ Soln. $y(x)$ must be defined in the interval $(-\frac{1}{2}, \frac{1}{2})$.

But, we can solve the IVP as follows

$$\int \frac{dy}{1+y^2} = \int dx \Rightarrow \tan^{-1}(y) = x + c$$

$$\Rightarrow y = \tan(x+c)$$

$$y(0) = 0 \Rightarrow c = 0 \Rightarrow \boxed{y = \tan x}$$



The solution $y = \tan x$ is defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$ which is bigger than $(-\alpha, \alpha)$.

Next question is about the uniqueness of solution.

We'll see under an addition condition we can guarantee the uniqueness & existence of solution to the IVP.

We'll discuss this in the next class.