Let $T : V \to V$ be a linear operator, where is a vector space over a field $\mathbb{F}$. A scalar $\lambda \in \mathbb{F}$ is called an <u>eigenvalue</u> of T if there exists a nonzero vector $v \in V$ s.t. $T(v) = \lambda v$.

Such nonzero v's are called <u>eigenvectors</u> of T corresponding to eigenvalue λ of T.

<u>Remark</u>: If $A \in M_{n \times n}(\mathbb{F})$, then we can define $T : \mathbb{F}^n \to \mathbb{F}^n$ by

$$T(X) = AX, \quad \text{where}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

So, eigenvalues and eigenvectors of T and A are the same.

Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear operator.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for V, then we get an $n \times n$ matrix $A = [T]_{\mathcal{B}}$. We know that

$$[T(v)]_{\mathcal{B}} = A [v]_{\mathcal{B}}$$

Also, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two ordered bases for V, then $[T]_{\mathcal{B}_1}$ and $[T]_{\mathcal{B}_2}$ are similar matrices.

$\therefore$ $\lambda I - [T]_{\mathcal{B}_1}$ and $\lambda I - [T]_{\mathcal{B}_2}$ are similar matrices.

Since, similar matrices have the same determinant, we can define the char. poly. of T as

$$p_T(x) = \det(\lambda I - [T]_{\mathcal{B}}),$$

where $\mathcal{B}$ is any ordered basis for V.

This is a polynomial of degree n.

Thus, to find the eigenvalues of a linear operator T on a finite dimensional vector space we choose any ordered basis of V and write the matrix of T w.r.t. this basis. We find the char. poly of this matrix and find the roots to get all eigenvalues of T.

Example: Let $T: \mathcal{P}_2 \to \mathcal{P}_2$ be the operator $T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$

Find the eigenvalues and eigenvectors of T.

Soln: Let $\mathcal{B} = \{1, x, x^2\}$

$$T(1) = 0 = 0.1 + 0.x + 0.x^2$$
$$T(x) = 1 = 1.1 + 0.x + 0.x^2$$
$$T(x^2) = 2x = 0.1 + 2.x + 0.x^2$$

$$\therefore \quad [T]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = A$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{pmatrix}$$
$$= \lambda^3$$

$\lambda^3 = 0 \iff \lambda = 0$

$\therefore$ $\lambda = 0$ is the only eigenvalue of T.

To find the eigenvectors:

$$T(a_0 + a_1 x + a_2 x^2) = 0 \cdot (a_0 + a_1 x + a_2 x^2)$$
$$a_1 + 2a_2 x = 0$$
$$\Rightarrow a_1 = 0, \; a_2 = 0$$

$\Rightarrow$

$\therefore$ $p(x) = a_0$ is an eigenvector of T for any $a_0 \neq 0$.

We can also use the matrix $[T]_B$ to find the eigenvectors as follows:

. First find the eigenvectors of A.

$$AX = 0 \cdot X = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_2 = 0$, $x_3 = 0$, $x_1 = a \in \mathbb{R}$

$\therefore$ $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $a \neq 0$ is an eigenvector of A.

$\therefore$ If v is an eigenvector of T, then $[v]_B = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ $v = a \cdot 1 + 0 \cdot x + 0 \cdot x^2 = a$

$\therefore$ $v = a \neq 0$ is an eigenvector of T.

Remark: The eigenvalues and eigenvectors are defined even for infinite dimensional vector spaces. However, we cannot define char. poly. in such cases.

e.g. $T: V \to V$ given by

$$T(p(x)) = p'(x)$$

where V is the vector space of all real polynomials.

If $p(x) = a_0 \neq 0$, then

$$T(p(x)) = 0 = 0 \cdot p(x)$$

$\therefore$ $\lambda = 0$ is an eigenvalue and any nonzero constant polynomial is an eigenvector.

Q: Can you show that these are the only eigenvectors? (i.e. no other λ is an eigenvalue)