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HTML Content

System of 1st order ODEs

Recall: Consider $\vec{X}' = A\vec{X}$, —— $(*)$

where A is $n \times n$ real matrix.

$$\vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

If $\vec{v}$ is an eigenvector of the matrix with real eigenvalue λ, then $\vec{X}(t) = e^{\lambda t}\vec{v}$

is a solution to $(*)$.

Our goal is to find n linearly independent solutions.

Case I: If the matrix A has n lin. indep. eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$, then we get n lin. indep. solutions given by

$$\vec{X}_i(t) = e^{\lambda_i t}\vec{v}_i, \quad i=1,2,\cdots,n$$

Hence, the general solution of $\vec{X}' = AX$ is given by

$$\boxed{\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n}$$

where $c_1, c_2, \cdots, c_n$ are arbitrary real constants.

This is the case when A is diagonalizable over $\mathbb{R}$.

<u>Remark</u>: $\lambda_1, \lambda_2, \cdots, \lambda_n$ need not be distinct.

<u>Example</u>: Solve $x_1' = x_1 + x_2$
$x_2' = 2x_2$

<u>Soln</u>: Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)$$

$\Rightarrow \lambda_1 = 1$ & $\lambda_2 = 2$ are the eigenvalues

<u>Eigenvector for $\lambda_1 = 1$</u>:

$$I - A = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$$

$$(I-A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_2 = 0$

$\therefore \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$.

So, $\vec{x}_1(t) = e^{t}\begin{pmatrix}1\\0\end{pmatrix}$ is a soln.

Eigenvector for $\lambda_2 = 2$:

$$(2I - A)\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$$

$\therefore$ $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector for $\lambda_2 = 2$

So, $\vec{x}_2(t) = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$ is a soln.

Hence, the general soln is

$$\vec{x}(t) = c_1 e^{t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}c_1 e^{t} + c_2 e^{2t}\\ c_2 e^{2t}\end{pmatrix}$$

$\therefore$ $x_1(t) = c_1 e^{t} + c_2 e^{2t}$; $x_2(t) = c_2 e^{2t}$.

Case II: Suppose the real matrix A has complex eigenvalues

$$\lambda = \alpha \pm i\beta \quad \text{with } \beta \neq 0$$

In this case the complex eigenvectors can be written as $\vec{u} \pm i\vec{v}$, where $\vec{u}$ and $\vec{v}$ real column vectors.

Then $e^{(\alpha+i\beta)t}(\vec{u}+i\vec{v})$

and $e^{(\alpha-i\beta)t}(\vec{u}-i\vec{v})$

are two lin. indep. complex-valued solutions.

To get real-valued valued solutions, we take the real & imaginary parts of the above solutions.

$$e^{(\alpha+i\beta)t}(\vec{u}+i\vec{v})$$

$$= e^{\alpha t} \cdot e^{i\beta t}(\vec{u}+i\vec{v})$$

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{u}+i\vec{v})$$

$$= e^{\alpha t}\left[(\cos\beta t)\vec{u} - (\sin\beta t)\vec{v}\right]$$
$$+ i\, e^{\alpha t}\left[(\cos\beta t)\vec{v} + (\sin\beta t)\vec{u}\right]$$

$\therefore$ The general soln. is given by

$$\boxed{\begin{aligned}\vec{X}(t) = {} & c_1 e^{\alpha t}\left[(\cos\beta t)\vec{u} - (\sin\beta t)\vec{v}\right] \\ & + c_2 e^{\alpha t}\left[(\cos\beta t)\vec{v} + (\sin\beta t)\vec{u}\right]\end{aligned}}$$

Example: Solve $\vec{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{X}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2+1$$

So, $\lambda = \pm i$ are the eigenvalues.

Check that $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector corresp. to $\lambda = i$.

$\therefore \quad e^{it}\begin{pmatrix}1\\ i\end{pmatrix}$ is a complex-valued solution.

$$e^{it}\begin{pmatrix}1\\ i\end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix}1\\ i\end{pmatrix}$$

$$= \begin{pmatrix}\cos t + i\sin t\\ -\sin t + i\cos t\end{pmatrix}$$

$$= \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + i\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

$\therefore$ The general solution is given by

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}}$$

or,

$$\boxed{\begin{aligned} x_1(t) &= c_1\cos t + c_2\sin t \\ x_2(t) &= -c_1\sin t + c_2\cos t \end{aligned}}$$

<u>Remark</u>: Case II is applicable if the matrix A is not diagonalizable over $\mathbb{R}$ but it is diagonalizable over $\mathbb{C}$.

Case III: A is not diagonalizable over $\mathbb{C}$.

Example: Solve $\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\vec{x}$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has eigenvalues 1, 1

$I - A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow$ rank $= 1$ $\therefore$ nullity $= 2-1=1$

$\therefore$ Eigenspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

So, $\vec{x}_1(t) = e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution.

Assume another solution to be of the form

$$\vec{x}_2(t) = te^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t}\vec{u},$$

where $\vec{u}$ is to be determined.

$$\vec{x}_2(t) = te^t\begin{pmatrix}1\\0\end{pmatrix} + e^t\vec{u}$$

$$\Rightarrow \quad \vec{x}_2{}'(t) = (te^t + e^t)\begin{pmatrix}1\\0\end{pmatrix} + e^t\vec{u}$$

Also,

$$A\vec{x}_2(t) = te^t \underbrace{A\begin{pmatrix}1\\0\end{pmatrix}}_{= \begin{pmatrix}1\\0\end{pmatrix}} + e^tA\vec{u}$$

$$= te^t\begin{pmatrix}1\\0\end{pmatrix} + e^tA\vec{u}$$

$\vec{x}_2(t)$ is a soln

$$\Leftrightarrow \quad (\cancel{te^t} + e^t)\begin{pmatrix}1\\0\end{pmatrix} + e^t\vec{u} = \cancel{te^t\begin{pmatrix}1\\0\end{pmatrix}} + e^tA\vec{u}$$

$$\Leftrightarrow \quad e^t\begin{pmatrix}1\\0\end{pmatrix} = e^tA\vec{u} - e^t\vec{u}$$

$$\Leftrightarrow \quad \begin{pmatrix}1\\0\end{pmatrix} = A\vec{u} - \vec{u}$$

$$\Leftrightarrow \quad (A - I)\vec{u} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$\Leftrightarrow \quad u_2 = 1 \quad \therefore \ \vec{u} = \begin{pmatrix}0\\1\end{pmatrix}$ is a choice.

$\therefore \vec{x}_2(t) = te^{t}\begin{pmatrix}1\\0\end{pmatrix} + e^{t}\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore$ The general soln. is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$
$$= c_1 e^{t}\begin{pmatrix}1\\0\end{pmatrix} + c_2\left[te^{t}\begin{pmatrix}1\\0\end{pmatrix} + e^{t}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

In general, if A has an eigenvalue λ with algebraic multiplicity 2 but geometric multiplicity 1,

$\vec{x}_1(t) = e^{\lambda t}\vec{v}$, $\vec{v}$ is an eigenvector for λ.

$\vec{x}_2(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}$,

where $\vec{u}$ satisfies

$$\boxed{(A - \lambda I)\vec{u} = \vec{v}}$$

eigenvector

is to be found.