

Lagrange multiplier method to solve constraint optimization problems

Problem: Maximize/minimize
 $f(x_1, x_2, \dots, x_n)$
 subject to m constraints
 $g_1(x_1, x_2, \dots, x_n) = 0$
 $g_2(x_1, x_2, \dots, x_n) = 0$
 \vdots
 $g_m(x_1, x_2, \dots, x_n) = 0$,
 where $m \leq n$.

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Maximize/minimize $f(x, y, z)$

subject to $g(x, y, z) = 0$

We write the Lagrange multiplier equation:

$$\underbrace{\nabla f}_{3 \text{ components}} = \lambda \underbrace{\nabla g}_{3 \text{ components}}, \text{ where } \lambda \in \mathbb{R}.$$

So, we get 3 equations and 4 unknowns x, y, z, λ .

There is one more eqn. $g(x, y, z) = 0$

We solve these 4 eqns in 4 unknowns to get the points where max/min can be attained.

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Example: Find the maximum and minimum values of the function

$$f(x, y) = x^3 + 4y^2$$

on the circle $x^2 + y^2 = 1$

Soln: Max./min. $f(x, y) = x^3 + 4y^2$

subject to $g(x, y) = x^2 + y^2 - 1 = 0$

$$\vec{\nabla} f = 3x^2 \hat{i} + 8y \hat{j}$$

$$\vec{\nabla} g = 2x \hat{i} + 2y \hat{j}$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 3x^2 \hat{i} + 8y \hat{j} = \lambda (2x \hat{i} + 2y \hat{j})$$

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$$\Rightarrow 3x^2 = 2\lambda x ; 8y = 2\lambda y$$

$$8y = 2\lambda y \Leftrightarrow y = 0 \text{ or } \lambda = 4$$

$$\text{If } \lambda = 4, \quad 3x^2 = 8x$$

$$\Rightarrow x = 0 \text{ or } x = \frac{8}{3}$$

not possible
since $x^2 + y^2 = 1$.

$$\text{When } x = 0, y^2 = 1 \Rightarrow y = \pm 1$$

$$(0, 1) ; (0, -1)$$

$$\text{If } \lambda \neq 4, \quad y = 0, \quad x = \pm 1$$

$$(1, 0), (-1, 0)$$

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$$\begin{aligned}
 f(0, 1) &= 4 \\
 f(0, -1) &= 4 \\
 f(1, 0) &= 1 \\
 f(-1, 0) &= -1
 \end{aligned}$$

\therefore Max. value of f is 4 attained at $(0, \pm 1)$
 \therefore Min. value of f is -1 attained $(-1, 0)$.
Ex: Try solving it using substitution method $y^2 = 1 - x^2$.

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Example: Find the shortest from the origin to the surface $x^2 - z^2 = 1$ in \mathbb{R}^3 .

Soln. We need to minimize $\sqrt{x^2 + y^2 + z^2}$ s.t. $x^2 - z^2 = 1$.

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$g(x, y, z) = x^2 - z^2 - 1 = 0$$

$$\nabla f = \lambda \nabla g$$

$$\nabla (x^2, 2y, 2z) = \lambda (2x, 0, -2z)$$

$$\Rightarrow 2x = 2\lambda x; \quad 2y = 0; \quad -2z = -2\lambda z$$

$$\Rightarrow x = \lambda x; \quad y = 0; \quad z = \lambda z$$

$$\Rightarrow x = 0; \quad y = 0; \quad z = 0$$

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$$zx = \bar{z}\bar{x} \Rightarrow x = 0 \text{ or } \bar{z} = 1$$
$$zz = -\bar{z}\bar{z} \Rightarrow z = 0 \text{ or } \bar{z} = -1$$
$$\text{Case: } \bar{z} = 1, z = 0 \Rightarrow \bar{x}^2 - 0^2 = 1 \Rightarrow x = \pm 1$$
$$(1, 0, 0); (-1, 0, 0)$$
$$\text{Case: } \bar{z} \neq 1, z = 0, \bar{x}^2 - \bar{z}^2 = 1 \Rightarrow \bar{z}^2 = -1, \text{ not possible}$$

Now, $f(1, 0, 0) = 1 = f(-1, 0, 0)$.
Shortest distance = 1.

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