Indian Institute of Technology Delhi - Abu Dhabi

AMTL101

Tutorial Sheet 8: Second Order ODEs

(1) Find the solutions of the following initial value problems:
  (a) $y'' - 2y' - 3y = 0,\ y(0) = 0,\ y'(0) = 1$
  (b) $y'' + 10y = 0,\ y(0) = \pi,\ y'(0) = \pi^2$

(2) Find all solutions of the following differential equations:
  (a) $4y'' - y = e^x$
  (b) $y'' + 4y = \cos x$
  (c) $y'' + 9y = \sin 3x$

(3) Verify that $y_1 = \frac{\cos x}{x}$ is a solution to the differential equation $xy'' + 2y' + xy = 0$, and then use reduction of order method to find a second linearly independent solution.

(4) Use the method of undetermined coefficients to find a particular solution of each of the following equations:
  (a) $y'' + 4y = \cos x$
  (b) $y'' - y' - 2y = x^2 + \cos x$
  (c) $y'' + 9y = x^2 e^{3x}$

(5) Solve the following initial value problems:
  (a) $x^2y'' - 2xy' + 2y = 0,\ y(1) = 1.5,\ y'(1) = 1$
  (b) $x^2y'' + 3xy' + y = 0,\ y(1) = 3,\ y'(1) = -4$
  (c) $x^2y'' - 3xy' + 4y = 0,\ y(1) = 0,\ y'(1) = 3$

(6) Use the variation of parameters method to solve the following equations:
  (a) $y'' + 9y = \csc 3x$
  (b) $x^2y'' - 2xy' + 2y = x^3 \sin x$
  (c) $y'' - 4y' + 4y = \dfrac{6e^{2x}}{x^4}$

Q1 (a) $y'' - 2y' - 3y = 0 \qquad y(0) = 0 \quad y'(0) = 1$

Char. eqn : $m^2 - 2m - 3 = 0$

$\Rightarrow m = -1 \ \& \ 3$

$y_h = c_1 e^{-t} + c_2 e^{3t}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(0) = 1 \Rightarrow -c_1 + 3c_2 = 1$

$4c_2 = 1 \Rightarrow c_2 = 1/4$

$\& \quad c_1 = -1/4.$

$\therefore y_h = -\frac{1}{4} e^{-t} + \frac{1}{4} e^{3t}.$

(b) $y'' + 10y = 0, \quad y(0) = \pi, \ y'(0) = \pi^2$

$\Rightarrow m^2 + 10 = 0$

$\Rightarrow m = \pm \sqrt{10}\, i$

$\Rightarrow y_h = c_1 \cos \sqrt{10}\, t + c_2 \sin \sqrt{10}\, t$

$\because y(0) = \pi \Rightarrow c_1 = \pi$

$\& \quad y_h' = -c_1 \sqrt{10} \sin \sqrt{10}\, t + c_2 \sqrt{10} \cos \sqrt{10}\, t$

$\because y_h'(0) = \pi^2 \Rightarrow c_2 \sqrt{10} = \pi^2$

$\Rightarrow c_2 = \pi^2 / \sqrt{10}$

$$\therefore \quad y_h = \pi \cos\sqrt{10}\,t + \frac{\pi^2}{\sqrt{10}} \sin\sqrt{10}\,t.$$

Q.2 (a) $4y'' - y = e^x$

Char. $4m^2 - 1 = 0$

$\Rightarrow \quad m^2 = 1/4$

$\Rightarrow \quad m = \pm 1/2$

$$\therefore y_h = c_1 e^{x/2} + c_2 e^{-x/2}$$

Let $y_p = Ae^x$

$y_p' = Ae^x \qquad y_p'' = Ae^x$

$\Rightarrow \quad 4Ae^x - Ae^x = e^x$

$\Rightarrow \quad 3Ae^x = e^x$

$\Rightarrow \quad A = 1/3$

$\Rightarrow \quad y_p = e^x/3$

$$\therefore \quad y = c_1 e^{x/2} + c_2 e^{-x/2} + e^x/3$$

(b) $y'' + 4y = \cos x$

$m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

Let $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$\Rightarrow \quad -A\cos x - B\sin x + 4A\cos x + 4B\sin x = \cos x$

$\Rightarrow \quad 4A - A = 1 \; ; \; 3B = 0$

$A = 1/3 \; ; \; B = 0$

$\Rightarrow \quad y_p = 1/3 \cos x$

Q. 3 $\quad xy'' + 2y' + xy = 0$

$$y = \frac{\cos x}{x}$$

$$\Rightarrow \quad y' = \frac{x(-\sin x) - \cos x}{x^2}$$

$$y'' = \frac{x^2\left[-\cancel{\sin x} - x\cos x + \cancel{\sin x}\right]}{x^4} - \frac{2x\left[-x\sin x - \cos x\right]}{x^4}$$

$$= \frac{-x^3\cos x + 2x^2\sin x + 2x\cos x}{x^4}$$

$$\therefore \quad xy'' + 2y' + xy = \frac{-x^3 \cancel{\cos x} + 2x^2 \sin x + 2x\cos x}{\cancel{x^3}}$$

$$+ \frac{2}{x^2}\left(-\cancel{x}\sin x - \cancel{\cos x}\right)$$

$$+ \cancel{\cos x}$$

$$= 0$$

Let $y_2 = u(x)\, y$

then $y_2' = uy' + u'y$

& $y_2'' = uy'' + u'y' + u'y' + u''y$

$= uy'' + 2u'y' + u''y$

$$\therefore \quad x\left[uy'' + 2u'y' + u''y\right] + 2\left[uy' + u'y\right]$$

$$+ x\,u(x)\,y = 0$$

$$\Rightarrow \quad u\left[xy'' + 2y' + xy\right] + 2xu'y' + xu''y$$

$$+ 2u'y = 0$$

$$\Rightarrow \quad u''(xy) + u'(2xy' + 2y) = 0$$

$$\Rightarrow \quad u''\cos x + 2u'\left[\frac{-x\sin x - \cos x}{x} + \frac{\cos x}{x}\right] = 0$$

$$\Rightarrow \quad u'' \cos x + u'(-2\sin x) = 0$$

Let $\quad u' = z$

$$\Rightarrow \quad \frac{dz}{dx} \cos x = 2 \sin x \, z$$

$$\Rightarrow \quad \int \frac{dz}{z} = \int 2 \frac{\sin x}{\cos x} \, dx$$

$$\Rightarrow \quad \ln z = 2 \ln|\sec x|$$

$$\Rightarrow \quad z = \sec^2 x$$

$$\Rightarrow \quad u' = \sec^2 x$$

$$\Rightarrow \quad u = \int \sec^2 x \, dx$$

$$\Rightarrow \quad u = \tan x$$

$$\therefore \quad y_2(x) = \tan x \cdot \frac{\cos x}{x}$$

$$\therefore \quad y_2(x) = \frac{\sin x}{x}.$$

Q. 4 $\quad y'' - y' - 2y = x^2 + \cos x$

$$\Rightarrow \quad m^2 - m - 2 = 0$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{1+8}}{2}$$

$$\Rightarrow \quad m = 2, -1$$

$$\Rightarrow \quad y_h(x) = c_1 e^{2x} + c_2 e^{-x}$$

Let $y_p(x) = Ax^2 + Bx + C + D\cos x + E\sin x$

$$y_p' = 2Ax + B - D\sin x + E\cos x$$

$$y_p'' = 2A - D\cos x - E\sin x$$

$$\Rightarrow 2A - D\cos x - E\sin x - 2Ax - B + D\sin x - E\cos x - 2Ax^2 - 2Bx - 2C - 2D\cos x - 2E\sin x = x^2 + \cos x$$

$$\Rightarrow \quad 2A - B - 2C = 0 \quad \text{—— } (1)$$

$$-2A = 1 \Rightarrow A = -1/2$$

$$-2A - 2B = 0 \Rightarrow B = 1/2$$

$$\& \quad C = \tfrac{1}{2}\left[2A - B\right] = \tfrac{1}{2}\left[-1 - \tfrac{1}{2}\right] = -3/4$$

&

$$-D-E-2D = -3D-E = 1$$

$$-E+D-2E = 0 \Rightarrow D-3E = 0$$

$$\Rightarrow 3D-9E = 0$$

$$\Rightarrow -10E = 1$$

$$\Rightarrow E = -1/10$$

$$\Rightarrow D = 3E = -3/10$$

$$\therefore y_p = -1/2\, x^2 + 1/2\, x - 3/4 - \frac{3}{10}\cos x - \frac{1}{10}\sin x$$

Q.5

(a) $x^2 y'' - 2x y' + 2y = 0$, $y(1) = 1.5$, $y'(1) = 1$

Let $y = x^m$

$$\Rightarrow y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

$$\Rightarrow m(m-1) x^m - 2m x^m + 2x^m = 0$$

$$\Rightarrow m^2 - m - 2m + 2 = 0$$

$$\Rightarrow m(m-1) - 2(m-1) = 0$$

$$\Rightarrow m = 2, 1$$

$$\Rightarrow \quad y_h(x) = c_1 x^2 + c_2 x$$

$$\because \quad y(1) = 1.5 \Rightarrow \quad c_1 + c_2 = 3/2$$

$$\& \quad y'(1) = 1 \Rightarrow \quad 2c_1 + c_2 = 1$$

$$\Rightarrow \quad c_1 = -1/2$$

$$\& \quad c_2 = 2$$

$$\therefore \quad y_h(x) = -\tfrac{1}{2} x^2 + 2x .$$

Q.6 (a) $y'' + 9y = \operatorname{Cosec} 3x .$

$$\Rightarrow \quad m^2 + 9 = 0$$

$$\Rightarrow \quad m = \pm 3 i$$

$$\therefore \quad y_h(x) = c_1 \cos 3x + c_2 \sin 3x .$$

Let $y_1 = \cos 3x \quad y_2 = \sin 3x$

$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3$$

$$\therefore \quad y_p(x) = u_1 y_1 + u_2 y_2$$

$$= y_1 \int \frac{-y_2\, g(x)}{W(y_1, y_2)} + y_2 \int \frac{y_1\, g(x)}{W(y_1, y_2)}$$

$$= \cos 3x \int \frac{-\sin 3x \cdot \operatorname{cosec} 3x}{3}\, dx$$

$$+ \sin 3x \int \frac{\cos 3x \cdot \operatorname{cosec} 3x}{3}\, dx$$

$$= -\frac{1}{3} x \cos 3x + \frac{\sin 3x}{9} \ln|\sin 3x|$$

(b) $x^2 y'' - 2x y' + 2y = x^3 \sin x$

Char. eq^n: $m(m-1) - 2m + 2 = 0$

$\Rightarrow \quad m(m-1) - 2(m-1) = 0$

$\Rightarrow \quad (m-2)(m-1) = 0$

$\Rightarrow \quad m = 2 \; ; \; m = 1.$

$\therefore \quad y_h(x) = c_1 x^2 + c_2 x.$

Let $y_1 = x^2, \quad y_2 = x$

$$\Rightarrow \quad W(y_1, y_2) = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix}$$

$$= x^2 - 2x^2$$
$$= -x^2$$

$$\therefore y_p(x) = y_1 \int \frac{-x \cdot x^3 \sin x}{-x^2 \cdot x^2}\, dx$$

$$+ y_2 \int \frac{x^2 \cdot x^3 \sin x}{-x^2 \cdot x^2}\, dx$$

$$= y_1 \int \sin x\, dx + y_2 \int -x \sin x\, dx$$

$$= -x^2 \cos x - x\left[-x \cos x + \int \cos x\, dx \right]$$

$$= -x^2 \cos x + x^2 \cos x - x \sin x$$

$$= -x \sin x.$$