$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

for any c.

$$= \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

Another kind of improper integral is when the domain is bounded but the function is unbounded.

e.g. ① $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$f(x) = \frac{1}{\sqrt{x}}$ is unbounded on $(0,1)$

and $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$.

$\therefore$ $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral (not a definite integral)

But, for $a>0$, $\int_a^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_a^1 = 2-2\sqrt{a}$

$$\Rightarrow \lim_{a\to 0+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} (2-2\sqrt{a}) = 2$$

$\therefore$ The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges and is equal to 2.

(2) $$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0+} \int_a^1 \frac{1}{x}\,dx$$
$$= \lim_{a\to 0+} (-\ln a) = +\infty.$$

$\therefore$ $\int_0^1 \frac{1}{x}\,dx$ diverges.

Exercise: Show that $\int_0^1 \frac{1}{x^p}\,dx$ converges if and only if $p<1$.

Such integrals are known as improper integral of the second kind.

Suppose $f(x)$ is integrable on $[a,c]$ for every $c<b$ and is unbounded on $[a,b]$.

We define $$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

Comparison test and limit comparison test are applicable for improper integrals of second kind also.

Example : (1) $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$

Let $0 < f(x) = \frac{e^x}{\sqrt{x}} \le \underbrace{\frac{e}{\sqrt{x}}}_{g(x)} \quad \forall x \in (0,1]$

$\int_0^1 g(x)\,dx = e\int_0^1 \frac{1}{\sqrt{x}}\,dx$, which converges.

By comparison test, $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ converges.

(2) $\int_0^1 \frac{e^{\sqrt{x}}-1}{x}\,dx$

$f(x) = \frac{e^{\sqrt{x}}-1}{x}$ is unbounded near 0.

$$= \frac{\left(\cancel{1}+\sqrt{x}+\frac{(\sqrt{x})^2}{2}+\frac{(\sqrt{x})^3}{3!}+\cdots\right)-\cancel{1}}{x}$$

$$= \frac{1+\frac{\sqrt{x}}{2}+\frac{(\sqrt{x})^2}{3!}+\cdots}{\sqrt{x}} \approx \frac{1}{\sqrt{x}} \text{ near } 0.$$

Let $g(x) = \frac{1}{\sqrt{x}}$

The $\frac{f(x)}{g(x)} = \left(\frac{e^{\sqrt{x}}-1}{x}\right)\sqrt{x} = \frac{e^{\sqrt{x}}-1}{\sqrt{x}}$

$\rightarrow 1$ as $x \to 0^+$

(by L'Hôpital's rule)

Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, by the limit comparison test, $\int_0^1 \frac{e^{\sqrt{x}}-1}{x}\,dx$ converges

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Gamma functions:

Consider $\int_0^\infty x^{\alpha-1}e^{-x}\,dx$ for $\alpha \in \mathbb{R}$.

Let's find out the values of α for which the integral converges.

If $\alpha \geq 1$

$f(x) = x^{\alpha-1}e^{-x}$ is bounded on $[0, b]$ for every $b > 0$.

$\therefore \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ is an improper integral of the first kind.

Let $f(x) = x^{\alpha-1}e^{-x}$ and $g(x) = \frac{1}{x^2}$

Then $\frac{f(x)}{g(x)} = \frac{x^{\alpha-1}e^{-x}}{1/x^2} = x^{\alpha+1}e^{-x} \to 0$ as $x \to \infty$

Also, $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

$\therefore$ By the limit comparison test,

$$\int_1^\infty x^{\alpha-1} e^{-x}\,dx \text{ converges.}$$

Now, $$\int_0^\infty x^{\alpha-1} e^{-x}\,dx = \underbrace{\int_0^1 x^{\alpha-1} e^{-x}\,dx}_{\text{definite integral}} + \underbrace{\int_1^\infty x^{\alpha-1} e^{-x}\,dx}_{\text{converges}}$$

$\therefore \int_0^\infty x^{\alpha-1} e^{-x}\,dx$ converges if $\alpha \geqslant 1$

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If $\alpha < 1$,

$$\int_0^\infty x^{\alpha-1} e^{-x}\,dx = \int_0^1 \underbrace{x^{\alpha-1}}_{\substack{\downarrow \\ \text{unbounded} \\ \text{near } 0}} e^{-x}\,dx + \int_1^\infty x^{\alpha-1} e^{-x}\,dx$$

$I_1 = \int_0^1 x^{\alpha-1} e^{-x}\,dx$ is an improper integral of the second kind.

$I_2 = \int_1^\infty x^{\alpha-1} e^{-x}\,dx$ is an improper integral of the first kind.

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For I_2, taking $g(x) = \frac{1}{x^2}$ and using LCT, I_2 converges.

<u>For I_1</u> $= \int_0^1 x^{\alpha-1} e^{-x} dx$.

Take $f(x) = x^{\alpha-1} e^{-x}$ and $g(x) = x^{\alpha-1}$

Then $\frac{f(x)}{g(x)} = e^{-x} \to 1$ as $x \to 0^+$

$\int_0^1 g(x)dx = \int_0^1 x^{\alpha-1} dx = \int_0^1 \frac{1}{x^{1-\alpha}} dx$,

converges if $1-\alpha < 1$ i.e. $\alpha > 0$

diverges if $1-\alpha \geq 1$ i.e. $\alpha \leq 0$

<u>Conclusion:</u>

$\int_0^\infty x^{\alpha-1} e^{-x} dx$ converges if and only if $\alpha > 0$

So, we define the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \text{for } \alpha > 0.$$

$\Gamma(\alpha)$ is finite for every $\alpha > 0$.

Properties of gamma function:

① $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$

② $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}\,dx$

By using integration by parts,

$$\int_0^\infty x^\alpha e^{-x}\,dx = -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\,dx$$

$$= 0 + \alpha \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$$= \alpha\Gamma(\alpha)$$

$\therefore \boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$

Since $\Gamma(1) = 1$, $\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$

$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2$

$\Gamma(4) = 3\Gamma(3) = 3\times 2 = 3!$

In general, $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$

or $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.