Facts:

(i) For any matrix A $(n \times n)$, the product of complex eigenvalues (counted with multiplicity) equals the determinant of A.

Pf: Let $p(x) = \det(xI - A)$

We know that the roots of $p(x)$ are the eigenvalues.

$\therefore \quad p(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

(λ_i's are the complex eigenvalues, they may not be distinct)

Then $p(0) = \underbrace{\det(-A)}_{=(-1)^n \det(A)} = (-1)^n \lambda_1\lambda_2\cdots\lambda_n$

$\Rightarrow \lambda_1\lambda_2\cdots\lambda_n = \det(A)$

(ii) Sum of the eigenvalues (counted with multiplicity) equals trace(A).

Proof for 2x2 matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p(x) = (x-a)(x-d) - bc$$
$$= x^2 - (a+d)x + ad - bc$$

$\therefore$ Sum of eigenvalues $= a+d = \operatorname{trace}(A)$

Prod. of eigenvalues $= ad - bc = \det(A)$

Theorem (Necessary and sufficient condition for diagonalizability):
Let A be any $n \times n$ matrix and suppose $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of A. Let E_i be the eigenspace corresponding to eigenvalue λ_i, for $i = 1, 2, \dots, k$
($E_i = \ker(\lambda_i I - A)$)
Then A is diagonalizable if and only if

$$\sum_{i=1}^{k} \dim(E_i) = n.$$

Defn: Let λ be an eigenvalue of A. Then the geometric multiplicity of A is the dimension of the eigenspace corresponding to λ.

Defn: The algebraic multiplicity of an eigenvalue λ is the number of times λ is repeated in the roots of char. poly.

e.g. If $p(x) = (x-1)(x-2)^3(x-3)^2$,
then the alg. mult. of $1, 2$ & 3 are $1, 3$ & 2, respectively.

Thm: The geometric multiplicity of each eigenvalue is less than or equal to the algebraic multiplicity.

Thm: A is diagonalizable if and only if the geom. mult. equals the alg. mult. for each eigenvalue.

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

Is A diagonalizable?

$$p(x) = \det \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x-2 & -3 \\ 0 & 0 & x-2 \end{pmatrix}$$

$$= (x-1)(x-2)^2$$

$\therefore$ Eigenvalues are 1, 2, 2.

Let's find the geom. mult. of 2.

$$2I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{rank} = 2$$

$\therefore \dim(\ker(2I - A)) = 3 - 2 = 1$

$\therefore$ geom. mult. $(2) = 1 <$ alg. mult. (2)

$\Rightarrow$ A is not diagonalizable.