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Contents

1 System of First Order ODEs

Consider the system of n differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= F_2(t, x_1, x_2, \ldots, x_n) \\
\vdots \qquad & \qquad \vdots \\
\frac{dx_n}{dt} &= F_n(t, x_1, x_2, \ldots, x_n),
\end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are functions of t and $F_1, F_2, \ldots, F_n$ are known functions.

Linear System: A linear system is a system of differential equations of the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + g_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + g_2(t) \\
\vdots \qquad & \qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + g_n(t).
\end{aligned}$$

This can be written in the matrix form as:

$$\frac{d\vec{X}}{dt} = A(t)\vec{X} + \vec{g}(t),$$

where

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad A(t) = (a_{ij}(t))\,;\ 1 \le i \le n,\ 1 \le j \le n,$$

and

$$\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

If $\vec{g}(t) = \vec{0}$, then the system is called homogeneous; otherwise, it is called non-homogeneous.

1.1 Constant Coefficient Systems

If the coefficient matrix is a constant matrix i.e. $A(t) = A$, then the system

$$\frac{d\vec{X}}{dt} = A\vec{X} + \vec{g}(t),$$

where $A \in M_{n\times n}(\mathbb{R})$ is called a constant coefficient system.

Theorem 1. *The solution space of the homogeneous system* $\frac{d\vec{X}}{dt} = A\vec{X}$ *is a vector space of dimension* n*.*

Remark 1. *From the above theorem, we see that in order to find the general solution of* $\frac{d\vec{X}}{dt} = A\vec{X}$*, we need to find* n *linearly independent solutions.*

How to find a non-zero solution of $\frac{d\vec{X}}{dt} = A\vec{X}$?
For $n = 1$, $\frac{dx}{dt} = ax$ has $x(t) = e^{at}$ as a solution. In general, let's assume $\vec{X}(t) = e^{\lambda t}\vec{v}$ is a solution for some $\lambda \in \mathbb{R}$ & $\vec{v} = (v_1, v_2, \ldots, v_n)^T$. So,

$$\begin{aligned} &\vec{X}(t) = e^{\lambda t}\vec{v} \\ \iff &\frac{d\vec{X}}{dt} = \lambda e^{\lambda t}\vec{v} \\ \iff &A\vec{X} = \lambda e^{\lambda t}\vec{v} \\ \iff &Ae^{\lambda t}\vec{v} = \lambda e^{\lambda t}\vec{v} \\ \iff &A\vec{v} = \lambda\vec{v}. \end{aligned}$$

Conclusion: If $\exists\ \lambda \in \mathbb{R}$ and $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$, then $\vec{X}(t) = e^{\lambda t}\vec{v}$ is a solution of $\frac{d\vec{X}}{dt} = A\vec{X}$. Since, we are looking for non-zero solutions, we need $A\vec{v} = \lambda\vec{v}$ for some non-zero vector $\vec{v}$, i.e. $\vec{v}$ is an eigenvector of the matrix A (with eigenvalue λ).

$\therefore$ If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\vec{X}(t) = e^{\lambda t}\vec{v}$ is a solution of the system $\frac{d\vec{X}}{dt} = A\vec{X}$.

Case 1: If the matrix A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we get n linearly independent solutions given by $\vec{X}_i(t) = e^{\lambda_i t}\vec{v}_i,\ i = 1, 2, \ldots, n$. Hence, the general solution of $\vec{X}' = A\vec{X}$ is given by

$$\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants. This is the case when A is diagonalizable over $\mathbb{R}$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_n$ need not be distinct.

Example 1. Solve the following system

$$x_1' = x_1 + x_2$$
$$x_2' = 2x_2.$$

Solution Let $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\vec{X}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \vec{X}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, then

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

$\therefore$ $\lambda_1 = 1$ & $\lambda_2 = 2$ are the eigenvalues of A. Now, for the eigenvector corresponding to eigenvalue 1, consider

$$\begin{aligned} (I - A) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 0 \\ \iff \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 0 \\ \iff v_2 &= 0 \end{aligned}$$

$\therefore$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$. So,

$$\vec{X}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution.

Similarly, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$. So,

$$\vec{X}_2(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Hence, the general solution is

$$\begin{aligned}\vec{X}(t) &= c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{2t} \\ c_2 e^{2t} \end{pmatrix}.\end{aligned}$$

$$\therefore \; x_1(t) = c_1 e^t + c_2 e^{2t}; \; x_2(t) = c_2 e^{2t}.$$

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Case 2: Suppose the real matrix A has complex eigenvalues $\lambda = \alpha \pm \iota\beta$ with $\beta \neq 0$. In this case the complex eigenvectors can be written as $\vec{u} \pm \iota\vec{v}$, where $\vec{u}$ and $\vec{v}$ are real column vectors. Then

$$e^{(\alpha+\iota\beta)t}(\vec{u} + \iota\vec{v}) \text{ and } e^{(\alpha-\iota\beta)t}(\vec{u} - \iota\vec{v})$$

are two linearly independent complex valued solutions. To get the real-valued solutions, we take the real and imaginary parts of the above solutions. Since,

$$\begin{aligned}e^{(\alpha+\iota\beta)t}(\vec{u} + \iota\vec{v}) &= e^{\alpha t}.e^{(\iota\beta)t}(\vec{u} + \iota\vec{v}) \\ &= e^{\alpha t}(\cos\beta t + \iota\sin\beta t)(\vec{u} + \iota\vec{v}) \\ &= e^{\alpha t}((\cos\beta t)\vec{u} - (\sin\beta t)\vec{v}) + \iota e^{\alpha t}((\cos\beta t)\vec{v} + (\sin\beta t)\vec{u})\end{aligned}$$

$\therefore$ The general solution is given by

$$\vec{X}(t) = c_1 e^{\alpha t}[(\cos\beta t)\vec{u} - (\sin\beta t)\vec{v}] + c_2 e^{\alpha t}[(\cos\beta t)\vec{v} + (\sin\beta t)\vec{u}].$$

Example 2. Solve $\vec{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{X}$.

Solution Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1,$$

so, $\lambda = \pm\iota$ are the eigenvalues and check that $\begin{pmatrix} 1 \\ \iota \end{pmatrix}$ is an eigenvector corresponding to $\lambda = \iota$.

$\therefore\ e^{it}\begin{pmatrix}1\\ \iota\end{pmatrix}$ is a complex-valued solution. Since,

$$\begin{aligned} e^{it}\begin{pmatrix}1\\ \iota\end{pmatrix} &= (\cos t + \iota\sin t)\begin{pmatrix}1\\ \iota\end{pmatrix}\\ &= \begin{pmatrix}\cos t + \iota\sin t\\ -\sin t + \iota\cos t\end{pmatrix}\\ &= \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + \iota\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\end{aligned}$$

$\therefore$ The general solution is given by

$$\vec{X}(t) = c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

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Remark 2. *Case 2 is applicable if the matrix A is not diagonalizable over $\mathbb{R}$ but it is diagonalizable over $\mathbb{C}$.*

Case 3: A is not diagonalizable over $\mathbb{C}$.

Example 3. Solve $\vec{X}' = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\vec{X}$.

Solution Here, $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 and geometric multiplicity 1, so, it has only one linearly independent eigenvector and that is $\begin{pmatrix}1\\ 0\end{pmatrix}$. So,

$$\vec{X}_1(t) = e^t\begin{pmatrix}1\\ 0\end{pmatrix}$$

is a solution. Assume another solution to be of the form

$$\vec{X}_2(t) = te^t\begin{pmatrix}1\\ 0\end{pmatrix} + e^t\vec{u},$$

where $\vec{u}$ is to be determined. Now,

$$\vec{X}_2'(t) = (te^t + e^t)\begin{pmatrix}1\\ 0\end{pmatrix} + e^t\vec{u}.$$

$$\because\ \vec{X}_2'(t) = A\vec{X}_2(t)$$

$$
\begin{aligned}
&\iff (te^t + e^t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\vec{u} = A\left[te^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\vec{u}\right] \\
&\iff (te^t + e^t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\vec{u} = te^t A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t A\vec{u} \\
&\iff (te^t + e^t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\vec{u} = te^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t A\vec{u} \\
&\iff e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^t A\vec{u} - e^t\vec{u} \\
&\iff \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\vec{u} - \vec{u} \\
&\iff (A - I)\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\iff u_2 = 1
\end{aligned}
$$

$\therefore\ \vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a choice.

$$\therefore \quad \vec{X}_2(t) = te^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$\therefore$ The general solution is

$$
\begin{aligned}
\vec{X}(t) &= c_1\vec{X}_1(t) + c_2\vec{X}_2(t) \\
&= c_1 e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\left[te^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].
\end{aligned}
$$

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In general, if A has an eigenvalue λ with algebraic multiplicity 2 but geometric multiplicity 1, then $\vec{X}_1(t) = e^{\lambda t}\vec{v}$, $\vec{v}$ is an eigenvector for λ, and $\vec{X}_2(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}$, where $\vec{u}$ satisfies $(A - \lambda I)\vec{u} = \vec{v}$.

2 Solving Non-Homogeneous Systems

Consider the system

$$\vec{X}' = A\vec{X} + \vec{g}(t),$$

where $A \in M_{n\times n}(\mathbb{R})$, $\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$, and $\vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$.

Theorem 2. *The general solution of the non-homogeneous system*

$$\vec{X}' = A\vec{X} + \vec{g}(t) \tag{$*$}$$

is given by

$$\vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t),$$

where $\vec{X}_h(t)$ is the general solution of $\vec{X}' = A\vec{X}$ and $\vec{X}_p(t)$ is a particular solution of $(*)$.

Proof. If $\vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t)$, then

$$\begin{aligned} \vec{X}'(t) &= \vec{X}_h'(t) + \vec{X}_p'(t) \\ &= A\vec{X}_h(t) + A\vec{X}_p(t) + \vec{g}(t) \\ &= A\left(\vec{X}_h(t) + \vec{X}_p(t)\right) + \vec{g}(t) \\ &= A\vec{X}(t) + \vec{g}(t) \end{aligned}$$

$\therefore\ \vec{X}(t)$ is a solution of $(\ *\ )$.

Also, if $\vec{X}(t)$ is any solution of $(*)$ then $\vec{X}(t) - \vec{X}_p(t)$ is a solution of the corresponding homogeneous system $\vec{X}' = A\vec{X}$.

$$\begin{aligned} \therefore\quad & \vec{X}(t) - \vec{X}_p(t) = \vec{X}_h(t) \\ \Longrightarrow\ & \vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t). \end{aligned}$$

$\square$

Fundamental Matrix

For the system $\vec{X}' = A\vec{X}$, a "fundamental matrix" $\tilde{X}(t)$ is an $n \times n$ matrix whose columns form a basis for the solution space of $\vec{X}' = A\vec{X}$ (i.e. the columns of $\tilde{X}(t)$ are n linearly independent solutions).

For example, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are n L.I. eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\vec{X}_i(t) = e^{\lambda_i t}\vec{v}_i$, $1 \le i \le n$ are n L.I. solutions of $\vec{X}' = A\vec{X}$.

$$\therefore\quad \tilde{X}(t) = \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix}$$

is a fundamental matrix.

Properties of fundamental matrix:

1. The matrix $\tilde{X}(t)$ is invertible (because the columns of $\tilde{X}(t)$ are L.I.).

2. $\tilde{X}(t)$ satisfies the matrix equation $\tilde{X}'(t) = A\tilde{X}(t)$.

   Proof. Let $\vec{X}_i(t)$ be the i^{th} column of $\tilde{X}(t)$ for $i = 1, 2, \ldots, n$. Then $\vec{X}_i'(t) = A\vec{X}_i(t)$. Also,

$$\begin{aligned}
\tilde{X}(t) &= \begin{pmatrix} \vec{X}_1(t) & \vec{X}_2(t) & \cdots & \vec{X}_n(t) \end{pmatrix} \\
\implies \tilde{X}'(t) &= \begin{pmatrix} \vec{X}_1'(t) & \vec{X}_2'(t) & \cdots & \vec{X}_n'(t) \end{pmatrix} \\
&= \begin{pmatrix} A\vec{X}_1(t) & A\vec{X}_2(t) & \cdots & A\vec{X}_n(t) \end{pmatrix} \\
&= A\tilde{X}(t)
\end{aligned}$$

   $\square$

3. The general solution of $\vec{X}' = A\vec{X}$ can be written as

$$\vec{X}(t) = \tilde{X}(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

   where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

   Proof. Since $\vec{X}_1(t), \vec{X}_2(t), \ldots, \vec{X}_n(t)$ are n L.I. Solutions of $\vec{X}' = A\vec{X}$, the general solution is given by

$$\begin{aligned}
\vec{X}(t) &= c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + \cdots + c_n\vec{X}_n(t) \\
&= \begin{pmatrix} \vec{X}_1(t) & \vec{X}_2(t) & \cdots & \vec{X}_n(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\
&= \tilde{X}(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.
\end{aligned}$$

   $\square$

3 Variation of Parameters Method for Non-Homogeneous Systems

Consider the non-homogeneous system

$$\vec{X}' = A\vec{X} + \vec{g}(t).$$

We know that the general solution to the corresponding homogeneous system
$\vec{X}' = A\vec{X}$ is given by

$$\vec{X}_h(t) = \tilde{X}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

where $\tilde{X}(t)$ is a fundamental matrix for the system.
In the variation of parameters method, we replace $c_1, c_2, \ldots, c_n$ by $u_1(t), u_2(t), \ldots, u_n(t)$ to get a particular solution $\vec{X}_p(t)$. So, assume

$$\vec{X}_p(t) = \tilde{X}(t)\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

$$i.e. \quad \vec{X}_p(t) = \tilde{X}(t)\vec{u}(t),$$

where $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ needs to be determined. Now,

$$\begin{aligned} \vec{X}_p'(t) &= \tilde{X}'(t)\vec{u}(t) + \tilde{X}(t)\vec{u}'(t) \\ &= A\tilde{X}(t)\vec{u}(t) + \tilde{X}(t)\vec{u}'(t) \end{aligned}$$

Since $\vec{X}_p(t)$ is a solution of $(*)$, we get

$$\begin{aligned} &\vec{X}_p'(t) = A\vec{X}_p(t) + \vec{g}(t) \\ \Longrightarrow\ &A\tilde{X}(t)\vec{u}(t) + \tilde{X}(t)\vec{u}'(t) = A\tilde{X}(t)\vec{u}(t) + \vec{g}(t) \\ \Longrightarrow\ &\vec{u}'(t) = (\tilde{X}(t))^{-1}\vec{g}(t). \end{aligned}$$

Integrate this to get $\vec{u}(t)$.

Summary: To solve $\vec{X}' = A\vec{X} + \vec{g}(t)$:

Step 1: Find n L.I. solutions $\vec{X}_1(t), \vec{X}_2(t), \ldots, \vec{X}_n(t)$ for the homogeneous system $\vec{X}' = A\vec{X}$. Write $\tilde{X}(t) = \begin{pmatrix} \vec{X}_1(t) & \vec{X}_2(t) & \cdots & \vec{X}_n(t) \end{pmatrix}$.

Step 2: $\vec{X}_p(t) = \tilde{X}(t)\vec{u}(t)$, where $\vec{u}(t)$ is given by $\vec{u}'(t) = (\tilde{X}(t))^{-1}\vec{g}(t)$. So, find $(\tilde{X}(t))^{-1}\vec{g}(t)$.

Step 3: Integrate to get $\vec{u}(t)$.

Step 4: The general solution is given by

$$\vec{X}(t) = \tilde{X}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \tilde{X}(t)\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

Example 4. Solve the system:

$$\begin{aligned} x_1' &= x_2 - 5\sin t \\ x_2' &= -4x_1 + 17\cos t. \end{aligned}$$

Solution This is equivalent to

$$\vec{X}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\vec{X} + \begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix},$$

where $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

So,

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \quad \vec{g}(t) = \begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix}.$$

Since,

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 \\ 4 & \lambda \end{pmatrix} = \lambda^2 + 4,$$

So, the eigenvalues are $\pm 2\imath$ and the corresponding eigenvectors:

$$\text{For } \lambda = 2\imath: \quad \begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies 2\iota u - v = 0$$

So, $\vec{v} = \begin{pmatrix} 1 \\ 2\iota \end{pmatrix}$ is an eigenvector.

Now,

$$\begin{aligned}
e^{\lambda t}\vec{v} &= e^{2it}\begin{pmatrix} 1 \\ 2\iota \end{pmatrix} \\
&= (\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 2\iota \end{pmatrix} \\
&= \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}
\end{aligned}$$

$$\therefore \quad \vec{X}_1(t) = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} \ \& \ \vec{X}_2(t) = \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

$$\therefore \quad \tilde{X}(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix}$$

$$\implies (\tilde{X}(t))^{-1} = \frac{1}{2}\begin{pmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}$$

$$\therefore \quad \vec{u}'(t) = (\tilde{X}(t))^{-1}\vec{g}(t)$$

$$\begin{aligned}
\implies \vec{u}'(t) &= \frac{1}{2}\begin{pmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -10\cos 2t\sin t - 17\sin 2t\cos t \\ -10\sin 2t\sin t - 5\cos 2t\sin t \end{pmatrix}
\end{aligned}$$

i.e.

$$u_1'(t) = -5\cos 2t\sin t - \frac{17}{2}\sin 2t\cos t$$

and

$$u_2'(t) = -5\sin 2t\sin t - \frac{5}{2}\cos 2t\sin t.$$

Integrate these to get $u_1(t)$ and $u_2(t)$.

After this, the general solution is

$$\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + u_1(t)\vec{X}_1(t) + u_2(t)\vec{X}_2(t).$$

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