Solving a system of linear equations

Elementary operations:

There are 3 types.

(i) Interchanging any two equations.

(ii) Multiply a nonzero constant throughout an equation.

(iii) Replace an equation by itself plus a constant multiple of another equation

Defn: (Equivalent linear systems)
Two linear systems are said to be equivalent if one can be obtained from the other by a finite number of elementary operations.

Thm: Two equivalent linear systems have the same set of solutions.

Consider the system $Ax = b$ and consider the augmented matrix

$$(A \mid b)$$

Defn: (Elementary Row operations)

The following three operations are called elementary row operations.

(i) Interchanging any two rows (denoted as $R_i \leftrightarrow R_j$)

interchanging ith & jth rows.

(ii) Multiply a row by a nonzero constant k ($R_i \to kR_i$)

(iii) Adding a multiple of a row to another row ($R_i \to R_i + \lambda R_j$)

Defn: (Row equivalent matrices)

Two matrices of the same size are said to be row equivalent if one can be obtained from the other by applying a finite no. of elem. row opers.

Example:

$$\begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 5 & 7 & 3 \\ 3 & 0 & 1 & 9 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 2 & 5 & 7 & 3 \\ 0 & 2 & 3 & 4 \\ 3 & 0 & 1 & 9 \end{pmatrix}$$

$$\underset{R_1 \to \frac{1}{2}R_1}{\sim} \begin{pmatrix} 1 & 5/2 & 7/2 & 7/2 \\ 0 & 2 & 3 & 4 \\ 3 & 0 & 1 & 9 \end{pmatrix} \underset{R_3 \to R_3 - 3R_1}{\sim} \begin{pmatrix} 1 & 5/2 & 7/2 & 3/2 \\ 0 & 2 & 3 & 4 \\ 0 & -\frac{15}{2} & -\frac{19}{2} & \frac{9}{2} \end{pmatrix}$$

Remark: If $(A|b)$ is the augmented matrix for the system $AX=b$, and $(A'|b')$ is another matrix which is row equivalent to $(A|b)$. Then the systems $A'X=b'$ and $AX=b$ have the same set of solutions.

Defn: (Row echelon form of a matrix)
A matrix is in row echelon form if it has the following three properties:

(i) All nonzero rows are above any zero row.

(ii) Each leading entry of a row (ie. the leftmost nonzero entry) is in a column to the right of the leading entry of the row above it.

(iii) All entries in a column below a leading entry are zero.

Defn. (Row reduced echelon form or RRE form)
If a matrix in row echelon form satisfies the following additional conditions, then it is said to be in RRE form:
(i) The leading entry in each nonzero row is 1.
(ii) Each leading entry 1 is the only nonzero entry in its column.

Example Apply elem row operations to transform the following matrix into its RRE form:

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{pmatrix}$$

Soln:

$$A \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{pmatrix} 3 & -9 & 12 & -9 \\ 3 & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_1]{} \begin{pmatrix} 3 & -9 & 12 & -9 \\ 0 & 2 & -4 & 4 \\ 0 & 3 & -6 & 6 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - \frac{3}{2}R_2]{} \begin{pmatrix} 3 & -9 & 12 & -9 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to \frac{1}{3}R_1} \begin{pmatrix} 1 & -3 & 4 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 + 3R_2]{} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in RRE form.