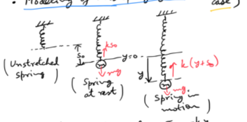
Applications of second order linear ODEs with constant coefficients

- Modeling of mass-spring system (undamped case)



Hooke's law: Spring force, $F_1 = -ky$,
where $k>0$ is the spring constant

At equilibrium: $\boxed{mg - ks_0 = 0}$

Newton's second law:
Force = mass × acceleration

$\Rightarrow mg - k(y+s_0) = my''$

$\Rightarrow \boxed{my'' + ky = 0}$ $(\because mg - ks_0 = 0)$

↳ Homogeneous linear ODE with constant coefficients.

Solution:

Char. eqn: $m\lambda^2 + k = 0$

$\Rightarrow \lambda^2 = -\frac{k}{m} < 0$

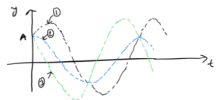
$\Rightarrow \lambda = \pm i\omega_0$,

where $\omega_0 = \sqrt{\frac{k}{m}}$

$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$
$= C\cos(\omega_0 t - \delta)$,

$C = \sqrt{A^2+B^2} \rightarrow$ amplitude

$\tan\delta = \frac{B}{A}$ $\delta \rightarrow$ phase angle.



$y(0) = A$: initial position

$y'(0) = \omega_0 B$: initial velocity $\rightarrow >0$ in (1), $=0$ in (2), <0 in (3)

Damped system

We add a damping force
$F_2 = -cy'$.

$my'' = -ky - cy'$

$\Rightarrow \boxed{my'' + cy' + ky = 0}$ Homog. linear with constant coefficients.

Char. eqn. $m\lambda^2 + c\lambda + k = 0$

$\Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

$= -\frac{c}{2m} \pm \frac{\sqrt{c^2-4mk}}{2m}$

$= -\alpha \pm \beta$.

Here $\alpha > 0$, β may be real or imaginary.

Case I (Overdamped case)

$\boxed{c^2 > 4mk}$

Then $\lambda = -\alpha \pm \beta$, $\beta > 0$

$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$

Also, $\beta^2 = \frac{c^2 - 4mk}{4m^2} = \alpha^2 - \frac{k}{m} < \alpha^2$

$\Rightarrow \beta < \alpha \Rightarrow \alpha - \beta > 0$

$\therefore$ $y(t) \rightarrow 0$ as $t \rightarrow \infty$

Case II (Critically damped)

$\boxed{c^2 = 4mk}$

$\lambda = -\alpha, -\alpha$

$y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$
$= (c_1 + c_2 t) e^{-\alpha t}$

$y(t) = 0$ for at most one $t > 0$

Case III (Underdamped case)

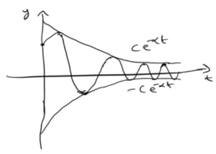
$\boxed{c^2 < 4mk}$

$\beta = i\omega^*$, $\omega^* = \frac{\sqrt{4mk - c^2}}{2m} > 0$
$= \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$

$\lambda = -\alpha \pm i\omega^*$

$y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$
$= Ce^{-\alpha t}\cos(\omega^* t - \delta)$,

$C = \sqrt{A^2+B^2}$



Forced Oscillation

We add an external force $F(t)$:

$my'' + cy' + ky = F(t)$

→ nonhomogeneous 2nd order linear ODE.

We'll assume $F(t) = F_0 \cos\omega t$

Case I (Undamped): $c = 0$

$my'' + ky = F_0 \cos\omega t$

$y_h = C\cos(\omega_0 t - \delta)$

We use method of undetermined coeffs. ~~to~~ find a particular soln:

$\boxed{\omega \neq \omega_0}$: $y_p = a\cos\omega t + b\sin\omega t$

$y_p' = -\omega a\sin\omega t + \omega b\cos\omega t$

$y_p'' = -\omega^2 a\cos\omega t - \omega^2 b\sin\omega t$

We get

$(k - m\omega^2) a\cos\omega t + (k - m\omega^2) b\sin\omega t = F_0 \cos\omega t$

$\Rightarrow b = 0$, $a = \frac{F_0}{k - m\omega^2}$

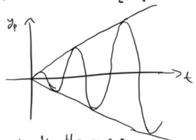
$$\boxed{y_p(t) = \frac{F_0}{k - m\omega^2}\cos\omega t}$$

$\boxed{\omega = \omega_0}$ $y_p = t(a\cos\omega_0 t + b\sin\omega_0 t)$

Solving, we get

$y_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$

$y(t) = C\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$



... the phenomenon