Examples of limits of some sequences:

① $\lim_{n\to\infty} n^{1/n} = 1$

Let $a_n = n^{1/n}$. Then $\ln a_n = \frac{1}{n}\ln n$

We know that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$$\Rightarrow \lim_{n\to\infty} \ln a_n = 0$$

Since $a_n = e^{\ln a_n}$,

$\lim_{n\to\infty} a_n = e^0 = 1$.

Theorem: Suppose $a_n \to L$ as $n\to\infty$ and let f be a function which is continuous at L. Then $f(a_n) \to f(L)$ as $n\to\infty$.

Proof: Let $\varepsilon > 0$ be given.

We want to find $N \in \mathbb{N}$ s.t.

$|f(a_n) - f(L)| < \varepsilon \quad \forall n > N$.

Since f is continuous at L, $\exists \delta > 0$ s.t. $|f(x) - f(L)| < \varepsilon$ whenever $|x - L| < \delta$

Since $a_n \to L$, $\exists N \in \mathbb{N}$ s.t.

$|a_n - L| < \delta \quad \forall n > N$

$\Rightarrow \quad |f(a_n) - f(L)| < \varepsilon \quad \forall n > N.$

② $\lim_{n\to\infty} x^{1/n} = 1 \quad \text{if } x > 0.$

Let $\quad b_n = \frac{1}{n}\ln x \to 0 \quad \text{as } n \to \infty$

$\Rightarrow \quad \underbrace{e^{b_n}}_{= x^{1/n}} \to e^0 = 1 \quad \text{as } n \to \infty$

③ $$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } x = 1 \\ \text{DNE} & \text{if } x = -1 \\ +\infty & \text{if } x > 1 \\ \text{DNE} & \text{if } x < -1. \end{cases}$$

DNE = does not exist.

④ $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$ for any number a.

Let $a_n = \left(1+\frac{a}{n}\right)^n$

Then $\ln a_n = n \ln\left(1+\frac{a}{n}\right)$

$= \dfrac{\ln(1+a/n)}{1/n}$ $\left[\frac{0}{0} \text{ form}\right]$

$\therefore \lim_{n\to\infty} \ln a_n \overset{\text{L'H rule}}{=} \lim_{n\to\infty} \dfrac{\frac{1}{1+\frac{a}{n}}\cdot\left(\cancel{-}\frac{a}{n^2}\right)}{\cancel{-}1/n^2} = a$

$\Rightarrow \lim_{n\to\infty} a_n = e^a$.

Recursive defn. of sequence

Example: ① $a_{n+1} = \frac{1}{2} a_n$ for $n \geq 1$
$a_1 = 1$.

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{2^2}$, $a_4 = \frac{1}{2^3}$,

$\ldots, a_n = \frac{1}{2^{n-1}}, \ldots$

② $a_1 = 1, a_2 = 1, a_{n+1} = a_n + a_{n-1}, n \geq 2$

$a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8,$
$a_7 = 13, a_8 = 21, \ldots$

This is known as the "Fibonacci sequence".

Monotonic sequences :

A sequence $\{a_n\}$ is called a "nondecreasing sequence" if $a_n \le a_{n+1}$ $\forall n \in \mathbb{N}$.
(i.e. $a_1 \le a_2 \le a_3 \le a_4 \le \cdots$)

A sequence $\{a_n\}$ is called "nonincreasing" if $a_n \ge a_{n+1}$ $\forall n \in \mathbb{N}$
(i.e. $a_1 \ge a_2 \ge a_3 \ge \cdots$)

$\{a_n\}$ is called a monotonic sequence if it is either nondecreasing or nonincreasing.

e.g.: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is nonincreasing (decreasing)

$\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ is increasing seq.

$\{n^2\}$ is increasing.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotonic.

$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ is not monotonic.