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Contents

Second-Order Linear ODEs

A second order ODE is called **linear** if it can be written as

$$y'' + p(t)y' + q(t)y = r(t). \tag{0.1}$$

It is called **homogeneous** if $r(t) = 0$, and **nonhomogeneous** otherwise. We shall assume the following important theorem about linear ODEs without proof.

Theorem 1 (Existence and Uniqueness Theorem). *Let $p(t)$, $q(t)$ and $r(t)$ be continuous functions on some open interval I. Let $t_0 \in I$. Then for any numbers y_0 and y_0', equation (0.1) with initial conditions $y(t_0) = y_0, y'(t_0) = y_0'$ has a unique solution on the interval I.*

Proof. Beyond the scope of this course. □

Remark 1. *The above theorem can be extended to higher order linear ODEs as well. Write the corresponding IVP and the existence-uniqueness theorem yourself.*

1 Homogeneous Second-Order Linear ODEs

Consider a general homogeneous second-order linear ODE of the form

$$y'' + p(t)y' + q(t)y = 0. \tag{1.1}$$

Theorem 2. *The set of all solutions of a homogeneous second-order linear ODE is a vector space.*

Proof. It is easy to verify that if y_1 and y_2 are any two solutions of (1.1) then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2. □

Remark 2. *Note that the above theorem is not valid for nonlinear ODEs or nonhomogeneous linear ODEs.*

Question: What is the dimension of the solution space of homogeneous second-order linear ODE?

Recall the definition of linear dependence and independence. First we claim that if $p(t)$ and $q(t)$ are continuous on an open interval I then there are at least two solutions of (1.1) which are linearly independent on I. To prove the claim let us take some $t_0 \in I$, and let $y_1(t)$ and $y_2(t)$ be the unique solutions of (1.1) satisfying the initial conditions $y_1(t_0) = 1, y_1'(t_0) = 0$ and $y_2(t_0) = 0, y_2'(t_0) = 1$, respectively. (Note that such $y_1(t)$ and $y_2(t)$ exist by Theorem 1.) It is easy to see in this case that $y_1(t)$ and $y_2(t)$ are linearly independent on I. Hence the dimension of the solution space is at least two.

We would show that the dimension is, in fact, equal to two. Let $y(t)$ be any solution of (1.1). Consider $Y(t) = y(t_0)\ y_1(t) + y'(t_0)\ y_2(t)$, where $y_1(t)$ and $y_2(t)$ are as defined above. Since $Y(t)$ is a linear combination of solutions $y_1(t)$ and $y_2(t)$, $Y(t)$ is also a solution. Furthermore, $Y(t_0) = y(t_0)$ and $Y'(t_0) = y'(t_0)$. By the uniqueness, $Y(t) = y(t)$ for all $t \in I$, and hence $y(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$. This proves that the dimension is equal to two. Thus we have obtained the following theorem.

Theorem 3. *If $p(t)$ and $q(t)$ are continuous on an open interval I, then the solution space of $y'' + p(t)y' + q(t)y = 0$ is two-dimensional.*

Exercise: Generalize the previous theorem to n-th order homogeneous linear ODEs with continuous coefficients.

Remark 3. *Note that the theorem is not true in general without the continuity assumptions.*

Summary: In order to find all solutions of a homogeneous second-order linear ODE of the form (1.1) with continuous coefficients it is enough to find any pair of linearly independent solutions.

Question: How to check whether two functions are linearly independent or not?

Suppose that $f_1(t), f_2(t), \ldots, f_n(t)$ are functions defined on an open interval I. Assume that $f_1(t), f_2(t), \ldots, f_n(t)$ are linearly dependent. Then there exist constants $c_1, c_2, \ldots, c_n$ not all zeros, such that $c_1 f_1(t) + c_2 f_2(t) +$

$\cdots + c_n f_n(t) = 0$ for all $t \in I$. If we assume that $f_1, f_2, \ldots, f_n$ are $(n-1)$ times differentiable on I, then we have

$$\begin{array}{ccccccccc}
c_1 f_1(t) & + & c_2 f_2(t) & + & \cdots & + & c_n f_n(t) & = & 0 \\
c_1 f_1'(t) & + & c_2 f_2'(t) & + & \cdots & + & c_n f_n'(t) & = & 0 \\
 & & & & \vdots & & & & \\
c_1 f_1^{(n-1)}(t) & + & c_2 f_2^{(n-1)}(t) & + & \cdots & + & c_n f_n^{(n-1)}(t) & = & 0
\end{array}$$

for all $t \in I$. Since the above homogeneous system has a nonzero solution $(c_1, c_2, \ldots, c_n)$, the determinant of the coefficient matrix must be zero for every $t \in I$. This motivates the following definition of the Wronskian function.

Definition 1 (Wronskian). *For n real-valued functions $f_1, f_2, \ldots, f_n$, which are $(n-1)$ times differentiable on an open interval I, the Wronskian $W(f_1, f_2, \ldots, f_n)$ as a function on I is defined by*

$$W(f_1, f_2, \ldots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}, \quad t \in I. \tag{1.2}$$

Now we have the following theorem.

Theorem 4. *If $(n-1)$ times differentiable functions $f_1, f_2, \ldots, f_n$ on an open interval I are linearly dependent on I, then $W(f_1, f_2, \ldots, f_n)(t) = 0$ for all $t \in I$. As a consecuence, $W(f_1, f_2, \ldots, f_n)(t_0) \neq 0$ for some $t_0 \in I$ implies that $f_1, f_2, \ldots, f_n$ are linearly independent on I.*

Remark 4. *(a) The converse of the above theorem is not true. For example, let $f_1(t) = t|t|$ and $f_2(t) = t^2$ on $t \in I = (-1, 1)$. Note that $W(f_1, f_2)(t) = 0$ for all $t \in I$ even though f_1 and f_2 are linearly independent on I.*
(b) It is possible that the Wronskian is zero at some points and nonzero at other points on an interval I. For instance, take $f_1(x) = x$ and $f_2(x) = x^2$.

We are interested in finding linearly independent solutions of second (or higher) order homogeneous linear ODEs. If we can somehow find two solutions y_1 and y_2 such that $W(y_1, y_2)(t) \neq 0$ for some $t \in I$, then y_1 and y_2 are linearly independent and hence form a basis for the solution space (for second-order). The following theorem is important for two reasons. Firstly, it tells that the Wronskian of two solutions is either identically zero on I or is never zero on I. Secondly, it gives the Wronskian (up to a constant multiple) without knowing the solutions.

Theorem 5 (Abel's Theorem). *If $y_1(t)$ and $y_2(t)$ are two solutions of $y'' + p(t)y' + q(t)y = 0$ with $p(t)$ and $q(t)$ continuous on an open interval I, then the Wronskian of y_1 and y_2 is given by*

$$W(y_1, y_2)(t) = c \exp\left(- \int_{t_0}^{t} p(t)dt \right),$$

for some constant c.

Proof. Let us write $W(t)$ for $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$.Then

$$\begin{aligned} W'(t) &= y_1(t)y_2''(t) - y_2(t)y_1''(t) \\ &= y_1(t)[-p(t)y_2'(t) - q(t)y_2(t)] - y_2(t)[-p(t)y_1'(t) - q(t)y_1(t)] \\ &= -p(t)W(t). \end{aligned}$$

Thus, $W(t) = c \exp\left(- \int_{t_0}^{t} p(t)dt \right)$ for some constant c. $\square$

Theorem 6. *Let $y_1(t)$ and $y_2(t)$ be two solutions of $y'' + p(t)y' + q(t)y = 0$ with $p(t)$ and $q(t)$ continuous on an open interval I, and let $t_0 \in I$. Then $W(y_1, y_2)(t_0) = 0$ implies that $y_1(t)$ and $y_2(t)$ are linearly dependent on I.*

Proof. $W(y_1, y_2)(t_0) = 0$ implies that there exist constant c_1 and c_2 not both zero, such that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0. \end{aligned}$$

We claim that $c_1 y_1(t) + c_2 y_2(t) = 0$ for all $t \in I$, which would imply $y_1(t)$ and $y_2(t)$ are linearly dependent on I. Consider the function $Y(t) = c_1 y_1(t) + c_2 y_2(t)$, $t \in I$. Then $Y(t)$ is a solution to the homogeneous ODE and satisfies the initial conditions $Y(t_0) = 0, Y'(t_0) = 0$. By uniqueness, $Y(t) \equiv 0$ on I. Hence we are done. $\square$

1.1 Method of Reduction of Order

Suppose we somehow know one nonzero solution $y_1(t)$ of $y'' + p(t)y' + q(t)y = 0$. Let us suppose that $y_2(t) = u(t)y_1(t)$ is another solution for an unknown function $u(t)$. Then $y_2'(t) = u'(t)y_1(t) + u(t)y_1'(t)$ and $y_2''(t) =$

$u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t)$. Substituting in the equation and using $y_1''(t)+p(t)y_1'(t)+q(t)y_1(t) = 0$, we get $u''(t)y_1(t)+u'(t)[2y_1'(t)+p(t)y_1(t)] = 0$. This is a first-order ODE in $u'(t)$ and can be solved, if $y_1(t)$ is never zero on I, to get

$$u'(t) = \frac{1}{(y_1(t))^2} \exp\left(-\int p(t)\, dt\right),$$

which can be integrated to get $u(t)$ and hence $y_2(t)$.

1.2 Homogeneous Linear ODEs with Constant Coefficients

Consider an ODE of the form $ay'' + by' + cy = 0$ with $a \neq 0$. Let us define $L(y) = ay'' + by' + cy$. In order to find the general solution of $L(y) = 0$ we need to somehow find two linearly independent solutions. It is not difficult to guess that a solution might be of the form e^{mt}. For $y = e^{mt}$, $L(y) = (am^2 + bm + c)e^{mt}$. Therefore, $y = e^{mt}$ is a solution for some m if and only if $am^2 + bm + c = 0$. We call $\boxed{am^2 + bm + c = 0}$ the *characteristic equation* corresponding to the ODE. Now there are three cases.

Case 1: The characteristic equation has two **real and distinct roots**, say m_1 and m_2. Then $y_1 = e^{m_1 t}$ and $y_2 = e^{m_2 t}$ are two linearly independent solutions. Thus, the general solution is given by

$$\boxed{y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.}$$

Case 2: The characteristic equation has **equal roots**, $m_1 = m_2 = -b/2a$. In this case $y_1(t) = e^{mt}$, where $m = -b/2a$. To find another solution $y_2(t)$ let us use the method of reduction of order. Assume $y_2(t) = u(t)e^{mt}$. Then $y_2'(t) = e^{mt}[u'(t) + mu(t)]$ and $y_2''(t) = e^{mt}[u''(t) + 2mu'(t) + m^2u(t)]$. Substituting and dividing by e^{mt}, we get

$$au''(t) + u'(t)[2am + b] + u(t)[am^2 + bm + c] = 0.$$

Using m is a repeated root, we have $am^2 + bm + c = 0$ and $2am + b = 0$. Therefore, $u''(t) = 0$ which is equivalent to $u(t) = c_1 t + c_2$. So $(c_1 t + c_2)e^{mt}$ is a solution for any c_1, c_2. In particular, $y_2(t) = te^{mt}$ is a solution. Thus, the general solution is given by

$$\boxed{y(t) = (c_1 + c_2 t)e^{mt}.}$$

Case 3: The characteristic equation has **complex conjugate roots**, say $m_1 = \lambda + i\mu$ and $m_2 = \lambda - i\mu$, where $\mu \neq 0$. In this case $e^{m_1 t}$ and $e^{m_2 t}$ are complex-valued solutions. To get real solutions, we take

$$y_1(t) = \frac{1}{2}(e^{m_1 t} + e^{m_2 t}) = e^{\lambda t}\cos(\mu t)$$

and

$$y_2(t) = \frac{1}{2i}(e^{m_1 t} - e^{m_2 t}) = e^{\lambda t}\sin(\mu t).$$

Since $\mu \neq 0$, $y_1(t)$ and $y_2(t)$ are linearly independent. Thus, the general solution is given by

$$\boxed{y(t) = e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)].}$$

1.3 Euler-Cauchy Equations

Consider an ODE of the form $at^2y'' + bty' + cy = 0$, $t > 0$ with $a \neq 0$. By looking at the form it is natural to guess a solution of the form $y = t^m$. We see that $y = t^m$, $t > 0$ is a solution if and only if $\boxed{am(m-1) + bm + c = 0}$. This is a quadratic equation and so there are three cases.

Case 1: The characteristic equation has two **real and distinct roots**, say m_1 and m_2. Then $y_1 = t^{m_1}$ and $y_2 = t^{m_2}$ are two linearly independent solutions. Thus, the general solution is given by

$$\boxed{y(t) = c_1t^{m_1} + c_2t^{m_2},\ t > 0.}$$

Case 2: The characteristic equation has **equal roots**, $m_1 = m_2 = m$. In this case $y_1(t) = t^m$ is a solution. To find another solution $y_2(t)$ let us use the method of reduction of order. Assume $y_2(t) = u(t)t^m$. Verify that in this case we can take $u(t) = \ln t$, so $y_2(t) = t^m \ln t$ is another solution. Thus, the general solution is given by

$$\boxed{y(t) = (c_1 + c_2\ln t)t^m,\ t > 0.}$$

Case 3: The characteristic equation has **complex conjugate roots**, say $m_1 = \lambda + i\mu$ and $m_2 = \lambda - i\mu$, where $\mu \neq 0$. In this case $t^{m_1} = t^\lambda e^{i\mu\ln t} = t^\lambda[\cos(\mu\ln t) + i\sin(\mu\ln t)]$ and $t^{m_2} = t^\lambda[\cos(\mu\ln t) - i\sin(\mu\ln t)]$ are complex-valued solutions. As in the constant coefficients case, the real and imaginary parts give two linearly independent real-valued solutions. Thus, the general solution is given by

$$\boxed{y(t) = t^\lambda[c_1\cos(\mu\ln t) + c_2\sin(\mu\ln t)],\ t > 0.}$$

Remark 5. *The Euler-Cauchy equation also has a general solution defined for $t < 0$, which is obtained by simply replacing t by $-t$ in the above formulas.*

Exercise 1. *Solve the IVP:* $4t^2y'' + 4ty' - y = 0, \quad y(-1) = 0, y'(-1) = 1.$

2 Nonhomogeneous Linear ODEs

Now we shall study nonhomogeneous linear ODEs of the form

$$y'' + p(t)y' + q(t)y = r(t), \quad t \in I, \tag{2.1}$$

where $p(t), q(t)$ and $r(t)$ are continuous real-valued functions on I.

Theorem 7. *The difference of any two solutions of the nonhomogeneous linear ODE* (2.1) *is a solution of the corresponding homogeneous linear ODE.*

Proof. Let $y(t) = y_1(t) - y_2(t)$ where y_1 and y_2 are solutions of (2.1). Then

$$\begin{aligned} y'' + p(t)y' + q(t)y &= [y_1'' + p(t)y_1' + q(t)y_1] - [y_2'' + p(t)y_2' + q(t)y_2] \\ &= r(t) - r(t) = 0. \end{aligned}$$

□

Now suppose we know a particular solution y_p of the nonhomogeneous equation. The existence (under the continuity assumption) of such a solution is guaranteed by the variation of parameters method which will be discussed later. For any solution y of the nonhomogeneous equation, the previous theorem says that $y - y_p$ is a solution of the corresponding homogeneous equation. Therefore, $y - y_p = c_1 y_1 + c_2 y_2$ for some constants c_1 and c_2 if y_1 and y_2 are a pair of linearly independent solutions of the homogeneous equation. This gives the following theorem.

Theorem 8. *The general solution of nonhomogeneous linear ODE is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where y_1 are y_2 are a pair of linearly independent solutions of the homogeneous part, and y_p is a particular solution of the actual nonhomogeneous equation.

Summary: In order to find all solutions of a second-order nonhomogeneous linear ODE we need to find two linearly independent solutions of the corresponding homogeneous equation, and also find just one solution of the nonhomogeneous equation. We have already learnt how to find two linearly independent solutions of homogeneous linear ODEs in some cases (constant coefficients and Euler-Cauchy equations, for instance). We will learn two methods to find a particular solution of nonhomogeneous equation. One is the **variation of parameters method** which is applicable in general, provided we know two linearly independent solutions of the corresponding homogeneous equation. The difficulty in this method is that one has to evaluate some indefinite integrals which might be cumbersome. The second method is the **method of undetermined coefficients** which is easier, whenever applicable. But, one must remember that the method of undetermined coefficients can be applied only when the homogeneous part is a constant coefficient equation and $r(t)$ is restricted to some special functions (polynomials, exponentials, sine or cosine, or a sum or product of these). These methods will be discussed in the lecture classes. Do enough examples to get used to these methods.

2.1 Variation of Parameters Method

Consider 2^{nd} order nonhomogeneous linear ODE of the form:

$$y'' + p(t)y' + q(t)y = r(t) \tag{1}$$

Suppose $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the corresponding homogeneous ODE:

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

Let $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ be a particular solution to equation (1) for some particular choice of functions $u_1(t)$ and $u_2(t)$. Then

$$y_p'(t) = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2').$$

Let's impose an extra condition

$$u_1'y_1 + u_2'y_2 = 0 \tag{3}$$

Then

$$\begin{aligned} y_p' &= u_1y_1' + u_2y_2' \\ \implies y_p'' &= u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' \end{aligned}$$

substituting y_p, y_p', y_p'' in (1), we get

$$\begin{aligned} &u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + p(t)[u_1y_1' + u_2y_2'] + q(t)[u_1y_1 + u_2y_2] = r(t) \\ \implies\ &u_1[y_1'' + p(t)y_1' + q(t)y_1] + u_2[y_2'' + p(t)y_2' + q(t)y_2] + u_1'y_1' + u_2'y_2' = r(t) \end{aligned}$$

$$\implies u_1' y_1' + u_2' y_2' = r(t) \tag{4}$$

(3) & (4) can be written as

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r(t) \end{pmatrix},$$

since

$$\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = W(y_1, y_2)(t) \neq 0,$$

we can find $u_1'(t)$ & $u_2'(t)$ uniquely as

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ r(t) & y_2'(t) \end{vmatrix}}{W(y_1, y_2)(t)} = -\frac{y_2(t) r(t)}{W(y_1, y_2)(t)}$$

and

$$u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & r(t) \end{vmatrix}}{W(y_1, y_2)(t)} = \frac{y_1(t) r(t)}{W(y_1, y_2)(t)}$$

Integrating the above expressions, we get $u_1(t)$ & $u_2(t)$ and hence

$$y_p(t) = -y_1(t) \int \frac{y_2(t) r(t)}{W(y_1, y_2)(t)} dt \ + y_2(t) \int \frac{y_1(t) r(t)}{W(y_1, y_2)(t)} dt.$$

Example 1. Solve $y'' + y = \sec t$.

Solution The corresponding homogeneous ODE is $y'' + y = 0$. Which has $y_1(t) = \cos t$ & $y_2(t) = \sin t$ as two linearly independent solutions. So,

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1,$$

and $r(t) = \sec t$.

$$\begin{aligned} \therefore \ y_p(t) &= -y_1(t) \int \frac{y_2(t) r(t)}{W(y_1, y_2)(t)} dt \ + y_2(t) \int \frac{y_1(t) r(t)}{W(y_1, y_2)(t)} dt \\ &= -\cos t \int \sin t \sec t \ dt \ + \sin t \int \cos t \sec t \ dt \\ &= \cos t \ln|\cos t| + t \sin t. \end{aligned}$$

$\therefore$ The general solution is

$$y = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t.$$

■

Remark 6. *Variation of parameters method can be applied if we know how to solve the corresponding homogeneous ODE, for example, if it is constant coefficient or Euler-Cauchy equation.*
For Euler-Cauchy equation: $at^2y'' + bty' + cy = g(t)$*, if we use the variation of parameters method,* $r(t) = \frac{g(t)}{at^2}$*, not* $g(t)$*.*

2.2 Method of Undetermined Coefficients

Example 2. Solve $y'' - y = e^{2t}$.

Solution The general solution of the corresponding homogeneous ODE is $y_h(t) = c_1e^t + c_2e^{-t}$. Let $y_p(t) = Ae^{2t}$ for some constant A. Then

$$y_p'(t) = 2Ae^{2t};\ y_p''(t) = 4Ae^{2t}$$

substituting y_p, y_p', y_p'', we get

$$4Ae^{2t} - Ae^{2t} = e^{2t}$$
$$\implies A = \frac{1}{3}$$

$\therefore\ y_p = \frac{1}{3}e^{2t}$ is a particular solution.

$$\therefore\ y = c_1e^t + c_2e^{-t} + \frac{1}{3}e^{2t}.$$

■

Example 3. Consider $y'' - y = e^t$. Here $y = Ae^t$ can't be a particular solution because it is a solution to the corresponding homogeneous equation. We try $y_p(t) = Ate^t$. Then

$$y_p' = A(te^t + e^t);\ y_p'' = A(te^t + 2e^t)$$

$$\therefore\ A[te^t + 2e^t - te^t] = e^t$$
$$\implies A = \frac{1}{2}$$

$\therefore\ y_p = \frac{1}{2}te^t$ is a particular solution and hence the general solution is

$$y = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t.$$

Remark 7. *This method is applicable when the homogeneous part is of constant coefficients linear ODE and the nonhomogeneous part is one of the following functions: exponential, polynomial, sine or cosine, sums or products of the above functions.*

Example 4. Solve $y'' - 2y' + y = e^t$.

Solution The characteristic equation of the corresponding homogeneous part is $m^2 - 2m + 1 = 0 \implies (m-1)^2 = 0$, $\therefore$ the general solution for the homogeneous part is

$$y_h(t) = c_1 e^t + c_2 t e^t.$$

Let $y_p(t) = At^2 e^t$ for some constant A. Then

$$\begin{aligned} y_p'(t) &= A(t^2 e^t + 2te^t) \\ y_p''(t) &= A(t^2 e^t + 4te^t + 2e^t) \\ \therefore\ Ae^t[(t^2 + 4t + 2) &- 2(t^2 + 2t) + t^2] = e^t \\ \implies A &= \frac{1}{2} \end{aligned}$$

$\therefore\ y_p(t) = \frac{1}{2}t^2 e^t$ and hence the general solution is

$$y = c_1 e^t + c_2 t e^t + \frac{1}{2}t^2 e^t.$$

$\blacksquare$

Example 5. Find a particular solution of $y'' - y = t^2$.

Solution Let $y_p(t) = At^2 + Bt + C$, then

$$y_p' = 2At + B;\ y_p'' = 2A$$

$$\begin{aligned} \therefore\ 2A - (At^2 + Bt + C) &= t^2 \\ \implies (2A - C) - Bt - (A+1)t^2 &= 0 \\ \implies 2A - C = 0,\ B = 0,\ A + 1 &= 0 \\ \implies A = -1,\ B = 0,\ C &= -2. \end{aligned}$$

$$\therefore\ y_p(t) = -t^2 - 2.$$

$\blacksquare$

Remark 8. ***(Modification rule for polynomial)*** *If $y = c$ (constant) is a solution of the homogeneous part but $y = t$ is not a solution, then we multiply the particular by t, i.e. if $r(t) = a_0 + a_1 t + \cdots + a_n t^n$ then $y_p(t) = t(b_0 + b_1 t + \cdots + b_n t^n)$ and then determine $b_0, b_1, \ldots, b_n$.*

Example 6. Find a particular solution of $y'' - y = \sin t$.

Solution The general solution of the corresponding homogeneous ODE is $y_h(t) = c_1 e^t + c_2 e^{-t}$. We take $y_p(t) = A\sin t + B\cos t$. Then

$$
\begin{aligned}
y_p' &= A\cos t - B\sin t \\
y_p'' &= -A\sin t - B\cos t \\
\therefore\ &- A\sin t - B\cos t - A\sin t - B\cos t = \sin t \\
\implies &- 2A\sin t - 2B\cos t = \sin t \\
\implies &A = -\frac{1}{2},\ B = 0.
\end{aligned}
$$

$$\therefore\ y_p(t) = -\frac{1}{2}\sin t.$$

■

Example 7. Find a particular solution of $y'' + y = \sin t$.

Solution The general solution of the corresponding homogeneous ODE is $y_h(t) = c_1 \sin t + c_2 \cos t$. Let $y_p(t) = t(A\sin t + B\cos t)$. Then

$$
\begin{aligned}
y_p' &= t(A\cos t - B\sin t) + A\sin t + B\cos t \\
y_p'' &= t(-A\sin t - B\cos t) + 2A\cos t - 2B\sin t \\
\therefore\ &t(-A\sin t - B\cos t) + 2(A\cos t - B\sin t) + t(A\sin t + B\cos t) = \sin t \\
\implies &A = 0,\ B = -\frac{1}{2}.
\end{aligned}
$$

$$\therefore\ y_p(t) = -\frac{1}{2}t\cos t.$$

■

Summary:

To solve a nonhomogeneous ODE with constant coefficients homogeneous part, we first solve the homogeneous ODE and then apply the following rule to find a particular solution:

$r(t)$	$y_p(t)$
$\beta e^{\alpha t}$	1. $Ae^{\alpha t}$ if $e^{\alpha t}$ is not a solution of the homogeneous part. 2. $Ate^{\alpha t}$ if $e^{\alpha t}$ is a solution but $te^{\alpha t}$ is not a solution of the homo. part. 3. $At^2e^{\alpha t}$ if both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homo. part.
$a_0 + a_1t + \cdots + a_nt^n$	1. $b_0 + b_1t + \cdots + b_nt^n$ if $y = c$ is not a solution of the homo. part. 2. $t(b_0 + b_1t + \cdots + b_nt^n)$ if $y = c$ is a solution but $y = t$ is not a solution. 3. $t^2(b_0 + b_1t + \cdots + b_nt^n)$ if both $y = c$ and $y = t$ are solutions.
$a\cos\alpha t + b\sin\alpha t$	1. $A\cos\alpha t + B\sin\alpha t$ if $\cos\alpha t$ is not a solution of the homo. part. 2. $t(A\cos\alpha t + B\sin\alpha t)$ if $\cos\alpha t$ is a solution of the homo. part.

Remark 9. ***(Sum Rule)*** *If $r(t)$ is a sum of functions of the above types, then we take particular solution as the sum of the corresponding functions.*

Example 8. Find a particular solution of $y'' - y = e^t \cos t$.

Solution The general solution of the corresponding homogeneous ODE is $y_h(t) = c_1e^t + c_2e^{-t}$. If we take $y_p(t) = e^t(A\cos t + B\sin t)$, then

$$\begin{aligned} y_p'(t) &= e^t(A\cos t + B\sin t) + e^t(-A\sin t + B\cos t) \\ &= e^t[(A+B)\cos t + (B-A)\sin t] \\ y_p''(t) &= e^t(2B\cos t - 2A\sin t) \end{aligned}$$

Now,

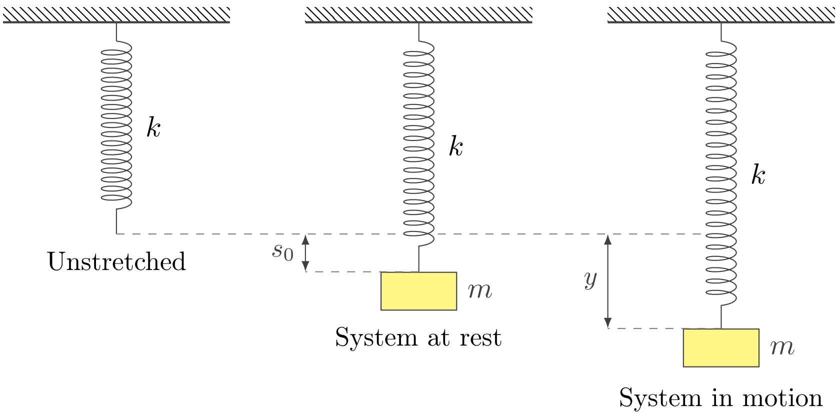
$$\begin{aligned} &y_p'' - y_p = e^t\cos t \\ \implies &e^t[2B\cos t - 2A\sin t - A\cos t - B\sin t] = e^t\cos t \\ \implies &2A + B = 0 \;\&\; 2B - A = 1 \\ \implies &A = -\frac{1}{5} \;\&\; B = \frac{2}{5}. \end{aligned}$$

$$\therefore\; y_p(t) = e^t\left(-\frac{1}{5}\cos t + \frac{2}{5}\sin t\right).$$

■

3 Applications of Second Order Linear ODEs with Constant Coefficients

3.1 Modeling of Mass-Spring System



Linear ODEs with constant coefficients have important applications in mechanics. We take an ordinary spring that resists extension as well as compression. We suspend it vertically from a fixed support and attach a mass at its lower end. We let $y = 0$ denote the position of the ball when the system is at rest.

Hooke's Law: The restoring force F is proportional to the spring's displacement x. I.e. $F \propto x$ or, $F = -kx$ Where $k > 0$ is termed as the force constant or spring constant.

Undamped System

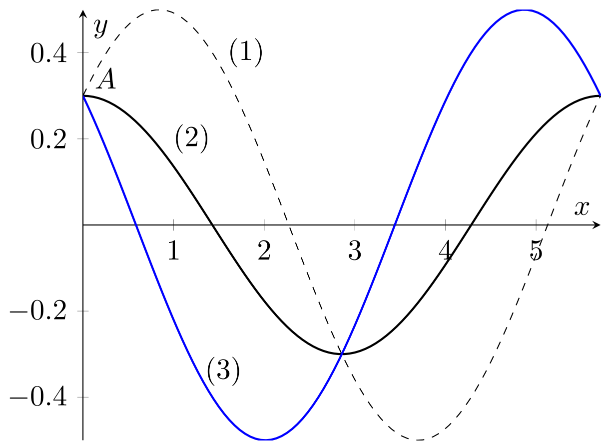
When the system is at rest or at equilibrium, then $mg - ks_0 = 0$. If we stretch the spring by some displacement $y > 0$, then by Newton's second law

$$\begin{aligned} &\text{Mass} \times \text{Acceleration} = \text{Force} \\ &\implies my'' = mg - k(y + s_0) \\ &\implies my'' + ky = 0. \end{aligned}$$

So, we have a homogeneous linear ODE with constant coefficients. The characteristic equation of the above ODE is $m\lambda^2 + k = 0 \implies \lambda = \pm i\omega_0$, where $\omega_0 = \sqrt{\frac{k}{m}}$.

$$\therefore\ y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

is the general solution.
It can also be written as $y(t) = C\cos(\omega_0 t - \delta)$, where $C = \sqrt{A^2+B^2}$ (amplitude) and $\tan\delta = \frac{B}{A}$ (phase angle). This motion is called harmonic oscillation.



Here, $y(0) = A$ is the initial position and $y'(0) = \omega_0 B$ is the initial velocity, (1) has positive initial velocity, (2) and (3) has 0 and negative initial velocity, respectively.

Damped System

We add a damping force $F_2 = -cy'$. Thus, the ODE of the damped mass-spring system is

$$my'' + cy' + ky = 0.$$

The characteristic equation of the above ODE is

$$\begin{aligned} m\lambda^2 + c\lambda + k = 0 \\ \implies \lambda &= \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \\ &= -\frac{c}{2m} \pm \frac{\sqrt{c^2-4mk}}{2m} \\ &= -\alpha \pm \beta \end{aligned}$$

Here, $\alpha > 0$ & β may be real or imaginary.

Case 1. (Overdamped case) If $c^2 > 4mk$, then $\lambda = -\alpha \pm \beta$ and $\beta > 0$. Also,

$$\beta^2 = \frac{c^2 - 4mk}{4m^2} = \alpha^2 - \frac{k}{m} < \alpha^2$$
$$\implies \beta < \alpha \implies \alpha - \beta > 0.$$

$$\therefore\ y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t} \to 0 \quad \text{as} \quad t \to \infty.$$

Case 2. (Critically damped) If $c^2 = 4mk$, then $\lambda = -\alpha, -\alpha$.

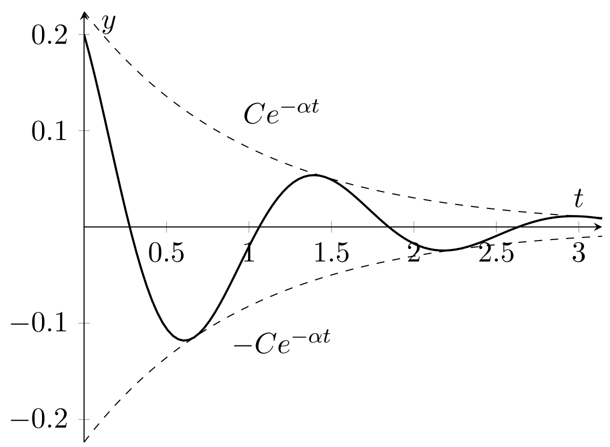
$$\therefore\ y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t} = (c_1 + c_2 t)e^{-\alpha t}.$$

In this case, $y(t) = 0$ for at most one $t > 0$.

Case 3. (Underdamped case) If $c^2 < 4mk$, then $\beta = i\omega$ where $\omega = \frac{\sqrt{4mk - c^2}}{2m} > 0$, therefore, $\lambda = -\alpha \pm i\omega$.

$$\therefore\ y(t) = e^{-\alpha t}(A\cos\omega t + B\sin\omega t) = Ce^{-\alpha t}\cos(\omega t - \delta),$$

where $C = \sqrt{A^2 + B^2}$.



Forced Oscillation

If we add an external force $F(t)$, then we have

$$my'' + cy' + ky = F(t),$$

which is a nonhomogeneous second-order linear ODE. We will assume $F(t) = F_0 \cos\omega t$.

When $c = 0$, we have $my'' + ky = F_0 \cos \omega t$. We know that the general solution of the corresponding homogeneous part is $y_h = C \cos(\omega_0 t - \delta)$. We use the method of undetermined coefficients to find a particular solution. Let $\omega \neq \omega_0$ and

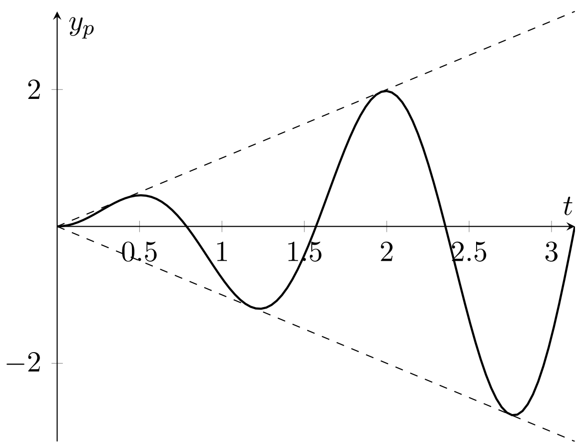
$$\begin{aligned} y_p &= a \cos \omega t + b \sin \omega t \\ \implies y_p' &= -\omega a \sin \omega t + \omega b \cos \omega t \\ \implies y_p'' &= -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t \end{aligned}$$

substituting in the equation, we get

$$\begin{aligned} &(k - m\omega^2) a \cos \omega t + (k - m\omega^2) b \sin \omega t = F_0 \cos \omega t \\ &\implies b = 0,\ a = \frac{F_0}{k - m\omega^2} \\ &\therefore y_p(t) = \frac{F_0}{k - m\omega^2} \cos \omega t. \end{aligned}$$

If $\omega = \omega_0$, then take $y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t)$. After substituting in the equation, we get

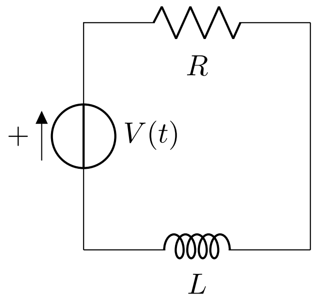
$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$



$$\therefore\ y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

3.2 Application to Electric Circuits

1. RL-Circuit



By Kirchhoff's voltage law,

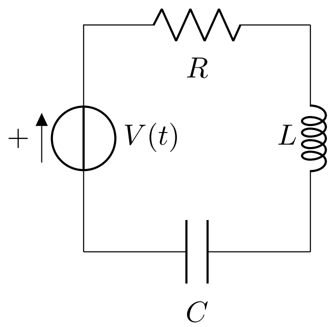
$$L\frac{dI}{dt} + RI = V(t)$$

$$i.e. \quad \frac{dI}{dt} + \frac{R}{L}I = \frac{V(t)}{L}.$$

This is a first-order linear ODE, and the solution is given by

$$I(t) = e^{-\frac{R}{L}t}\left[\int e^{\frac{R}{L}t}.\frac{V(t)}{L}dt + c\right].$$

2. RLC-Circuit



Let Q be the charge and I be the current, then the voltage drop across the capacitor is $\frac{Q}{C}$,

$$\therefore \quad L\frac{dI}{dt} + RI + \frac{Q}{C} = V(t).$$

Since $I = \frac{dQ}{dt}$, we get

$$LQ'' + RQ' + \frac{1}{C}Q = V(t).$$

This is a second-order linear ODE.

Free Oscillations: If $V(t) = 0$, then we have

$$LQ'' + RQ' + \frac{1}{C}Q = 0$$

a second-order homogeneous ODE with constant coefficients. So, the characteristic equation is

$$Lm^2 + Rm + \frac{1}{C} = 0$$
$$\implies m = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

Case 1. (Underdamped Case) If $R < \sqrt{\frac{4L}{C}}$, then $m = -\frac{R}{2L} \pm i\omega_0$, where $\omega_0 = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$, therefore the solution is given by

$$Q(t) = e^{-\frac{R}{2L}t}(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t).$$

Case 2. (overdamped Case) If $R > \sqrt{\frac{4L}{C}}$, then

$$m_1 = \frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L}; \; m_2 = \frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

Note that $m_1, m_2 < 0$, therefore the solution is

$$Q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

Case 3. (Critically damped Case) If $R = \sqrt{\frac{4L}{C}}$, then $m_1 = m_2 = -\frac{R}{2L}$, therefore the solution is

$$Q(t) = e^{-\frac{R}{2L}t}(c_1 + c_2 t).$$

If $R \neq 0$, then $Q(t) \to 0$ as $t \to \infty$ in each case.
Forced Oscillations: If $V(t) \neq 0$, then we have

$$LQ'' + RQ' + \frac{1}{C}Q = V(t), \; Q(0) = Q_0, \; Q'(0) = I_0$$

a non-homogeneous linear ODE. So, the general solution will be $Q(t) = Q_h(t) + Q_p(t)$, where $Q_h(t)$ is the general solution of the corresponding homogeneous equation and $Q_p(t)$ is a particular solution. For $V(t) = V_0 \cos \omega t$, we can find $Q_p(t)$ using the method of undetermined coefficients. This is analogous to the mechanical mass-spring system discussed before.

$$\begin{aligned} &\text{RLC-Circuit:} \quad LQ'' + RQ' + \frac{1}{C}Q = V(t) \\ &\text{Mass-Spring:} \quad my'' + cy' + ky = F(t) \end{aligned}$$

Analogy:

Mass-spring system	Electrical system
Mass m	Inductance L
Damping constant c	Resistance R
Spring constant k	Reciprocal $\frac{1}{C}$ of capacitance
Driving force $F(t)$	Voltage $V(t)$

4 Higher Order Linear ODEs

An n^{th} order linear ODE is an equation of the form

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = r(t).$$

It is called homogeneous if $r(t) = 0$ and non-homogeneous otherwise. To solve n^{th} order homogeneous linear ODE, we need to find n linearly independent solutions.

- If $W(y_1, y_2, \ldots, y_n)(t) \neq 0$, then $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent, where

$$W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}.$$

Example 9. Let $y_1 = 1,\ y_2 = t,\ y_3 = t^2$, then

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & t \end{vmatrix} = t.$$

4.1 n^{th} Order Homogeneous ODE with Constant Coefficients

Consider $a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. We assume $y = e^{mt}$ is a solution. This gives the characteristic equation:

$$a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0.$$

If $m = m_1$ is a root of the characteristic equation, then $y = e^{m_1t}$ is a solution of the ODE. So, we need to find the roots of this polynomial of degree n.

1. Suppose the polynomial has n real and distinct roots $m_1, m_2, \ldots, m_n$. Then the general solution is given by

$$y = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \cdots + c_n e^{m_n t}.$$

2. Suppose a root m is repeated k times, $k \le n$, then $e^{mt},\ te^{mt},\ \ldots,\ t^{k-1}e^{mt}$ are k linearly independent solutions corresponding to this root.

3. If $m_1, m_2 = \alpha \pm i\beta$ are two complex conjugate roots, then $y_1 = e^{\alpha t}\cos\beta t$ and $y_2 = e^{\alpha t}\sin\beta t$ are two linearly independent solutions and if $\alpha \pm i\beta$ is repeated twice then $y_1 = e^{\alpha t}\cos\beta t,\ y_2 = e^{\alpha t}\sin\beta t,\ y_3 = te^{\alpha t}\cos\beta t,\ y_4 = te^{\alpha t}\sin\beta t$ are four linearly independent solutions.

Example 10. Solve $y^{(4)} + 2y'' + y = 0$.

Solution The characteristic equation is $m^4 + 2m^2 + 1 = 0 \implies (m^2+1)^2 = 0$, therefore the roots of the characteristic equation are $\pm i,\ \pm i$. So, the general solution is given by

$$y(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t.$$

■

4.2 n^{th} Order Euler-Cauchy Equation

Consider $a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = 0$. We assume $y = x^m$ is a solution. This gives the characteristic equation:

$$a_0m(m-1)\ldots(m-n+1) + a_1m(m-1)\ldots(m-n+2) + \cdots + a_n = 0.$$

For example, the characteristic equation of the 3^{rd} order Euler-Cauchy equation is

$$a_0m(m-1)(m-2) + a_1m(m-1) + a_2m + a_3 = 0.$$

1. Suppose the polynomial has n real and distinct roots $m_1, m_2, \ldots, m_n$. Then the general solution is given by

$$y = c_1x^{m_1} + c_2x^{m_2} + \cdots + c_nx^{m_n}.$$

2. Suppose a root m is repeated k times, then $x^m,\ x^m\ln x,\ \ldots,\ x^m(\ln x)^{k-1}$ are k linearly independent solutions corresponding to this root.

3. If $m_1, m_2 = \alpha \pm i\beta$ are two complex conjugate roots, then $y_1 = x^\alpha\cos(\beta\ln x)$ and $y_2 = x^\alpha\sin(\beta\ln x)$ are two linearly independent solutions.

4.3 n^{th} Order Non-homogeneous Linear ODE

For non-homogeneous linear ODE of n^{th} order, the solution is $y = y_h + y_p$, where y_h is the general solution of the corresponding homogeneous ODE and y_p is a particular solution of the non-homogeneous ODE. To find y_p, we can use either the method of undetermined coefficients (whenever applicable) or the variation of parameters method.

Variation of parameters method for n^{th} order ODE

Suppose $y_1, y_2, \ldots, y_n$ are n linearly independent solutions to the corresponding homogeneous equation, then $y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n$ is a particular solution, where $u_i(t)$ is given by

$$u_i'(t) = \frac{W_i(t)}{W(t)}r(t),$$

where

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

and

$$W_i(t) = \begin{vmatrix} y_1(t) & \cdots & 0 & \cdots & y_n(t) \\ y_1'(t) & \cdots & 0 & \cdots & y_n'(t) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(t) & \cdots & 1 & \cdots & y_n^{(n-1)}(t) \end{vmatrix},$$

i.e. we replace i^{th} column of $W(t)$ with $(0, 0, \ldots, 1)^t$ to find $W_i(t)$. Then integrate the above equation to find $u_i(t)$ and hence $y_p(t)$.

5 Power Series Method

Example 11. Solve $y'' = y$ using the power series method.

Solution We assume that the solution $y(x)$ can be written as a power series in x, i.e.

$$y(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n.$$

Then by the term-by-term differentiation,

$$
\begin{aligned}
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
\& \quad y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

$y(x)$ is a solution to $y'' = y$ iff

$$
\begin{aligned}
& \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \\
\Longrightarrow & \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n = \sum_{n=0}^{\infty} a_n x^n \\
\Longrightarrow & \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0 \\
\Longrightarrow & (n+2)(n+1) a_{n+2} - a_n = 0 \\
\Longrightarrow & a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad \forall\ n = 0, 1, 2, \ldots\ .
\end{aligned}
$$

Putting $n = 0, 2, 4, \ldots$, we get

$$
\begin{aligned}
a_2 &= \frac{a_0}{2 \times 1} = \frac{a_0}{2!} \\
a_4 &= \frac{a_2}{4 \times 3} = \frac{a_0}{4!} \\
\text{inductively,} \quad a_{2n} &= \frac{a_0}{(2n)!}, \quad n = 0, 1, 2, \ldots\ .
\end{aligned}
$$

Putting $n = 1, 3, 5, \ldots$, we get

$$
\begin{aligned}
a_3 &= \frac{a_1}{3 \times 2} = \frac{a_1}{3!} \\
a_5 &= \frac{a_3}{5 \times 4} = \frac{a_1}{5!} \\
\text{inductively,} \quad a_{2n+1} &= \frac{a_1}{(2n+1)!}, \quad n = 1, 2, 3, \ldots\ .
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad y &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{a_1}{(2n+1)!} x^{2n+1} \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}
$$

Note that

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
&= \frac{e^x + e^{-x}}{2}, \\
\text{and} \quad y_2(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
&= \frac{e^x - e^{-x}}{2}
\end{aligned}
$$

$$
\therefore \; y(x) = a_0 \left(\frac{e^x + e^{-x}}{2} \right) + a_1 \left(\frac{e^x - e^{-x}}{2} \right)
$$

$$
= c_1 e^x + c_2 e^{-x},
$$

where $c_1 = \frac{a_0 + a_1}{2}$ and $c_2 = \frac{a_0 - a_1}{2}$. ■

Example 12. Solve $y'' + y = 0$ by using the power series method.

Solution Let $y = \sum\limits_{n=0}^{\infty} a_n x^n$. Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\
\therefore \quad & \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0 \\
\implies a_{n+2} &= \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \dots , \\
\implies a_{2n} &= \frac{(-1)^n a_0}{(2n)!}, \quad n = 0, 1, 2, \dots , \\
\& \quad a_{2n+1} &= \frac{(-1)^n a_1}{(2n+1)!}, \quad n = 0, 1, 2, \dots ,
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad y &= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1} \\
&= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}
\end{aligned}
$$

$$\therefore \quad y = a_0 \cos x + a_1 \sin x.$$

■

Question 1. *When is the power series method applicable?*

Theorem 9. *Consider the linear ODE of the form $y'' + p(x)y' + q(x)y = r(x)$. Suppose $p(x), q(x)$ and $r(x)$ can be expressed as power series in x (or about $x = a$). Then the solution $y(x)$ can be written as a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ $\left(\text{or } \sum\limits_{n=0}^{\infty} a_n (x-a)^n \right)$.*

Remark 10.

1. *The above theorem is valid for any n^{th} order linear ODE.*
2. *Generally, we use this method when $p(x), q(x), r(x)$ are polynomials.*

Example 13. Solve $y'' + xy = 0$ by using the power series method.

Solution Let $y = \sum\limits_{n=0}^{\infty} a_n x^n$. Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\
\therefore \quad & \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0 \\
\Longrightarrow \ & \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \\
\Longrightarrow \ & 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) a_{n+3} + a_n] x^{n+1} = 0 \\
\Longrightarrow \ & a_2 = 0 \ \& \ (n+3)(n+2) a_{n+3} + a_n = 0; \\
\Longrightarrow \ & a_{n+3} = \frac{-a_n}{(n+3)(n+2)}, \ n = 0, 1, \dots \ .
\end{aligned}
$$

Since $a_2 = 0$, we get $a_5 = 0 = a_8 = a_{11} = \cdots$,

$$\therefore \ a_{2+3k} = 0 \quad \text{for } k = 0, 1, 2, \dots \ .$$

Putting $n = 0, 3, 6, \dots$, we get,

$$
\begin{aligned}
a_3 &= \frac{-a_0}{3 \times 2} = \frac{-a_0}{6}, \\
a_6 &= \frac{-a_3}{6 \times 5} = \frac{a_0}{180};
\end{aligned}
$$

and putting $n = 1, 4, 7, \dots$, we get,

$$
\begin{aligned}
a_4 &= \frac{-a_1}{4 \times 3} = \frac{-a_1}{12}, \\
a_7 &= \frac{-a_4}{7 \times 6} = \frac{a_1}{504};
\end{aligned}
$$

$\therefore$ The solution is:

$$y = a_0 \left[1 - \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots \right] + a_1 \left[x - \frac{1}{12} x^4 + \frac{1}{504} x^7 + \cdots \right].$$

∎

Exercise 2. *Solve the following using the power series method:*

1. $y'' - xy' + y = 0.$

2. $y'' - y' = 0.$

3. $(2x^2 - 3x + 1)y'' + 2xy' - 2y = 0.$