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HTML Content

Existence & Uniqueness of first order IVP :

Defn (Lipschitz condition) :
A function $f(x,y)$ of two variables is said to satisfy the "Lipschitz condition" on a region $R \subseteq \mathbb{R}^2$ if $\exists$ a constant M such that

$$|f(x,y_1)-f(x,y_2)| \le M|y_1-y_2|$$

for all $(x,y_1),(x,y_2) \in R$.

Remark: If $f(x,y)$ is continuous on R and $\frac{\partial f}{\partial y}$ is also continuous on R, then f satisfies the Lipschitz condition.

Pf: Let $(x,y_1),(x,y_2) \in R$.
Then by the mean value thm,

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x, y^*)\ (y_1 - y_2)$$

for some y^* between y_1 & y_2

$$\Rightarrow |f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,y^*)\right| |y_1 - y_2|$$

Since $\frac{\partial f}{\partial y}$ is continuous on R,

$\exists M$ s.t. $\left|\frac{\partial f}{\partial y}\right| \le M$ on R.

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le M |y_1 - y_2|$$

$\therefore$ f satisfies the Lipschitz condition.

<u>Remark:</u> $f(x,y) = x + |\sin y|$

satisfies the Lipschitz condition on $\mathbb{R}^2$

but $\frac{\partial f}{\partial y}$ does not exist on the x-axis $(y = 0)$

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &= \big|\,|\sin y_1| - |\sin y_2|\,\big| \\
&\le |\sin y_1 - \sin y_2| \\
&\le |y_1 - y_2|
\end{aligned}
$$

$\therefore$ f satisfies the Lipschitz condition

Theorem (Existance & Uniqueness)

Consider the IVP: $\dfrac{dy}{dx} = f(x,y)$;

$y(x_0) = y_0$.

Suppose $f(x,y)$ is continuous on a rectangle $R = \{(x,y) : |x - x_0| \le a,\ |y - y_0| \le b\}$

and f satisfies the Lipschitz condition on R. Then the IVP has a unique solution $y(x)$ defined on some open interval containing $x = x_0$.

Example: (Non-uniqueness)

Consider $\frac{dy}{dx} = \sqrt{|y|}$; $y(0) = 0$.

Clearly, $y \equiv 0$ is a solution to the above IVP.

Note that

$$y = \begin{cases} \frac{x^2}{4}, & x \geq 0 \\ -\frac{x^2}{4}, & x < 0 \end{cases}$$

is also a solution to the IVP.

Verification:

$$\frac{dy}{dx} = \begin{cases} \frac{x}{2}, & x \geq 0 \\ -\frac{x}{2}, & x < 0 \end{cases}$$

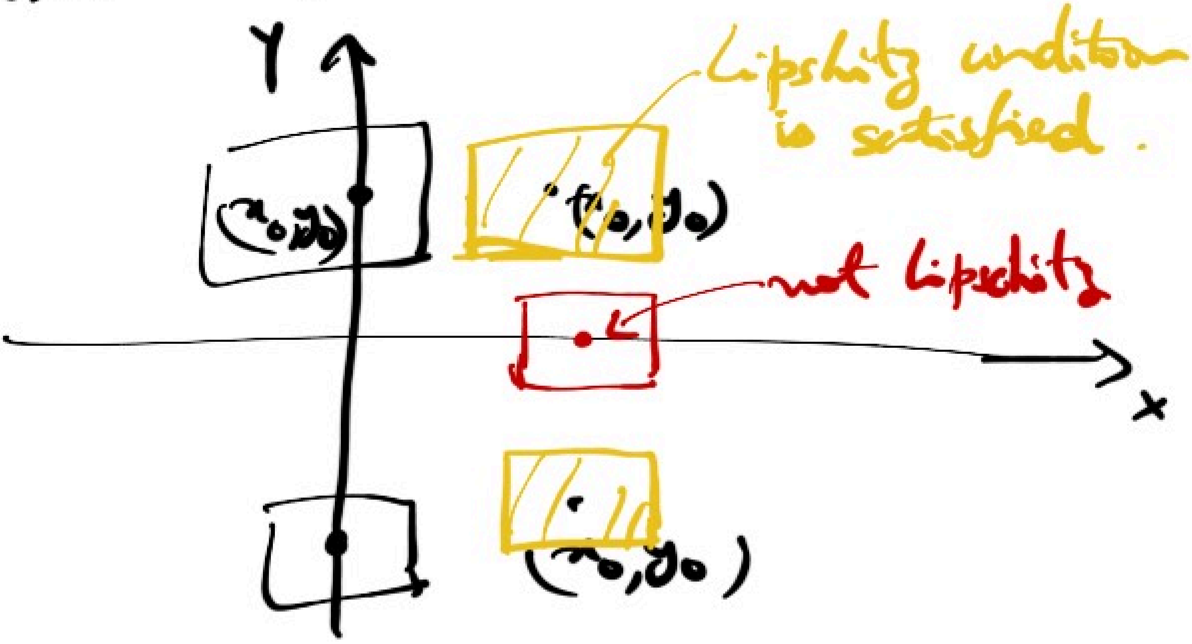
Also, $\sqrt{|y|} = \sqrt{\frac{x^2}{4}} = \frac{|x|}{2} = \begin{cases} \frac{x}{2}, & x \geq 0 \\ -\frac{x}{2}, & x < 0 \end{cases}$

Also, $y(0) = 0$.

Example: $\frac{dy}{dx} = \sqrt{|y|}$; $y(x_0) = y_0$.

If $y_0 \neq 0$, then $f(x,y) = \sqrt{|y|}$ satisfies the Lipschitz condition on a small enough rectangle containing (x_0, y_0).

Hence, the IVP has a unique solution if $y_0 \neq 0$.



Example: Consider the IVP:

$$y\frac{dy}{dx} = x \quad ; \quad y(0) = \beta .$$

Find the values of β for which the IVP has

(a) a unique solution

(b) more than one solution

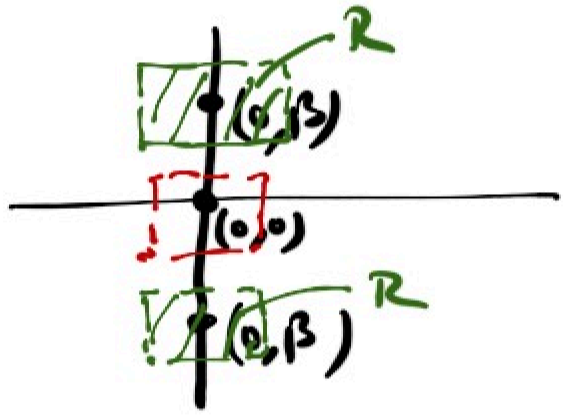
(c) no solution.

Solution:

$f(x,y) = \frac{x}{y}$ is continuous for (x,y) except on the line $y=0$

Also, $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$ is continuous except when $y=0$

If $\beta \neq 0$, then the existence-uniqueness guarantees a unique solution to the IVP.



Here, we can find the solution as follows:

Integrating $y\frac{dy}{dx} = x$, we get

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$y(0) = \beta \Rightarrow \quad c = \frac{\beta^2}{2}$$

$$\therefore \quad y^2 = x^2 + \beta^2$$

$$\Rightarrow \quad y = \sqrt{x^2+\beta^2} \text{ or } -\sqrt{x^2+\beta^2}$$

(Are both the solutions?)

We have already seen that if $\beta \neq 0$, then it has a unique solution.

For $y = \sqrt{x^2+\beta^2}$, $y(0) = \sqrt{\beta^2} = |\beta|$

If $\beta > 0$, then $y = \sqrt{x^2+\beta^2}$ is a soln.

If $\beta < 0$, then $y = \sqrt{x^2+\beta^2}$ is not a soln to the IVP.

If $\beta < 0$, $y = -\sqrt{x^2+\beta^2}$ is a soln. to the IVP.

If $\beta > 0$, $y = -\sqrt{x^2+\beta^2}$ is NOT a soln. to the IVP.

For $\beta = 0$: The existence-uniqueness theorem does not give us any information.

However, $y = x$ and $y = -x$ are clearly solutions to the IVP: $y\frac{dy}{dx} = x$; $y(0) = 0$.

Exercise: Consider the IVP:

$$(x^2 - 4x)\frac{dy}{dx} = (2x-4)y \quad ; \quad y(x_0) = y_0.$$

Find (x_0, y_0) for which the IVP has

(a) no soln. (b) unique soln.

(c) more than one soln.