Vector Spaces

Field : A field is a set of objects, called scalars, with two binary operations: addition (+) and multiplication (.), that satisfy the following properties:

Axioms for addition:

- Closure property : For $a, b \in \mathbb{F}$, $a+b \in \mathbb{F}$
- Commutativity : $a+b = b+a \quad \forall a, b \in \mathbb{F}$
- Associativity : $a+(b+c) = (a+b)+c$
- Existence of additive identity : $\exists\, 0 \in \mathbb{F}$ s.t. $a+0 = a \quad \forall a \in \mathbb{F}$.
- Additive inverse : For any $a \in \mathbb{F}$, $\exists\, -a \in \mathbb{F}$ s.t. $a+(-a) = 0$.

Axioms for multiplication:

- Closure : For $a, b \in \mathbb{F}$, $a.b \in \mathbb{F}$.
- Commutativity : $a.b = b.a \quad \forall a, b \in \mathbb{F}$
- Associativity : $a.(b.c) = (a.b).c$
- Existence of multiplicative identity : $\exists\, 1 \in \mathbb{F}$ $(1 \neq 0)$ s.t. $a.1 = a \quad \forall a \in \mathbb{F}$
- Existence of multiplicative inverse : For every $a \neq 0$, $\exists\, a^{-1} \in \mathbb{F}$ s.t. $a.a^{-1} = 1$

Distributive law : $a.(b+c) = a.b + a.c$.

Examples of field:

1. $\mathbb{Q}$, the set of rational numbers
2. $\mathbb{R}$, the set of real numbers
3. $\mathbb{C}$, the set of complex numbers
4. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$
5. $\mathbb{F} = \{0, 1\}$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Ex: Prove that $a \cdot 0 = 0$ for any $a \in \mathbb{F}$.

Soln: $a \cdot 0 = a \cdot (0 + 0)$ $(\because 0 = 0 + 0)$

$= a \cdot 0 + a \cdot 0$ (by distri law)

Adding the additive inverse of $a \cdot 0$ i.e. $-(a \cdot 0)$, we get

$$\underbrace{a \cdot 0 + (-(a \cdot 0))}_{\substack{\shortparallel \\ 0}} = \underbrace{(a \cdot 0 + a \cdot 0) + (-(a \cdot 0))}_{\substack{\shortparallel \\ a \cdot 0 + (\underbrace{a \cdot 0 + -a \cdot 0}_{\substack{\shortparallel \\ 0}})}}$$

$\Rightarrow \quad 0 = a \cdot 0$

Vector Space

A vector space consists of :

(i) a field $\mathbb{F}$ of scalars.

(ii) a set V of objects, called vectors.

(iii) a binary operation, called vector addition on V i.e. for any $u, v \in V$, $\exists$ a vector $(u+v) \in V$ satisfying

(a) $+$ is commutative : $u+v = v+u \quad \forall u, v \in V$

(b) $+$ is associative : $(u+v)+w = u+(v+w) \quad \forall u, v, w \in V$.

(c) Existence of zero vector : $\exists\, 0 \in V$ s.t. $u + 0 = u \quad \forall u \in V$

(d) For each $u \in V$, $\exists\, (-u) \in V$ s.t. $u + (-u) = 0$.

(iv) an operation called scalar multiplication for $c \in \mathbb{F}$ and $u \in V$, $\exists\, c.u \in V$ s.t.

(a) $c.(u+v) = c.u + c.v$

(b) $(c_1 + c_2).u = c_1.u + c_2.u$

(c) $(c_1 . c_2).u = c_1.(c_2.u)$

(d) $1.u = u \quad \forall u \in V$.

We say V is a vector space over the field $\mathbb{F}$.

Examples:

① $V = \mathbb{R}$ is a vector space over $\mathbb{F} = \mathbb{R}$ with the usual addition & mult. of real numbers.

In fact, any $\mathbb{F}$ is a vector space over $\mathbb{F}$.

② $V = \mathbb{R}^n = \{(x_1, x_2, \cdots x_n) : x_i \in \mathbb{R}\}$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

$$c \cdot (x_1, x_2, \cdots, x_n) = (cx_1, cx_2, \cdots, cx_n)$$

What is the zero vector?

$$\vec{0} = (0, 0, \cdots, 0)$$