Cayley-Hamilton theorem

Suppose A is any $n\times n$ matrix and let $p(x) = \det(xI - A)$ be the characteristic polynomial of A. Then the matrix $p(A)$ is the zero matrix.

(If $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
then $p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$)

. The proof is not trivial and we will skip it.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$p(x) = \det(xI - A) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix}$$
$$= (x-a)(x-d) - bc$$
$$= x^2 - (a+d)x + ad - bc$$

$\therefore\ p(A) = A^2 - (a+d)A + (ad-bc)I$

You can verify that $p(A) = 0$.

Applications of Cayley-Hamilton thm.

① Calculating inverse of a matrix :

Let A be $n\times n$ matrix and

$$\phi(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

be the char. poly. of A.

Note that A is invertible iff $a_0 \neq 0$
(because $a_0 = \phi(0) = \det(-A)$)

By the Cayley-Hamilton thm,

$$\phi(A) = A^n + a_{n-1}A^{n-1} + \dots + a_2 A^2 + a_1 A + a_0 I$$
$$= 0$$

ie. $$a_0 I = -(A^n + a_{n-1}A^{n-1} + \dots + a_2 A^2 + a_1 A)$$
$$= -(A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2 A + a_1 I) A$$

If $a_0 \neq 0$, $$I = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2 A + a_1 I) A$$

$$\Rightarrow \boxed{A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2 A + a_1 I)}$$

Example: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Then $p(x) = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & -1 \\ -1 & -1 & x \end{vmatrix}$

$$= (x-1)\left[x(x-1) - 1\right]$$

$$= (x-1)(x^2 - x - 1)$$

$$= x^3 - 2x^2 + 1$$

Since $p(0) = 1 \neq 0$, A is invertible.

Also, $A^3 - 2A^2 + I = 0$

$\Rightarrow \quad I = 2A^2 - A^3 = A(2A - A^2)$

$\Rightarrow \quad A^{-1} = 2A - A^2$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

(2) <u>Calculating powers of A or $g(A)$ for any polynomial $g(x)$</u>:

By the division algorithm,

$g(x) = p(x)\,q(x) + r(x)$,

where $p(x)$ is the char. poly.

and either $r(x) = 0$

or $\deg r(x) < \underset{= n}{\deg p(x)}$

$\therefore$ $r(x)$ is a polynomial of degree at most $(n-1)$.

$g(A) = \underbrace{p(A)}_{\substack{\shortparallel \\ 0}} q(A) + r(A)$

by Cayley-Hamilton thm

$\therefore$ $g(A) = r(A)$

To calculate $r(A)$, we only need $A, A^2, \dots, A^{n-1}$.

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

We saw, $p(x) = x^3 - 2x^2 + 1$

If we want to compute A^{100},

$g(x) = x^{100}$.

We write $g(x) = x^{100} = (x^3 - 2x^2 + 1)\,q(x) + r(x)$,

where $r(x) = ax^2 + bx + c$

$$x^{100} = (x^3 - 2x^2 + 1)\,q(x) + ax^2 + bx + c$$
$$= (x-1)(x^2 - x - 1)\,q(x) + ax^2 + bx + c$$
$$= (x-1)\left(x - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(x - \left(\frac{1-\sqrt{5}}{2}\right)\right)$$
$$q(x) + ax^2 + bx + c$$

Put $x = 1$: $1 = a + b + c$

Put $x = \frac{1+\sqrt{5}}{2}$: $\left(\frac{1+\sqrt{5}}{2}\right)^{100} = a\left(\frac{1+\sqrt{5}}{2}\right)^2 + b\left(\frac{1+\sqrt{5}}{2}\right) + c$

Put $x = \frac{1-\sqrt{5}}{2}$: $\left(\frac{1-\sqrt{5}}{2}\right)^{100} = a\left(\frac{1-\sqrt{5}}{2}\right)^2 + b\left(\frac{1-\sqrt{5}}{2}\right) + c$

We get 3 linear eqns. in a, b, c, which can be solved to get a, b, c.

After that
$$A^{100} = r(A) = aA^2 + bA + cI.$$

<u>Example</u>: Calculate A^{50} for
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$p(x) = \det(xI - A) = \begin{vmatrix} x & -1 \\ 1 & x \end{vmatrix} = x^2 + 1$$

$\therefore\ A^2 + I = 0$ $\left(\therefore \Rightarrow A^2 = -I,\ A^{50} = (A^2)^{50} = -I\right)$

$$x^{50} = (x^2+1)\,g(x) + ax + b$$

Put $x = i$: $i^{50} = ai + b$

i.e. $ia + b = -1$ — (i)

Put $x = -i$: $-ia + b = (-i)^{50} = -1$ — (ii)

(i) + (ii) $\Rightarrow$ $2b = -2 \Rightarrow b = -1$

$\therefore$ from (i), $ia - 1 = -1 \Rightarrow a = 0$

$\therefore$ $x^{50} = (x^2+1)\,g(x) - 1$

$\Rightarrow$ $A^{50} = 0 - I = -I$.