AMTL 100 (CALCULUS)
Midterm Exam Solution

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1. Find the following limits

(a) $\displaystyle\lim_{x\to 1} \frac{2x^2+2-(3x+1)\sqrt{x}}{x-1}$ [2]

(b) $\displaystyle\lim_{x\to\infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right)$ [2]

Solution: (a) By the L'Hôpital's rule,

$$\lim_{x\to 1} \frac{2x^2+2-(3x+1)\sqrt{x}}{x-1} = \lim_{x\to 1} \frac{4x - 3\sqrt{x} - \frac{3x+1}{2\sqrt{x}}}{1} = -1.$$

(b)

$$\lim_{x\to\infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right) = \lim_{x\to\infty} \frac{2x^3}{(x^2-1)(x^2+1)} = \lim_{x\to\infty} \frac{\frac{2}{x}}{(1-\frac{1}{x^2})(1+\frac{1}{x^2})} = 0.$$

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2. Use the formal definition of limit to prove that $\displaystyle\lim_{x\to 1} \frac{1}{x} = 1.$ [3]

Solution: For a given $\epsilon > 0$, we have to find a suitable $\delta > 0$ such that

$$0 < |x-1| < \delta \implies |\frac{1}{x} - 1| < \epsilon.$$

We want

$$|\frac{1}{x} - 1| = \frac{|x-1|}{|x|} < \epsilon.$$

If we had $|x|$ greater than or equal to $1/2$, then

$$|x-1| < \delta \implies \frac{|x-1|}{|x|} < 2\delta,$$

so choosing $\delta = \epsilon/2$ would work. Now, $|x-1| < \delta$ implies $1-\delta < x < 1+\delta$. Thus, if $\delta \le 1/2$, then $x \ge 1/2$, as we wanted.
Hence, we can take $\delta = \min\{1/2, \epsilon/2\}$.

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3. Find the local and global extrema (if any) of the following function. [5]

$$f(x) = \frac{x^2}{4-x^2}, \quad -2 < x \le 1$$

Solution: First, we find the derivative

$$f'(x) = \frac{8x}{(4-x^2)^2}.$$

So, $f'(x) = 0$ if and only if $x = 0$.
Also, $f'(x) < 0$ if $-2 < x < 0$ and $f'(x) > 0$ if $0 < x < 1$.

By the first derivative test, we conclude that the function f has a local as well as global minimum at $x = 0$ (in the given domain).
Also, since f is increasing in the interval $[0, 1]$, it has a local maximum at the end-point $x = 1$.
Finally, since $\lim_{x\to -2^+} f(x) = \infty$, f has no global maximum.

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4. Consider the sequence given by

$$a_1 = 0,\ a_{n+1} = \sqrt{8 + 2a_n} \quad \text{for } n \geq 1$$

(a) Prove that the sequence is bounded. [2]

(b) Prove that the sequence is increasing. [2]

(c) Conclude that the sequence converges and find its limit. [3]

Solution: (a) Clearly, $a_n \geq 0$ for all n. Also, if we assume that, for some n, $a_n < 4$, then

$$a_{n+1} < \sqrt{8 + 2 \times 4} = 4.$$

Since $a_1 < 4$, by the principle of mathematical induction, $a_n < 4$ for all n.
Hence, the given sequence is bounded.

(b) We have $a_1 < a_2$. If we assume that $a_n < a_{n+1}$, then $8+2a_n < 8+2a_{n+1}$, which implies that $a_{n+1} < a_{n+2}$.
Thus, by the principle of mathematical induction, $a_n < a_{n+1}$ for all n.
Hence, the given sequence is increasing.

(c) By parts (a) and (b), we know that the sequence is a bounded monotonic sequence and hence it must converge. Let $\lim_{n\to\infty} = L$. Then $\lim_{n\to\infty} a_{n+1} = L$. Therefore, we must have

$$L = \sqrt{8 + 2L}.$$

This implies $L^2 - 2L - 8 = 0$, that is, $(L - 4)(L + 2) = 0$. Since each term is non-negative, the limit cannot be -2. Therefore, $L = 4$.

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5. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ [3]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3} + \sqrt{n+2}}$ [3]

(c) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ [2]

Solution: (a) Let $a_n = \frac{1}{n}\sin(\frac{1}{n})$ and $b_n = \frac{1}{n^2}$.
Then $\frac{a_n}{b_n} = \frac{\sin(1/n)}{1/n} \to 1$ as $n \to \infty$.
Since $\sum_{n=1}^{\infty} b_n$ converges, by the limit comparison test, we conclude that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let $a_n = \frac{1}{\sqrt{n+3}+\sqrt{n+2}}$ and $b_n = \frac{1}{\sqrt{n}}$.
Then $\frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n+3}+\sqrt{n+2}} \to \frac{1}{2}$ as $n \to \infty$.

Since $\sum_{n=1}^{\infty} b_n$ diverges, by the limit comparison test, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

(c) We have $\lim_{n\to\infty} \cos(1/n) = \cos(0) = 1 \neq 0$. By the divergence test, we conclude that $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges.

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6. Use Taylor's theorem to estimate the error when $\ln(1+x)$ is approximated by $x - \frac{x^2}{2} + \frac{x^3}{3}$ for $|x| \le \frac{1}{10}$. [3]

Solution: Let $f(x) = \ln(1+x)$. Then $f'(x) = 1/(1+x)$, $f''(x) = -1/(1+x)^2$, $f'''(x) = 2/(1+x)^3$ and $f^4(x) = -6/(1+x)^4$.
Therefore, $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 2$.
By the Taylor's theorem, $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(c)}{4!}x^4$ for some c between 0 and x.
Therefore, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{4(1+c)^4}x^4$ for some c between 0 and x.
Thus, $|\text{error}| = \frac{|x|^4}{4|1+c|^4} \le \frac{|x|^4|}{4(1-|c|)^4} \le \frac{(1/10)^4}{4(1-1/10)^4} = \frac{1}{4\times 9^4}$, since $|x| \le 1/10$ implies $|c| \le 1/10$.
