Indian Institute of Technology Delhi - Abu Dhabi

AMTL101

Tutorial Sheet 12: Systems of ODEs

(1) Solve the following systems of ODEs.

(a) $x_1' = x_1 + 2x_2,\ x_2' = \frac{1}{2}x_1 + x_2$

(b) $x_1' = -8x_1 - 2x_2,\ x_2' = 2x_1 - 4x_2$

(c) $x_1' = x_2,\ x_2' = -x_1 + x_3,\ x_3' = -x_2$

(2) Solve the following IVPs.

(a) $x_1' = 2x_1 + 2x_2,\ x_2' = 5x_1 - x_2,\ x_1(0) = 0,\ x_2(0) = 7$

(b) $x_1' = -14x_1 + 10x_2,\ x_2' = -5x_1 + x_2,\ x_1(0) = -1,\ x_2(0) = 1$

(3) Solve the following systems of ODEs.

(a) $x_1' = x_2 + e^{3t},\ x_2' = x_1 - 3e^{3t}$

(b) $x_1' = -x_1 + x_2 + 10\cos t,\ x_2' = -3x_1 - x_2 - 10\sin t$

(c) $x_1' = x_1 + 4x_2 - 2\cos t,\ x_2' = x_1 + x_2 - \cos t + \sin t$

Solution 1 :

(a) $x_1' = x_1 + 2x_2$

$x_2' = \frac{1}{2}x_1 + x_2$

$$\Rightarrow \quad \vec{X}'(t) = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix} \vec{X}(t)$$

Let $A = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}$

$$\Rightarrow \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1/2 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 2\lambda + 1 - 1$$

$$= \lambda(\lambda - 2)$$

$\therefore$ e.values of A are $0, 2$.

Eigenvector corresponding to e.value 0 :

$$Au = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\approx \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad u_1 + 2u_2 = 0$$

$\therefore \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an e.vector with $\lambda = 0$.

$$\Rightarrow \quad \vec{x}_1(t) = e^{0 \cdot t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a solution.

Eigenvector corresponding to evalue 2 :

$$(A - 2I)\, v = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\approx \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad -v_1 + 2v_2 = 0$$

$\therefore \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an e.vector with $\lambda = 2$.

$\Rightarrow \quad \vec{x}_2(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a Solution.

$\therefore$ The general Solution of the given

System is

$$\vec{X}(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$$

$$= c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2c_1 + 2c_2 e^{2t} \\ c_1 + c_2 e^{2t} \end{bmatrix}$$

$$\Rightarrow \quad x_1(t) = -2c_1 + 2c_2 e^{2t}$$

$$x_2(t) = c_1 + c_2 e^{2t}.$$

<u>Solution 2</u> :

(a) $x_1' = 2x_1 + 2x_2$; $x_1(0) = 0$

$x_2' = 5x_1 - x_2$; $x_2(0) = 7$

$$\Rightarrow \quad \vec{X}' = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \vec{X}$$

So, $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$

$$\Rightarrow \quad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix}$$

$$= -2 - \lambda + \lambda^2 - 10$$

$$= \lambda^2 - \lambda - 12$$

$$= (\lambda - 4)(\lambda + 3)$$

$$\therefore \quad \lambda = 4, -3$$

$$\therefore \quad \vec{X}(t) = c_1 e^{4t} \vec{u} + c_2 e^{-3t} \vec{v}$$

where $(A - 4I)\vec{u} = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\approx \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad -u_1 + u_2 = 0$$

$$\therefore \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

and $(A + 3I)\vec{v} = 0$

$$\Rightarrow \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\approx \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad 5v_1 + 2v_2 = 0$$

$$\therefore \quad \vec{v} = \begin{pmatrix} 1 \\ -5/2 \end{pmatrix}.$$

$$\therefore \quad \vec{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -5/2 \end{pmatrix}.$$

$$= \begin{bmatrix} c_1 e^{4t} + c_2 e^{-3t} \\ c_1 e^{4t} - 5/2\, c_2 e^{-3t} \end{bmatrix}$$

$$\Rightarrow \quad x_1(t) = c_1 e^{4t} + c_2 e^{-3t}$$

$$x_2(t) = c_1 e^{4t} - 5/2\, c_2 e^{-3t}$$

$$\because \quad x_1(0) = 0 \Rightarrow c_1 + c_2 = 0$$

$$\& \quad x_2(0) = 7 \Rightarrow c_1 - 5/2\, c_2 = 7$$

$$\Rightarrow \quad 7/2\, c_2 = -7$$

$$\Rightarrow \quad c_2 = -2$$

$$\Rightarrow \quad c_1 = 2$$

$$\therefore \quad \vec{x}(t) = \begin{bmatrix} 2e^{4t} - 2e^{-3t} \\ 2e^{4t} + 5e^{-3t} \end{bmatrix}.$$

Solution 3.

(a) $x_1' = x_2 + e^{3t}$

$x_2' = x_1 - 3e^{3t}$

i.e. $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix}$$

So, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{g}(t) = \begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix}$

e. Values of A are : $\lambda = \pm 1$.

Also, e. vector corresponding to $\lambda = 1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = -1$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\therefore$ The general solution corresponding to the homogeneous system is

$$\vec{X}_h(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Particular Solution :

Let $\vec{X}_1(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$

& $\vec{X}_2(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

$\therefore$ The fundamental matrix :

$$\tilde{X}(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}$$

$$\Rightarrow \left(\tilde{X}(t)\right)^{-1} = -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$\therefore \quad \vec{u}\,'(t) = \left(\tilde{X}(t)\right)^{-1} \cdot \vec{g}(t)$$

$$= -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix} \cdot \begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2e^{2t} \\ -4e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2t} \\ 2e^{4t} \end{bmatrix}$$

$$\therefore \quad \vec{u}(t) = \begin{bmatrix} -\frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} \end{bmatrix}$$

$$\Rightarrow \quad \vec{X}_p(t) = \tilde{X}(t) \cdot \vec{u}(t)$$

$$= \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -e^{3t} \end{bmatrix}$$

$$\therefore \quad \vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -e^{3t} \end{pmatrix}.$$

$$= \begin{pmatrix} c_1 e^{t} + c_2 e^{-t} \\ c_1 e^{t} - c_2 e^{-t} - e^{3t} \end{pmatrix}.$$