Curved surface area of a cone:

Consider a cone with radius of its base r and height h.

We want to derive the formula for its curved surface area.

If we cut the cone along an edge and open it up, we get a sector of a circle of radius l and arc length $2\pi r$.

If the area of this sector is S, then

$$\frac{S}{\cancel{\pi} l^2} = \frac{\theta}{2\cancel{\pi}} \Rightarrow \boxed{S = \frac{1}{2}\theta l^2}$$

Also, $$\frac{\theta}{2\pi} = \frac{2\pi r}{2\pi l} \Rightarrow \theta = \frac{2\pi r}{l}$$

$\therefore$ Area $$S = \frac{1}{2} \times \frac{2\pi r}{l} \times l^2$$

i.e. $$\boxed{S = \pi r l}$$

Surface area of frustum of a cone

Area of frustum is

$$A = \pi r_2 (l_1 + l) - \pi r_1 l_1$$

Also, $$\frac{l_1}{l_1 + l} = \frac{r_1}{r_2}$$

$$\Rightarrow \quad l_1 r_2 = (l_1 + l) r_1$$

$$\Rightarrow \quad l_1 = \left(\frac{r_1}{r_2 - r_1}\right) l$$

$$\Rightarrow \quad l_1 + l = \left(\frac{r_1}{r_2 - r_1} + 1\right) l = \left(\frac{r_2}{r_2 - r_1}\right) l$$

$$\therefore \quad A = \pi r_2 \cdot \left(\frac{r_2}{r_2 - r_1}\right) l - \pi r_1 \left(\frac{r_1}{r_2 - r_1}\right) l$$

$$= \pi l \left(\frac{r_2^2 - r_1^2}{r_2 - r_1}\right)$$

$$= \pi l (r_1 + r_2) = 2\pi \left(\frac{r_1 + r_2}{2}\right) l$$

Improper integrals

Improper integral of the first kind:

Suppose f is bounded on $[a, \infty)$ and f is Riemann integrable on $[a, b]$ $\forall b > a$.

Then we define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx .$$

If the limit exists and is finite, then we say the improper integral converges. Otherwise it diverges.

Comparison test:

Let $0 \leq f(x) \leq g(x)$ $\forall x \geq a$. Then

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

e.g. $\displaystyle\int_1^\infty \frac{1}{x^2(1+x)}\,dx$

Since $0 < \dfrac{1}{x^2(1+x)} \leq \dfrac{1}{x^2}$ $\forall x \geq 1$,

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, therefore, by the comparison test, $\displaystyle\int_1^\infty \frac{1}{x^2(1+x)}\,dx$ converges.

Limit comparison test

Let $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

(i) If $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are either both convergent or both divergent.

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(ii) If $L=0$ and $\int_a^\infty g(x)\,dx$ converges,

then $\int_a^\infty f(x)\,dx$ converges.

(iii) If $L=\infty$ and $\int_a^\infty g(x)\,dx$ diverges,

then $\int_a^\infty f(x)\,dx$ diverges.

Examples:

① $\int_1^\infty \frac{1}{\sqrt{1+x}}\,dx$ $\qquad f(x)=\frac{1}{\sqrt{1+x}}\ ;\ g(x)=\frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{1+x}}=1$$

Since $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, by LCT,

$\int_1^\infty \frac{1}{\sqrt{1+x}}\,dx$ diverges.

Absolute convergence

We say $\int_a^\infty f(x)\,dx$ converges absolutely

if $\int_a^\infty |f(x)|\,dx$ converges.

Theorem: If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

Proof: $\quad 0 \le f(x) + |f(x)| \le 2|f(x)|$

If $\int_a^\infty |f(x)|dx$ converge, by comparison test, $\int_a^\infty (f(x) + |f(x)|)\,dx$ converges.

Since $f(x) = (f(x) + |f(x)|) - |f(x)|$,

$\int_a^\infty f(x)\,dx$ converges.

Example:

$\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent but not absolutely convergent.

$$\int_1^\infty \left|\frac{\sin x}{x}\right| dx = \underbrace{\int_1^\pi \frac{|\sin x|}{x}\,dx}_{<\infty} + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx$$

On the interval $[n\pi, (n+1)\pi]$,

$$\frac{|\sin x|}{x} \geq \frac{|\sin x|}{(n+1)\pi}$$

$$\therefore \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx$$

$$= \frac{1}{(n+1)\pi}\int_0^{\pi} \sin x\,dx, \quad \text{(by periodicity of } \sin x\text{)}$$

$$= \frac{2}{(n+1)\pi}$$

$$\therefore \int_1^{\infty} \frac{|\sin x|}{x}\,dx \geq \int_1^{\pi} \frac{\sin x}{x}\,dx + \sum_{n=1}^{\infty} \frac{2}{(n+1)\pi}$$

$$= \infty$$

$\therefore \int_1^{\infty} \frac{|\sin x|}{x}\,dx$ diverges.

To show: $\int_1^{\infty} \frac{\sin x}{x}\,dx$ converges.

$$\int_1^{b} \frac{\sin x}{x}\,dx = \int_1^{b} \frac{1}{x}\,d(1-\cos x)$$

$$= \frac{1}{x}(1-\cos x)\Big|_1^b - \int_1^b \left(\frac{-1}{x^2}\right)(1-\cos x)\,dx \quad \text{(by integration by parts)}$$

$$= \underbrace{\left(\frac{1-\cos b}{b}\right)}_{\to 0 \text{ as } b\to\infty} - \left(\frac{1-\cos 1}{1}\right) + \underbrace{\int_1^b \left(\frac{1-\cos x}{x^2}\right)dx}_{<\infty \text{ as } b\to\infty}$$