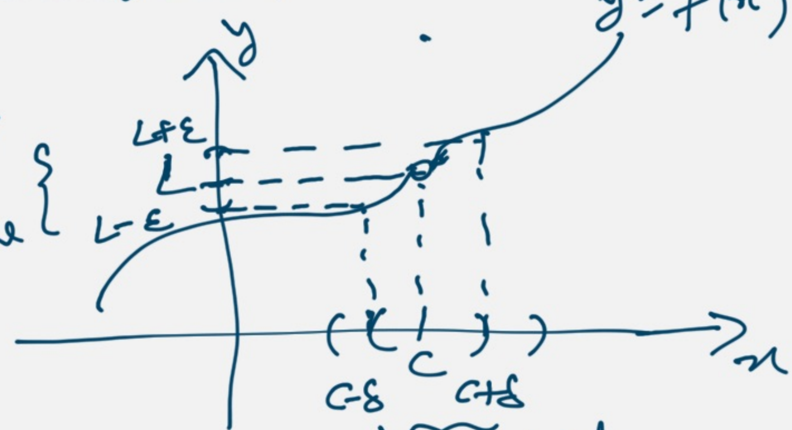
Formal defn. of limit:

We want $f(x)$ to lie here



We restrict x to be in this interval $(c-\delta, c+\delta)$ except $x = c$

$$x \in (c-\delta, c+\delta), \ x \neq c$$

$$\iff 0 < |x - c| < \delta$$

Want: $f(x) \in (L-\varepsilon, L+\varepsilon)$

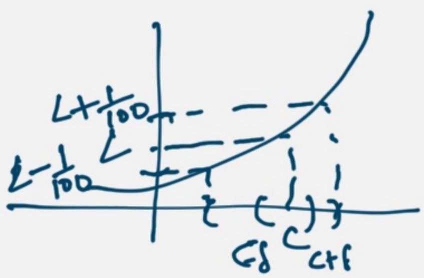
$$\iff |f(x) - L| < \varepsilon .$$

Defn: Let $f(x)$ be a function defined on an open interval about c, except possibly at $x = c$ itself. We say a real number L is limit of $f(x)$ as x approaches c, written as

$\lim_{x\to c} f(x) = L$ if

for any given $\varepsilon>0$, there exists a real number $\delta>0$ (depending on ε) such that

$$\boxed{0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon}.$$



$\varepsilon = \frac{1}{100}$

$\delta = ?$

Example: Prove that

$$\lim_{x\to 2}(3x-5) = 1$$

using ε-δ definition.

Solution: For a given $\varepsilon>0$, we need to find $\delta>0$ s.t.

$$0<|x-2|<\delta \Rightarrow |(3x-5)-1|<\varepsilon$$

We find δ by working backwards from: $|(3x-5)-1|<\varepsilon$

i.e. $|3x-6|<\varepsilon$ i.e. $3|x-2|<\varepsilon$

So we can take $\delta = \varepsilon/3$.

Then $|x-2| < \delta = \varepsilon/3$

$\Rightarrow 3|x-2| < 3\delta = \varepsilon$

$\Rightarrow |(3x-5)-1| < \varepsilon$

$\Rightarrow \lim_{x \to 2} (3x-5) = 1$

Example: $\lim_{x \to 1} \frac{1}{x} = 1$

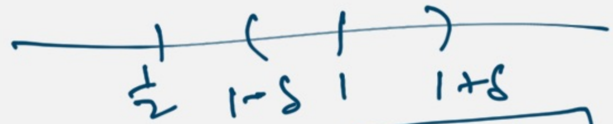
Soln: given $\varepsilon > 0$, we need to find $\delta > 0$ such that

$0 < |x-1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \varepsilon$

$\left|\frac{1}{x} - 1\right| < \varepsilon \Leftrightarrow \frac{|x-1|}{|x|} < \varepsilon$

If we had $|x|$ greater or equal to some positive number, say $1/2$, then

$\frac{|x-1|}{|x|} \leq 2|x-1| < 2\delta$ if $|x-1| < \delta$



If $1-\delta \geq \frac{1}{2}$, i.e. $\boxed{\delta \leq \frac{1}{2}}$

then $x > \frac{1}{2}$ for all $|x-1| < \delta$

So, we can choose $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$

then $|x-1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \varepsilon$