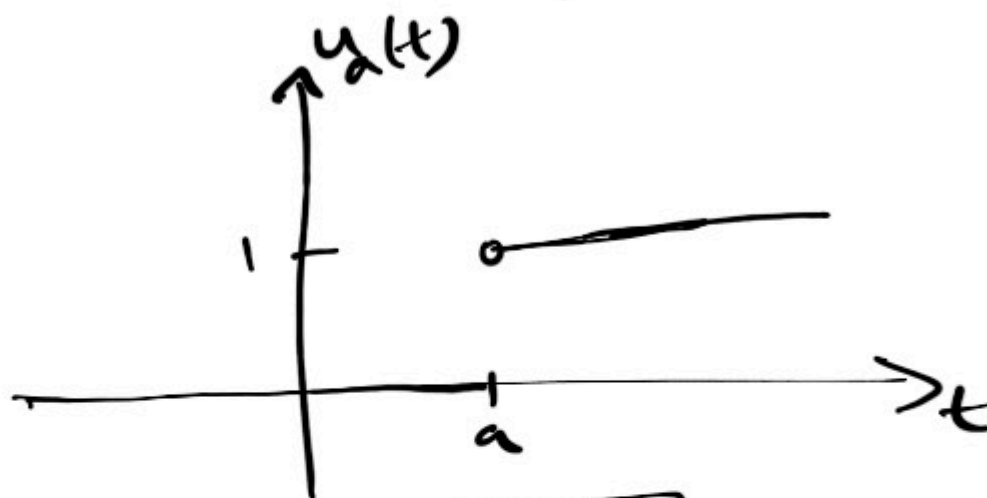


Heaviside functionFor $a \geq 0$, we define

$$u_a(t) = u(t-a) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$

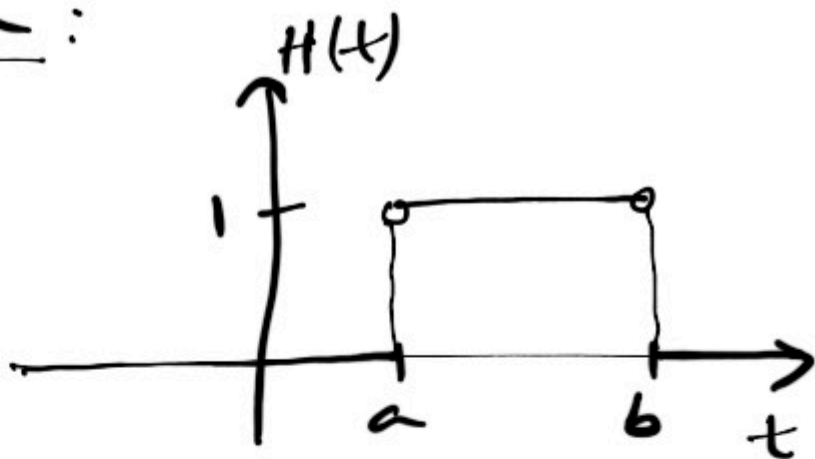


$$\mathcal{L}(u(t-a))(s) = \frac{e^{-as}}{s}$$

$$\mathcal{L}(f(t-a)u(t-a))(s) = e^{-as}\mathcal{L}(f)(s)$$

$$f(t-a)u(t-a) = \begin{cases} 0, & t \leq a \\ f(t-a), & t > a \end{cases}$$

Hat function:



For $0 \leq a < b$,

$$H(t) = \begin{cases} 1, & \text{if } a < t < b \\ 0, & \text{otherwise} \end{cases}$$

$$H(t) = u(t-a) - u(t-b)$$

Example: Let $f(t) = \begin{cases} 2, & 0 < t < 1 \\ \frac{\pi}{2}, & 1 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$

Find $\mathcal{L}(f)(s)$.

Solution: First we express $f(t)$ in terms of Heaviside function.

$$f(t) = \begin{cases} 2, & 0 < t < 1 \\ t^2/2, & 1 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$$

$$= 2[1 - u(t-1)] + \frac{t^2}{2}[u(t-1) - u(t-\frac{\pi}{2})] + \cos t \cdot u(t-\frac{\pi}{2})$$

$$\begin{aligned} \therefore \mathcal{L}(f)(s) &= 2(\mathcal{L}(1) - \mathcal{L}(u(t-1))) \\ &\quad + \frac{1}{2}\mathcal{L}(t^2 u(t-1)) - \frac{1}{2}\mathcal{L}(t^2 u(t-\frac{\pi}{2})) \\ &\quad + \mathcal{L}(\cos t \cdot u(t-\frac{\pi}{2})) \\ &= 2\left[\frac{1}{s} - \frac{e^{-s}}{s}\right] + \dots \end{aligned}$$

$$\begin{aligned}\text{Now, } t^2 u(t-1) &= \{(t-1)+1\}^2 u(t-1) \\ &= (t-1)^2 u(t-1) + 2(t-1)u(t-1) \\ &\quad + u(t-1)\end{aligned}$$

$$\begin{aligned}\therefore \mathcal{L}(t^2 u(t-1)) &= \mathcal{L}((t-1)^2 u(t-1)) \\ &\quad + 2\mathcal{L}((t-1)u(t-1)) \\ &\quad + \mathcal{L}(u(t-1))\end{aligned}$$

$$= e^{-s} \mathcal{L}(t^2) + 2e^{-s} \mathcal{L}(t) + \frac{e^{-s}}{s}$$

$$= e^{-s} \cdot \frac{2}{s^3} + 2e^{-s} \cdot \frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$\begin{aligned}\text{Similarly } t^2 u(t-\frac{\pi}{2}) &= \{(t-\frac{\pi}{2})+\frac{\pi}{2}\}^2 u(t-\frac{\pi}{2}) \\ &= (t-\frac{\pi}{2})^2 u(t-\frac{\pi}{2}) + \pi(t-\frac{\pi}{2})u(t-\frac{\pi}{2}) \\ &\quad + \frac{\pi^2}{4} u(t-\frac{\pi}{2})\end{aligned}$$

$$\therefore \mathcal{L}(t^2 u(t-\frac{\pi}{2})) = e^{-\frac{\pi}{2}s} \cdot \frac{2}{s^3} + \pi e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2} + \frac{\pi^2}{4} \frac{e^{-\frac{\pi}{2}s}}{s}$$

$$\text{Also, } \cos t \, u(t - \frac{\pi}{2}) = \sin(\frac{\pi}{2} - t) \, u(t - \frac{\pi}{2}) \\ = -\sin(t - \frac{\pi}{2}) \, u(t - \frac{\pi}{2})$$

$$\therefore \mathcal{L}(\cos t \, u(t - \frac{\pi}{2})) = -\mathcal{L}(\sin(t - \frac{\pi}{2}) \, u(t - \frac{\pi}{2})) \\ = -e^{-\frac{\pi}{2}s} \mathcal{L}(\sin t)(s) \\ = -e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2 + 1}$$

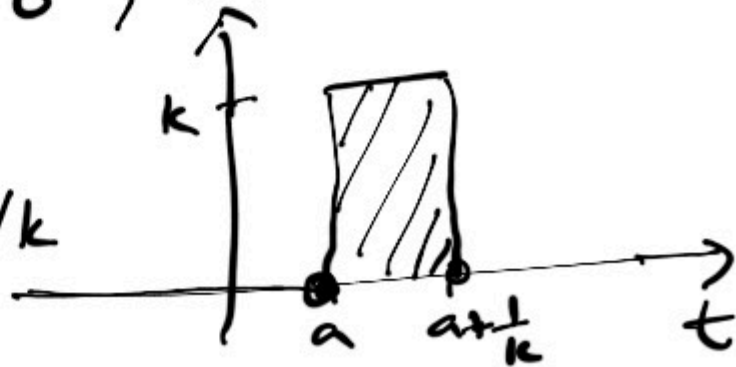
Dirac delta "function"

For $k \in \mathbb{N}$, let

$$f_k(t-a) = \begin{cases} k, & a \leq t \leq a + \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

Note that

$$\int_0^{\infty} f_k(t-a) dt = 1 \quad \forall k$$



define the Dirac-delta function as

$$\delta(t-a) = \lim_{k \rightarrow \infty} f_k(t-a)$$

$$= \begin{cases} \infty & , t=a \\ 0 & , t \neq a \end{cases}$$



Property:

$$\textcircled{1} \int_0^{\infty} \delta(t-a) dt = 1$$

$$\textcircled{2} \mathcal{L}(\delta(t-a)) = ?$$

$$\delta(t-a) = \lim_{k \rightarrow \infty} f_k(t-a)$$

$$f_k(t-a) = \begin{cases} k & , a \leq t \leq a + \frac{1}{k} \\ 0 & , \text{otherwise} \end{cases}$$

$$= k \left[u(t-a) - u\left(t - \left(a + \frac{1}{k}\right)\right) \right]$$

$$\begin{aligned}
 \therefore \mathcal{L}(f_k(t-a)) &= k \left[\mathcal{L}(u(t-a)) - \mathcal{L}(u(t-(a+\frac{1}{k}))) \right] \\
 &= k \left[\frac{e^{-as}}{s} - \frac{e^{-(k+\frac{1}{k})s}}{s} \right] \\
 &= e^{-as} \left[\frac{1 - e^{-\frac{1}{k}s}}{s/k} \right]
 \end{aligned}$$

$$\begin{aligned}
 \therefore \mathcal{L}(\delta(t-a)) &= \lim_{k \rightarrow \infty} \mathcal{L}(f_k(t-a)) \\
 &= \lim_{k \rightarrow \infty} e^{-as} \left[\frac{1 - e^{-s/k}}{s/k} \right]
 \end{aligned}$$

(0/0 form)

L'Hôpital's rule

$$= e^{-as} \lim_{k \rightarrow \infty} \frac{+e^{-s/k} \cdot \frac{s}{k^2}}{+s/k^2}$$

$$= e^{-as}$$

$$\therefore \boxed{\mathcal{L}(\delta(t-a)) = e^{-as}}$$

Solving an IVP involving Dirac delta function.

Example: $y'' + 3y' + 2y = \delta(t-1)$,
 $y(0) = 0$, $y'(0) = 0$.

Taking the Laplace transform, we get

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(\delta(t-1))$$

$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + 3[s Y(s) - y(0)] + 2Y(s) = e^{-s}$$

$$\Rightarrow (s^2 + 3s + 2) Y(s) = e^{-s}$$

$$\Rightarrow Y(s) = e^{-s} \cdot \frac{1}{(s+1)(s+2)}$$

$$= e^{-s} \left[\frac{1}{s+1} - \frac{1}{s+2} \right]$$

$$\Rightarrow Y(s) = \frac{\bar{e}^{-s}}{s+1} - \frac{\bar{e}^{-s}}{s+2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{\bar{e}^{-s}}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{\bar{e}^{-s}}{s+2}\right]$$

$$= \mathcal{L}^{-1}\left[\bar{e}^{-s} \mathcal{L}(\bar{e}^t)(s)\right]$$

$$- \mathcal{L}^{-1}\left[\bar{e}^{-s} \mathcal{L}(\bar{e}^{2t})(s)\right]$$

$$= \bar{e}^{(t-1)} u(t-1) - \bar{e}^{2(t-1)} u(t-1)$$

because $\mathcal{L}^{-1}(\bar{e}^{-as} \mathcal{L}(f)) = f(t-a) u(t-a)$.

$$\therefore y(t) = \begin{cases} 0, & t < 1 \\ \bar{e}^{(t-1)} - \bar{e}^{2(t-1)}, & t > 1 \end{cases}$$