Indian Institute of Technology Delhi - Abu Dhabi

AMTL101

Tutorial Sheet 9: Applications to Second Order ODEs and Higher Order ODEs

(1) Model the RC-circuit and LC-circuit. Also, find the current $I(t)$ in each of the circuits for constant voltage supply.

(2) Consider the mass-spring system with $m = 1$ kg, $c = 4$ kg/sec, $k = 24$ kg/sec^2 and $F(t) = 10\cos(\omega t)$ Newton. Determine ω such that you get the steady-state vibration of maximum possible amplitude. Determine this amplitude. Then find the general solution with this ω and check whether the results are in agreement.

(3) Solve the following ODEs:
   (a) $y''' + 25y' = 0$
   (b) $y^{(4)} + 2y'' + y = 0$
   (c) $y^{(4)} + 10y'' + 9y = 0$

(4) Solve the following initial value problems:
   (a) $y^{(4)} + 4y = 0,\ y(0) = 1/2,\ y'(0) = -3/2,\ y''(0) = 5/2,\ y'''(0) = -7/2$
   (b) $y^{(4)} - 9y'' - 400y = 0,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 41,\ y'''(0) = 0$

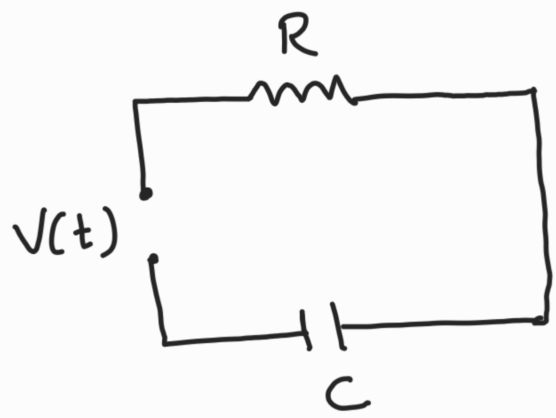
(5) Solve the following ODEs:
   (a) $y''' + 3y'' + 3y' + y = e^x - x - 1$
   (b) $y''' + 2y'' - y' - 2y = 1 - 4x^3$
   (c) $y''' + 4y' = \sin x$
   (d) $y''' + 4y' = \sin 2x$

(6) Use the variation of parameters method to solve the following equations:

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3\ln x$$

Solution 1 :

RC - Circuit :



$$\because \quad RI + \frac{Q}{C} = V(t)$$

$$\& \quad I = \frac{dQ}{dt}$$

$$\Rightarrow \quad RQ' + \frac{1}{C}Q = V(t)$$

$$\Rightarrow \quad Q' + \frac{1}{RC}Q = \frac{V(t)}{R}$$

$$\therefore \quad I.F. = e^{\frac{1}{RC}t}$$

$$\text{and} \quad Q \cdot e^{\frac{1}{RC}t} = \int \frac{V(t)}{R} \cdot e^{\frac{1}{RC}t} + \alpha$$

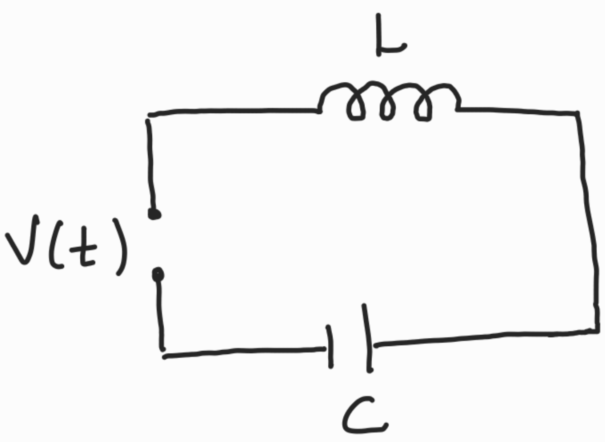
$$\Rightarrow \quad Q(t) = \frac{e^{-\frac{1}{RC}t}}{R} \int V(t) \cdot e^{\frac{1}{RC}t} + \alpha e^{-\frac{1}{RC}t} .$$

If $V(t)$ is constant, say C_0, then

$$Q(t) = C_0 C + \alpha e^{-\frac{1}{RC}t}$$

$$\Rightarrow \quad I(t) = Q'(t) = -\frac{\alpha}{RC} e^{-\frac{1}{RC}t}.$$

LC - Circuit :



$$L\frac{dI}{dt} + \frac{1}{C}Q = V(t)$$

$$\Rightarrow \quad L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = V(t)$$

$$\Rightarrow \quad Q'' + \frac{1}{LC}Q = \frac{1}{L}V(t).$$

(i) Free oscillation: If $V(t) = 0$

$$\Rightarrow \quad Q'' + \frac{1}{LC} Q = 0$$

$$\therefore \quad m^2 = -\frac{1}{LC} \Rightarrow m = \pm \frac{1}{\sqrt{LC}} i$$

$$\therefore \quad Q_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

where, $\omega_0 = \frac{1}{\sqrt{LC}}$

(iii) Forced Oscillation : If $V(t) \neq 0$.

Suppose $V(t) = V_0 \cos \omega t$, $\omega \neq \omega_0$.

Let $Q_p(t) = A \cos \omega t + B \sin \omega t$

$$\Rightarrow Q_p'(t) = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$Q_p''(t) = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$\Rightarrow -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t + \frac{1}{LC} A \cos \omega t$$

$$+ \frac{1}{LC} B \sin \omega t = \frac{V_0 \cos \omega t}{L}$$

$$\Rightarrow \quad -A\omega^2 + \frac{A}{LC} = \frac{V_0}{L} \quad \text{———} (1)$$

& $-B\omega^2 + \dfrac{B}{LC} = 0$ —— (2)

$$\Rightarrow \quad B\left(-\omega^2 + \frac{1}{LC}\right) = 0$$

$$\Rightarrow \quad B = 0 \quad \left(\because \omega \neq \omega_0\right)$$

and $A\left(-\omega^2 + \omega_0^2\right) = \dfrac{V_0}{L}$

$$\Rightarrow \quad A = \frac{V_0}{L(\omega_0^2 - \omega^2)}$$

$$\Rightarrow \quad Q_p(t) = \frac{V_0}{L(\omega_0^2 - \omega^2)} \cos \omega t$$

$$\therefore \quad Q(t) = Q_n(t) + Q_p(t).$$

(iii) If $V(t)$ is constant, then

$$Q'' + \frac{1}{LC} Q = \frac{C_0}{L}$$

by case (i),

$$Q_n(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

Let $Q_p(t) = A_0$

$$\Rightarrow \quad Q_p'' = 0$$

$$\Rightarrow \quad \frac{1}{LC} A_0 = \frac{C_0}{L}$$

$$\Rightarrow \quad A_0 = C_0 C$$

$$\therefore \quad Q_p(t) = C_0 C$$

$$\therefore \quad Q(t) = Q_n(t) + Q_p(t)$$

$$= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + C_0 C$$

$$\Rightarrow \quad I(t) = Q'(t)$$

$$= -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t.$$

Solution 2 :

$m = 1$ kg , $c = 4$ kg/sec , $K = 24$ kg/sec^2

$F(t) = 10 \cos \omega t$

$$\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m^2} , \quad \omega_0 = \sqrt{K/m}$$

$$= 24 - \frac{16}{2}$$

$$= 16$$

$$\Rightarrow \quad \omega_{max} = 4$$

& $$c^*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

$$= \frac{2 \cdot 1 \cdot 10}{4\sqrt{4 \cdot 1 \cdot 24 - 16}}$$

$$= \frac{\sqrt{5}}{4}$$

Now, $my'' + cy' + ky = F_0 \cos \omega t$

$$\Rightarrow \quad y'' + 4y' + 24y = 10 \cos 4t$$

Char. eqn : $m^2 + 4m + 24 = 0$

$$\Rightarrow \quad m = \frac{-4 \pm \sqrt{16 - 96}}{2}$$

$$= \frac{-4 \pm 4\sqrt{5}i}{2}$$

$$= -2 \pm 2\sqrt{5}\, i$$

$$\therefore \quad y_h(t) = e^{-2t}\left[C_1 \cos 2\sqrt{5}t + C_2 \sin 2\sqrt{5}t \right]$$

Let $y_p(t) = A \cos 4t + B \sin 4t$

$$\therefore \quad y_p'(t) = -4A \sin 4t + 4B \cos 4t$$

$$y_p''(t) = -16A \cos 4t - 16B \sin 4t$$

Substituting in the equation, we get

$$-16A\cos 4t - 16B\sin 4t - 16A\sin 4t$$

$$+16B\cos 4t + 24A\cos 4t + 24B\sin 4t$$

$$= 10\cos 4t$$

$$\Rightarrow \quad -16B - 16A + 24B = 0$$

$$\Rightarrow \quad -16A + 8B = 0 \quad \text{---(i)}$$

and $\quad -16A + 16B + 24A = 10$

$$\Rightarrow \quad 8A + 16B = 10$$

$$\Rightarrow \quad 16A + 32B = 20 \quad \text{---(ii)}$$

from (i) & (ii),

$$40B = 20$$

$$\Rightarrow \quad B = 1/2$$

$$\Rightarrow \quad A = B/2 = 1/4$$

$$\therefore \quad y_p(t) = \frac{1}{4}\cos 4t + \frac{1}{2}\sin 4t$$

$$\therefore \quad y(t) = y_h(t) + y_p(t)$$

$$= e^{-2t}\left[c_1 \cos 2\sqrt{5}t + c_2 \sin 2\sqrt{5}t\right]$$

$$+ \frac{1}{4}\cos 4t + \frac{1}{2}\sin 4t$$

as $t \to \infty$, $y_h(t) \longrightarrow 0$

and $y(t)$ approaches $y_p(t)$.

$$\text{amplitude} = \sqrt{\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{16} + \frac{1}{4}} = \frac{\sqrt{5}}{4}.$$

Solution 3 :

(a) $y''' + 25y' = 0$

Char. eqn : $m^3 + 25m = 0$

$$\Rightarrow \quad m(m^2 + 25) = 0$$

$$\Rightarrow \quad m = 0, \pm 5i$$

$$y_h(t) = c_1 + c_2 \cos 5t + c_3 \sin 5t.$$

(b) $\quad y^{(4)} + 2y'' + y = 0$

Char. eq^n: $\quad m^4 + 2m^2 + 1 = 0$

$$\Rightarrow (m^2+1)^2 = 0$$

$$\Rightarrow \quad m = \pm i, \pm i$$

$$\therefore \quad y_h(t) = c_1 \cos t + c_2 \sin t$$

$$+ t\,(c_3 \cos t + c_4 \sin t)$$

Solution 4 :

(a) $\quad y^{(4)} + 4y = 0$

Char. eq^n: $\quad m^4 + 4 = 0$

$$\Rightarrow \quad m^2 = \pm 2i$$

$$\Rightarrow \quad m^2 = 2i = 2e^{i\pi/2}$$

$$\Rightarrow \quad m = \pm\sqrt{2}\, e^{i\pi/4} = \pm\sqrt{2}\left[\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right]$$

$$= \pm(1+i)$$

also $$m^2 = -2i = 2e^{-i\pi/2}$$

$$\Rightarrow \quad m = \pm\sqrt{2}\, e^{-i\pi/4} = \pm\sqrt{2}\left[\tfrac{1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}}\right]$$

$$= \pm(1-i)$$

$$\therefore \quad y_h(t) = e^{t}\left[c_1 \cos t + c_2 \sin t\right]$$

$$+ e^{-t}\left[c_3 \cos t + c_4 \sin t\right]$$

now, $y(0) = 1/2$, $y'(0) = -3/2$

$y''(0) = 5/2$, $y'''(0) = -7/2$

$$\because \quad y(0) = c_1 + c_3 = 1/2 \quad \text{——} (1)$$

$$y_h'(t) = e^{t}\left[-c_1 \sin t + c_2 \cos t\right]$$

$$+ e^{t}\left[c_1 \cos t + c_2 \sin t\right]$$

$$+ e^{-t}\left[-c_3 \sin t + c_4 \cos t\right]$$

$$- e^{-t}\left[c_3 \cos t + c_4 \sin t\right]$$

$$= e^{t}\left[(c_1+c_2)\cos t + (c_2-c_1)\sin t\right]$$
$$+ e^{-t}\left[(-c_3+c_4)\cos t + (-c_3-c_4)\sin t\right]$$

$\because \; y'(0) = -3/2$

$$\Rightarrow \quad c_1 + c_2 - c_3 + c_4 = -3/2 \quad \text{——} (2)$$

$$y_h''(t) = e^{t}\left[2c_2\cos t - 2c_1 \sin t\right]$$
$$+ e^{-t}\left[2c_4\cos t + 2c_3\sin t\right]$$

$y''(0) = 5/2$

$$\Rightarrow \quad 2c_2 + 2c_4 = 5/2 \quad \text{——} (3)$$

$$\& \quad y_h'''(t) = e^{t}\left[2(c_2-c_1)\cos t + (-2c_1-2c_2)\sin t\right]$$
$$+ e^{-t}\left[2(c_3-c_4)\cos t + (-2c_4-2c_3)\sin t\right]$$

$\because \; y'''(0) = -7/2$

$$\Rightarrow \quad 2(c_2-c_1) + 2(c_3-c_4) = -7/2$$

$$\Rightarrow \quad -c_1 + c_2 + c_3 - c_4 = -\frac{7}{4} \quad \text{—}(4)$$

So, we have a system of 4 equations in 4 unknowns, which can be solved for c_1, c_2, c_3 & c_4.

Solution 5.

(c) $y''' + 4y' = \sin x$.

char. eq^n: $m^3 + 4m = 0$

$$\Rightarrow m(m^2+4) = 0$$

$$\Rightarrow m = 0, \pm 2i$$

$$\therefore \quad y_h(x) = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

Let $y_p(x) = A\cos x + B\sin x$

$$y_p'(x) = -A\sin x + B\cos x$$

$$y_p''(x) = -A\cos x - B\sin x$$

$$y_p'''(x) = A\sin x - B\cos x$$

$$\because \quad y_p''' + 4y_p' = \sin x$$

$$\Rightarrow \quad A\sin x - B\cos x + 4\left[-A\sin x + B\cos x\right] = \sin x$$

$$\Rightarrow \quad -3A\sin x + 3B\cos x = \sin x$$

$$\Rightarrow \quad A = -\frac{1}{3}, \quad B = 0$$

$$\therefore \quad y_p(x) = -\frac{1}{3}\cos x$$

$$\Rightarrow \quad y(x) = y_h(x) + y_p(x)$$

$$= C_1 + C_2\cos 2x + C_3\sin 2x - \frac{1}{3}\cos x.$$

Solution 6 :

$$x^3 y''' + x^2 y'' - 2x y' + 2y = x^3 \ln x .$$

Char. eq^n :

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$$

$$\Rightarrow \quad [m(m-2) + m - 2](m-1) = 0$$

$$\Rightarrow \quad (m+1)(m-2)(m-1) = 0$$

$$\Rightarrow \quad m = 1, 2, -1$$

$$\therefore \quad y_h(x) = c_1 x + c_2 x^2 + \frac{c_3}{x}$$

Let $y_1 = x$, $y_2 = x^2$, $y_3 = \frac{1}{x}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -\frac{1}{x^2} \\ 0 & 2 & 2/x^3 \end{vmatrix}$$

$$= x\left[\frac{4x}{x^3} + 2/x^2\right]$$

$$-1\left[2x^2/x^3 - 2/x\right]$$

$$= x\cdot\frac{6}{x^2} = 6/x$$

$$\therefore\ y_p(x) = u_1 y_1 + u_2 y_2 + u_3 y_3$$

where,

$$u_1' = \frac{W_1\, r(x)}{W}, \quad r(x) = \ln x$$

$$W_1 = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix}$$

$$= -1 - 2 = -3$$

$$u_1' = \frac{-3}{6}\cdot x\cdot \ln x$$

$$\Rightarrow \quad u_1' = -\frac{x}{2} \ln x$$

$$\Rightarrow \quad u_1 = -\frac{1}{2} \int x \ln x \, dx$$

$$= -\frac{1}{2} \cdot \left[\ln x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right.$$

$$= -\frac{1}{2} \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4} \right]$$

$$u_2' = \frac{W_2 \, g(x)}{W}$$

$$W_2 = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix}$$

$$= -1 \left[-\frac{1}{x} - \frac{1}{x} \right]$$

$$= 2/x$$

$$\therefore \quad u_2' = \frac{2}{x} \cdot \frac{x}{6} \ln x$$

$$= \frac{1}{3} \ln x$$

$$\therefore \quad u_2 = \frac{1}{3}\int \ln x \, dx$$

$$= \frac{1}{3}\left[x \ln x - \int \frac{1}{x} \cdot x \, dx \right]$$

$$= \frac{1}{3}\left[x \ln x - x \right]$$

$$= \frac{x}{3}(\ln x - 1)$$

and $$u_3' = \frac{W_3 \, g(x)}{W}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2$$

$$\therefore \quad u_3' = \frac{x^2}{6} \cdot x \cdot \ln x$$

$$= \frac{x^3 \ln x}{6}$$

$$\Rightarrow \quad u_3 = \int \frac{x^3}{6} \ln x \, dx$$

$$= \frac{1}{6}\left[\ln x \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4} dx\right]$$

$$= \frac{1}{6}\left[\frac{x^4}{4} \ln x - \frac{x^4}{16}\right]$$

$$= \frac{x^4}{24}\left(\ln x - \frac{1}{4}\right)$$

$$\therefore \quad y_p = -\frac{x^3}{4}\left(\ln x - \frac{1}{2}\right)$$

$$+ \frac{x^3}{3}\left(\ln x - 1\right)$$

$$+ \frac{x^3}{24}\left(\ln x - \frac{1}{4}\right)$$

$$= \left(-\frac{1}{4} + \frac{1}{3} + \frac{1}{24}\right) x^3 \ln x$$

$$+ \left(\frac{1}{8} - \frac{1}{3} - \frac{1}{96}\right) x^3$$

$$= \left(\frac{-6+8+1}{24}\right) x^3 \ln x$$

$$+ \left(\frac{12-32-1}{96}\right) x^3$$

$$= \frac{1}{8} x^3 \ln x - \frac{7}{32} x^3$$

$$= x^3 \left(\frac{4 \ln x - 7}{32}\right).$$

$$\therefore \quad y(x) = y_h(x) + y_p(x).$$