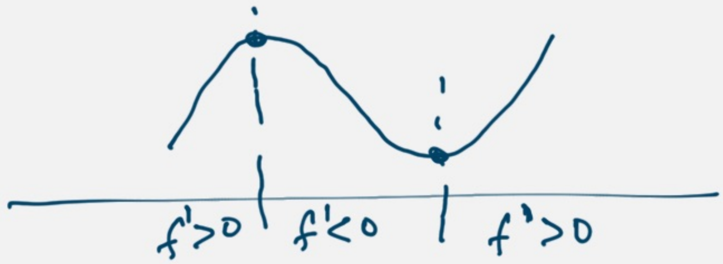
First derivative test for local extrema (minima or maxima):



(i) If f' changes sign from +ve to −ve at the point c, then f has a local max. at $x=c$.

(ii) If f' changes from −ve to +ve, then f has a local minimum at that point.

Remark: The function f may not be differentiable at $x=c$ to apply the first derivative test. We only require f to be differentiable to the left and to the right of $x=c$.

e.g. $f(x) = |x| = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$

$f'(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$

Concavity of a function

Concave up

Concave down

f' is decreasing

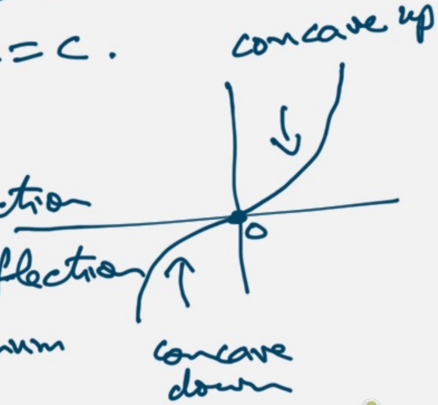
Thm: If $f'' > 0$ on an interval, then f is concave up.
If $f'' < 0$ on an interval, then f is concave down.

Point of inflection:

A point $(c, f(c))$ is called a point of inflection for the function f if $f'(c)$ exists and the concavity changes around $x = c$.

e.g. $f(x) = x^3$

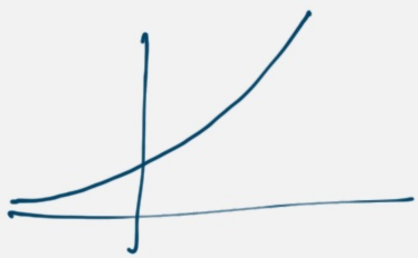
$(0,0)$ is a pt. of inflection

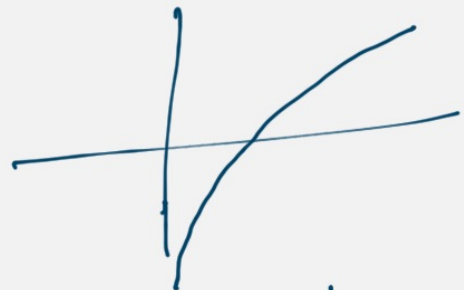


Note that at a pt. of inflection there is no local extremum

$f(x) = e^x$: concave up

$g(x) = \ln x$: concave down





$f''(x) = e^x > 0$

$g'(x) = \frac{1}{x}$

$g''(x) = -\frac{1}{x^2} < 0$ for $x > 0$

Second derivative test:

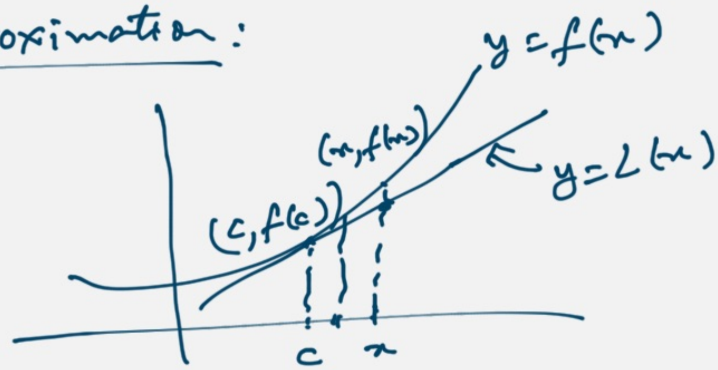
Suppose f is twice differentiable at $x=c$ and suppose $f'(c) = 0$.

(i) If $f''(c) < 0$, then f has a local maximum at c.

(ii) If $f''(c) > 0$, then f has a local minimum at c.

(iii) If $f''(c) = 0$, the test fails. We ~~may~~ have local max./local min/ neither.

Linear approximation:



$$L(x) = f(c) + f'(c)(x-c)$$

$$f(x) \approx L(x)$$

Error, $E(x) = |f(x) - L(x)|$

Use the linear approx. to find an approx. value of (i) $\sqrt{4.1}$ (ii) $\sqrt{4.01}$

Take $f(x) = \sqrt{x}$, $c = 4$

$$f(x) \approx f(c) + f'(c)(x-c)$$

$$f(c) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(c) = \frac{1}{4}$$

$$f(x) \approx 2 + \frac{1}{4}(x-c)$$

Put $x = 4.1$, $\sqrt{4.1} \approx 2 + \frac{1}{4} \times 0.1$

$= 2.025$

$$\sqrt{4.01} = f(4.01) \approx 2 + \frac{1}{4} \times 0.01$$
$$= 2.0025$$