**Indian Institute of Technology Delhi-Abu Dhabi**
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# Contents

# Chapter 1

# Matrices

## 1 Definition and Some Examples

A rectangular array of numbers is called a matrix.
The horizontal arrays of a matrix are called its "Rows" and the vertical arrays are called its "Columns". A matrix having $m$ rows and $n$ columns is said to have the order $m \times n$.

A matrix $A$ of order $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $a_{ij}$ is the entry at the intersection of the $i^{\text{th}}$ row and $j^{\text{th}}$ column. In a more concise manner, we also denote the matrix $A$ by $[a_{ij}]$ by suppressing its order.

**Remark 1.** *Some books also use* $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ *to represent a matrix.*

**Example 1.** Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$.

Then $a_{11} = 1, a_{12} = 3, a_{13} = 7, a_{21} = 4, a_{22} = 5$, and $a_{23} = 6$.

A matrix having only one column is called a column vector; and a matrix with only one row is called a row vector.

Whenever a vector is used, it should be understood from the context whether it is a row vector or a column vector.

**Definition 1.** *(Equality of two Matrices) Two matrices* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *having the same order* $m \times n$ *are equal if* $a_{ij} = b_{ij}$ *for each* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$*.*

In other words, two matrices are said to be equal if they have the same order and their corresponding entries are equal.

## 1.1 Some Special Matrices

1. A matrix in which each entry is zero is called a zero-matrix, denoted by $\mathbf{0}$. For example,

$$\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. A matrix having the number of rows equal to the number of columns is called a square matrix. Thus, its order is $m \times m$ (for some $m$ ) and is represented by $m$ only.

3. In a square matrix, $A = [a_{ij}]$, of order $n$, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal entries and form the principal diagonal of $A$.

4. A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. In other words, the non-zero entries appear only on the principal diagonal. For example, the zero matrix $\mathbf{0}_n$ and $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

A diagonal matrix $D$ of order $n$ with the diagonal entries $d_1, d_2, \ldots, d_n$ is denoted by $D = \operatorname{diag}(d_1, \ldots, d_n)$.
If $d_i = d$ for all $i = 1, 2, \ldots, n$ then the diagonal matrix $D$ is called a scalar matrix.

5. A square matrix $A = [a_{ij}]$ with $a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$
is called the identity matrix, denoted by $I_n$.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. A square matrix $A = [a_{ij}]$ is said to be an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

A square matrix $A = [a_{ij}]$ is said to be a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

A square matrix $A$ is said to be triangular if it is an upper or a lower triangular matrix.

For example $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ is an upper triangular matrix. An upper triangular matrix is represented by $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.

## 2 Operations on Matrices

**Definition 2.** *(Transpose of a Matrix) The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is defined as the $n \times m$ matrix $B = [b_{ij}]$, with $b_{ij} = a_{ji}$ for $1 \le i \le m$ and $1 \le j \le n$. The transpose of $A$ is denoted by $A^t$.*

That is, by the transpose of an $m \times n$ matrix $A$, we mean a matrix of order $n \times m$ having the rows of $A$ as its columns and the columns of $A$ as its rows.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$.

Thus, the transpose of a row vector is a column vector and vice versa.

**Theorem 1.** *For any matrix $A$, we have $\left(A^t\right)^t = A$.*

*Proof.* Let $A = [a_{ij}], A^t = [b_{ij}]$ and $\left(A^t\right)^t = [c_{ij}]$. Then, the definition of transpose gives

$$c_{ij} = b_{ji} = a_{ij} \quad \text{for all} \ \ i, j$$

and the result follows. □

**Definition 3.** *(Addition of Matrices) let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the sum $A + B$ is defined to be the matrix $C = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$.*

Note that we define the sum of two matrices only when the order of the two matrices is the same.

**Definition 4.** *(Multiplying a Scalar to a Matrix) Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then for any $k \in \mathbb{R}$, we define $kA = [ka_{ij}]$.*

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ and $k = 5$, then $5A = \begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}$.

**Theorem 2.** *Let $A, B$ and $C$ be matrices of order $m \times n$, and let $k, \ell \in \mathbb{R}$. Then*

1. $A + B = B + A$ *(commutativity).*
2. $(A + B) + C = A + (B + C)$ *(associativity).*
3. $k(\ell A) = (k\ell)A$.
4. $(k + \ell)A = kA + \ell A$.

**Definition 5.** *(Additive Inverse and Identity) Let $A$ be an $m \times n$ matrix.*

1. *Then there exists a matrix $B$ with $A + B = \mathbf{0}$. This matrix $B$ is called the additive inverse of $A$, and is denoted by $-A = (-1)A$.*
2. *Also, for the matrix $\mathbf{0}_{m\times n}$, $A + \mathbf{0} = \mathbf{0} + A = A$. Hence, the matrix $\mathbf{0}_{m\times n}$ is called the additive identity.*

**Definition 6.** *For a square matrix $A$ of order $n$, we define trace of $A$, denoted by $\operatorname{tr}(A)$ as*

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots a_{nn}.$$

**Theorem 3.** *For two square matrices, $A$ and $B$ of the same order, we have*

1. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
2. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

## 2.1 Multiplication of Matrices

**Definition 7.** *(Matrix Multiplication / Product) Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times r$ matrix. The product $AB$ is a matrix $C = [c_{ij}]$ of order $m \times r$, with*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Observe that the product $AB$ is defined if and only if the number of columns of $A$ is equal to the number of rows of $B$.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ then

$$\begin{aligned} AB &= \begin{bmatrix} 1+0+3 & 2+0+0 & 1+6+12 \\ 2+0+1 & 4+0+0 & 2+12+4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 & 19 \\ 3 & 4 & 18 \end{bmatrix}. \end{aligned}$$

Note that in this example, while $AB$ is defined, the product $BA$ is not defined. However, for square matrices $A$ and $B$ of the same order, both the product $AB$ and $BA$ are defined.

**Definition 8.** *Two square matrices $A$ and $B$ are said to commute if $AB = BA$.*

**Remark 2.** *In general, the matrix product is not commutative. For example, consider the following two matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then check that the matrix product*

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA$$

**Theorem 4.** *Suppose that the matrices $A, B$ and $C$ are so chosen that the matrix multiplications are defined.*

1. *Then $(AB)C = A(BC)$. That is, the matrix multiplication is associative.*

2. *For any $k \in \mathbb{R}, (kA)B = k(AB) = A(kB)$.*

3. *Then $A(B + C) = AB + AC$. That is, multiplication distributes over addition.*

4. *If $A$ is an $n \times n$ matrix then $AI_n = I_nA = A$.*

## 3 Some More Special Matrices

**Definition 9.** 1. *A matrix $A$ over $\mathbb{R}$ is called* ***symmetric*** *if $A^t = A$ and* ***skew-symmetric*** *if $A^t = -A$.*

2. *A matrix $A$ is said to be* ***orthogonal*** *if $AA^t = A^tA = I$.*

3. *The matrices $A$ for which a positive integer $k$ exists such that $A^k = \mathbf{0}$ are called* ***nilpotent*** *matrices. The least positive integer $k$ for which $A^k = \mathbf{0}$ is called the order of nilpotency.*

4. *The matrices that satisfy the condition that $A^2 = A$ are called* ***idempotent*** *matrices.*

**Example 2.** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$. Then $A$ is a symmetric matrix and $B$ is a skew-symmetric matrix.

**Example 3.** Let $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$. Then $A$ is an orthogonal matrix.

**Example 4.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A$ is an idempotent matrix.

## 4 Submatrix of a Matrix

**Definition 10.** *A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix.*

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, then a few submatrices of $A$ are

$$[1],\ [2],\ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ [15],\ \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix},\ A$$

But the matrices $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not submatrices of $A$.

### 4.1 Block Matrices

Let $A$ be an $n \times m$ matrix and $B$ be an $m \times p$ matrix. Suppose $r < m$. Then, we can decompose the matrices $A$ and $B$ as $A = [PQ]$ and $B = \begin{bmatrix} H \\ K \end{bmatrix}$; where $P$ has order $n \times r$ and $H$ has order $r \times p$. That is, the matrices $P$ and $Q$ are submatrices of $A$ and $P$ consists of the first $r$ columns of $A$ and $Q$ consists of the last $m - r$ columns of $A$. Similarly, $H$ and $K$ are submatrices of $B$ and $H$ consists of the first $r$ rows of $B$ and $K$ consists of the last $m - r$ rows of $B$.

**Theorem 5.** *Let $A = [a_{ij}] = [PQ]$ and $B = [b_{ij}] = \begin{bmatrix} H \\ K \end{bmatrix}$ be defined as above. Then*

$$AB = PH + QK.$$

## 5 Matrices over Complex Numbers

Here the entries of the matrix are complex numbers.

**Definition 11.** 1. *(Conjugate of a Matrix) Let $A$ be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate of $A$, denoted by $\bar{A}$, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ij}}$.*

*For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then*

$$\bar{A} = \begin{bmatrix} 1 & 4-3i & -i \\ 0 & 1 & -i-2 \end{bmatrix}$$

2. *Let $A$ be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate Transpose of $A$, denoted by $A^*$, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ji}}$.*

*For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then*

$$A^* = \begin{bmatrix} 1 & 0 \\ 4-3i & 1 \\ -i & -i-2 \end{bmatrix}$$

3. *A square matrix $A$ over $\mathbb{C}$ is called Hermitian if $A^* = A$.*

4. *A square matrix $A$ over $\mathbb{C}$ is called skew-Hermitian if $A^* = -A$.*

5. *A square matrix $A$ over $\mathbb{C}$ is called unitary if $A^*A = AA^* = I$.*

6. *A square matrix $A$ over $\mathbb{C}$ is called Normal if $AA^* = A^*A$.*

**Remark 3.** *If $A = [a_{ij}]$ with $a_{ij} \in \mathbb{R}$, then $A^* = A^t$.*

# 6 Exercise

1. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   Geometrically interpret $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y} = B\mathbf{x}$.

2. Let $A$ and $B$ be two $m \times n$ matrices and let $\mathbf{x}$ be an $n \times 1$ column vector.
   (a) Prove that if $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$, then $A$ is the zero matrix.
   (b) Prove that if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$, then $A = B$.

3. Let $A$ be an $n \times n$ matrix such that $AB = BA$ for all $n \times n$ matrices $B$. Show that $A = \alpha I$ for some $\alpha \in \mathbb{R}$.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$. Show that there exist infinitely many matrices $B$ such that $BA = I_2$. Also, show that there does not exist any matrix $C$ such that $AC = I_3$.

5. Suppose $A + B = A$. Then show that $B = \mathbf{0}$.

6. Suppose $A + B = \mathbf{0}$. Then show that $B = (-1)A = [-a_{ij}]$.

7. Let $A$ and $B$ be two matrices. If the matrix addition $A+B$ is defined, then prove that $(A+B)^t = A^t + B^t$. Also, if the matrix product $AB$ is defined then prove that $(AB)^t = B^t A^t$.

8. Let $A = [a_1, a_2, \ldots, a_n]$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Compute the matrix products $AB$ and $BA$.

9. Let $n$ be a positive integer. Compute $A^n$ for the following matrices:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Can you guess a formula for $A^n$ and prove it by induction?

10. Find examples for the following statements.
(a) Suppose that the matrix product $AB$ is defined. Then the product $BA$ need not be defined.
(b) Suppose that the matrix products $AB$ and $BA$ are defined. Then the matrices $AB$ and $BA$ can have different orders.
(c) Suppose that the matrices $A$ and $B$ are square matrices of order $n$. Then $AB$ and $BA$ may or may not be equal.

11. Show that for any square matrix $A$, $S = \frac{1}{2}\left(A + A^t\right)$ is symmetric, $T = \frac{1}{2}\left(A - A^t\right)$ is skew-symmetric, and $A = S + T$.

12. Show that the product of two lower triangular matrices is a lower triangular matrix. A similar statement holds for upper triangular matrices.

13. Let $A$ and $B$ be symmetric matrices. Show that $AB$ is symmetric if and only if $AB = BA$.

14. Show that the diagonal entries of a skew-symmetric matrix are zero.

15. Let $A, B$ be skew-symmetric matrices with $AB = BA$. Is the matrix $AB$ symmetric or skew-symmetric?

16. Let $A$ be a symmetric matrix of order $n$ with $A^2 = \mathbf{0}$. Is it necessarily true that $A = \mathbf{0}$ ?

17. Let $A$ be a nilpotent matrix. Show that there exists a matrix $B$ such that $B(I + A) = I = (I + A)B$.

18. Give examples of Hermitian, skew-Hermitian and unitary matrices that have entries with non-zero imaginary parts.

19. Show that for any square matrix $A$, $S = \frac{A+A^*}{2}$ is Hermitian, $T = \frac{A-A^*}{2}$ is skew-Hermitian, and $A = S + T$.

20. Show that if $A$ is a complex triangular matrix and $AA^* = A^*A$ then $A$ is a diagonal matrix.