Quiz 2 Solutions
AMTL101

1. (a) The characteristic polynomial of A is:

$$\begin{aligned} P(x) &= |xI - A| \\ &= \begin{vmatrix} x+4 & 0 & 2 \\ 1/2 & x-1 & 1/2 \\ -15 & 0 & x-7 \end{vmatrix} \\ &= (x-1)[(x+4)(x-7)+30] \\ &= (x-1)\left(x^2-3x+2\right) \\ &= (x-1)(x-1)(x-2) \\ \therefore\ P(x) &= (x-1)^2(x-2). \end{aligned}$$

(b) Since the eigenvalues of A are the roots of the characteristic polynomial, $\therefore$ eigenvalues are 1,1 & 2 .
(C) A matrix is diagonalizable if and only if algebraic multiplicity (A.M.) is equal to geometric multiplicity (G.M.) for each eigenvalue. Since G.M. $\leq$ A.M., so, it is enough to check for the eigen value $\lambda = 1$.
Now,

$$\begin{aligned} \text{G. M.}(1) &= \dim\left(E_1\right) \\ &= \text{ Nullity } (A-I) \\ &= 3 - \text{Rank}(A-I) \end{aligned}$$

Since,

$$A - I = \begin{bmatrix} -5 & 0 & -2 \\ -1/2 & 0 & -1/2 \\ 15 & 0 & 6 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 2/5 \\ 1 & 0 & 1 \\ 15 & 0 & 6 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 0 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, $\operatorname{rank}(A - I) = 2$

$$\therefore \text{G. M.}(1) = 1 < \text{A. M.}(1).$$

Thus, A is not diagonalizable.

2. (a) We have

$$\left[2y + \frac{y^2}{x} + e^x\left(1 + \frac{1}{x}\right)\right] dx + (x + 2y)dy = 0$$

So, $M = 2y + \frac{y^2}{x} + e^x\left(1 + \frac{1}{x}\right)$ & $N = x + 2y$

$$\begin{aligned} \therefore M_y - N_x &= 2 + \frac{2y}{x} - 1 \\ &= \frac{2y}{x} + 1 \\ &= \frac{2y + x}{x} \end{aligned}$$

$$\because \frac{M_y - N_x}{N} = \frac{1}{x}$$
$$\therefore I.F. = e^{\int \frac{M_y - N_x}{N} dx}$$
$$= e^{\int \frac{1}{x} dx}$$
$$= e^{\ln x}$$
$$= x$$

(b) Now,

$$x\left[2y + \frac{y^2}{x} + e^x\left(1 + \frac{1}{x}\right)\right] dx + x(x + 2y)dy = 0$$

is an exact ODE, and $u(x, y) = c$ is the general solution, where,

$$u(x, y) = \int x\left[2y + \frac{y^2}{x} + e^x\left(1 + \frac{1}{x}\right)\right] dx + h(y)$$
$$= yx^2 + xy^2 + xe^x - e^x + e^x + h(y)$$
$$= yx^2 + xy^2 + xe^x + h(y)$$

Since, $\frac{\partial u}{\partial y} = x(x + 2y)$

$$\Rightarrow x^2 + 2xy + h'(y) = x^2 + 2xy$$
$$\Rightarrow \quad h'(y) = 0$$
$$\Rightarrow h(y) = c$$
$$\therefore \; u(x, y) = yx^2 + xy^2 + xe^x = c$$

is the general solution of the given ODE.

(c) If $y(1) = 0 \Rightarrow 0 + 0 + e = c$

$$\Rightarrow c = e$$

$$\begin{aligned}
\therefore \quad & yx^2 + xy^2 + xe^x = e \\
\Rightarrow \quad & y^2 + xy + e^x = \frac{e}{x} \\
\Rightarrow \quad & \left(y + \frac{x}{2}\right)^2 = \frac{e}{x} - e^x + \frac{x^2}{4} \\
\Rightarrow \quad & y = \pm\sqrt{\frac{e}{x} - e^x + \frac{x^2}{4}} - x/2
\end{aligned}$$

Since, $y(1) = 0$,

$$\Rightarrow \quad y = \sqrt{\frac{e}{x} - e^x + \frac{x^2}{4}} - \frac{x}{2}.$$