Eigenvalues and eigenvectors

Let A be $n \times n$ matrix with real or complex entries.
A scalar λ is called an eigenvalue of the matrix A if there exists some nonzero $X \in \mathbb{R}^n$ such that

$$AX = \lambda X$$

In this all such X $(\neq 0)$ are called eigenvectors of A corresponding to the eigenvalue λ.

Note that if X is an eigenvector of A corresp. to eigenvalue λ, then cX is also an eigenvector for any $c \neq 0$.

Also, if $AX_1 = \lambda X_1$ & $AX_2 = \lambda X_2$

then $A(X_1 + X_2) = AX_1 + AX_2$
$= \lambda X_1 + \lambda X_2 = \lambda(X_1 + X_2)$

Defn: The eigenspace of A corresponding to an eigenvalue λ is defined as

$$E_\lambda = \{ X \in \mathbb{R}^n : AX = \lambda X \}$$

E_λ consists of all eigenvectors corresp. to eigenvalue λ and the zero vector.

- E_λ is a subspace of $\mathbb{R}^n$.

In fact, $E_\lambda = \text{Null space}(A - \lambda I)$
$(AX = \lambda X \iff (A - \lambda I)X = 0)$

Examples:

① $A = 0$, the zero matrix

For any $0 \neq X \in \mathbb{R}^n$, $AX = 0 = \lambda X$ (zero vector in $\mathbb{R}^n$)

$\Rightarrow \lambda = 0$.

$\therefore \lambda = 0$ is the only eigenvalue of A

The corresp. eigenspace is $\mathbb{R}^n$.

② $A = I$, the identity matrix

$AX = IX = X$

$\Rightarrow \lambda = 1$ is the only eigenvalue.

③ A = diagonal matrix

$$= \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots$$

$$A\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \lambda_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A.

Thm: λ is an eigenvalue of A
$\iff \det(A-\lambda I)=0$

Proof: λ is an eigenvalue of A
$\iff AX=\lambda X$ for some nonzero X.
$\iff (A-\lambda I)X=0$ " " "
$\iff A-\lambda I$ is not invertible
$\iff \det(A-\lambda I)=0$.

Defn (Characteristic polynomial)
For $A \in M_{n\times n}(\mathbb{R})$, we define the characteristic polynomial of A as
$p(x) = \det(xI-A) \leftarrow$ polynomial of degree n.

From the previous theorem, we see that the eigenvalues of A are the roots of the characteristic polynomial.
i.e. λ is an eigenvalue of $A \iff p(\lambda)=0$.

Examples:

① For $A = 0$,
$p(x) = \det(xI - A) = \det(xI) = x^n$

② For $A = I$,
$p(x) = \det(xI - I) = (x-1)^n$

③ For $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$,

$$p(x) = \begin{vmatrix} x-\lambda_1 & & & \\ & x-\lambda_2 & & \\ & & \ddots & \\ & & & x-\lambda_n \end{vmatrix} = (x-\lambda_1)\cdots(x-\lambda_n)$$

④ $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

$$p(x) = \det(xI - A) = \det\begin{pmatrix} x-2 & -3 \\ -3 & x-2 \end{pmatrix}$$
$$= (x-2)^2 - 9 = x^2 - 4x - 5$$
$$= (x+1)(x-5)$$

$\therefore$ The eigenvalues are -1 and 5.

To find the eigenvectors:

For $\lambda = -1$: $\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow x + y = 0 \Leftrightarrow y = -x$

$X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresp. to eigenvalue $\lambda = -1$

For $\lambda = 5$:

$\lambda I - A = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$x - y = 0 \iff y = x$

$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresp. to eigenvalue 5.

(5) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$xI - A = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$

$p(x) = \det \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = x^2 + 1$

Here, the char. poly. of A has no real roots. So, it has no real eigenvalue.

So, no vector in $\mathbb{R}^2$ is an eigenvector