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HTML Content

Derivatives of Laplace Transforms

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = ?$$

$$F'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[-t f(t)\right] dt$$

$$= \mathcal{L}(-t f(t))(s)$$

$$\therefore \boxed{F'(s) = \mathcal{L}(-t f(t))(s)}$$

or $$\boxed{\mathcal{L}(t f(t)) = -F'(s)}$$

$$\mathcal{L}(t^2 f(t))(s) = \mathcal{L}(t\,(tf(t)))(s)$$

$$= -\frac{d}{ds}\left[\mathcal{L}(tf(t))(s)\right]$$

$$= -\frac{d}{ds}\left(-F'(s)\right) = F''(s)$$

In general,

$$\boxed{\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)}$$

for $n = 1, 2, 3, \cdots$

<u>Example</u>: Find $\mathcal{L}(t\cos\omega t)$

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} = F(s)$$

$$\Rightarrow \mathcal{L}(t\cos\omega t) = -F'(s)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Exercise: Find $\mathcal{L}(t\sin\omega t)$, $\mathcal{L}(t e^{at})$, $\mathcal{L}(t\sinh(at))$, $\mathcal{L}(t\cosh(at))$.

Laplace transform of derivatives

$$\mathcal{L}(f')(s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sF(s)$$

$$\therefore \boxed{\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)}$$

$$\Rightarrow \mathcal{L}(f'')(s) = s\mathcal{L}(f')(s) - f'(0)$$

$$= s\left[s\mathcal{L}(f)(s) - f(0)\right] - f'(0)$$

$$\boxed{\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)}$$

In general,

$$\boxed{\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

for $n = 1, 2, 3, \cdots$

Laplace transform of integrals

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \ ?$$

Let $g(t) = \int_0^t f(\tau)\,d\tau$

Then $g'(t) = f(t)$ and $g(0) = 0$.

$\therefore \ \mathcal{L}(g')(s) = \mathcal{L}(f)(s)$

$\Rightarrow \ s\,\mathcal{L}(g)(s) - \overset{0}{g(0)} = F(s)$

$\Rightarrow \ \mathcal{L}(g)(s) = \dfrac{F(s)}{s}$

$$\therefore \ \boxed{\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}}$$

Example: Find $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)$

Let $F(s) = \frac{1}{s^2+\omega^2} = \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right)$

$= \frac{1}{\omega}\mathcal{L}(\sin\omega t)(s)$

$\Rightarrow f(t) = \frac{1}{\omega}\sin(\omega t)$

$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\,d\tau$

$= \int_0^t \frac{1}{\omega}\sin(\omega\tau)\,d\tau$

$\therefore \boxed{\mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = \frac{1}{\omega^2}(1-\cos\omega t)}$

Application of Laplace transform to solve IVPs

Example: Solve $y'' - y = t$; $y(0)=1$, $y'(0)=1$

Taking the Laplace transform,

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t)$$

$$\Rightarrow \left[s^2 \mathcal{L}(y) - s\,y(0) - y'(0) \right] - \mathcal{L}(y) = \frac{1}{s^2}$$

$$\Rightarrow (s^2-1)\mathcal{L}(y)(s) - s - 1 = \frac{1}{s^2}$$

$$\Rightarrow (s^2-1)\mathcal{L}(y)(s) = \frac{1}{s^2} + (s+1)$$

$$\Rightarrow \mathcal{L}(y)(s) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}$$

$$= \frac{1}{s^2-1} - \frac{1}{s^2} + \frac{1}{s-1}$$

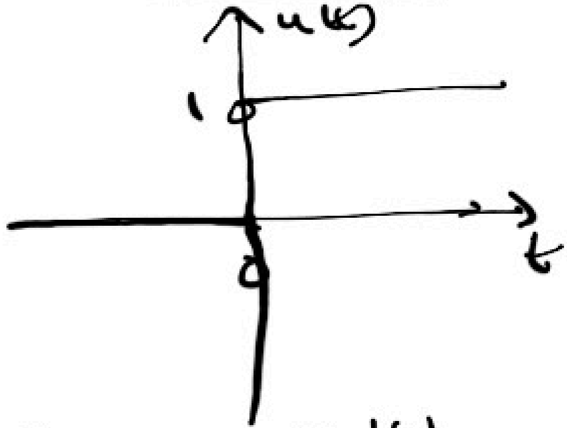
Taking $\mathcal{L}^{-1}$:

$$y(t) = \sinh(t) - t + e^t$$

$$= \frac{e^t - e^{-t}}{2} - t + e^t = \frac{3}{2}e^t - \frac{1}{2}e^{-t} - t$$

Heaviside function (Unit step function)

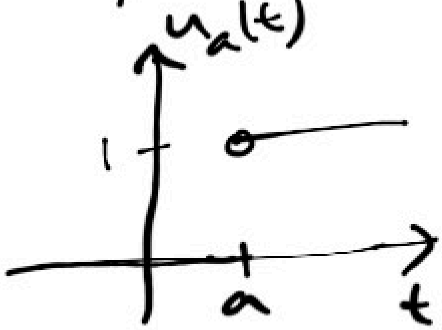
$$u(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$$



For any $a > 0$,

$$u_a(t) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$

$$= u(t-a)$$



$$\mathcal{L}\left(u(t-a)\right) = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \frac{e^{-as}}{s}$$

$$\boxed{\mathcal{L}\left(u_a(t)\right)(s) = \mathcal{L}\left(u(t-a)\right)(s) = \frac{e^{-as}}{s}}$$

Let $\tilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t \le a \\ f(t-a) & \text{if } t > a \end{cases}$

Then $$\mathcal{L}(\tilde{f})(s) = \int_0^\infty e^{-st}\tilde{f}(t)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= e^{-as}\int_a^\infty e^{-s(t-a)} f(t-a)\,dt$$

$$= e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-as}\mathcal{L}(f)(s)$$

$$\therefore \boxed{\mathcal{L}\big(f(t-a)\,u(t-a)\big) = e^{-as}\mathcal{L}(f)}$$