Sum of subspaces

Suppose W_1 and W_2 are subspaces of a vector space V. Then the sum $W_1 + W_2$ is defined as

$$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 , w_2 \in W_2 \}.$$

Thm: $W_1 + W_2$ is a subspace of V.

Pf: $0 = 0 + 0 \in W_1 + W_2$.

Let $u, v \in W_1 + W_2$ and $\alpha, \beta \in \mathbb{F}$

Then $u = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$

$v = w_1' + w_2'$, $w_1' \in W_1$, $w_2' \in W_2$.

$\therefore \alpha u + \beta v = (\underbrace{\alpha w_1 + \beta w_1'}_{\in W_1}) + (\underbrace{\alpha w_2 + \beta w_2'}_{\in W_2}) \in W_1 + W_2$.

Thm: $W_1 + W_2 = \text{span}(W_1 \cup W_2)$

Pf: $W_1 \subseteq W_1 + W_2$, $W_2 \subseteq W_1 + W_2$

$\therefore W_1 \cup W_2 \subseteq W_1 + W_2$

$\Rightarrow \text{span}(W_1 \cup W_2) \subseteq W_1 + W_2$

(because $\text{span}(W_1 \cup W_2)$ is the smallest subspace containing $W_1 \cup W_2$)

Conversely, if $w_1 + w_2 \in W_1 + W_2$,

then $w_1 + w_2 = 1 \cdot w_1 + 1 \cdot w_2 \in \text{span}(W_1 \cup W_2)$

$\Rightarrow W_1 + W_2 \subseteq \text{span}(W_1 \cup W_2)$

Hence, $W_1 + W_2 = \text{span}(W_1 \cup W_2)$.

Thm: Let V be a finite dimensional vector space and W_1, W_2 are subspaces of V.

Then $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Proof: Let $B_1 = \{v_1, v_2, \cdots, v_k\}$ be a basis for $W_1 \cap W_2$ (If $W_1 \cap W_2 = \{0\}$, then $B_1 = \phi$)

Since $W_1 \cap W_2$ is contained in W_1, and $\{v_1, v_2, \cdots, v_k\}$ is linearly indep subset of W_1, there exists a basis

$B_2 = \{v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_m\}$

for W_1.

Similarly, there is a basis

$B_3 = \{v_1, v_2, \cdots, v_k, w_1, w_2, \cdots, w_l\}$

for W_2.

Claim: $B = \{v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_m, w_1, w_2, \cdots, w_l\}$ is a basis for $W_1 + W_2$.

- B is linearly indep.:

Let $a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m + c_1 w_1 + \cdots + c_l w_l = 0$

$\Rightarrow a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m = -c_1 w_1 - \cdots - c_l w_l$. —(1)

The L.H.S. $\in W_1$ and the R.H.S $\in W_2$.

$\therefore \quad -(c_1 w_1 + \cdots + c_\ell w_\ell) \in W_1 \cap W_2$

$\Rightarrow \quad -(c_1 w_1 + \cdots + c_\ell w_\ell) = d_1 v_1 + \cdots + d_k v_k$

$\Rightarrow \quad d_1 v_1 + \cdots + d_k v_k + c_1 w_1 + \cdots + c_\ell w_\ell = 0$

Since $\{v_1, \cdots, v_k, w_1, \cdots, w_\ell\}$ is lin. indep.,

$d_1 = 0 = \cdots = d_k \; , \; c_1 = 0 = \cdots = c_\ell$.

$\therefore$ From ①,

$a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m = 0$

$\Rightarrow \quad a_1 = \cdots = a_k = 0 = b_1 = \cdots = b_m$

$(\because \{v_1, \cdots, v_k, u_1, \cdots, u_m\}$ is L.I.$)$

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- $\text{span}(B) = W_1 + W_2$.

Let $w_1' + w_2' \in W_1 + W_2$

Then $w_1' = \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_m u_m$

$w_2' = \gamma_1 v_1 + \cdots + \gamma_k v_k + \delta_1 w_1 + \cdots + \delta_\ell w_\ell$.

$\Rightarrow \quad w_1' + w_2' = (\alpha_1 + \gamma_1) v_1 + \cdots + (\alpha_k + \gamma_k) v_k + \beta_1 u_1 + \cdots + \beta_m u_m + \delta_1 w_1 + \cdots + \delta_\ell w_\ell$

$\in \text{span}(B)$.

$\therefore$ B is a basis for $W_1 + W_2$

$\Rightarrow \dim(W_1 + W_2) = k + m + \ell = (k+m) + (k+\ell) - k$

$= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

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Defn: The sum $(W_1 + W_2)$ is called a direct sum if $W_1 \cap W_2 = \{0\}$.

Notation: $W_1 \oplus W_2$ denotes the direct sum.

Note that: $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$

Thm: $V = W_1 \oplus W_2$ iff every vector $v \in V$ can be written uniquely as sum of vectors in W_1 & W_2.

Pf: $(\Rightarrow)$ Suppose $v = w_1 + w_2 = w_1' + w_2'$,

$w_1, w_1' \in W_1$; $w_2, w_2' \in W_2$.

$\Rightarrow w_1 - w_1' = w_2' - w_2 \in W_1 \cap W_2 = \{0\}$

$\Rightarrow w_1 = w_1'$ & $w_2 = w_2'$.

$(\Leftarrow)$ Suppose any $v \in V$ can be written uniquely as $v = w_1 + w_2$, where $w_1 \in W_1$, $w_2 \in W_2$.

To show: $V = W_1 \oplus W_2$

For this, we need to show $W_1 \cap W_2 = \{0\}$.

Let $w \in W_1 \cap W_2$.

Then $w = w + 0 \in W_1 + W_2$

Also, $w = 0 + w \in W_1 + W_2$

By uniqueness, $w = 0$

If W_1, W_2, W_3 are three subspaces, we can consider $W_1+W_2+W_3$ which is also a subspace.

<u>Q</u>: $\dim(W_1+W_2+W_3) \overset{?}{=} \dim W_1 + \dim W_2 + \dim W_3$
$- \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3)$
$- \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$

This is <u>NOT</u> correct.

Take $V = \mathbb{R}^2$, $W_1 =$ x-axis, $W_2 =$ y-axis,
$W_3 = \{y = x\}$ line.

Then $W_1 + W_2 + W_3 = \mathbb{R}^2$ $\qquad$ L.H.S. $= 2$
$W_i \cap W_j = \{0\}$ $\qquad$ R.H.S. $= 3$.