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Contents

Linear Transformations

1 Linear Transformation

Definition 1. *Let V and W be vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a linear transformation if*

1. *$T(u + v) = T(u) + T(v) \quad \forall\ u, v \in V$;*

2. *$T(\alpha v) = \alpha T(v) \quad \forall\ v \in V,\ \alpha \in \mathbb{F}$.*

Equivalently, T is a linear transformation if $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$, for all $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$.

Proposition 1. *$T(0_v) = 0_w$, where 0_v & 0_w are the zero vectors in V & W, respectively.*

Proof.

$$0_v = 0_v + 0_v$$
$$\implies T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$$

adding the additive inverse $-T(0_v)$, we get

$$
\begin{aligned}
0_w &= (T(0_v) + T(0_v)) + (-T(0_v)) \\
&= T(0_v) + (T(0_v) + (-T(0_v))) \\
&= T(0_v) + 0_w \\
&= T(0_v).
\end{aligned}
$$

$\square$

Remark 1. *If $T(0) \neq 0$, then T is not a linear transformation.*

Example 1. Is $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?

Solution Since $T(0) = 1 \neq 0$, T is not a linear transformation. ∎

Example 2. 1. (Zero transformation) $T : V \to W$ defined as $T(v) = 0 \quad \forall\, v \in V$ is a linear transformation.

2. (Identity transformation) $T : V \to V$ defined as $T(v) = v \quad \forall\, v \in V$ is a linear transformation.

Example 3. Is $T(x) = x^2$ a linear transformation from $\mathbb{R}$ to $\mathbb{R}$?

Solution Since $T(2x) = 4x^2 \neq 2T(x)$ if $x \neq 0$, T is not a linear transformation. ∎

Example 4. $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = \alpha x$, where $\alpha \in \mathbb{R}$ is a linear transformation.

Solution Since $T(c_1 x + c_2 y) = \alpha(c_1 x + c_2 y) = c_1(\alpha x) + c_2(\alpha y) = c_1 T(x) + c_2 T(y)$.

$\therefore$ T is a linear transformation. ∎

Exercise 1. *Show that any linear transformation $T : \mathbb{R} \to \mathbb{R}$ is of the form $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.*

Solution $T(x) = T(x.1) = xT(1)$ (because T is linear), let $\alpha = T(1)$, then

$$T(x) = \alpha x \quad \forall\, x \in \mathbb{R}.$$

∎

Exercise 2. *Write all the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}$.*

Solution For $(x, y) \in \mathbb{R}^2$,

$$(x, y) = x(1, 0) + y(0, 1)$$
$$\therefore\; T(x, y) = xT(1, 0) + yT(0, 1)$$
$$= \alpha x + \beta y$$

where, $\alpha = T(1, 0)$ & $\beta = T(0, 1)$. ∎

Example 5. Let V be the space of all real polynomials. define $T : V \to V$ as $T(p(x)) = p(x + 1)$. Since, $T(\alpha p(x) + \beta q(x)) = \alpha p(x + 1) + \beta q(x + 1) = \alpha T(p(x)) + \beta T(q(x))$, T is a linear transformation.

Exercise 3. *Let V be the space of all real polynomials. Show that $T : V \to V$ defined as $T(p(x)) = p'(x)$, is a linear transformation.*

Exercise 4. *Let V be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $T : V \rightarrow V$ defined as $T(f(x)) = \int\limits_{0}^{x} f(t)dt$, is a linear transformation.*

Let V & W be vector spaces over the same field $\mathbb{F}$. Let

$$L(V,W) = \{T : V \rightarrow W \,|\, T \text{ is a linear transformation}\}.$$

Note that if $T_1, T_2 \in L(V,W)$, then $T_1 + T_2 \in L(V,W)$. Also, if $T \in L(V,W)$, then $\alpha T \in L(V,W)$ for any $\alpha \in \mathbb{F}$. That means $L(V,W)$ is a vector space over $\mathbb{F}$.

2 Null Space and Range Space

Definition 2. *(Null Space or Kernel)*
Let $T : V \rightarrow W$ be a linear transformation. Then nullspace(T) *or* Ker(T) *is* $\{v \in V : T(v) = 0\}$.

Since $T(0) = 0$, $0 \in \text{Ker}(T)$.

Proposition 2. Ker$(T) = \{0\}$ *iff T is one-to-one or injective.*

Proof. Suppose T is one-to-one and $v \in \text{Ker}(T)$, then $T(v) = 0 = T(0) \implies v = 0$ ($\because T$ is one-to-one), i.e. Ker$(T) = \{0\}$.
Conversely, suppose Ker$(T) = \{0\}$. Assume $T(v_1) = T(v_2) \implies T(v_1 - v_2) = T(v_1) - T(v_2) = 0 \implies v_1 - v_2 \in \text{Ker}(T) \implies v_1 - v_2 = 0$, *i.e.* $v_1 = v_2$. $\square$

Proposition 3. *For any linear transformation $T : V \rightarrow W$,* nullspace(T) *is a subspace of V.*

Proof. Since $0 \in \text{nullspace}(T) \implies \text{nullspace}(T) \neq \phi$. Let $v_1, v_2 \in \text{nullspace}(T)$ and $a_1, a_2 \in \mathbb{F}$. We need to show that $a_1 v_1 + a_2 v_2 \in \text{nullspace}(T)$.
Since $T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2) = a_1.0 + a_2.0 = 0 \implies a_1 v_1 + a_2 v_2 \in \text{nullspace}(T)$. $\therefore$ nullspace(T) is a subspace of T. $\square$

Definition 3. *(Nullity of a linear transformation) The dimension of the null space of T is called the nullity of T.*

From proposition 2, T is injective if and only if nullity$(T) = 0$.

Definition 4. *(**Range space of** T) If $T : V \to W$ is a linear transformation,*

$$\text{range}(T) := \{w \in W : \ w = T(v) \text{ for some } v \in V\}.$$

Note that range(T) is a subspace of W.

Definition 5. *(**Rank of** T) The dimension of the range space of T is called the rank of T.*

Note that T is onto (or surjective) iff range(T)=W iff rank(T)=dim(W) if W is finite dimensional.

Theorem 1. *(**Rank-nullity Theorem**) Let V be a finite-dimensional vector space and let $T : V \to W$ be a linear transformation. Then*

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Let nullity$(T) = k$ and $\dim(V) = n \geq k$. Let $\beta_1 = \{v_1, v_2, \ldots, v_k\}$ be a basis for nullspace(T). This basis can be extended to a basis for V, say $\beta_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$.
Claim: rank$(T) = n-k$. To show this we will show that $\beta_3 = \{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for range(T).

- β_3 is linearly independent.
  Let $c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n) = 0$.

$$\implies T(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0$$
$$\implies c_{k+1}v_{k+1} + \cdots + c_nv_n \in \text{nullspace}(T)$$
$$\implies c_{k+1}v_{k+1} + \cdots + c_nv_n = c_1v_1 + \cdots + c_kv_k$$
$$\implies c_1v_1 + \cdots + c_kv_k - c_{k+1}v_{k+1} - \cdots - c_nv_n = 0$$
$$\implies c_i = 0 \quad \forall i \quad (\because \{v_1, v_2, \ldots, v_n\} \text{ is L.I.}).$$

  Thus, β_3 is linearly independent.

- span$(\beta_3) = $ range(T).
  Let $w = T(v) \in$ range(T), then

$$v = a_1v_1 + \cdots + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n.$$
$$\implies T(v) = a_1T(v_1) + \cdots + a_kT(v_k) + a_{k+1}T(v_{k+1}) + \cdots + a_nT(v_n)$$
$$= a_{k+1}T(v_{k+1}) + \cdots + a_nT(v_n) \quad (\because \{v_1, \ldots, v_k\} \subseteq \text{nullspace}(T))$$
$$\in \text{span}(\beta_3).$$

$$\therefore \operatorname{rank}(T) = n - k$$
$$\implies \operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

$\square$

Theorem 2. *For any $A \in M_{m \times n}(\mathbb{F})$, $\operatorname{rowrank}(A) = \operatorname{columnrank}(A)$.*

Proof. Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by $T(X) = AX$, where $X = (x_1, x_2, \ldots, x_n)^t$. Note that T is a linear transformation. Also, $\operatorname{range}(T) = \{T(X) : X \in \mathbb{F}^n\} = \{AX : X \in \mathbb{F}^n\} = \operatorname{columnspace}(A) \implies \operatorname{columnrank}(A) = \operatorname{rank}(T)$. Also, $\operatorname{nullspace}(T) = \{X \in \mathbb{F}^n : T(X) = 0\} = \{X \in \mathbb{F}^n : AX = 0\} = $ Solution space(A).
$\implies \operatorname{nullity}(T) = \dim(\text{Solution space}(A)) = n - k$, where $k = \operatorname{rowrank}(A)$.
By rank-nullity theorem,

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$$
$$\implies \operatorname{Col.rank}(A) + n - \operatorname{rowrank}(A) = n$$
$$\implies \operatorname{rowrank}(A) = \operatorname{columnrank}(A).$$

$\square$

Exercise 5. *Does there exist an onto linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$?*

Solution   Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation, then by rank-nullity theorem:
$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(\mathbb{R}^2) = 2$$

$\implies \operatorname{rank}(T) \leq 2 \implies \dim(\operatorname{range}(T)) \leq 2 < \dim(\mathbb{R}^3)$. Thus, $\operatorname{range}(T)$ is a proper subset of $\mathbb{R}^3$, i.e. T is not onto. $\blacksquare$

Proposition 4.   *1. There is no onto linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ if $m > n$.*

   2. There is no one-to-one linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ if $n > m$.

Theorem 3. *Let $T : V \to W$ be a linear transformation and let V & W be finite-dimensional, and $\dim V = \dim W$. Then T is 1-1 iff T is onto.*

Proof. We know that T is $1 - 1 \iff \operatorname{nullity}(T) = 0 \iff \operatorname{rank}(T) = \dim V = \dim W$ (by the rank-nullity theorem), since W is finite-dimensional, $\operatorname{rank}(T) = \dim W \iff \operatorname{range}(T) = W$, i.e. T is onto. $\square$

In particular, if $T : V \to V$ is a linear transformation and $\dim V < \infty$, then T is $1-1$ iff T is onto.

Remark 2. *The above result is not true if V is infinite-dimensional.*

Example 6. Let V be the vector space of all real sequences. Let $T : V \to V$ be given by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Note that T is $1-1$ but not onto. And $S : V \to V$ defined by $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ is onto but not $1-1$.

Example 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (x + y - z, x - y + z, y - z)$. Find a basis for the $N(T)$-null space of T and for the $R(T)$-range(T).

Solution

$$N(T) = \{(x, y, z) \in \mathbb{R}^3 : T(x, y, z) = (0, 0, 0)\}$$
$$= \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0, \ x - y + z = 0 \ \& \ y - z = 0\}$$
$$= \text{Solution space of the system of linear equations } AX = 0,$$

where $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$. By performing elementary row operations on

A, we get the row reduced echelon form of A as $\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. So,

we get $x = 0$ and $y = z$.

$$\therefore \ N(T) = \{(0, \lambda, \lambda) : \ \lambda \in \mathbb{R}\}.$$

$\therefore \ \beta = \{(0, 1, 1)\}$ is a basis for $N(T)$.
Now,

$$R(T) = \{T(x, y, z) : \ (x, y, z) \in \mathbb{R}^3\}$$

Since,

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$
$$\implies T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$$

$\implies$ the set $\{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$ spans $R(T)$.

$$T(1, 0, 0) = (1, 1, 0);$$
$$T(0, 1, 0) = (1, -1, 1);$$
$$T(0, 0, 1) = (-1, 1, -1)$$

Note that $T(0, 0, 1) = -T(0, 1, 0)$, therefore

$$
\begin{aligned}
R(T) &= \text{span}\{T(1,0,0), T(0,1,0), T(0,0,1)\} \\
&= \text{span}\{T(1,0,0), T(0,1,0)\} \\
&= \text{span}\{(1,1,0), (1,-1,1)\}
\end{aligned}
$$

Since the two vectors are not multiples of each other, they form a linearly independent set, therefore the set $\{T(1,0,0), T(0,1,0)\} = \{(1,1,0),\ (1,-1,1)\}$ forms a basis for $R(T)$. ■

Example 8. Let V be the vector space of all polynomials of degree less than or equal to 3. Consider $T : V \to V$ given by $T(p(x)) = p'(x)$. Find a basis for $N(T)$ and $R(T)$.

Solution

$$
\begin{aligned}
N(T) &= \{p(x) \in V :\ T(p(x)) = 0\} \\
&= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 :\ a_1 + 2a_2 x + 3a_3 x^2 = 0\} \\
&= \{a_0\}
\end{aligned}
$$

$$\therefore\ \beta = \{1\} \text{ is a basis for } N(T).$$

Since, $\{1, x, x^2, x^3\}$ is a basis for V, $\{T(1), T(x), T(x^2), T(x^3)\}$ spans $R(T)$.

$$
\begin{aligned}
\therefore\ R(T) &= \text{span}\{0, 1, 2x, 3x^2\} \\
&= \text{span}\{1, 2x, 3x^2\}
\end{aligned}
$$

Also, $\{1, 2x, 3x^2\}$ is linearly independent, $\therefore$ $\{1, 2x, 3x^2\}$ or $\{1, x, x^2\}$ is a basis for $R(T)$. ■

3   Ordered Basis and Coordinate Vector

Definition 6. *(**Coordinate vector w.r.t. an ordered basis**)*
*Let V be a finite-dimensional vector space, say $\dim V = n$. An **ordered basis** is a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V with a fixed ordering. Then any $v \in V$ can be written uniquely as $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, where $a_1, a_2, \ldots, a_n \in \mathbb{F}$. We define the coordinate vector of v w.r.t. β as:*

$$
[v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n.
$$

3.1 Change of Bases Matrix

Suppose $\beta_1 = \{v_1, v_2, \ldots, v_n\}$ and $\beta_2 = \{w_1, w_2, \ldots, w_n\}$ are two ordered bases for V. How are $[v]_{\beta_1}$ & $[v]_{\beta_2}$ related?

Example 9. Let $V = \mathbb{R}^3$, and

$$\beta_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\},$$
$$\beta_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$

Let $v = (x, y, z) \in \mathbb{R}^3$. Since, $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$,

$$\therefore \ [v]_{\beta_1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Also,

$$(x, y, z) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1)$$
$$= (a + b + c, b + c, c)$$
$$\implies c = z;$$
$$b + c = y \implies b = y - z;$$
$$a + b + c = x \implies a = x - y.$$

$$\therefore \ [v]_{\beta_2} = \begin{pmatrix} x - y \\ y - z \\ z \end{pmatrix}.$$

We want to find the change of bases matrix $P \in M_{3 \times 3}(\mathbb{R})$ such that

$$[v]_{\beta_2} = P[v]_{\beta_1}.$$

Putting $v = (1, 0, 0)$, we get

$$[(1, 0, 0)]_{\beta_2} = P \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1^{st} \text{ column of } P.$$

Therefore, to find the 1st column of P, we need to find the coordinate vector of the first vector in β_1 w.r.t. β_2. Similarly, the 2nd & 3rd columns are the coordinate vectors of 2nd & 3rd vectors in β_1 w.r.t. β_2.

More generally, if

$$\beta_1 = \{v_1, v_2, \ldots, v_n\}$$
$$\text{and} \quad \beta_2 = \{w_1, w_2, \ldots, w_n\}$$

are two ordered bases for V. We find $[v_1]_{\beta_2}, [v_2]_{\beta_2}, \ldots, [v_n]_{\beta_2}$ and let

$$P = \left(\begin{array}{cccc} [v_1]_{\beta_2} & [v_2]_{\beta_2} & \cdots & [v_n]_{\beta_2} \end{array} \right)_{n \times n}$$

then for any $v \in V$,

$$[v]_{\beta_2} = P[v]_{\beta_1}.$$

Proposition 5. *The change of bases matrix P is always invertible.*

Proof. Let P be the change of bases matrix from β_1 to β_2 and let Q be the change of bases matrix from β_2 to β_1. Then

$$[v]_{\beta_2} = P[v]_{\beta_1},$$
$$\text{and} \quad [v]_{\beta_1} = Q[v]_{\beta_2}$$
$$\therefore [v]_{\beta_2} = PQ[v]_{\beta_2} \quad \forall \, v \in V$$
$$\implies PQ = I.$$

$\square$

4 Matrix Representation of a Linear Transformation

Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Let $\beta = \{v_1, v_2, \ldots, v_m\}$ be an ordered basis for V and $\beta' = \{w_1, w_2, \ldots, w_n\}$ be an ordered basis for W. Then for each $j \in \{1, 2, \ldots, m\}$,

$$T(v_j) = \sum_{i=1}^{n} a_{ij} w_i.$$

Consider the matrix $A = (a_{ij}) \in M_{n \times m}(\mathbb{R})$.

Now, for any $v \in V$,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

$$\therefore T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_m T(v_m)$$

$$= \sum_{j=1}^{m} \alpha_j T(v_j)$$

$$= \sum_{j=1}^{m} \alpha_j \left(\sum_{i=1}^{n} a_{ij} w_i \right)$$

$$\implies T(v) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} \alpha_j \right) w_i$$

$\therefore \sum_{j=1}^{m} a_{ij} \alpha_j$ is the ith coordinate of the coordinate vector $[Tv]_{\beta'}$.

$$\therefore \quad [Tv]_{\beta'} = A[v]_{\beta}.$$

We denote this matrix A by $[T]_{\beta}^{\beta'}$. The 1st column of $[T]_{\beta}^{\beta'}$ is nothing but $[T(v_1)]_{\beta'}$.

Example 10. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (2x + z, y + 3z)$, let $\beta = \{(1, 1, 0), (1, 0, 1), (1, 1, 1)\}$ and $\beta' = \{(2, 3), (3, 2)\}$. Find $[T]_{\beta}^{\beta'}$.

Solution Since,

$$T(1, 1, 0) = (2, 1) = -\frac{1}{5}(2, 3) + \frac{4}{5}(3, 2)$$

$$T(1, 0, 1) = (3, 3) = \frac{3}{5}(2, 3) + \frac{3}{5}(3, 2)$$

$$T(1, 1, 1) = (3, 4) = \frac{6}{5}(2, 3) + \frac{1}{5}(3, 2)$$

$$\therefore \ [T]_{\beta}^{\beta'} = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

$\blacksquare$