

Thm: If W_1 and W_2 are subspaces of V , then $W_1 \cap W_2$ is also a subspace.

Pf: Let $u, v \in W_1 \cap W_2$

Then $u, v \in W_i$ for $i = 1, 2$.

Since W_i is a subspace,

$\alpha u + \beta v \in W_i$ for $i = 1, 2$.

$\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$

In fact, intersection of any arbitrary collection of subspaces is a subspace.

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Q: Is the union of two subspaces also a subspace?

Ans: No.

e.g. take $V = \mathbb{R}^2$

$W_1 = x\text{-axis} = \{(x, 0) : x \in \mathbb{R}\}$

$W_2 = y\text{-axis} = \{(0, y) : y \in \mathbb{R}\}$

W_1 & W_2 are both subspaces of \mathbb{R}^2 .

But, $W_1 \cup W_2$ is NOT a subspace of \mathbb{R}^2

because $(1, 0) \in W_1 \cup W_2$, $(0, 1) \in W_1 \cup W_2$

but $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

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Thm: $W_1 \cup W_2$ is a subspace if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof: If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then clearly $W_1 \cup W_2$ is a subspace.

Conversely, suppose $w_1 \notin W_2$ and $w_2 \notin W_1$. We need to prove that $W_1 \cup W_2$ is not a subspace.

Since $w_1 \notin W_2$, $\exists w_1 \in W_1, w_1 \notin W_2$.

Since $w_2 \notin W_1$, $\exists w_2 \in W_2, w_2 \notin W_1$.

Let $v = w_1 + w_2$

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Claim: $v \notin W_1 \cup W_2$.

Suppose $v = w_1 + w_2 \in W_1$

Then $v - w_1 \in W_1 \Rightarrow w_2 \in W_1$, which is a contradiction

Illy, if $v = w_1 + w_2 \in W_2$,
then $v - w_2 \in W_2 \Rightarrow w_1 \in W_2$,
which is a contradiction.

$\therefore v \notin W_1 \cup W_2$

$\Rightarrow W_1 \cup W_2$ is not a subspace.

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Defn: (Subspace spanned by a subset).
 Let S be any subset of a vector space V . The subspace spanned by S is the intersection of all subspaces containing S .

Note that the subspace spanned by S is the smallest subspace containing S .
 Notation: $\langle S \rangle$
 • If $S = \emptyset$, the subspace spanned by S is $\{0\}$.

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Defn: (Linear span of a subset)
 Let S be a nonempty subset of a vector space V . The linear span of S consists of all possible linear combinations of vectors from S , i.e., it consists of all vectors of the form

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n, \text{ where } a_i \in \mathbb{F}, v_i \in S \text{ for } i=1, 2, \dots, n.$$

Remark: Note that S may be an infinite subset of V .

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The linear span of S consists of all finite linear combinations of vectors from S .
 Notation: $\text{span}(S)$.

Then: (i) $\text{span}(S)$ is a subspace of V
 (ii) $S \subseteq \text{span}(S)$.
 (iii) $\text{span}(S) = \langle S \rangle$, the intersection of all subspaces containing S .

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Linear dependence & independence

Defn: Let S be any subset of a vector space V . We say that S is "linearly dependent" if the zero vector can be written as a nontrivial linear combination of some vectors from S , i.e. $\exists v_1, v_2, \dots, v_n \in S, a_1, a_2, \dots, a_n \in \mathbb{F}$ with at least one of the a_i 's nonzero such that $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$.

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