

### Jacobian in polar coordinates

$$x = r \cos \theta \quad ; \quad y = r \sin \theta$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\therefore \iint f(x, y) dx dy = \iint f(r \cos \theta, r \sin \theta) r dr d\theta$$

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### Change of variables for triple integrals

$$x = x(u, v, w)$$

$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

$$\iiint f(x, y, z) dx dy dz = \iiint f(u, v, w) |J| du dv dw$$

$$\text{where } J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

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Example:

Evaluate  $I = \iiint_{\Omega} (x^2y + 3xyz) dV$ ,

where  $\Omega = \{(x, y, z) : 1 \leq x \leq 2, 0 \leq xy \leq 2, 0 \leq z \leq 1\}$

Soln: Let  $u = x, v = xy, w = z$   
 i.e.  $x = u; y = \frac{v}{u}; z = w$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{u}$$

The domain  $\Omega$  is transformed into  
 $\{(u, v, w) : 1 \leq u \leq 2, 0 \leq v \leq 2, 0 \leq w \leq 1\}$

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Integrand  $= x^2y + 3xyz$   
 $= uv + 3vw$

$$\therefore I = \int_{w=0}^1 \int_{v=0}^2 \int_{u=1}^2 (uv + 3vw) \cdot \frac{1}{u} du dv dw$$

$$= \int_{w=0}^1 \int_{v=0}^2 (vu + 3vw \ln u) \Big|_{u=1}^2 dv dw$$

$$= \int_{w=0}^1 \int_{v=0}^2 (v + 3 \ln 2 vw) dv dw$$

$$= \int_{w=0}^1 [1 + 3(\ln 2)w] 2 dw$$

$$= 2 + 3 \ln 2$$

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## Cylindrical coordinates

$$(\rho, \theta, z)$$

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

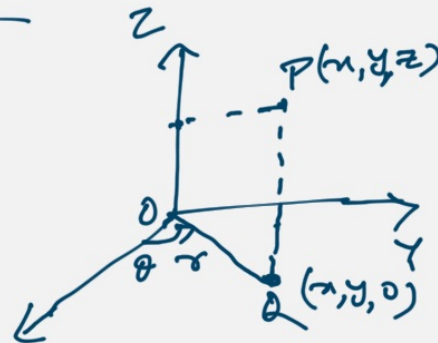
$$z = z$$

$$\text{or, } \rho = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \begin{vmatrix} \cos \theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho$$



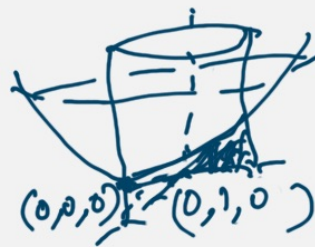
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$$\iiint f(x, y, z) dx dy dz$$

$$= \iiint f(\rho \cos \theta, \rho \sin \theta, z) \rho d\rho d\theta dz$$

Example: Find the volume of the cylinder  $x^2 + (y-1)^2 = 1$  bounded by  $z = x^2 + y^2$  and  $z = 0$ .

Soln:



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$$\text{Volume} = \iiint_R \int_{z=0}^{x^2+y^2} dz \, dA,$$

where  $R$  is the projection of the solid onto the  $xy$  plane.

We use the cylindrical coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z,$$

$$x^2 + (y-1)^2 \leq 1$$

$$\text{i.e. } x^2 + y^2 - 2y \leq 0$$

$$\text{i.e. } r^2 - 2r \sin \theta \leq 0$$

$\therefore r$  varies from 0 to  $2 \sin \theta$

$\therefore \theta$  varies from 0 to  $\pi$

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$$\therefore V = \int_{\theta=0}^{\pi} \int_{r=0}^{2 \sin \theta} \int_{z=0}^{x^2+y^2} r \, dr \, d\theta \, dz$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2 \sin \theta} r^3 \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{4} (2 \sin \theta)^4 \, d\theta = 4 \int_{\theta=0}^{\pi} \sin^4 \theta \, d\theta$$

$$= 4 \int_{\theta=0}^{\pi} \left( \frac{1 - \cos 2\theta}{2} \right)^2 \, d\theta$$

$$= \dots$$

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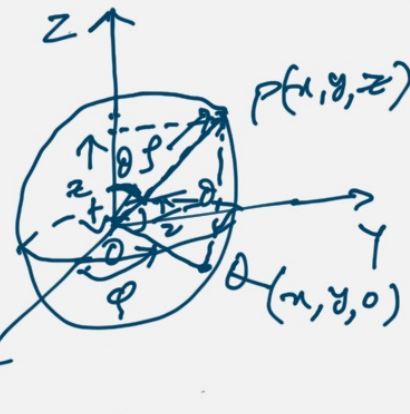
### Spherical coordinates:

Let  $P(x, y, z)$  be any point and  $Q(x, y, 0)$

Let  $OP = \rho$ ,  
 $\theta =$  the angle of the ray  $OP$  with the  $z$ -axis

$\phi =$  the angle of the ray  $OQ$  from the  $x$ -axis.

Then  $0 \leq \rho < \infty$ ;  $0 \leq \theta \leq \pi$ ;  $0 \leq \phi \leq 2\pi$



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We have

$$z = \rho \cos \theta$$

$$\text{and } OQ = \rho \sin \theta$$

$$\text{Now, } x = OQ \cos \phi, \quad y = OQ \sin \phi$$

$$\therefore \begin{cases} x = \rho \sin \theta \cos \phi \\ y = \rho \sin \theta \sin \phi \\ z = \rho \cos \theta \end{cases}$$

$$\rho = \begin{vmatrix} \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -\rho \sin \theta \sin \phi \\ \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\ \cos \theta & -\rho \sin \theta & 0 \end{vmatrix}$$

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$$\begin{aligned}
 \therefore J &= \cos \theta \begin{vmatrix} r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \end{vmatrix} \\
 &\quad + r \sin \theta \begin{vmatrix} \sin \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \sin \theta \cos \varphi \end{vmatrix} \\
 &= \cos \theta \cdot r^2 \cos \theta \sin \theta (\cos^2 \varphi + \sin^2 \varphi) \\
 &\quad + r \sin \theta \cdot r \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) \\
 &= r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= r^2 \sin \theta.
 \end{aligned}$$

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$$\iiint f(x, y, z) dx dy dz$$

$$= \iiint F(r, \theta, \varphi) \cdot r^2 \sin \theta dr d\theta d\varphi.$$

Example: Calculate the volume of sphere of radius  $R$  using spherical coords.

Soln:  $V = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^R r^2 \sin \theta dr d\theta d\varphi$

$$= \frac{R^3}{3} \times 2\pi \int_0^{\pi} \sin \theta d\theta = \frac{4\pi}{3} R^3.$$

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