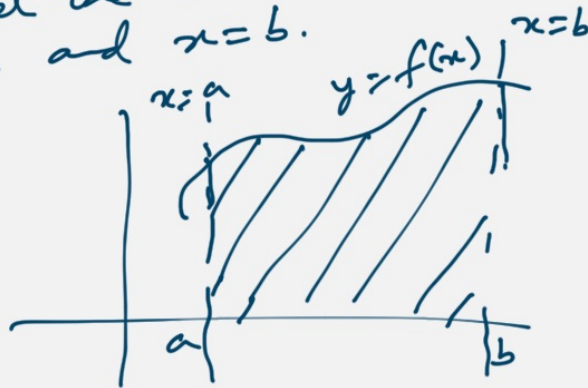
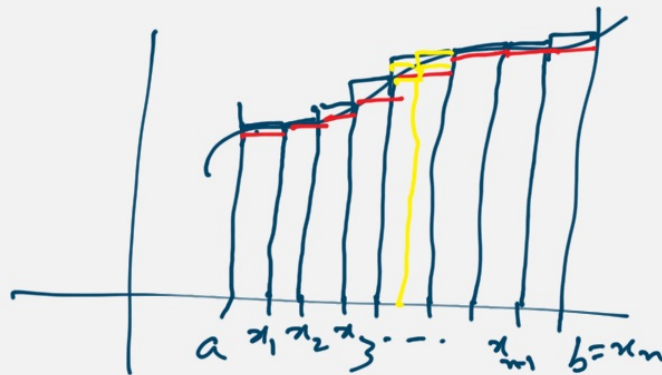


Definite Integrals

Suppose we have $f(x)$ defined on the interval $[a, b]$. We want to find the area under the curve $y = f(x)$ between $x = a$ and $x = b$.



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Let $P = \{a = x_0, x_1, x_2, x_3, \dots, x_{n-1}, x_n = b\}$

be a partition of $[a, b]$

i.e. $x_0 < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n$

The i th subinterval is $[x_{i-1}, x_i]$,
 $1 \leq i \leq n$

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$$\text{let } M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Define the upper and the lower Riemann sum by

$$U(P, f) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$

$$\text{and } L(P, f) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$

Then $L(P, f) \leq \text{area of the region} \leq U(P, f)$

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Definition (Riemann integrability)

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. For any partition

$$P = \{x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$$

we define the lower and upper Riemann sum as

$$L(P, f) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$

$$U(P, f) = \sum_{i=1}^n M_i (x_i - x_{i-1}),$$

where

$$m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$$

$$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

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Next, we define the lower and upper Riemann integral as

$$\int_a^b f(x) dx = \sup \left\{ L(P, f) : P \text{ is a partition of } [a, b] \right\}$$

$$\int_a^b f(x) dx = \inf \left\{ U(P, f) : P \text{ is a partition of } [a, b] \right\}$$

i.e., the lower integral is the supremum of all possible lower sums; and the upper integral is the infimum of all possible upper sums.

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We say that f is Riemann integrable (or that the definite integral exists) if the lower and the upper integrals are equal, i.e.

$$\int_a^b f(x) dx = \int_a^b f(x) dx.$$

In this case, we denote the value by $\int_a^b f(x) dx$ and call it the definite integral of $f(x)$ between a and b .

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Example of a function which is not Riemann integrable:

Let $f: [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $L(P, f) = 0$ and $U(P, f) = 1$ for every partition P .

$\therefore \int_0^1 f(x) dx = 0$; $\int_0^1 f(x) dx = 1$.

$\therefore f$ is not Riemann integrable.

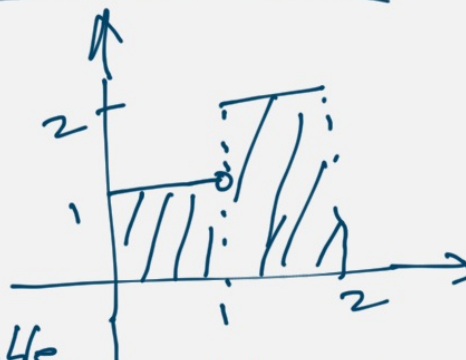
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Theorem: Every continuous function on a closed & bounded interval $[a, b]$ is Riemann integrable.

$f: [0, 2] \rightarrow \mathbb{R}$
 $f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$

f is discontinuous at $x=1$ but it is Riemann integrable.

Fact: Any function with finitely many discontinuities is Riemann integrable.



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