Example: Consider the sequence
$$x_1 = a > 0 \; ; \; x_{n+1} = \frac{x_n^2 + 3}{4} , n \geq 1$$
Find the limit, if it exists.

$$x_{n+1} - x_n = \frac{x_n^2 + 3}{4} - x_n$$

$$= \frac{1}{4}(x_n^2 - 4x_n + 3)$$

$$= \frac{1}{4}(x_n - 1)(x_n - 3)$$

$$< 0 \quad \text{if} \quad 1 < x_n < 3$$
$$> 0 \quad \text{if} \quad x_n < 1 \text{ or } x_n > 3.$$

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$$x_{n+1} = \frac{x_n^2 + 3}{4}$$

If $0 < x_n < 1$, then $0 < x_{n+1} < 1$

If $1 < x_n < 3$, then $1 < x_{n+1} < 3$

If $x_n > 3$, then $x_{n+1} > 3$

∴ If $0 < a < 1$, i.e. $0 < x_1 < 1$, $0 < x_n < 1 \; \forall n$

If $1 < a < 3$, then $1 < x_n < 3 \quad \forall n$

If $a > 3$, then $x_n > 3 \; \forall n.$

If $a = 1$, then $x_n = 1 \quad \forall n$

If $a = 3$, then $x_n = 3 \; \forall n.$

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If $0 < a < 1$, then $x_{n+1} - x_n > 0$ $\forall n$

$\Rightarrow \{x_n\}$ is increasing.

Also, $\{x_n\}$ is bounded

$\therefore \lim\limits_{n \to \infty} x_n$ exists.

If $\lim\limits_{n \to \infty} x_n = L$, then

$$L = \frac{L^2 + 3}{4} \Rightarrow L = 1 \text{ or } L = 3.$$

$\therefore$ If $0 < a < 1$, $0 < x_n < 1$ $\forall n$

$\Rightarrow L \leq 1$ $\therefore \boxed{L = 1}$

If $1 < a < 3$, then $x_{n+1} - x_n < 0$

$\Rightarrow \{x_n\}$ is decreasing

$\therefore \lim\limits_{n \to \infty} x_n = \inf\limits_{n \in \mathbb{N}} x_n$

$\Rightarrow \lim\limits_{n \to \infty} x_n = 1$

If $a > 3$, then $x_{n+1} - x_n > 0$

$\Rightarrow \{x_n\}$ is incr.

If $\{x_n\}$ is bounded, then

$\lim\limits_{n \to \infty} x_n = \sup\limits_{n \in \mathbb{N}} x_n > 3$,

which is not possible.

$\therefore \lim\limits_{n \to \infty} x_n = \infty$.

Conclusion:

If $0 < a < 3$, then $\lim_{n \to \infty} x_n = 1$

If $a = 3$, then $\lim_{n \to \infty} x_n = 3$

If $a > 3$, then $\lim_{n \to \infty} x_n = \infty$.

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