<u>Defn</u> (Null space or kernel of a linear transformation)

Let $T : V \to W$ be a linear transf.
Then null space(T) or $\ker(T)$ is
$\{ v \in V : T(v) = 0 \}$.

Since $T(0) = 0$, $0 \in \ker(T)$.

Note that $\ker(T) = \{0\}$ iff T is one-to-one or injective

<u>Pf</u>: Suppose T is 1-1 and $v \in \ker(T)$
Then $T(v) = 0 = T(0)$
$\Rightarrow v = 0$ ($\because T$ is 1-1)
$\therefore \ker(T) = \{0\}$.

Created with Doceri

Conversely, suppose $\ker(T) = \{0\}$
Assume $T(v_1) = T(v_2)$
$\Rightarrow T(v_1 - v_2) = T(v_1) - T(v_2) = 0$
$\Rightarrow v_1 - v_2 \in \ker(T) = \{0\}$
$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$.

<u>Prop</u>: For any linear transf. $T : V \to W$, null space (T) is a subspace of V.

<u>Pf</u>: $0 \in$ nullspace(T) $\Rightarrow$ null space$(T) \neq \emptyset$.
Let $v_1, v_2 \in$ nullspace(T)
& $a_1, a_2 \in \mathbb{F}$.

Created with Doceri

To show: $a_1 v_1 + a_2 v_2 \in \text{nullspace}(T)$.

$$T(a_1v_1 + a_2v_2) = a_1 T(v_1) + a_2 T(v_2)$$
$$= a_1 \cdot 0 + a_2 \cdot 0 = 0$$
$$\Rightarrow a_1 v_1 + a_2 v_2 \in \text{nullspace}(T)$$

$\therefore$ null space (T) is a subspace (T).

<u>Defn</u> (Nullity of a linear transf.)

$\text{Nullity}(T) = \text{dimension of nullspace}(T)$.

- T is injective $\Leftrightarrow$ nullity $(T) = 0$.

Created with Doceri

<u>Defn</u> (Range space of T)

If $T : V \to W$ is a linear transf.,

$$\text{range}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \}$$

Note that range (T) is a subspace of W. (Exercise)

<u>Defn</u> (rank of T):

$$\text{rank}(T) = \dim(\text{range}(T)).$$

- T is onto (or surjective) iff $\text{range}(T) = W$ iff $\text{rank}(T) = \dim(W)$ if W is finite dimensional

Created with Doceri

Rank-nullity Theorem

Let V be a finite dimensional vector space and let $T: V \to W$ be a linear transf. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof: Let $\text{nullity}(T) = k$ and $\dim(V) = n \geq k$

Let $B_1 = \{v_1, v_2, \ldots, v_k\}$ be a basis for nullspace(T).

This basis can be extended to a basis for V, say $B_2 = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$

Created with Doceri

Claim: $\text{rank}(T) = n-k$

To show this we'll show that $B_3 = \{T(v_{k+1}), \ldots, T(v_n)\}$ is basis for range(T).

- B_3 is linearly indep.

Let $c_{k+1} T(v_{k+1}) + \cdots + c_n T(v_n) = 0$

$\Rightarrow T(c_{k+1} v_{k+1} + \cdots + c_n v_n) = 0$

$\Rightarrow c_{k+1} v_{k+1} + \cdots + c_n v_n \in$ nullspace(T)

$\Rightarrow c_{k+1} v_{k+1} + \cdots + c_n v_n = c_1 v_1 + \cdots + c_k v_k$

$\Rightarrow c_{k+1} v_{k+1} + \cdots$

Created with Doceri

$\Rightarrow c_1 v_1 + \cdots + c_k v_k - c_{k+1} v_{k+1} - \cdots - c_n v_n = 0$

$\Rightarrow c_i = 0 \;\; \forall i \quad (\because \{v_1, \ldots, v_n\}$ is L.I.$)$

$\Rightarrow c_{k+1} = \cdots = c_n = 0$

- $\text{span}(B_3) = \text{range}(T)$

Let $w = T(v) \in \text{range}(T)$

Then $v = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_n v_n$

$\Rightarrow T(v) = a_1 \cancel{T(v_1)}^{0} + \cdots + a_k \cancel{T(v_k)}^{0} + a_{k+1} T(v_{k+1}) + \cdots + a_n T(v_n)$

$\in \text{span}(B_3)$

Created with Doceri

Thm: For any $A \in M_{m\times n}(\mathbb{F})$,

row rank(A) = col. rank(A).

Proof: Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x) = Ax,$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Note that T is a linear transf.

Also, $\text{range}(T) = \{T(x) : x \in \mathbb{F}^n\}$

$= \{Ax : x \in \mathbb{F}^n\}$

$= \text{col. space}(A)$

$=$

Created with Doceri

$\Rightarrow$ col. rank(A) = rank(T)

Also, null space$(T) = \{ X \in \mathbb{F}^n : T(X) = 0 \}$

$= \{ X \in \mathbb{F}^n : AX = 0 \}$

$=$ soln. space(A)

$\Rightarrow$ nullity(T) = dim of soln space(A)

$= n - k$,

where k = row rank(A)

By rank-nullity thm,

rank(T) + nullity(T) = dim V

col. rank(A) + n − row rank(A) = n

$\Rightarrow$ col. rank = row rank.

Created with Doceri