<u>Example:</u> Consider the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

set of rational numbers

Show that f is continuous at $x=0$.

Since $f(0)=0$, we need to show that

$$\lim_{x \to 0} f(x) = 0.$$

Let $\varepsilon > 0$ be given.

We need to find $\delta > 0$ such that

$$|x-0| < \delta \Rightarrow |f(x)-0| < \varepsilon.$$

Now,

$$|f(x)| = \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

If we take $\delta = \varepsilon$, then

$$|x| < \delta,\ x \in \mathbb{Q} \Rightarrow |f(x)| = |x| < \delta = \varepsilon$$

$$|x| < \delta,\ x \notin \mathbb{Q} \Rightarrow |f(x)| = 0 < \varepsilon$$

$$\therefore \lim_{x \to 0} f(x) = 0 = f(0)$$

$\Rightarrow$ f is continuous at $x=0$.

Remark: Continuity is needed in the intermediate value theorem.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$f(-1) = -1 < 0$

$f(1) = 1 > 0$

There is no $x \in [-1, 1]$ such that $f(x) = 0$.

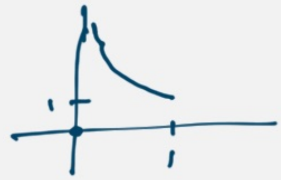
Theorem: Let $f(x)$ be a continuous function on the closed interval $[a,b]$. Then the function attains its minimum and maximum values in the interval $[a,b]$.

$f(x) = \frac{1}{x}$ on $(0,1)$ has no maximum value

"Closed interval" is necessary.

Continuity is required:

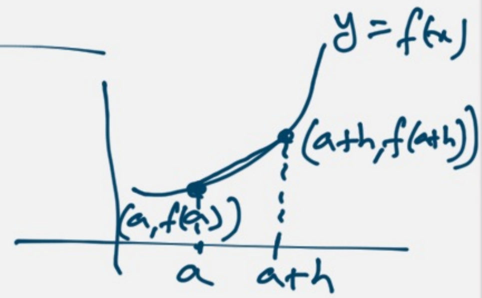
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$



f has no maximum value on $[0,1]$

However, f is <u>not</u> continuous on $[0,1]$ since f is discontinuous at $x=0$.

Derivatives



Slope of the chord joining $(a, f(a))$ and $(a+h, f(a+h))$ is

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

The derivative of f at $x=a$ is defined as

$$f'(a) := \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \text{ if the limit exists.}$$

Defn: We say f is differentiable at $x=a$ if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

and $f'(a)$ is called the derivative of f at $x=a$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

Ex: Find the derivative of $f(x)=\sin x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x(\cos h-1)}{h} + \cos x \frac{\sin h}{h} \right]$$

$$= \sin x \lim_{h\to 0} \frac{\cos h-1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

Using $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and

$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$, we get

$f'(x) = \cos x$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \times \frac{1 + \cos h}{1 + \cos h}$$

$$= \lim_{h \to 0} \frac{-\sin^2 h}{h(1 + \cos h)} = -\lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{\sin h}{1 + \cos h}$$

$$= -1 \times \frac{0}{1+1} = 0$$