<u>Theorem</u> (Local minima/maxima test)
Suppose $f(x)$ is differentiable $2n$ times and $f^{(2n)}(x)$ is continuous at $x=a$.
Then
(i) if $f'(a) = 0 = f''(a) = \cdots = f^{(2n-1)}(a)$ and $f^{(2n)}(a) > 0$, then f has a local minimum at $x=a$.
(ii) if $f'(a) = f''(a) = \cdots = f^{(2n-1)}(a) = 0$ and $f^{(2n)}(a) < 0$, then f has a local maximum at $x=a$.
(iii) if $f^{(k)}(a) = 0$ for $k = 1, 2, \ldots, 2n-2$ and $f^{(2n-1)}(a) \neq 0$, ~~th~~ f has

neither min nor max. at $x=a$.

<u>Proof</u>: By the Taylor's thm.,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(2n-1)}(a)}{(2n-1)!}(x-a)^{2n-1} + \frac{f^{(2n)}(c)}{(2n)!}(x-a)^{2n}$$

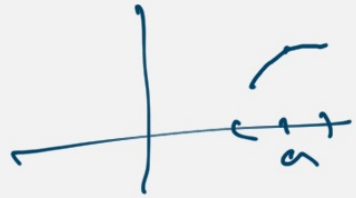
for some c between a and x.

$$\Rightarrow f(x) = f(a) + \frac{f^{(2n)}(c)}{(2n)!}(x-a)^{2n} \quad (*)$$

(i) if $f^{(2n)}(a) > 0$, then by the continuity of $f^{(2n)}(x)$, $f^{(2n)}(c) > 0$ if x is close to a.

$\therefore$ (*) implies

$f(x) > f(a)$ in a neighborhood of a.



$\Rightarrow$ $x = a$ is a point of local minimum.

(ii) Similar to (i)

(iii)
$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(2n-2)}(a)}{(2n-2)!}(x-a)^{2n-2} + \frac{f^{(2n-1)}(c)}{(2n-1)!}(x-a)^{2n-1}$$
$$= f(a) + \frac{f^{(2n-1)}(c)}{(2n-1)!}(x-a)^{2n-1}$$

If $f^{(2n-1)}(a) \neq 0$, then $f^{(2n-1)}(c) \neq 0$

L' Hôpital's Rule

Thm: Suppose $f(x)$ and $g(x)$ are differentiable n times. Also, assume $f^{(n)}$, $g^{(n)}$ are continuous at a, and $f^{(k)}(a)=0$, $g^{(k)}(a)=0$ for $k=0,1,2,\dots,n-1$.

If $g^{(n)}(a)\neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

Proof: By Taylor's thm.,

$$f(x) = f(a)^{\to 0} + f'(a)^{\to 0}(x-a) + \frac{f''(a)^{\to 0}}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)^{\to 0}}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c_1)}{n!}(x-a)^n$$

$$\Rightarrow f(x) = \frac{f^{(n)}(c_1)}{n!}(x-a)^n$$

Similarly, $g(x) = \dfrac{g^{(n)}(c_2)}{n!}(x-a)^n$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f^{(n)}(c_1)}{g^{(n)}(c_2)},$$ c_1 & c_2 are between a & x.

As $x \to a$, $c_1 \to a$ and $c_2 \to a$

$$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

<u>Remark</u>: The L'Hôpital's rule also works for $\frac{\infty}{\infty}$ form and also for limit $x \to \infty$ or $x \to -\infty$.

<u>Example</u>:

$$\lim_{x \to \infty} \frac{x^3}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{3x^2}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{6x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{6}{e^x} = \lim_{x \to \infty} 6e^{-x} = 0$$