Row space, column space & null space of a matrix:

Let $A \in M_{m \times n}(\mathbb{F})$

There are m rows $R_1, R_2, \dots, R_m$ and n column $C_1, C_2, \dots, C_n$ of A.

Each row of A can be thought as an element of $\mathbb{F}^n$.

Each column of A can be thought as an element of $\mathbb{F}^m$.

Defn: The row space of A is the space spanned by the rows $R_1, \dots, R_m$.

So, row space is a subspace $\mathbb{F}^n$.

Defn: The column space of A is the subspace of $\mathbb{F}^m$ spanned by the columns $C_1, C_2, \dots, C_n$.

For example, if $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}$

$$\text{Row space}(A) = \text{span}\{(1,2,3), (2,-1,4)\} \subseteq \mathbb{R}^3.$$

$$\text{Column space}(A) = \text{span}\{(1,2), (2,-1), (3,4)\} = \mathbb{R}^2$$

Defn: The row rank and the column rank of A are the dimensions of the row space and the column space, respectively.

row rank(A) = dim(row space(A))

column rank (A) = dim (column space(A))

Remark: $0 \le$ row rank$(A) \le \min\{m, n\}$

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(Since rowspace$(A) \subseteq \mathbb{F}^n$,
row rank $(A) \le n$

Also, since row space $(A) = \text{span}\{R_1, R_2, \cdots, R_m\}$,
row rank $(A) \le m$)

We'll prove later that
$\boxed{\text{row rank}(A) = \text{column rank}(A).}$

Remark: For $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$,

$$AX = x_1 C_1 + x_2 C_2 + \cdots + x_n C_n \in \text{span}\{C_1, C_2, \cdots, C_n\}$$

$\therefore$ Column space $(A) = \{AX : X \in \mathbb{F}^n\}$

row space (A) = column space (A^t)

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Q: How to find row rank(A) and a basis for row space (A) ?

Note that if A is row equivalent to B, then row space(A) = row space(B).

∴ If R is the RRE form of A, then row space(A) = row space(R).

Also, the nonzero row of an RRE matrix are linearly indep.

∴ The nonzero rows of R forms a basis for row space (A) .

∴ row rank (A) = # of nonzero rows in the RRE form of A.
= rank (A) .

Defn: The null space of A is defined as

$$\text{Null space}(A) = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : AX = 0 \right\}$$

= the solution space of the homog. system $AX = 0$.

$\subseteq \mathbb{F}^n$

∴ dim (null space (A)) = dim (soln. space of $AX=0$)

$= n - r$, where

r is the no. of nonzero rows in the RRE form of A.

n is the no. of columns of A.

To find a basis for null space (A), we identify the $(n-r)$ free variables and then find solutions by putting one of the free variables equal to 1 and the rest 0.

Example: Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & -1 & 4 & 1 \\ 4 & 1 & 5 & 1 \\ 2 & -3 & 3 & 1 \end{pmatrix}$

Find a basis for null space (A).

Soln: $A \xrightarrow[\substack{R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -7 & 1 & 1 \\ 0 & -7 & 1 & 1 \\ 0 & -7 & 1 & 1 \end{pmatrix}$

$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -7 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow{R_2 \to -\frac{1}{7}R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{9}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \text{RRE matrix}$$

Free variables are x_3 & x_4.

Putting $x_3 = \lambda$ and $x_4 = \mu$, we get

$$x_1 = -\frac{9}{7}\lambda - \frac{2}{7}\mu$$

$$x_2 = \frac{1}{7}\lambda + \frac{1}{7}\mu$$

$\therefore$ Null space $(A) = \left\{ \left(-\frac{9}{7}\lambda - \frac{2}{7}\mu, \frac{1}{7}\lambda + \frac{1}{7}\mu, \lambda, \mu\right) : \lambda, \mu \in \mathbb{R} \right\}$

$$\left(-\frac{9}{7}\lambda - \frac{2}{7}\mu, \frac{1}{7}\lambda + \frac{1}{7}\mu, \lambda, \mu\right) = \lambda\left(-\frac{9}{7}, \frac{1}{7}, 1, 0\right) + \mu\left(-\frac{2}{7}, \frac{1}{7}, 0, 1\right)$$

$\therefore$ A basis for null space (A) is

$$B = \left\{ \left(-\frac{9}{7}, \frac{1}{7}, 1, 0\right), \left(-\frac{2}{7}, \frac{1}{7}, 0, 1\right) \right\}$$