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HTML Content

Laplace Transforms

Definition: Let $f:[0,\infty)\to\mathbb{R}$ be any function. We define the Laplace transform of f as

$$F(s) := (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)\,dt,$$

provided the improper integral converges.

Examples:

① $f(t) = 1 \quad \forall t$

$$F(s) = \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_{t=0}^{t=b}$$

$$= \lim_{b\to\infty} \frac{1}{s}\left(1 - e^{-bs}\right)$$

$$= \frac{1}{s}, \text{ if } s>0$$

$$\therefore \boxed{\mathcal{L}(1)(s) = \frac{1}{s} \text{ for } s>0}$$

② $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt = \frac{1}{s-a}, \text{ if } s > a$$

$$\therefore \boxed{\mathcal{L}(e^{at})(s) = \frac{1}{s-a}, \quad s > a}$$

③ $f(t) = t^n, \quad n \in \mathbb{N}$

$$F(s) = \int_0^\infty e^{-st} t^n \, dt$$

$$= \underbrace{t^n \frac{e^{-st}}{-s} \Big|_0^\infty}_{\overset{\shortparallel}{0} \text{ if } s>0} - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} \, dt$$

$$\Rightarrow F(s) = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt = \frac{n}{s} \mathcal{L}(t^{n-1})(s)$$

$$\therefore \boxed{\mathcal{L}(t^n)(s) = \frac{n}{s} \mathcal{L}(t^{n-1})(s)}$$

$\therefore$ By induction,

$$\boxed{\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}}, \quad s>0$$

Properties of Laplace transforms:

① **Linearity**

$$\mathcal{L}\big(af(t)+bg(t)\big)(s) = a\mathcal{L}(f(t))(s) + b\mathcal{L}(g(t))(s)$$

Pf: $L.H.S. = \int_0^\infty e^{-st}\big(af(t)+bg(t)\big)\,dt$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s).$$

Using this, we can find the Laplace transform of any polynomials.

e.g. $\mathcal{L}(t^3-2t^2+t+3)(s)$

$$= \mathcal{L}(t^3)(s) - 2\mathcal{L}(t^2)(s) + \mathcal{L}(t)(s) + 3\mathcal{L}(1)(s)$$

$$= \frac{3!}{s^4} - \frac{2\times 2!}{s^3} + \frac{1!}{s^2} + 3\cdot\frac{1}{s}.$$

Laplace transforms of hyperbolic sine & hyperbolic cosine functions:

$$\sinh(at) := \frac{e^{at} - e^{-at}}{2}$$

$$\cosh(at) := \frac{e^{at} + e^{-at}}{2}$$

$$\therefore \mathcal{L}(\sinh(at))(s) = \frac{1}{2}\left[\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\boxed{\mathcal{L}(\sinh(at))(s) = \frac{a}{s^2-a^2}} \text{ if } s > |a|$$

$$\mathcal{L}(\cosh(at))(s) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2-a^2}, \text{ if } s > |a|$$

Laplace transforms of sine & cosine :

- For $f(t) = \cos(\omega t)$,

$$F(s) = \int_0^\infty e^{-st}\cos(\omega t)\,dt$$

$$= \frac{e^{-st}}{-s}\cos(\omega t)\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(-\omega)\sin(\omega t)\,dt$$

$$= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)\,dt, \quad s>0$$

$$= \frac{1}{s} - \frac{\omega}{s}\Big[\cancel{\frac{e^{-st}}{-s}\sin(\omega t)\Big|_0^\infty}^{\,0} - \int_0^\infty \frac{e^{-st}}{-s}\,\omega\cos(\omega t)\,dt\Big]$$

$$= \frac{1}{s} - \frac{\omega^2}{s^2}F(s)$$

$$\Rightarrow \left(1+\frac{\omega^2}{s^2}\right)F(s) = \frac{1}{s}$$

$$\Rightarrow F(s) = \frac{s}{s^2+\omega^2}$$

$$\therefore \boxed{\mathcal{L}(\cos(\omega t))(s) = \frac{s}{s^2+\omega^2}}$$

Similarly,

$$\boxed{\mathcal{L}(\sin(\omega t)(s)) = \frac{\omega}{s^2+\omega^2}}$$

Inverse Laplace transforms:

Defn: If $F(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is said to be the inverse Laplace transform of $F(s)$, and is denoted by $\mathcal{L}^{-1}(F)(t)$.

e.g. (i) $\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{1}{(n-1)!}\mathcal{L}^{-1}\left(\frac{(n-1)!}{s^n}\right)$

$= \frac{t^{n-1}}{(n-1)!}$ for $n = 1, 2, 3, \ldots$

(ii) $\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)$

$$= \frac{1}{2}\sin(2t)$$

<u>More properties of Laplace transforms:</u>

- <u>(s-shifting property)</u>

Let $F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$

Then $F(s-a) = ?$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\{e^{at} f(t)\}dt$$

$$= \mathcal{L}\left(e^{at} f(t)\right)(s)$$

$$\Rightarrow \boxed{\mathcal{L}\left(e^{at} f(t)\right)(s) = \mathcal{L}(f(t))(s-a)}$$

Also, $\mathcal{L}^{-1}\left(F(s-a)\right) = e^{at} f(t)$

$$\boxed{\mathcal{L}^{-1}\left(F(s-a)\right)(t) = e^{at}\mathcal{L}^{-1}(F(s))(t)}$$

e.g. Find $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right)$

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+2^2}$$

Let $F(s) = \frac{1}{s^2+2^2} = \frac{1}{2}\mathcal{L}(\sin(2t))(s)$

Then $\mathcal{L}^{-1}(F(s))(t) = \frac{1}{2}\sin(2t)$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = \mathcal{L}^{-1}\left(F(s+1)\right)(t)$$
$$= e^{-t}\mathcal{L}^{-1}(F(s))(t)$$
$$= \frac{e^{-t}}{2}\sin(2t)$$