Cauchy condensation Test :

Suppose $a_n \geq 0$ and $a_{n+1} \leq a_n$ $\forall n$
i.e. $\{a_n\}$ is a non increasing seq. of nonnegative numbers. Then
$\sum_{n=1}^{\infty} a_n$ converges if and only if
$\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

(Note that $\sum_{n=1}^{\infty} 2^n a_{2^n} = 2a_2 + 4a_4 + 8a_8 + \cdots$)

example :
Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where
p is any real number.

Here $a_n = \frac{1}{n^p}$

If $p \leq 0$, then $\lim_{n\to\infty} a_n \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n$ diverges.

Let $p > 0$. Then $a_n > 0$ $\forall n$.
Also, $a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n$

$\therefore \{a_n\}$ is a decreasing seq. of positive numbers.

So we can apply the Cauchy condensation test.

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \leftarrow \text{geometric series. with } r = \frac{1}{2^{p-1}}$$

If $\frac{1}{2^{p-1}} < 1$ i.e. $2^{p-1} > 1$ i.e. $p > 1$,

then $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

If $p \leq 1$, then $\frac{1}{2^{p-1}} \geq 1$

$\Rightarrow \sum_{n=1}^{\infty} 2^n a_{2^n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

<u>Question</u>: Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ converge or diverge?

$a_n = \frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n \cdot n^{1/n}}$

Let $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = \frac{1}{n^{1/n}} \to 1$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

by the LCT, $\sum_{n=1}^{\infty} a_n$ diverges.

Example: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\left\{\frac{1}{n \ln n}\right\}_{n=2}^{\infty}$ is decreasing, positive.

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \ln(2^n)}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n(\ln 2)} = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

$\therefore$ By Cauchy condensation test,

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Ratio Test:

Suppose $a_n \neq 0$ $\forall n$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$.

Then (i) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\rho = 1$, the test fails.

Examples:

(1) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$a_n = \frac{2^n}{n!}$; $a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2 \cdot 2^n}{(n+1)\, n!}$

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}$

$\therefore \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by Ratio test.

(2) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{n!\; n!}{(2n)!}$

$= \frac{(2n+2)(2n+1)}{(n+1)\,(n+1)} \to 4 > 1$

$\therefore$ The series diverges.

③ $a_n = \begin{cases} n/2^n & \text{if } n \text{ odd} \\ 1/2^n & \text{if } n \text{ even} \end{cases}$

Does $\sum a_n$ converge?

Try ratio test!