

## Infinite Sequences

A sequence is a list of numbers  
 $a_1, a_2, a_3, \dots, a_n, \dots$   
 in a given order.

Mathematically, a sequence is a function  
 from the set of natural numbers  $\mathbb{N}$   
 to the set of real numbers  $\mathbb{R}$ .

$$f: \mathbb{N} \rightarrow \mathbb{R}$$

$$f(n) = a_n$$

We denote a sequence by  $\{a_1, a_2, a_3, \dots, a_n, \dots\}$   
 or  $\{a_n\}_{n=1}^{\infty}$

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Sometimes we start the sequence  
 from  $n=2$ ,  $n=3$ , etc.

e.g.  $a_n = \frac{n}{\ln n}$ ,  $n=2, 3, 4, \dots$

We can rewrite the same sequence

as  $a_n = \frac{n+1}{\ln(n+1)}$ ,  $n=1, 2, 3, \dots$

Examples:

$$\{2^n\}_{n=1}^{\infty}, \{(-1)^n\}_{n=1}^{\infty}, \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}, \{c\}_{n=1}^{\infty}$$

↑ constant seq.

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## Convergence of a sequence

Definition: A sequence  $\{a_n\}$  is said to converge to  $L$  or  $\lim_{n \rightarrow \infty} a_n = L$

if given any  $\varepsilon > 0$ , there exists a corresponding natural number  $N$  s.t.

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N.$$

e.g.  $\lim_{n \rightarrow \infty} c = c \quad (a_n = c \quad \forall n)$   
 $\varepsilon > 0, N = 1$ . Then  $|a_n - c| = 0 < \varepsilon \quad \forall n > 1$   
 $\therefore \lim_{n \rightarrow \infty} a_n = c$

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e.g. Show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

Let  $\varepsilon > 0$ . We need to find  $N \in \mathbb{N}$  s.t.  $|\frac{1}{n} - 0| < \varepsilon$  whenever  $n > N$ .

If  $n > N$ ,  $\frac{1}{n} < \frac{1}{N}$   
 So, we choose  $N$  s.t.  $\frac{1}{N} < \varepsilon$  i.e.  $N > \frac{1}{\varepsilon}$

Theorem: The Sum Rule, Difference Rule, Constant Multiple Rule, Product Rule, Quotient Rule hold true for limits of sequences.

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Quotient rule:

If  $\lim_{n \rightarrow \infty} a_n = L$  &  $\lim_{n \rightarrow \infty} b_n = M$ ,  
 then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$ , provided  $M \neq 0$ .

Sandwich theorem:

If  $b_n \leq a_n \leq c_n$  and  $\lim_{n \rightarrow \infty} b_n = L$ ,  
 $\lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} a_n = L$ .

e.g.  $a_n = \frac{\sin(n)}{n}$

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Since  $-1 \leq \sin(n) \leq 1$ ,

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \quad \forall n$$

By sandwich thm,  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$ .

Notation:  $a_n \rightarrow L$  means  $\lim_{n \rightarrow \infty} a_n = L$

Example of a sequence which does not converge:

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①  $a_n = (-1)^n$  does not converge to any  $L$ .

②  $b_n = \sqrt{n}$  does not converge to any number  $L$ .

$$\lim_{n \rightarrow \infty} b_n = \infty$$

Defn: We say  $\lim_{n \rightarrow \infty} a_n = \infty$  or  $a_n \rightarrow \infty$  (an diverges to  $\infty$ )

if given  $M > 0$ ,  $\exists N \in \mathbb{N}$  s.t.  
 $a_n > M$  whenever  $n > N$ .

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Theorem: Let  $a_n = f(n)$  for some function  $f(x)$ .

If  $\lim_{x \rightarrow \infty} f(x) = L$ , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

Remark: L'Hôpital's rule can be applied to sequences as well.

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Example: Show that  $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$

Soln: let  $f(x) = \frac{\ln x}{x}$ ,  $x \in (0, \infty)$

Then since  $\ln x \rightarrow \infty$  and  $x \rightarrow \infty$  as  $x \rightarrow \infty$ , by L'Hopital's rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0.$$

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Remark: The converse of the previous theorem is not true.

e.g. The sequence  $a_n = \sin(n\pi) = 0$   $\forall n$

converges to 0 but  $f(x) = \sin(\pi x)$  does not converge as  $x \rightarrow \infty$ .

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Example: Find  $\lim_{n \rightarrow \infty} \left( \frac{n+1}{n-1} \right)^n$

Let  $f(x) = \left( \frac{x+1}{x-1} \right)^x$

Then  $\ln f(x) = x \ln \left( \frac{x+1}{x-1} \right)$   
 $= \frac{\ln \left( \frac{x+1}{x-1} \right)}{\frac{1}{x}} \quad \left( \frac{0}{0} \text{ form} \right)$

$\therefore \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1} - \frac{1}{x-1}}{-\frac{1}{x^2}} = 2$   
 Show this

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$\Rightarrow \lim_{x \rightarrow \infty} f(x) = e^2$

Since  $a_n = f(n)$ ,  $\lim_{n \rightarrow \infty} a_n = e^2$ .

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