Method of undetermined coefficients

- This method is applicable when the homogeneous part is of constant coefficients linear ODE and the nonhomogeneous part is one of the following functions:
  - exponential
  - polynomial
  - sine or cosine
  - sums or products of above functions

Example: $y'' - 2y' + y = e^t$

$y_h(t) = c_1 e^t + c_2 t e^t$

(because char. eqn. $m^2 - 2m + 1 = 0$ i.e. $(m-1)^2 = 0$

$\therefore\ e^t,\ te^t$)

$y_p(t) = A t^2 e^t$

$y_p'(t) = A(t^2 e^t + 2te^t)$

$y_p''(t) = A(t^2 e^t + 4te^t + 2e^t)$

$\therefore\ A e^t \left[(\cancel{t^2} + 4\cancel{t} + 2) - 2(\cancel{t^2} + 2\cancel{t}) + \cancel{t^2}\right] = e^t$

$\Rightarrow\ 2A = 1 \ \Rightarrow\ A = \frac{1}{2}$

$\therefore\ y_p(t) = \frac{1}{2} t^2 e^t$

Example: $y'' - y = t^2$

$y_h(t) = c_1 e^t + c_2 e^{-t}$

What should be $y_p(t)$?

$y_p(t) = At^2 + Bt + C$ (polynomial of the same degree as $r(t) = t^2$)

$y_p'(t) = 2At + B$

$y_p''(t) = 2A$

$\therefore\ 2A - (At^2 + Bt + C) = t^2$

$\Rightarrow\ (2A - C) - Bt - (A+1)t^2 = 0$

$\Rightarrow\ 2A - C = 0,\ B = 0,\ A + 1 = 0$

$\Rightarrow\ A = -1,\ B = 0,\ C = -2$

$\therefore\ \boxed{y_p(t) = -t^2 - 2}$

$\boxed{y = c_1 e^t + c_2 e^{-t} - t^2 - 2}$

Modification Rule for polynomial:

If $y =$ constant is a solution of the homog. part but $y = t$ is not a solution, then we multiply the particular by t

i.e. if $r(t) = a_0 + a_1 t + \dots + a_n t^n$

then $y_p(t) = t(b_0 + b_1 t + \dots + b_n t^n)$

and then determine $b_0, b_1, \dots, b_n$.

Example: $y'' - y = \sin(t)$

$y_h(t) = c_1 e^t + c_2 e^{-t}$

We take $y_p(t) = A \sin t + B \cos t$

Then $y_p' = A \cos t - B \sin t$

$y_p'' = -A \sin t - B \cos t$

$\therefore\ -A \sin t - B \cos t - A \sin t - B \cos t = \sin t$

$\Rightarrow\ -2A \sin t - 2B \cos t = \sin t$

$\Rightarrow\ -2A = 1$ and $-2B = 0$

$\Rightarrow\ A = -\frac{1}{2}$ and $B = 0$

$\therefore\ y_p = -\frac{1}{2} \sin t$

Example: $y'' + y = \sin t$

$y_h(t) = c_1 \sin t + c_2 \cos t$

$y_p(t) = t(A \sin t + B \cos t)$

$y_p'(t) = t(A \cos t - B \sin t) + A \sin t + B \cos t$

$y_p''(t) = t(-A \sin t - B \cos t) + (A \cos t - B \sin t) + A \cos t - B \sin t$

$\therefore\ t(-\cancel{A \sin t} - \cancel{B \cos t}) + 2(A \cos t - B \sin t) + t(\cancel{A \sin t} + \cancel{B \cos t}) = \sin t$

$\Rightarrow\ A = 0,\ -2B = 1$ i.e. $B = -\frac{1}{2}$

$\therefore\ \boxed{y_p(t) = -\frac{1}{2} t \cos t}$

Summary: To solve a nonhomog. ODE with constant coefficients homog. part, we first solve the homog. ODE and then apply the following rule to find a particular solution:

	$r(t)$	$y_p(t)$
(i)	$\beta e^{\alpha t}$	• $A e^{\alpha t}$ if $e^{\alpha t}$ is not a soln. of the homog. part. • $A t e^{\alpha t}$ if $e^{\alpha t}$ is a soln. but $t e^{\alpha t}$ is not a soln. • $A t^2 e^{\alpha t}$ if both $e^{\alpha t}$ and $t e^{\alpha t}$ are solns.
(ii)	$a_0 + a_1 t + \dots + a_n t^n$	• $b_0 + b_1 t + \dots + b_n t^n$ if $y =$ constant is not a soln of homog. • $t(b_0 + b_1 t + \dots + b_n t^n)$ if $y =$ constant is a soln but $y = t$ is not a soln. • $t^2(b_0 + b_1 t + \dots + b_n t^n)$ if $y = c$ & $y = t$ are both solns.
(iii)	$r(t) = a \cos(\alpha t) + b \sin(\alpha t)$	• $A \cos(\alpha t) + B \sin(\alpha t)$ if $\cos \alpha t$ is not a soln. of the homog. • $t(A \cos \alpha t + B \sin \alpha t)$ if $\cos \alpha t$ is a soln.

Sum Rule: If $r(t)$ is a sum of functions of the above types, then we take particular solution as the sum of the corresponding functions.

Example: $y'' - y = e^t \cos t$

$y_h(t) = c_1 e^t + c_2 e^{-t}$

$y_p(t) = e^t (A \cos t + B \sin t)$

$y_p'(t) = e^t (A \cos t + B \sin t) + e^t(-A \sin t + B \cos t)$

$= e^t \left[(A+B) \cos t + (B-A) \sin t\right]$

$y_p''(t) = e^t \left[(A+B) \cos t + (B-A) \sin t - (A+B) \sin t + (B-A) \cos t\right]$

$= e^t \left[2B \cos t - 2A \sin t\right]$

$y_p'' - y_p = e^t \cos t$

$\Rightarrow\ e^t (2B \cos t - 2A \sin t - A \cos t - B \sin t) = e^t \cos t$

$\Rightarrow\ 2A + B = 0$ and $2B - A = 1$

$\Rightarrow\ A = -\frac{1}{5},\ B = \frac{2}{5}$

$y_p(t) = e^t\left(-\frac{1}{5} \cos t + \frac{2}{5} \sin t\right)$.