Indian Institute of Technology Delhi - Abu Dhabi

AMTL101

Tutorial Sheet 10: Picard's Iteration and Existence-Uniqueness Theorem

(1) Find the n-th Picard's iterate y_n for the following IVP: $y' = x^2 + y,\ y(0) = 0$.

(2) Apply the Picard's iteration method to $y' = 2y^2,\ y(0) = 1$.

(3) Consider the IVP:

$$(x^2 - 1)y' = 4y,\ y(x_0) = y_0.$$

 (a) Find the values of (x_0, y_0) for which a unique solution is guaranteed by the existence-uniqueness theorem.

 (b) Show that if $(x_0, y_0) = (1, 0)$, then the IVP has infinitely many solutions.

(4) Find all the initial conditions, such that corresponding IVP, with the ODE

$$(x^2 - 4x)y' = (2x - 4)y$$

has no solution, a unique solution and more than one solution, respectively.

Solution 1 : $y' = x^2 + y \quad , \quad y(0) = 0$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$= 0 + \int_{0}^{x} t^2\, dt = \frac{x^3}{3}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$= 0 + \int_{0}^{x} \left(t^2 + \frac{t^3}{3}\right) dt$$

$$= \frac{x^3}{3} + \frac{x^4}{12}$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\, dt$$

$$= 0 + \int_{0}^{x} \left(t^2 + \frac{t^3}{3} + \frac{t^4}{12}\right) dt$$

$$= \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60}$$

$$\therefore \quad y_n(x) = \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \cdots + \frac{x^{n+2} \cdot 2}{(n+2)!}$$

$$= 2\left[\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^{n+2}}{(n+2)!}\right]$$

as $n \to \infty$,

$$y_n(x) \longrightarrow 2\left[e^x - 1 - x - \frac{x^2}{2!}\right]$$

The IVP can be solved as :

$$y' - y = x^2$$

$$\Rightarrow \quad I.F. = e^{-\int 1\,dx} = e^{-x}$$

$$\therefore \quad y.e^{-x} = \int e^{-x}.x^2\,dx + c$$

$$\Rightarrow \quad y.e^{-x} = -x^2e^{-x} + \int 2x e^{-x} + c$$

$$= -x^2 e^{-x} - 2xe^{-x} + \int 2e^{-x} + c$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$\Rightarrow \quad y = -x^2 - 2x - 2 + ce^x$$

$$\because \quad y(0) = 0 \Rightarrow c = 2$$

$$\Rightarrow \quad y = -x^2 - 2x - 2 + 2e^x$$

$$= 2\left[e^x - 1 - x - \frac{x^2}{2!}\right]$$

Solution 2: $\quad y' = 2y^2, \; y(0) = 1$

Here, $f(x,y) = 2y^2, \; x_0 = 0, \; y_0 = 1$

$$\therefore \quad y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$= 1 + \int_0^x 2\, dt = 1 + 2x$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$= 1 + \int_0^x 2(1+2t)^2\, dt$$

$$= 1 + 2\int_0^x \left(1 + 4t + 4t^2\right) dt$$

$$= 1 + 2\left[x + 2x^2 + \frac{4x^3}{3}\right]$$

$$= 1 + 2x + 4x^2 + \frac{8}{3}x^3$$

$$y_3(x) = 1 + 2\int_0^x \left[1 + 2t + 4t^2 + \frac{8}{3}t^3\right]^2 dt$$

$$\left[1 + 2t + 4t^2 + \frac{8}{3}t^3\right]^2 = (1+2t)^2 + \left(4t^2 + \frac{8}{3}t^3\right)^2$$

$$+ 2(1+2t)\left(4t^2 + \frac{8}{3}t^3\right)$$

$$\Rightarrow \quad y_3(x) = 1 + 2\int_0^x (1+2t)^2 dt + 4\int_0^x (1+2t)\left(4t^2 + \frac{8}{3}t^3\right)$$

$$+ \; 2\int_0^x \left(4t^2+\frac{8}{3}t^3\right)^2 dt$$

$$= 1 + 2x + 4x^2 + \frac{8}{3}x^3 + 16\frac{x^3}{3} + \cdots$$

$$= 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

$$\therefore \quad y_n(x) = 1 + (2x) + (2x)^2 + \cdots + (2x)^n + \cdots$$

$$\Rightarrow \quad y_n(x) \longrightarrow \frac{1}{1-2x} \quad \text{as } n \to \infty .$$

Solution 3 : $(x^2-1)y' = 4y$, $y(x_0) = y_0$

$$\Rightarrow \quad y' = \frac{4y}{x^2-1}$$

$\because$ $f(x,y) = \dfrac{4y}{x^2-1}$ is continuous except

$x = \pm 1$ and $\dfrac{\partial f}{\partial y} = \dfrac{4}{x^2-1}$ is continuous

for all (x,y) except $x = \pm 1$,

$\therefore$ by existence-uniqueness theorem, the IVP has a unique Solution for all (x_0, y_0)

Such that $x_0 \neq \pm 1$.

(b) If $(x_0, y_0) = (1, 0)$,

$$(x^2-1)\, y' = 4y$$

$$\Rightarrow \quad \frac{1}{y}\, dy = \frac{4}{x^2-1}\, dx$$

$$\Rightarrow \quad \frac{1}{y}\, dy = \frac{4}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \quad \ln y = 2\left\{\ln|x-1| - \ln|x+1|\right] + \ln c$$

$$\Rightarrow \quad \ln y = 2 \ln \left|\frac{x-1}{x+1}\right| + \ln c$$

$$\Rightarrow \quad y = c\left(\frac{x-1}{x+1}\right)^2$$

Also, $y(1) = 0$

$\therefore \quad y = c\left(\frac{x-1}{x+1}\right)^2$ is a Solution of

the IVP for every $c \in \mathbb{R}$.

Solution 4: $(x^2-4x)y' = (2x-4)y \; , \; y(x_0) = y_0$

$$\Rightarrow \quad y' = \frac{(2x-4)y}{x(x-4)}$$

Here, $f(x,y) = \dfrac{(2x-4)y}{x(x-4)}$, $\dfrac{\partial f}{\partial y} = \dfrac{2x-4}{x(x-4)}$

Since, $f(x,y)$ & $\dfrac{\partial f}{\partial y}$ are continuous everywhere except $x=0$ and 4, $\therefore$ by existence & uniqueness theorem, the IVP has a unique solution for all (x_0, y_0) such that $x_0 \notin \{0,4\}$.

Now,

$$\int \frac{dy}{y} = \int \frac{2x-4}{x(x-4)}\,dx = \int \frac{2(x-4)}{x(x-4)} + \frac{4}{x(x-4)}$$

$$\Rightarrow \quad \ln y = 2\ln x + \int\left(\frac{1}{x-4} - \frac{1}{x}\right) + \ln c$$

$$= 2\ln x + \ln|x-4| - \ln x + \ln c$$

$$= \ln c\, x(x-4)$$

$$\Rightarrow \quad y = c\,x(x-4)$$

$\therefore \quad y(0) = 0 = y(4)$

$\therefore$ if $(x_0, y_0) = (0,0)$ or $(4,0)$

then the IVP has infinitely many solutions,

and if $x_0 = 0$ or 4 but $y_0 \neq 0$ then the IVP

has no solution.