Gauss Elimination Method

To solve a system $AX = b$ we convert the augmented matrix $(A|b)$ into an upper triangular matrix (or row echelon matrix) be performing elementary row operations.

Example: Solve the system

$$\begin{aligned} x+y+z &= 3 \\ x+2y+2z &= 5 \\ 3x+4y+4z &= 11 \end{aligned}$$

Sol: $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{array}\right)$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

i.e.
$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \end{aligned}$$

The solution set, $S = \{(1, 2-z, z) : z \in \mathbb{R}\}$

Definition (Pivot column)

For a matrix in echelon form, the first nonzero entry in each nonzero row is called a pivot entry. The columns corresponding to the pivot entries are called the pivot columns.

In the previous example

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns are 1 & 2.

Definition (Basic & free variables)

Consider the system $AX = b$ consisting of m equations in n variables $x_1, x_2, \ldots, x_n$. Let $(C|d)$ be a row echelon matrix equivalent to $(A|b)$. Then the variables corresponding to the pivot columns in $(C|d)$ are called the basic variables. The variables which are not basic are free variables.

In the previous example x and y are basic variables and z is the free variables.

To solve a system the free variables are assigned arbitrary values and the basic variables can be written in terms of the free variables.

<u>Remark</u>: If there are r nonzero rows in row reduced echelon form (or echelon form) then there will r basic variables and $(n-r)$ free variables.

<u>Example</u>:

$$\begin{aligned} x_1 + x_2 + x_3 + 3x_4 &= 7 \\ x_1 + x_2 + 2x_3 + 5x_4 &= 11 \\ x_3 + 2x_4 &= 4 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 7 \\ 1 & 1 & 2 & 5 & 11 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$R_1 \to R_1 - R_2 \quad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_1 & x_3 are basic variables

x_2 & x_4 are free variables.

Let $x_2 = \lambda$ and $x_4 = \mu$

$$x_1 + x_2 + x_4 = 3$$

$$x_3 + 2x_4 = 4$$

$\therefore \quad x_1 = 3 - \lambda - \mu \; ; \; x_3 = 4 - 2\mu$

$S = \{(3-\lambda-\mu, \lambda, 4-2\mu, \mu) : \lambda, \mu \in \mathbb{R}\}$

Gauss - Jordan Elimination

In this we convert the augmented matrix into its row reduced echelon form.

Definition (Row rank/rank of a matrix) The row rank of a matrix is the number of nonzero rows in the RRE form of the matrix.

For a system $Ax = b$:

(i) it has no solutions if
$\text{rank}(A|b) \neq \text{rank}(A)$

(ii) it has a unique solution if
$\text{rank}(A|b) = \text{rank}(A) = n$

\# of variables

(iii) it has infinitely many solns if
$\text{rank}(A|b) = \text{rank}(A) < n$.