Coordinate vector w.r.t. an ordered basis & change of bases matrix :

Example: Let $V = \mathbb{R}^3$,

$B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$B_2 = \{(1,0,0), (1,1,0), (1,1,1)\}$

For $v = (x,y,z) \in \mathbb{R}^3$

$[v]_{B_1} = ?$ $\quad [v]_{B_2} = ?$

$v = (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$\therefore [v]_{B_1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$v = (x,y,z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$= (a+b+c, b+c, c)$

$\Rightarrow$ $c = z$

$b + c = y \Rightarrow b = y - z$

$a + b + c = x \Rightarrow a = x - y$

$[v]_{B_2} = \begin{pmatrix} x-y \\ y-z \\ z \end{pmatrix}$

We want to find the change of bases matrix $P \in M_{3\times 3}(\mathbb{R})$ such that

$[v]_{B_2} = P [v]_{B_1}$

Putting $v = (1,0,0)$, we get

$$[(1,0,0)]_{B_2} = P\begin{pmatrix}1\\0\\0\end{pmatrix} = \text{1st column of } P.$$

$\therefore$ To find the 1st column of P we need to find the coordinate vector of the 1st vector in B_1 w.r.t. B_2.

Similarly, the 2nd & 3rd columns are the coord. vectors of 2nd & 3rd vectors in B_1 w.r.t. B_2.

More generally, if

$$B_1 = \{v_1, v_2, \ldots, v_n\}$$

and $B_2 = \{w_1, w_2, \ldots, w_n\}$ are two ordered bases for V.

We find $[v_1]_{B_2}, [v_2]_{B_2}, \ldots, [v_n]_{B_2}$,

then $P = \left([v_1]_{B_2} \; [v_2]_{B_2} \cdots [v_n]_{B_2} \right)_{n\times n}$

Then for any $v \in V$,

$$\boxed{[v]_{B_2} = P[v]_{B_1}}.$$

Prop: The change of bases matrix P is always invertible.

Pf: Let P be the change of bases matrix from B_1 to B_2 and let Q be the change of bases matrix from B_2 to B_1.

Then $[v]_{B_2} = P[v]_{B_1} \quad \forall v \in V$

and $[v]_{B_1} = Q[v]_{B_2} \quad \forall v \in V$

$\therefore \quad [v]_{B_2} = PQ[v]_{B_2} \Rightarrow Q = P^{-1}$.

$\Rightarrow PQ = I$

Matrix representation of a linear transf.

Let V & W be finite dimensional vector spaces and let $T: V \to W$ be a linear transformation.

Let $B = \{v_1, v_2, \cdots, v_m\}$ be an ordered basis for V and $B' = \{w_1, w_2, \cdots, w_n\}$ be an ordered basis for W.

$$T(v_1) = \sum_{i=1}^{n} a_{i1} w_i$$

$$T(v_2) = \sum_{i=1}^{n} a_{i2} w_i$$

$$\vdots$$

$$T(v_m) = \sum_{i=1}^{n} a_{im} w_i$$

Consider the matrix

$$A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}} \in M_{n \times m}(\mathbb{R})$$

Now for any $v \in V$,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$$

$$\therefore \quad T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_m T(v_m)$$

$$= \sum_{j=1}^{m} \alpha_j T(v_j)$$

$$= \sum_{j=1}^{m} \alpha_j \left(\sum_{i=1}^{n} a_{ij} w_i \right)$$

$$\Rightarrow \quad T(v) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} \alpha_j \right) w_i$$

$\therefore \quad \sum_{j=1}^{m} a_{ij} \alpha_j$ is the ith coordinate of the coord. vector $[Tv]_{B'}$

$$\therefore \quad [Tv]_{B'} = A [v]_B$$

We denote this matrix A by $[T]_B^{B'}$

The 1st column of $[T]_B^{B'}$ is nothing but $[T(v_1)]_{B'}$

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = (2x + z, y + 3z),$$

$B = \{(1,1,0), (1,0,1), (1,1,1)\}$

$B' = \{(2,3), (3,2)\}$

Find $[T]_B^{B'}$.

$$T(1,1,0) = (2,1) = a(2,3) + b(3,2)$$

$$\Leftarrow \left.\begin{array}{l} 2a + 3b = 2 \\ 3a + 2b = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a = -\frac{1}{5} \\ b = 4/5 \end{array}$$

$$T(1,0,1) = (3,3) = \frac{3}{5}(2,3) + \frac{3}{5}(3,2)$$

$$T(1,1,1) = (3,4) = \frac{6}{5}(2,3) + \frac{1}{5}(3,2)$$

$$\therefore \quad [T]_B^{B'} = \begin{pmatrix} -1/5 & 3/5 & 6/5 \\ 4/5 & 3/5 & 1/5 \end{pmatrix}$$