Dimension of a vector space

Fact: Every vector space has a basis.
We'll not prove this result. The proof involves a result from set theory known as the "Zorn's lemma".

Defn (Finite dimensional vector space):
A vector space is said to be finite dimensional if it has a finite basis.
A vector space is called infinite dimensional if it has no finite basis.

To show a vector space is infinite dimensional one only needs to show that there is an infinite linearly independent subset.

- If V is a finite dimensional vector space, then any two bases of V has the same number of vectors.

Defn: For a finite dimensional vector space, the dimension of V, denote by $\dim(V)$, is the number of vectors in any basis.
For infinit dim. v. sp., we say $\dim(V) = \infty$.

Examples:

① $V = \mathbb{R}^n$ over $\mathbb{R}$.

$\dim(V) = n$ because

$B = \{(1,0,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)\}$

is a basis.

② $V_1 = \mathbb{C}^n$ over $\mathbb{C}$

$\dim(V_1) = n$

$V_2 = \mathbb{C}^n$ over $\mathbb{R}$

$\dim(V_2) = 2n$

$B = \{(1,0,\dots,0), \dots, (0,0,\dots,0,1), (i,0,\dots,0), (0,i,0,\dots,0), \dots, (0,0,\dots,0,i)\}$

③ $\dim(M_{m\times n}(\mathbb{R})) = mn$

④ $\dim(\mathbb{F}[x]) = \infty$

Thm: Suppose $B = \{v_1, v_2, \dots, v_n\}$ be a basis for a finite dimensional vector space V. Then any vector $v \in V$ can be written as a linear combination of $v_1, v_2, \dots, v_n$ in a unique way.

Proof: Since B is a basis of V, $\operatorname{span}(B) = V$.

$\therefore$ Any $v \in V$ is a linear combination of $v_1, v_2, \dots, v_n$.

Suppose $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$
$= b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$.

To show: $a_i = b_i$ for $i = 1, 2, \cdots, n$.

We have $(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_n - b_n) v_n = 0$

Now since $\{v_1, v_2, \cdots, v_n\}$ is lin. indep.,

$a_i - b_i = 0$ for $i = 1, 2, \cdots, n$

$\Rightarrow a_i = b_i$ for $i = 1, 2, \cdots, n$.

Q: Let V be any vector space.
Is $S = \phi$ linearly dependent or independent?

Ans: ϕ is linearly independent.

- For $W = \{0\}$, the zero subspace,
$B = \phi$ is a basis.
$\therefore \dim(\{0\}) = 0$.
$\text{span}(\phi) = \{0\}$.

Thm: Let W be a subspace of a finite dimensional vector space V. Then W is finite dimensional and $\dim(W) \leq \dim(V)$. Also, if $\dim(W) = \dim(V)$, then $W = V$.

Example: Let $V = M_{n\times n}(\mathbb{R})$ and

$$W = \{ A \in V : A^t = A \} \to \text{all symmetric matrices.}$$

Then W is a subspace of V.

$\dim W = ?$

For $V = M_{2\times 2}(\mathbb{R})$,

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for W.

Any $A \in W$ can be written as

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\dim W = 3$.

For $n\times n$, $\dim W = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.