Gradient :

The gradient of a real-valued function $f(x,y)$ is given by

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} .$$

For a fn. $f(x,y,z)$ of three variables,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Example: Let $f(x,y) = \dfrac{x^2+y^2}{2}$.

Then $\vec{\nabla} f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} = x\hat{i} + y\hat{j}$

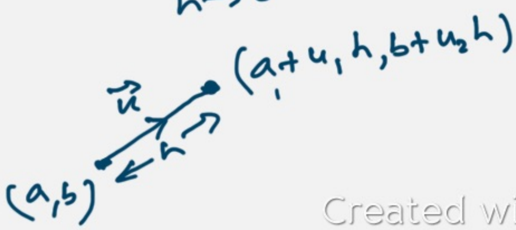
$$\vec{\nabla} f(1,1) = \hat{i} + \hat{j} .$$

Directional derivatives

Let $\vec{u} = u_1\hat{i} + u_2\hat{j}$ be a unit vector (ie. $u_1^2 + u_2^2 = 1$)

If $f(x,y)$ is any function, we define the directional derivative of f in the direction of $\vec{u}$ by

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$



In particular, if $\vec{u} = \hat{i}$, then

$$D_{\hat{i}} f(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$= \frac{\partial f}{\partial x}(a,b)$$

Similarly,

$$D_{\hat{j}} f(a,b) = \frac{\partial f}{\partial y}(a,b)$$

Relationship between directional derivative and gradient:

If $f(x,y)$ is "differentiable" at the point (a,b), then

$$\boxed{D_{\vec{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}}$$

dot product

$\vec{u} = u_1 \hat{i} + u_2 \hat{j}$

$\vec{v} = v_1 \hat{i} + v_2 \hat{j}$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

$\vec{u} \cdot \vec{v} = uv \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Suppose $f(x,y)$ is a real-valued fn.
How to find the directions in which the function f increases/decrease most rapidly?

We have $D_{\vec{u}}f = \vec{\nabla} f \cdot \vec{u}$
$= |\vec{\nabla} f|\,|\vec{u}| \cos\theta$
$= |\vec{\nabla} f| \cos\theta$

$\therefore$ $D_{\vec{u}}f$ is maximum when $\cos\theta = 1$,
i.e., $\theta = 0$; i.e. $\vec{u}$ is in the direction of $\vec{\nabla} f$.

Also, $D_{\vec{u}}f$ is minimum when $\cos\theta = -1$, i.e. $\theta = \pi$.
i.e. $\vec{u}$ is in opposite direction of $\vec{\nabla} f$.

Conclusion: f increases most rapidly in the direction of $\vec{\nabla} f$, and it decreases most rapidly in the direction of $-\vec{\nabla} f$.
Also, the direction in which there is no change is perpendicular to $\vec{\nabla} f$.

Example: $f(x,y) = \frac{x^2+y^2}{2}$
Find the direction in which f increases most rapidly at $(1,1)$.

$\vec{\nabla} f = x\hat{i} + y\hat{j}$; $\vec{\nabla} f(1,1) = \hat{i} + \hat{j}$.

$\therefore$ The direction of max. increase is $\hat{u} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$.
& the max. increase $|\vec{\nabla} f| = \sqrt{2}$

Chain rule:

Let $z = f(u, v)$, where
$u = u(x, y)$ & $v = v(x, y)$.

Then
$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}}$$
$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}}$$

Example: Let $z = \ln(u^2+v^2)$, where
$u = e^{x+y^2}$ & $v = x^2+y$.

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

Soln: $z = f(u,v) = \ln(u^2+v^2)$

$\frac{\partial f}{\partial u} = \frac{2u}{u^2+v^2}$; $\frac{\partial f}{\partial v} = \frac{2v}{u^2+v^2}$

$\frac{\partial u}{\partial x} = e^{x+y^2}\cdot 1$; $\frac{\partial u}{\partial y} = e^{x+y^2}\cdot 2y$

$\frac{\partial v}{\partial x} = 2x$; $\frac{\partial v}{\partial y} = 1$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{2u}{u^2+v^2} \cdot e^{x+y^2} + \frac{2v}{u^2+v^2} \cdot 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{2u}{u^2+v^2} \cdot e^{x+y^2} \cdot 2y + \frac{2v}{u^2+v^2} \cdot 1 .$$

<u>Derivative of implicitly defined function</u>

Let $y = y(x)$ be defined by

$$F(x,y) = 0 .$$

Then $\dfrac{dy}{dx} = \dfrac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

<u>Proof</u>: Differentiating $F(x, y(x)) = 0$ w.r.t. x, we get (using chain rule),

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Example : $F(x,y) = e^y - e^x + xy = 0$

$$\frac{\partial F}{\partial x} = -e^x + y$$

$$\frac{\partial F}{\partial y} = e^y + x$$

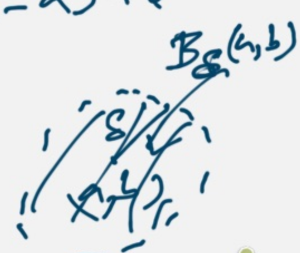
$$\therefore \quad \frac{dy}{dx} = \frac{-(-e^x + y)}{e^y + x} = \frac{e^x - y}{e^y + x}$$

Local maxima and minima of function of two variables :

Defn: We say $f(x,y)$ has a local minimum at (a,b) if $\exists\ \delta > 0$

s.t. $f(a,b) \le f(x,y)$

for all (x,y) s.t. $\sqrt{(x-a)^2 + (y-b)^2} < \delta$



Similarly, local max. at (a,b)

if $f(a,b) \ge f(x,y)$

$\forall (x,y) \in B_\delta(a,b)$