

Lagrange multiplier method to solve constraint optimization problems

Problem: Maximize/minimize
 $f(x_1, x_2, \dots, x_n)$
 subject to m constraints
 $g_1(x_1, x_2, \dots, x_n) = 0$
 $g_2(x_1, x_2, \dots, x_n) = 0$
 \vdots
 $g_m(x_1, x_2, \dots, x_n) = 0$,
 where $m < n$.

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Maximize/minimize $f(x, y, z)$

subject to $g(x, y, z) = 0$

We write the Lagrange multiplier equation:

$$\nabla f = \lambda \nabla g, \text{ where } \lambda \in \mathbb{R}.$$

3 components

So, we get 3 equations and
 4 unknowns x, y, z, λ .

There is one more eqn. $g(x, y, z) = 0$

We solve these 4 eqns in 4 unknowns
 to get the points where max/min can
 be attained.

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Example: Find the maximum and minimum values of the function
 $f(x, y) = x^3 + 4y^2$
 on the circle $x^2 + y^2 = 1$

Soln: Max./min. $f(x, y) = x^3 + 4y^2$
 subject to $g(x, y) = x^2 + y^2 - 1 = 0$
 $\vec{\nabla} f = 3x^2 \hat{i} + 8y \hat{j}$
 $\vec{\nabla} g = 2x \hat{i} + 2y \hat{j}$
 $\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 3x^2 \hat{i} + 8y \hat{j} = \lambda (2x \hat{i} + 2y \hat{j})$

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$$\Rightarrow 3x^2 = 2\lambda x ; \quad 8y = 2\lambda y$$

$$8y = 2\lambda y \Leftrightarrow y = 0 \text{ or } \lambda = 4$$

If $\lambda = 4$, $3x^2 = 8x$
 $\Rightarrow x = 0$ or $x = \frac{8}{3}$
 not possible since $x^2 + y^2 = 1$.

$$\text{When } x = 0, y^2 = 1 \Rightarrow y = \pm 1$$

$$(0, 1) ; (0, -1)$$

$$\text{If } \lambda \neq 4, y = 0, x = \pm 1$$

$$(1, 0), (-1, 0)$$

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$$f(0, 1) = 4$$

$$f(0, -1) = 4$$

$$f(1, 0) = 1$$

$$f(-1, 0) = -1$$

\therefore Max. value of f is 4 attained at $(0, \pm 1)$

Min. value of f is -1 attained $(-1, 0)$.

Ex: Try solving it using substitution method $y^2 = 1 - x^2$.

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Example: Find the shortest from the origin to the surface $x^2 - z^2 = 1$ in \mathbb{R}^3 .

Soln. We need to minimize $\sqrt{x^2 + y^2 + z^2}$ s.t. $x^2 - z^2 = 1$.

$$\text{Let } f(x, y, z) = x^2 + y^2 + z^2$$

$$g(x, y, z) = x^2 - z^2 - 1 = 0$$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow (2x, 2y, 2z) = \lambda (2x, 0, -2z)$$

$$\Rightarrow 2x = 2\lambda x; \quad 2y = 0; \quad 2z = -2\lambda z$$

$$\Rightarrow y = 0$$

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$$2x = 2\lambda x \Rightarrow x=0 \text{ or } \lambda=1$$

$$2z = -2\lambda z \Rightarrow z=0 \text{ or } \lambda=-1$$

Case: $\lambda=1$, $z=0 \Rightarrow x^2 - 0^2 = 1$
 $\Rightarrow x = \pm 1$

$(1, 0, 0); (-1, 0, 0)$

Case: $\lambda \neq 1$, $x=0$, $x^2 - z^2 = 1$
 $\Rightarrow z^2 = -1$, not possible

Now, $f(1, 0, 0) = 1 = f(-1, 0, 0)$.

Shortest distance = 1.

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