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HTML Content

Tut 10 Prob (2) :

$$y' = 2y^2 \; ; \; y(0) = 1$$

Picard's iterations :

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$\vdots$$

Here, $f(x,y) = 2y^2 \; ; \; x_0 = 0 \; ; \; y_0 = 1$

$$y_1(x) = 1 + \int_0^x f(t, 1)\, dt$$

$$= 1 + \int_0^x 2\, dt = 1 + 2x$$

$$y_2(x) = 1 + \int_0^x f(t, y_1(t))\, dt$$

$$= 1 + \int_0^x 2(y_1(t))^2\, dt$$

$$y_2(x) = 1 + \int_0^x 2(1+2t)^2\,dt$$

$$= 1 + \left.\frac{(1+2t)^3}{3}\right|_0^x$$

$$= 1 + \frac{1}{3}\left[(1+2x)^3 - 1\right]$$

$$= \frac{1}{3}\left[(1+2x)^3 + 2\right]$$

$$y_3(x) = 1 + \int_0^x f(t, y_2(t))\,dt$$

$$= 1 + \int_0^x 2(y_2(t))^2\,dt$$

$$= 1 + 2\int_0^x \frac{1}{9}\left[(1+2t)^3 + 2\right]^2 dt$$

i

$$y_2(x) = \frac{1}{3}\left[1 + 6x + 12x^2 + 8x^3 + 2\right]$$

$$= 1 + 2x + 4x^2 + \frac{8}{3}x^3 .$$

Solving directly:

$$\frac{dy}{dx} = 2y^2 \quad ; \quad y(0)=1$$

$$\int \frac{dy}{y^2} = \int 2\,dx$$

$$\Rightarrow \quad \frac{-1}{y} = 2x + c$$

$$y(0)=1 \Rightarrow \quad -1 = c$$

$$\therefore \quad \frac{-1}{y} = 2x-1$$

$$\Rightarrow \quad \boxed{y = \frac{1}{1-2x}}$$

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

for $|2x| < 1$
i.e. $|x| < \frac{1}{2}$

Assume $y_n(x) = 1 + 2x + (2x)^2 + \cdots + (2x)^n$

This is true for $n = 1, 2$

$$y_{n+1}(x) = 1 + \int_0^x f(t, y_n(t))\,dt$$

$$= 1 + \int_0^x 2(y_n(t))^2\,dt$$

This was incorrect!

$$= 1 + 2\int_0^1 \left[(1 + 2t + \cdots + (2t)^n\right]^2 dt$$

$$= 1 + 2\int_0^1 \sum_{k=0}^{n} (2t)^{2k} + 2\sum \cdots$$

$\vdots$

$$= 1 + 2x + (2x)^2 + \cdots + (2x)^{n+1}$$

Here it is difficult to guess a formula for $y_n(x)$ in general.

So, just find the first few iterates.

Tut 10 Prob (4)

$$(x^2-4x)\frac{dy}{dx} = (2x-4)y \ ; \ y(x_0)=y_0$$

Find (x_0, y_0) s.t. the IVP has

(a) no soln.

(b) a unique soln.

(c) more than one soln.

Solution: Putting $y(x_0)=y_0$ in the given ODE gives

$$(x_0^2-4x_0)\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = (2x_0-4)y_0$$

Note that the L.H.S. $=0$ if $x_0=0$ or $x_0=4$

For $x_0=0$, the R.H.S. $= -4y_0$

For $x_0=4$, the R.H.S. $= 4y_0$

So, for a soln. to exist $y_0=0$ if either $x_0=0$ or $x_0=4$.

Hence if $x_0 = 0$ & $y_0 \neq 0$

or if $x_0 = 4$ & $y_0 \neq 0$

then the IVP <u>has no soln.</u>

Remaining cases are

(i) $(x_0, y_0) = (0, 0)$

(ii) $(x_0, y_0) = (4, 0)$

(iii) $x_0 \neq 0$ and $x_0 \neq 4$.

Let's consider <u>case (iii)</u>:

$x_0 \neq 0$ & $x_0 \neq 4$.

Then the ODE can be written as

$$\frac{dy}{dx} = \underbrace{\frac{(2x-4)\,y}{x(x-4)}}_{f(x,y)}, \quad x \neq 0, x \neq 4$$

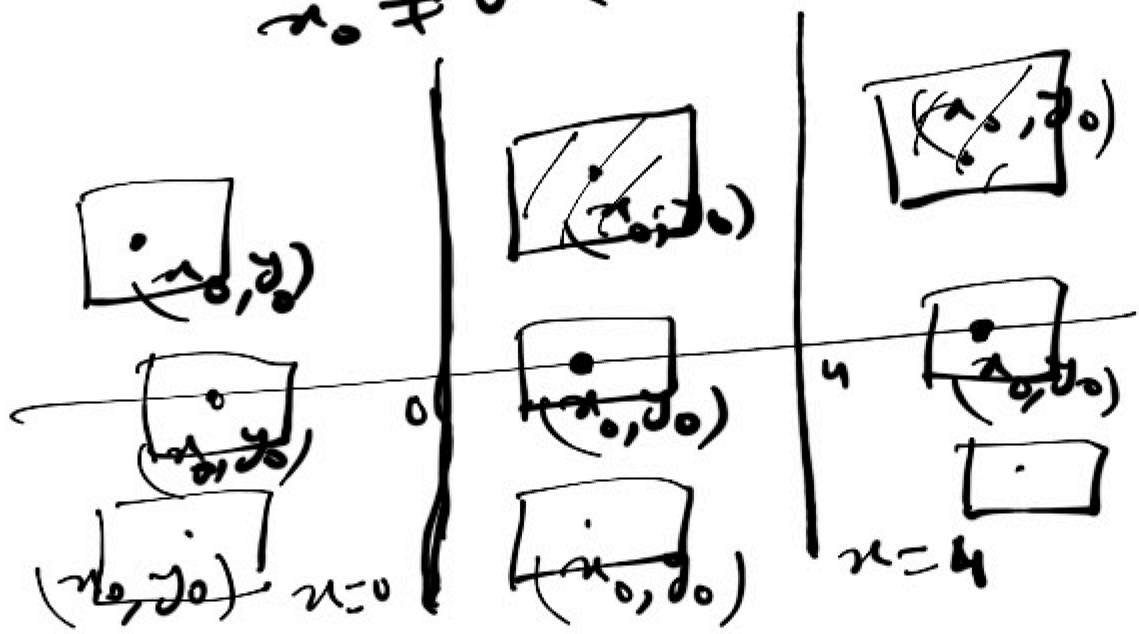
$$f(x,y) = \frac{(2x-4)y}{x(x-4)}$$

is continuous everywhere in $\mathbb{R}^2$ except the lines $x=0$ & $x=4$

Also, $\dfrac{\partial f}{\partial y} = \dfrac{2x-4}{x(x-4)}$

is cont. everywhere except on the line $x=0$ & $x=4$.

The initial pt. is (x_0, y_0) with $x_0 \neq 0$ & $x_0 \neq 4$



Since (x_0, y_0) does not lie on the lines $x = 0$ or $x = 4$, we can always find a rectangle R containing (x_0, y_0) such that $f(x,y)$ is continuous and $\frac{\partial f}{\partial y}$ is continuous on the rectangle R.

$\therefore$ By the existence-uniqueness thm; the IVP has a unique soln. if $\boxed{x_0 \neq 0, x_0 \neq 4}$.

If $(x_0, y_0) = (0,0)$ or $(x_0, y_0) = (4, 0)$ then the existence thm. fails.

We need to directly see if the IVP has more than one solns. or not.

For $(x_0, y_0) = (0,0)$:

$$x(x-4)\frac{dy}{dx} = (2x-4)y \; ; \; y(0)=0.$$

Clearly, $y \equiv 0$ is a soln.

To find other soln.

$$\int \frac{dy}{y} = \int \frac{2x-4}{x^2-4x}\, dx$$

$$\Rightarrow \ln|y| = \ln|x^2-4x| + c$$

$$\Rightarrow \boxed{y = k\,(x^2-4x)}$$

This satisfies $y(0)=0$ for any $k \in \mathbb{R}$

$\boxed{y = k(x^2-4x)}$ is a soln. to the IVP for every $k \in \mathbb{R}$

Same for $(x_0, y_0) = (4, 0)$

So, if $(x_0, y_0) = (0, 0)$ or

$(x_0, y_0) = (4, 0)$

~~the~~ the IVP has infinitely many solns.