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Contents

Basis and Dimension of a Vector Space

1 Linear Span of a Subset

Definition 1. ***(Linear Combination)*** *A vector β in V is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in V provided there exist scalars $c_1, c_2, \ldots, c_n$ in $\mathbb{F}$ such that*

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n.$$

Definition 2. ***(Linear Span of a Subset)*** *Let S be a nonempty subset of a vector space V. The linear span of S denoted by span(S), consists of all possible linear combinations of vectors from S i.e. it consists of all vectors of the form*

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n,$$

where $c_i \in \mathbb{F}$, $\alpha_i \in S$, $n \in \mathbb{N}$.

Remark 1. *S may be an infinite subset of V. The linear span of S consists of all finite linear combinations of vectors from S.*

Remark 2. *For convenience, we define span$(\phi) = \{0\}$.*

Theorem 1. *Let S be a nonempty subset of a vector space V. Then*

1. *span(S) is a subspace of V,*
2. *$S \subseteq$ span(S),*
3. *span(S) is equal to the subspace spanned by S, that is, the intersection of all subspaces containing S.*

2 Linear Dependence and Independence

Definition 3. *Let S be any subset of a vector space V. We say that S is* ***linearly dependent*** *if the zero vector can be written as a nontrivial linear combination of some vectors from S, i.e. $\exists\ v_1, v_2, \ldots, v_n \in S,\ a_1, a_2, \ldots, a_n \in \mathbb{F}$ with at least one of the $a_i^{\prime}s$ nonzero such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. A set which is not linearly dependent is called* ***linearly independent****.*

Observations:

1. Any set which contains a linearly dependent set is linearly dependent.
2. Any subset of a linearly independent set is linearly independent.
3. Any set which contains the zero vector is linearly dependent.
4. A set of vectors is linearly independent if and only if each finite subset of S is linearly independent, i.e., if and only if for any distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ of S, $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$ implies each $c_i = 0$.

Example 1. In $\mathbb{R}^3$, the set of vectors $\{(3, 0, -3),\ (-1, 1, 2),\ (4, 2, -2),\ (2, 1, 1)\}$ is linearly dependent, whereas the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent.

Solution Since

$$2(3, 0, -3) + 2(-1, 1, 2) - (4, 2, -2) + 0.(2, 1, 1) = (0, 0, 0),$$

So, the above set is linearly dependent.
Now, suppose

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

$$\implies c_1 = c_2 = c_3 = 0,$$

therefore, this set is linearly independent. ■

Definition 4. *A subset S of a vector space V generates or spans V if $span(S) = V$, i.e. any vector in V can be written as a linear combination of the vectors of S.*

Example 2. Show that the vectors $(1, 1, 0),\ (1, 0, 1),\ (0, 1, 1)$ span $\mathbb{R}^3$.

Solution Let (a_1, a_2, a_3) be any arbitrary vector in $\mathbb{R}^3$. We need to show that there exist scalars $r, s, t \in \mathbb{R}$ such that

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (a_1, a_2, a_3)$$

comparing both sides, we get the following three equations in three unknowns:

$$\begin{aligned} r + s &= a_1 \\ r + t &= a_2 \\ s + t &= a_3. \end{aligned}$$

By solving the above system, we get

$$\begin{aligned} r &= \frac{1}{2}(a_1 + a_2 - a_3), \\ s &= \frac{1}{2}(a_1 - a_2 + a_3), \\ t &= \frac{1}{2}(-a_1 + a_2 + a_3). \end{aligned}$$

Therefore, the vectors $(1,1,0),\ (1,0,1),\ (0,1,1)$ span $\mathbb{R}^3$. ■

Theorem 2. *Let S be a linearly independent subset of a vector space V, and let x be an element of V that is not in S. Then $S \cup \{x\}$ is linearly dependent if and only if $x \in span(S)$.*

Proof. If $S \cup \{x\}$ is linearly dependent, then there are vectors $x_1, \ldots, x_n$ in $S \cup \{x\}$ and scalars $a_1, \ldots, a_n$, not all zero, such that $a_1x_1 + \cdots + a_nx_n = 0$. Because S is linearly independent, one of the x_i, say x_1, equals x and hence $a_1 \neq 0$. Thus $a_1x + a_2x_2 + \cdots + a_nx_n = 0$, and so

$$x = a_1^{-1}\left(-a_2x_2 - \cdots - a_nx_n\right)$$

Since x is a linear combination of $x_2, \ldots, x_n$, which are elements of S, $x \in \text{span}(S)$.

Conversely, suppose that $x \in \text{span}(S)$. Then there exist vectors $x_1, \ldots, x_n$ in S and scalars $a_1, \ldots, a_n$ such that $x = a_1x_1 + \cdots + a_nx_n$. So $0 = a_1x_1 + \cdots + a_nx_n + (-1)x$, and since $x \neq x_i$ for $i = 1, \ldots, n$, $\{x_1, \ldots, x_n, x\}$ is linearly dependent. Thus $S \cup \{x\}$ is linearly dependent. □

3 Bases and Dimension

Definition 5. *A basis β for a vector space V is a linearly independent subset of V that spans V.*

Example 3. We have defined *span*$(\phi) = \{0\}$ and that ϕ is linearly independent, so ϕ is a basis for the vector space $\{0\}$.

Example 4. Let $\mathbb{F}$ be any field. In $\mathbb{F}^n$, let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1)$; we have seen that $\{e_1, e_2, \ldots, e_n\}$ is a linearly independent set and spans $\mathbb{F}^n$; therefore, it is a basis for $\mathbb{F}^n$.

Remark 3. $\{e_1, e_2, \ldots, e_n\}$ *is called the standard basis for* $\mathbb{F}^n$.

Example 5. In $M_{m\times n}(\mathbb{F})$, let M^{ij} denote the matrix whose only nonzero entry is a 1 in the i^{th} row and j^{th} column. Then

$$\{M^{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

is a basis for $M_{m\times n}(\mathbb{F})$.

Example 6. The set $\{1, x, x^2, \ldots\}$ is a basis for $P(\mathbb{F})$, the vector space of all polynomials over the field $\mathbb{F}$.

Remark 4. *Observe that the above example shows that a basis need not be finite.*

Definition 6. ***(Finite Dimensional Vector Space)*** *A vector space is said to be finite dimensional if it has a finite basis.*

Theorem 3. *Let V be a vector space and $\beta = \{x_1, \ldots, x_n\}$ be a subset of V. Then β is a basis for V if and only if each vector y in V can be uniquely expressed as a linear combination of vectors in β, i.e., can be expressed in the form*

$$y = a_1x_1 + \cdots + a_nx_n$$

for unique scalars $a_1, \ldots, a_n$.

Proof. Let β be a basis for V. If $y \in V$, then $y \in \text{span}(\beta)$ since $\text{span}(\beta) = V$. Thus y is a linear combination of the elements of β. Suppose that $y = a_1x_1 + \cdots + a_nx_n$ and $y = b_1x_1 + \cdots + b_nx_n$ are two such representations of y. Subtracting the second equality from the first gives

$$0 = (a_1 - b_1)\,x_1 + \cdots + (a_n - b_n)\,x_n$$

Since β is linearly independent, it follows that $a_1-b_1 = \cdots = a_n-b_n = 0$. Thus $a_1 = b_1, \ldots, a_n = b_n$, so that y is uniquely expressible as a linear combination of the elements of β.

Conversely, if each vector y in V can be uniquely expressed as a linear combination of vectors in β, that implies $\operatorname{span}\beta = V$ and because $0 = 0.x_1 + 0.x_2 + \cdots + 0.x_n$, therefore, β is a basis.

$\square$

Theorem 4. *Let V be a vector space which is spanned by a finite set of vectors $v_1, v_2, \cdots, v_n$. Then any independent set of vectors in V is finite and contains at most n elements.*

Corollary 1. *If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.*

This corollary allows us to define the dimension of a finite-dimensional vector space.

Definition 7. *(**Dimension**) The dimension of a finite-dimensional vector space is the number of elements in a basis for V. We denote the dimension of a finite dimensional vector space V by $\dim V$.*

Theorem 5. *Let V be a finite-dimensional vector space and let $n = \dim V$. Then*

1. *any subset of V which contains more than n vectors is linearly dependent;*
2. *no subset of V which contains fewer than n vectors can span V.*

Remark 5. *In other words, if V is a finite-dimensional vector space and $\dim V = n$, then any linearly independent subset of V containing exactly n vectors is a basis for V and any subset containing exactly n vectors which spans V is a basis for V.*

Example 7. From the above examples, it is easy to see that:

1. The vector space $\{0\}$ has dimension zero.
2. The vector space $\mathbb{F}^n$ has dimension n.
3. The vector space $M_{m\times n}(\mathbb{F})$ has dimension mn.

4. The vector space $P_n(\mathbb{F})$ has dimension $n+1$.

5. The vector space $P(\mathbb{F})$ is infinite-dimensional.

Example 8. The vectors $(1,-3,2),\ (4,1,0)$ and $(0,2,-1)$ form a basis for $\mathbb{R}^3$.

Solution If

$$a_1(1,-3,2)+a_2(4,1,0)+a_3(0,2,-1)=(0,0,0),$$

then a_1, a_2 and a_3 must satisfy the system of equations:

$$\begin{aligned} a_1+4a_2&=0\\ -3a_1+a_2+2a_3&=0\\ 2a_1-a_3&=0 \end{aligned}$$

but, the only solution of this system is $a_1=a_2=a_3=0$. Hence The vectors $(1,-3,2),\ (4,1,0)$ and $(0,2,-1)$ are linearly independent and by theorem 5, form a basis for $\mathbb{R}^3$. Note that we do not have to check that the given vectors span $\mathbb{R}^3$. ■

The dimension of a vector space also depends on its field of scalars.

Example 9. Over the field of complex numbers, the vector space of complex numbers has dimension 1, but over the field of real numbers, the vector space of complex numbers has dimension 2.

4 The Dimension of Subspaces

Theorem 6. *Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and* $\dim(W)\le\dim(V)$. *Moreover, if* $\dim(W)=\dim(V)$, *then* $W=V$.

Proof. Let dim(V) $=n$. If $W=\{0\}$, then W is finite-dimensional and $\dim(W)=0\leqslant n$. Otherwise, W contains a nonzero element x_1; so $\{x_1\}$ is a linearly independent set. Continuing in this way, choose elements $x_1, x_2, \ldots, x_k$ in W such that $\{x_1, x_2, \ldots, x_k\}$ is linearly independent. Since no linearly independent subset of V can contain more than n elements, this process must stop at a stage where $k\leqslant n$ and $\{x_1, x_2, \ldots, x_k\}$ is linearly independent but adjoining any other element of W produces a linearly dependent set. Theorem 2 now implies that $\{x_1, x_2, \ldots, x_k\}$ generates W, and hence it is a basis for W. Therefore, $\dim(W) = k \leqslant n$.

If $\dim(\mathrm{W}) = n$, then a basis for W is a linearly independent subset of V containing n elements, then theorem 5 implies that this basis for W is also a basis for V, so $W = V$. $\square$

Example 10. Let $W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_3 = 0,\ a_2 = a_4\}$. It is easy to show that W is a subspace of $\mathbb{R}^4$. Observe that any vector (a_1, a_2, a_3, a_4) in W can be written as

$$\begin{aligned}(a_1, a_2, a_3, a_4) &= (a_1, a_2, -a_1, a_2)\\ &= (a_1, 0, -a_1, 0) + (0, a_2, 0, a_2)\\ &= a_1(1, 0, -1, 0) + a_2(0, 1, 0, 1)\end{aligned}$$

that means the vectors $\{(1, 0, -1, 0),\ (0, 1, 0, 1)\}$ span W and since the vectors are linearly independent, therefore, the set $\{(1, 0, -1, 0),\ (0, 1, 0, 1)\}$ forms a basis for W. Thus $\dim(W) = 2$.

Example 11. Find a basis for the subspace of all diagonal $n \times n$ matrices in $M_{n\times n}(\mathbb{F})$.

Solution Let $D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$ be any diagonal matrix.

Let $\beta = \{M^{11}, M^{22}, \ldots, M^{nn}\}$, where M^{ii} is the matrix in which the only nonzero entry is the ii^{th} entry and is equal to 1. Then D can be written as

$$D = d_{11}M^{11} + d_{22}M^{22} + \cdots + d_{nn}M^{nn},$$

also, the set β is a linearly independent set, therefore, β forms a basis for the subspace of all diagonal $n \times n$ matrices. Thus dimension of the subspace of all diagonal $n \times n$ matrices is equal to n. $\blacksquare$

5 Exercise

1. Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?

2. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

3. Are the vectors $(1, 1, 2, 4)$, $(2, -1, -5, 2)$, $(1, -1, -4, 0)$, $(2, 1, 1, 6)$ linearly independent in $\mathbb{R}^4$?

4. Show that the vectors $(1, 0, -1)$, $(1, 2, 1)$, $(0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of these vectors.

5. Let V be the set of all 2×2 matrices A with complex entries which satisfy $a_{11} + a_{22} = 0$.

   (a) Show that V is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

   (b) Find a basis for this vector space.

   (c) Let W be the set of all matrices A in V such that $a_{21} = -\overline{a_{12}}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.