Variation of Parameters method

Consider 2nd order nonhomogeneous linear ODE of the form:

$$y'' + p(t)\,y' + q(t)\,y = r(t) \quad \text{— ①}$$

Suppose $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the corresponding homogeneous ODE:

$$y'' + p(t)\,y' + q(t)\,y = 0 \quad \text{— ②}$$

Let $y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$ be a particular solution to ① for some particular choice of fns. $u_1(t)$ and $u_2(t)$.

Then $y_p'(t) = \underbrace{(u_1' y_1 + u_2' y_2)} + (u_1 y_1' + u_2 y_2')$

Let's impose an extra condition

$$\boxed{u_1' y_1 + u_2' y_2 = 0} \quad \text{— ③}$$

Then $y_p' = u_1 y_1' + u_2 y_2'$

$\Rightarrow y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$

Substituting y_p, y_p', y_p'' in ①, we get

$$u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + p(t)\left[u_1 y_1' + u_2 y_2'\right] + q(t)(u_1 y_1 + u_2 y_2) = r(t)$$

$$\Rightarrow u_1 \big[\underbrace{y_1'' + p(t)\,y_1' + q(t)\,y_1}_{=0}\big] + u_2 \big[\underbrace{y_2'' + p(t)\,y_2' + q(t)\,y_2}_{=0}\big] + u_1' y_1' + u_2' y_2' = r(t)$$

$$\Rightarrow \boxed{u_1' y_1' + u_2' y_2' = r(t)} \quad \text{— ④}$$

③ & ④ can be written as

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

Since $\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = W(y_1, y_2)(t) \neq 0$,

we can find $u_1'(t)$ & $u_2'(t)$ uniquely as

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ r(t) & y_2'(t) \end{vmatrix}}{W(y_1, y_2)(t)} = \frac{-y_2(t)\,r(t)}{W(y_1, y_2)(t)}$$

and

$$u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & r(t) \end{vmatrix}}{W(y_1, y_2)(t)} = \frac{y_1(t)\,r(t)}{W(y_1, y_2)(t)}$$

Integrating we get $u_1(t)$ & $u_2(t)$ and hence

$$\boxed{y_p(t) = -y_1(t) \int \frac{y_2(t)\,r(t)}{W(y_1, y_2)(t)}\,dt + y_2(t) \int \frac{y_1(t)\,r(t)}{W(y_1, y_2)(t)}\,dt}$$

Example: Solve $y'' + y = \sec(t)$

The corresp. homog. ODE $y'' + y = 0$ has $y_1(t) = \cos(t)$ & $y_2(t) = \sin(t)$ are two lin. indep. solns

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and $r(t) = \sec t$

$$u_1'(t) = \frac{-y_2(t)\,r(t)}{W(y_1, y_2)(t)} = \frac{-\sin t \sec t}{} = -\tan t$$

$$\Rightarrow u_1(t) = -\int \tan t\,dt = \ln|\cos t|$$

$$u_2'(t) = \frac{y_1(t)\,r(t)}{W(y_1, y_2)(t)} = \frac{\cos t \sec t}{1} = 1$$

$$\Rightarrow u_2(t) = t$$

$$\therefore y_p(t) = u_1 y_1 + u_2 y_2 = \cos t \ln(\cos t) + t \sin t$$

$\therefore$ The general soln is

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t$$

Exercise: Solve
① $y'' + y = \tan t$
② $y'' - y = e^{2t}$
③ $y'' - y = e^{t}$

- Variation of parameters method can be applied if we know how to solve the corresponding homog. ODE, for example, if it is constant coefficient or Euler-Cauchy eqn.

For Euler-Cauchy:

$$a t^2 y'' + b t y' + c y = g(t)$$

If we use the variation of parameter method, $r(t) = \dfrac{g(t)}{a t^2}$ and NOT $g(t)$.

Method of undetermined coefficients

Example ① Solve: $y'' - y = e^{2t}$

$$y_h(t) = c_1 e^{t} + c_2 e^{-t}$$

Let $y_p(t) = A e^{2t}$ for some constant A.

Then $y_p' = 2A e^{2t}$

$y_p'' = 4A e^{2t}$

$\therefore 4A e^{2t} - A e^{2t} = e^{2t}$

$\Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$

$\therefore y_p = \frac{1}{3} e^{2t}$ is a particular soln.

$\therefore y = c_1 e^{t} + c_2 e^{-t} + \frac{1}{3} e^{2t}$.

② $y'' - y = e^{t}$

Here $y = A e^{t}$ cannot be a particular soln. because it is a soln. to the corresp. homog. eqn.

We try $y_p(t) = A t e^{t}$

Then $y_p' = A(t e^{t} + e^{t})$

$y_p'' = A(t e^{t} + 2e^{t})$

$\therefore A\left(\cancel{te^{t}} + 2e^{t} - \cancel{te^{t}}\right) = e^{t}$

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$\therefore y_p = \frac{1}{2} t e^{t}$ is a particular soln.

$$\therefore \boxed{y = c_1 e^{t} + c_2 e^{-t} + \frac{1}{2} t e^{t}}$$