Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Let $s_n = \sum_{k=1}^{n} a_k$.

Then $a_n = s_n - s_{n-1}$ for $n \geq 2$

Since $\sum_{n=1}^{\infty} a_n$ converges,

$\lim_{n\to\infty} s_n = S$ exists.

$\Rightarrow \lim_{n\to\infty} s_{n-1} = S$

$\therefore \lim_{n\to\infty} a_n = S - S = 0$

The converse of the previous theorem is not true.

e.g. $a_n = \frac{1}{n} \to 0$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Corollary (Divergence test)

If $\{a_n\}$ does not converge to 0

(i.e. either $\lim_{n\to\infty} a_n$ does not exist

or $\lim_{n\to\infty} a_n = \ell \neq 0$), then

$\sum_{n=1}^{\infty} a_n$ diverges.

e.g. ① Prove that $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Sol: $\lim_{n\to\infty} (-1)^n$ does not exist, by the divergence test, $\sum_{n=1}^{\infty} (-1)^n$ diverges.

② $\sum_{n=1}^{\infty} \sin(n)$ diverges since $\sin(n) \not\to 0$.

③ $\sum_{n=1}^{\infty} 1$ diverges since $a_n = 1 \to 1 \neq 0$

④ $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges because $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

⑤ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} a_n = 0$ in both cases.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Theorem (Comparison test)

(i) If $0 \le a_n \le b_n$ $\forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ must converge.

Example: ① $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Since $0 < \frac{1}{n^3} < \frac{1}{n^2}$ and we have seen that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

② $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

$0 < \frac{1}{2^n+n} < \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

$\therefore$ $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges.

(ii) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

e.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} > \frac{1}{n} > 0$ & $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Theorem (Limit comparison test)

Let $a_n > 0$ and $b_n > 0$ for all n.

Let $\lim_{n\to\infty} \frac{a_n}{b_n} = l$. Then

(i) If $0 < l < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

(ii) If $l = 0$, then $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

(iii) If $l = \infty$, then $\sum_{n=1}^{\infty} b_n$ diverge

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Example: ① $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Take $a_n = \frac{1}{2^n - 1}$; $b_n = \frac{1}{2^n}$

Then $\frac{a_n}{b_n} = \frac{2^n}{2^n - 1} \to 1$ as $n \to \infty$

Since $\sum b_n$ converges, by LCT, $\sum a_n$ converges.

② $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

$a_n = \frac{1}{n^2 - n}$; $b_n = \frac{1}{n^2}$

$\frac{a_n}{b_n} = \frac{n^2}{n^2 - n} = \frac{1}{1 - 1/n} \to 1$

By LCT, $\sum a_n$ converges

as $\sum b_n = \sum \frac{1}{n^2}$ converges.