Taylor series

For a function $f(x)$ which is infinitely differentiable at $x=a$, the Taylor series generated by f at $x=a$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

This is a power series $\sum_{k=0}^{\infty} a_k (x-a)^k$, where $a_k = \frac{f^{(k)}(a)}{k!}$.

If $a=0$, we call the series as the Maclaurin series.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Remark: If $f(x)$ is a polynomial, then the Taylor series is the same as $f(x)$.

Examples:

① $f(x) = e^x$

What is the Maclaurin series of $f(x)$?

$f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \quad \forall k$

$\Rightarrow f^{(k)}(0) = e^0 = 1 \quad \forall k.$

$\therefore$ Maclaurin series of f is

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

By the ratio test, the above series converges for all $x \in \mathbb{R}$.

② $f(x) = \frac{1}{1-x}, \quad x \neq 1$

$f(0) = 1 \quad ; \quad f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1,$

$f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2$

In general, $f^{(k)}(0) = k!$

$\therefore$ Maclaurin series is

$$\sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k.$$

This series converge for $|x| < 1$.

Also, $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ if $|x| < 1$.

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③ Similarly, the Maclaurin series for $\sin x$, $\cos x$, $\ln(1+x)$ are

$\sin x$: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ converges for all x

$\cos x$: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ converges for all x

$\ln(1+x)$: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

converges if $|x|<1$

An example for which Maclaurin series is not equal to $f(x)$ for every $x \neq 0$.

Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

It can be shown (using induction) that $f^{(k)}(0) = 0$ for all $k = 0, 1, 2, \cdots$

$\therefore$ The Maclaurin series of f is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^k}{k!} = 0$

But $f(x) \neq 0 \quad \forall\, x \neq 0$

Convergence of Taylor series:

When is $f(x)$ equal to its Taylor series?

By Taylor's theorem,

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

and $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some c between a and x.

If $R_n(x) \to 0$, then

$P_n(x) \to f(x)$ as $n \to \infty$.

$$\therefore \quad f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

if $R_n(x) \to 0$ as $n \to \infty$.

Example: $f(x) = e^x$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c}{(n+1)!} x^{n+1}$$

for some c between 0 & x.

If $x \leq 0$, then $c \leq 0$ $\therefore e^c \leq 1$

If $x > 0$, then $0 < c < x \Rightarrow e^c < e^x$

$$\therefore \quad |R_n(x)| \leq \begin{cases} \dfrac{|x|^{n+1}}{(n+1)!} & \text{if } x \leq 0 \\ \dfrac{e^x |x|^{n+1}}{(n+1)!} & \text{if } x > 0 \end{cases}$$

$\Rightarrow \quad R_n(x) \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}$.

$$\therefore \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x \in \mathbb{R}.$$

Binomial Series:

Recall: For $m \in \mathbb{N}$,

$$(1+x)^m = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \cdots + \binom{m}{m} x^m$$
$$= \sum_{k=0}^{m} \binom{m}{k} x^k,$$

where $\binom{m}{k} = \dfrac{m!}{k!(m-k)!} = \dfrac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$

For $-1 < x < 1$ and $m \in \mathbb{R}$.

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k ,$$

where $\binom{m}{0} = 1$, $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$,

.. $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$