Some more properties of elementary row operations :

① Every elementary row operation is invertible :

- Inverse of $(R_i \leftrightarrow R_j)$ is again $(R_i \leftrightarrow R_j)$
- Inverse of $(R_i \to \lambda R_i)$ $(\lambda \neq 0)$ is $(R_i \to \frac{1}{\lambda} R_i)$
- Inverse of $(R_i \to R_i + \mu R_j)$ is $(R_i \to R_i - \mu R_j)$

$$(\rho_1 \circ \rho_2 \circ \cdots \circ \rho_k)^{-1} = \rho_k^{-1} \circ \cdots \circ \rho_2^{-1} \circ \rho_1^{-1}$$

② $\det(A) \neq 0 \Rightarrow \det(\rho(A)) \neq 0$ for any elem. row oper. ρ.

Remark: A square matrix A is invertible if and only if the RRE form of A is the identity matrix I.

- If ρ is an elem. row operation, then $\rho(A) = \rho(I) A$.
  (applying ρ on A is equivalent to applying ρ on I and multiplying it by A)

e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

- $(f_1 \circ f_2 \circ \cdots \circ f_k)(A) = (f_1 \circ f_2 \circ \cdots \circ f_k)(I)\, A$.

How to find the inverse of an invertible matrix?

Suppose A is invertible.
Then its RRE form is I.
$\Rightarrow \exists$ elem. row opers. $f_1, f_2, \cdots, f_k$ s.t.
$(f_1 \circ f_2 \circ \cdots f_k)(A) = I$.

$$\Rightarrow \underbrace{(f_1 \circ f_2 \circ \cdots \circ f_k)(I)}_{B}\, A = I$$

$\Rightarrow BA = I \quad \Rightarrow B = A^{-1}$

Thus in order to calculate the inverse of A we convert A into its RRE form I and apply the same operations on I to get A^{-1}.

- Start with the matrix $(A \mid I)$
- Apply elem. row operations on this augmented matrix to convert it into $(I \mid B)$.
- $B = A^{-1}$.

Example: Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. Find A^{-1}

Soln: $\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$

$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right)$

$\xrightarrow{R_3 \to \frac{1}{2} R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$

$\xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$

Hence, $A^{-1} = \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Another explanation why the previous method gives A^{-1}:

If A is an invertible matrix,

$\exists\ B$ s.t. $AB = I$.

Let the columns of B be $X_1, X_2, \dots, X_n$

Then $I = AB = A\begin{pmatrix} X_1 & X_2 & \cdots & X_n \end{pmatrix}$

$$= \begin{pmatrix} AX_1 & AX_2 & \cdots & AX_n \end{pmatrix}$$

$\therefore\ AX_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad AX_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$