

Linear dependence

Let S be a subset of a vector space V .
Then S is called linearly dependent if there exist distinct $v_1, v_2, \dots, v_n \in S$ and $a_1, a_2, \dots, a_n \in \mathbb{F} \setminus \{0\}$ such that

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0.$$

Remarks:

- ① Any subset containing the zero vector is linearly dependent.
- ② Any superset of a linearly dependent set is linearly dependent.

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Defn: A subset S is called linearly independent if it is not linearly dependent.

For a finite set $S = \{v_1, v_2, \dots, v_n\}$,
 S is linearly independent if

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

$$\Rightarrow a_1 = 0 = a_2 = \dots = a_n.$$

Example: Check whether $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is linearly independent in \mathbb{R}^3 ?

Soln: Let $a_1(1, 1, 0) + a_2(1, 0, 1) + a_3(0, 1, 1) = (0, 0, 0).$

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Then $(a_1 + a_2, a_1 + a_3, a_2 + a_3) = (0, 0, 0)$

$$\Rightarrow \begin{cases} a_1 + a_2 = 0 \\ a_1 + a_3 = 0 \\ a_2 + a_3 = 0 \end{cases} \quad \left. \begin{array}{l} \text{This is a homog.} \\ \text{system of linear} \\ \text{eqns in unknowns} \\ a_1, a_2, a_3. \end{array} \right\}$$

If this system has zero as the only soln., then S is linearly indep. Otherwise it is linearly dependent.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + R_2, R_3 \rightarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{array}{l} R_2 \rightarrow -R_2 \\ R_3 \rightarrow \frac{1}{2}R_3 \end{array} \xrightarrow{\quad} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 + R_3 \end{array} \xrightarrow{\quad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore a_1 = 0, a_2 = 0, a_3 = 0$
 $\therefore S$ is linearly independent.

Remark: If S is a subset of \mathbb{R}^n consisting of m vectors, where $m > n$, then S must be linearly dependent.

- Any subset of a linearly independent set is linearly independent.

Basis of a vector space

Let V be a vector space over a field \mathbb{F} . A subset B of V is called a basis of V if

- B is linearly independent
- $\text{span}(B) = V$.

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Example: $V = \mathbb{R}^n$ over \mathbb{R} .

$$B = \{ (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1) \}$$

Is B linearly indep.?

$$a_1 (1, 0, \dots, 0) + a_2 (0, 1, 0, \dots, 0) + \dots + a_n (0, 0, \dots, 0, 1) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, a_2, \dots, a_n) = (0, 0, \dots, 0)$$

$$\Rightarrow a_i = 0 \quad \forall i$$

$\therefore B$ is linearly indep.

$$\text{Also, } (x_1, x_2, \dots, x_n) = x_1 (1, 0, \dots, 0) + x_2 (0, 1, \dots, 0) + \dots + x_n (0, 0, \dots, 0, 1) \in \text{span}(B)$$

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Remarks:

- ① A basis is a maximal linearly indep. subset. Also, any maximal linearly indep. subset is a basis.
- ② A basis is a minimal spanning subset. Also, any minimal spanning subset is a basis.

More examples of basis:

$$V = M_{2 \times 2}(\mathbb{R})$$

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

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$$V = \{ \text{all polynomials of degree at most } n \}$$

$$= \{ a_0 + a_1x + a_2x^2 + \dots + a_nx^n : a_i \in \mathbb{R} \}$$

$$= \text{span} \{ 1, x, x^2, \dots, x^n \}$$

Also, $\{ 1, x, x^2, \dots, x^n \}$ is linearly indep.

$$\therefore B = \{ 1, x, x^2, \dots, x^n \} \text{ is a basis of } V.$$

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• Let $V = \mathbb{F}[x]$ = the vector space of all polynomials.

$$B = \{1, x, x^2, \dots, x^n, \dots\}$$

• B is a basis of V .

Note that $1 + x + x^2 + \dots \notin \text{span}(B)$

• $V = \{ (x_1, x_2, \dots, x_n, \dots) : x_i \in \mathbb{R} \forall i \}$
 = the vector of all infinite sequences.

$$S = \{ (1, 0, \dots, 0, \dots), (0, 1, 0, 0, \dots), \dots \}$$

What is $\text{span}(S)$?

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$\text{span}(S)$ = all sequences which are eventually zero

$$= \{ (x_1, x_2, \dots, x_n, 0, 0, \dots) : n \in \mathbb{N}, x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \}$$

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