

Jacobian in polar coordinates

$$\begin{aligned}
 x &= r \cos \theta \quad ; \quad y = r \sin \theta \\
 J &= \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
 &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) \\
 &= r
 \end{aligned}$$

$\therefore \iint f(x, y) dxdy = \iint f(r \cos \theta, r \sin \theta) r dr d\theta$

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Change of variables for triple integrals

$$\begin{aligned}
 x &= x(u, v, w) \\
 y &= y(u, v, w) \\
 z &= z(u, v, w)
 \end{aligned}$$

$$\iiint f(x, y, z) dxdydz = \iiint f(x(u, v, w), y(u, v, w), z(u, v, w)) |J| du dv dw,$$

$$\text{where } J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

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Example:
 Evaluate $I = \iiint_D (x^2y + 3xyz) dV$,
 where $D = \{(x, y, z) : 1 \leq x \leq 2, 0 \leq xy \leq 2, 0 \leq z \leq 1\}$

Soln: Let $u = x, v = xy, w = z$
 i.e. $x = u; y = \frac{v}{u}; z = w$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{u} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{u}$$

The domain $\{(u, v, w) : 1 \leq u \leq 2, 0 \leq v \leq 2, 0 \leq w \leq 1\}$ is transformed into $\{u, v, w\} : 1 \leq u \leq 2, 0 \leq v \leq 2, 0 \leq w \leq 1\}$

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$$\begin{aligned} \text{Integrand} &= x^2y + 3xyz \\ &= uv + 3vw \\ \therefore I &= \iiint_{w=0}^{z=2} \int_{v=0}^{2/u} (uv + 3vw) \cdot \frac{1}{u} du dv dw \\ &= \int_{w=0}^{z=2} \int_{v=0}^{2/u} (v(u + 3w \ln u)) \Big|_{u=1}^2 dv dw \\ &= \int_{w=0}^{z=2} \int_{v=0}^{2/u} (v + 3 \ln^2 v w) dv dw \\ &= \int_{w=0}^{z=2} [v + 3(\ln^2 v) w]_0^2 dw \\ &= 2w + 3w \ln^2 w \Big|_0^2 \end{aligned}$$

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Cylindrical coordinates

$$(r, \theta, z)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

or, $r = \sqrt{x^2 + y^2}$

$$\theta = \tan^{-1}(\frac{y}{x})$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

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$$\iiint f(r, \theta, z) dr d\theta dz$$

$$= \iiint f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

Example: Find the volume of the cylinder $x^2 + (y-1)^2 = 1$ bounded by $z = x^2 + y^2$ and $z = 0$.

Soln:

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$$\text{Volume} = \iiint_R^{x+y^2} dz dA ,$$

where R is the projection of the solid onto the xy -plane.

We use the cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

$$x^2 + (y-1)^2 \leq 1$$

$$\text{i.e. } x^2 + y^2 - 2y \leq 0$$

$$\text{i.e. } r^2 - 2r\sin\theta \leq 0$$

$$\therefore r \text{ varies from } 0 \text{ to } 2\sin\theta$$

$$\therefore \theta \text{ varies from } 0 \text{ to } \pi$$

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$$\therefore V = \iiint_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} \int_{z=0}^{x+y^2} r dr d\theta dz$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} r^3 dr d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{4}(2\sin\theta)^4 d\theta = 4 \int_{\theta=0}^{\pi} \sin^4\theta d\theta$$

$$= 4 \int_{\theta=0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= \dots$$

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Spherical coordinates :

Let $P(x, y, z)$ be any point and O be $(x, y, 0)$

Let $OP = r$,
 $\theta =$ the angle of the ray OP with the
 +ve z -axis
 $\phi =$ the angle of the ray OA
 from the +ve x -axis.

Then $0 \leq r < \infty$; $0 \leq \theta \leq \pi$; $0 \leq \phi \leq 2\pi$

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We have

$$z = r \cos \theta$$

and $OA = r \sin \theta$

Now, $x = OA \cos \phi$, $y = OA \sin \phi$

\therefore

$x = r \sin \theta \cos \phi$	$y = r \sin \theta \sin \phi$
$z = r \cos \theta$	

$J = \begin{vmatrix} \sin \theta \cos \phi & r \sin \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \sin \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \cos \theta & 0 \end{vmatrix}$

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$$\begin{aligned}
 \therefore J &= \omega \theta \begin{vmatrix} r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \end{vmatrix} \\
 &\quad + r \sin \theta \begin{vmatrix} r \sin \theta \cos \varphi & -r \sin \theta \sin \varphi \\ r \sin \theta \sin \varphi & r \sin \theta \cos \varphi \end{vmatrix} \\
 &= \omega \theta r^2 \cos \theta \sin \theta (\cos^2 \varphi + \sin^2 \varphi) \\
 &\quad + r \sin \theta \cdot r \sin \theta (\cos^2 \varphi + \sin^2 \varphi) \\
 &= r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= r^2 \sin \theta .
 \end{aligned}$$

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$$\begin{aligned}
 &\iiint f(x, y, z) dx dy dz \\
 &= \iiint f(r, \theta, \varphi) \cdot r^2 \sin \theta dr d\theta d\varphi .
 \end{aligned}$$

Example: Calculate the volume of sphere
of radius R using spherical coords.

$$\begin{aligned}
 \text{Soln: } V &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^R r^2 \sin \theta dr d\theta d\varphi \\
 &= \frac{R^3}{3} \times 2\pi \int_0^{\pi} \sin \theta d\theta = \frac{4\pi}{3} R^3 .
 \end{aligned}$$

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