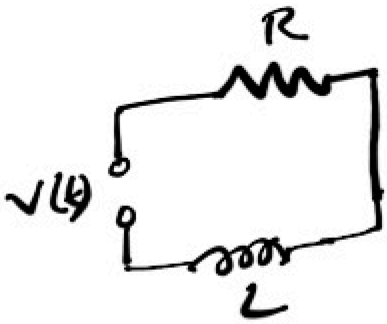
Written 14/04/25, 11:52

HTML Content

Application to Electric Circuits

RL - circuit

Kirchoff's voltage law



$$\Rightarrow \quad L\frac{dI}{dt} + RI = V(t)$$

i.e. $$\frac{dI}{dt} + \frac{R}{L} I = \frac{V(t)}{L}$$

This is a first order linear ODE and the solution is given by

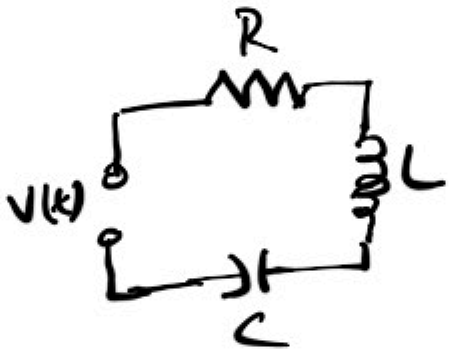
$$I(t) = e^{-\frac{R}{L}t}\left[\int e^{\frac{R}{L}t}\frac{V(t)}{L}\,dt + c\right]$$

RLC - circuit

Q = charge

I = current

Voltage drop across capacitor = $\frac{Q}{C}$



$$\therefore \quad L\frac{dI}{dt} + RI + \frac{Q}{C} = V(t)$$

Since $I = \frac{dQ}{dt}$, we get

$$\boxed{L Q'' + R Q' + \frac{1}{C} Q = V(t)}$$

This is a second order linear ODE.

Free oscillations:

$V(t) = 0$

$L Q'' + R Q' + \frac{1}{C} Q = 0$

$\to$ 2nd order homog. ODE with constant coeffs.

Char. eqn. $\quad L m^2 + R m + \frac{1}{C} = 0$

$$\Rightarrow \quad m = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

Underdamped case: $\quad R < \sqrt{\frac{4L}{C}}$

$$m = -\frac{R}{2L} \pm i\omega_0 \, , \quad \omega_0 = \frac{\sqrt{4L/C - R^2}}{2L}$$

$\left(= \frac{1}{\sqrt{LC}} \text{ if } R = 0 \right)$

$$\boxed{Q(t) = e^{-\frac{R}{2L}t} \left(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \right)}$$

Overdamped case: $R > \sqrt{\frac{4L}{C}}$

$m_1, m_2 < 0$

$$\boxed{Q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}}$$

Critically damped case: $R = \sqrt{\frac{4L}{C}}$

$$m_1 = m_2 = \frac{-R}{2L}$$

$$\therefore \boxed{Q(t) = e^{-\frac{R}{2L}t}(c_1 + c_2 t)}$$

- If $R \neq 0$, $Q(t) \to 0$ as $t \to \infty$ in each case.

Forced oscillations:

$$LQ'' + RQ' + \frac{1}{C}Q = V(t), \quad Q(0) = Q_0, \; Q'(0) = I_0$$

This is a nonhomog. linear ODE.

$$Q(t) = Q_h(t) + Q_p(t),$$

where $Q_h(t)$ is the general soln. of the homog.

$Q_p(t)$ is a particular soln.

For $V(t) = V_0 \cos\omega t$ find $Q_p(t)$ using method of undetermined coeffs.

This is analogous to the mechanical mass-spring system discussed before.

- $LQ'' + RQ' + \frac{1}{C}Q = V(t)$: RLC
- $my'' + cy' + ky = F(t)$: mass-spring

<u>Analogy</u>:

Mechanical system		Electrical system	
Mass	m	Inductance	L
Damping constant	c	Resistance	R
Spring constant	k	Reciprocal of capacitance	$1/C$
Driving force	$F(t)$	Voltage	$V(t)$

Higher order linear ODEs

Second order linear : $y'' + p(t)y' + q(t)y = r(t)$

nth order linear : $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = r(t)$

Homogeneous : if $r(t) = 0$

Non homogeneous : if $r(t) \neq 0$.

To solve nth order homogeneous linear ODE, we need to find n linearly independent solutions.

- If $W(y_1, y_2, \cdots, y_n)(t) \neq 0$, then $y_1(t), \cdots, y_n(t)$ are linearly indep.

$$W(y_1, y_2, \cdots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & & & \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Example: $y_1 = 1$, $y_2 = t$, $y_3 = t^2$

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & t \end{vmatrix}$$

$$= t$$

nth order homog. ODE with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$.

We assume $y = e^{mt}$ is a soln.

This gives the char. eqn:

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0$$

Need to find the roots of this polynomial of degree n.

If $m = m_1$ is a root of the the char. eqn., then $y = e^{m_1 t}$ is a soln. of the ODE.

Suppose the polynomial has n real and distinct roots $m_1, m_2, \dots, m_n$. Then the general soln. is given by

$$\boxed{y = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \dots + c_n e^{m_n t}}$$

Repeated roots case:

Suppose a root m is repeated two times. Then e^{mt} & te^{mt} are two lin. indep. solns. corresp. to this root.

Suppose m is repeated 3 times. Then e^{mt}, te^{mt}, $t^2 e^{mt}$ are 3 lin. indep solns corresp. to this root.

Complex conjugate roots:

If $m_1, m_2 = \alpha \pm i\beta$ are roots,

then $y_1 = e^{\alpha t} \cos\beta t$

& $y_2 = e^{\alpha t} \sin\beta t$

are two lin. indep. solns.

If $\alpha \pm i\beta$ is repeated twice,

then $y_1 = e^{\alpha t} \cos\beta t$, $y_2 = e^{\alpha t} \sin\beta t$,

$y_3 = t e^{\alpha t} \cos\beta t$, $y_4 = t e^{\alpha t} \sin\beta t$

are four lin. indep. solns.

Example: Solve $y^{(4)} + 2y'' + y = 0$

Char. eqn. $m^4 + 2m^2 + 1 = 0$

i.e. $(m^2+1)^2 = 0$

The roots are $\pm i, \pm i$

The general soln. is given by

$$\boxed{y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t}$$

---

nth order Euler-Cauchy eqn.

2nd order : $a x^2 y'' + b x y' + c y = 0$

nth order : $a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = 0$

Use $y = x^m$ to get the char eqn.

$$\boxed{a_0 m(m-1)\cdots(m-n+1) + a_1 m(m-1)\cdots(m-n+2) + \cdots + a_n = 0}$$

For 3rd order : $a x^3 y''' + b x^2 y'' + c x y' + d y = 0$

$y = x^m$

$$\boxed{a\, m(m-1)(m-2) + b\, m(m-1) + c m + d = 0}$$

$\to$ cubic eqn.

For distinct roots $m_1, m_2, \dots, m_n$, the soln. is

$$y = c_1 x^{m_1} + c_2 x^{m_2} + \dots + c_n x^{m_n}.$$

If m is repeated two times, x^m & $x^m \ln x$ are lin. indep. solns.

If m is repeated three times, x^m, $x^m \ln x$, $x^m (\ln x)^2$ are three lin. indep. solns.

and so on.

For nonhomog. linear ODE of nth order, the soln. is

$$y = y_h + y_p,$$

where y_h is the general soln. of the corresp. homog. ODE and y_p is a particular soln. of the nonhomog. ODE.

- To find y_p we can use either the method of undetermined coeffs. (whenever applicable) or the variation of parameters method.

- <u>Variation of parameters method.</u>

Suppose $y_1, y_2, \dots, y_n$ are n lin. indep. solns. to the homog.

Then $y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$, where $u_i(t)$ is given by

$$u_i'(t) = \frac{W_i(t)}{W(t)} r(t),$$

where $W(t) = \begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & & \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$

$$W_1(t) = \begin{vmatrix} 0 & y_2(t) & \cdots & y_n(t) \\ 0 & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \\ 0 & & & y_n^{(n-1)}(t) \\ 1 & y_2^{(n-1)}(t) & \cdots & \end{vmatrix}$$

$$W_2(t) = \begin{vmatrix} y_1(t) & 0 & y_3(t) & \cdots & y_n(t) \\ \vdots & \vdots & \vdots & & \vdots \\ & 0 & & & \\ y_1^{(n-1)}(t) & 1 & y_3^{(n-1)}(t) & & y_n^{(n-1)}(t) \end{vmatrix}$$

and so on.

Integrate to find $u_i(t)$ and hence $y_p(t)$.