

Thm: If  $W_1$  and  $W_2$  are subspaces of  $V$ ,  
then  $W_1 \cap W_2$  is also a subspace.

Pf: Let  $u, v \in W_1 \cap W_2$

Then  $u, v \in W_i$  for  $i = 1, 2$ .

Since  $W_i$  is a subspace,

$\alpha u + \beta v \in W_i$  for  $i = 1, 2$ .

$\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$

In fact, intersection of any arbitrary  
collection of subspaces is a subspace.

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Q: Is the union of two subspaces  
also a subspace?

Ans: No.

e.g. take  $V = \mathbb{R}^2$

$W_1 = x\text{-axis} = \{(x, 0) : x \in \mathbb{R}\}$

$W_2 = y\text{-axis} = \{(0, y) : y \in \mathbb{R}\}$

$W_1$  &  $W_2$  are both subspaces of  $\mathbb{R}^2$ .

But,  $W_1 \cup W_2$  is NOT a subspace of  $\mathbb{R}^2$   
because  $(1, 0) \in W_1 \cup W_2$ ,  $(0, 1) \in W_1 \cup W_2$   
but  $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$ .

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Thm:  $W_1 \cup W_2$  is a subspace if and only if either  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

Proof: If  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ , then clearly  $W_1 \cup W_2$  is a subspace.

Conversely, suppose  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1$ . We need to prove that  $W_1 \cup W_2$  is not a subspace.

Since  $W_1 \not\subseteq W_2$ ,  $\exists w_1 \in W_1, w_1 \notin W_2$ .

Since  $W_2 \not\subseteq W_1$ ,  $\exists w_2 \in W_2, w_2 \notin W_1$ .

Let  $v = w_1 + w_2$

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Claim:  $v \notin W_1 \cup W_2$ .

Suppose  $v = w_1 + w_2 \in W_1$

Then  $v - w_1 \in W_1 \Rightarrow w_2 \in W_1$ , which is a contradiction

Similarly, if  $v = w_1 + w_2 \in W_2$ , then  $v - w_2 \in W_2 \Rightarrow w_1 \in W_2$ , which is a contradiction.

$\therefore v \notin W_1 \cup W_2$

$\Rightarrow W_1 \cup W_2$  is not a subspace.

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Defn: (Subspace spanned by a subset).  
 Let  $S$  be any subset of a vector space  $V$ . The subspace spanned by  $S$  is the intersection of all subspaces containing  $S$ .  
 Note that the subspace spanned by  $S$  is the smallest subspace containing  $S$ .  
 Notation:  $\langle S \rangle$   
 • If  $S = \emptyset$ , the subspace spanned by  $S$  is  $\{0\}$ .

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Defn: (Linear span of a subset)  
 Let  $S$  be a nonempty subset of a vector space  $V$ . The linear span of  $S$  consists of all possible linear combinations of vectors from  $S$ , i.e., it consists of all vectors of the form  $a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ , where  $a_i \in \mathbb{F}$ ,  $v_i \in S$  for  $i=1, 2, \dots, n$ .  
Remark: Note that  $S$  may be an infinite subset of  $V$ .

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The linear span of  $S$  consists of all finite linear combinations of vectors from  $S$ .  
 Notation:  $\text{span}(S)$ .

Then: (i)  $\text{span}(S)$  is a subspace of  $V$   
 (ii)  $S \subseteq \text{span}(S)$ .  
 (iii)  $\text{span}(S) = \langle S \rangle$ , the intersection of all subspaces containing  $S$ .

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### Linear dependence & independence

Defn: Let  $S$  be any subset of a vector space  $V$ . We say that  $S$  is "linearly dependent" if the zero vector can be written as a nontrivial linear combination of some vectors from  $S$ , i.e.  
 $\exists v_1, v_2, \dots, v_n \in S, a_1, a_2, \dots, a_n \in \mathbb{F}$   
 with at least one of the  $a_i$ 's nonzero  
 such that  $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ .

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