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HTML Content

Solving integral equations using Laplace transforms:

Example ① Solve the following equation:

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$$

Note that $\displaystyle\int_0^t y(\tau)\sin(t-\tau)\,d\tau = y * \sin t$

$\therefore \quad y(t) - y*\sin t = t$

Taking Laplace transform, we get

$$Y(s) - Y(s)\cdot\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s)\cdot\frac{s^2}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\Rightarrow \quad y(t) = t + \frac{t^3}{3!} = t + \frac{t^3}{6}.$$

Example ②: Solve:

$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh(t)$$

$$\Rightarrow \quad y(t) - (1+t) * y(t) = 1 - \sinh(t)$$

Taking Laplace transform, we get

$$Y(s) - \left(\frac{1}{s} + \frac{1}{s^2}\right) Y(s) = \frac{1}{s} - \frac{1}{s^2-1}$$

$$\Rightarrow \quad Y(s)\left[\frac{s^2 - s - 1}{s^2}\right] = \frac{s^2 - s - 1}{s(s^2-1)}$$

$$\Rightarrow \quad Y(s) = \frac{s}{s^2-1} = \mathcal{L}(\cosh(t))(s)$$

$$\Rightarrow \quad \boxed{y(t) = \cosh(t)}$$

Example: Find the inverse Laplace transform of

$$F(s) = \ln\left(1 + \frac{\omega^2}{s^2}\right)$$

Solution:

$$F(s) = \ln(s^2 + \omega^2) - \ln(s^2)$$
$$= \ln(s^2 + \omega^2) - 2\ln s$$

$$\Rightarrow F'(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}$$

$$\Rightarrow \mathcal{L}^{-1}(F'(s)) = \mathcal{L}^{-1}\left(\frac{2s}{s^2 + \omega^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$
$$= 2\cos(\omega t) - 2$$

Since, $\mathcal{L}^{-1}(F'(s)) = -tf(t)$,

$$-tf(t) = 2(\cos(\omega t) - 1)$$

$$\Rightarrow \boxed{f(t) = \frac{2(1 - \cos(\omega t))}{t}}$$

Example: Find the inverse Laplace transform of $F(s) = \cot^{-1}\left(\frac{s}{\omega}\right)$.

Soln: $F'(s) = -\frac{1}{1+\frac{s^2}{\omega^2}} \cdot \frac{1}{\omega}$

$= -\frac{\omega}{s^2+\omega^2}$

$\Rightarrow \mathcal{L}^{-1}\left(-F'(s)\right) = \mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)$

$\Rightarrow \quad tf(t) = \sin(\omega t)$

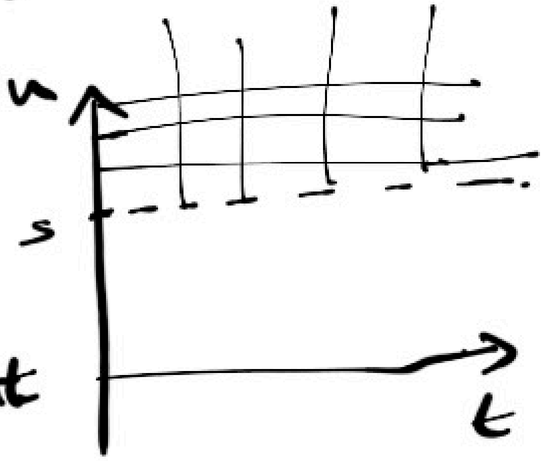
$\Rightarrow \quad \boxed{f(t) = \frac{\sin(\omega t)}{t}}$

Prop: $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)\,du$,

where $F(s) = \mathcal{L}(f(t))(s)$.

Proof: $F(u) = \int_0^\infty e^{-ut} f(t)\,dt$

$$\Rightarrow \int_s^\infty F(u)\,du = \int_s^\infty \int_0^\infty e^{-ut} f(t)\,dt\,du$$



Interchanging the order of integration,

$$\int_s^\infty F(u)\,du = \int_0^\infty f(t) \int_s^\infty e^{-ut}\,du\,dt$$

$$= \int_0^\infty f(t) \left\{ \left(\frac{e^{-ut}}{-t} \right) \Bigg|_{u=s}^{u=\infty} \right\} dt$$

$$= \int_0^\infty f(t) \frac{e^{-st}}{t}\,dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= \mathcal{L}\left(\frac{f(t)}{t} \right)$$

Example: Find $\mathcal{L}\left(\frac{\sin t}{t}\right)$

Solution: $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)\,du$

For $f(t) = \sin t$, $F(u) = \frac{1}{u^2+1}$.

$$\therefore \mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{u^2+1}\,du$$

$$= \tan^{-1}u \Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$\therefore \boxed{\mathcal{L}\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)}$$

System of First Order ODEs:

$$\frac{dx_1}{dt} = F_1(t, x_1, x_2, \dots, x_n)$$

$$\frac{dx_2}{dt} = F_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = F_n(t, x_1, x_2, \dots, x_n),$$

where $x_1, x_2, \dots, x_n$ are functions of t and $F_1, F_2, \dots, F_n$ are known functions.

Linear system:

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + g_1(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + g_n(t)$$

This can be written in the matrix form as:

$$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{g}(t),$$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

$$A(t) = \left(a_{ij}(t)\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

$$\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

Homogeneous: if $\vec{g}(t) = \vec{0}$

Non-homogeneous: if $\vec{g}(t) \neq \vec{0}$

IVP for system:
System of ODEs + initial condition

$$\vec{x}(t_0) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Constant coefficient system:

$A(t) = A$, a constant matrix.

i.e. $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{g}(t)$,

where $A \in M_{n\times n}(\mathbb{R})$.

Theorem: The solution space of the homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

is a vector space of dimension n.

Remark: From the above theorem, we see that in order to find the general soln. of $\frac{d\vec{x}}{dt} = A\vec{x}$, we need to find n lin. indep. solutions.

How to find a nonzero solution of $\frac{d\vec{x}}{dt} = A\vec{x}$?

For $n=1$, $\frac{dx}{dt} = ax$ has $x(t) = e^{at}$ as a solution.

In general, let's assume

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a soln. for some $\lambda \in \mathbb{R}$ & $\vec{v} \in \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

$$\Rightarrow \frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v}$$

$$\frac{d\vec{x}}{dt} = Ax \iff \lambda \cancel{e^{\lambda t}}\vec{v} = A\cancel{e^{\lambda t}}\vec{v}$$

$$\iff \boxed{A\vec{v} = \lambda\vec{v}}$$

Conclusion: If $\exists\, \lambda \in \mathbb{R}$ and $\vec{v}$ s.t. $A\vec{v} = \lambda \vec{v}$, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution of $\frac{d\vec{x}}{dt} = A\vec{x}$

Since, we are looking for nonzero solutions, we need $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.

i.e. $\vec{v}$ is an eigenvector of the matrix A (with eigenvalue λ).

$\therefore$ $\boxed{\text{If } \vec{v} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda, \text{ then } \vec{x}(t) = e^{\lambda t}\vec{v} \text{ is a solution of } \frac{d\vec{x}}{dt} = A\vec{x}}$