Examples of vector space

① $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.
($\mathbb{F}$ is any field).

② $\mathbb{C}^n$ is a vector space over $\mathbb{R}$.

Note that $\mathbb{R}^n$ is NOT a vector space over $\mathbb{C}$.

③ $M_{m\times n}(\mathbb{F})$ = the set of all $m\times n$ matrices with entries from $\mathbb{F}$.

is a vector space over $\mathbb{F}$.

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④ $\mathbb{F}[x]$ = set of all polynomials with coefficients from $\mathbb{F}$.

$a_0 + a_1 x + \cdots + a_n x^n$

$\mathbb{F}[x]$ is a vector space over $\mathbb{F}$.

. The zero polynomial is the zero vector.

⑤ $V = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ is a function$\}$

$(f+g)(x) = f(x) + g(x)$

$(c\cdot f)(x) = cf(x)$

V is a vector space.

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⑥ $V = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$

V is a vector space over $\mathbb{R}$.

⑦ $V = \{ (x_1, x_2, \dots, x_n, \dots) : x_i \in \mathbb{R} \}$

For $x = (x_1, x_2, \dots, x_n, \dots)$

$y = (y_1, y_2, \dots, y_n, \dots)$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, \dots)$

$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n, \dots)$

V is a vector space over $\mathbb{R}$.

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Subspace of a vector space

Let V be a vector space over $\mathbb{F}$. A nonempty subset W of V is called a subspace of V if W is itself a vector space over $\mathbb{F}$ under the operations restricted to W.

Remark: For W to be a subspace of V, it must be closed under vector addition and scalar multiplication, i.e.,

if $w_1, w_2 \in W$, then $w_1 + w_2 \in W$.

& if $w \in W$, then $\alpha . w \in W$ $\forall \alpha \in \mathbb{F}$.

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Theorem: A nonempty subset W of V is a subspace of V if and only if W is closed under addition and scalar multiplication

if and only if $c_1 w_1 + c_2 w_2 \in W$ whenever $w_1, w_2 \in W$, $c_1, c_2 \in \mathbb{F}$.

Example: ① $V = \mathbb{F}[x]$, the vector space of polynomials.

$W = \{ p(x) \in V : \text{degree}(p(x)) \leq n \text{ or } p(x) = 0 \}$

Then W is a subspace of V.

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② $V = M_{n\times n}(\mathbb{R})$

$W = \{ A \in V : \text{trace}(A) = 0 \}$

Is W a subspace of V?

. $0 \in W$ $(\because \text{trace}(0) = 0)$

. Let $A, B \in W$, $\alpha, \beta \in \mathbb{R}$.

Then $\text{trace}(\alpha A + \beta B)$

$= \text{trace}(\alpha A) + \text{trace}(\beta B)$

$= \alpha \,\text{trace}(A) + \beta \,\text{trace}(B)$

$= \alpha . 0 + \beta . 0 = 0$

$\Rightarrow \alpha A + \beta B \in W$. $\therefore$ W is a subspace.

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<u>Remark</u> :• For any vector space V,

$W = \{0\}$ is a subspace of V.

(called the zero subspace)

Also, $W = V$ is a subspace of V.

• For $V = \mathbb{R}$ over the field $\mathbb{R}$,

$\{0\}$ and $\mathbb{R}$ are the only subspaces.

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