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HTML Content

Convolution

Ques: Is $\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g)$?

Ans: No

e.g. (i) $f = 1$, $g = 1$

$\mathcal{L}(f) = \frac{1}{s}$; $\mathcal{L}(g) = \frac{1}{s}$

$\mathcal{L}(fg) = \mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(f)\mathcal{L}(g) = \frac{1}{s^2}$

So, $\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g)$

(ii) $f = e^t$, $g = 1$

$\mathcal{L}(f) = \mathcal{L}(e^t) = \frac{1}{s-1}$

$\mathcal{L}(g) = \mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(fg) = \mathcal{L}(e^t) = \frac{1}{s-1} \neq \frac{1}{s-1} \cdot \frac{1}{s}$

Ques: $\mathcal{L}^{-1}(F(s)G(s)) = ?$

if $\mathcal{L}^{-1}(F(s)) = f(t)$ & $\mathcal{L}^{-1}(G(s)) = g(t)$.

We have seen by above examples that $\mathcal{L}^{-1}\left(F(s)G(s)\right) \neq f(t)\,g(t)$.

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$G(s) = \int_0^\infty e^{-su} g(u)\, du$$ Put $u = t - \tau$ Then $du = dt$

$$= \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt$$

Multiplying, we get

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt$$

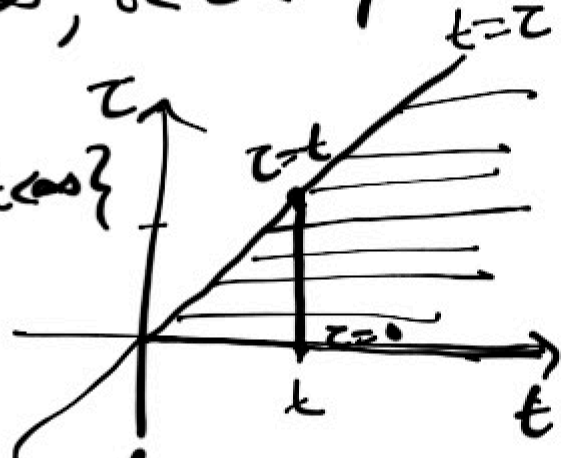
$$= \int_0^\infty \left(e^{-s\tau} f(\tau)\, e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) d\tau$$

$$= \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\, dt\; d\tau \qquad \text{——} (*)$$

This is a double integral over the region R given below:

$$R = \{ (t,\tau) : \tau \le t < \infty ,\ 0 < \tau < \infty \}$$

$$= \{ (t,\tau) : 0 \le \tau \le t ,\ 0 \le t < \infty \}$$



Changing the order of integrals in (*), we get

$$F(s)\,G(s) = \int_0^\infty e^{-st} \left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau \right) dt$$

$$= \mathcal{L}\big((f*g)(t) \big)(s),$$

where $(f*g)(t) := \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

$f*g$ is called the convolution of f and g.

$$\boxed{(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau}$$

Exercise: Use the above formula for $f*g$ and show that

$$\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g).$$

Hence, $$\boxed{\mathcal{L}^{-1}\big(F(s)G(s)\big) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s))}$$

Example: Find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right)$

We know that $\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t)$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \frac{1}{\omega}\sin(\omega t) * \frac{1}{\omega}\sin(\omega t)$$

$$= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin\omega(t-\tau)\, d\tau$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \frac{1}{2\omega^2}\int_0^t \left[\cos(2\omega\tau-\omega t) - \cos(\omega t)\right] d\tau$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \frac{1}{2\omega^3}\left[\sin(\omega t) - \omega t\cos(\omega t)\right]}$$

Properties of convolution:

① $f * g = g * f$ (commutativity)

Pf: $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

Put $t-\tau = u$ ie $\tau = t-u$

Then $d\tau = -du$

When $\tau = 0$, $u = t$

When $\tau = t$, $u = 0$

$$\therefore (f*g)(t) = \int_t^0 f(t-u)\, g(u)\, (-du)$$

$$= \int_0^t g(u)\, f(t-u)\, du = (g*f)(t)$$

② $f*(g+h) = f*g + f*h$

③ $(f*g)*h = f*(g*h)$ (Associativity)

④ $f*0 = 0$, where 0 denotes the zero function

⑤ $f*1 \neq f$

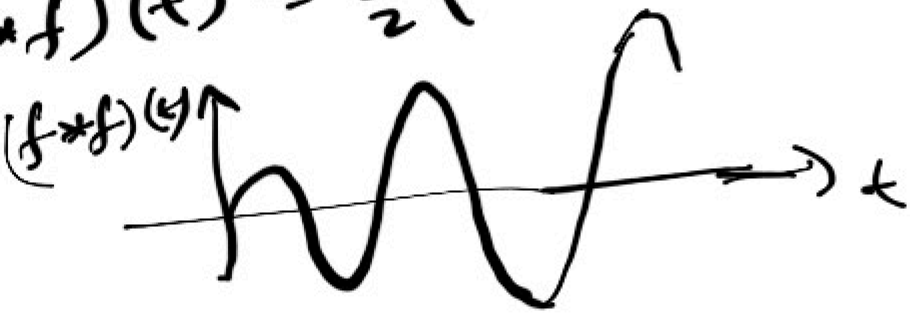
e.g. $f(t) = t$

$$(f*1)(t) = \int_0^t \tau \cdot 1 \, d\tau = \frac{t^2}{2} \neq f(t)$$

⑥ $(f*f)(t)$ need not be a nonnegative function.

e.g. $f(t) = \sin t$

$$(f*f)(t) = \frac{1}{2}\left[\sin t - t\cos t\right]$$



Using convolution to solve nonhomog.

ODEs:

Example: Solve: $y'' + 3y' + 2y = r(t)$,

$y(0) = 0$, $y'(0) = 0$,

$$r(t) = \begin{cases} 1, & \text{if } 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

Method 1: $r(t) = u(t-1) - u(t-2)$.

Take the Laplace transform & find $Y(s)$. Then take the inverse Laplace transform to get $y(t)$. (Exercise)

Method 2: (Using convolution):

Taking Laplace transform, we get

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(r)$$

$$\Rightarrow s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 3\left[sY(s) - \cancel{y(0)}^{0}\right]$$

$$+ 2Y(s) = R(s)$$

$$\Rightarrow (s^2 + 3s + 2)Y(s) = R(s)$$

$$\Rightarrow Y(s) = \frac{1}{(s+1)(s+2)} \cdot R(s)$$

Now, $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$

$$= \mathcal{L}\left(e^{-t} - e^{-2t}\right)$$

Taking the inverse Laplace transform,

$y(t) = (r * q)(t)$; where $q(t) = e^{-t} - e^{-2t}$

$$= \int_0^t r(\tau)\, q(t-\tau)\, d\tau$$

For $0 < t < 1$, $x(t) = 0$

$\therefore \quad y(t) = \int_0^t 0 \cdot g(t-\tau)\,d\tau = 0$

For $1 < t < 2$,

$$y(t) = \int_1^t 1 \cdot \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}.$$

For $t > 2$,

$$y(t) = \int_1^2 1 \cdot \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

$= \cdots\cdots$