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Contents

Vector Spaces and Subspaces

1 Introduction

Let $\mathbb{R}^n$ denote the Euclidean space. We observe that: For any two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, and scalars $c_1, c_2 \in \mathbb{R}$,

1. Addition: Their sum $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ is also a vector in $\mathbb{R}^n$.

2. Scalar Multiplication: For any scalar $c \in \mathbb{R}$, the product $c.\mathbf{u} = (cu_1, cu_2, \ldots, cu_n)$ is also a vector in $\mathbb{R}^n$.

3. Commutativity of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. Associativity of Addition: For any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

5. Existence of Additive Identity: There exists a zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$ such that for any vector $\mathbf{u}$ in $\mathbb{R}^n$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

6. Existence of Additive Inverse: For any vector $\mathbf{u}$ in $\mathbb{R}^n$, there exists a vector $-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$ in $\mathbb{R}^n$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

7. $1.\mathbf{u} = \mathbf{u}$ for every $\mathbf{u} \in \mathbb{R}^n$.

8. $(c_1c_2).\mathbf{u} = c_1.(\mathbf{c_2u})$.

9. $c.(\mathbf{u} + \mathbf{v}) = c.\mathbf{u} + c.\mathbf{v}$.

10. $(c_1 + c_2).\mathbf{u} = c_1.\mathbf{u} + c_2.\mathbf{u}$.

These observations motivate the definition of a vector space, which generalizes the properties of Euclidean space to more abstract settings.

2 vector spaces

2.1 Field

Definition 1. *A field is a non-empty set $\mathbb{F}$ of objects, called scalars, with two binary operations, addition $(+)$ and multiplication $(\cdot)$, that satisfy the following properties:*

Axioms for Addition

1. *Closure: For all $a, b \in \mathbb{F}$, $a + b \in \mathbb{F}$.*

2. *Commutativity: For all $a, b \in \mathbb{F}$, $a + b = b + a$.*

3. *Associativity: For all $a, b, c \in \mathbb{F}$, $(a + b) + c = a + (b + c)$.*

4. *Existence of Additive Identity: There exists an element $0 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, $a + 0 = a$.*

5. *Existence of Additive Inverse: For each $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$.*

Axioms for Multiplication

1. *Closure: For all $a, b \in \mathbb{F}$, $a \cdot b \in \mathbb{F}$.*

2. *Commutativity: For all $a, b \in \mathbb{F}$, $a \cdot b = b \cdot a$.*

3. *Associativity: For all $a, b, c \in \mathbb{F}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*

4. *Existence of Multiplicative Identity: There exists an element $1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, $a \cdot 1 = a$.*

5. *Existence of Multiplicative Inverse: For each non-zero $a \in \mathbb{F}$, there exists an element $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.*

Distributive Property

For all $a, b, c \in \mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example 1. The set of real numbers $\mathbb{R}$, the set of rational numbers $\mathbb{Q}$ forms a field under natural addition and multiplication.

Example 2. The set of complex numbers: $\mathbb{C} = \{a + b\iota \mid a, b \in \mathbb{R}, \iota^2 = -1\}$ with the operations addition and multiplication defined as

$$(a + b\iota) + (c + d\iota) = (a + c) + (b + d)\iota,$$

$$(a + b\iota).(c + d\iota) = (ac - bd) + (ad + bc)\iota$$

forms a field.

2.2 Vector Space

Definition 2. *A vector space consists of the following:*

1. *a field $\mathbb{F}$ of scalars;*
2. *a set V of objects, called vectors;*
3. *a rule or operation, called vector addition, which associates each pair of vectors u, v in V with a unique vector $u + v \in V$ in such a way that*
   (a) *addition is commutative, $u + v = v + u$;*
   (b) *addition is associative, $u + (v + w) = (u + v) + w$;*
   (c) *there is a unique vector $0 \in V$, such that $u + 0 = u$ for all $u \in V$;*
   (d) *for each vector $u \in V$ there is a unique vector $-u \in V$ such that $u + (-u) = 0$;*
4. *a rule or operation, called scalar multiplication, which associates each scalar $c \in \mathbb{F}$ and vector $u \in V$ with a unique vector $cu \in V$ in such a way that*
   (a) *$1u = u$ for every u in V;*
   (b) *$(c_1c_2)u = c_1(c_2u)$;*
   (c) *$c(u + v) = cu + cv$;*
   (d) *$(c_1 + c_2)u = c_1u + c_2u$.*

Remark 1. *The elements of $\mathbb{F}$ are called scalars, and that of V are called vectors. If $\mathbb{F} = \mathbb{R}$, the vector space is called a real vector space. If $\mathbb{F} = \mathbb{C}$, the vector space is called a complex vector space.*

Example 3. Let $\mathbb{F}$ be any field, and let V be the set of all n-tuples $\{(x_1, x_2, \ldots, x_n);\ x_i \in \mathbb{F}\}$. For any $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$ in V and $c \in \mathbb{F}$, define

$$u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n);$$

$$cu = (cx_1, cx_2, \ldots, cx_n),$$

it is easy to verify that the above defined operations satisfy all the conditions of vector addition and scalar multiplication therefore V is a vector space over $\mathbb{F}$ with respect to the above operations.

Example 4. The set of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is a vector space, which we denote by $M_{m\times n}(\mathbb{F})$, under the following operations of addition and scalar multiplication: For $A, B \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}.$$

Example 5. Let S be any nonempty set and $\mathbb{F}$ be any field. Let $\mathcal{F}(S, \mathbb{F})$ denote the set of all functions from S into $\mathbb{F}$. The set $\mathcal{F}(S, \mathbb{F})$ is a vector space under the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, \mathbb{F})$ and $c \in \mathbb{F}$ by

$$(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c(f(s))$$

for each $s \in S$.

Note that two functions $f, g \in \mathcal{F}(S, \mathbb{F})$ are defined to be equal if $f(s) = g(s)$ for each $s \in S$.

Definition 3. *A polynomial with coefficients from a field* $\mathbb{F}$ *is an expression of the form*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n *is a nonnegative integer and the coefficients* $a_i \in \mathbb{F}$.

- If $a_n = \cdots = a_0 = 0$, then $P(x) = 0$ is called the zero polynomial.
- The set of all polynomials over a field $\mathbb{F}$ is denoted by $P(\mathbb{F})$.

Definition 4. *The degree of a polynomial is defined to be the largest exponent of* x *whose coefficient is nonzero.*

- The degree of the zero polynomial is defined to be -1. Note that the polynomials of degree 0 are of the form $f(x) = c$ for some nonzero scalar c.

- A polynomial may be regarded as a function $f : \mathbb{F} \to \mathbb{F}$ which have a rule of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some nonnegative integer n and fixed scalars $a_i \in \mathbb{F}$. A function of this type is called a polynomial function on F.

Example 6. The set of all polynomials of degree atmost n with coefficients from a field $\mathbb{F}$ is a vector space, denoted by $P_n(\mathbb{F})$, under the following operations: For

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

in $P_n(\mathbb{F})$ and $c \in \mathbb{F}$,

$$(f + g)(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

and

$$(cf)(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0.$$

Exercise 1. *Let $S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For (a_1, a_2), $(b_1, b_2) \in S$ and $c \in \mathbb{R}$, define*

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \quad \textit{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Show that S is not a vector space under these operations.

Theorem 1. *In any vector space V the following statements are true:*

1. *If $x + z = y + z$, then $x = y$.*

2. *$0x = 0$ for each $x \in V$.*

3. *$(-a)x = -(ax) = a(-x)$ for each $a \in \mathbb{F}$ and each $x \in V$.*

4. *$a.0_v = 0_v$ for each $a \in \mathbb{F}$, where 0_v is the zero vector of V.*

3 Subspaces

Definition 5. *Let V be a vector space over the field $\mathbb{F}$. A subspace of V is a subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication defined on V.*

Example 7. If V is any vector space, then V is a subspace of V; the subset consisting of the zero vector alone is a subspace of V, called the zero subspace of V.

It is not necessary to verify all the vector space conditions in order to prove that a subset of a vector space is in fact a subspace.

Theorem 2. *A non-empty subset W of V is a subspace of V if and only if for each pair of vectors w_1, w_2 in W and each scalar c in $\mathbb{F}$ the vector $cw_1 + w_2$ is again in W.*

Proof. Suppose that W is a non-empty subset of V such that $cw_1 + w_2$ belongs to W for all vectors w_1, w_2 in W and all scalars c in $\mathbb{F}$. Since W is non-empty, there is a vector w in W, and hence $(-1)w + w = 0$ is in W. Then if w is any vector in W and c any scalar, the vector $cw = cw + 0$ is in W. In particular, $(-1)w = -w \in W$. Finally, if $w_1, w_2 \in W$, then $w_1 + w_2 = 1w_1 + w_2 \in W$. Thus W is a subspace of V.

Conversely, if W is a subspace of V, $w_1, w_2 \in W$, and c is a scalar, then $cw_1 + w_2 \in W$ by the definition of subspace.

□

Example 8. Let $\mathbb{F}$ be any field, and let W be the set of all n-tuples $\{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : \; x_1 = 0\}$. Then W is a subspace of $\mathbb{F}^n$.

Example 9. Let n be a non-negative integer, and let $P_n(\mathbb{F})$ consist of all polynomials in $P(\mathbb{F})$ having degree at most n. Then $P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.

Example 10. Let $C[0,1]$ be the collection of all continuous real-valued functions defined on $[0,1]$. Then $C[0,1]$ is a subspace of the vector space $\mathcal{F}[[0,1], \mathbb{R}]$ defined in example 5.

Example 11. Let $M_{n\times n}(\mathbb{F})$ be the vector space of all $n \times n$ matrices. Then the following subsets are subspaces of $M_{n\times n}(\mathbb{F})$:

1. the set of all $n \times n$ symmetric matrices.

2. the set of all $n \times n$ diagonal matrices.

3. the set of all $n \times n$ matrices having trace equal to 0.

Theorem 3. *Let V be a vector space over the field $\mathbb{F}$. The intersection of any collection of subspaces of V is a subspace of V.*

Proof. Let $\{W_\alpha\}$ be a collection of subspaces of V, and let $W = \cap_\alpha W_\alpha$ be their intersection. Let $u, v \in W$ and $c \in \mathbb{F}$, then $u, v \in W_\alpha$ for each α, and because each W_α is a subspace, the vector $cu + v$ is in every W_α. Thus $cu + v \in W$. So, by theorem 2, W is a subspace of V. □

From Theorem 3 it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, that is, a subspace which contains S and which is contained in every other subspace containing S.

Definition 6. ***(Subspace spanned by a set)*** *Let S be a set of vectors in a vector space V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{v_1, v_2, \ldots, v_n\}$, we shall simply call W the subspace spanned by the vectors $v_1, v_2, \ldots, v_n$.*

4 Exercise

1. Show that the set defined as

$$Q\left[\sqrt{2}\right] = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$$

   is a field.

2. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{C}\}$. For $(a_1, a_2),\ (b_1, b_2) \in V$ and $c \in \mathbb{C}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

   Is V a vector space under these operations?

3. Let V denote the set of all $m \times n$ matrices with entries from real numbers, and let $\mathbb{F}$ be the field of rational numbers. Is V a vector space over $\mathbb{F}$ under the usual definitions of matrix addition and scalar multiplication?

4. Let $W = \{(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n : a_1 + a_2 + \cdots + a_n = 0\}$. Is W a subspace of $\mathbb{F}^n$?

5. Is the set $W = \{f \in P(\mathbb{F}) : f = 0 \text{ or } f \text{ has degree } n \geq 1\}$ a subspace of $P(\mathbb{F})$?

6. Prove that the only subspaces of R are R and the zero subspace.