AMTL 100 (CALCULUS)
Midterm Exam

Date: 21/10/2024   Total Marks: 30   Time: 2 hours

1. Find the following limits

   (a) $\lim\limits_{x \to 1} \dfrac{2x^2 + 2 - (3x+1)\sqrt{x}}{x - 1}$ [2]

   (b) $\lim\limits_{x \to \infty} \left(\dfrac{x^3}{x^2 - 1} - \dfrac{x^3}{x^2 + 1} \right)$ [2]

2. Use the formal definition of limit to prove that $\lim\limits_{x \to 1} \dfrac{1}{x} = 1$.   [3]

3. Find the local and global extrema (if any) of the following function.   [5]

$$f(x) = \frac{x^2}{4 - x^2}, \quad -2 < x \leq 1$$

4. Consider the sequence given by

$$a_1 = 0, \; a_{n+1} = \sqrt{8 + 2a_n} \; \text{ for } n \geq 1$$

   (a) Prove that the sequence is bounded.   [2]

   (b) Prove that the sequence is increasing.   [2]

   (c) Conclude that the sequence converges and find its limit.   [3]

5. Determine whether the following series converge or diverge.

   (a) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n} \sin\left(\dfrac{1}{n} \right)$ [3]

   (b) $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n+3} + \sqrt{n+2}}$ [3]

   (c) $\sum\limits_{n=1}^{\infty} \cos\left(\dfrac{1}{n} \right)$ [2]

6. Use Taylor's theorem to estimate the error when $\ln(1 + x)$ is approximated by $x - \frac{x^2}{2} + \frac{x^3}{3}$ for $|x| \leq \frac{1}{10}$.   [3]