Proof of Sandwich theorem:

Given $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c.

& $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

To prove: $\lim_{x \to c} f(x) = L$

Let $\varepsilon > 0$ be given.
We need to find $\delta > 0$ s.t.
$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
i.e. $L - \varepsilon < f(x) < L + \varepsilon$

Since $\lim_{x \to c} g(x) = L$, there exists
$\delta_1 > 0$ s.t.
$0 < |x - c| < \delta_1 \Rightarrow L - \varepsilon < g(x) < L + \varepsilon$ — (i)

Since $\lim_{x \to c} h(x) = L$, $\exists\ \delta_2 > 0$ s.t.
$0 < |x - c| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon$ — (ii)

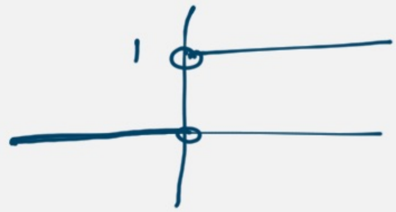
Take $\delta = \min\{\delta_1, \delta_2\}$
Then if $0 < |x - c| < \delta$, from (i) & (ii),
$L - \varepsilon < g(x) \le f(x) \le h(x) < L + \varepsilon$
$\Rightarrow L - \varepsilon < f(x) < L + \varepsilon$. Thus we are done.

Example:

Let $f(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x > 0 \end{cases}$

$\lim_{x \to 0^+} f(x) = 1$

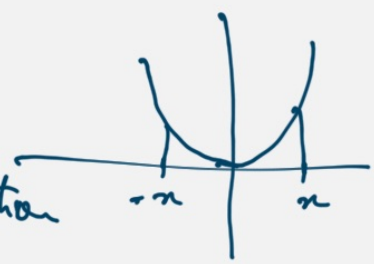
$\lim_{x \to 0^-} f(x) = 0$



$f(x) = x^2$

$f(-x) = f(x)$

$\therefore$ f is an even function

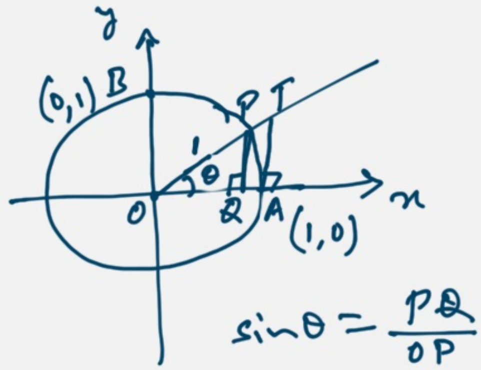


$f(x) = \frac{\sin x}{x}$

$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x}$

$= \frac{\sin x}{x} = f(x)$

Let $0 < \theta < \frac{\pi}{2}$

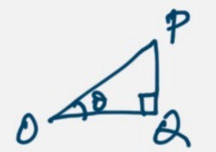


Area ΔOAP

$$= \frac{1}{2} OA \times PQ$$

$$= \frac{1}{2} \times 1 \times \sin\theta$$

$$\sin\theta = \frac{PQ}{OP}$$



Area of ΔOAT

$$= \frac{1}{2} \times OA \times AT$$

$$= \frac{1}{2} \times 1 \times \tan\theta$$

Area of sector OAP

$$= \frac{\theta}{2\pi} \times \pi (1)^2 = \frac{1}{2}\theta$$

From the figure,

area $\Delta OAP <$ area sector $OAP <$ area ΔOAT

$$\Rightarrow \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}$$

$$\Rightarrow \boxed{\cos\theta < \frac{\sin\theta}{\theta} < 1}$$ for any $0 < \theta < \frac{\pi}{2}$

Since $\lim_{\theta \to 0^+} \cos\theta = 1$, by Sandwich thm;

$$\boxed{\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1}$$