

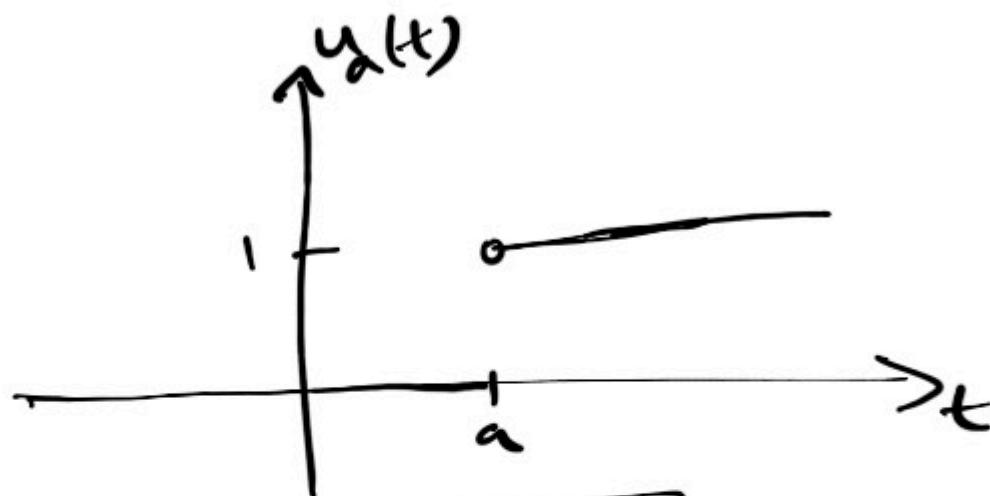
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HTML Content

Heaviside function

For $a \geq 0$, we define

$$u_a(t) = u(t-a) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$

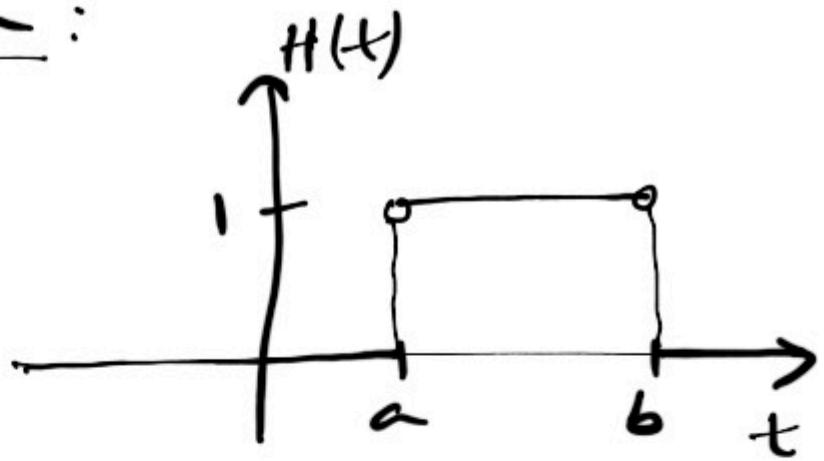


$$\mathcal{L}(u(t-a))(s) = \frac{e^{-as}}{s}$$

$$\mathcal{L}(f(t-a)u(t-a))(s) = e^{-as} \mathcal{L}(f)(s)$$

$$f(t-a)u(t-a) = \begin{cases} 0, & t \leq a \\ f(t-a), & t > a \end{cases}$$

Hat function:



For $a < b$,

$$H(t) = \begin{cases} 1, & \text{if } a < t < b \\ 0, & \text{otherwise} \end{cases}$$

$$H(t) = u(t-a) - u(t-b)$$

Example: Let $f(t) = \begin{cases} 2, & 0 < t < 1 \\ \frac{t}{\pi/2}, & 1 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$

Find $\mathcal{L}(f)(s)$.

Solution: First we express $f(t)$ in terms of Heaviside function.

$$f(t) = \begin{cases} 2, & 0 < t < 1 \\ t^2/2, & 1 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$$

$$= 2[1 - u(t-1)] + \frac{t^2}{2}[u(t-1) - u(t-\frac{\pi}{2})] + \cos t u(t-\frac{\pi}{2})$$

$$\therefore L(f)(s) = 2(L(1) - L(u(t-1))) + \frac{1}{2}L(t^2 u(t-\frac{\pi}{2})) + L(\cos t u(t-\frac{\pi}{2})) = 2\left[\frac{1}{s} - \frac{e^{-s}}{s}\right] + \dots$$

$$\text{Now, } t^2 u(t-1) = \{(t-1)+1\}^2 u(t-1)$$

$$= (t-1)^2 u(t-1) + 2(t-1) u(t-1)$$

$$+ u(t-1)$$

$$\therefore \mathcal{L}(t^2 u(t-1)) = \mathcal{L}\left((t-1)^2 u(t-1)\right)$$

$$+ 2\mathcal{L}\left((t-1) u(t-1)\right)$$

$$+ \mathcal{L}(u(t-1))$$

$$= e^{-s} \mathcal{L}(t^2) + 2 e^{-s} \mathcal{L}(t)$$

$$+ \frac{e^{-s}}{s}$$

$$= e^{-s} \cdot \frac{2}{s^3} + 2 e^{-s} \cdot \frac{1}{s^2} + \frac{e^{-s}}{s}$$

Similarly $t^2 u(t-\frac{\pi}{2}) = \left\{(t-\frac{\pi}{2})+\frac{\pi}{2}\right\}^2 u(t-\frac{\pi}{2})$

$$= (t-\frac{\pi}{2})^2 u(t-\frac{\pi}{2}) + \pi (t-\frac{\pi}{2}) u(t-\frac{\pi}{2})$$

$$+ \frac{\pi^2}{4} u(t-\frac{\pi}{2})$$

$$\therefore \mathcal{L}\left(t^2 u(t-\frac{\pi}{2})\right) = e^{-\frac{\pi}{2}s} \cdot \frac{2}{s^3} + \pi e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2}$$

$$+ \frac{\pi^2}{4} \frac{e^{-\frac{\pi}{2}s}}{s}$$

$$\begin{aligned}
 \text{Also, } \cos t u(t - \frac{\pi}{2}) &= \sin\left(\frac{\pi}{2} - t\right) u(t - \frac{\pi}{2}) \\
 &= -\sin(t - \frac{\pi}{2}) u(t - \frac{\pi}{2}) \\
 \therefore \mathcal{L}\left(\cos t u(t - \frac{\pi}{2})\right) &= -\mathcal{L}\left(\sin(t - \frac{\pi}{2}) u(t - \frac{\pi}{2})\right) \\
 &= -e^{-\frac{\pi}{2}s} \mathcal{L}(\sin t)(s) \\
 &= -e^{-\frac{\pi}{2}s} \cdot \frac{1}{s^2 + 1}
 \end{aligned}$$

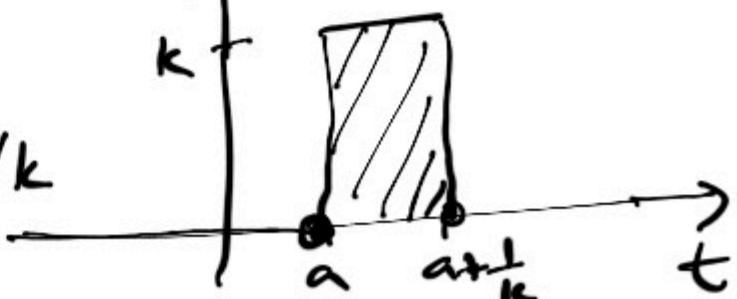
Dirac delta "function"

For $k \in \mathbb{N}$, let

$$f_k(t-a) = \begin{cases} k, & a \leq t \leq a + \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

Note that

$$\int_0^\infty f_k(t-a) dt = 1 \quad \forall k$$

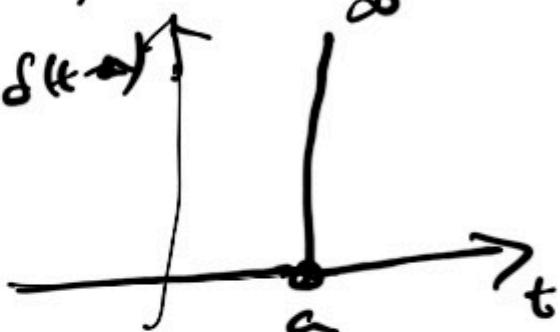


define the Dirac-delta function as

$$\delta(t-a) = \lim_{k \rightarrow \infty} f_k(t-a)$$

$$= \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}$$

$$\delta(t-a) \uparrow$$



Property:

$$\textcircled{1} \quad \int_0^\infty \delta(t-a) dt = 1$$

$$\textcircled{2} \quad \mathcal{L}\{\delta(t-a)\} = ?$$

$$\delta(t-a) = \lim_{k \rightarrow \infty} f_k(t-a)$$

$$f_k(t-a) = \begin{cases} k, & a \leq t \leq a + \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

$$= k [u(t-a) - u(t-(a+\frac{1}{k}))]$$

$$\begin{aligned}\therefore \mathcal{L}(f_k(t-a)) &= k \left[\mathcal{L}(u(t-a)) - \mathcal{L}(u(t-\frac{a}{k})) \right] \\ &= k \left[\frac{e^{-as}}{s} - \frac{e^{-(t+\frac{1}{k})s}}{s} \right] \\ &= e^{-as} \left[\frac{1 - e^{-\frac{t}{k}s}}{s/k} \right]\end{aligned}$$

$$\begin{aligned}\therefore \mathcal{L}(\delta(t-a)) &= \lim_{k \rightarrow \infty} \mathcal{L}(f_k(t-a)) \\ &= \lim_{k \rightarrow \infty} e^{-as} \left[\frac{1 - e^{-\frac{s}{k}a}}{\frac{s}{k}} \right]\end{aligned}$$

L'Hôpital's rule (0/0 form)

$$= e^{-as} \lim_{k \rightarrow \infty} \frac{1 - e^{-\frac{s}{k}a}}{\frac{s}{k}} + \frac{e^{-as} \cdot \cancel{\frac{s}{k}}}{\cancel{+ s/a}}$$

$$\therefore \boxed{\mathcal{L}(\delta(t-a)) = e^{-as}}$$

Solving an IVP involving
Dirac delta function.

Example: $y'' + 3y' + 2y = \delta(t-1)$,
 $y(0) = 0, y'(0) = 0$.

Taking the Laplace transform, we get

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(\delta(t-1))$$
$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + 3[s Y(s) - y(0)] + 2Y(s) = e^{-s}$$

$$\Rightarrow (s^2 + 3s + 2) Y(s) = e^{-s}$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{(s+1)(s+2)}$$

$$= e^{-s} \left[\frac{1}{s+1} - \frac{1}{s+2} \right]$$

$$\begin{aligned} \Rightarrow y(s) &= \frac{\bar{e}^s}{s+1} - \frac{\bar{e}^s}{s+2} \\ \Rightarrow y(t) &= \mathcal{L}^{-1}\left[\frac{\bar{e}^s}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{\bar{e}^s}{s+2}\right] \\ &= \mathcal{L}^{-1}\left[e^s \mathcal{L}(e^{-t})(s)\right] \\ &\quad - \mathcal{L}^{-1}\left[e^s \mathcal{L}(e^{-2t})(s)\right] \\ &= e^{-(t-1)} u(t-1) - e^{-2(t-1)} u(t-1) \end{aligned}$$

because $\mathcal{L}^{-1}(e^{-as} \mathcal{L}(f)) = f(t-a) u(t-a)$

$$\therefore y(t) = \begin{cases} 0, & t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t \geq 1 \end{cases}$$