Theorem: Assume that $M(x,y)$ and $N(x,y)$ have continuous partial derivatives. Then $M\,dx + N\,dy = 0$ is an exact ODE if and only if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Proof: $(\Rightarrow)$ Already proved in the last class.

$(\Leftarrow)$ Assume $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

We want to find $u(x,y)$ s.t.

$$\frac{\partial u}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y} = N(x,y).$$

Integrating $\dfrac{\partial u}{\partial x} = M(x,y)$ w.r.t. x, we get

$$u(x,y) = \int M(x,y)\,dx + h(y),$$

where $h(y)$ is any fn. of y.

Differentiating the above w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + h'(y)$$

$$= \int \frac{\partial M}{\partial y}\,dx + h'(y)$$

$$=$$

We want: $\dfrac{\partial u}{\partial y} = N(x,y)$

So, $\int \frac{\partial M}{\partial y} dx + h'(y) = N(x,y)$

$\Leftrightarrow \quad h'(y) = N(x,y) - \int \frac{\partial M}{\partial y} dx \quad (*)$

If we find $h(y)$ satisfying the above, then the eqn. is exact.

We can find $h(y)$ satisfying $(*)$ provided $\frac{\partial}{\partial x}(\text{R.H.S.}) = 0$

i.e. $\frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\int \frac{\partial M}{\partial y} dx\right) = 0$

i.e. $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$, which is true.

Example: Solve $2xy\, dx + (x^2+y^2)dy = 0$

Here, $M = 2xy$; $N = x^2+y^2$

$\therefore \frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the eqn. is exact.

We'll a $u(x,y)$ s.t.

$\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

$\frac{\partial u}{\partial x} = 2xy \Rightarrow u(x,y) = x^2y + h(y)$

$\frac{\partial u}{\partial y} = N = x^2+y^2 \Rightarrow x^2 + h'(y) = x^2+y^2$

$\Rightarrow h'(y) = y^2$

So, we can take $h(y) = \frac{y^3}{3}$.

$\therefore \quad u(x,y) = x^2 y + \frac{y^3}{3}$.

$\therefore$ The general soln. is $u(x,y) = c$

i.e. $\boxed{x^2 y + \frac{y^3}{3} = c}$

Q. If $M\,dx + N\,dy = 0$ is not exact, can we still solve it in some special cases?

Example: Consider $y\,dx - x\,dy = 0$

$M = y, \quad N = -x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$

$\therefore$ The eqn. is not exact.

Can we multiply the given eqn. by some fn. so that it becomes exact?

Multiplying by $\frac{1}{x^2}$, we get

$\frac{y\,dx - x\,dy}{x^2} = 0$ i.e. $d\left(\frac{y}{x}\right) = 0$

$\therefore \quad \frac{y}{x} = c$ is the general soln.

Integrating factors for non-exact eqns:

Suppose $M\,dx + N\,dy = 0$ is not exact

but we can some fn. $\mu(x,y)$ s.t.

$$\mu M\,dx + \mu N\,dy = 0 \quad \text{is exact.}$$

We must have $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$

$$\Leftrightarrow \frac{\partial \mu}{\partial y} M + \mu M_y = \frac{\partial \mu}{\partial x} N + \mu N_x$$

$$\Leftrightarrow \frac{\partial \mu}{\partial y} M - \frac{\partial \mu}{\partial x} N = \mu (N_x - M_y)$$

Suppose μ is a fn. of x only.

Then $\frac{\partial \mu}{\partial y} = 0$ & $\frac{\partial \mu}{\partial x} = \mu'(x)$.

Then $-\mu'(x) N = \mu(x)(N_x - M_y)$

$$\Leftrightarrow \mu'(x) = \mu(x)\left(\frac{M_y - N_x}{N}\right)$$

If $\frac{M_y - N_x}{N}$ is a fn. of x only,

then we can find $\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right)$

Similarly, if $\frac{M_y - N_x}{M}$ is a fn. of y only,

then $\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\,dy\right)$

is an integrating factor.

Summary:

- First check if $M_y - N_x = 0$ or not.
- If $M_y - N_x = 0$, then the eqn. is exact.
- If $M_y - N_x \neq 0$, check if either

(i) $\dfrac{M_y - N_x}{N}$ is a fn. of x, then

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\, dx\right)$$

is an integ. factor.

or (ii) $\dfrac{M_y - N_x}{M}$ is a fn. of y only, then

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\, dy\right)$$ is an integ. factor.

Example: $x \sin y\, dx + (x+1)\cos y\, dy = 0$

Try to find an integ. factor to make it exact and then solve it.