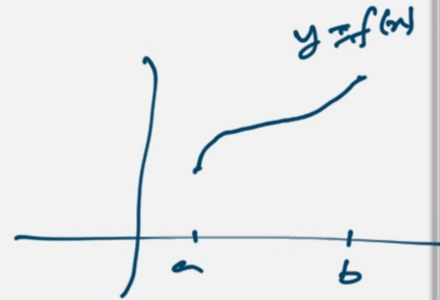
Arc length

For a curve given by $y = f(x)$ between $x = a$ and $x = b$, the arc length is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Parametric form:

Suppose a curve is defined in parametric form as $x = x(t)$ and $y = y(t)$ for $t_1 \le t \le t_2$. Then the arc length is given by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Consider the circle given $x^2 + y^2 = r^2$. This can be written in parametric form as $x = r\cos\theta$, $y = r\sin\theta$,

for $0 \le \theta \le 2\pi$

$\therefore$ Arc length (perimeter of the circle) is

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

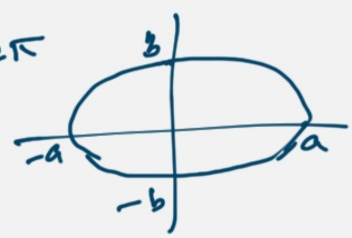
$$= \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta = 2\pi r.$$

Example: Find the perimeter of the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln: $x = a\cos t$; $0 \le t \le 2\pi$

$y = b\sin t$



$\frac{dx}{dt} = -a\sin t$

$\frac{dy}{dt} = b\cos t$

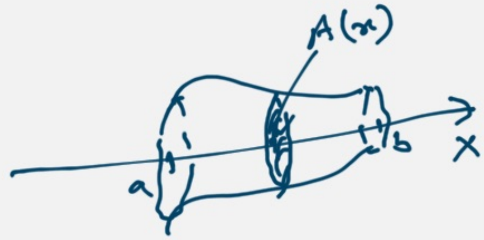
$\therefore$ Perimeter, $s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$

Find volume of solids using integrals:

① Method of slicing

Suppose the solid lies between $x=a$ & $x=b$; and the cross-sectional area at x is given by $A(x)$. Then the volume is

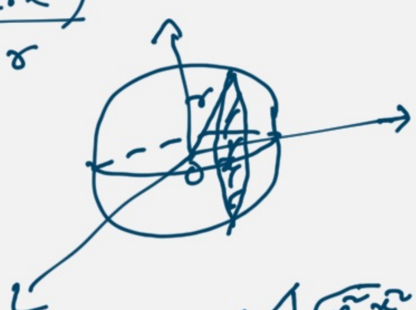
$$V = \int_a^b A(x)\,dx$$



Example: (Volume of sphere)

Eqn. of a sphere of radius r is

$$x^2+y^2+z^2=r^2$$



At any x, the area of cross section

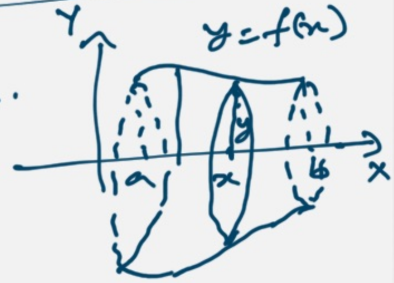
$$A(x) = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi\left(r^2-x^2\right)$$

$$\therefore \text{Volume, } V = \int_{-r}^{r} \pi\left(r^2-x^2\right)dx = 2\pi\int_0^r \left(r^2-x^2\right)dx = \frac{4\pi}{3}r^3$$

Exercise: Show that the volume of a right circular cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$.

② Method of revolution about an axis

Suppose $y = f(x)$, $a \leq x \leq b$ is rotated about the x-axis.

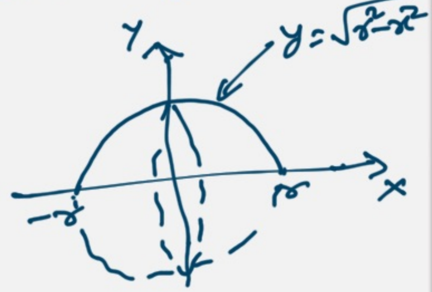


Then the volume of the solid generated is

$$V = \int_a^b \underbrace{\pi [f(x)]^2}_{=A(x)} dx$$

Example: Volume of sphere using revolution method.

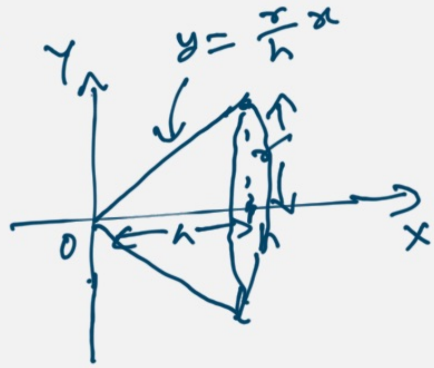
Sphere can be obtained by rotating $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$ about the x-axis.



$$\therefore \; V = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{4}{3}\pi r^3 .$$

(Volume of a cone):



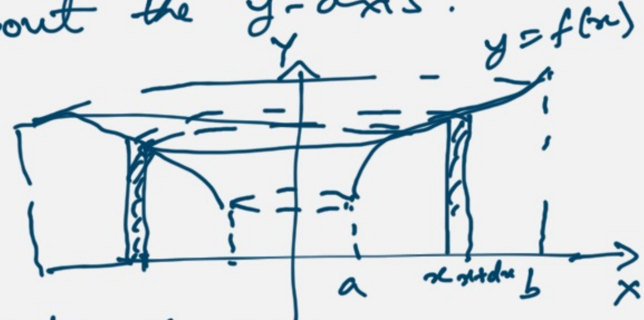
$$V = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$$

③ Cylindrical shell method

Suppose $y = f(x)$, $a \le x \le b$ is rotated about the y-axis.



Volume of cylindrical shell of radius x & thickness dx & height $f(x)$

$$dV = 2\pi x f(x)\,dx$$

$$\therefore\ V = \int_a^b 2\pi x f(x)\,dx$$