Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx \quad , \alpha > 0$$

- $\Gamma(1) = 1$
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$
- $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof: $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-1/2} e^{-x}\,dx$

Substitute $x = t^2$. Then $dx = 2t\,dt$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1} e^{-t^2} . 2t\,dt = 2\int_0^\infty e^{-t^2}\,dt$$

Let $I = \int_0^\infty e^{-x^2}\,dx$

Then $I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$

$$= \int_0^\infty\int_0^\infty e^{-x^2} . e^{-y^2}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

We use polar coords.: $x^2+y^2 = r^2$, $dx\,dy = r\,dr\,d\theta$

$$\therefore \quad I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^{\infty} = \frac{\pi}{4}$$

$$\Rightarrow \quad I = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \Gamma\left(\tfrac{1}{2}\right) = 2I = \sqrt{\pi}$$

Beta function:

For $m>0$, $n>0$,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

It can be shown that the integral converges for $m>0$, $n>0$.

Properties:

① $\beta(m,n) = \beta(n,m)$

② $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (We'll not prove this)

Examples:

① Evaluate $I = \int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\,dx$

Put $t = \sqrt{x}$ i.e. $x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int_0^1 t^3 (1-t)^{1/2} \cdot 2t\,dt$$

$$= 2\int_0^1 t^4 (1-t)^{1/2}\,dt$$

$$= 2\,\beta\left(5, \tfrac{3}{2}\right) = 2\,\frac{\Gamma(5)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{13}{2}\right)}$$

$$= \frac{\cancel{2} \times 4! \times \frac{1}{\cancel{2}}\cancel{\Gamma\left(\frac{1}{2}\right)}}{\frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\,\cancel{\Gamma\left(\frac{1}{2}\right)}}$$

$$I = \frac{2^6 \times 4!}{11 \times 9 \times 7 \times 5 \times 3} = \frac{64 \times \cancel{24}^{8}}{\cancel{99}_{33} \times 105} = \frac{512}{3465}$$

② $\int_0^\infty x^{2/3} e^{-\sqrt{x}}\,dx$

Put $t = \sqrt{x}$; $dx = 2t\,dt$

$$\int_0^\infty x^{2/3} e^{-\sqrt{x}}\,dx = \int_0^\infty t^{4/3} e^{-t} \cdot 2t\,dt$$

$$= 2\int_0^\infty t^{7/3} e^{-t}\,dt = 2\,\Gamma\left(\tfrac{10}{3}\right)$$

$$= 2 \times \tfrac{7}{3}\,\Gamma\left(\tfrac{7}{3}\right)$$

$$= 2 \times \tfrac{7}{3} \times \tfrac{4}{3} \times \tfrac{1}{3}\,\Gamma\left(\tfrac{1}{3}\right) = \tfrac{56}{27}\,\Gamma\left(\tfrac{1}{3}\right)$$

How to calculate $\int_0^\infty \frac{\sin x}{x}\,dx$?

Let $I(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx \quad , \alpha \geq 0$

Then $I(0) = \int_0^\infty \frac{\sin x}{x}\,dx$

$$\frac{d}{d\alpha}I(\alpha) = \frac{d}{d\alpha}\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx$$

$$= \int_0^\infty \frac{\partial}{\partial\alpha}\left[e^{-\alpha x}\frac{\sin x}{x}\right]dx$$

$$= \int_0^\infty -\cancel{x}\, e^{-\alpha x}\cdot\frac{\sin x}{\cancel{x}}\,dx$$

$$\therefore \quad I'(\alpha) = \int_0^\infty -e^{-\alpha x}\sin x\,dx$$

$$= (\sin x)\cdot\frac{e^{-\alpha x}}{\alpha}\Big|_0^\infty - \int_0^\infty \cos x\cdot\frac{e^{-\alpha x}}{\alpha}\,dx$$

$$= 0 - \frac{1}{\alpha}\int_0^\infty \cos x\cdot e^{-\alpha x}\,dx$$

$$= -\frac{1}{\alpha}\left[\cos x\cdot\frac{e^{-\alpha x}}{-\alpha}\Big|_0^\infty - \int_0^\infty (-\sin x)\cdot\frac{e^{-\alpha x}}{-\alpha}\,dx\right]$$

$$= -\frac{1}{\alpha^2} - \frac{1}{\alpha^2}I'(\alpha)$$

$$\Rightarrow \left(1+\frac{1}{\alpha^2}\right)I'(\alpha) = -\frac{1}{\alpha^2} \Rightarrow I'(\alpha) = \frac{-1}{\alpha^2+1}$$

$$\Rightarrow \quad I(\alpha) = -\tan^{-1}\alpha + C$$

Now, $$I(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin x}{x}\, dx$$

$$\because \lim_{\alpha \to \infty} I(\alpha) = 0$$

$$\therefore \quad 0 = -\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2}$$

$$\therefore \quad I(\alpha) = \frac{\pi}{2} - \tan^{-1}(\alpha)$$

$$\therefore \int_0^\infty \frac{\sin x}{x}\, dx = I(0) = \frac{\pi}{2}.$$

Function of several variables

$f : D \to \mathbb{R}$

$f(x,y)$ for $(x,y) \in D \subseteq \mathbb{R}^2$

What is the meaning $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$?

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if given $\varepsilon > 0$, $\exists\, \delta > 0$

s.t. $|f(x,y) - L| < \varepsilon$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$