

Mid-Term Solutions
AMTL 101

March 7, 2025

Solution 1. Since, rank of a matrix is the number of non-zero rows in the row-reduced echelon form of the matrix, and by the definition of row reduced echelon form, zero rows must be at the bottom and the leading entry must be 1 and only non-zero entry in it's column. So, the possible 3×3 real RRE matrices of rank 2 are:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where a & b are arbitrary real numbers.

Solution 2. Consider a matrix A whose columns are the given vectors:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \\ 1 & 0 & 1 & d \end{pmatrix} \\ &\xrightarrow[\substack{R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1}]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & -1 & c-a \\ 0 & -1 & 0 & d-a \end{pmatrix} \\ &\xrightarrow{R_4 \rightarrow R_4 - R_2} \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & -1 & c-a \\ 0 & 0 & 0 & d-b \end{pmatrix} \end{aligned}$$

We know that the vectors are linearly dependent iff $\text{rank}(A) < 4$ (i.e. the number of non-zero rows in the echelon form is strictly less than 4) iff $d = b$.

Solution 3. The augmented matrix for the given system of linear equations

is:

$$\begin{aligned}
 [A : b] &= \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 1 & 2 & 4 & 3 & 5 \\ 1 & 3 & 2 & a & 4 \\ 1 & 2 & 1 & 0 & b \end{pmatrix} \\
 &\xrightarrow[\substack{R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1}]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & a-2 & 0 \\ 0 & 1 & -2 & -2 & b-4 \end{pmatrix} \\
 &\xrightarrow[\substack{R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 - R_2}]{} \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & a-4 & -2 \\ 0 & 0 & -3 & -3 & b-5 \end{pmatrix} \\
 &\xrightarrow[\substack{R_4 \rightarrow R_4 - R_3}]{} \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & a-4 & -2 \\ 0 & 0 & 0 & 1-a & b-3 \end{pmatrix}
 \end{aligned}$$

1. *no solution:* if $\text{rank}(A) < \text{rank}[A : b]$ iff $a = 1$ & $b \neq 3$.

2. *unique solution:* if $\text{rank}(A) = \text{rank}[A : b] = 4$ iff $a \neq 1$ & $b \in \mathbb{R}$.

3. *infinitely many solutions:* if $\text{rank}(A) = \text{rank}[A : b] < 4$ iff $a = 1$ & $b = 3$.

4

Now, if $a = 1$ & $b = 3$,

$$\begin{aligned}
 [A : b] &\cong \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\xrightarrow{R_3 \rightarrow -\frac{1}{3}R_3} \begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\xrightarrow{\substack{R_2 \rightarrow R_2 - R_3 \\ R_1 \rightarrow R_1 - 3R_3}} \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & -1 & \frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

So, the row equivalent system is:

$$\begin{aligned}
 x_1 - x_4 &= \frac{5}{3} \\
 x_2 &= \frac{1}{3} \\
 x_3 + x_4 &= \frac{2}{3}
 \end{aligned}$$

let $x_4 = \lambda$, we get

$$\begin{aligned}
 x_1 &= \frac{5}{3} + \lambda \\
 x_2 &= \frac{1}{3} \\
 x_3 &= \frac{2}{3} - \lambda
 \end{aligned}$$

so, the solution set of the given system in case of $a = 1$ & $b = 3$ is:

$$\left\{ \left(\frac{5}{3} + \lambda, \frac{1}{3}, \frac{2}{3} - \lambda, \lambda \right) : \lambda \in \mathbb{R} \right\}.$$

Solution 4. (a)

$$W_1 = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0, y + z = 0\};$$

$$W_2 = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z = 0, y + z + w = 0\};$$

$$W_1 \cap W_2 = \left\{ (x, y, z, w) \in \mathbb{R}^4 : \begin{array}{ll} x + y + z + w = 0, & y + z = 0, \\ x + y + z = 0, & y + z + w = 0 \end{array} \right\}$$

i.e. $W_1 \cap W_2$ is the solution set of the system of linear equations $AX = 0$, where

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \\ &\cong \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, we get $x = 0 = w$ and $y + z = 0$.

$$\therefore W_1 \cap W_2 = \{(0, y, -y, 0) : y \in \mathbb{R}\}.$$

Thus, $\{(0, 1, -1, 0)\}$ is a basis for $W_1 \cap W_2$.

(b)

$$\begin{aligned} W_1 &= \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0, y + z = 0\} \\ &= \{(x, y, z, w) \in \mathbb{R}^4 : x + w = 0, y + z = 0\} \\ &= \{(x, y, -y, -x) : x, y \in \mathbb{R}\} \\ &= \{x(1, 0, 0, -1) + y(0, 1, -1, 0) : x, y \in \mathbb{R}\} \\ &= \text{span}\{(1, 0, 0, -1), (0, 1, -1, 0)\} \end{aligned}$$

Since the two vectors are not multiples of each other, they form a linearly independent set, therefore the set

$$\beta_1 := \{(1, 0, 0, -1), (0, 1, -1, 0)\}$$

is a basis for W_1 . Thus, $\dim W_1 = 2$.

Similarly, W_2 is the solution set of the system of linear equations:

$$\begin{aligned} x + y + z &= 0, \\ y + z + w &= 0 \end{aligned}$$

by solving the above system, we get $w = x$ & $z = -y - x$.

$$\begin{aligned} \therefore W_2 &= \{(x, y, -y - x, x) : x, y \in \mathbb{R}\} \\ &= \{x(1, 0, -1, 1) + y(0, 1, -1, 0) : x, y \in \mathbb{R}\} \\ &= \text{span}\{(1, 0, -1, 1), (0, 1, -1, 0)\} \end{aligned}$$

Since the two vectors are not multiples of each other, they form a linearly independent set, therefore the set

$$\beta_2 := \{(1, 0, -1, 1), (0, 1, -1, 0)\}$$

is a basis for W_2 . Thus, $\dim W_2 = 2$.

Since,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$\therefore \dim(W_1 + W_2) = 2 + 2 - 1 = 3.$$

(c) Note that $\beta_1 \cup \beta_2$ is a spanning set for $W_1 + W_2$, i.e.

$$\begin{aligned} W_1 + W_2 &= \text{span}\{(1, 0, 0, -1), (0, 1, -1, 0), (1, 0, -1, 1), (0, 1, -1, 0)\} \\ &= \text{span}\{(1, 0, 0, -1), (0, 1, -1, 0), (1, 0, -1, 1)\} \end{aligned}$$

Since, $a(1, 0, 0, -1) + b(0, 1, -1, 0) + c(1, 0, -1, 1) = 0 \implies a = b = c = 0$, the vectors are linearly independent, therefore the set

$$\beta_3 := \{(1, 0, 0, -1), (0, 1, -1, 0), (1, 0, -1, 1)\}$$

is a basis for $W_1 + W_2$.

Solution 5. (a) True: Clearly, $0 \in W$. Let $X_1, X_2 \in W$ and $\alpha \in \mathbb{R}$, since $A(\alpha X_1 + X_2)B = \alpha AX_1B + AX_2B = \alpha BX_1A + BX_2A = B(\alpha X_1 + X_2)A$, $\therefore (\alpha X_1 + X_2) \in W$. Thus, W is a subspace of $M_{n \times n}(\mathbb{R})$.

(b) False: Let $V = \mathbb{R}^2$ and

$$W_1 = \{(x, 0) : x \in \mathbb{R}\},$$

$$W_2 = \{(0, y) : y \in \mathbb{R}\},$$

$$W_3 = \{(x, x) : x \in \mathbb{R}\}.$$

Note that $\dim(W_i) = 1 \forall i = 1, 2, 3$, and $W_i \cap W_j = \{0\} \forall i \neq j$. But the dimension of $W_1 + W_2 + W_3 = 2$.

(c) False: Consider

$$a(u - 2v) + b(4v - 2w) + c(w - u) = 0.$$

$$\implies (a - c)u + (-2a + 4b)v + (-2b + c)w = 0.$$

Since, $\{u, v, w\}$ is linearly independent,

$$a - c = 0$$

$$-2a + 4b = 0$$

$$-2b + c = 0$$

by solving the above system, we get,

$$c = a, b = \frac{a}{2}.$$

In fact, the system has infinitely many solutions. Therefore, there exist scalars a, b, c not all zero such that $a(u - 2v) + b(4v - 2w) + c(w - u) = 0$.

Thus, the set $\{u - 2v, 4v - 2w, w - u\}$ is linearly dependent.

Solution 6. (a) False: Observe that $T(\alpha z) = \overline{\alpha z} = \bar{\alpha} \bar{z} \neq \alpha \bar{z} = \alpha T(z)$ when α is a complex number with non-zero imaginary part and $z \neq 0$.

(b) True: Let $p(x), q(x) \in \mathbb{R}[x]$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} T((\alpha p + q)(x)) &= x^2((\alpha p + q)(x)) + (\alpha p + q)(1) \\ &= x^2(\alpha p(x) + q(x)) + (\alpha p(1) + q(1)) \\ &= \alpha(x^2 p(x) + p(1)) + (x^2 q(x) + q(1)) \\ &= \alpha T(p(x)) + T(q(x)) \end{aligned}$$

$\therefore T$ is a linear transformation.