Linear Transformations

Let V and W are vector spaces over the same field $\mathbb{F}$. A function
$$T: V \to W$$
is called a linear transformation if

(i) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$.

(ii) $T(\alpha v) = \alpha T(v) \quad \forall v \in V, \alpha \in \mathbb{F}$.

Equivalently, T is a linear transformation if $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

Prop: $T(0_V) = 0_W$, where 0_V & 0_W are the zero vectors in V & W, respectively

Pf: $0_V = 0_V + 0_V$

$\Rightarrow T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$

Adding the additive inverse $-T(0_V)$, we get

$$0_W = \left(T(0_V) + T(0_V)\right) + \left(-T(0_V)\right)$$
$$= T(0_V) + \left(T(0_V) + -T(0_V)\right)$$
$$= T(0_V) + 0_W = T(0_V).$$

Remark: If $T(0) \neq 0$, then T is not a linear transformation.

e.g. ① Is $T : \mathbb{R} \to \mathbb{R}$ given by
$$T(x) = x+1$$
a linear transformation?

Ans: Since $T(0) = 1 \neq 0$, T is not a linear transformation.

② Is $T(x) = x^2$ a linear transf. from $\mathbb{R}$ to $\mathbb{R}$?

$$T(2x) = (2x)^2 = 4x^2 \neq 2T(x)$$
if $x \neq 0$.

or, $T(x+y) = (x+y)^2 \neq x^2+y^2$

③ $T : \mathbb{R} \to \mathbb{R}$ given by
$$T(x) = \alpha x, \text{ where } \alpha \in \mathbb{R}$$
is a linear transformation

$$T(x+y) = \alpha(x+y) = \alpha x + \alpha y = T(x) + T(y)$$
$$T(cx) = \alpha(cx) = c(\alpha x) = cT(x)$$

$\therefore$ T is a linear transformation.

Ex: Show that any linear transformation $T : \mathbb{R} \to \mathbb{R}$ is of the form $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.

Soln: $T(x) = T(x.1)$

$= x\,T(1)$ (because T is linear)

$= \alpha x$ for $\alpha = T(1)$

Ex: Write all linear transformation $T: \mathbb{R}^2 \to \mathbb{R}$.

For $(x, y) \in \mathbb{R}^2$,

$(x, y) = x(1,0) + y(0,1)$

$\therefore T(x,y) = x\underbrace{T(1,0)}_{\in \mathbb{R}} + y\underbrace{T(0,1)}_{\in \mathbb{R}}$

$= \alpha x + \beta y$, $\alpha = T(1,0)$, $\beta = T(0,1)$

Some more examples of linear transf:

① $T: V \to W$, $T(v) = 0 \quad \forall v \in V$

is a linear transf.

(Zero transformation)

② $T: V \to V$, $T(v) = v \quad \forall v \in V$

(Identity transformation)

③ Let V = space of all real polynomials

(i) $T: V \to V$

$T(p(x)) = p(x+1)$

$$T(\underbrace{\alpha p(x)+\beta q(x)}_{f(x)}) = f(x+1)$$
$$= \alpha p(x+1)+\beta q(x+1)$$
$$= \alpha T(p(x))+\beta T(q(x))$$

$\Rightarrow$ T is a linear transf.

(ii) $T : \mathbb{F}[x] \to \mathbb{F}[x]$

$T(p(x)) = p'(x)$ $\leftarrow$ derivative of $p(x)$.

T is a linear transf. (verify)

④ Let V = the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

$T : V \to V$

$$T(f(x)) = \int_0^x f(t)\,dt$$

Verify that T is a linear transf.

Let V & W be vector spaces over the same field $\mathbb{F}$.

Let $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is a linear transf}\}$

Note that if $T_1, T_2 \in \mathcal{L}(V,W)$, then

$$T_1 + T_2 \in \mathcal{L}(V,W)$$

Also, if $T \in \mathcal{L}(V,W)$, then $\alpha T \in \mathcal{L}(V,W)$ for any $\alpha \in \mathbb{F}$.

- $\mathcal{L}(V,W)$ is a vector space over $\mathbb{F}$.