Mean value theorem for integrals

Suppose $f(x)$ is continuous on $[a,b]$.
Then $\exists\ \xi \in [a,b]$ s.t

$$\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{is called the average of } f \text{ over the interval } [a,b].} = f(\xi).$$

Proof: Since f is continuous on $[a,b]$,
$\exists\ m, M$ s.t.

$$m \le f(x) \le M \qquad \left(\begin{array}{l} m = \min_{x\in[a,b]} f(x) \\ M = \max_{x\in[a,b]} f(x) \end{array} \right)$$

$$\Rightarrow \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{(b-a)}\int_a^b f(x)\,dx \le M$$

Since f is cont. by the IVT,
$\exists\ \xi \in [a,b]$ s.t

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx = f(\xi).$$

First fundamental theorem of Calculus:

Suppose f is continuous on $[a, b]$ and let $\varphi(x) = \int_a^x f(s)\,ds$ for $a \le x \le b$.

Then φ is differentiable and

$$\varphi'(x) = f(x).$$

Proof:

$$\varphi(x+h) - \varphi(x) = \int_a^{x+h} f(s)\,ds - \int_a^x f(s)\,ds$$

$$= \left(\cancel{\int_a^x f(s)\,ds} + \int_x^{x+h} f(s)\,ds \right) - \cancel{\int_a^x f(s)\,ds}$$

$$= \int_x^{x+h} f(s)\,ds$$

$$\Rightarrow \quad \frac{\varphi(x+h) - \varphi(x)}{h} = \frac{1}{h} \int_x^{x+h} f(s)\,ds$$

By the MVT,

$$\frac{1}{h} \int_x^{x+h} f(s)\,ds = f(\xi) \text{ for some } \xi \in [x, x+h]$$

Now, as $h \to 0$, $\xi \to x$ (by sandwich thm)

Since f is cont., $f(\xi) \to f(x)$.

$$\therefore \lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = f(x)$$

$\therefore$ φ is diff'ble & $\varphi'(x) = f(x)$.

Example: Find the limit

$$\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

Soln: By L'Hôpital's rule,

$$\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\int_0^x \sin(t^2)\,dt}{2x}$$

$$= \lim_{x\to 0} \frac{\sin(x^2)}{2x}$$

$$= \lim_{x\to 0} \frac{\sin(x^2)}{x^2}\cdot\frac{x}{2}$$

$$= 1\times 0 = 0.$$

Thm (Second fundamental thm. of Calculus):

Suppose $f(x)$ is a continuous fn. on $[a,b]$ and let $F(x)$ be any function such that $F'(x) = f(x)$

(F is called an antiderivative of f)

Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: Let $\varphi(x) = \int_a^x f(t)\,dt$

Then by the first fund. thm. of Calculus,

$\varphi'(x) = f(x)$ (Note that cont. of f is used here)

$$\Rightarrow \quad \varphi'(x) - F'(x) = 0 \qquad \forall x \in [a,b]$$

$$\Rightarrow \quad \frac{d}{dx}\left[\varphi(x) - F(x)\right] = 0$$

$$\Rightarrow \quad \varphi(x) - F(x) = c$$

$$\Rightarrow \quad \varphi(x) = F(x) + c$$

$$\Rightarrow \quad \int_a^x f(t)\,dt = F(x) + c$$

Putting $x = a$, we get $0 = F(a) + c$

$\Rightarrow c = -F(a)$

$\therefore \quad \int_a^x f(t)\,dt = F(x) - F(a)$

Put $x = b$: $\int_a^b f(t)\,dt = F(b) - F(a)$

Example: $\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2$

$= \ln 2 - \ln 1 = \ln 2.$

Thm: (Change of variable formula)

Let $u(t)$, $u'(t)$ be continuous on $[a,b]$ and let f be continuous on $u([a,b])$.

Then $$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(y)\,dy$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$

Then $F'(x) = f(x)$. (1st fund. thm)

$$\therefore \frac{d}{dt} F(u(t)) = F'(u(t)) . u'(t) \quad \text{(by Chain Rule)}$$

$$= f(u(t)) . u'(t)$$

By the 2nd fund. thm.,

$$\int_a^b f(u(t))\, u'(t)\,dt = F(u(b)) - F(u(a))$$

$$= \int_a^{u(b)} f(y)\,dy - \int_a^{u(a)} f(y)\,dy$$

$$= \int_{u(a)}^{u(b)} f(y)\,dy$$