

Quiz 3 Solutions AMTL101

1. $x^2y'' + xy' - 4y = 0; \quad y(1) = 3, \quad y'(1) = 2.$

Substituting $y = x^m$ in the above IVP, we get the characteristic equation as:

$$\begin{aligned} m(m-1) + m - 4 &= 0 \\ \implies m^2 - 4 &= 0 \\ \implies m &= \pm 2. \end{aligned}$$

\therefore The general solution is:

$$\begin{aligned} y(x) &= c_1x^2 + c_2x^{-2}. \\ \because y(1) &= 3, \\ \implies c_1 + c_2 &= 3 \end{aligned} \tag{1}$$

and $y'(1) = 2$,

$$\begin{aligned} \implies 2c_1 - 2c_2 &= 2 \\ \implies c_1 - c_2 &= 1 \end{aligned} \tag{2}$$

From (1) and (2),

$$\begin{aligned} c_1 &= 2, \quad c_2 = 1 \\ \therefore y(x) &= 2x^2 + \frac{1}{x^2}. \end{aligned}$$

2. $y'' - 2y' + y = x^2 + 2e^x.$

The characteristic equation of the corresponding homogeneous part is:

$$\begin{aligned} m^2 - 2m + 1 &= 0 \\ \implies (m-1)^2 &= 0 \\ \implies m &= 1, 1. \end{aligned}$$

\therefore The general solution of the corresponding homogeneous equation is:

$$y_h(x) = c_1e^x + c_2xe^x.$$

Let

$$y_p(x) = Ax^2 + Bx + C + Dx^2e^x$$

be a particular solution of the given ODE. Then

$$\begin{aligned} y'_p(x) &= 2Ax + B + Dx^2e^x + 2Dxe^x \\ y''_p(x) &= 2A + Dx^2e^x + 4Dxe^x + 2De^x \end{aligned}$$

Substituting in the equation, we get

$$\begin{aligned} Ax^2 + (-4A + B)x + (2A - 2B + C) + 2De^x &= x^2 + 2e^x \\ \implies A = 1, D = 1, B = 4, C = 6. \\ \therefore y_p(x) &= x^2 + 4x + 6 + x^2e^x. \end{aligned}$$

So, the general solution of the given non-homogeneous equation is:

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1e^x + c_2xe^x + x^2 + 4x + 6 + x^2e^x. \end{aligned}$$

$$3. \quad y'' - 2y' + 5y = e^x \sec 2x.$$

The characteristic equation of the corresponding homogeneous part is:

$$\begin{aligned} m^2 - 2m + 5 &= 0 \\ \implies m &= \frac{2 \pm \sqrt{4 - 20}}{2} \\ &= 1 \pm 2i. \end{aligned}$$

$$\therefore y_h(x) = e^x [c_1 \cos 2x + c_2 \sin 2x].$$

So, $y_1(x) = e^x \cos 2x$ & $y_2(x) = e^x \sin 2x$ are two L.I. Solutions of the corresponding homogeneous equation. So, by the variation of parameters method,

$$y_p = u_1y_1 + u_2y_2$$

where

$$u_1 = - \int \frac{y_2 r(x)}{W(y_1, y_2)} dx,$$

and

$$u_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Now,

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{vmatrix} \\ &= 2e^{2x} \cos^2 2x + 2e^{2x} \sin^2 2x \\ &= 2e^{2x} \end{aligned}$$

$$\begin{aligned} \therefore u_1 &= - \int \frac{e^x \sin 2x \cdot e^x \sec 2x}{2e^{2x}} dx \\ &= -\frac{1}{2} \int \tan 2x dx \\ &= \frac{1}{4} \ln |\cos 2x| \end{aligned}$$

$$\text{and } u_2 = \int \frac{e^x \cos 2x \cdot e^x \sec 2x}{2e^{2x}} dx = \frac{1}{2} x.$$

$$\therefore y_p(x) = \frac{1}{4} \ln |\cos 2x| \cdot e^x \cos 2x + \frac{x}{2} e^x \sin 2x.$$

So, the general solution of the given non-homogeneous equation is:

$$y(x) = e^x [c_1 \cos 2x + c_2 \sin 2x] + \frac{1}{4} \ln |\cos 2x| \cdot e^x \cos 2x + \frac{x}{2} e^x \sin 2x.$$