Example : Consider the sequence

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \quad , n \in \mathbb{N}$$

We'll show $\{x_n\}$ is increasing and bounded from above by 3.
This will prove that the sequence $\{x_n\}$ is convergent and the limit is denoted by e.

$$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0$$

$\Rightarrow \quad x_{n+1} > x_n \quad \forall n.$

$\therefore \quad \{x_n\}$ is increasing.

Also, $\quad k! > 2^{k-1} \quad \forall k \geq 3$

$$\left(k! = k(k-1)(k-2)\cdots 3.2 > \underbrace{2.2\ldots 2}_{(k-1)\text{times}} = 2^{k-1} \right)$$

$$\therefore \; x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 + 2\left(1 - \frac{1}{2^n}\right) < 3$$

$$\left[a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \right]$$

$$S = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$(1-r)S = a - ar^n$$
$$\Rightarrow S = \frac{a(1-r^n)}{1-r} \; ; \; r \neq 1$$

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

We'll skip the proof.

e.g. $\{(-1)^n\}$ is bounded but not convergent.

However, it has convergent subsequence

e.g. $\{x_{2n}\}_{n=1}^{\infty}$, $\{x_{2n-1}\}_{n=1}^{\infty}$

Q. Does the sequence $x_n = \sin(n)$, $n \in \mathbb{N}$ have any convergent subsequence?

Ans: Yes, by the Bolzano-Weierstrass theorem, since $-1 \le \sin(n) \le 1$.

Example:

Consider the sequence: $x_1 = a > 0$, $x_{n+1} = \dfrac{x_n^2 + 3}{4}$, $n \geq 1$.

Find the limit of $\{x_n\}$, if it exists.

Easy cases:

If $a = 1$, $x_n = 1 \ \forall n \Rightarrow \lim_{n \to \infty} x_n = 1$

If $a = 3$, $x_n = 3 \ \forall n \Rightarrow \lim_{n \to \infty} x_n = 3$.

Hint:

$$
\begin{aligned}
x_{n+1} - x_n &= \frac{x_n^2+3}{4} - x_n \\
&= \frac{1}{4}\left(x_n^2 - 4x_n + 3\right) \\
&= \frac{1}{4}\left(x_n - 1\right)\left(x_n - 3\right) \\
&> 0 \quad \text{if } x_n < 1 \text{ or } x_n > 3 \\
&< 0 \quad \text{if } 1 < x_n < 3
\end{aligned}
$$

Case I: $a < 1$

Case II: $1 < a < 3$

Case III: $a > 3$