System of linear equations

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (more generally, $\mathbb{F}$ can be any "field")

A system of m linear equations in n unknowns $x_1, x_2, x_3, \dots, x_n$ is of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m ,
\end{aligned}
$$

where $a_{ij} \in \mathbb{F}$ for $1 \le i \le m$, $1 \le j \le n$

$b_k \in \mathbb{F}$ for $1 \le k \le m$

This system can be written in matrix form as:

$$AX = B \text{ , where}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{F})$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ is the column vector of the unknowns

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} .$$

A is called the "coefficient matrix"

The augmented matrix $(A|B)$ is the matrix obtained by adding the column vector B to the coefficient matrix A.

$(A|B) \in M_{m\times(n+1)}(\mathbb{F})$

The augmented matrix $(A|B)$ gives the system of linear eqns uniquely.

e.g. $(A|B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right)$ corresponds to the system $\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \end{cases}$

The system $AX = B$ is called a homogeneous system if $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i.e. $AX = 0$

If $B \neq 0$, then it is called a non-homogeneous system.

The solution set of the system $AX = B$ consists of all $Y \in M_{n\times 1}(\mathbb{F})$ such that $AY = B$.

Sometime we write Y as an n-tuple $(y_1, y_2, \cdots, y_n)$

Example (1) Consider the system

$$x + y = 2$$
$$x - y = 0$$

Here $(A|B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right)$

We can easily see that $x=1, y=1$ is the unique soln, or $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $(1, 1)$ is the unique soln.

(2)
$$x + y = 2$$
$$2x + 2y = 3$$

This system has no solution.

(3)
$$x + 2y = 3$$
$$2x + 4y = 6$$

This has infinitely many solutions.

Solution set, $S = \{ (3-2\lambda, \lambda) : \lambda \in \mathbb{F} \}$

For a homogeneous system, $AX = 0$,
$X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is always a solution.

Thm: Any homogeneous system $AX = 0$ has either a unique solution $X = 0$ or infinitely many solutions.

Proof: Suppose it has a nonzero solution X_1.
Then $X = \lambda X_1$ is also a solution for any $\lambda \in \mathbb{F}$.
Thus $AX = 0$ has infinitely many solns.

Thm: Any system of linear eqns $AX = B$ has either no solutions or a unique solution or infinitely many solutions.

Pf: Suppose $X_1 \neq X_2$ are ~~two~~ solutions.
Then $X = X_1 + \lambda(X_1 - X_2)$ is a solution of $AX = B$ for any $\lambda \in \mathbb{F}$

$$\left(AX = AX_1 + \lambda(AX_1 - AX_2) = B + \lambda(B - B) = B \right)$$