**Q.1** (a) $y' = x + y$ , $y(0) = 0$ .

$$\Rightarrow \quad y' - y = x$$

$$\therefore \quad \text{I.F.} = e^{\int -1 \cdot dx} = e^{-x}$$

$$\Rightarrow \quad y \cdot e^{-x} = \int x \cdot e^{-x} \, dx + c$$

$$\Rightarrow \quad y = e^{x}\left[-x \cdot e^{-x} - e^{-x}\right] + c e^{x}$$

$$= -x - 1 + c e^{x}$$

$$\because \quad y(0) = 0 \quad \Rightarrow \quad c - 1 = 0$$

$$\Rightarrow \quad c = 1$$

$$\therefore \quad y = e^{x} - x - 1 .$$

(b) Here, $f(x, y) = x + y$ , $x_0 = 0 = y_0$ .

$$\therefore \quad y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0) \, dt$$

$$= \int_{0}^{x} t \, dt = \frac{x^2}{2} .$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

$$= \int_0^x \left[t + \frac{t^2}{2}\right] dt$$

$$= \frac{x^2}{2} + \frac{x^3}{3 \cdot 2}$$

$$y_3(x) = \int_0^x \left[t + t^2/2 + \frac{t^3}{3 \cdot 2}\right] dt$$

$$= \frac{x^2}{2} + \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3 \cdot 2}$$

in general, $y_n(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{n+1!}$

So, $\lim_{n \to \infty} y_n(x) = \sum_{n=2}^{\infty} \frac{x^n}{n!}$

$$= e^x - x - 1.$$

Q.2

$$y'' - 2xy' - 2y = 0$$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 - 2a_0) + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} - 2n a_n - 2 a_n \right] x^n = 0$$

$$\Rightarrow \quad 2a_2 - 2a_0 = 0 \quad \text{i.e.} \quad a_2 = a_0 ,$$

$$\& \quad (n+2)(n+1) a_{n+2} - 2n a_n - 2 a_n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{2(n+1) a_n}{(n+2)(n+1)} \quad \forall \; n \geq 1 .$$

$$\Rightarrow \quad a_{n+2} = \frac{2\, a_n}{n+2} \qquad \forall\, n \geqslant 1.$$

Now, $\quad a_2 = a_0,$

$$a_4 = \frac{2\, a_2}{4} = \frac{a_0}{2}$$

$$a_6 = \frac{2\, a_4}{6} = \frac{a_0}{6}$$

$$a_8 = \frac{2\, a_6}{8} = \frac{a_0}{24}$$

i.e. $\quad a_{2n} = \dfrac{a_0}{n!}$

Also, $\quad a_3 = \dfrac{2\, a_1}{3}$

$$a_5 = \frac{2\, a_3}{5} = \frac{2^2}{5\cdot 3}\, a_1$$

$$a_7 = \frac{2\, a_5}{7} = \frac{2^3}{7\cdot 5\cdot 3}\, a_1$$

$$\therefore \quad a_{2n+1} = \frac{2^n}{(2n+1)\cdot(2n-1)\cdots 3}\, a_1$$

$$\therefore \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{a_0}{n!} x^{2n} + \sum_{n=0}^{\infty} \frac{2^n a_1}{(2n+1)\cdot(2n-1)\cdots 3} \cdot x^{2n+1},$$

**Q.3** $\quad y'' - 2y' = \delta(t-1), \quad t > 0$

$y(0) = 1, \quad y'(0) = 0.$

Taking Laplace transforms both side,

$$L(y'') - 2L(y') = L(\delta(t-1))$$

$$\Rightarrow [s^2 L(y) - s] - 2[sL(y) - 1] = e^{-s}$$

$$\Rightarrow (s^2 - 2s) L(y) = e^{-s} + s - 2$$

$$\Rightarrow \quad L(y) = \frac{e^{-s}}{s(s-2)} + \frac{1}{s}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{e^{-s}}{s(s-2)}\right] + L^{-1}\left[\frac{1}{s}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{e^{-s}}{s-2} - \frac{e^{-s}}{s}\right] + 1$$

$$\because \quad L(u(t-a) f(t-a)) = e^{-as} L(f)(s)$$

$$\Rightarrow \quad L^{-1}\left(\frac{e^{-s}}{s-2}\right) = u(t-1)\, e^{2(t-1)}$$

$$\therefore \quad y(t) = \frac{1}{2}\left[u(t-1)\, e^{2(t-1)} - u(t-1)\right] + 1$$

$$= \frac{1}{2} u(t-1)\left(e^{2(t-1)} - 1\right) + 1$$

$$= \begin{cases} 1 & ; \quad t < 1 \\ \frac{1}{2} e^{2(t-1)} + 1/2 & ; \quad t > 1 \end{cases}$$

**Q.4**

$$f(t) = \begin{cases} t & , \quad 0 < t < 1 \\ 1 & , \quad 1 < t < 2 \\ 3-t & , \quad 2 < t < 3 \\ 0 & , \quad t > 3 \end{cases}$$

$$\Rightarrow \quad f(t) = t\left[1 - u(t-1)\right] + \left[u(t-1) - u(t-2)\right]$$

$$+ (3-t)\left[u(t-2) - u(t-3)\right]$$

$$\therefore \quad L(f(t)) = \frac{1}{s^2} + \frac{d}{ds}\left(\frac{e^{-s}}{s}\right) + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$+ 3\left[\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right] + \frac{d}{ds}\left[\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right]$$

$$= \frac{1}{s^2} + \frac{-se^{-s} - e^{-s}}{s^2} + \frac{1}{s}\left[e^{-s} + 2e^{-2s} - 3e^{-3s}\right]$$

$$+ \left[\frac{-2se^{-2s} - e^{-2s}}{s^2} - \frac{-3se^{-3s} - e^{-3s}}{s^2}\right].$$

**Q.5** (a) $F(s) = \dfrac{1}{s^2 - 2s - 3}$

$$= \frac{1}{(s+1)(s-3)}$$

$$= \frac{1}{4}\left[\frac{1}{s-3} - \frac{1}{s+1}\right]$$

$$\therefore \; L^{-1}(F(s)) = \frac{1}{4}\left[e^{3t} - e^{-t}\right]$$

(b) $G(s) = \tan^{-1}\left(\dfrac{3}{s}\right)$

$$\because \; G'(s) = \frac{1}{1 + \left(\frac{3}{s}\right)^2} \cdot \left(-\frac{3}{s^2}\right)$$

$$= \frac{-3}{s^2 + 3^2}$$

$$\therefore \quad L^{-1}(G'(s)) = -\sin 3t$$

$$\Rightarrow \quad -t\, g(t) = -\sin 3t$$

$$\Rightarrow \quad g(t) = \frac{\sin 3t}{t}.$$

**Q.6** (a) $A = \begin{bmatrix} -4 & 12 \\ -3 & 8 \end{bmatrix}$

$$\because \quad |A - \lambda I| = \begin{vmatrix} -4-\lambda & 12 \\ -3 & 8-\lambda \end{vmatrix}$$

$$= (-4-\lambda)(8-\lambda) + 36$$

$$= -32 + 4\lambda - 8\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$\therefore \quad |A - \lambda I| = 0 \Rightarrow \lambda = 2, 2$$

Now, $A - 2I = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$

$$\approx \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\therefore (A-2I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow v_1 - 2v_2 = 0$$

$\therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.

$$\therefore \vec{X}_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let $\vec{X}_2(t) = te^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{2t} \cdot \vec{u}$

where, $(A-2I)\vec{u} = \vec{v}$

$$\Rightarrow \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[A : \vec{v}] = \begin{bmatrix} -6 & 12 & 2 \\ -3 & 6 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -6 & 12 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -6u_1 + 12u_2 = 2$$

$$\therefore \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \text{ is a choice.}$$

$$\Rightarrow \quad \vec{x}_2(t) = te^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$$

(b) $$\vec{x}_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$$

$$\vec{x}_2(t) = \begin{pmatrix} 2te^{2t} - \frac{1}{3}e^{2t} \\ te^{2t} \end{pmatrix}$$

$$\therefore \tilde{x}(t) = \begin{bmatrix} 2e^{2t} & 2te^{2t} - \frac{1}{3}e^{2t} \\ e^{2t} & te^{2t} \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 2 & 2t - \frac{1}{3} \\ 1 & t \end{bmatrix}$$

$$\therefore (\tilde{x}(t))^{-1} = 3e^{-2t}\begin{bmatrix} t & -2t + \frac{1}{3} \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (\tilde{X}(t))^{-1} \cdot \vec{g}(t) = 3 \begin{bmatrix} t & -2t + 1/3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6t \\ 1-t \end{bmatrix}$$

$$= 3 \begin{bmatrix} 6t^2 - 2t + 2t^2 + 1/3 - \frac{t}{3} \\ -6t + 2 - 2t \end{bmatrix}$$

$$= 3 \begin{bmatrix} 8t^2 - \frac{7}{3}t + \frac{1}{3} \\ -8t + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 24t^2 - 7t + 1 \\ -24t + 6 \end{bmatrix}$$

$$\because \quad \vec{u}'(t) = (\tilde{X}(t))^{-1} \cdot \vec{g}(t)$$

$$\Rightarrow \quad \vec{u}(t) = \begin{bmatrix} 8t^3 - 7/2\, t^2 + t \\ -12t^2 + 6t \end{bmatrix}$$

$$\therefore \quad \vec{X}_p(t) = \tilde{X}(t) \cdot \vec{u}(t)$$

$$= e^{2t} \begin{bmatrix} 2 & 2t - 1/3 \\ 1 & t \end{bmatrix} \begin{bmatrix} 8t^3 - 7/2\, t^2 + t \\ -12t^2 + 6t \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -8t^3 + 9t^2 \\ -4t^3 + 5/2\, t^2 + t \end{bmatrix}$$

Q.7 $\quad y' = \frac{10}{3} x y^{2/5}, \quad y(x_0) = y_0$

(a) $\because$ $f(x,y) = \frac{10}{3} x y^{2/5}$ is continuous for all $(x,y) \in \mathbb{R}^2$, $\therefore$ by Existence theorem, a solution exist $\forall\, x_0, y_0$.

(b) $\because$
$$\frac{\partial f}{\partial y} = \frac{10}{3} x \cdot \frac{2}{5} y^{(2/5 - 1)}$$
$$= \frac{4}{3} x \cdot y^{-3/5}$$

$\because$ $\frac{\partial f}{\partial y}$ is continuous for all $(x,y) \in \mathbb{R}^2$ except $y = 0$, $\therefore$ the IVP has a unique solution for all $x_0 \in \mathbb{R}$ and $y_0 \neq 0$.

(c) If $(x_0, y_0) = (0,0)$,

clearly, $y \equiv 0$ is a solution to the IVP.

Also, $\quad y' = \frac{10}{3} x y^{2/5}$

$$\Rightarrow \quad \frac{dy}{y^{2/5}} = \frac{10}{3} x \, dx$$

$$\Rightarrow \int \frac{dy}{y^{2/5}} = \frac{10}{3}\int x\,dx + c$$

$$\Rightarrow \frac{5}{3} y^{3/5} = \frac{5}{3} x^2 + c$$

$$\Rightarrow y^{3/5} = x^2 + \frac{3}{5}c$$

$$\because \quad y(0) = 0 \Rightarrow c = 0$$

$$\Rightarrow y = x^{10/3}$$

$\therefore$ $y = x^{10/3}$ is also a Solution to the IVP.

**Q.8** Let $T: \mathbb{R}^3 \to \mathbb{R}^5$.

then by Rank-nullity theorem,

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(\mathbb{R}^3) = 3$$

$$\Rightarrow \text{Rank}(T) = 3 - \text{Nullity}(T)$$

$$\leq 3$$

$$< 5 = \dim(\mathbb{R}^5)$$

$$\therefore \dim(\text{Range}(T)) < \dim(\mathbb{R}^5)$$

ie. $\text{Range}(T) \subsetneq \mathbb{R}^5$, i.e. $T$ is not onto.

Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined as

$$T(x_1, x_2, \dots, x_5) = (x_1, x_2, x_3)$$

then $T$ is Linear and observe that

$$\text{Range}(T) = \mathbb{R}^3, \text{ i.e. } T \text{ is onto.}$$