Thm: If f is differentiable at $x=c$, then f must be continuous at c.

Proof: Since f is differentiable at c,

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists.}$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \begin{pmatrix}\text{Put} \\ x-c=h\end{pmatrix}$$

Now,
$$f(x) = [f(x)-f(c)] + f(c)$$
$$= \left[\frac{f(x)-f(c)}{x-c}\right](x-c) + f(c)$$
for any $x \neq c$

$$\therefore \lim_{x\to c} f(x) = \left[\lim_{x\to c} \frac{f(x)-f(c)}{(x-c)}\right] \lim_{x\to c}(x-c) + f(c)$$
$$= f'(c)\cdot 0 + f(c) = f(c)$$

$\Rightarrow$ f is continuous at c.

Remark: The converse of the previous theorem is not true.

Example: $f(x) = |x|$ is continuous at $x=0$ but not differentiable at $x=0$.

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = -1$$

$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist

$\Rightarrow f$ is not diff. at $x=0$.

Exercise: Check the continuity and differentiability of the following functions at $x=0$.

(i) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

(ii) $g(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

Some rules of differentiability:

(i) $(f+g)'(x) = f'(x) + g'(x)$

(ii) $(f-g)'(x) = f'(x) - g'(x)$

(iii) $(cf)'(x) = cf'(x)$

(iv) Product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(v) Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

(vi) Chain rule:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example: $f(x) = \sin(x^2)$

$f'(x) = \cos(x^2) \cdot 2x$

(vii) $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

Let $g(x) = f^{-1}(x)$

Then $f(g(x)) = x$

Differentiating both sides, we get

$$f'(g(x)) \cdot g'(x) = 1$$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

$$\Rightarrow \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

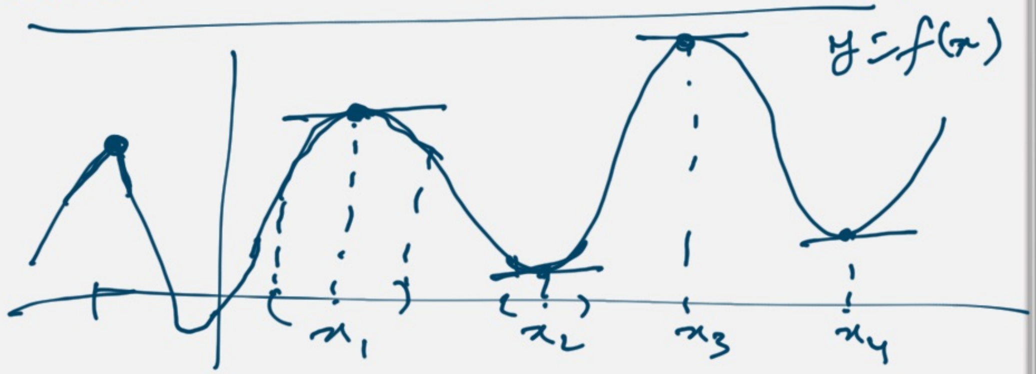
Example: Derivative of $\ln x$:

Let $f(x) = e^x$. Then $f^{-1}(x) = \ln x$

$$\therefore \frac{d}{dx}[\ln x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}}$$

$$= \frac{1}{x}$$

Local minima and local maxima :



x_1 & x_3 are local max. of f

x_2 & x_4 are local min. of f .

Defn: We say $x = a$ is a local minimum of f if there exists $\delta > 0$ s.t.

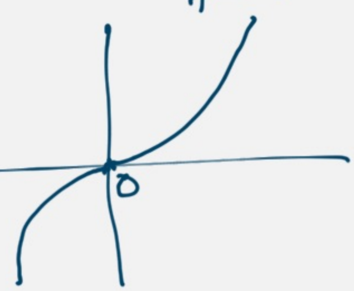
$$f(a) \leq f(x) \quad \forall\, x \in (a-\delta, a+\delta)$$

Similarly $x = a$ is a local maximum if $\exists\ \delta > 0$ s.t.

$$f(a) \geq f(x) \quad \forall\, x \in (a-\delta, a+\delta)$$

Theorem: If $x = a$ is a local min. or local max. of f and if f is differentiable at $x = a$, then $\boxed{f'(a) = 0}$.

$f(x) = x^3$



$f'(0) = 0$ for $f(x) = x^3$

But $x = 0$ is neither a local min. nor a local max. of f.

Rolle's Theorem :

Suppose $f(x)$ is continuous on $[a, b]$
and it is differentiable on (a, b).
If $f(a) = f(b)$, then $\exists\ c \in (a, b)$
such that $f'(c) = 0$