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HTML Content

Solving non-homogeneous systems of ODEs :

Consider the system

$$\vec{x}' = AX + \vec{g}(t),$$

where $A \in M_{n\times n}(\mathbb{R})$ given

$$\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$$\vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \to \text{to be found.}$$

Theorem: The general solution of the nonhomogeneous system

$$\vec{x}' = A\vec{x} + \vec{g}(t) \quad (*)$$

is given by

$$\vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t),$$

where $\vec{X}_h(t)$ is the general soln. of $\vec{X}' = AX$
and $\vec{X}_p(t)$ is a particular soln. of $(*)$.

Proof: If $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$, then

$$\vec{x}'(t) = \vec{x}_h'(t) + \vec{x}_p'(t)$$
$$= A\vec{x}_h(t) + A\vec{x}_p(t) + \vec{g}(t)$$
$$= A\left(\vec{x}_h(t) + \vec{x}_p(t)\right) + \vec{g}(t)$$
$$= A\vec{x}(t) + \vec{g}(t)$$

$\therefore$ $\vec{x}(t)$ is a soln. of $(*)$

Also, if $\vec{x}(t)$ is any soln. of $(*)$

then $\vec{x}(t) - \vec{x}_p(t)$ is a soln. of the corresp. homog. system

$$\vec{x}' = A\vec{x}.$$

$\therefore$ $\vec{x}(t) - \vec{x}_p(t) = \vec{x}_h(t)$

$\Rightarrow$ $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t).$

Fundamental matrix:

For the system $\vec{x}' = A\vec{x}$, a "fundamental matrix" $\tilde{X}(t)$ is an $n \times n$ matrix whose columns form a basis for the solution space of $\vec{x}' = A\vec{x}$

(i.e. the columns of $\tilde{X}(t)$ are n lin. indep. solutions)

For example, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are n L.I. eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$, $1 \le i \le n$ are n L.I. solns. of $\vec{x}' = A\vec{x}$.

$$\therefore \tilde{X}(t) = \left[e^{\lambda_1 t}\vec{v}_1 \; e^{\lambda_2 t}\vec{v}_2 \cdots e^{\lambda_n t}\vec{v}_n \right]$$

is a fundamental matrix.

Properties of fundamental matrix:

① The matrix $\tilde{X}(t)$ is invertible.
(because the columns of $\tilde{X}(t)$ are L.I.)

② $\tilde{X}(t)$ satisfies the matrix eqn.

$$\tilde{X}'(t) = A\tilde{X}(t).$$

Pf: Let $\vec{x}_i(t)$ be the ith column of $\tilde{X}(t)$ for $i = 1, 2, \dots, n$.

Then $\vec{x}_i'(t) = A\vec{x}_i(t)$

Also, $\tilde{X}(t) = \left(\vec{x}_1(t)\ \vec{x}_2(t) \dots \vec{x}_n(t)\right)$

$$\Rightarrow \tilde{X}'(t) = \left(\vec{x}_1'(t)\ \vec{x}_2'(t) \dots \vec{x}_n'(t)\right)$$
$$= \left(A\vec{x}_1(t)\ A\vec{x}_2(t) \dots A\vec{x}_n(t)\right)$$
$$= A\tilde{X}(t)$$

③ The general of $\vec{x}' = Ax$ can be written as

$$\vec{x}(t) = \tilde{x}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, c_2, \dots, c_n$ are arbitrary real constants.

<u>Pf</u>: Since $\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)$ are n L.I. solns. of $\vec{x}' = Ax$, the general soln. is given by

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \dots + c_n\vec{x}_n(t) \\ &= \left(\vec{x}_1(t) \cdots \vec{x}_n(t)\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \tilde{x}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}\end{aligned}$$

Variation of parameters method for nonhomogeneous system

Consider the nonhomog. system

$$\vec{x}\,' = A\vec{x} + \vec{g}(t) \quad \text{—— } (*)$$

We know that the general soln. to the corresp. homog. system

$\vec{x}\,' = A\vec{x}$ is given by

$$\vec{x}_h(t) = \tilde{X}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

where $\tilde{X}(t)$ is a fundamental matrix for the system.

In variation of parameters method we replace $c_1, c_2, \cdots, c_n$ by $u_1(t), u_2(t), \cdots, u_n(t)$ to get a particular soln. $\vec{x}_p(t)$.

So, assume

$$\vec{x}_p(t) = \tilde{X}(t)\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

i.e. $\vec{x}_p(t) = \tilde{X}(t)\,\vec{u}(t)$

$\left(\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} \text{ needs to be determined} \right)$

$$\vec{x}_p{}'(t) = \tilde{X}'(t)\,\vec{u}(t) + \tilde{X}(t)\,\vec{u}'(t)$$

$$= A\tilde{X}(t)\,\vec{u}(t) + \tilde{X}(t)\,\vec{u}'(t)$$

$\vec{x}_p(t)$ is a sol. of $(*)$

$$\vec{x}_p{}'(t) = A\vec{x}_p(t) + \vec{g}(t)$$

$\Leftrightarrow$ $\vec{x}_p{}'(t) =$

$\Leftrightarrow$ $\cancel{A\tilde{X}(t)\vec{u}(t)} + \tilde{X}(t)\,\vec{u}'(t) = \cancel{A\tilde{X}(t)\vec{u}(t)} + \vec{g}(t)$

$\Leftrightarrow$ $\boxed{\vec{u}'(t) = \left(\tilde{X}(t)\right)^{-1}\vec{g}(t)}$

Integrate this to get $\vec{u}(t)$.

Summary: To solve $\vec{x}' = A\vec{x} + \vec{g}(t)$:

Step 1: Find n L.I. solns. $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$ for the homog. system $\vec{x}' = A\vec{x}$.

Write $\tilde{X}(t) = \left(\vec{x}_1(t) \cdots x_n(t)\right)$

Step 2: $\vec{x}_p(t) = \tilde{X}(t)\,\vec{u}(t)$, where $\vec{u}(t)$ is given by

$$\vec{u}'(t) = \left(\tilde{X}(t)\right)^{-1}\vec{g}(t).$$

So, find $\left(\tilde{X}(t)\right)^{-1}\vec{g}(t)$.

Step 3: Integrate to get $\vec{u}(t)$.

Step 4: The general soln. is given by

$$\boxed{\vec{x}(t) = \tilde{X}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \tilde{X}(t)\begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}}$$

Example: Solve the system:

$$x_1' = x_2 - 5\sin t$$
$$x_2' = -4x_1 + 17\cos t$$

Soln: This is equivalent to

$$\vec{x}\,' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix},$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

So, $A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix}$.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 \\ 4 & \lambda \end{pmatrix} = \lambda^2 + 4$$

So, the eigenvalues are $\pm 2i$

For $\lambda = 2i$: $\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad 2iu - v = 0$$

So, $\vec{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$ is an eigenvector.

$$e^{\lambda t}\vec{v} = e^{2it}\begin{pmatrix}1\\2i\end{pmatrix}$$

$$= (\cos 2t + i\sin 2t)\begin{pmatrix}1\\2i\end{pmatrix}$$

$$= \begin{pmatrix}\cos 2t\\-2\sin 2t\end{pmatrix} + i\begin{pmatrix}\sin 2t\\2\cos 2t\end{pmatrix}$$

$$\therefore \quad \vec{x}_1(t) = \begin{pmatrix}\cos 2t\\-2\sin 2t\end{pmatrix}$$

$$\vec{x}_2(t) = \begin{pmatrix}\sin 2t\\2\cos 2t\end{pmatrix}$$

$$\therefore \quad \tilde{X}(t) = \begin{pmatrix}\cos 2t & \sin 2t\\-2\sin 2t & 2\cos 2t\end{pmatrix}$$

$$(\tilde{X}(t))^{-1} = \frac{1}{2}\begin{pmatrix}2\cos 2t & -\sin 2t\\2\sin 2t & \cos 2t\end{pmatrix}$$

$$\therefore \quad \vec{u}'(t) = (\tilde{X}(t))^{-1}\vec{g}(t)$$

7) $\vec{u}'(t) = \frac{1}{2}\begin{pmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} -5\sin t \\ 17\cos t \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} -10\cos 2t\sin t - 17\sin 2t\cos t \\ -10\sin 2t\sin t - 5\cos 2t\sin t \end{pmatrix}$$

i.e. $u_1'(t) = -5\cos 2t\sin t - \frac{17}{2}\sin 2t\cos t$

$u_2'(t) = -5\sin 2t\sin t - \frac{5}{2}\cos 2t\sin t$

Integrate these to get $u_1(t)$ and $u_2(t)$

After this, the general soln is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + u_1(t)x_1(t) + u_2(t)x_2(t).$$