Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(x,y,z) = (x+y-z,\ x-y+z,\ y-z)$$

Find a basis for the null space of T and for range(T).

Soln: $N(T) = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$

$$= \left\{(x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x+y-z=0 \\ x-y+z=0 \\ y-z=0 \end{array}\right\}$$

= soln. space of the system of linear eqns. $AX = 0$,

where $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $z = \lambda$. Then $x = 0$, $y = z = \lambda$

$\therefore N(T) = \{(0, \lambda, \lambda) : \lambda \in \mathbb{R}\}$

$\therefore B = \{(0,1,1)\}$ is a basis for $N(T)$.

$T(1,0,0) = (1,1,0)$

$T(0,1,0) = (1,-1,1)$

$T(0,0,1) = (-1,1,-1)$

Since $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$, $\{T(1,0,0), T(0,1,0), T(0,0,1)\}$ spans $R(T)$ = range space (T).

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + R_2]{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{2} R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{RRE matrix}$$

$\therefore$ $\{(1,0,\frac{1}{2}), (0,1,-1/2)\}$ is a basis for $R(T)$.

Example: Let V be the vector space of all polynomials of degree ≤ 3.
Consider $T: V \to V$ given by
$$T(p(x)) = p'(x) \leftarrow \text{derivative}$$
Find a basis for $N(T)$ and $R(T)$.

Soln. $N(T) = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_1 + 2a_2 x + 3a_3 x^2 = 0 \}$

$= \{ a_0 \}$

$\therefore \; B = \{1\}$ is a basis for $N(T)$.

$\{1, x, x^2, x^3\}$ is a basis for V

$\therefore \; \{T(1), T(x), T(x^2), T(x^3)\}$ span $R(T)$

$T(1) = 0$
$T(x) = 1$
$T(x^2) = 2x$
$T(x^3) = 3x^2$

$\therefore \; R(T) = \text{span}\{0, 1, 2x, 3x^2\}$
$= \text{span}\{1, 2x, 3x^2\}$

Also, $\{1, 2x, 3x^2\}$ is linearly indep.

$\therefore \; \{1, 2x, 3x^2\}$ is a basis for $R(T)$.

or, $\{1, x, x^2\}$ is a basis for $R(T)$.

Defn: (coordinate vector w.r.t. an ordered basis)

Let V be a finite dimensional vector space, say $\dim V = n$.

An ordered is a basis $B = \{v_1, v_2, \dots, v_n\}$ for B with a fixed ordering.

Then any $v \in V$ can be written uniquely as

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n,$$

where $a_1, a_2, \dots, a_n \in \mathbb{F}$.

We define the coordinate vector of V w.r.t. B as $[v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$.

Change of basis matrix:

Suppose $B_1 = \{v_1, v_2, \dots, v_n\}$

and $B_2 = \{w_1, w_2, \dots, w_n\}$

be two ordered bases for V.

How are $[v]_{B_1}$ & $[v]_{B_2}$ related?

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$
$$= b_1 w_1 + b_2 w_2 + \dots + b_n w_n$$