

Infinite Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order.

Mathematically, a sequence is a function from the set of natural numbers \mathbb{N} to the set of real numbers \mathbb{R} .

$$f : \mathbb{N} \rightarrow \mathbb{R}$$

$$f(n) = a_n$$

We denote a sequence by $\{a_1, a_2, a_3, \dots, a_n, \dots\}$

$$\text{or } \{a_n\}_{n=1}^{\infty}$$

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Sometimes we start the sequence from $n=2, n=3, \dots$

$$\text{e.g. } a_n = \frac{n}{\ln n}, n=2, 3, 4, \dots$$

We can rewrite the same sequence

$$\text{as } a_n = \frac{n+1}{\ln(n+1)}, n=1, 2, 3, \dots$$

Examples:

$$\{2^n\}_{n=1}^{\infty}, \{(-1)^n\}_{n=1}^{\infty}, \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}, \{c\}_{n=1}^{\infty}$$

↑ constant seq.

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Convergence of a sequence

Definition: A sequence $\{a_n\}$ is said to converge to L or $\lim_{n \rightarrow \infty} a_n = L$

if given any $\epsilon > 0$, there exists a corresponding natural number N s.t.

$|a_n - L| < \epsilon$ whenever $n > N$.

$$|a_n - L| < \epsilon \quad \text{whenever } n > N \quad (a_n = c \quad \forall n)$$

e.g. $\lim_{n \rightarrow \infty} c = c$ ($a_n = c \quad \forall n$)
 $\epsilon > 0$, $N = 1$. Then $|a_n - c| = 0 < \epsilon \quad \forall n \geq 1$
 $\therefore \lim_{n \rightarrow \infty} a_n = c$

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e.g. Show that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$

s.t. $|\frac{1}{n} - 0| < \epsilon$ whenever $n > N$.

If $n > N$, $\frac{1}{n} < \frac{1}{N}$

So, we choose N s.t. $\frac{1}{N} < \epsilon$ i.e. $N > \frac{1}{\epsilon}$

Theorem: The Sum Rule, Difference Rule, Constant Multiple Rule, Product Rule, Quotient Rule hold true for limits of sequences.

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Quotient rule:
 If $\lim_{n \rightarrow \infty} a_n = L$ & $\lim_{n \rightarrow \infty} b_n = M$,
 then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$, provided $M \neq 0$.

Sandwich theorem:
 If $b_n \leq a_n \leq c_n$ and $\lim_{n \rightarrow \infty} b_n = L$,
 $\lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} a_n = L$.

e.g. $a_n = \frac{\sin(n)}{n}$

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Since $-1 \leq \sin(n) \leq 1$,

$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \quad \forall n$$

By sandwich thm, $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$.

Notation: $a_n \rightarrow L$ means $\lim_{n \rightarrow \infty} a_n = L$

Example of a sequence which does not converge:

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- ① $a_n = (-1)^n$ does not converge to any L .
- ② $b_n = \sqrt{n}$ does not converge to any number L .

$$\lim_{n \rightarrow \infty} b_n = \infty$$

Defn: We say $\lim_{n \rightarrow \infty} a_n = \infty$ or $a_n \rightarrow \infty$ (an diverges to ∞) if given $M > 0$, $\exists N \in \mathbb{N}$ s.t. $a_n > M$ whenever $n > N$.

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Theorem: Let $a_n = f(n)$ for some function $f(x)$.

If $\lim_{x \rightarrow \infty} f(x) = L$, then

$$\lim_{n \rightarrow \infty} a_n = L.$$

Remark: L'Hopital's rule can be applied to sequences as well.

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Example: Show that $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$

Soln: Let $f(x) = \frac{\ln x}{x}$, $x \in (0, \infty)$

Then since $\ln x \rightarrow \infty$ and $x \rightarrow \infty$ as $x \rightarrow \infty$, by L'Hopital's rule,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0.$$

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Remark: The converse of the previous theorem is not true.

e.g. The sequence $a_n = \frac{\sin(n\pi)}{n} = 0 \forall n$

converges to 0 but
 $f(x) = \sin(\pi x)$ does not
 converge as $x \rightarrow \infty$.

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Example: Find $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1}\right)^n$

Let $f(x) = \left(\frac{x+1}{x-1}\right)^x$

Then $\ln f(n) = n \ln \left(\frac{n+1}{n-1}\right)$

$$= \frac{\ln \left(\frac{n+1}{n-1}\right)}{1/n} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\therefore \lim_{n \rightarrow \infty} \ln f(n) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} - \frac{1}{n-1}}{-1/n^2} = 2$$

show this

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$$\Rightarrow \lim_{n \rightarrow \infty} f(n) = e^2$$

$$\text{Since } a_n = f(n), \lim_{n \rightarrow \infty} a_n = e^2.$$

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