Example:

Evaluate $I = \iiint\limits_{\Omega} \frac{1}{\sqrt{1+x^2+y^2+z^2}}\, dx\,dy\,dz$,

where $\Omega : x^2+y^2+z^2 \le 1$.

Soln: We use the spherical coords.

$x = \rho \sin\theta \cos\phi$ $\qquad x^2+y^2+z^2 = \rho^2$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\theta$

$J = \rho^2 \sin\theta$.

$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{\rho=0}^{1} \frac{1}{\sqrt{1+\rho^2}} \cdot \rho^2 \sin\theta \, d\rho\, d\theta\, d\phi$

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$\Rightarrow I = 2\pi \int_0^1 \frac{\rho^2}{\sqrt{1+\rho^2}}\, d\rho \int_0^{\pi} \sin\theta\, d\theta$

$= 4\pi \int_0^1 \frac{\rho^2}{\sqrt{1+\rho^2}}\, d\rho$ $\qquad$ Put $\rho = \tan x$

$= 4\pi \int_0^{\pi/4} \frac{\tan^2 x}{\sec x} \sec^2 x\, dx$

$= 4\pi \left[\int_0^{\pi/4} \sec^3 x\, dx - \int_0^{\pi/4} \sec x\, dx \right]$

$\int \sec^3 x\, dx = \int \sec x \, d(\tan x)$

$= \sec x \tan x - \int \sec x \tan x . \tan x\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\Rightarrow \int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x| \right]$

$\therefore I = 4\pi \left[\frac{1}{\sqrt{2}} - \frac{1}{2} \ln(\sqrt{2}+1) \right]$.

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Problem: Evaluate $I = \iiint_\Omega x \, dV$,

where $\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x^2+y^2+z^2 \leq 4 \}$

Soln: We use the spherical coords.

$x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$, $z = \rho\cos\theta$

$dV = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$.

$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{2} \rho \sin\theta \cos\phi \cdot \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

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$$\therefore I = \left(\int_0^2 \rho^3 d\rho \right) \left(\int_0^{\pi/2} \sin^2\theta \, d\theta \right) \left(\int_0^{\pi/2} \cos\phi \, d\phi \right)$$

$$= 4 \times \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2} \right) d\theta \times 1$$

$$= 4 \left[\frac{\pi}{4} - \frac{\sin 2\theta}{4} \Big|_0^{\pi/2} \right]$$

$$= \pi$$

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Prob: Show that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for any } x>0$$

Soln: We know $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x

$$\Rightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

(We can use ratio test to show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges :

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0 \text{ as } n\to\infty \text{)}$$

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Prob: Find $\lim_{n\to\infty} (n!)^{1/n}$

Claim: $n! > \left(\frac{n}{e}\right)^n$

$$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} > \frac{n^n}{n!}$$

$$\Rightarrow n! > \frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n$$

$$\therefore (n!)^{1/n} > \frac{n}{e} \to \infty \text{ as } n\to\infty$$

$$\therefore \lim_{n\to\infty} (n!)^{1/n} = \infty.$$

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Prob: Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n\ n\ n\ \cdots n\cdot n}$$

$$= \underbrace{\left(\frac{n}{n}\right)}_{=1} \underbrace{\left(\frac{n-1}{n}\right)}_{<1} \underbrace{\left(\frac{n-2}{n}\right)}_{<1} \cdots \underbrace{\left(\frac{2}{n}\right)}_{<1} \cdot \frac{1}{n}$$

$$< \frac{1}{n}$$

By sandwich thm. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

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Prob: Does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

Sol: $a_n = \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\to \underbrace{\frac{1}{e}}_{<1} \text{ as } n\to\infty.$$

By ratio test $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

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