Infinite series

$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}+\cdots$

$1+(-1)+1+(-1)+1+(-1)+\cdots$

<u>Defn</u>: If $\{a_n\}$ is a sequence of real numbers, $\sum_{n=1}^{\infty} a_n$ is called an infinite series. $s_n = \sum_{k=1}^{n} a_k = a_1+a_2+\cdots+a_n$

$\{s_n\}$ is called the sequence of partial sums

$s_1 = a_1$
$s_2 = a_1+a_2$
$s_3 = a_1+a_2+a_3$
$\vdots$
$s_n = a_1+a_2+\cdots+a_n$
$\vdots$

If $s_n \to L \in \mathbb{R}$, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to L.

If $\{s_n\}$ does not converge, we say the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $s_n \to L$, we write
$$\sum_{n=1}^{\infty} a_n = L.$$

Examples:

① geometric series:

Let $a \neq 0$ and $r \in \mathbb{R}$.

Let $a_n = ar^{n-1}$.

i.e. $a_1 = a$, $a_2 = ar$, $a_3 = ar^2, \ldots$

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r} \quad \text{if } r \neq 1$$

If $r = 1$, $s_n = na \to \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$

$\therefore$ The series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges if $r = 1$.

If $r \neq 1$
$$s_n = \frac{a(1-r^n)}{(1-r)}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$\therefore$ $s_n \to \frac{a}{1-r}$ $\Rightarrow$ The series converges.

$$\therefore \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r| < 1$$

If $|r|>1$, then r^n diverges.
$(|r^n| \to \infty$ as $n\to\infty)$

$\therefore$ The series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges if $|r|>1$.

If $r=-1$, $S_n = \dfrac{a(1-(-1)^n)}{1-(-1)}$

$$= \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The series diverges.

Conclusion: For $a \neq 0$, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if and only if $\boxed{|r|<1}$ and in this case

$$S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

(2) $a_n = \frac{1}{n(n+1)}$, $n = 1, 2, 3, \ldots$

Does the series $\sum_{n=1}^{\infty} a_n$ converge?

$$a_n = \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore\ s_n = a_1 + a_2 + \cdots + a_n$$
$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} \to 1 - 0 = 1$$

$$\therefore\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(3) Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
$$= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> 2 \cdot \frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> 4 \cdot \frac{1}{8} = \frac{1}{2}} + \cdots$$

$$\therefore\ s_{2^n} > 1 + \frac{n}{2} \quad \forall n \geq 2.$$
$$\to +\infty \text{ as } n \to \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

④ $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$< 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

$$= 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$= 1 + 1 = 2$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.