Power Series

An infinite series of the form
$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$
where x is a variable and a_n's and c are constants, is called a power series centered at $x=c$.

This power series may converge or diverge for a given value of x.

. Definitely converges for $x=c$.

If $c=0$, the power series is
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Theorem: If a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges for some $x = x_0$, then it converges absolutely for all x such that $|x-c| < |x_0 - c|$.

Proof: Since $\sum_{n=0}^{\infty} a_n (x_0 - c)^n$ converges,
$$\lim_{n\to\infty} a_n (x_0-c)^n = 0.$$

$\therefore \{a_n(x_0-c)^n\}_{n=0}^{\infty}$ is a convergent sequence.

$\therefore \exists M$ s.t. $|a_n(x_0-c)^n| \le M \ \forall n$ — (i)

(Since convergent sequences are bounded)

Let $|x-c| < |x_0-c|$.

Then $\displaystyle\sum_{n=0}^{\infty} |a_n(x-c)^n| = \sum_{n=0}^{\infty} |a_n||x-c|^n$

$\displaystyle\le \sum_{n=0}^{\infty} \frac{M}{|x_0-c|^n}|x-c|^n$ [by (i)]

$\displaystyle= M \sum_{n=0}^{\infty} \left|\frac{x-c}{x_0-c}\right|^n$

Since $\left|\frac{x-c}{x_0-c}\right| < 1$, the series $\displaystyle\sum_{n=0}^{\infty} M\left|\frac{x-c}{x_0-c}\right|^n$ is convergent.

$\therefore$ By the comparison test, $\displaystyle\sum_{n=0}^{\infty} |a_n(x-c)^n|$ is convergent.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ converges for $x=1$ (but not absolutely convergent)

(Using alternating series test)

$\therefore$ By the previous theorem

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ converges absolutely for all x with $|x|<1$

Note that the series diverges for $x=-1$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ converges if and only if $x \in (-1, 1]$.

Radius of convergence:

$R \geq 0$ if called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ if the series converges for $|x-c|<R$ and diverges for $|x-c|>R$.

If the series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is convergent only when $x=c$, then $R=0$.

If the series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is convergent for all $x \in \mathbb{R}$, then we say $R=\infty$.

Examples :

① $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

Let $a_n = \frac{1}{n!} x^n$

Then $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$

$\therefore \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ for any x.

By the Ratio Test, the series converges for any x.

$\therefore R = \infty$

(2) $\sum_{n=0}^{\infty} n!\, x^n$

$|n!\, x^n| = n!\, |x|^n \to \infty$ as $n \to \infty$ for any $x \neq 0$.

$\therefore \sum_{n=0}^{\infty} n!\, x^n$ diverges if $x \neq 0$

Alternatively, use ratio test:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = (n+1)|x| \to \infty$ if $x \neq 0$

$\therefore$ The series diverges.

$\therefore$ Radius of convergences, $R = 0$.

(3) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

converges if $|x| < 1$

diverges if $|x| > 1$

$\therefore R = 1$

What happens at the end-points?

$x = 1$: $\sum_{n=0}^{\infty} 1$ diverges.

$x = -1$: $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Interval of convergence is $(-1, 1)$.

Theorem (Ratio test)
For a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, the radius of convergence is given $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.

Proof: Let $b_n = a_n (x-c)^n$

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |x-c|$$

The series $\sum_{n=0}^{\infty} b_n$ converges if

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| < 1$$

i.e. $\left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) |x-c| < 1$

i.e. $|x-c| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.

Also, diverges if $|x-c| > \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$\therefore \quad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.

Theorem (Root test)

The radius of convergence is given by $R = \dfrac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}}$,

provided the limit exists.

Example:

Find the radius of convergence for

(i) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^n}{n}$ $\quad R = 1$

(ii) $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ $\quad R = \infty$

(iii) $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$ $\quad R = \infty$

(Exercise).

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Theorem (Term by term differentiation)

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n .$$

Then the function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, defined on the interval $(c-R, c+R)$, is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

Similarly, we can do term by term integration.

Example:

① $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{if } |x| < 1$$

$$\therefore \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$\Rightarrow \ln(1+x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + C$$

Putting $x = 0$, gives $C = 0$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

② Series for $\tan^{-1}x$.

$$\tan^{-1}x = \int_0^x \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \quad \text{for } |x|<1$$

By term-by term integration,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$