Q. Does there exist an onto linear transf. $T: \mathbb{R}^2 \to \mathbb{R}^3$?

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transf.

Then by rank-nullity thm.,

$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^2) = 2$

$\Rightarrow \quad \text{rank}(T) \le 2$

$\Rightarrow \quad \dim(\text{range}(T)) \le 2 < \dim(\mathbb{R}^3)$

$\Rightarrow \quad \text{range}(T)$ is a proper subset of $\mathbb{R}^3$

$\Rightarrow \quad T$ is not onto.

In fact, more generally

Prop: There is no onto linear transf. $T: \mathbb{R}^n \to \mathbb{R}^m$ if $m > n$.

Similarly,

Prop: There is no one-to-one linear transf. $T: \mathbb{R}^n \to \mathbb{R}^m$ if $n > m$.

Thm: Let $T: V \to W$ be a linear transformation and let V & W be finite dimensional, and $\dim V = \dim W$. Then T is 1-1 iff T is onto.

Proof: T is 1-1
$\iff$ nullity$(T) = 0$
$\iff$ rank$(T) = \dim V$ (by rank-nullity theorem)
$= \dim W$
$\iff$ range$(T) = W$ ($\because W$ is finite dim.)
$\iff$ T is onto.

In particular if $T: V \to V$ is a linear transf. and $\dim V < \infty$, then T is 1-1 iff T is onto.

Remark: The above result is not true if V is infinite dimensional.

Example: Let V be the vector space of all real sequences. Let $T: V \to V$ be given by
$T(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots)$
Note that T is 1-1 but not onto.

$S: V \to V$

$S(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$

S is onto but not 1-1.