Indian Institute of Technology Delhi-Abu Dhabi
Course: AMTL 101 Part-II (Differential Equation)
Semester II, 2024-2025
Instructor: Prof. Amit Priyadarshi

Contents

1 Picard's Iteration Method

This method is used to find successive approximations to the unknown solution of a first-order initial value problem. Consider the IVP:

$$\frac{dy}{dx} = f(x, y) \; ; \quad y(x_0) = y_0 \tag{1}$$

This is equivalent to the following integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{2}$$

Integrating (1), we get

$$\int_{x_0}^{x} \frac{dy}{dx}\, dx = \int_{x_0}^{x} f(t, y(t))dt$$

$$\implies y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

$\therefore$ (1) $\implies$ (2). Now, differentiating (2), we get

$$\frac{dy}{dx} = 0 + f(x, y(x)) = f(x, y).$$

Also,

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t, y(t))dt = y_0,$$

$\therefore$ (2) $\implies$ (1). Since $y(t)$ is unknown, we can't use that in the integral, so we use $y(t) = y_0$ in the integral to get a function $y_1(x)$, i.e.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt.$$

Then we use $y(t) = y_1(t)$ in the integral to get the next approximation:

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt.$$

In general,

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

$y_1(x), y_2(x), \ldots, y_n(x), \ldots$ are called the Picard iterations. Note that all $y_n(x)$ satisfy the initial condition $y_n(x_0) = y_0$. But none of them may satisfy the ODE. However, under some conditions on $f(x, y)$, $y_n(x)$ converges to the unique solution $y(x)$ to the IVP.

Example 1. Consider the IVP $\frac{dy}{dx} = x + y$; $y(0) = 0$. Note that this can be solved easily to get $y(x) = e^x - x - 1$. Let's find the Picard iterations. Here, $f(x, y) = x + y$ & $x_0 = 0, y_0 = 0$.

$$\begin{aligned}
y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0) dt \\
&= 0 + \int_{0}^{x} f(t, 0) dt \\
\therefore y_1(x) &= \int_{0}^{x} t dt = \frac{x^2}{2}. \\
y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t)) dt \\
&= 0 + \int_{0}^{x} (t + y_1(t)) dt \\
&= \int_{0}^{x} \left(t + \frac{t^2}{2} \right) dt = \frac{x^2}{2} + \frac{x^3}{6}. \\
\text{Similarly,} \quad y_3(x) &= 0 + \int_{0}^{x} f(t, y_2(t)) dt \\
&= \int_{0}^{x} \left(t + \frac{t^2}{2} + \frac{t^3}{6} \right) dt = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}. \\
\text{By induction,} \quad y_n(x) &= \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}, \quad n \geq 1, \\
&= e^x - x - 1.
\end{aligned}$$

In practice, we can use $y_n(x)$ for large enough n to approximate the solution $y(x)$ to the given IVP when we are not able to find the actual solution.

2 Existence and Uniqueness of First-Order IVP

Some examples:

1. The IVP $|y'| + |y| = 0$; $y(0) = 1$ has no solutions because $|y'| + |y| = 0 \implies |y| = 0 \implies y = 0 \implies y(0) = 0 \neq 1$.

2. The IVP $y' = 2x$; $y(0) = 1$ has a unique solution $y = x^2 + 1$.

3. Consider the IVP $xy' = y - 1$; $y(0) = 1$, then $y = 1 + cx$ is a solution for every $c \in \mathbb{R}$.

Hence, an IVP may have no solutions, a unique solution, or more than one solution.

Theorem 1. ***(Sufficient conditions for existence of solutions)***
Consider the IVP: $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$. *Suppose the function* $f(x, y)$ *is continuous on a rectangle* $R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}$. *Assume* $|f(x, y)| \leq k \ \forall \ (x, y) \in R$ *for some constant* k. *Then the IVP has at least one solution* $y(x)$. *Also, the solution must be defined for* $x \in (x_0 - \alpha, x_0 + \alpha)$, *where* $\alpha = \min\{a, \frac{b}{k}\}$.

Remark 1. *The* α *that we get in the theorem need not be the maximum possible for a given IVP.*

Example 2. Consider the IVP $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$. Let's find the maximum possible α given by Theorem 1. Here, $f(x, y) = 1 + y^2$ is continuous everywhere. So, we can take a and b to be as large as we want.
If $R = \{(x, y) : |x| \leq a, |y| \leq b\}$, then

$$|f(x, y)| = 1 + y^2 \leq 1 + b^2 \ \forall \ (x, y) \in R.$$

So, $k = 1 + b^2$, then $\frac{b}{k} = \frac{b}{1+b^2} \leq \frac{1}{2}$,

$$\therefore \alpha = \min\left\{a, \frac{b}{k}\right\} \leq \frac{b}{k} \leq \frac{1}{2}.$$

So, the maximum value α that we get using the existence theorem is $\alpha = \frac{1}{2}$.
$\therefore$ Solution $y(x)$ must be defined in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
But, by solving the IVP, we get $y = \tan x$. The solution $y = \tan x$ is defined on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ which is bigger than $(-\alpha, \alpha)$.

Next question is about the uniqueness of solution. We will see that under an additional condition we can guarantee the uniqueness and existence of solution to the IVP.

Definition 1. ***(Lipschitz condition)*** *A function $f(x, y)$ of two variables is said to satisfy the "Lipschitz condition" on a region $R \subseteq \mathbb{R}^2$ if $\exists$ a constant M such that*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in R$.

Remark 2. *If $f(x, y)$ is continuous on R and $\frac{\partial f}{\partial y}$ is also continuous on R, then f satisfies the Lipschitz condition.*

Proof. Let $(x, y_1),\ (x, y_2) \in R$. Then by the mean value theorem,

$$f(x, y_1) - f(x, y_2) = \frac{\partial f}{\partial y}(x, y^*)\ \ (y_1 - y_2)$$

for some y^* between y_1 & y_2.

$$\implies |f(x, y_1) - f(x, y_2)| = \left|\frac{\partial f}{\partial y}(x, y^*)\right| |y_1 - y_2|\,,$$

since $\frac{\partial f}{\partial y}$ is continuous on R, $\exists\ M$ such that $\left|\frac{\partial f}{\partial y}\right| \leq M$ on R.

$$\implies |f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|.$$

$\therefore$ f satisfies the Lipschitz condition. □

The converse of the above remark is not true in general.

Example 3. Let $f(x, y) = x + |\sin y|$. Then f satisfies the Lipschitz condition on $\mathbb{R}^2$, because

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= \big||\sin y_1| - |\sin y_2|\big| \\ &\leq |\sin y_1 - \sin y_2| \\ &\leq |y_1 - y_2|. \end{aligned}$$

But, $\frac{\partial f}{\partial y}$ does not exist on the x-axis.

Theorem 2. ***(Existence and uniqueness)*** *Consider the IVP :*

$$\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0.$$

Suppose $f(x, y)$ is continuous on a rectangle $R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}$ and f satisfies the Lipschitz condition on R. Then the IVP has a unique solution $y(x)$ defined on some open interval containing $x = x_0$.

Example 4. (Non-uniqueness) Consider $\frac{dy}{dx} = \sqrt{|y|}$; $y(0) = 0$. Clearly, $y \equiv 0$ is a solution to the above IVP. Let

$$y = \begin{cases} \frac{x^2}{4}, & x \geq 0 \\ -\frac{x^2}{4}, & x < 0 \end{cases}$$

$$\implies \frac{dy}{dx} = \begin{cases} \frac{x}{2}, & x \geq 0 \\ -\frac{x}{2}, & x < 0 \end{cases}$$

Also, $\sqrt{|y|} = \sqrt{\frac{x^2}{4}} = \frac{|x|}{2} = \begin{cases} \frac{x}{2}, & x \geq 0 \\ -\frac{x}{2}, & x < 0 \end{cases}$,

$$\therefore \ \frac{dy}{dx} = \sqrt{|y|}.$$

Also, $y(0) = 0$, $\therefore$ $y(x)$ is also a solution to the IVP.

Example 5. Consider $\frac{dy}{dx} = \sqrt{|y|}$; $y(x_0) = y_0$. If $y_0 \neq 0$, then $f(x,y) = \sqrt{|y|}$ satisfies the Lipschitz condition on a small enough rectangle containing (x_0, y_0). Hence, the IVP has a unique solution if $y_0 \neq 0$.

Example 6. Consider the IVP $y\frac{dy}{dx} = x$; $y(0) = \beta$. Find the values of β for which the IVP has a unique solution, more than one solution, or no solution.

Solution Here, $f(x,y) = \frac{x}{y}$ is continuous for (x,y) except on the line $y = 0$. Also, $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$ is continuous except when $y = 0$. If $\beta \neq 0$, then the existence-uniqueness theorem guarantees a unique solution to the IVP. Here, we can find the solution as follows:

$$\begin{aligned} y\frac{dy}{dx} &= x \\ \implies \int y dy &= \int x dx \\ \implies \frac{y^2}{2} &= \frac{x^2}{2} + c \\ \because y(0) = \beta \implies c &= \frac{\beta^2}{2} \end{aligned}$$

$$\therefore \ y^2 = x^2 + \beta^2 \implies y = \pm\sqrt{x^2 + \beta^2}.$$

We have already seen that if $\beta \neq 0$, then the IVP has a unique solution. Since $y = \sqrt{x^2 + \beta^2} \implies y(0) = |\beta|$, $\therefore$ if $\beta > 0$, then $y = \sqrt{x^2 + \beta^2}$ is a solution but $y = -\sqrt{x^2+\beta^2}$ is not a solution to the IVP. Similarly, if $\beta < 0$, then $y = -\sqrt{x^2+\beta^2}$ is a solution to the IVP but $y = \sqrt{x^2+\beta^2}$ is not a solution to the IVP.

For $\beta = 0$, the existence-uniqueness theorem does not give us any information. However, $y = x$ and $y = -x$ are clearly solutions to the IVP: $y\frac{dy}{dx} = x \ ; \ y(0) = 0$. ■

Exercise 1. *Consider the IVP:*

$$(x^2 - 4x)\frac{dy}{dx} = (2x-4)y \ ; \ y(x_0) = y_0.$$

Find (x_0, y_0) for which the IVP has no solution, a unique solution, or more than one solution.

3 Laplace Transforms

Definition 2. *Let $f : [0,\infty) \to \mathbb{R}$ be any function. We define the Laplace transform of f as*

$$F(s) = (Lf)(s) = \int_0^\infty e^{-st} f(t)dt,$$

provided the improper integral converges.

Example 7. Let $f(t) = 1 \quad \forall\, t$, then

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}.1dt = \lim_{b\to\infty} \int_0^b e^{-st}dt \\ &= \lim_{b\to\infty} \frac{1}{s}\left(1 - e^{-bs}\right) \\ &= \frac{1}{s} \quad \text{for } s > 0, \end{aligned}$$

$$\therefore \ L(1)(s) = \frac{1}{s} \quad \text{for } s > 0.$$

Example 8. Let $f(t) = e^{at}$, then

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} e^{at} dt \\
&= \int_0^\infty e^{-(s-a)t} dt \\
&= \frac{1}{s-a} \quad \text{for } s > a,
\end{aligned}$$

$$\therefore \; L(e^{at})(s) = \frac{1}{s-a} \quad \text{for } s > a.$$

Example 9. Let $f(t) = t^n, \; n \in \mathbb{N}$, then

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} t^n dt \\
&= t^n \left. \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} dt \\
&= 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt, \quad \text{if} \quad s > 0, \\
&= \frac{n}{s} \; L(t^{n-1})(s)
\end{aligned}$$

$$\therefore \; L(t^n)(s) = \frac{n}{s} \; L(t^{n-1})(s).$$

By induction,

$$L(t^n)(s) = \frac{n!}{s^{n+1}}, \; s > 0.$$

3.1 Properties of Laplace Transforms

1. Linearity:

$$L\left(af(t) + bg(t)\right)(s) = aL(f(t))(s) + bL(g(t))(s).$$

Proof.

$$\begin{aligned}
L\left(af(t)+bg(t)\right)(s) &= \int_0^\infty e^{-st}(af(t)+bg(t))dt \\
&= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt \\
&= aL(f)(s)+bL(g)(s).
\end{aligned}$$

□

Using this, we can find the Laplace transform of any polynomial. For example,

$$\begin{aligned}
L(t^3-2t^2+t+3)(s) &= L(t^3)(s)-2L(t^2)(s)+L(t)(s)+3L(1)(s) \\
&= \frac{3!}{s^4}-2\left(\frac{2!}{s^2}\right)+\frac{1!}{s^2}+3.\frac{1}{s}.
\end{aligned}$$

Laplace transforms of hyperbolic sine and hyperbolic cosine functions:

We know that

$$\begin{aligned}
\sinh(at) &:= \frac{e^{at}-e^{-at}}{2} \\
\therefore\ L(\sinh(at))(s) &= \frac{1}{2}\left[L(e^{at})-L(e^{-at})\right] \\
&= \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]
\end{aligned}$$

$$\therefore\ L(\sinh(at))(s) = \frac{a}{s^2-a^2} \quad \text{if} \quad s>|a|.$$

Similarly,

$$\begin{aligned}
\cosh(at) &:= \frac{e^{at}+e^{-at}}{2} \\
\therefore\ L(\cosh(at))(s) &= \frac{1}{2}\left[L(e^{at})+L(e^{-at})\right] \\
&= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]
\end{aligned}$$

$$\therefore\ L(\cosh(at))(s) = \frac{s}{s^2-a^2} \quad \text{if} \quad s>|a|.$$

Laplace transforms of sine and cosine functions:

Let $f(t) = \cos(\omega t)$, then

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\cos(\omega t)dt \\
&= \frac{e^{-st}}{-s}\cos(\omega t)\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(-\omega)\sin(\omega t)dt \\
&= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)dt, \quad \text{if} \quad s > 0, \\
&= \frac{1}{s} - \frac{\omega}{s}\left[\frac{e^{-st}}{-s}\sin(\omega t)\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\omega\cos(\omega t)dt\right] \\
&= \frac{1}{s} - \frac{\omega^2}{s^2}F(s) \\
\implies \quad & \left(1 + \frac{\omega^2}{s^2}\right)F(s) = \frac{1}{s}
\end{aligned}$$

$$\therefore \; F(s) = L(\cos(\omega t))(s) = \frac{s}{s^2 + \omega^2}.$$

Similarly,

$$L(\sin(\omega t))(s) = \frac{\omega}{s^2 + \omega^2}.$$

3.2 Inverse Laplace Transforms

Definition 3. *If $F(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is said to be the inverse Laplace transform of $F(s)$, and is denoted by $L^{-1}(F)(t)$.*

Example 10. From example 9,

$$\begin{aligned}
L^{-1}\left(\frac{1}{s^n}\right) &= \frac{1}{(n-1)!}L^{-1}\left(\frac{(n-1)!}{s^n}\right) \\
&= \frac{t^{n-1}}{(n-1)!} \quad \text{for} \quad n \in \mathbb{N}.
\end{aligned}$$

Example 11.

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin(2t).$$

3.3 Some More Properties of Laplace Transforms

s-Shifting Property of Laplace Transforms:

Let $F(s) = L(f)(s) = \int_0^\infty e^{-st} f(t)dt$, then

$$\begin{aligned}
F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)dt \\
&= \int_0^\infty e^{-st}\left(e^{at} f(t)\right) dt \\
&= L(e^{at} f(t))(s)
\end{aligned}$$

$$\implies L(e^{at} f(t))(s) = L(f)(s-a).$$

Also,

$$L^{-1}(F(s-a)) = e^{at} f(t),$$

$$\therefore\ L^{-1}(F(s-a))(t) = e^{at} L^{-1}(F(s))(t).$$

Example 12. Find $L^{-1}\left(\frac{1}{s^2+2s+5}\right)$.

Solution

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+2^2}$$

Let $F(s) = \frac{1}{s^2+2^2}$, then from example 11,

$$\begin{aligned}
F(s) &= \frac{1}{2} L(\sin(2t))(s) \\
\implies F(s+1) &= \frac{1}{2} L(e^{-t}\sin(2t))(s) \\
&= L\left(\frac{1}{2} e^{-t}\sin(2t)\right)(s)
\end{aligned}$$

$$\therefore\ L^{-1}\left(\frac{1}{s^2+2s+5}\right) = L^{-1}(F(s+1))(t) = \frac{1}{2} e^{-t}\sin(2t).$$

■

Derivatives of Laplace Transforms

Let $F(s) = L(f)(s) = \int\limits_0^\infty e^{-st} f(t)dt$, then

$$\begin{aligned} F'(s) &= \frac{d}{ds}\left[\int\limits_0^\infty e^{-st} f(t)dt\right] \\ &= \int\limits_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]dt \\ &= \int\limits_0^\infty -te^{-st} f(t)dt \\ &= \int\limits_0^\infty e^{-st}[-tf(t)]dt \\ &= L(-tf(t))(s) \end{aligned}$$

$$\therefore\ F'(s) = L(-tf(t))(s) \quad \text{or} \quad L(tf(t))(s) = -F'(s).$$

$$\begin{aligned} \therefore\ L(t^2 f(t))(s) &= L(t(tf(t)))(s) \\ &= -\frac{d}{ds}\left[L(tf(t))(s)\right] \\ &= -\frac{d}{ds}(-F'(s)) \\ &= F''(s). \end{aligned}$$

In general,

$$L\left(t^n f(t)\right)(s) = (-1)^n F^{(n)}(s) \quad \text{for} \quad n = 1, 2, 3, \dots\ .$$

Example 13. Find $L(t\cos\omega t)$.

Solution We know that $L(\cos\omega t) = \frac{s}{s^2+\omega^2} = F(s)$, therefore,

$$\begin{aligned} L(t\cos\omega t) &= -F'(s) \\ &= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) \\ &= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}. \end{aligned}$$

■

Laplace Transform of Derivatives

$$\begin{aligned}
L(f')(s) &= \int_0^\infty e^{-st} f'(t)dt \\
&= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)dt \\
&= 0 - f(0) + s\int_0^\infty e^{-st}f(t)dt \\
&= -f(0) + sF(s)
\end{aligned}$$

$$\therefore\ L(f')(s) = sL(f)(s) - f(0).$$

$$\begin{aligned}
\implies L(f'')(s) &= sL(f')(s) - f'(0) \\
&= s[sL(f)(s) - f(0)] - f'(0) \\
\therefore\ L(f'')(s) &= s^2L(f)(s) - sf(0) - f'(0)
\end{aligned}$$

In general,

$$L(f^{(n)})(s) = s^nL(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \quad \forall\ n \in \mathbb{N}.$$

Laplace Transform of Integrals

Let

$$g(t) = \int_0^t f(\tau)d\tau,$$

then $g'(t) = f(t)$ and $g(0) = 0$.

$$\begin{aligned}
\therefore\ L(g')(s) &= L(f)(s) \\
\implies sL(g)(s) - g(0) &= F(s) \\
\implies L(g)(s) &= \frac{F(s)}{s}
\end{aligned}$$

$$\therefore\ L\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}.$$

Example 14. Find $L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)$.

Solution Let $F(s) = \frac{1}{s^2+\omega^2} = \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}L(\sin\omega t)(s)$,

$$\implies f(t) = \frac{1}{\omega}\sin(\omega t)$$

$$\therefore\ L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau$$

$$= \int_0^t \frac{1}{\omega}\sin(\omega\tau)d\tau$$

$$\therefore\ L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = \frac{1}{\omega^2}(1-\cos\omega t).$$

■

3.4 Application of Laplace Transforms to Solve IVPs

Example 15. Solve $y'' - y = t;\ y(0) = 1,\ y'(0) = 1.$

Solution Taking the Laplace transform,

$$\begin{aligned}
&L(y'') - L(y) = L(t)\\
&\Longrightarrow [s^2L(y) - sy(0) - y'(0)] - L(y) = \frac{1}{s^2}\\
&\Longrightarrow (s^2-1)L(y)(s) - s - 1 = \frac{1}{s^2}\\
&\Longrightarrow (s^2-1)L(y)(s) = \frac{1}{s^2} + s + 1\\
&\Longrightarrow L(y)(s) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}\\
&\qquad\qquad = \frac{1}{s^2-1} - \frac{1}{s^2} + \frac{1}{s-1}
\end{aligned}$$

Taking the inverse Laplace transform L^{-1},

$$\begin{aligned}
y(t) &= \sinh t - t + e^t\\
&= \frac{3}{2}e^t - \frac{1}{2}e^{-t} - t.
\end{aligned}$$

■

3.5 Heaviside Function (Unit Step Function)

Define

$$u(t) = \begin{cases} 0\ , & t \le 0 \\ 1\ , & t > 0 \end{cases}$$

So, for any $a > 0$,

$$u_a(t) = \begin{cases} 0\ , & t \le a \\ 1\ , & t > a \end{cases}$$
$$= u(t-a).$$

$$\therefore\ L(u(t-a)) = \int_0^\infty e^{-st} u(t-a)dt$$
$$= \int_a^\infty e^{-st} dt$$
$$= \frac{e^{-as}}{s}.$$

Let

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if} \quad t \le a \\ f(t-a) & \text{if} \quad t > a \end{cases}$$

Then

$$L(\tilde{f})(s) = \int_0^\infty e^{-st} \tilde{f}(t)dt$$
$$= \int_a^\infty e^{-st} f(t-a)dt$$
$$= e^{-as} \int_a^\infty e^{-s(t-a)} f(t-a)dt$$
$$= e^{-as} \int_0^\infty e^{-s\tau} f(\tau)d\tau$$
$$= e^{-as} L(f)(s)$$

$$\therefore\ L(f(t-a)u(t-a)) = e^{-as} L(f).$$

Hat Function:

For $0 \le a < b$,

$$H(t) := \begin{cases} 1 & , \quad a < t < b \\ 0 & , \quad \text{otherwise} \end{cases}$$

i.e.

$$H(t) = u(t-a) - u(t-b).$$

Example 16. Let

$$f(t) = \begin{cases} 2 & , \quad 0 < t < 1 \\ \frac{t^2}{2} & , \quad 1 < t < \frac{\pi}{2} \\ \cos t & , \quad t > \frac{\pi}{2} \end{cases} .$$

Find $L(f)(s)$.

Solution First we express $f(t)$ in terms of the Heaviside function.

$$\begin{aligned} f(t) &= \begin{cases} 2 & , \quad 0 < t < 1 \\ \frac{t^2}{2} & , \quad 1 < t < \frac{\pi}{2} \\ \cos t & , \quad t > \frac{\pi}{2} \end{cases} \\ &= 2[1 - u(t-1)] + \frac{t^2}{2}\left[u(t-1) - u\left(t - \frac{\pi}{2}\right)\right] + \cos t \; u\left(t - \frac{\pi}{2}\right) \\ \therefore \; L(f)(s) &= 2(L(1) - L(u(t-1))) + \frac{1}{2}L(t^2 u(t-1)) - \frac{1}{2}L\left(t^2 u\left(t - \frac{\pi}{2}\right)\right) \\ &\quad + L\left(\cos t \; u\left(t - \frac{\pi}{2}\right)\right) \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} t^2 u(t-1) &= [(t-1)+1]^2 u(t-1) \\ &= (t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1) \\ \therefore \; L(t^2 u(t-1)) &= L((t-1)^2 u(t-1)) + 2L((t-1)u(t-1)) + L(u(t-1)) \\ &= e^{-s}L(t^2) + 2e^{-s}L(t) + \frac{e^{-s}}{s} \\ &= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s^2} + \frac{e^{-s}}{s} \end{aligned}$$

Similarly,

$$\begin{aligned} t^2 u\left(t - \frac{\pi}{2}\right) &= \left[\left(t - \frac{\pi}{2}\right) + \frac{\pi}{2}\right]^2 u\left(t - \frac{\pi}{2}\right) \\ &= \left(t - \frac{\pi}{2}\right)^2 u\left(t - \frac{\pi}{2}\right) + \pi\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right) + \frac{\pi^2}{4} u\left(t - \frac{\pi}{2}\right) \\ \therefore \; L\left(t^2 u\left(t - \frac{\pi}{2}\right)\right) &= e^{-\frac{\pi}{2}s}\frac{2}{s^3} + \pi e^{-\frac{\pi}{2}s}\frac{1}{s^2} + \frac{\pi^2}{4}e^{-\frac{\pi}{2}s}\frac{1}{s} \end{aligned}$$

Also,

$$
\begin{aligned}
\cos t \; u\left(t-\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}-t\right) u\left(t-\frac{\pi}{2}\right) \\
&= -\sin\left(t-\frac{\pi}{2}\right) u\left(t-\frac{\pi}{2}\right) \\
\therefore\; L\left(\cos t \; u\left(t-\frac{\pi}{2}\right)\right) &= -L\left(\sin\left(t-\frac{\pi}{2}\right) u\left(t-\frac{\pi}{2}\right)\right) \\
&= -e^{-\frac{\pi}{2}s} L(\sin t)(s) \\
&= -e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}.
\end{aligned}
$$

■

3.6 Dirac Delta Function

For $k \in \mathbb{N}$, let

$$
f_k(t-a) = \begin{cases} k & , \quad a \le t \le a + \frac{1}{k} \\ 0 & , \quad \text{otherwise} \end{cases}
$$

Note that

$$
\int_0^\infty f_k(t-a)dt = 1 \quad \forall\; k.
$$

Define the Dirac-delta function as

$$
\begin{aligned}
\delta(t-a) &= \lim_{k\to\infty} f_k(t-a) \\
&= \begin{cases} \infty & , \quad t = a \\ 0 & , \quad t \ne a \end{cases}
\end{aligned}
$$

Properties:

1. $\int\limits_0^\infty \delta(t-a)dt = 1.$

2. Note that

$$\begin{aligned}
f_k(t-a) &= \begin{cases} k & , \quad a \le t \le a + \frac{1}{k} \\ 0 & , \quad \text{otherwise} \end{cases} \\
&= k\left[u(t-a) - u\left(t - \left(a + \frac{1}{k}\right)\right)\right] \\
\therefore \; L(f_k(t-a)) &= k\left[L(u(t-a)) - L\left(u\left(t - \left(a + \frac{1}{k}\right)\right)\right)\right] \\
&= k\left[\frac{e^{-as}}{s} - \frac{e^{-\left(a+\frac{1}{k}\right)s}}{s}\right] \\
&= e^{-as}\left[\frac{1 - e^{-\frac{s}{k}}}{\frac{s}{k}}\right] \\
\therefore \; L(\delta(t-a)) &= \lim_{k\to\infty} L(f_k(t-a)) \\
&= \lim_{k\to\infty} e^{-as}\left[\frac{1 - e^{-\frac{s}{k}}}{\frac{s}{k}}\right] \\
&= e^{-as}
\end{aligned}$$

$$\therefore \; L(\delta(t-a)) = e^{-as}.$$

Example 17. Solve the IVP:

$$y'' + 3y' + 2y = \delta(t-1), \; y(0) = 0, y'(0) = 0.$$

Solution Taking the Laplace transform, we get

$$L(y'') + 3L(y') + 2L(y) = L(\delta(t-1))$$

$$\begin{aligned}
&\Longrightarrow [s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-s} \\
&\Longrightarrow (s^2 + 3s + 2)Y(s) = e^{-s}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \; Y(s) &= e^{-s}.\frac{1}{(s+1)(s+2)} \\
&= \frac{e^{-s}}{s+1} - \frac{e^{-s}}{s+2} \\
\Longrightarrow \; y(t) &= L^{-1}\left[\frac{e^{-s}}{s+1}\right] - L^{-1}\left[\frac{e^{-s}}{s+2}\right] \\
&= L^{-1}[e^{-s}L(e^{-t})(s)] - L^{-1}[e^{-s}L(e^{-2t})(s)] \\
&= e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)
\end{aligned}$$

$$\therefore\ y(t) = \begin{cases} 0 & , \quad t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & , \quad t > 1 \end{cases}.$$

■

3.7 Convolution

Question: Is $L(fg) = L(f)L(g)$?

Example 18. Let $f = 1;\ g = 1$, then $L(f) = \frac{1}{s} = L(g)$. Also, $L(fg) = L(1) = \frac{1}{s}$,

$$\therefore\ L(fg) \neq L(f)L(g).$$

Example 19. Let $f = e^t;\ g = 1$, then $L(f) = \frac{1}{s-1};\ L(g) = \frac{1}{s}$, but $L(fg) = L(e^t) = \frac{1}{s-1} \neq L(f)L(g)$.

If $L^{-1}(F(s)) = f(t)$ & $L^{-1}(G(s)) = g(t)$, then we have seen by the above examples that $L^{-1}(F(s)G(s)) \neq f(t)g(t)$. Let

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau$$

$$G(s) = \int_0^\infty e^{-su} g(u) du$$

Put $u = t - \tau$, then $du = dt$.

$$\implies G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau) dt.$$

Multiplying, we get

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} f(\tau) d\tau \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau) dt \\ &= \int_0^\infty \left(e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau) dt \right) d\tau \\ &= \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau) dt\ d\tau \qquad (*) \end{aligned}$$

This is a double integral over the region R given below :

$$\begin{aligned} R &= \{(t,\tau):\ \tau \le t < \infty,\ 0 < \tau < \infty\} \\ &= \{(t,\tau):\ 0 \le \tau \le t,\ 0 \le t < \infty\} \end{aligned}$$

Changing the order of integrals in $(*)$, we get

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-st} \left(\int_0^t f(\tau)g(t-\tau)d\tau \right) dt \\ &= L\left((f*g)(t)\right)(s), \end{aligned}$$

where

$$(f*g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau.$$

Definition 4.

$$(f*g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$$

is called the convolution of f and g.

Exercise 2. *Use the above formula for $f*g$ and show that $L(f*g) = L(f)L(g)$. Hence,*

$$L^{-1}(F(s)G(s)) = L^{-1}(F(s)) * L^{-1}(G(s)).$$

Example 20. Find $L^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right)$.

Solution We know that $L^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t)$,

$$\begin{aligned} \therefore\ L^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) &= \frac{1}{\omega}\sin(\omega t) * \frac{1}{\omega}\sin(\omega t) \\ &= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin\omega(t-\tau)d\tau \\ &= \frac{1}{2\omega^2}\int_0^t [\cos(2\omega\tau-\omega t) - \cos(\omega t)]d\tau \\ \Longrightarrow\ L^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) &= \frac{1}{2\omega^3}[\sin(\omega t) - \omega t\cos(\omega t)]. \end{aligned}$$

$\blacksquare$

Properties of Convolution:

1. $f * g = g * f$.

   Proof. $(f * g)(t) = \int\limits_0^t f(\tau)g(t-\tau)d\tau$.

   Put $t - \tau = u \implies \tau = t - u \implies d\tau = -du$, and when $\tau = 0,\ u = t$; when $\tau = t,\ u = 0$,

$$\begin{aligned}\therefore\ (f * g)(t) &= \int_t^0 f(t-u)g(u)(-du) \\ &= \int_0^t g(u)f(t-u)du \\ &= (g * f)(t).\end{aligned}$$

   □

2. $f * (g + h) = f * g + f * h$.

3. $(f * g) * h = f * (g * h)$.

4. $f * 0 = 0$, where 0 denotes the zero function.

5. $f * 1 \neq f$ (for example let $f(t) = t$).

6. $(f * f)(t)$ need not be a non-negative function. Let $f(t) = \sin t$, then

$$(f * f)(t) = \frac{1}{2}[\sin t - t\cos t].$$

Example 21. Solve $y'' + 3y' + 2y = r(t),\ y(0) = 0, y'(0) = 0$, where

$$r(t) = \begin{cases} 1 & , \quad 1 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Solution **Method 1:** Observe that $r(t) = u(t-1) - u(t-2)$. Take the Laplace transform and find $Y(s)$. Then take the inverse Laplace transform to get $y(t)$.

Method 2: (Using convolution) Taking the Laplace transform, we get

$$
\begin{aligned}
&L(y'') + 3L(y') + 2L(y) = L(r)\\
\implies &[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = R(s)\\
\implies &(s^2 + 3s + 2)Y(s) = R(s)\\
\implies &Y(s) = \frac{1}{(s+1)(s+2)}.R(s)
\end{aligned}
$$

Now,

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} = L(e^{-t} - e^{-2t}).$$

Taking the inverse Laplace transform, we get

$$
\begin{aligned}
y(t) &= (r * q)(t); \quad \text{where} \quad q(t) = e^{-t} - e^{-2t},\\
&= \int_0^t r(\tau)q(t-\tau)d\tau
\end{aligned}
$$

For $0 < t < 1$, $r(t) = 0$,

$$\therefore\ y(t) = \int_0^t 0.q(t-\tau)d\tau = 0.$$

For $1 < t < 2$,

$$
\begin{aligned}
y(t) &= \int_1^t 1.\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau\\
&= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}.
\end{aligned}
$$

For $t > 2$,

$$
\begin{aligned}
y(t) &= \int_1^2 1.\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau\\
&= e^{-(t-2)} - \frac{e^{-2(t-2)}}{2} - e^{-(t-1)} + \frac{e^{-2(t-1)}}{2}.
\end{aligned}
$$

$\blacksquare$