Ordinary Differential Equations (ODE)

What is an ODE ?

$$F(x, y, y', \dots, y^{(n)}) = 0,$$

where x is an indep. variable,

$y = y(x)$ is a fn. of x,

y' = 1st derivative of y

$\vdots$

$y^{(n)}$ = nth derivative of y.

This is an ODE of order n.

First order ODE :

$F(x, y, y') = 0 \rightarrow$ Implicit form

$y' = f(x, y) \rightarrow$ Explicit form.

Solution of an ODE: A solution to an ODE is a function defined on some interval and satisfies the ODE.

(For nth order ODE, a solution must be at least n-times differentiable)

$y = h(x)$, $x \in I \leftarrow$ open interval.

Q. Does every ODE have a solution?

Ans No

$\left(\frac{dy}{dx}\right)^2 + 1 = 0$ has no (real) solutions

Q. If an ODE has a solution, is it unique?

Ans: No.

$\frac{dy}{dx} = 0$ has $y = c$ as a solution for every $c \in \mathbb{R}$.

Initial value problem (IVP)

An IVP is an ODE: $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ together with initial conditions $y(x_0) = y_0$, $y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$.

First order IVP:

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

Important questions:

(i) Existence of solutions of an IVP.

(ii) Uniqueness of solutions.

Methods of solving 1st order ODEs

(1) Separable ODEs: Suppose the ODE can be written as

$$g(y)\frac{dy}{dx} = h(x)$$

i.e. $g(y)\,dy = h(x)\,dx$

Integrating both sides, we get

$$\int g(y)\,dy = \int h(x)\,dx + C$$

- Sometimes the ODE can be converted into a separable ODE by substitution, for example, $u = \frac{y}{x}$.

e.g. $$\frac{dy}{dx} = \frac{x+2y}{2x+y} = \frac{1+2\frac{y}{x}}{2+\frac{y}{x}}$$

Put $u = \frac{y}{x}$, i.e. $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$\therefore$ $$u + x\frac{du}{dx} = \frac{1+2u}{2+u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{1-u^2}{2+u}$$

$$\Rightarrow \int \frac{2+u}{1-u^2}\,du = \int \frac{dx}{x}$$

② Linear 1st order ODE:

$$\frac{dy}{dx} + p(x)\, y = g(x)$$

Homogeneous: if $g(x) = 0$

Non-homogeneous: if $g(x) \neq 0$

How to solve it?

We multiply the ODE by an integrating factor $\mu(x) = e^{\int p(x)dx}$:

$$\underbrace{e^{\int p(x)dx} \frac{dy}{dx} + e^{\int p(x)dx} \cdot p(x) y}_{\frac{d}{dx}[\mu(x)\, y]} = \mu(x) g(x)$$

$$\Rightarrow \quad \mu(x)\, y = \int \mu(x) g(x) dx + C$$

$$\Rightarrow \quad y = \frac{1}{\mu(x)} \left[\int \mu(x) g(x)\, dx + C \right]$$

- 1st order linear ODEs always have solutions provided $p(x)$ and $g(x)$ are continuous on an interval I. In that case, the solutions must exist on interval I.

③ Exact ODEs :

Consider 1st order ODE of the form:

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

This is called an "exact" ODE if

$$M(x,y)\,dx + N(x,y)\,dy = d(u(x,y))$$

for some $u(x,y)$.

Then, we have $d(u(x,y)) = 0$

which gives $u(x,y) = c$

as solutions for any constant c.

e.g.: $y\,dx + x\,dy = 0$

$$y\,dx + x\,dy = d(xy)$$

∴ The above eqn. is exact

and $xy = c$ gives the soln.

Conditions for exactness:

Necessary condition:

By the chain rule,

$$\frac{d}{dx}\left[u(x,y)\right] = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

i.e. $d(u(x,y)) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

If $M = \frac{\partial u}{\partial x}$ & $N = \frac{\partial u}{\partial y}$,

then the ODE $M\,dx + N\,dy$ is exact.

In this case,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \& \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

So, $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ "exactness condition"

We'll see that this is also a sufficient condition.