Indian Institute of Technology Delhi - Abu Dhabi

AMTL101

Tutorial Sheet 11: Laplace Transforms

---

(1) Find the Laplace transforms of the following functions.

$$\cos^2 \omega t,\ e^t \cosh 3t,\ \sin 2t \cos 2t,\ e^{-\alpha t} \cos \beta t.$$

(2) Find the inverse Laplace transforms of the following functions.

$$\frac{5s}{s^2-25},\ \frac{1-7s}{(s-3)(s-1)(s+2)},\ \frac{2s^3}{s^4-1},\ \frac{2}{s^2+s+\frac{1}{2}}.$$

(3) Solve the following IVPs using Laplace transform.

(a) $y'' - y' - 2y = 0;\ y(0) = 8, y'(0) = 7.$

(b) $y'' + 2y' - 3y = 6e^{-2t};\ y(0) = 2, y'(0) = -14.$

(4) Find the Laplace transforms of the following functions (u is the unit step function).

$$tu(t-1),\ e^{-2t}u(t-3),\ 4u(t-\pi)\cos t.$$

(5) Find the inverse Laplace transforms of the following functions.

$$\frac{e^{-3s}}{s^3},\ \frac{3(1-e^{-\pi s})}{s^2+9},\ \frac{se^{-2s}}{s^2+\pi^2}.$$

(6) Solve the following IVPs using Laplace transform.

(a) $y'' + 6y' + 8y = e^{-3t} - e^{-5t};\ y(0) = 0, y'(0) = 0.$

(b) $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1;\ y(0) = 0, y'(0) = 0.$

(c) $y'' + 4y' + 5y = \delta(t-1);\ y(0) = 0, y'(0) = 3.$

(7) Find inverse Laplace transforms of the following functions.

$$\ln\left(\frac{s+2}{s+3}\right),\ \cot^{-1}\frac{s}{\pi}.$$

(8) Compute convolution of the following functions.

$$1 * \sin t,\ \cos 2t * \sin 2t,\ u(t-1) * t^2,\ u(t-3) * e^{2t}.$$

(9) Solve the following integral equations using Laplace transform.

(a) $y(t) = 2t - 4\int_0^t y(\tau)(t-\tau)d\tau$

(b) $y(t) = 1 - \sinh t + \int_0^t (1+\tau)y(t-\tau)d\tau$

Solution 1 :

(a) $\cos^2 \omega t = \frac{1+\cos 2\omega t}{2}$

$$\Rightarrow L(\cos^2 \omega t) = \frac{1}{2}\left[L(1) + L(\cos 2\omega t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4\omega^2}\right]$$

(b) $e^t \cosh(3t)$

$$\because \quad L(e^{at} f(t)) = F(s-a)$$

$$\& \quad L(\cosh 3t) = \frac{s}{s^2-9}$$

$$\Rightarrow L(e^t \cosh 3t) = \frac{s-1}{(s-1)^2-9}$$

(c) $\sin 2t \cos 2t = \frac{1}{2}\sin 4t$

$$\Rightarrow L(\sin 2t \cos 2t) = \frac{1}{2} L(\sin 4t)$$

$$= \frac{1}{2}\,\frac{4}{s^2+4^2}$$

(d) $e^{-\alpha t} \cos \beta t$

$$\because \ L(\cos \beta t) = \frac{s}{s^2+\beta^2} = F(s)$$

$$\Rightarrow L(e^{-\alpha t} \cos \beta t) = F(s-(-\alpha))$$

$$= \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

Solution 2:

(a) $$L^{-1}\left[\frac{5s}{s^2-25}\right] = 5L^{-1}\left[\frac{s}{s^2-5^2}\right]$$

$$= 5 \cosh(5t)$$

(b) $$L^{-1}\left[\frac{1-7s}{(s-3)(s-1)(s+2)}\right]$$

$$\frac{1-7s}{(s-3)(s-1)(s+2)} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\Rightarrow 1-7s = A(s-1)(s+2) + B(s-3)(s+2) + C(s-3)(s-1)$$

$$= A[s^2+s-2] + B[s^2-s-6] + C[s^2-4s+3]$$

$$= (A+B+C)s^2 + (A-B-4C)s$$
$$-2A-6B+3C$$

$$\Rightarrow \quad A+B+C = 0$$
$$A-B-4C = -7$$
$$-2A-6B+3C = 1$$

$$\Rightarrow \quad A = -2, \quad B = 1, \quad C = 1$$

$$\therefore \quad L^{-1}\left[\frac{1-7s}{(s-3)(s-1)(s+2)}\right]$$

$$= (-2)\, L^{-1}\left(\frac{1}{s-3}\right) + L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{1}{s+2}\right)$$

$$= -2e^{3t} + e^{t} + e^{-2t}$$

(c) $L^{-1}\left(\frac{2s^3}{s^4-1}\right)$

$$\because \quad \frac{2s^3}{s^4-1} = \frac{2s^3}{(s^2-1)(s^2+1)}$$

$$= \frac{s}{s^2-1} + \frac{s}{s^2+1}$$

$$\Rightarrow \quad L^{-1}\left(\frac{2s^3}{s^4-1}\right) = L^{-1}\left(\frac{s}{s^2-1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= \cosh t + \cos t .$$

(d) $$L^{-1}\left(\frac{2}{s^2+s+1/2}\right)$$

$$= L^{-1}\left(\frac{2}{\left(s+1/2\right)^2 + \left(\frac{1}{2}\right)^2}\right)$$

$$\because \quad L\left(\sin \tfrac{1}{2}t\right) = \frac{1/2}{s^2 + \left(1/2\right)^2}$$

$$\Rightarrow \quad L\left(e^{-1/2\, t} \sin \tfrac{1}{2}t\right) = \frac{1/2}{\left(s+1/2\right)^2 + \left(1/2\right)^2}$$

$$\therefore \quad L^{-1}\left(\frac{2}{s^2+s+1/2}\right) = 4\, e^{-1/2\, t} \sin \tfrac{1}{2}t .$$

Solution 3 : (b) $y'' + 2y' - 3y = 6e^{-2t}$

$$y(0) = 2, \quad y'(0) = -14$$

$$L(y'') + 2L(y') - 3L(y) = 6L(e^{-2t})$$

$$\Rightarrow \left[s^2 Y(s) - s \cdot y(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right]$$

$$-3Y(s) = 6L(e^{-2t})$$

$$\Rightarrow (s^2 + 2s - 3)Y(s) - 2s + 14 - 4 = \frac{6}{s+2}$$

$$\Rightarrow (s^2 + 2s - 3)Y(s) = \frac{6}{s+2} + 2s - 10$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)(s+3)} \cdot \frac{6}{(s+2)} + \frac{2(s-5)}{(s-1)(s+3)}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{6}{(s-1)(s+3)(s+2)}\right] + L^{-1}\left[\frac{2(s-5)}{(s-1)(s+3)}\right]$$

now, $$\frac{6}{(s-1)(s+3)(s+2)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$\Rightarrow \quad 6 = A(s+3)(s+2) + B(s-1)(s+2) + C(s-1)(s+3)$$

$$\Rightarrow \quad 6 = A[s^2+5s+6] + B[s^2+s-2] + C[s^2+2s-3]$$

$$\Rightarrow \quad A+B+C = 0$$

$$5A+B+2C = 0$$

$$6A-2B-3C = 6$$

$$\Rightarrow \quad A = 1/2, \; B = 3/2, \; C = -2$$

$$\Rightarrow L^{-1}\left[\frac{6}{(s-1)(s+3)(s+2)}\right] = \frac{1}{2}e^{t} + \frac{3}{2}e^{-3t} - 2e^{-2t}$$

Also, $$\frac{2(s-5)}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

$$\Rightarrow \quad 2s-10 = A(s+3) + B(s-1)$$

$$= (A+B)s + 3A - B$$

$$\Rightarrow \quad A+B = 2 \; ; \quad 3A-B = -10$$

$$\Rightarrow \quad A = -2, \quad B = 4$$

$$\Rightarrow \quad L^{-1}\left[\frac{2(s-5)}{(s-1)(s+3)}\right] = -2e^{t} + 4e^{-3t}$$

$$\therefore \quad y(t) = -\tfrac{3}{2}e^{t} + \frac{11}{2}e^{-3t} - 2e^{-2t}.$$

Solution 4 : (a) $t\,u(t-1)$

$$\because \quad L(u(t-1)) = \frac{e^{-s}}{s}$$

and $\quad L(t f(t)) = -F'(s)$

$$\Rightarrow \quad L(t\,u(t-1)) = -\frac{-se^{-s} - e^{-s}}{s^2}$$

$$= \frac{(s+1)\,e^{-s}}{s^2}$$

(b) $L(e^{-2t}u(t-3))$

$$\because \quad L(u(t-3)) = \frac{e^{-3s}}{s}$$

and $\quad L(e^{at}f(t)) = F(s-a)$

$$\Rightarrow \quad L(e^{-2t}u(t-3)) = \frac{e^{-3(s+2)}}{s+2}$$

(c) $\quad 4\,u(t-\pi)\cos t$

$$= -4\,u(t-\pi)\cos(\pi-t)$$

$$= -4\,u(t-\pi)\cos(t-\pi)$$

$$\because \quad L(f(t-a)\,u(t-a)) = e^{-as}L(f)(s)$$

$$\Rightarrow \quad L(4\,u(t-\pi)\cos t) = -4L(u(t-\pi)\cos(t-\pi))$$

$$= -4e^{-\pi s}\cdot\frac{s}{s^2+1}$$

Solution 5 :

(a) $$L^{-1}\left(\frac{e^{-3s}}{s^3}\right) = \frac{1}{2}L^{-1}\left(e^{-3s}\cdot\frac{2}{s^3}\right)$$

$$= \frac{1}{2}\,u(t-3)\cdot(t-3)^2$$

(b) $\frac{3(1-e^{-\pi s})}{s^2+9} = \frac{3}{s^2+9} - \frac{3}{s^2+9}e^{-\pi s}$

$$L^{-1}\left(\frac{3}{s^2+3^2}\right) = \sin 3t$$

& $L^{-1}\left(e^{-\pi s}\cdot\frac{3}{s^2+3^2}\right) = L^{-1}\left(e^{-\pi s}\cdot L(\sin 3t)\right)$

$= u(t-\pi)\sin 3(t-\pi)$

$= -u(t-\pi)\sin(3\pi-3t)$

$= -u(t-\pi)\sin 3t$

$\Rightarrow L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = \sin 3t\left(1+u(t-\pi)\right)$

(c) $L^{-1}\left(\frac{s\,e^{-2s}}{s^2+\pi^2}\right) = L^{-1}\left(e^{-2s}\cdot L(\cos \pi t)\right)$

$= u(t-2)\cos\pi(t-2)$

$= u(t-2)\cos\pi t.$

Solution 6 :

(a) $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0 = y'(0)$

$$L(y'') + 6L(y') + 8L(y) = L(e^{-3t}) - L(e^{-5t})$$

$$\Rightarrow \left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 6\left[s Y(s) - y(0)\right]$$

$$+ 8Y(s) = \frac{1}{s+3} - \frac{1}{s+5}$$

$$\Rightarrow \left[s^2 + 6s + 8\right] Y(s) = \frac{1}{s+3} - \frac{1}{s+5}$$

$$\Rightarrow Y(s) = \frac{1}{(s+2)(s+4)(s+3)} - \frac{1}{(s+2)(s+4)(s+5)}$$

Now, $$\frac{1}{(s+2)(s+4)(s+3)} = \frac{A}{s+2} + \frac{B}{s+4} + \frac{C}{s+3}$$

$$\Rightarrow 1 = A(s+4)(s+3) + B(s+2)(s+3)$$

$$+ C(s+2)(s+4)$$

$$\Rightarrow 1 = A(s^2 + 7s + 12) + B(s^2 + 5s + 6)$$

$$+ C(s^2 + 6s + 8)$$

$$\Rightarrow \quad A + B + C = 0$$

$$7A + 5B + 6C = 0$$

$$12A + 6B + 8C = 1$$

$$\Rightarrow \quad A = 1/2, \; B = 1/2, \; C = -1.$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{(s+2)(s+4)(s+3)}\right) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} - e^{-3t}.$$

Similarly,

$$\frac{1}{(s+2)(s+4)(s+5)} = \frac{A}{s+2} + \frac{B}{s+4} + \frac{C}{s+5}$$

$$\Rightarrow \quad 1 = A[s^2 + 9s + 20] + B[s^2 + 7s + 10] + C[s^2 + 6s + 8]$$

$$\Rightarrow \quad A + B + C = 0$$

$$9A + 7B + 6C = 0$$

$$20A + 10B + 8C = 1$$

$$\Rightarrow \quad A = \frac{1}{6}, \quad B = -1/2, \quad C = 1/3$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s+2)(s+4)(s+5)}\right) = \frac{1}{6}e^{-2t} - \frac{1}{2}e^{-4t} + \frac{1}{3}e^{-5t}$$

$$\therefore \quad y(t) = \frac{1}{3}e^{-2t} + e^{-4t} - e^{-3t} + \frac{1}{3}e^{-5t}.$$

(b) $$y'' + 3y' + 2y = \begin{cases} 4t & ; \quad 0 < t < 1 \\ 8 & ; \quad t > 1 \end{cases}$$

$$y(0) = 0, \quad y'(0) = 0$$

Let $$x(t) = \begin{cases} 4t, & 0 < t < 1 \\ 8, & t > 1 \end{cases}$$

$$= 4t\left(1 - u(t-1)\right) + 8u(t-1)$$

$$\Rightarrow \quad L(x(t)) = 4L(t) - 4L\left(t\,u(t-1)\right) + 8L\left(u(t-1)\right)$$

$$= \frac{4}{s^2} + 4\frac{d}{ds}\left(\frac{e^{-s}}{s}\right) + 8\frac{e^{-s}}{s}$$

$$= \frac{4}{s^2} + 4\,\frac{s(-e^{-s}) - e^{-s}}{s^2} + 8\frac{e^{-s}}{s}$$

$$= \frac{4 - 4se^{-s} - 4e^{-s} + 8se^{-s}}{s^2}$$

$$= \frac{4se^{-s} - 4e^{-s} + 4}{s^2}$$

Now, $\quad L(y'') + 3L(y') + 2L(y) = L(x(t))$

$$\Rightarrow \quad (s^2 + 3s + 2)\,Y(s) = L(x(t))$$

$$\Rightarrow \quad Y(s) = \frac{1}{(s+1)(s+2)} \cdot L(x(t))$$

$$= \frac{4e^{-s}}{s(s+1)(s+2)} - \frac{4e^{-s}}{s^2(s+1)(s+2)}$$

$$+ \frac{4}{s^2(s+1)(s+2)}$$

Now, $L^{-1}\left(\dfrac{4\, e^{-s}}{s(s+1)(s+2)}\right)$

$\because$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{(s+1)(s+2)}\right) = e^{-t} - e^{-2t}$$

$$\therefore \quad L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = \int_0^t e^{-u} - e^{-2u}\, du$$

$$= \left[-e^{-u}\right]_0^t + \left[\frac{e^{-2u}}{2}\right]_0^t$$

$$= 1 - e^{-t} + \frac{e^{-2t}}{2} - \frac{1}{2}$$

$$= 1/2 - e^{-t} + \frac{1}{2} e^{-2t}$$

$$\therefore \quad L^{-1}\left(\frac{4\, e^{-s}}{s(s+1)(s+2)}\right)$$

$$= 4\, u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}\right)$$

Also, $L^{-1}\left(\dfrac{1}{s^2(s+1)(s+2)}\right) = \int_0^t \left(\frac{1}{2} - e^{-u} + \frac{1}{2}e^{-2u}\right)du$

$$= \frac{1}{2}t + \left[e^{-t} - 1\right] - \frac{1}{4}\left[e^{-2t} - 1\right]$$

$$= \frac{t}{2} + e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4}$$

$\therefore L^{-1}\left(\dfrac{4e^{-s}}{s^2(s+1)(s+2)}\right)$

$$= 4u(t-1)\left(\frac{t-1}{2} + e^{-(t-1)} - \frac{1}{4}e^{-2(t-1)} - \frac{3}{4}\right)$$

So, the solution is

$$y(t) = 4u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right)$$

$$- 4u(t-1)\left(\frac{t-1}{2} + e^{-(t-1)} - \frac{1}{4}e^{-2(t-1)} - \frac{3}{4}\right)$$

$$+ 4\left(\frac{t}{2} + e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4}\right)$$

$$= \begin{cases} 2t + 4e^{-t} - e^{-2t} - 3 & ; \ 0 < t < 1 \\ -8e^{-(t-1)} + 3e^{-2(t-1)} + 4e^{-t} - e^{-2t} & ; \ t > 1 \end{cases}$$

Method 2:

$$Y(s) = \frac{1}{(s+1)(s+2)} \cdot L(x(t))$$

$$= L\left(e^{-t} - e^{-2t}\right) \cdot L(x(t))$$

$$\therefore \quad y(t) = (x * g)(t) \quad ; \quad g(t) = e^{-t} - e^{-2t}$$

$$= \int_0^t x(\tau)\, g(t-\tau)\, d\tau$$

if $t < 1$, $x(\tau) = 4\tau$

$$\Rightarrow \quad y(t) = \int_0^t 4\tau \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

if $t > 1$, $x(t) = 8$

$$\Rightarrow \quad y(t) = \int_0^1 4\tau \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

$$+\int_1^t 8\left(e^{-(t-\tau)}-e^{-2(t-\tau)}\right)d\tau .$$

(c) $y''+4y'+5y = \delta(t-1)$, $y(0)=0$, $y'(0)=3$

$$L(y'')+4L(y')+5L(y) = L(\delta(t-1))$$

$$\Rightarrow \left[s^2Y(s)-sy(0)-y'(0)\right]+4\left[sY(s)-y(0)\right]$$

$$+\ 5Y(s) = e^{-s}$$

$$\Rightarrow (s^2+4s+5)Y(s) = e^{-s}+3$$

$$\Rightarrow \quad Y(s) = \frac{e^{-s}+3}{s^2+4s+5}$$

$$= \frac{e^{-s}}{(s+2)^2+1}+\frac{3}{(s+2)^2+1}$$

$$\because \quad L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$\Rightarrow L^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}\sin t$$

$$\therefore \quad y(t) = u(t-1)\cdot e^{-2(t-1)}\sin(t-1) + 3e^{-2t}\sin t$$

Solution 7 : (a) $\ln\left(\frac{s+2}{s+3}\right)$

Let $F(s) = \ln\left(\frac{s+2}{s+3}\right)$

$$\Rightarrow \quad F'(s) = \frac{1}{(s+2)(s+3)}$$

$$\because \quad L(t\,f(t)) = -F'(s)$$

$$\Rightarrow \quad t\,f(t) = -L^{-1}(F'(s))$$

$$= -\left(e^{-2t} - e^{-3t}\right)$$

$$\Rightarrow \quad f(t) = \frac{-e^{-2t} + e^{-3t}}{t}$$

(b) $\quad F(s) = \cot^{-1}\left(\frac{s}{\pi}\right)$

$$F'(s) = \frac{-1}{1+\left(\frac{s}{\pi}\right)^2}\cdot\frac{1}{\pi}$$

$$= \frac{-\pi^2}{s^2+\pi^2} \cdot \frac{1}{\pi}$$

$$\because \quad L(t f(t)) = -F'(s)$$

$$= \frac{\pi}{s^2+\pi^2}$$

$$\Rightarrow \quad t f(t) = L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right)$$

$$= \sin \pi t$$

$$\Rightarrow \quad f(t) = \frac{1}{t} \sin \pi t$$

Solution 8 :

(a) $$1 * \sin t = \int_0^t 1 \cdot \sin(t-\tau)\, d\tau$$

$$= 1 - \cos t .$$

(b) $$\cos 2t * \sin 2t = \int_0^t \cos 2\tau \sin 2(t-\tau)\, d\tau$$

$$= \int_0^t \frac{1}{2}\left[\sin 2t - \sin(4\tau - 2t)\right] d\tau$$

(c) $$u(t-1) * t^2 = \int_0^t u(\tau-1)\cdot(t-\tau)^2\, d\tau$$

$$= \int_1^t (t-\tau)^2\, d\tau, \quad \text{for } t > 1.$$

(d) $$u(t-3) * e^{2t} = \int_0^t u(\tau-3)\cdot e^{2(t-\tau)}\, d\tau$$

$$= \int_3^t e^{2(t-\tau)}\, d\tau, \quad \text{for } t > 3.$$

Solution 9 :

(a) $$y(t) = 2t - 4\int_0^t y(\tau)(t-\tau)\, d\tau$$

$$\Rightarrow \quad y(t) + 4\,(y(t) * t) = 2t$$

$$\Rightarrow \quad L(y)(s) + 4\, L\big(y(t) * t\big)(s) = 2\, L(t)(s)$$

$$\Rightarrow \quad Y(s) + 4\,Y(s)\cdot\frac{1}{s^2} = \frac{2}{s^2}$$

$$\Rightarrow \quad \left(1+\frac{4}{s^2}\right)Y(s) = 2/s^2$$

$$\Rightarrow \quad Y(s) = \frac{2}{s^2}\cdot\frac{s^2}{s^2+4} = \frac{2}{s^2+4}$$

$$\therefore \quad y(t) = L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= \sin 2t\,.$$