经典场论笔记

作者：温晨煜

组织：清华大学致理书院

版本：1.0

封面：John Atkinson Grimshaw, *Whitby*

目录

第一章 初识经典场论

狭义相对论相互作用有上限 $\Longrightarrow$ 近距作用 $\Longrightarrow$ 场

1.1 从粒子到场

对于 n 自由度粒子

$$\begin{cases} S(q_i(t), p_i(t)) = \int \mathrm{d}t \left[\sum_{i=1}^{n} p_i \dot{q}_i - H(q_i, p_i)\right] \\ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ [A, B] = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \end{cases}$$

连续化指标，S 和 H 成为泛函

$$\begin{cases} S[q_\sigma(t), p_\sigma(t)] = \int \mathrm{d}t \left[\int \mathrm{d}\sigma p_\sigma \dot{q}_\sigma - H[q_\sigma, p_\sigma]\right] \\ \dot{q}_\sigma = \frac{\delta H}{\delta p_\sigma}, \quad \dot{p}_\sigma = -\frac{\delta H}{\delta q_\sigma} \\ [A, B] = \int \mathrm{d}\sigma \left(\frac{\delta A}{\delta q_\sigma}\frac{\delta B}{\delta p_\sigma} - \frac{\delta A}{\delta p_\sigma}\frac{\delta B}{\delta q_\sigma}\right) \end{cases}$$

改写得到标量场

$$\phi(\sigma, t) = q_\sigma(t), \quad \pi(\sigma, t) = p_\sigma(t)$$

$$\begin{cases} S[\phi(\sigma, t), \pi(\sigma, t)] = \int \mathrm{d}t\mathrm{d}\sigma \pi(\sigma, t)\dot{\phi}(\sigma, t) - \int \mathrm{d}t H[\phi(\sigma), \pi(\sigma)] \\ \dot{\phi}(\sigma, t) = \frac{\delta H}{\delta \pi(\sigma)}, \quad \dot{\pi}(\sigma, t) = -\frac{\delta H}{\delta \phi(\sigma)} \\ [A, B] = \int \mathrm{d}\sigma \left(\frac{\delta A}{\delta \phi(\sigma)}\frac{\delta B}{\delta \pi(\sigma)} - \frac{\delta A}{\delta \pi(\sigma)}\frac{\delta B}{\delta \phi(\sigma)}\right) \end{cases}$$

将 σ 取为空间位置 $\boldsymbol{x}$

局域性要求 Hamiltonian 能够写成如下形式

$$H = \int \mathrm{d}^3\boldsymbol{x}\mathscr{H}(\pi, \phi, \nabla\phi)$$

$$S[\phi(x), \pi(x)] = \int \mathrm{d}^4x \left[\pi(x)\dot{\phi}(x) - \mathscr{H}(\pi, \phi, \nabla\phi)\right]$$

从 $\dot{\phi}(x) = \frac{\partial \mathscr{H}}{\partial \pi}$ 反解出 $\phi(x)$，代入 $S[\phi(x), \pi(x)]$

$$S[\phi(x)] = \int \mathrm{d}^4x\mathscr{L}(\phi, \dot{\phi}, \nabla\phi)$$

根据最小作用量原理，

$$\delta S[\phi(x)] = 0$$

认为无穷远时间和无穷远处 $\delta\phi = 0$，得到场方程

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) + \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial \nabla\phi}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

例题 1.1 场的相互作用势能 $\mathscr{U}(\phi)$

$$\mathscr{H} = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2\right] + \mathscr{U}(\phi)$$
$$\mathscr{L} = \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2\right] - \mathscr{U}(\phi)$$

场方程

$$\partial_t^2\phi - \nabla^2\phi = -\frac{\partial \mathscr{U}}{\partial \phi}$$

1.2 粒子作为场方程的解

考虑 $1+1$ 维时空 (t, σ)

$$\partial_t^2\phi - \partial_\sigma^2\phi = -\frac{\partial \mathscr{U}}{\partial \phi}$$

假设 $\mathscr{U}(\phi)$ 有最小值且最小值为 0，定义真空场位形

$$\Omega = \{\phi | \partial_t\phi = \partial_\sigma\phi = 0, \quad \mathscr{U}(\phi) = 0\}$$

总能量有限，要求无穷远处为真空场位形

$$\phi_\pm = \lim_{\sigma\to\pm\infty}\phi(\sigma) \in \Omega$$

- 若 $\phi_+ = \phi_-$，场位形拓扑上等价于真空场位形
- 若 $\phi_+ \neq \phi_-$，场位形不能连续变形成真空场位形，拓扑上不等价于真空场位形，每对 (ϕ_-, ϕ_+) 给出场位形的一个拓扑等价类，这样的场位形称为扭结，场方程的扭结解是一种孤立子，对应粒子

静态场下能量

$$\begin{aligned} E &= V[\phi(\sigma)] \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\sigma \left[\frac{1}{2}(\partial_\sigma\phi)^2 + \mathscr{U}(\phi)\right] \\ &\geq \int_{-\infty}^{+\infty} \sqrt{2\mathscr{U}(\phi)}\partial_\sigma\phi \mathrm{d}\sigma \\ \Longrightarrow \quad E &\geq \left|\int_{\phi_-}^{\phi_+} \sqrt{2\mathscr{U}(\phi)}\mathrm{d}\phi\right| \end{aligned}$$

引入 $W(\phi)$ 使

$$\mathscr{U}(\phi) = \frac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}\phi}\right)^2$$

$$E \geq |W(\phi_+) - W(\phi_-)|$$

该能量下界称为 Bogomolny 能限，对应静态场位形满足 Bogomolny 方程

$$\partial_\sigma \phi = \pm\sqrt{2\mathscr{U}(\phi)}$$

例题 1.2　sine-Gordon 模型

$$\mathscr{U}(\phi) = 1 - \cos\phi$$

真空场位形

$$\phi = 2\pi n, \quad n \in \mathbb{Z}$$

每个拓扑等价类由拓扑荷 N 标记

$$N = \frac{\phi_+ - \phi_-}{2\pi} \in \mathbb{Z}$$

Bogomolny 能限

$$E \geq 8|N|$$

Bogomolny 方程

$$\begin{cases} \partial_\sigma \phi = 2\sin\frac{\phi}{2} \implies \phi(\sigma) = 4\arctan \mathrm{e}^{\sigma - a}, \quad N = 1 \\ \partial_\sigma \phi = -2\sin\frac{\phi}{2} \implies \phi(\sigma) = 4\arctan \mathrm{e}^{-\sigma - a}, \quad N = -1 \\ \text{特解}\phi = 2\pi n, \quad n \in \mathbb{Z}, \quad N = 0 \end{cases}$$

对于 N 为其他值的解，由于扭结之间有相互作用能，无法达到 Bogomolny 能限，解是不稳定的。$1+1$ 维 sine-Gordon 模型是可积场论，可以精确得到所有的多扭结解

第二章 相对性原理与场论

由于 Lorentz 协变性，作用量是标量；由于局域性，作用量可以写成

$$S = \int \mathrm{d}^4 x \mathscr{L}$$

$\mathscr{L}$ 是标量

考虑标量场论，场变量 ϕ，$\mathscr{L} = \mathscr{L}(\phi, \partial_\mu \phi)$，场方程

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathscr{L}}{\partial \phi}$$

最简单 Lagrangian 密度

$$\mathscr{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \mathscr{U}(\phi)$$

对于复标量场

$$\mathscr{L} = -\partial_\mu \overline{\phi} \partial^\mu \phi - \mathscr{U}(\overline{\phi} \phi)$$

满足 $U(1)$ 对称性，在如下变换下不变

$$\phi \to \mathrm{e}^{\mathrm{i}\theta} \phi, \quad \overline{\phi} \to \mathrm{e}^{-\mathrm{i}\theta} \overline{\phi}$$

第三章 对称性与对称性自发破缺

3.1 Noether 定理

考虑 n 个标量场 $\phi^a(x), a = 1, \cdots, n$ 组成的系统，作用量 $S[\phi]$ 在连续对称变换 $g(\theta)$ 下保持不变，考察 $\theta = \varepsilon(x)$ 的无穷小变换

$$\phi^a(x) \to \tilde{\phi}^a(x) = \phi^a(x) + \varepsilon(x) F^a(\phi(x))$$

$$\delta S = \int \mathrm{d}^4 x J^\mu \partial_\mu \varepsilon = -\int \mathrm{d}^4 x \varepsilon \partial_\mu J^\mu$$

真实场位形满足最小作用量原理

$$\delta S = -\int \mathrm{d}^4 x \varepsilon \partial_\mu J^\mu = 0$$

由 $\varepsilon(x)$ 任意性，得到四维流守恒方程

$$\partial_\mu J^\mu(x) = 0$$

守恒荷

$$Q = \int \mathrm{d}^3 \boldsymbol{x} \rho(\boldsymbol{x}, t)$$

3.2 对称性

3.2.1 内部对称性

1. $U(1)$ 对称性
   考虑复标量场系统

   $$S[\phi] = -\int \mathrm{d}^4 x [\partial_\mu \overline{\phi} \partial^\mu \phi + \mathscr{U}(\overline{\phi}\phi)]$$

   在如下变换下不变

   $$\phi(x) \to \mathrm{e}^{\mathrm{i}\theta} \phi(x), \quad \overline{\phi}(x) \to \mathrm{e}^{-\mathrm{i}\theta} \overline{\phi}(x)$$

   取无穷小变换 $\theta = \varepsilon(x)$

   $$\delta S = -\mathrm{i} \int \mathrm{d}^4 x [\phi \partial^\mu \overline{\phi} - \overline{\phi} \partial^\mu \phi] \partial_\mu \varepsilon$$

   守恒流

   $$J^\mu = \mathrm{i}[\overline{\phi} \partial^\mu \phi - \phi \partial^\mu \overline{\phi}]$$

2. $U(N)$ 对称性

考虑有 N 个复标量场 $\phi^i(x), i = 1, \cdots, N$ 的系统，组成列向量

$$\Phi = \begin{bmatrix} \phi^1 \\ \phi^2 \\ \cdots \\ \phi^N \end{bmatrix}$$

在幺正矩阵 U 变换下不变

$$\Phi(x) \to U\Phi(x), \cdots \Phi^\dagger(x) \to \Phi^\dagger(x)U^\dagger$$
$$U = \mathrm{e}^{\mathrm{i}\theta^a T_a}, \quad a = 1, 2, \cdots, N^2$$

θ^a 为参数，T_a 为 Hermite 矩阵。取无穷小变换 $\theta^a = \varepsilon^a(x)$

$$\delta S = \mathrm{i} \int \mathrm{d}^4x \left[\Phi^\dagger T_a \partial^\mu \Phi - (\partial^\mu \Phi) T_a \Phi\right] \partial_\mu \varepsilon^a$$

守恒流

$$J_a^\mu = \mathrm{i}\left[\Phi^\dagger T_a \partial^\mu \Phi - (\partial^\mu \Phi) T_a \Phi\right]$$

3.3 时空平移对称性

考虑实标量场构成的系统，时空平移

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

场变换

$$\phi(x) \to \phi'(x) = \phi(x')$$

取无穷小变换 $a^\mu = \varepsilon^\mu(x)$，Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu \varepsilon^\mu(x), \quad \left|\frac{\partial x}{\partial x'}\right| = 1 - \partial_\mu \varepsilon^\mu(x)$$
$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu{}_\mu \mathscr{L}\right] \partial_\nu \epsilon^\mu$$

守恒流为能动张量

$$T^\nu{}_\mu = -\left[\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu{}_\mu \mathscr{L}\right]$$

流守恒方程

$$\partial_\mu T^\nu{}_\mu = 0$$

3.4 Lorentz 对称性

考虑实标量场构成的系统，Lorentz 变换

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

取无穷小变换 $\varepsilon^\mu{}_\nu(x)$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu$$

$$\det\Lambda = 0 \implies \varepsilon^\mu{}_\mu = 0$$

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta$$

$$\implies \varepsilon^{\mu\nu}\text{反对称}$$

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x \left[(T^{\mu\nu} - T^{\nu\mu})\varepsilon_{\mu\nu}\right] + \frac{1}{2}\int \mathrm{d}^4x \left[(x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu})\partial_\rho\varepsilon_{\mu\nu}\right]$$

$$\varepsilon_{\mu\nu}\text{为常数时}\delta S = 0 \implies T^{\mu\nu} = T^{\nu\mu}$$

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x \left[(x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu})\partial_\rho\varepsilon_{\mu\nu}\right]$$

守恒流

$$M^{\rho\mu\nu} = x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu}$$

流守恒方程

$$\partial_\rho M^{\rho\mu\nu} = 0$$

3.5 对称性自发破缺

连续对称性自发破缺后的系统中，存在以光速传播的波动，量子化后得到零质量粒子，称为 Nambu-Goldstone 玻色子

考虑如下复标量场模型

$$\mathscr{L} = -\partial_\mu\bar{\phi}\partial^\mu\phi - \mathscr{U}(\bar{\phi}\phi), \quad \mathscr{U}(\bar{\phi}\phi) = \frac{g}{4}(|\phi|^2 - u)^2$$

系统具有 $U(1)$ 对称性

真空场位形满足

$$\partial_t\phi = \nabla\phi = 0$$

且 $\mathscr{U}(\bar{\phi}\phi)$ 取最小值

- $u < 0$：$\mathscr{U}(\bar{\phi}\phi)$ 只有唯一的最小值点 $\phi = 0$
- $u > 0$：$\mathscr{U}(\bar{\phi}\phi)$ 有无穷多最小值点，满足 $|\phi| = \sqrt{u}$，即 $\phi = \sqrt{u}\mathrm{e}^{\mathrm{i}\alpha}$，而真空场位形只会处于其中一个点，真空不再具有 $U(1)$ 对称性，称为对称破缺相

二者之间发生相变

$u > 0$ 下，设真空场位形

$$\phi_0 = \sqrt{u}$$
$$\phi = (\sqrt{u} + \rho)e^{i\theta}$$
$$\mathscr{L} = -(\sqrt{u} + \rho)^2 \partial_\mu \theta \partial^\mu \theta - \partial_\mu \rho \partial^\mu \rho - gu\rho^2(x) - g\sqrt{u}\rho^3(x) - \frac{g}{4}\rho^4(x)$$

在真空上 $\rho = 0$

$$\mathscr{L} = -u\partial_\mu \theta \partial^\mu \theta$$

由场方程

$$\partial_\mu \partial^\mu \theta = 0$$

$\theta(x)$ 以光速传播

第四章 规范对称性和 Maxwell 方程

$$\text{整体对称性} \stackrel{\text{局域化}}{\longrightarrow} \text{规范对称性}$$

4.1 局域化 $U(1)$ 整体对称性

考虑 $U(1)$ 不变的复标量场论

$$\mathscr{L} = -\partial_\mu\bar{\phi}\partial^\mu\phi - \mathscr{U}(\bar{\phi}\phi)$$
$$J^\mu = \mathrm{i}[\phi\partial^\mu\bar{\phi} - \bar{\phi}\partial^\mu\phi]$$

为了使 $U(1)$ 对称性局域化，引入矢量场 A_μ 使得作用量变为

$$S[\phi] = -\int \mathrm{d}^4x\left[\partial_\mu\bar{\phi}\partial^\mu\phi + \mathscr{U}(\bar{\phi}\phi)\right] + \int \mathrm{d}^4x J^\mu A_\mu$$

A_μ 的规范变换为

$$A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x)$$

从而可以使 $\delta S = 0$

守恒流需要修正

$$J^\mu_A = \mathrm{i}e(\phi\overline{\mathrm{D}_\mu\phi} - \bar{\phi}\mathrm{D}_\mu\phi)$$

协变导数

$$\mathrm{D}_\mu\phi = (\partial_\mu - \mathrm{i}eA_\mu)\phi$$

作用量

$$S_m = --\int \mathrm{d}^4x[\overline{\mathrm{D}_\mu\phi}\mathrm{D}^\mu\phi + \mathscr{U}(\bar{\phi}\phi)]$$
$$\delta S_m = \int \mathrm{d}^4x J^\mu_A\delta A_\mu$$

可以看出 A_μ 为仿射联络

4.2 规范场的动力学：Maxwell 方程

如下 $F_{\mu\nu}$ 在规范变换下不变

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

利用 $F_{\mu\nu}$ 构造最简单标量，取作用量为

$$S_g = -\int \mathrm{d}^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
$$\delta S_g = \int \mathrm{d}^4x \partial_\mu F^{\mu\nu} \delta A_\nu$$

整个系统

$$S = S_m + S_g = -\int \mathrm{d}^4x \left[\overline{\mathrm{D}\phi} \mathrm{D}^\mu \phi + \mathscr{U}(\overline{\phi}\phi) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$
$$\delta S = \int \mathrm{d}^4x \left[\partial_\mu F^{\mu\nu} + J_A^\nu \right] \delta A_\nu = 0$$

得到

$$-\partial_\mu F^{\mu\nu} = J_A^\nu$$
$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

即为 Maxwell 方程组

引入 Lorenz 规范

$$\partial_\mu A^\mu = 0$$

自由空间中，A^μ 满足

$$\partial_\mu \partial^\mu A^\nu = 0$$

证明光是电磁波

4.3 可观测量规范不变性

如下非局域量规范不变

$$\overline{\phi}(b) \mathrm{e}^{\mathrm{i}e \int_a^b A_\mu \mathrm{d}x^\mu} \phi(a)$$

a、b 重合时为

$$\mathrm{e}^{\mathrm{i}e \oint A_\mu \mathrm{d}x^\mu}$$

4.4 电磁场的能量、动量、角动量

4.4.1 实标量场

能动张量

$$T^{\mu\nu} = -\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi(x) - \eta^{\mu\nu}\mathscr{L}\right]$$

$$T^{0i}(x) = -\pi(x)\partial^i(x)$$

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0\phi)}$$

$$J^{ij} = \int \mathrm{d}^3\boldsymbol{x}\,\left[x^i T^{0j} - x^j T^{0i}\right]$$

角动量矢量

$$L_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$$

$$\boldsymbol{L} = -\int \mathrm{d}^3\boldsymbol{x}\,\left[\pi(x)(\boldsymbol{x}\times\nabla)\phi(x)\right]$$

称为轨道角动量

4.4.2 电磁场

$$T^{\mu\nu} = -\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu A^\rho)}\partial^\nu A^\rho - \eta^{\mu\nu}\mathscr{L}\right] = F^\mu{}_\rho\partial^\nu A^\rho - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

添加一项 $\partial^\rho(-F^\mu{}_\rho A^\nu) = -F^\mu{}_\rho\partial^\rho A^\nu$ 进行对称化

$$T^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

角动量矢量

$$\begin{aligned}
L_i &= \int \mathrm{d}^3\boldsymbol{x}\,\left[\epsilon_i{}^{jk}x_j\epsilon_k{}^{lm}E_l\epsilon_m{}^{no}\partial_n A_o\right] \\
&= \int \mathrm{d}^3\boldsymbol{x}\epsilon_i{}^{jk}x_j\left[E_l\partial_k A^l - E_l\partial^l A_k\right] \\
&= \int \mathrm{d}^3\boldsymbol{x}\,\left[E_l(\boldsymbol{x}\times\nabla)_i A^l - \epsilon_i{}^{jk}x_j E_l\partial^l A_k\right]
\end{aligned}$$

分部积分，结合 $\partial^l E_l = 0$ 得到

$$L_i = \int \mathrm{d}^3\boldsymbol{x}\,\left[E_l(\boldsymbol{x}\times\nabla)_i A^l + (\boldsymbol{E}\times\boldsymbol{A})_i\right]$$

第二项称为内禀角动量

4.4.3 电磁场与复标量场耦合系统

$$\mathscr{L} = -\left[\overline{\mathrm{D}_\mu\phi}\mathrm{D}^\mu\phi + \mathscr{U}(\bar\phi\phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]$$

$$T^{\mu\nu} = F^\mu{}_\rho\partial^\nu A^\rho - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \overline{\mathrm{D}^\mu\phi}\mathrm{D}^\nu\phi + \mathrm{D}^\mu\phi\overline{\mathrm{D}^\nu\phi} - \eta^{\mu\nu}\left[\overline{\mathrm{D}_\rho\phi}\mathrm{D}^\rho\phi + \mathscr{U}(\bar\phi\phi)\right] + J^\mu_a A^\nu$$

添加修正 $\partial^\rho(-F^\mu{}_\rho A^\nu) = -J_A^\mu A^\nu - F^\mu{}_\rho \partial^\rho A^\nu$

$$T^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \overline{\mathrm{D}^\mu\phi}\mathrm{D}^\nu\phi + \mathrm{D}^\mu\phi\overline{\mathrm{D}^\nu\phi} - \eta^{\mu\nu}\left[\overline{\mathrm{D}_\rho\phi}\mathrm{D}^\rho\phi + \mathscr{U}(\bar{\phi}\phi)\right]$$

4.5 外微分形式 Maxwell 方程组

$$\begin{cases} \mathrm{d}F = 0 \\ \star\mathrm{d}(\star F) = J_A \end{cases}$$

第五章 超导的有效理论以及涡旋解

5.1 Ginzburg-Landau 理论

Ginzburg-Landau 理论原始版本是非相对论理论，在静态场位形下无需区分非相对论与相对论，考虑相对论性的复标量场与电磁场耦合系统

$$S = \int \mathrm{d}^4 x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \overline{\mathrm{D}_\mu \phi} \mathrm{D}^\mu \phi - \frac{g}{2} (|\phi|^2 - u)^2 \right] \quad u > 0$$

$$T^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \overline{\mathrm{D}^\mu \phi} \mathrm{D}^\nu \phi + \mathrm{D}^\mu \phi \overline{\mathrm{D}^\nu \phi} - \eta^{\mu\nu} \left[\overline{\mathrm{D}_\rho \phi} \mathrm{D}^\rho \phi + \frac{g}{2} (|\phi|^2 - u)^2 \right]$$

能量密度

$$\mathscr{H} = T^{00} = \frac{1}{2} (\boldsymbol{E}^2 + \boldsymbol{B}^2) + |\mathrm{D}_0 \phi|^2 + \sum_{i=1}^{3} |\mathrm{D}_i \phi|^2 + \frac{g}{2} (|\phi|^2 - u)^2$$

选取轴规范

$$A^0 = 0$$

静态场位形下，$\boldsymbol{E} = 0, \mathrm{D}_0 \phi = 0$，Ginzburg-Landau 能量泛函

$$H = \int \mathrm{d}^3 \boldsymbol{x} \left[\frac{1}{2} \boldsymbol{B}^2 + \sum_{i=1}^{3} |\mathrm{D}_i \phi|^2 + \frac{g}{2} (|\phi|^2 - u)^2 \right]$$

最小作用量原理

$$\delta H = 0$$

ϕ 称为序参量，耦合常数 $e = \frac{q_{cp}}{\hbar}$，q_{cp} 为 Cooper 对的电荷量

记除磁能之外的能量为

$$H_m = \int \mathrm{d}^3 \boldsymbol{x} \left[+ \sum_{i=1}^{3} |\mathrm{D}_i \phi|^2 + \frac{g}{2} (|\phi|^2 - u)^2 \right]$$

$$\delta H_m = - \int \mathrm{d}^3 \boldsymbol{x} \boldsymbol{J}_A \cdot \delta \boldsymbol{A}$$

$$J_A^i = \mathrm{i} e (\phi \overline{\mathrm{D}_i \phi} - \overline{\phi} \mathrm{D}_i \phi)$$

设

$$\phi(x) = |\phi|\mathrm{e}^{\mathrm{i}\theta(x)}$$

$$H_m = \int \mathrm{d}^3\boldsymbol{x} \left[(\partial_i|\phi|)^2 + |\phi|^2(\partial_i\theta - eA_i)^2 + \frac{g}{2}(|\phi|^2 - u)^2\right]$$

$$\boldsymbol{J}_A = 2e|\phi|^2(\nabla\theta - e\boldsymbol{A})$$

$$\delta H = -\int \mathrm{d}^3\boldsymbol{x} J_A^i \delta A_i = \frac{1}{e}\int \mathrm{d}^3\boldsymbol{x} J_A^i \delta(\partial_i\theta) = -\frac{1}{e}\int \mathrm{d}^3\boldsymbol{x} (\partial_i J_A^i)(\delta\theta) = 0$$

固定 $|\phi|$

$$\begin{aligned}\delta H &= \int \mathrm{d}^3\boldsymbol{x} \left[B_i\delta B^i + \frac{1}{e}J_A^i\delta(\partial_i\theta) - J_A^i\delta A_i\right] \\ &= \int \mathrm{d}^3\boldsymbol{x} \left[B_i\epsilon^{ijk}\delta(\partial_j A_k) - J_A^i\delta A_i + \frac{1}{e}J_A^i\delta(\partial_i\theta)\right] \\ &= \int \mathrm{d}^3\boldsymbol{x} \left[-\epsilon^{ijk}(\partial_j B_i)\delta A_k - J_A^i\delta A_i + \frac{1}{e}J_A^i\delta(\partial_i\theta)\right] \\ &= \int \mathrm{d}^3\boldsymbol{x} \left[(\epsilon^{ijk}\partial_j B_k - J_A^i)\delta A_i + \frac{1}{e}J_A^i\delta(\partial_i\theta)\right] \\ \Longrightarrow \quad & \nabla\cdot\boldsymbol{J}_A = 0, \quad \nabla\times\boldsymbol{B} = \boldsymbol{J}_A\end{aligned}$$

系统能量最小时，

$$|\phi| = \sqrt{u}, \quad \partial_i\theta - eA_i = 0$$

$$\boldsymbol{J}_A = 0, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{e}\nabla\times\nabla\theta = 0$$

此时称为超导相，而 $|\phi| = 0, \partial_i\theta \neq eA_i$ 的相称为正常相

1. 对相位场和磁矢势场微扰

$$\nabla\cdot\delta\boldsymbol{J}_A = 0, \quad \nabla\times\delta\boldsymbol{B} = \delta\boldsymbol{J}_A$$

$$\Longrightarrow \begin{cases}\nabla^2\delta\theta = e\nabla\cdot\delta\boldsymbol{A} \\ \nabla\times(\nabla\times\delta\boldsymbol{A}) = 2eu(\nabla\delta\theta - e\delta\boldsymbol{A})\end{cases}$$

选取规范

$$\nabla\cdot\delta\boldsymbol{A} = 2eu\delta\theta$$

$$\begin{cases}\nabla^2\delta\theta = 2e^2u\delta\theta \\ \nabla^2\delta\boldsymbol{A} = 2e^2u\delta\boldsymbol{A}\end{cases}$$

解为指数衰减，特征长度为穿透深度

$$\lambda = \frac{1}{\sqrt{2e^2u}}$$

2. 对模长场微扰

$$
\begin{gathered}
|\phi| = \sqrt{u} + \rho \\
H_m \approx \int \mathrm{d}^3\boldsymbol{x}\left[(\partial_i\rho)^2 + 2gu\rho^2\right] \\
\Longrightarrow \nabla^2\rho = 2gu\rho
\end{gathered}
$$

解为指数衰减，特征长度为关联长度

$$
\xi = \frac{1}{\sqrt{2gu}}
$$

3. 引入第三个参数

$$
\kappa = \frac{\lambda}{\xi} = \sqrt{\frac{g}{e^2}}
$$

$\kappa < 1$ 为第一类超导体，$\kappa > 1$ 为第二类超导体

正常相变为超导相能量密度减少

$$
\Delta = \frac{1}{2}gu^2 = \frac{1}{8e^2\lambda^2\xi^2}
$$

若正常相下存在外磁场 B，当 $\frac{1}{2}B^2 < \Delta$ 时系统才会倾向于处于超导相

5.2 涡旋解

超导体中，可能存在管状区域，中心处于正常态，超导体内部绕管状区域的闭合回路上

$$
\Phi = \oint_L \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \frac{1}{e}\oint_L \nabla\theta\mathrm{d}\boldsymbol{l} = \frac{\Delta\theta}{e} = \frac{2N\pi}{e}
$$

得到磁通量子化，量子化单位为

$$
\Phi_0 = \frac{2\pi}{e}
$$

Bogomolnyi 证明了仅对于第二类超导体，通量 $N\Phi_0$ 的涡旋线在分解为 N 个通量为 Φ_0 的涡旋线时更加稳定。设单位横截面积涡旋线数密度为 n，在半径 ξ 的管状区域内接近正常态，为了使管状区域不重叠，限制 $n < \frac{1}{\pi\xi^2}$，产生涡旋线单位体积需要能量 $n\pi\xi^2\Delta$，

单位体积涡旋态能量

$$
W_V = \begin{cases} n\pi\xi^2\Delta + \frac{1}{2}B^2(1 - n\pi\lambda^2), & n < \frac{1}{\pi\lambda^2} \\ n\pi\xi^2\Delta, & n > \frac{1}{\pi\lambda^2} \end{cases}
$$

正常态单位体积比超导态多出的能量

$$
W_N \approx \Delta
$$

超导态为了排除磁场，单位体积需要能量

$$
W_S = \frac{1}{2}B^2
$$

- 对于第一类超导体，$n < \frac{1}{\pi\xi^2} < \frac{1}{\pi\lambda^2}$
  - 当 $B < \sqrt{2\Delta}$ 时，$W_V > \frac{1}{2}B^2 + n\pi(\xi^2 - \lambda^2)\Delta > W_S$
  - 当 $B > \sqrt{2\Delta}$ 时，$W_V > \Delta[1 + n\pi(\xi^2 - \lambda^2)] > W_N$

  不存在涡旋态
- 对于第二类超导体，磁场分为三个区域，临界值

$$B_{c1} = \sqrt{2\Delta}\frac{\xi}{\lambda}, \quad B_{c2} = \sqrt{2\Delta}\frac{\lambda}{\xi}$$

  根据 Bogomolnyi 证明的结论，$n = \frac{B}{\Phi_0}$
  - 当 $B < B_{c1}$ 时，$n < \frac{B_{c1}}{\Phi_0} = \frac{1}{4\pi\lambda^2} \sim \frac{1}{\pi\lambda^2}$

$$W_V = n\pi\xi^2\Delta + \frac{1}{2}B^2(1 - n\pi\lambda^2) = \frac{1}{2}B^2 + n\pi(\xi^2\Delta - \frac{1}{2}B^2\lambda^2) > \frac{1}{2}B^2 = W_S$$

$$W_N = \Delta = \frac{1}{2}B_{c1}^2\frac{\lambda^2}{\xi^2} > \frac{1}{2}B^2 = W_S$$

    系统处于超导态
  - 当 $B > B_{c1}$ 时，$n > \frac{B_{c1}}{\Phi_0} = \frac{1}{4\pi\lambda^2} \sim \frac{1}{\pi\lambda^2}$

$$W_V = n\pi\xi^2\Delta = \frac{1}{2}eB\xi^2\Delta \sim \frac{B}{B_{c2}}W_N \sim \frac{B_{c1}}{B}W_S$$

    - 当 $B_{c1} < B < B_{c2}$ 时，$W_V < W_N, W_V < W_S$，系统处于涡旋态
    - 当 $B > B_{c2}$ 时，$W_V < W_S, W_V > W_N$，系统处于正常态

  考虑在 x_3 方向上具有对称性的涡旋解，只考虑第二类超导体，

静态场位形能量

$$H = \int \mathrm{d}^2x \left[\frac{1}{2}B^2 + \sum_{i=1}^{2}|\mathrm{D}_i\phi|^2 + \frac{g}{2}(|\phi|^2 - u)^2\right]$$

$$\begin{aligned}
|\mathrm{D}_1\phi|^2 + |\mathrm{D}_2\phi|^2 &= |(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 \mp \mathrm{i}\overline{\mathrm{D}_1\phi}\mathrm{D}_2\phi \pm \mathrm{i}\overline{\mathrm{D}_2\phi}\mathrm{D}_1\phi \\
&= |(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 \mp \mathrm{i}\epsilon^{ij}\overline{\mathrm{D}_i\phi}\mathrm{D}_j\phi
\end{aligned}$$

$$\begin{aligned}
\mp\mathrm{i}\epsilon^{ij}\overline{\mathrm{D}_i\phi}\mathrm{D}_j\phi &= \mp\mathrm{i}\epsilon^{ij}(\partial_i\overline{\phi})\mathrm{D}_j\phi \pm \epsilon^{ij}eA_i\overline{\phi}\mathrm{D}_j\phi \\
&= \mp\mathrm{i}\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi) \pm \mathrm{i}\epsilon^{ij}\overline{\phi}(\partial_i\mathrm{D}_j\phi) \pm \epsilon^{ij}eA_i\overline{\phi}\mathrm{D}_j\phi \\
&= \mp\mathrm{i}\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi) \pm \mathrm{i}\epsilon^{ij}\overline{\phi}(\mathrm{D}_i\mathrm{D}_j\phi)
\end{aligned}$$

$$\epsilon^{ij}\mathrm{D}_i\mathrm{D}_j = \frac{1}{2}\epsilon^{ij}[\mathrm{D}_i, \mathrm{D}_j] = -\mathrm{i}eB$$

$$\begin{aligned}
H &= \int \mathrm{d}^2x \left[|(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 \pm eB(|\phi|^2 - u) + \frac{1}{2}B^2 + \frac{g}{2}(|\phi|^2 - u)^2\right] \\
&\quad + \int \mathrm{d}^2x \left[\pm euB \mp \mathrm{i}\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi)\right] \\
&= \int \mathrm{d}^2x \left[|(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 + \frac{1}{2}\left(B \pm e(|\phi^2 - u)^2\right)^2 + \frac{g - e^2}{2}(|\phi|^2 - u)^2\right] \\
&\quad + \int \mathrm{d}^2x \left[\pm euB \mp \mathrm{i}\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi)\right]
\end{aligned}$$

$$\int \mathrm{d}^2x\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi) = \int \partial_i(\overline{\phi}\mathrm{D}_j\phi)\mathrm{d}x^i \wedge \mathrm{d}x^j = \int \mathrm{d}(\overline{\phi}\mathrm{D}_j\phi\mathrm{d}x^j) = \oint \overline{\phi}\mathrm{D}_j\phi\mathrm{d}x^j$$

无穷远处，$|\phi| \to \sqrt{u}, \mathrm{D}_i\phi \to 0$

$$\int \mathrm{d}^2x\epsilon^{ij}\partial_i(\overline{\phi}\mathrm{D}_j\phi) = 0$$

$$H = \int \mathrm{d}^2x \left[|(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 + \frac{1}{2}\left(B \pm e(|\phi^2 - u)^2\right)^2 + \frac{g-e^2}{2}(|\phi|^2 - u)^2\right] \pm 2\pi N u$$

Bogomolny 能限

$$H \geq 2\pi|N|u$$

对于临界超导体 $g = e^2$

$$H = \int \mathrm{d}^2x \left[|(\mathrm{D}_1 \pm \mathrm{i}\mathrm{D}_2)\phi|^2 + \frac{1}{2}\left(B \pm e(|\phi^2 - u)^2\right)^2\right] \pm 2\pi N u$$

能量最小时满足 BPS 方程

$$(\mathrm{D}_1 + \mathrm{i}\mathrm{D}_2)\phi = 0, \quad B + e(|\phi|^2 - u) = 0$$

引入极坐标 (r, α)，在无穷远处 θ 变化 N 圈，可取 $\theta(x) = N\alpha$，假设 $|\phi|$ 只与 r 有关，得到

$$\begin{aligned}
A_i &= \frac{1}{e}\left[\epsilon_{ij}\partial_j \ln|\phi|(r) + \partial_i\theta\right] \\
B &= \epsilon_{ij}\partial_i A_j \\
&= \frac{1}{e}\epsilon_{ij}\epsilon_{jk}\partial_i\partial_k \ln|\phi| + \frac{1}{e}\epsilon_{ij}\partial_i\partial_j\theta \\
&= -\frac{1}{e}\partial_i^2 \ln|\phi| + \frac{1}{e}\epsilon_{ij}\partial_i\partial_j\theta
\end{aligned}$$

由 $\epsilon_{ij}\partial_i\partial_j\alpha = 2\pi\delta^2(x)$ 得

$$B = -\frac{1}{e}\partial_i^2 \ln|\phi| + \frac{1}{e}2\pi N\delta^2(x)$$

代入另一个方程得到

$$\nabla^2 \ln|\phi|(r) = 2\pi N\delta^2(x) + e^2(|\phi|^2 - u)$$

- $r \to \infty$

$$|\phi| \to \sqrt{u}, \quad A_i \to \frac{1}{e}\partial_i\theta$$

- $r \to 0$

$$\nabla^2 \ln|\phi|(r) = 2\pi N\delta^2(x) \implies |\phi| \to Ar^N$$

$$B \to eu$$

第六章 带电粒子与电磁场的耦合

6.1 多自由粒子系统

考虑 N 个相对论性自由粒子组成的系统，作用量

$$S[x(s)] = -\sum_n m_n \int \mathrm{d}\tau_n = -\sum_n m_n \int \mathrm{d}s_n \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x_n^\mu}{\mathrm{d}s_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}s_n}}$$

仿射参数选为 $s_n = \tau_n$

$$\delta S = -\sum_n m_n \int \delta(\mathrm{d}\tau_n) = \sum_n m_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\eta_{\mu\nu}\mathrm{d}(\delta x_n^\nu) = -\sum_n m_n \int \mathrm{d}\tau_n \frac{\mathrm{d}^2 x_n^\mu}{\mathrm{d}\tau_n^2}\eta_{\mu\nu}\delta x_n^\nu = 0$$

$$\implies m_n \frac{\mathrm{d}^2 x_n^\mu}{\mathrm{d}\tau_n^2} = 0$$

在时空平移 $x_n^\mu \to x_n^{'\mu} = x_n^\mu + a^\mu$ 下系统不变，取无穷小变换

$$a^\mu = \varepsilon^\mu(x_n^\mu)$$

$$\begin{aligned}
\delta S &= \sum_n m_n \int \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\eta_{\mu\nu}\mathrm{d}(\delta x_n^\nu) \\
&= \sum_n m_n \int \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\eta_{\mu\nu}\partial_\rho \varepsilon^\nu \mathrm{d}x_n^\rho \\
&= \sum_n m_n \int \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\partial_\rho \varepsilon_\mu \mathrm{d}x_n^\rho \\
&= \sum_n m_n \int \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\partial_\nu \varepsilon_\mu \mathrm{d}\tau_n \\
&= \int \mathrm{d}^4 x \left[\sum_n m_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\partial_\nu \varepsilon_\mu \delta^4(x - x_n(\tau_n))\right]\partial_\nu \varepsilon_\mu(x)
\end{aligned}$$

能动张量

$$\begin{aligned}
T^{\mu\nu} &= \sum_n m_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\delta^4(x - x_n(\tau_n)) \\
&= \sum_n m_n \int \mathrm{d}s_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}s_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\partial_\nu \delta^4(x - x_n(s_n)) \\
&= \sum_n \frac{p_n^\mu p_n^\nu}{p_n^0}\delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))
\end{aligned}$$

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= \sum_n m_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\partial_\mu \delta^4(x-x_n(\tau_n)) \\
&= -\sum_n m_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\frac{\partial}{\partial x_n^\mu}\delta^4(x-x_n(\tau_n)) \\
&= -\sum_n \int \mathrm{d}\tau_n p_n^\nu \frac{\mathrm{d}}{\mathrm{d}\tau_n}\delta^4(x-x_n(\tau_n)) \\
&= \sum_n \int \mathrm{d}\tau_n \frac{\mathrm{d}p_n^\nu}{\mathrm{d}\tau_n}\delta^4(x-x_n(\tau_n))
\end{aligned}
$$

6.2 带电粒子与电磁场的耦合

若前述粒子是带电粒子，与电磁场耦合的作用量

$$
S[x(s)] = -\sum_n m_n \int \mathrm{d}\tau_n + \sum_n q_n \int A_\mu \mathrm{d}x_n^\mu
$$

$$
\begin{aligned}
\sum_n q_n \delta \int A_\mu \mathrm{d}x_n^\mu &= \sum_n q_n \int \left[(\delta A_\nu)\mathrm{d}x_n^\nu + A_\mu \mathrm{d}(\delta x_n^\mu)\right] \\
&= \sum_n q_n \int \left[(\partial_\mu A_\nu)\delta x_n^\mu \mathrm{d}x_n^\nu - \mathrm{d}A_\mu (\delta x_n^\mu)\right] \\
&= \sum_n q_n \int \left[\partial_\mu A_\nu \delta x_n^\mu \mathrm{d}x_n^\nu - \partial_\nu A_\mu \mathrm{d}x_n^\nu \delta x_n^\mu\right] \\
&= \sum_n q_n \int \mathrm{d}\tau_n F_{\mu\nu}\frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\delta x_n^\mu
\end{aligned}
$$

$$
\begin{aligned}
\delta S &= \sum_n \int \mathrm{d}\tau_n \left[-m_n \frac{\mathrm{d}^2 x_n^\mu}{\mathrm{d}\tau_n^2} + q_n F^\mu{}_\nu \frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}\right]\delta(x_n)_\mu \\
&\Longrightarrow \frac{\mathrm{d}p_n^\mu}{\mathrm{d}\tau_n} = q_n F^\mu{}_\nu \frac{\mathrm{d}x_n^\nu}{\mathrm{d}\tau_n}
\end{aligned}
$$

对规范势 A_μ 变分

$$
\begin{aligned}
\delta S &= \sum_n q_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\delta A_\mu \\
&= \int \mathrm{d}^4 x \left[\sum_n q_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\delta^4(x-x_n(\tau_n))\right]\delta A_\mu
\end{aligned}
$$

$$
\begin{aligned}
J^\mu(x) &= \sum_n q_n \int \mathrm{d}\tau_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau_n}\delta^4(x - x_n(\tau_n)) \\
&= \sum_n q_n \int \mathrm{d}s_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}s_n}\delta^4(x - x_n(s_n)) \\
&= \sum_n q_n \int \mathrm{d}x_n^0 \frac{\mathrm{d}x_n^\mu}{\mathrm{d}x_n^0}\delta^4(x - x_n(x_n^0)) \\
&= \sum_n q_n u_n^\mu \delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))
\end{aligned}
$$

多粒子系统能动张量

$$
\partial_\mu T^{\mu\nu} = \sum_n \int \mathrm{d}\tau_n q_n F^\nu{}_\rho \frac{\mathrm{d}x_n^\rho}{\mathrm{d}\tau_n}\delta^4(x - x_n(\tau_n)) = F^\nu{}_\rho J^\rho
$$

电磁场能动张量

$$
T_{\mathrm{em}}^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}
$$

$$
\begin{aligned}
\partial_\mu T_{\mathrm{em}}^{\mu\nu} &= \partial_\mu F^\mu{}_\rho F^{\nu\rho} + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{4}\partial^\nu(F_{\mu\rho}F^{\mu\rho}) \\
&= \partial_\mu F^\mu{}_\rho F^{\nu\rho} + \frac{1}{2}F_{\mu\rho}(\partial^\mu F^{\nu\rho} - \partial^\rho F^{\nu\mu}) - \frac{1}{2}F_{\mu\rho}\partial^\nu F^{\mu\rho} \\
&= \partial_\mu F^\mu{}_\rho F^{\nu\rho} + \frac{1}{2}F_{\mu\rho}(\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu}) \\
&= \partial_\mu F^\mu{}_\rho F^{\nu\rho} \\
&= -F^\nu{}_\rho J^\rho
\end{aligned}
$$

总能动张量守恒

$$
\partial_\mu T^{\mu\nu} = 0
$$

第七章 运动电荷的电磁场

7.1 推迟势

Lorenz 规范 $\partial_\mu A^\mu = 0$ 下，规范势满足

$$-\partial_\nu \partial^\nu A^\mu(x) = \mu_0 J^\mu(x)$$

定义 Green 函数 $G(x,x')$

$$-\partial_\nu \partial^\nu G(x,x') = \delta^4(x-x')$$

由时空平移对称性

$$G(x,x') = G(x-x') \implies -\partial_\nu \partial^\nu G(x) = \delta^4(x)$$

$$A^\mu(x) = \mu_0 \int \mathrm{d}^4x' G(x-x') J^\mu(x')$$

由电荷守恒 $\partial_\mu J^\mu = 0$ 可以推出这样的解满足 Lorenz 规范

进行 Fourier 变换

$$\delta^4(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{\mathrm{i}k_\mu x^\mu}$$

$$G(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{G}(k) \mathrm{e}^{\mathrm{i}k_\mu x^\mu}$$

$$\tilde{G}(k) = \frac{1}{k_\mu k^\mu} = -\frac{1}{\omega^2 - |\boldsymbol{k}|^2}$$

$$G(x) = -\int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega^2 - |\boldsymbol{k}|^2}$$

对 ω 的积分，被积函数奇点 $\omega = \pm|\boldsymbol{k}|$，选取从实轴上方绕过奇点的围道，得到推迟 Green 函数 $G_{\mathrm{ret}}(\boldsymbol{x},t)$

- $t < 0$ 时，选取上半平面无穷远半圆构成闭合围道，由 Cauchy 定理，

$$G_{\mathrm{ret}}(\boldsymbol{x},t) = 0$$

- $t > 0$ 时，选取下半平面无穷远半圆构成闭合围道 C，设 C_{δ_1} 和 C_{δ_2} 分别为以 $\pm|\boldsymbol{k}|$ 为圆心的小圆弧由留数定理，

$$\oint_C \frac{\mathrm{d}z}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}zt}}{z^2 - |\boldsymbol{k}|^2} = \frac{1}{2\pi}(-2\pi\mathrm{i})\mathrm{Res} \left.\frac{\mathrm{e}^{-\mathrm{i}zt}}{z^2 - |\boldsymbol{k}|^2}\right|_{z=\pm|\boldsymbol{k}|} = -\mathrm{i}\left[\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|} - \frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right]$$

由大圆弧引理

$$\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{d}z}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}zt}}{z^2 - |\boldsymbol{k}|^2} = 0$$

由小圆弧引理

$$\int_{C_{\delta_1}}\frac{\mathrm{d}z}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}zt}}{z^2-|\boldsymbol{k}|^2}=\mathrm{i}(-\pi)\frac{1}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}=-\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{4|\boldsymbol{k}|}$$
$$\int_{C_{\delta_2}}\frac{\mathrm{d}z}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}zt}}{z^2-|\boldsymbol{k}|^2}=\mathrm{i}(-\pi)\left(-\frac{1}{2\pi}\frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right)=\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{4|\boldsymbol{k}|}$$
$$\Longrightarrow\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega^2-|\boldsymbol{k}|^2}=-\mathrm{i}\left[\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}-\frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right]$$

$$\begin{aligned}
G(x)&=\mathrm{i}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}-\frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right]\\
&=\frac{\mathrm{i}}{(2\pi)^3}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|r\cos\theta}\left[\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}-\frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right]|\boldsymbol{k}|^2\sin\theta\mathrm{d}|\boldsymbol{k}|\mathrm{d}\theta\mathrm{d}\varphi\\
&=\frac{\mathrm{i}}{(2\pi)^2}\int_0^{+\infty}\frac{1}{\mathrm{i}|\boldsymbol{k}|r}\left[\frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}-\frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}}{2|\boldsymbol{k}|}\right]\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|r}-\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|r}\right)|\boldsymbol{k}|^2\mathrm{d}|\boldsymbol{k}|\\
&=\frac{1}{8\pi^2r}\int_0^{+\infty}\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|(r-t)}-\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|(r+t)}-\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(r+t)}+\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(r-t)}\right)\mathrm{d}|\boldsymbol{k}|\\
&=\frac{1}{8\pi^2r}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|(r-t)}-\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|(r+t)}\right)\mathrm{d}|\boldsymbol{k}|\\
&=\frac{1}{4\pi r}[\delta(r-t)-\delta(r+t)]\\
&=\frac{1}{2\pi}\delta(t^2-r^2)\\
&=\frac{1}{2\pi}\delta(x^2)=\frac{1}{4\pi r}\delta(t-r)
\end{aligned}$$

综上所述，得到推迟 Green 函数

$$G_{\mathrm{ret}}(\boldsymbol{x},t)=\frac{1}{2\pi}\delta(x^2)\theta(t)$$

$$\begin{aligned}
A^\mu(x)&=\mu_0\int\mathrm{d}^4x'G_{\mathrm{ret}}(x-x')J^\mu(x')\\
&=\frac{\mu_0}{4\pi}\int\mathrm{d}^4x'\frac{\delta(t-t'-R)}{R}J^\mu(x')\\
&=\frac{\mu_0}{4\pi}\int\mathrm{d}^3\boldsymbol{x}'\frac{J^\mu(\boldsymbol{x},t-R)}{R}
\end{aligned}$$

7.2 带电粒子电磁场

7.2.1 Lienard-Wiechert 势

设带电荷 q 的运动粒子，世界线 $\tilde{x}^\mu(\tau)$，速度 $\boldsymbol{v}=\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}t}$

$$J^\mu(x)=q\int\mathrm{d}\tau\frac{\mathrm{d}\tilde{x}^\mu(\tau)}{\mathrm{d}\tau}\delta^4(x-\tilde{x}(\tau))$$

$$
\begin{aligned}
A^{\mu}(x) &= \frac{\mu_0 q}{4\pi}\int \mathrm{d}^4x' \frac{\delta(t-t'-R)}{R}\int \mathrm{d}\tau \frac{\mathrm{d}\tilde{x}^{\mu}(\tau)}{\mathrm{d}\tau}\delta^4(x'-\tilde{x}(\tau)) \\
&= \frac{\mu_0 q}{4\pi}\int \mathrm{d}\tau \frac{\mathrm{d}\tilde{x}^{\mu}(\tau)}{\mathrm{d}\tau}\frac{\delta(t-\tilde{t}-R(\tilde{t}))}{R(\tilde{t})}
\end{aligned}
$$

推迟时间 t_{ret} 满足

$$
t_{\mathrm{ret}} = t - R(t_{\mathrm{ret}})
$$

注意到

$$
\begin{aligned}
\delta(t-\tilde{t}-R(\tilde{t})) &= \frac{\delta(\tilde{t}-t_{\mathrm{ret}})}{\left|\frac{\partial}{\partial \tilde{t}}(t-\tilde{t}-R(\tilde{t}))\right|} \\
\frac{\partial}{\partial \tilde{t}}(\tilde{t}+R(\tilde{t})-t) = 1+\frac{\mathrm{d}R(\tilde{t})}{\mathrm{d}\tilde{t}} &= 1-\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}\tilde{t}}\cdot\nabla R = 1-\boldsymbol{v}(\tilde{t})\cdot\boldsymbol{n}(\tilde{t})
\end{aligned}
$$

还原到国际单位制，得到

$$
\begin{aligned}
\phi(\boldsymbol{x},t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{R}}\right]_{\mathrm{ret}} \\
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\boldsymbol{v}}{R-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{R}}\right]_{\mathrm{ret}}
\end{aligned}
$$

取世界线参数 $\tau=\tilde{t}$，求出电磁场，注意到

$$
\nabla\delta(t-\tilde{t}-R(\tilde{t})) = -\nabla R\frac{\partial}{\partial t}(\delta(t-\tilde{t}-R(\tilde{t}))) = -\boldsymbol{n}(\tilde{t})\frac{\partial}{\partial t}(\delta(t-\tilde{t}-R(\tilde{t})))
$$

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{x},t) &= -\nabla\phi-\frac{\partial \boldsymbol{A}}{\partial t} \\
&= -\frac{q}{4\pi\varepsilon_0}\nabla\int \mathrm{d}\tilde{t}\frac{\delta(t-\tilde{t}-R(\tilde{t}))}{R(\tilde{t})}-\frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\int \mathrm{d}\tilde{t}\boldsymbol{v}(\tilde{t})\frac{\delta(t-\tilde{t}-R(\tilde{t}))}{R(\tilde{t})} \\
&= \frac{q}{4\pi\varepsilon_0}\left\{\left[\frac{\boldsymbol{n}}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})R^2}\right]_{\mathrm{ret}}+\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\boldsymbol{n}-\frac{\boldsymbol{v}}{c}}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})R}\right]_{\mathrm{ret}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{x},t) &= \nabla\times\boldsymbol{A} \\
&= \frac{\mu_0 q}{4\pi}\nabla\times\int \mathrm{d}\tilde{t}\boldsymbol{v}(\tilde{t})\frac{\delta(t-\tilde{t}-R(\tilde{t}))}{R(\tilde{t})} \\
&= \frac{\mu_0 q}{4\pi}\int \mathrm{d}\tilde{t}\left[\frac{\nabla\delta(t-\tilde{t}-R(\tilde{t}))}{R(\tilde{t})}\times\boldsymbol{v}(\tilde{t})-\frac{\boldsymbol{n}(\tilde{t})}{R^2(\tilde{t})}\delta(t-\tilde{t}-R(\tilde{t}))\times\boldsymbol{v}(\tilde{t})\right] \\
&= \frac{\mu_0 q}{4\pi}\int \mathrm{d}\tilde{t}\left[-\frac{\boldsymbol{n}(\tilde{t})}{R(\tilde{t})}\frac{\partial}{\partial t}(\delta(t-\tilde{t}-R(\tilde{t})))\times\boldsymbol{v}(\tilde{t})-\frac{\boldsymbol{n}(\tilde{t})}{R^2(\tilde{t})}\delta(t-\tilde{t}-R(\tilde{t}))\times\boldsymbol{v}(\tilde{t})\right] \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\frac{\boldsymbol{v}}{c}\times\boldsymbol{n}}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})R^2}\right]_{\mathrm{ret}}+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\frac{\boldsymbol{v}}{c}\times\boldsymbol{n}}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})R}\right]_{\mathrm{ret}}\right\}
\end{aligned}
$$

7.2.2 Heaviside-Feynman 公式

推迟时间

$$t_{\text{ret}} = t - R(t_{\text{ret}})$$
$$\frac{\mathrm{d}t}{\mathrm{d}t_{\text{ret}}} = 1 + \frac{\mathrm{d}R(t_{\text{ret}})}{\mathrm{d}t_{\text{ret}}}$$

注意到

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}t}\cdot\nabla R = -\boldsymbol{v}\cdot\boldsymbol{n}$$
$$\frac{\mathrm{d}t}{\mathrm{d}t_{\text{ret}}} = [1-\boldsymbol{v}\cdot\boldsymbol{n}]_{\text{ret}}$$
$$\Longrightarrow \frac{\mathrm{d}t_{\text{ret}}}{\mathrm{d}t} = 1 - \frac{\mathrm{d}R(t_{\text{ret}})}{\mathrm{d}t} = \frac{1}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]_{\text{ret}}}$$
$$\boldsymbol{v}_{\text{ret}} = -\frac{\mathrm{d}\boldsymbol{R}_{\text{ret}}}{\mathrm{d}t_{\text{ret}}} = -[1-\boldsymbol{v}\cdot\boldsymbol{n}]_{\text{ret}}\frac{\mathrm{d}\boldsymbol{R}_{\text{ret}}}{\mathrm{d}t}$$
$$\Longrightarrow \left[\frac{\boldsymbol{v}}{1-\boldsymbol{v}\cdot\boldsymbol{n}}\right]_{\text{ret}} = -\frac{\mathrm{d}\boldsymbol{R}_{\text{ret}}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \frac{1}{R}\left[-\boldsymbol{v}+\boldsymbol{n}(\boldsymbol{v}\cdot\boldsymbol{n})\right] = \frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v})}{R}$$
$$\boldsymbol{n}_{\text{ret}}\times\frac{\mathrm{d}\boldsymbol{n}_{\text{ret}}}{\mathrm{d}t} = \left[\frac{\boldsymbol{v}\times\boldsymbol{n}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})R}\right]_{\text{ret}}$$

代入得到

$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{4\pi\varepsilon_0}\left\{\left[\frac{\boldsymbol{n}}{R^2}\right]_{\text{ret}} + \frac{R_{\text{ret}}}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\boldsymbol{n}}{R^2}\right]_{\text{ret}} + \frac{1}{c^2}\frac{\mathrm{d}^2\boldsymbol{n}_{\text{ret}}}{\mathrm{d}t^2}\right\}$$
$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\boldsymbol{n}}{R}\right]_{\text{ret}}\times\frac{\mathrm{d}\boldsymbol{n}_{\text{ret}}}{\mathrm{d}t} + \frac{\boldsymbol{n}_{\text{ret}}}{c}\times\frac{\mathrm{d}^2\boldsymbol{n}_{\text{ret}}}{\mathrm{d}t^2}\right\}$$
$$\boldsymbol{B} = \boldsymbol{n}_{\text{ret}}\times\frac{\boldsymbol{E}}{c}$$

7.2.3 运动电荷的电磁辐射

将电磁场表达式中对 t 的导数换成对 t_{ret} 的导数

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})R^2} + \frac{1}{1-\boldsymbol{v}\cdot\boldsymbol{n}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{n}-\boldsymbol{v}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})R}\right)\right]_{\text{ret}} \\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})R^2} + \frac{\dot{\boldsymbol{n}}-\dot{\boldsymbol{v}}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})^2R} - \frac{\boldsymbol{n}-\boldsymbol{v}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})^3R^2}\left(R\frac{\mathrm{d}}{\mathrm{d}t}(1-\boldsymbol{v}\cdot\boldsymbol{n}) + (1-\boldsymbol{v}\cdot\boldsymbol{n})\frac{\mathrm{d}R}{\mathrm{d}t}\right)\right]_{\text{ret}}\end{aligned}$$

代入 $\frac{\mathrm{d}R}{\mathrm{d}t} = -\boldsymbol{v}\cdot\boldsymbol{n}$, $\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v})}{R}$ 得到

$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\boldsymbol{n}-\frac{\boldsymbol{v}}{c})(1-\frac{v^2}{c^2})}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})^3R^2} + \frac{\boldsymbol{n}\times((\boldsymbol{n}-\frac{\boldsymbol{v}}{c})\times\dot{\boldsymbol{v}})}{c^2(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})^3R}\right]_{\text{ret}}$$

第二项为辐射项

$$\boldsymbol{E}_a = \frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}\times((\boldsymbol{n}-\frac{\boldsymbol{v}}{c})\times\boldsymbol{a})}{c^2(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})^3R}\right]_{\text{ret}}$$

能流密度

$$\frac{1}{\mu_0 c}\left[E^2\boldsymbol{n}_{\text{ret}}-(\boldsymbol{E}\cdot\boldsymbol{n}_{\text{ret}})\boldsymbol{E}\right]$$

辐射场

$$\boldsymbol{S}=\frac{1}{\mu_0 c}E_a^2\boldsymbol{n}_{\text{ret}}=\frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left[\frac{\boldsymbol{n}\times((\boldsymbol{n}-\frac{\boldsymbol{v}}{c})\times\boldsymbol{a})}{c^2(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})^3R}\right]_{\text{ret}}^2\boldsymbol{n}_{\text{ret}}$$

发射的能流

$$\mathrm{d}P(t)=\frac{\mathrm{d}W}{\mathrm{d}t}=R^2\mathrm{d}\Omega\boldsymbol{S}(t)\boldsymbol{n}_{\text{ret}}$$

观察者接收的能流

$$\mathrm{d}P(t_{\text{ret}})=\frac{\mathrm{d}W}{\mathrm{d}t_{\text{ret}}}=\frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t_{\text{ret}}}=\frac{q^2}{16\pi^2\varepsilon_0 c^3}\left[\frac{|\boldsymbol{n}\times((\boldsymbol{n}-\frac{\boldsymbol{v}}{c})\times\boldsymbol{a})|^2}{(1-\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{n})^5}\right]_{\text{ret}}\mathrm{d}\Omega$$

1. 非相对论极限：Larmor 公式

$$\mathrm{d}P(t_{\text{ret}})=\frac{q^2}{16\pi^2\varepsilon_0 c^3}\boldsymbol{a}_{\text{ret}}^2\sin^2\theta\mathrm{d}\Omega$$

   积分得到

$$P(t_{\text{ret}})=\frac{q^2}{6\pi\varepsilon_0 c^3}\boldsymbol{a}_{\text{ret}}^2$$

2. 相对论情形，直接计算很复杂，采取另一种方法

   带电粒子本征系 S' 中，辐射动量 $\mathrm{d}\boldsymbol{p}'=0,\mathrm{d}\boldsymbol{x}'=0$

$$P(t)=\frac{\mathrm{d}W}{\mathrm{d}t}=\frac{\gamma(\mathrm{d}W'+\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{p}')}{\gamma(\mathrm{d}t'+\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{x}'}{c^2})}=\frac{\mathrm{d}W'}{\mathrm{d}t'}=P'(t')$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\frac{\gamma(\mathrm{d}\boldsymbol{p}'+\boldsymbol{v}\frac{\mathrm{d}W}{c^2})}{\gamma(\mathrm{d}t'+\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{x}'}{c^2})}=\frac{\boldsymbol{v}}{c^2}\frac{\mathrm{d}W'}{\mathrm{d}t'}=\frac{\boldsymbol{v}}{c^2}\frac{\mathrm{d}W}{\mathrm{d}t}$$

   辐射功率是协变的，在本征系中为 Larmor 公式，推广为协变形式

$$P(t_{\text{ret}})=\frac{q^2}{6\pi\varepsilon_0 c^3}(a^\mu a_\mu)_{\text{ret}}\tag{7.1}$$

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}-\frac{1}{c^2}\left(\frac{\mathrm{d}E}{\mathrm{d}\tau}\right)^2=(m\gamma)^2\left[\frac{\mathrm{d}(\gamma\boldsymbol{v})}{\mathrm{d}t}\cdot\frac{\mathrm{d}(\gamma\boldsymbol{v})}{\mathrm{d}t}-c^2\left(\frac{\gamma}{t}\right)^2\right]\tag{7.2}$$

$$P(t)=\frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^6\left[\boldsymbol{a}^2-\left(\frac{\boldsymbol{v}\times\boldsymbol{a}}{c}\right)^2\right]\tag{7.3}$$

7.3 低速带电粒子体系的有效作用量

Lagrangian

$$L=\sum_i\left[-m_i\sqrt{1-\boldsymbol{v}_i^2}-q_i\phi+q_i\boldsymbol{A}\cdot\boldsymbol{v}_i\right]$$

按 $\frac{v}{c}$ 的幂次进行 Taylor 展开，忽略 $m_i c^2$，零阶项

$$L^{(0)} = \sum_i \frac{1}{2} m_i \boldsymbol{v}_i^2 - \sum_{i>j} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{R_{ij}}$$

展开到 $\frac{v^2}{c^2}$ 项，需要考虑推迟效应

$$\begin{aligned}
\phi(\boldsymbol{x}, t) &= \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}^3\boldsymbol{x}' \frac{\rho(\boldsymbol{x}', t - \frac{R}{c})}{R} \\
&= \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}^3\boldsymbol{x}' \frac{\rho(\boldsymbol{x}', t)}{R} - \frac{1}{4\pi\varepsilon_0 c} \frac{\partial}{\partial t} \int \mathrm{d}^3\boldsymbol{x}' \rho(\boldsymbol{x}', t) + \frac{1}{8\pi\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} \int \mathrm{d}^3\boldsymbol{x}' R\rho(\boldsymbol{x}', t) \\
&= \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}^3\boldsymbol{x}' \frac{\rho(\boldsymbol{x}', t)}{R} + \frac{1}{8\pi\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} \int \mathrm{d}^3\boldsymbol{x}' R\rho(\boldsymbol{x}', t) \\
\boldsymbol{A}(\boldsymbol{x}, t) &= \frac{1}{4\pi\varepsilon_0 c^2} \int \mathrm{d}^3\boldsymbol{x}' \frac{\boldsymbol{J}(\boldsymbol{x}', t - \frac{R}{c})}{R}
\end{aligned}$$

如果只有一个电荷 q，

$$\begin{aligned}
\phi(\boldsymbol{x}, t) &= \frac{q}{4\pi\varepsilon_0 R} + \frac{q}{8\pi\varepsilon_0 c^2} \frac{\partial^2 R}{\partial t^2} \\
\boldsymbol{A}(\boldsymbol{x}, t) &= \frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2 R}
\end{aligned}$$

取规范变换参数

$$\begin{aligned}
\varepsilon(\boldsymbol{x}, t) &= \frac{q}{8\pi\varepsilon_0 c^2} \frac{\partial R}{\partial t} \\
\phi' &= \phi - \frac{\partial \varepsilon}{\partial t} = \frac{q}{4\pi\varepsilon_0 R} \\
\boldsymbol{A}' &= \boldsymbol{A} + \nabla\varepsilon = \frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2 R} + \frac{q}{8\pi\varepsilon_0 c^2} \nabla \frac{\partial R}{\partial t} \\
&= \frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2 R} + \frac{q}{8\pi\varepsilon_0 c^2} \frac{\partial \boldsymbol{n}}{\partial t} = \frac{q}{8\pi\varepsilon_0 c^2 R} [\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}]
\end{aligned}$$

这样就得到了电磁势，代入 Lagrangian

$$L = \sum_i \left[\frac{1}{2} m_i \boldsymbol{v}_i^2 + \frac{1}{8} m_i \frac{\boldsymbol{v}_i^4}{c^2}\right] - \frac{1}{4\pi\varepsilon_0} \sum_{i>j} \frac{q_i q_j}{R_{ij}} + \frac{1}{8\pi\varepsilon_0 c^2} \sum_{i>j} \frac{q_i q_j}{R_{ij}} \left[\boldsymbol{v}_i \cdot \boldsymbol{v}_j + (\boldsymbol{v}_i \cdot \boldsymbol{n}_{ij})(\boldsymbol{v}_j \cdot \boldsymbol{n}_{ij})\right]$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2}} \approx m\boldsymbol{v}$$

Hamiltonian

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} + \frac{1}{4\pi\varepsilon_0} \sum_{i>j} \frac{q_i q_j}{R_{ij}} - \sum_i \frac{\boldsymbol{p}_i^4}{8c^2 m_i^3} - \frac{1}{8\pi\varepsilon_0 c^2} \sum_{i>j} \frac{q_i q_j}{m_i m_j R_{ij}} \left[\boldsymbol{p}_i \cdot \boldsymbol{p}_j + (\boldsymbol{p}_i \cdot \boldsymbol{n}_{ij})(\boldsymbol{p}_j \cdot \boldsymbol{n}_{ij})\right]$$

第八章 介质中的电磁场与偶极辐射

8.1 偶极耦合

考虑质量 m、电荷 $+q$、位于 x_1^μ 的粒子，绕着电荷 $-q$、位于 x_2^μ 的核运动

作用量

$$S_I = -q\int \boldsymbol{A}_\mu \mathrm{d}x_2^\mu + q\int \boldsymbol{A}_\mu \mathrm{d}x_1^\mu = \int \mathrm{d}t\,[q\phi(x_2) - q\boldsymbol{A}(x_2)\cdot\dot{\boldsymbol{x}}_2] + \int \mathrm{d}t\,[-q\phi(x_1) + q\boldsymbol{A}(x_1)\cdot\dot{\boldsymbol{x}}_1]$$

假设核和粒子只在空间原点附近微小运动，在空间原点处做 Taylor 展开

$$S_I \approx \int \mathrm{d}t\,[q\boldsymbol{A}(t,0)\cdot(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2)] + \int \mathrm{d}t\,[-q\nabla\phi(t,0)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)] + \int \mathrm{d}t\left[q\partial_i A_j(t,0)(x_1^i\dot{x}_1^j - x_2^i\dot{x}_2^j)\right]$$

对第一项分部积分，得到

$$\int \boldsymbol{E}(t,0)\cdot\boldsymbol{p}$$

对第三项进行规范变换，

$$A_j \to A_j - \partial_j\varepsilon$$

$$\varepsilon = \frac{1}{2}A_i(t,\boldsymbol{x})x^i$$

$$\int \mathrm{d}t\partial_i A_j(t,0)x^i\dot{x}^j \to \frac{1}{2}\int \mathrm{d}tF_{ij}(t,0)x^i\dot{x}^j = \int \mathrm{d}t\frac{1}{2}\epsilon_{ijk}B^k(t,0)x^i\dot{x}^j$$

磁矩

$$m_k = \frac{1}{2}q\epsilon_{ijk}(x_1^i\dot{x}_1^j - x_2^i\dot{x}_2^j)$$

总作用量

$$S_I \approx \int \mathrm{d}t\,[\boldsymbol{p}\cdot\boldsymbol{E}(t,0) + \boldsymbol{m}\cdot\boldsymbol{B}(t,0)]$$

容易推广到多粒子体系

定义极化-磁化张量

$$M^{\mu\nu} \to \begin{bmatrix} 0 & -cP_1 & -cP_2 & -cP_3 \\ cP_1 & 0 & M_3 & -M_2 \\ cP_2 & -M_3 & 0 & M_1 \\ cP_3 & M_2 & -M_1 & 0 \end{bmatrix}$$

$$S_I = \int \mathrm{d}^4x\,[\boldsymbol{P}\cdot\boldsymbol{E} + \boldsymbol{M}\cdot\boldsymbol{B}] = \frac{1}{2}\int \mathrm{d}^4x M_{\mu\nu}F^{\mu\nu}$$

8.2 介质中的电磁场

$$S_I = \frac{1}{2}\int \mathrm{d}^4 x M_{\mu\nu}F^{\mu\nu}$$

对 A_μ 进行变分并分部积分得到

$$\delta S_I = -\int \mathrm{d}^4 x(\partial_\mu M^{\mu\nu})\delta A_\nu$$

诱导出电流

$$J_I^\nu = -\partial_\mu M^{\mu\nu}$$

$$\rho_I = -\nabla\cdot\boldsymbol{P}, \quad \boldsymbol{J}_I = \frac{\partial\boldsymbol{P}}{\partial t} + \nabla\times\boldsymbol{M}$$

自由电流 J_f^μ 与电磁场耦合作用量 S_f，电磁场自身作用量 $S_{em} = -\frac{1}{4\mu_0}\int \mathrm{d}^4 x F_{\mu\nu}F^{\mu\nu}$，整个系统作用量

$$S = S_{em} + S_I + S_f$$

对 A_μ 变分得到 Maxwell 方程

$$-\partial_\mu F^{\mu\nu} = \mu_0(J_f^\nu + J_I^\nu)$$

定义

$$H^{\mu\nu} = \frac{1}{\mu_0}F^{\mu\nu} - M^{\mu\nu}$$

得到

$$-\partial_\mu H^{\mu\nu} = J_f^\nu$$

加上 Bianchi 恒等式就得到 Maxwell 方程组

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

定义电位移矢量和磁场强度

$$H^{0i} = cD_i, \quad H^{ij} = \epsilon^{ijk}H_k$$

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

矢量形式 Maxwell 方程组

$$\begin{cases}\nabla\cdot\boldsymbol{D} = \rho_f \\ \nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{J}_f \\ \nabla\cdot\boldsymbol{B} = 0 \\ \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\end{cases}$$

本构关系

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{B}), \quad \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{E}, \boldsymbol{B})$$

- 对于均匀各向同性线性介质，在介质参考系内

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}, \quad \boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{M} = \kappa \boldsymbol{H}, \quad \boldsymbol{B} = \mu \boldsymbol{H}$$

- 对于导电介质，还有欧姆定律

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}$$

8.2.1 线性介质中电磁场的能量和动量

能量-动量张量

$$T^{\mu\nu}_{\text{em}} = \frac{1}{\mu_0}\left[F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right]$$
$$\partial_\mu T^{\mu\nu}_{\text{em}} = -F^{\nu}{}_{\rho}J^{\rho}$$

假设没有自由电流

$$\partial_\mu T^{\mu\nu}_{\text{em}} = -F^{\nu}{}_{\rho}J^{\rho}_{I}$$

$$\begin{aligned}
\partial_\mu T^{\mu\nu}_{\text{em}} &= F^{\nu}{}_{\rho}\partial_\mu M^{\mu\rho} \\
&= \partial_\mu(F^{\nu}{}_{\rho}M^{\mu\rho}) - (\partial^{\mu}F^{\nu\rho})M_{\mu\rho} \\
&= \partial_\mu(F^{\nu}{}_{\rho}M^{\mu\rho}) - \frac{1}{2}(\partial^{\mu}F^{\nu\rho} - \partial^{\rho}F^{\nu\mu})M_{\mu\rho} \\
&= \partial_\mu(F^{\nu}{}_{\rho}M^{\mu\rho}) = \frac{1}{2}(\partial^{\mu}F^{\rho\nu} + \partial^{\rho}F^{\nu\mu})M_{\mu\rho}
\end{aligned}$$

$$\partial_\mu\left[T_{\text{em}} - F^{\nu}{}_{\rho}M^{\mu\rho}\right] = -\frac{1}{2}(\partial^{\nu}F^{\mu\rho})M_{\mu\rho} = -\frac{1}{4}\partial^{\nu}(F^{\mu\rho}M_{\mu\rho}) + \frac{1}{4}F^{\mu\rho}(\partial^{\nu}M_{\mu\rho}) - \frac{1}{4}(\partial^{\nu}F^{\mu\rho})M_{\mu\rho}$$
$$\partial_\mu\left[T_{\text{em}} - F^{\nu}{}_{\rho}M^{\mu\rho}\right] + \frac{1}{4}\partial^{\nu}(F^{\mu\rho}M_{\mu\rho}) = \frac{1}{4}\left[F^{\mu\rho}(\partial^{\nu}M_{\mu\rho}) - (\partial^{\nu}F^{\mu\rho})M_{\mu\rho}\right]$$

8.3 偶极子的场

8.3.1 静态偶极子的场

假设偶极子位于原点

$$\begin{aligned}
L &= \int \mathrm{d}^3\boldsymbol{x}\left[\frac{1}{2}\varepsilon_0\boldsymbol{E}^2 - \frac{1}{2\mu_0}\boldsymbol{B}^2 + (\boldsymbol{p}\cdot\boldsymbol{E})\delta^3(\boldsymbol{x}) + (\boldsymbol{m}\cdot\boldsymbol{B})\delta^3(\boldsymbol{x})\right] \\
&= \int \mathrm{d}^3\boldsymbol{x}\left[\frac{1}{2}\varepsilon_0(\nabla\phi)^2 - \frac{1}{2\mu_0}(\nabla\times\boldsymbol{A})^2 - (\boldsymbol{p}\cdot\nabla\phi)\delta^3(\boldsymbol{x}) + (\boldsymbol{m}\cdot(\nabla\times\boldsymbol{A}))\delta^3(\boldsymbol{x})\right]
\end{aligned}$$

对 ϕ 和 $\boldsymbol{A}$ 变分，得到

$$\varepsilon_0\nabla^2\phi = \boldsymbol{p}\cdot\nabla\delta^3(\boldsymbol{x})$$
$$-\frac{1}{\mu_0}\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{m}\times\nabla\delta^3(\boldsymbol{x})$$

取 Coulomb 规范

$$\frac{1}{\mu_0}\nabla^2\boldsymbol{A} = \boldsymbol{m}\times\nabla\delta^3(\boldsymbol{x})$$

注意到

$$\delta^3(\boldsymbol{x}) = -\nabla^2\left(\frac{1}{4\pi|\boldsymbol{x}|}\right)$$

得到

$$\phi = \frac{\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\varepsilon_0|\boldsymbol{x}|^3}$$
$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{|\boldsymbol{x}|^3}$$
$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{3(\boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n}-\boldsymbol{p}}{|\boldsymbol{x}|^3}$$
$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\boldsymbol{n})\boldsymbol{n}-\boldsymbol{m}}{|\boldsymbol{x}|^3}$$

8.3.2 偶极辐射

假设电偶极矩与磁偶极矩随时间变化，诱导电流

$$\rho_I(\boldsymbol{x},t) = -\boldsymbol{p}(t)\cdot\nabla\delta^3(\boldsymbol{x})$$
$$\boldsymbol{J}_I(\boldsymbol{x},t) = \dot{\boldsymbol{p}}(t)\delta^3(\boldsymbol{x}) - \boldsymbol{m}(t)\times\nabla\delta^3(\boldsymbol{x})$$

代入推迟势公式

$$\begin{aligned}
\phi(\boldsymbol{x},t) &= \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\boldsymbol{x}'\frac{-\boldsymbol{p}(t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})\cdot\nabla'\delta^3(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\boldsymbol{x}'\delta^3(\boldsymbol{x}')\nabla'\cdot\frac{\boldsymbol{p}(t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= -\frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\boldsymbol{x}'\delta^3(\boldsymbol{x}')\nabla\cdot\frac{\boldsymbol{p}(t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= -\frac{1}{4\pi\varepsilon_0}\nabla\cdot\frac{\boldsymbol{p}(t-\frac{|\boldsymbol{x}|}{c})}{|\boldsymbol{x}|}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\boldsymbol{x}'\frac{\dot{\boldsymbol{p}}(t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})\delta^3(\boldsymbol{x}') - \boldsymbol{m}(t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})\times\nabla'\delta^3(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= \frac{\mu_0}{4\pi|\boldsymbol{x}|}\dot{\boldsymbol{p}}(t-\frac{|\boldsymbol{x}|}{c}) + \frac{\mu_0}{4\pi}\nabla\times\frac{\boldsymbol{m}(t-\frac{|\boldsymbol{x}|}{c})}{|\boldsymbol{x}|}
\end{aligned}$$

记 $\boldsymbol{p}(t-\frac{|\boldsymbol{x}|}{c}=[\boldsymbol{p}])$

$$\partial_i[p_j]=-[\dot{p}_j]\partial_i\frac{|\boldsymbol{x}|}{c}=-\frac{1}{c}n_i[\dot{p}_j]$$

保留到 $\frac{1}{|\boldsymbol{x}|}$ 阶

$$\boldsymbol{E}=\frac{\mu_0}{4\pi}\frac{(\boldsymbol{n}\cdot[\ddot{\boldsymbol{p}}])\boldsymbol{n}-[\ddot{\boldsymbol{p}}]}{|\boldsymbol{x}|}$$

第九章 磁单极子和 θ 项

9.1 电磁对偶

无源 Maxwell 方程组

$$\begin{cases} \nabla\cdot\boldsymbol{E}=0 \\ \nabla\times\boldsymbol{B}=\frac{\partial\boldsymbol{E}}{\partial t} \\ \nabla\cdot\boldsymbol{B}=0 \\ \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \end{cases}$$

在如下变换下保持不变

$$\begin{cases} \boldsymbol{E}\to\boldsymbol{B} \\ \boldsymbol{B}\to-\boldsymbol{E} \end{cases}$$

在外微分形式下容易看出

$$\mathrm{d}(\star F)=0,\quad \mathrm{d}F=0$$

有源 Maxwell 方程组需要引入磁荷 g、磁荷密度 ρ_m 和磁流密度 $\boldsymbol{J}_m$

$$\begin{cases} \nabla\cdot\boldsymbol{E}=\rho_e \\ \nabla\times\boldsymbol{B}=\frac{\partial\boldsymbol{E}}{\partial t}+\boldsymbol{J}_e \\ \nabla\cdot\boldsymbol{B}=\rho_m \\ \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}-\boldsymbol{J}_m \end{cases}$$

在如下变换下保持不变

$$\begin{cases} \boldsymbol{E}\to\boldsymbol{B},\quad \rho_e\to\rho_m,\quad \boldsymbol{J}_e\to\boldsymbol{J}_m \\ \boldsymbol{B}\to-\boldsymbol{E},\quad \rho_m\to-\rho_e,\quad \boldsymbol{J}_m\to-\boldsymbol{J}_e \end{cases}$$

Lorentz 力公式

$$\boldsymbol{F}=e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})+g(\boldsymbol{B}-\boldsymbol{v}\times\boldsymbol{E})$$

在场论中，引入磁荷需要引入带磁荷的物质场并与电磁场耦合，但带磁荷的场与带电荷的场不容易兼容，两者没有共同的基本场变量

9.2 Dirac 理论

仅引入粒子而不引入带电荷或磁荷的场

考虑一个位于原点的磁单极子，磁荷为 g

$$\nabla \cdot \boldsymbol{B} = g\delta^3(\boldsymbol{x})$$
$$\boldsymbol{B} = \frac{g}{4\pi r^2}\boldsymbol{e}_r$$

在 $\mathbb{R}^3\backslash\{\mathbf{0}\}$ 上，

$$\nabla \cdot \boldsymbol{B} = 0$$

由 Poincaré 引理，在 $\mathbb{R}^3\backslash\{\mathbf{0}\}$ 的任何一个拓扑非平庸的开子集 U 上，可以找到矢势 $\boldsymbol{A}^U$ 使 $\boldsymbol{B} = \nabla \times \boldsymbol{A}^U$，因此可以分别在北半球和南半球定义

$$\boldsymbol{A}^N = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\boldsymbol{e}_\varphi$$
$$\boldsymbol{A}^S = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\boldsymbol{e}_\varphi$$

$\boldsymbol{A}^N$ 在 $-z$ 轴没有定义，$\boldsymbol{A}^S$ 在 $+z$ 轴没有定义

协变导数

$$\mathrm{D}_\mu = \partial_\mu - \mathrm{i}\frac{e}{\hbar}A_\mu$$

物质场 $U(1)$ 规范变换

$$\mathrm{e}^{\mathrm{i}\frac{e}{\hbar}\varepsilon(x)}$$

矢势规范变换

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} + \nabla\varepsilon$$

在赤道上

$$A_\varphi^N = A_\varphi^S + \frac{1}{r\sin\theta}\partial_\varphi\varepsilon, \quad \varepsilon = \frac{g\varphi}{2\pi}$$
$$\Longrightarrow \varepsilon(2\pi) = \varepsilon(0) + g$$

根据 $U(1)$ 规范变换

$$\frac{e}{\hbar}\varepsilon(2\pi) = \frac{e}{\hbar}\varepsilon(0) + 2\pi n$$
$$\Longrightarrow \varepsilon(2\pi) = \varepsilon(0) + \frac{2\pi n\hbar}{e}$$

综上，得到 Dirac 量子化条件

$$eg = 2\pi n\hbar, \quad n \in \mathbb{Z}$$

- 若电荷是量子化的，则磁通也是量子化的
$$\int_{S^2}\boldsymbol{B}\cdot\boldsymbol{S} = g = \frac{2\pi n\hbar}{e}$$
  最小磁通
$$\Phi_0 = \frac{2\pi\hbar}{e}$$

- 若电荷之间比例为无理数，则 Dirac 量子化条件无法满足，不存在磁单极子

对于包含多个磁单极子的情形，仍然有

$$\boldsymbol{A}^N = \boldsymbol{A}^S + \nabla\varepsilon$$

$$\int_{S^2} \boldsymbol{B}\cdot\boldsymbol{S} = \varepsilon(2\pi) - \varepsilon(0) = \frac{2\pi n\hbar}{e}$$

易知对任意闭合曲面 Σ 均成立

$$\int_\Sigma F = \frac{2\pi n\hbar}{e} \iff \int_\Sigma \frac{\frac{e}{\hbar}F}{2\pi} = n \in \mathbb{Z}$$

$c_1 = \frac{\frac{e}{\hbar}F}{2\pi}$ 被称为第一陈类

对于双荷子 (e_1, g_1) 和 (e_2, g_2)，要满足 Dirac-Zwanziger 量子化条件

$$e_1g_2 - e_2g_1 \in 2\pi\hbar\mathbb{Z}$$

9.3 θ 项

最简单的电磁作用量

$$S = S_{\text{Maxwell}} + S_\theta$$

$$S_{\text{Maxwell}} = -\frac{1}{4}\int \mathrm{d}^4x F^{\mu\nu}F_{\mu\nu}$$

$$\begin{aligned}S_\theta &= \theta\frac{e^2}{4\pi^2\hbar}\int \mathrm{d}^4x\boldsymbol{E}\cdot\boldsymbol{B} \\ &= \theta\frac{e^2}{4\pi^2\hbar}\int \mathrm{d}^4x\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} \\ &= \theta\frac{e^2}{4\pi^2\hbar}\int \mathrm{d}^4x\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &= \theta\frac{e^2}{\hbar}\frac{1}{2}\int\left(\frac{F}{2\pi}\right)\wedge\left(\frac{F}{2\pi}\right)\end{aligned}$$

四维时空 M 分解为两个二维曲面 Σ_1 和 Σ_2 的 Cartesian 积

$$M = \Sigma_1 \times \Sigma_2$$

$$\int_{\Sigma_1}\frac{\frac{e}{\hbar}F}{2\pi} = n_1, \quad \int_{\Sigma_2}\frac{\frac{e}{\hbar}F}{2\pi} = n_2, \quad n_1, n_2 \in \mathbb{Z}$$

$$\frac{1}{2}\int_M\left(\frac{F}{2\pi}\right)\wedge\left(\frac{F}{2\pi}\right) = \frac{1}{2}\left(\int_{\Sigma_1}\frac{F}{2\pi}\int_{\Sigma_2}\frac{F}{2\pi} + \int_{\Sigma_2}\frac{F}{2\pi}\int_{\Sigma_1}\frac{F}{2\pi}\right) = n_1n_2\left(\frac{\hbar}{e}\right)^2$$

$$S_\theta = \hbar\theta n_1n_2$$

根据量子场论路径积分表述，θ 项贡献

$$\mathrm{e}^{\mathrm{i}\frac{S_\theta}{\hbar}} = \mathrm{e}^{\mathrm{i}\theta n_1n_2}$$

由单值性，θ 为周期 2π 的角变量

9.3.1 轴子电动力学

假设 θ 依赖于时空坐标 x，称为轴子场

$$S=\int \mathrm{d}^4x\left[-\frac{1}{4}\int \mathrm{d}^4xF^{\mu\nu}F_{\mu\nu}+\frac{e^2}{32\pi^2\hbar}\theta(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]$$

$$\begin{aligned}\delta S&=\int \mathrm{d}^4x\left[\partial_\mu F^{\mu\nu}\delta A_\nu+\frac{e^2}{16\pi^2\hbar}\theta(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\delta F_{\rho\sigma}\right]\\&=\int \mathrm{d}^4x\left[\partial_\mu F^{\mu\nu}\delta A_\nu+\frac{e^2}{4\pi^2\hbar}\theta(x)\tilde{F}^{\rho\sigma}\partial_\rho(\delta A_\sigma)\right]\\&=\int \mathrm{d}^4x\left[\partial_\mu F^{\mu\nu}-\frac{e^2}{4\pi^2\hbar}\partial_\mu(\theta(x)\tilde{F}^{\mu\nu})\right]\delta A_\nu\end{aligned}$$

得到轴子电动力学场方程

$$\partial_\mu\left[F^{\mu\nu}-\frac{e^2}{4\pi^2\hbar}\theta(x)\tilde{F}^{\mu\nu}\right]=0$$

注意到

$$\partial_\mu\tilde{F}^{\mu\nu}=0$$

$$\Longrightarrow\ \partial_\mu F^{\mu\nu}=\frac{e^2}{4\pi^2\hbar}(\partial_\mu\theta)\tilde{F}^{\mu\nu}$$

- 将 $\frac{e^2}{4\pi^2\hbar}\theta(x)\tilde{F}^{\mu\nu}$ 看作磁化极化张量 $M^{\mu\nu}$

  本构关系

$$\begin{cases}\boldsymbol{D}=\boldsymbol{E}+\frac{\alpha}{\pi}\theta\boldsymbol{B}\\\boldsymbol{H}=\boldsymbol{B}-\frac{\alpha}{\pi}\theta\boldsymbol{E}\end{cases}$$

  场方程 $\partial_\mu H^{\mu\nu}=0$ 给出

$$\nabla\cdot\boldsymbol{D}=0,\quad \nabla\times\boldsymbol{H}=\frac{\partial\boldsymbol{D}}{\partial t}$$

  $\partial_\mu\tilde{F}^{\mu\nu}=0$ 给出

$$\nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$$

- 将 $-\frac{e^2}{4\pi^2\hbar}(\partial_\mu\theta)\tilde{F}^{\mu\nu}$ 看作电流四矢量 J_I^ν

$$J_I^\nu=-\frac{\alpha}{\pi}(\partial_\mu\theta)\tilde{F}^{\mu\nu}$$

$$\begin{cases}\rho_I=-\frac{\alpha}{\pi}\nabla\theta\cdot\boldsymbol{B}\\\boldsymbol{J}_I=\frac{\alpha}{\pi}(\dot{\theta}\boldsymbol{B}+\nabla\theta\times\boldsymbol{E})\end{cases}$$

  场方程给出

$$\nabla\cdot\boldsymbol{E}=\rho_I,\quad \nabla\times\boldsymbol{B}-\frac{\partial\boldsymbol{E}}{\partial t}=\boldsymbol{J}_I$$

$\partial_\mu \tilde{F}^{\mu\nu} = 0$ 给出

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

9.3.2 拓扑绝缘体

拓扑绝缘体内部可以实现轴子电动力学，如 Bi_2Se_3 和 Bi_2Te_3，在拓扑绝缘体内部 $\theta = \pi$，外部 $\theta = 0$

拓扑绝缘体有拓扑磁电效应，考虑填满 $z < 0$ 空间的拓扑绝缘体材料，在材料内部施加电场 $\boldsymbol{E} = E\boldsymbol{e}_y$，电流密度 $J_x = -\frac{\alpha}{\pi}\partial_z \theta E_y$，面电流密度

$$K_x = \int J_x \mathrm{d}z = \alpha E_y$$

Hall 电导

$$\sigma_{xy} = \alpha = \frac{1}{2}\frac{e^2}{2\pi\hbar}$$

而量子 Hall 效应结果指出，Hall 电导为 $\frac{e^2}{2\pi\hbar}$ 的有理数倍

也可以用边值关系求解

9.3.3 Witten 效应

将磁荷 g 的磁单极子放到真空中，在外侧球对称地包裹 $\theta \neq 0$ 的介质，在介质中既有电场又有磁场，则对于介质，相当于磁单极子携带电荷

$$q = \int \mathrm{d}^3\boldsymbol{x}\rho_I = -\frac{e^2}{4\pi^2\hbar}\frac{\theta}{\pi}g$$

取最小磁荷 $g = \frac{2\pi\hbar}{e}$

$$q = -\frac{\theta}{2\pi}e$$

$\boldsymbol{E}$ 在空间反演下是奇的，在时间反演下是偶的，而 $\boldsymbol{B}$ 在空间反演下是偶的，在时间反演下是奇的

$$S_\theta = \theta\frac{e^2}{4\pi^2\hbar}\int \mathrm{d}^4x \boldsymbol{E}\cdot\boldsymbol{B}$$

在空间和时间反演下都是奇的，除非 $\theta = 0, \pi$，因此拓扑绝缘体分为这两种

第十章 Yang-Mills 理论与’t Hooft-Polyakov 磁单极

Yang-Mills 理论即非 Abel 规范场论，而’t Hooft-Polyakov 磁单极是一类非 Abel 规范场的孤子解

10.1 Yang-Mills 理论

$U(1)$ 整体对称性局域化得到电磁场理论，$U(N)$ 整体对称性局域化得到 Yang-Mills 理论，是非线性的

$SU(N)$ 群的元素 U 可以写成

$$U = \mathrm{e}^{\mathrm{i}\theta^a T_a}$$

Hermite 矩阵 T_a 满足

$$\mathrm{Tr}(T_a) = 0$$

归一化条件

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$$

独立 Hermite 矩阵又 $N^2 - 1$ 个

$$\begin{aligned}\mathrm{Tr}([T_a, T_b]) &= 0\\ \implies [-\mathrm{i}T_a, -\mathrm{i}T_b] &= C_{abc}(-\mathrm{i}T_c)\\ C_{abc} &= -2\mathrm{i}\mathrm{Tr}([T_a, T_b]T_c)\end{aligned}$$

- 称物质场取 $SU(N)$ 规范群的基础表示，若物质场为 N 维复标量场 Φ，局域 $SU(N)$ 规范变换

  $$\Phi \to \Phi' = U(x)\Phi = \mathrm{e}^{\mathrm{i}\epsilon_a(x)T_a}\Phi$$

  协变导数

  $$\mathrm{D}_\mu = \partial_\mu + A_\mu, \quad A_\mu = -\mathrm{i}T_a A_\mu^a$$

  规范势 A_μ 变换规则

  $$A_\mu \to A'_\mu = UA_\mu U^{-1} + U\partial_\mu U^{-1}$$

  当规范变换参数 $\epsilon_a(x)$ 为无穷小量时，

  $$\delta A_\mu = A'_\mu - A_\mu = \partial_\mu\varepsilon + [A_\mu, \varepsilon], \quad \varepsilon = -\mathrm{i}T_a\epsilon_a(x)$$

定义规范场强

$$F_{\mu\nu} = -\mathrm{i}T_a F^a_{\mu\nu} = [\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$
$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + C_{abc} A^a_\mu A^b_\nu$$
$$F = \mathrm{d}A + A \wedge A$$

变换规则

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$$

- 称物质场取 $SU(N)$ 规范群的伴随表示，若物质场为 $N \times N$ 矩阵 ϕ

$$\phi(x) = -\mathrm{i}T_a \phi^a(x)$$

变换规则

$$\phi(x) \to U(x)\phi(x)U^{-1}(x)$$

协变导数

$$\mathrm{D}_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$$

规范场作用量取为

$$S_g = a \int \mathrm{d}^4x \frac{1}{2} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + b \int \mathrm{d}^4x \mathrm{Tr}(F \wedge F)$$

$a = \frac{1}{e^2}$，e 为规范场耦合常数，注意到

$$\mathrm{Tr}(F \wedge F) = \mathrm{d}\mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A)$$

第二项积分无贡献

$$S_g = \int \mathrm{d}^4x \frac{1}{2e^2} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\int \mathrm{d}^4x \frac{1}{4e^2} F^a_{\mu\nu} F^{a\mu\nu}$$

对 A_μ 变分

$$\delta F_{\mu\nu} = \mathrm{D}_\mu \delta A_\nu - \mathrm{D}_\nu \delta A_\mu$$
$$\mathrm{D}_\mu \delta A_\nu = \partial_\mu \delta A_\nu + [A_\mu, \delta A_\nu]$$
$$\partial_\mu \mathrm{Tr}(A_1 A_2) = \mathrm{Tr}((\mathrm{D}_\mu A_1) A_2) + \mathrm{Tr}(A_1 \mathrm{D}_\mu A_2)$$

分部积分得到

$$\delta S_g = \frac{2}{e^2} \int \mathrm{d}^4x \mathrm{Tr}(-\mathrm{D}_\mu F^{\mu\nu} \delta A_\nu) = \frac{1}{e^2} \int \mathrm{d}^4x (\mathrm{D}_\mu F^{\mu\nu})^a \delta A^a_\nu$$

- 若物质场取基础表示，作用量

$$S_m = -\int \mathrm{d}^4x \left[(\mathrm{D}_\mu \Phi)^\dagger \mathrm{D}^\mu \Phi + \mathscr{U}(\Phi^\dagger \Phi) \right]$$

对 A^a_μ 变分

$$\delta S_m = \mathrm{i} \int \mathrm{d}^4x \left[(\mathrm{D}^\mu \Phi)^\dagger T_a \Phi - \Phi^\dagger T_a \mathrm{D}^\mu \Phi \right] \delta A^a_\mu$$

得到规范场场方程

$$-\frac{1}{e^2}(\mathrm{D}_\mu F^{\mu\nu})^a = \mathrm{i}\left[(\mathrm{D}^\mu\Phi)^\dagger T_a\Phi - \Phi^\dagger T_a \mathrm{D}^\mu\Phi\right]$$

对 Φ 变分得到物质场场方程

$$\mathrm{D}_\mu \mathrm{D}^\mu \Phi = \frac{\partial \mathscr{U}}{\partial \Phi^\dagger}$$

$$\mathrm{D}_\mu \mathrm{D}^\mu \Phi^\dagger = \frac{\partial \mathscr{U}}{\partial \Phi}$$

、

- 若物质场取伴随表示，作用量

$$S_m = \int \mathrm{d}^4 \left[\frac{1}{e^2}\mathrm{Tr}(\mathrm{D}_\mu\phi \mathrm{D}^\mu\phi) - \mathscr{U}(\phi^a\phi^a)\right]$$

$$\delta S_m = \frac{2}{e^2}\int \mathrm{d}^4\mathrm{Tr}([\phi, \mathrm{D}^\mu\phi]\delta A_\mu) + \int \mathrm{d}^4\left[\frac{1}{e^2}(\mathrm{D}_\mu \mathrm{D}^\mu\phi)^a - \frac{\partial\mathscr{U}}{\partial\phi^a}\right]\delta\phi^a$$

得到规范场场方程和物质场场方程

$$\mathrm{D}_\mu F^{\mu\nu} = [\phi, \mathrm{D}^\nu\phi]$$

$$\frac{1}{e^2}(\mathrm{D}_\mu \mathrm{D}^\mu\phi)^a = \frac{\partial\mathscr{U}}{\partial\phi^a}$$

10.2 ’t Hooft-Polyakov 磁单极

考虑具有 $SU(2)$ 规范对称性的系统，作用量

$$S = \int \mathrm{d}^4x\left[\frac{1}{2e^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{e^2}\mathrm{Tr}(\mathrm{D}_\mu\phi \mathrm{D}^\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - v^2)^2\right]$$

能动量张量

$$T^{\mu\nu} = \frac{1}{e^2}\left[F^{a\mu}{}_\rho F^{a\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F^a_{\rho\sigma}F^{a\rho\sigma}\right] + \frac{1}{e^2}(\mathrm{D}^\mu\phi)^a(\mathrm{D}^\nu\phi)^a + \eta^{\mu\nu}\left[\frac{1}{e^2}\mathrm{Tr}(\mathrm{D}_\mu\phi \mathrm{D}^\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - v^2)^2\right]$$

$$\mathscr{H} = T^{00} = \frac{1}{2e^2}\left[\boldsymbol{E}^a\cdot\boldsymbol{E}^a + \boldsymbol{B}^a\cdot\boldsymbol{B}^a + (\mathrm{D}_0\phi)^a(\mathrm{D}_0\phi)^a + (\mathrm{D}_i\phi)^a(\mathrm{D}_i\phi)^a\right] + \mathscr{U}(|\phi|)$$

$$\mathscr{U}(|\phi|) = \frac{\lambda}{4}(|\phi|^2 - v^2)^2, \quad E^a_i = F^{a0i}, \quad B^a_i = \frac{1}{2}\varepsilon_{ijk}F^{ajk}$$

ϕ 称为 Higgs 场

- 真空场位形

$$\mathscr{H} = 0 \iff F^{a\mu\nu} = \mathrm{D}^\mu\phi = \mathscr{U}(|\phi|) = 0$$

真空解 ϕ 为常数

$$|\phi|^2 = v^2$$

在 $U(1)$ 变换下，真空解保持不变

$$\mathrm{e}^{-\alpha\frac{\phi}{v}}\phi\mathrm{e}^{\alpha\frac{\phi}{v}} = \phi$$

在 $SU(2)$ 规范变换下，真空解不能保持不变，形成一个解的等价类

- 有限能量解

 定义 Higgs 真空

$$\mathscr{M}_H = \{\phi|\mathscr{U}(|\phi|) = 0\} = S^2$$

 对于有限能量解，在无穷远处场位形趋于 Higgs 真空，ϕ 构成了两者之间的映射

$$\phi : S^2_\infty \to \mathscr{M}_H = S^2$$

 根据覆盖 S^2 的次数，可以进行拓扑学分类，称为 S^2 的第二同伦群 $\pi_2(S^2) = \mathbb{Z}$，覆盖次数

$$n = \frac{1}{8\pi v^3}\int_{S^2_\infty} \varepsilon_{abc}\phi^a \mathrm{d}\phi^b \wedge \mathrm{d}\phi^c$$

 在无穷远处，同样有 $U(1)$ 对称性

$$U(1) = \mathrm{e}^{-\alpha\frac{\phi}{v}}$$

 覆盖 $n \neq 0$ 时，A_μ 必须取非 0 场位形，总能量有限，

$$(D_\mu\phi)^a = \partial_\mu\phi^a + \varepsilon_{abc}A^b_\mu\phi^c \sim 0$$

 无穷远处

$$\phi^a\phi^a \sim v^2,\quad \phi^a\partial_\mu\phi^a \sim 0$$

$$A^a_\mu \sim \frac{1}{v^2}\varepsilon_{abc}\phi^b\partial_\mu\phi^c + \frac{1}{v}\phi^a a_\mu(x)$$

$$\mathscr{F}_{\mu\nu} = \frac{\phi^a}{v}F^a_{\mu\nu} \sim \partial_\mu a_\nu - \partial_\nu a_\mu + \frac{1}{v^3}\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c$$

 最后一项额外项带来磁荷

$$g = \int_{S^2_\infty}\mathscr{F} = 4\pi n$$

 $\mathscr{F}^{\mu\nu}$ 满足无源 Maxwell 方程组，从无穷远处的 $U(1)$ 对称性看来，这样的解为磁单极子，称为’t Hooft-Polyakov 磁单极

- 求解磁单极解

 在 $n = 1$ 时才有静态解，考虑 $n = 1$ 情形，选取 $A_0 = D_0\phi = E^a_i = 0$

$$\mathscr{H} = \frac{1}{2e^2}\left[\boldsymbol{B}^a\cdot\boldsymbol{B}^a + (\mathrm{D}_i\phi)^a(\mathrm{D}_i\phi)^a\right] + \mathscr{U}(|\phi|)$$

 Higgs 场取如下形式

$$\phi^a = \frac{x^a}{r^2}h(r)$$

$$h(r) = \begin{cases} 0, & r \to 0 \\ vr, & r \to \infty \end{cases}$$

相应的规范场取如下形式

$$A_i^a = -\varepsilon_{aij}\frac{x^j}{r^2}(1-k(r))$$

$$k(r) = \begin{cases} 0, & r \to 0 \\ 1, & r \to \infty \end{cases}$$

代入场方程即可求解

- Bogomolnyi 能限

能量密度

$$\mathscr{H} = \frac{1}{2e^2}\left[(E_i^a - \mathrm{D}_i\phi^a\sin\theta)^2 + (B_i^a - \mathrm{D}_i\phi^a\cos\theta)^2 + (\mathrm{D}_0\phi)^2\right] + \mathscr{U}(|\phi|) + \frac{1}{e^2}\left[E_i^a\mathrm{D}_i\phi^a\sin\theta + B_i^a\mathrm{D}_i\phi^a\cos$$

注意到 $\mathrm{D}_iB_i = 0$

$$\frac{1}{v}\int \mathrm{d}^3\boldsymbol{x}\, B_i^a\mathrm{D}_i\phi^a = \frac{1}{v}\int_{S_\infty^2}(\boldsymbol{B}^a\phi^a)\cdot\mathrm{d}\boldsymbol{S} = g$$

$$\frac{1}{v}\int \mathrm{d}^3\boldsymbol{x}\, E_i^a\mathrm{D}_i\phi^a = \frac{1}{v}\int_{S_\infty^2}(\boldsymbol{E}^a\phi^a)\cdot\mathrm{d}\boldsymbol{S} = q$$

总能量

$$\begin{aligned}\mathscr{E} &= \int \mathrm{d}^3\boldsymbol{x}\,\mathscr{H} \\ &= \frac{v}{e^2}[q\sin\theta + g\cos\theta] + \int \mathrm{d}^3\boldsymbol{x}\left\{\frac{1}{2e^2}\left[(E_i^a - \mathrm{D}_i\phi^a\sin\theta)^2 + (B_i^a - \mathrm{D}_i\phi^a\cos\theta)^2 + (\mathrm{D}_0\phi)^2\right] + \mathscr{U}(|\phi|)\right\} \\ &\geq \frac{v}{e^2}[q\sin\theta + g\cos\theta] \\ &\geq \frac{v}{e^2}\sqrt{q^2+g^2}\end{aligned}$$

在 Bogomolnyi 能限下有如下 BPS 方程

$$E_i^a = \mathrm{D}_i\phi^a\sin\theta_m,\quad B_i^a = \mathrm{D}_i\phi^a\cos\theta_m,\quad \mathrm{D}_0\phi^a = 0$$

$$\tan\theta_m = \frac{q}{g}$$

对于’t Hooft-Polyakov 磁单极，$q = \theta_m = 0$，BPS 方程化为

$$E_i^a = 0,\quad \mathrm{D}_0\phi^a = 0,\quad B_i^a = D_i\phi^a$$

对于 $n = 1$ 的磁单极子，解得

$$h(r) = vr\coth(vr) - 1,\quad k(r) = \frac{vr}{\sinh(vr)}$$