Volatility Forecasting with the Multifractal Model of Asset Returns

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Abstract

This paper presents an empirical application of the Multifractal Model of Asset Returns (MMAR) to intraday stock prices, with a goal of generating accurate volatility forecasts. Intraday stock volatility exhibits long tails, persistence, and strong evidence of moment scaling. This allows us to apply the MMAR. A forecasting method for the MMAR is implemented through Monte Carlo simulation, and this forecasting method is compared to Generalized Autoregressive Conditional Heteroskedasticity (GARCH) alternatives over several testing samples. The MMAR significantly outperformed the GARCH models. This suggests that the framework of multifractality has a large potential for further development and application within finance.

1. Literature Review

The study of fractal geometry has a long history, with many key players. In addition, fractals and the mathematics that govern them can be studied from many different approaches. Of all the minds that have studied fractal geometry, perhaps none have contributed as much to the field as Benoit B. Mandelbrot. To him, fractal geometry is the “study of roughness, of the irregular and jagged” (Mandelbrot, 2006, p. 116). A unique characteristic of fractal geometry is the ability to express a variety complex behavior in a few simple rules and formulae (p. 117).

The roots of the current Multifractal Model of Asset Returns (MMAR) can be traced back to Mandelbrot (1963), which discussed fat tails in the distribution of returns, as well as Levy-stable processes. Long memory, a feature of the MMAR, was investigated in Mandelbrot and Wallis (1968). Fractional Brownian motion (FBM), one of the two components of the MMAR, was discussed in Mandelbrot and Ness (1968). Long memory in absolute returns is a feature of FBM. The second component of the MMAR is the concept of trading time, or time deformation. Trading time is expressed in the MMAR by multifractal measures. Indeed, it is this component of the MMAR that gives the model its multifractal behavior; a FBM alone does not exhibit multifractal characteristics. Mandelbrot (1989) provides a technical discussion of multifractal measures, with application to geophysics. In addition, it gives a summary of prior work conducted into multifractals.

The MMAR was introduced in series of three papers written by Mandelbrot and his two graduate students. Mandelbrot, Calvet, and Fisher (1997), the first paper in the series, lays out the foundations of the MMAR and its main properties. It compares the MMAR to its peers, and briefly introduces other multifractal formalisms. Calvet, Fisher, and Mandelbrot (1997), the second paper in the series, discusses several elements of multifractals in greater depth than the first paper. Topics contained within the paper include Hölder exponents, the multifractal spectrum, and multifractal measures. Fisher, Calvet, and Mandelbrot (1997), the last paper in the series, is an empirical paper. It describes the construction of the MMAR and outlines the application of the MMAR to modelling the Deutschemark / US Dollar exchange rate, at daily and intraday frequencies. The paper concludes that the MMAR can replicate certain empirical features of the Deutschemark / US Dollar exchange rate more accurately than GARCH alternatives. Specifically, the MMAR is shown to better reproduce scaling characteristics that are present in the moments of the empirical returns. Calvet and Fisher (2002) provide a good review of the MMAR and GARCH literature, as well as a thorough overview of multifractal processes and the construction of the MMAR. It discusses the results found in Fisher et al. (1997), and presents evidence of moment scaling behavior in equity returns at the daily level.

Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models were first introduced in Engle (1982) and Bollerslev (1986). Since then, many variants of the standard GARCH model (S-GARCH) have been introduced. The shared element between GARCH models is a conditional distribution of returns that have a finite, time-varying second moment. Integrated GARCH (I-GARCH) was introduced in Nelson (1990). Glosten-

Jagannathan-Runkle GARCH (GJR-GARCH) was introduced in Glosten, Jagannathan, and Runkle (1993). Bollerslev, Engle, and Nelson (1994) provide a thorough review of the existing literature on GARCH models up to that time, and introduce another variant, exponential GARCH (E-GARCH). Fractionally-Integrated GARCH (FIGARCH), introduced in Baillie, Bollerslev, and Mikkelsen (1996), marks a significant enhancement to GARCH modelling. First, FIGARCH exhibits long memory in squared returns. Second, FIGARCH does not converge to Brownian behavior as the sampling interval increases in length. FIGARCH is not explored in this paper, due to time and computational constraints.

2. Model Descriptions

In this section, I briefly introduce the theoretical backgrounds of the MMAR and GARCH models, respectively. Step-by-step construction of the MMAR will be described in Section 3. Some specific details that are not needed to understand the MMAR at a basic level will be included in Section 3, and will be excluded from this section. The purpose of the current section is to provide some insight into the different approaches used by the two models.

a. Introduction to the MMAR

The first step to understanding the MMAR is to gain some level of familiarity with the concept of multifractals. Multifractals, as a label, can be applied to both objects and processes. Perhaps the principle that best links different sorts of multifractals together is scale-consistency. Mandelbrot et al. (1997) defines the property of scale-consistency as "a well-defined scaling rule" that "relates returns over different sampling intervals". The use of the word "scale" or "scaling" in the context of financial prices refers to returns over varying interval lengths. This is in contrast to physical applications, when "scale" most likely refers to the resolution from which an object or image is viewed. A simpler, but perhaps mildly inaccurate definition of scale-consistency is a state where some relationship holds over varying time scales. Collecting data over several time scales is a common theme behind Mandelbrot's work, reflecting the idea that "reliance upon a single time scale leads to inefficiency, or worse, forecasts that vary with the time-scale of the chosen data" (Mandelbrot et al., 1997).

Taking the idea of scale-consistency one step further, multifractality can be defined as a condition where there exists a "scaling property in moments of the process" (Mandelbrot et al., 1997). The scaling property is presented below, for illustrative purposes only:

$$E(|X(t)|^q) = c(q)t^{\tau(q)+1}, \tag{2.1}$$

where *X(t)* is a random process as a function of time. In addition, *τ(q)* and *c(q)* are both deterministic functions of *q*. Of these two, *τ(q)* plays a more important role in the construction of the MMAR. This moment scaling property is a key characteristic of the type of processes to which the MMAR can be applied.

The MMAR itself is a fractional Brownian Motion (FBM), which is compounded by a multifractal trading time. The equation is as follows:

$$X(t) = B_H[\theta(t)], \tag{2.2}$$

where B_H is an FBM with self-affinity index H, and $\theta(t)$ is a multifractal trading time process. *X(t)* is a logarithmic return process:

$$X(t) = \ln(P_t) - \ln(P_0) \tag{2.3}$$

The concept of trading time deserves some further explanation. Essentially, trading time is an alternative scale to everyday clock time. The figurative interpretation is that, whereas clock time runs at a constant speed, trading time varies. Like a DVD that speeds up and slows down at random intervals, trading time passes inconsistently. Periods when big market-moving events occur one after another would correspond to quick passage of trading time. In other periods, when market action is calmer and more reserved, trading time slows down.

In the MMAR, trading time is represented by a multifractal measure, which itself is constructed through a mathematical process known as a multiplicative cascade. Precise details on the replication of trading time for the MMAR will be found in Section 3. A multiplicative cascade is an iterative procedure that divides mass across intervals. In the case of the MMAR, the "mass" being portioned out is time.

As an iterative procedure, a multiplicative cascade proceeds in an infinite amount of steps. At step k = 1, the mass is evenly distributed across an interval. At step k = 2, let us give 60% of the original mass to the left half, and 40% to the right half. Then repeat this process over and over again. The graphic below, from Mandelbrot (2006, p. 215), illustrates the result of this multiplicative cascade.

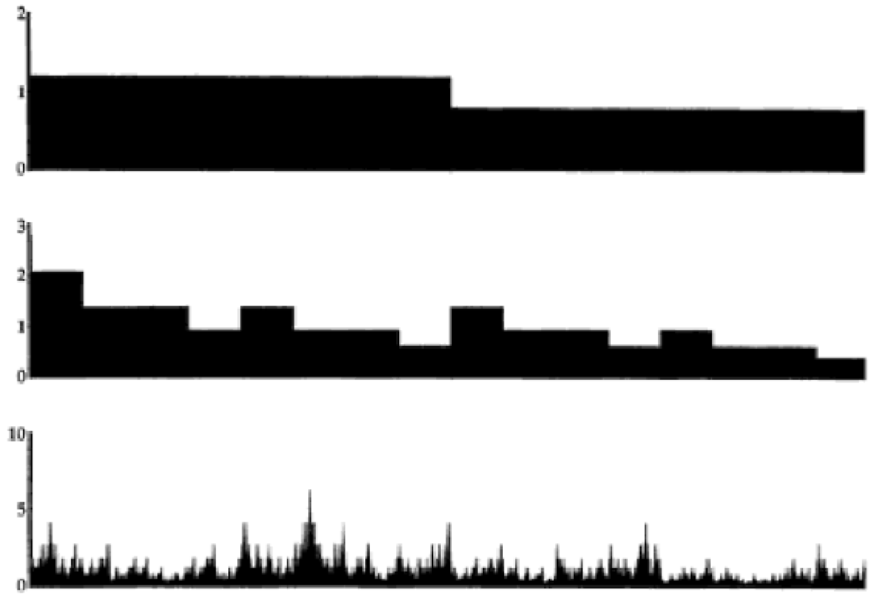

Figure 1. Successive steps of a simple multiplicative cascade.

After enough steps, the multiplicative cascade results in a complex distribution of mass, with peaks and valleys. It does this by repeating a simple rule. Multifractal measures are constructed in this way. Total trading time elapsed $\theta(t)$ corresponds to the cumulative distribution function of a multifractal measure.

Equation (2.2) is the first assumption of the MMAR. The additional assumptions, as listed in Mandelbrot et al. (1997), are as follows:

Assumption 2. *The trading time $\theta(t)$ is the c.d.f. of a multifractal measure defined on $[0,T]$. That is, $\theta(t)$ is a multifractal process with continuous, non-decreasing paths, and stationary increments.*

Assumption 3. *$B_H(t)$ and $\theta(t)$ are independent.*

This section introduced several key concepts that make up the foundation for the MMAR. Additional technical details and proofs for these principles can be found in Mandelbrot et al. (1997) and Calvet et al. (1997).

b. Introduction of GARCH

The original GARCH (q,p) model describes returns as follows:

$$r_t = \mu + \varepsilon_t, \tag{2.4}$$

where ε_t are innovations in returns, that are distributed with a variance σ_t^2:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 \tag{2.5}$$

GARCH variants either modify this original construction, introduce new properties, or do both. The main idea behind GARCH modelling is that volatility is time-varying and dependent, to some degree, on previous values.

The most distinguishing feature of the MMAR is the aforementioned property of scale consistency. GARCH models do not contain this property found in actual financial prices. Mandelbrot et al. (2007) provides deeper comparisons between GARCH, MMAR, and earlier models, accompanied with tables.

3. Data Adjustments

a. Dataset and Return Calculation

The returns of twenty stocks were analyzed in this project. The list of tickers is in the Appendix. The dataset used for each stock was the NYSE Trade and Quote (TAQ) dataset, accessed through Wharton Research Data Services (WRDS). The TAQ dataset collects timestamped bid and ask quotes, across various exchanges, down to the frequency of a millisecond. The WRDS website contains a SAS program, dated 7 September 2010, which computes the national best bid and offer (NBBO) quotes for each second of the trading day.[1] This program is reproduced in the appendix. According to the WRDS website, the program "calculates the inside quote by determining whether a given quote is eligible for NBBO consideration". The algorithm calculates the "last NBBO every second". The empirical results in this paper uses natural logarithmic returns, calculated with the midpoint of the best bid and best offer quotes at any particular timestamp.

b. Seasonality

The empirical results in this paper use intraday returns. Fisher et al. (1997) states that seasonality within intraday price series violates the stationarity requirement of the MMAR, as seasonal components of the return series are non-scaling. The solution is to apply a seasonal adjustment filter to the raw dataset.

A measure of market activity is average absolute fifteen minute returns. Figure 2 shows the ratio of average logarithmic return for each time interval, divided by the average logarithmic return across all time intervals. This data for XOM was collected

[1] https://wrds-web.wharton.upenn.edu/wrds/research/applications/microstructure/NBBO%20derivation/index.cfm (Requires subscription to WRDS).

over the entire year of 2012. A value of 1 on the plot indicates that, on average, the period's absolute logarithmic return was equal to the average absolute logarithmic return across all time intervals.

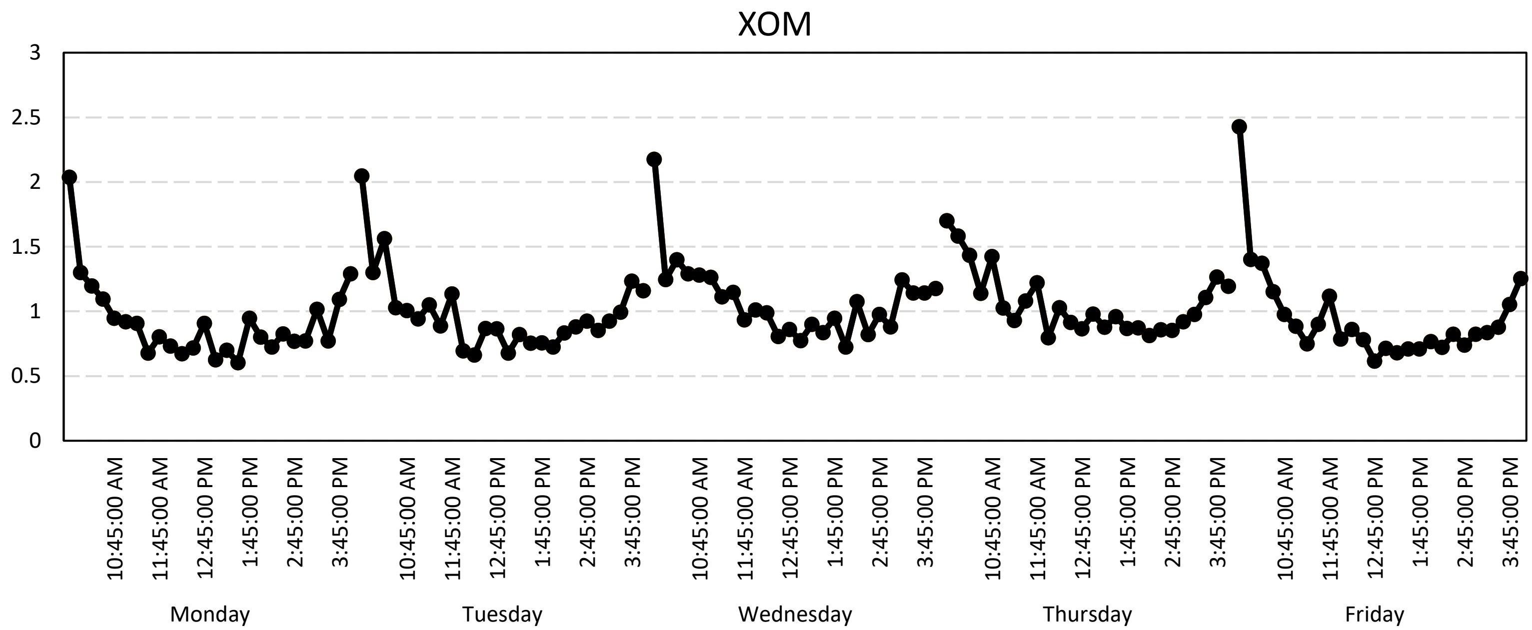

Figure 2. Seasonality Plot for Exxon Mobil (XOM)

Figure 2 shows that the beginning and end of the trading day are typically marked by abnormally higher market action. Therefore, the empirical results of this paper apply a simple seasonal adjustment filter, which consists of eliminating all data collected prior to 10:30:00 hours and all data collected after 15:30:00 hours, each day. This filter shortens the length of each trading day to five hours, and should do a reasonable job of reducing the effects of seasonality on the underlying data. Fisher et al. (1997) explores other seasonality filters, and provides more in-depth discussion on the issue of seasonality.

4. Construction of the MMAR

This section serves as a guide for replicating the MMAR, for a single price series.

a. The Partition Function

Recall from Section II that moment scaling is a symptom of a multifractal process. Therefore, the first step to the MMAR is checking for this scaling property by applying the partition function to the data.

A price series $P(t)$ on the time interval $[0,T]$ is divided into N intervals of length Δt. The value of N depends on the value of Δt. The partition function for a chosen value of Δt and a chosen value of q is

$$S_q(T, \Delta t) = \sum_{i=0}^{N-1} \left| \ln\left(\frac{P(i\Delta t + \Delta t)}{P(i\Delta t)}\right) \right|^q$$

(4.1)

After calculating values of the partition function over various values of Δt and various values of q, partition plots are created by plotting values of $\log_{10} S_q(T, \Delta t)$ against $\log_{10} \Delta t$. Linearity in the partition plots are indicators of moment scaling in the data.

Fisher et al. (1997) checked for linearity, through visual inspection of the partition plots. Partition plots from the empirical data used in this paper are evaluated with the r-squared statistic of the OLS regression for each of the partition plots. The grand mean of the r-squared statistic, calculated across various estimation intervals and across 20 stocks, was 0.66. This is convincing evidence of scaling behavior in equity prices, and marks an important discovery of this paper.

The slopes of the OLS regressions for each of the partition plots are used in estimates for the next component of the MMAR, which is $\tau_P(q)$, or the scaling function of the price process. Below is a sample partition plot for Exxon Mobil stock for a few values of q. Each partition has been shifted such that the first value of zero. This is done solely for the purpose of displaying these partitions on a single chart.

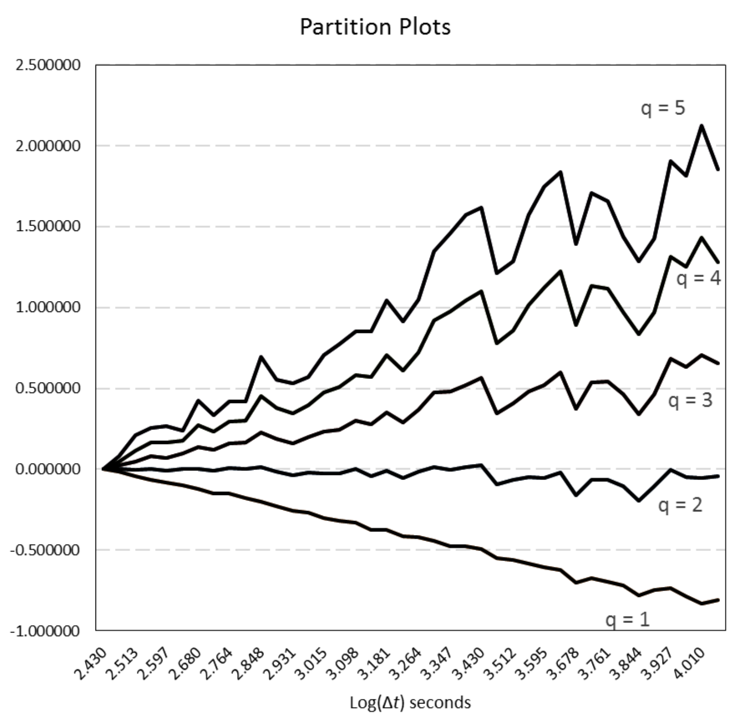

Figure 3.

Linearity is present in the above range for Δt. However, this is not the case of all Δt. When looking at intraday data, moment scaling breaks down at shorter return intervals. This phenomenon is known as high frequency crossover. Figure 4 uses the same input data as Figure 3, but extends the x-axis to the left.

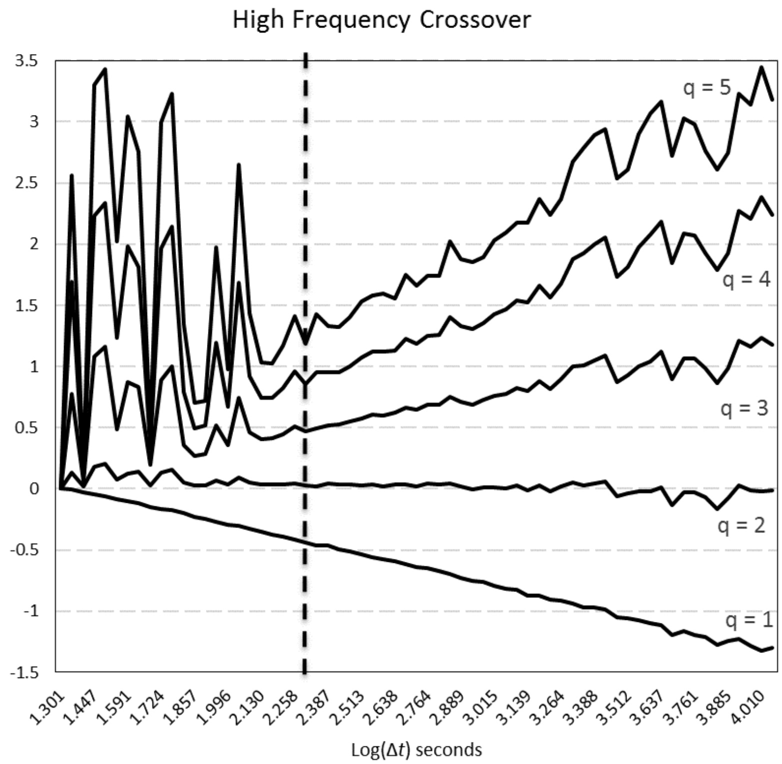

Figure 4.

For these partition plots, the high frequency crossover appears to occur at Δt of around 200 seconds. This means that over shorter time intervals, moment scaling breaks down. This presents a problem with performing out-of-sample tests, as the crossover point is liable to change with each input dataset. This limitation to the MMAR should be reflected in the results. Therefore, empirical results in this paper generates forecasts for a single return interval length. Most of the time, this interval length should lie above the crossover point, but if it doesn't for a particular sample, model estimation will still proceed. Any loss in forecast accuracy will be attributed to this limitation of the MMAR. Fisher et al. (1997) provides more theoretical background for usage of the partition function to capture moment scaling.

b. The Scaling Function

The scaling function of the price process is $\tau_P(q)$. The subscript indicates that this function is distinct from the scaling function of the multifractal process underlying the return series. The sample scaling function $\widehat{\tau_P}(q)$ is directly estimated from the sample partition plots. For each value of q, the slope of the OLS regression line for the associated partition plot is the value of $\widehat{\tau_P}(q)$. An estimated scaling function is shown below.

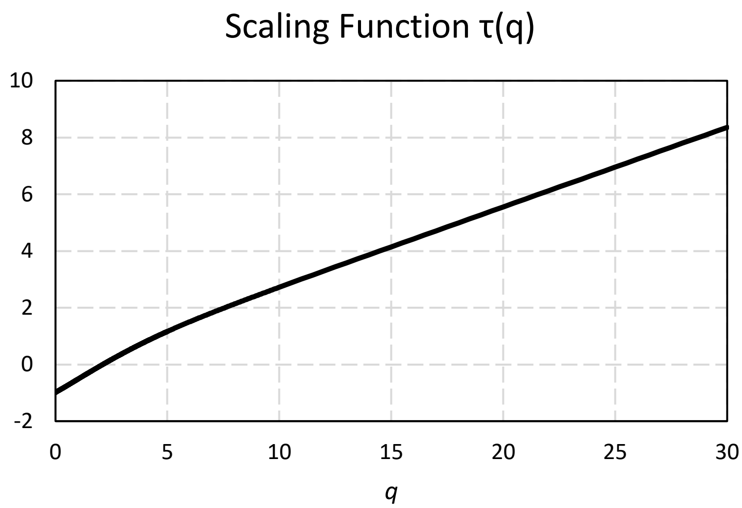

Figure 5.

The scaling function is concave over all positive values of *q*. The 1997 MMAR papers dives deeper into the theoretical details of the scaling function, but for practical purposes, the scaling function can be viewed as a stepping stone to derive the multifractal spectrum of the price process. However, the self-affinity index *H*, of the FBM that will be used to complete the model, is derived from the scaling function as

$$\tau_P\left(\frac{1}{H}\right) = 0 \tag{4.2}$$

Since the sample scaling function was estimated with discrete values of *q*, the self-affinity index *H* is estimated as the inverse of the value of *q* which minimizes the absolute value of the sample scaling function.

c. The Multifractal Spectrum

The multifractal spectrum $f(\alpha)$ is one of the central components of the MMAR and multifractal analysis. Its development and modifications span many years, and the original 1997 MMAR papers are very in depth on this subject. As a result, this paper will avoid the more complex facets of the spectrum. The spectrum is a function of α, which are local Hölder exponents. The local Hölder exponents measure the "local regularity of a process" in the form of an exponential relationship (Fisher et al., 1997). A major point to grasp is that multifractal processes contain a range of local Hölder exponents, whereas other processes may contain only one Hölder exponent. There exist at least three formal interpretations of the multifractal spectrum, but an easy interpretation of the spectrum is that it is the limit of a histogram of Hölder exponents. In a loose sense, it indicates the relative frequency of Hölder exponents found within the process. Lower values of α correspond to the most irregular periods of the process. The sample multifractal spectrum of the return process is estimated via a Legendre transform of the sample scaling function:

$$\widehat{f_P}(\alpha) = \alpha q - \widehat{\tau_P}(q), \tag{4.3}$$

where α is the derivative of $\widehat{\tau_P}(q)$ with respect to q.

To understand the role played by the spectrum in constructing the MMAR, the idea of multifractal time and multifractal measure, which are interchangeable in the MMAR, must be revisited. There exist four main variants of multifractal measures, each defined by the type of distribution used to construct it. Each variant has its own unique multifractal spectrum, which can then be transformed into the corresponding multifractal spectrum of the price process in order to be compared to the data being analyzed. This transformation from the spectrum of a multifractal process to the spectrum of the price process is accomplished by the following equation:

$$f_P(\alpha) = f_\theta(\frac{\alpha}{H}) \tag{4.4}$$

where f_θ is the multifractal spectrum of the multifractal process, and H is the self-affinity index estimated from Equation (4.2).

In the chart below, the spectrum for this particular sample has been fitted to a spectrum of a multifractal measure whose multipliers are sampled from the lognormal distribution.

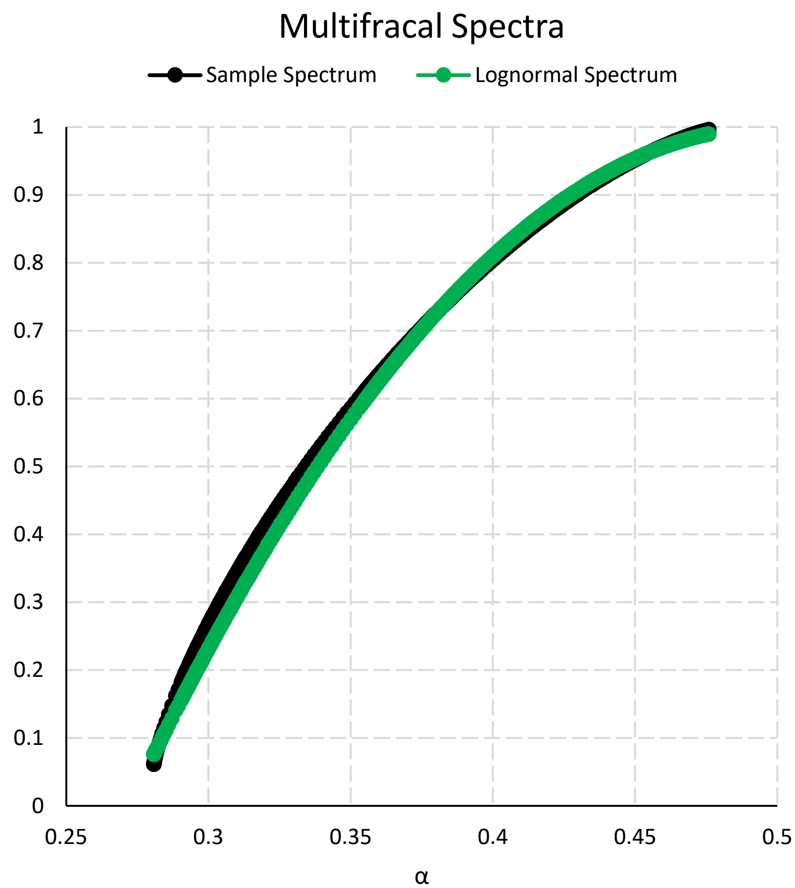

Figure 6.

The fit for this sample spectrum is reasonable. Sample spectra from the empirical results of this paper are also fit relatively well to one of the four variants. All empirical results in this paper only look at the left side of the spectrum. The right side of the spectrum is constructed by including negative values of *q* when analyzing the partition functions, which would amplify errors in imprecisely measured data. Intraday, tick-by-tick data is inevitably imprecise, so all analysis will be performed on the left side of the spectrum only.

d. Constructing Multifractal Measures

After fitting the sample spectrum to one of the four types of multifractal measures, the next step is constructing the appropriate multifractal measure. Multifractal measures are randomly generated, by sampling from a distribution. The quantities being sampled are multipliers, which are used to construct the measure. Each of the four types of multifractal measures are associated with a particular distribution, from which the multipliers are drawn. Figure 1 provides a visualization for the construction of a multifractal measure.

First, start with a single interval that has a value of 1. This is step k = 0.

Second, divide that interval into *b* intervals. The value assigned to each of those *b* intervals is calculated by multiplying the value of the original interval (this would be 1), by a randomly sampled multiplier. A new multiplier is sampled for each of the *b* intervals. This is step k = 1.

Third, divide each of the final intervals from step k = 1 into *b* intervals. The value assigned to each of these new intervals is calculated by multiplying the value of the associated "parent" interval from step k = 1 by a new multiplier. This completes step k = 2.

This is then continued recursively. A multifractal measure is the limit as the above process is continued indefinitely, and a good stopping point for an approximate multifractal measure would be step k = 10. The final step is to integrate the result. Figure 7 provides an example of an integrated multifractal measure.

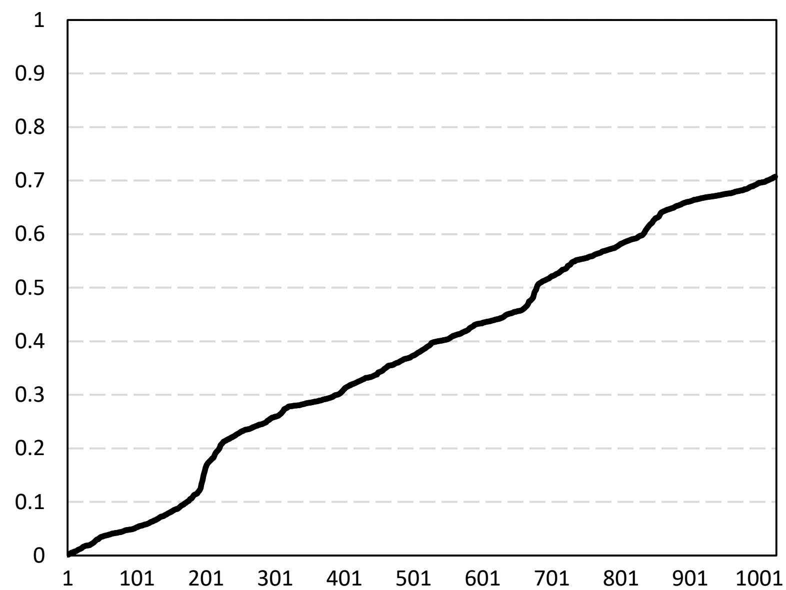

Figure 7.

In the context of the MMAR, the multifractal measure is interchangeable with multifractal time. It is used to indicate the total trading time elapsed, at any given data point. Contrast Figure 7 to clock time elapsed, which would simply be a diagonal line with 45 degrees of slope.

Compounding trading time with a FBM is explained in the next section. For simplicity, all multifractal measures constructed for the empirical results of this paper use a value of $b = 2$. Derivations of the multifractal spectrum and the sampling distributions for each type of multifractal measure are included in the Appendix.

5. Testing on Historical Data

This section will document the testing procedure, as well as relevant details regarding implementation of the MMAR and GARCH models. Lastly, results will be presented.

a. Testing Procedure

The input interval used to calibrate the models was a 50 day period, with an initial date of 3 January 2012 and an end date of 14 March 2012. Recall that each trading day has been shortened to only last five hours, from 10:30:00 to 15:30:00. The forecast interval was a 25 day period, with an initial date of 15 March 2012 and an end date of 19 April 2012. The volatility of 10 minute returns during the forecast interval is calculated and compared to the estimated volatility.

b. Testing the MMAR

The first set of choices that needs to be made involve the partition function. The values of Δt chosen ranged from 400 seconds to 9000 seconds, and were spaced by a factor of 1.1. This allows for the plots on $\log_{10} \Delta t$ to be approximately evenly spaced. The values of q chosen for the partition function ranged from 0.01 to 30.00. For each stock, a volatility forecast for 10 minute returns was generated by simulating 10,000 returns via Monte Carlo simulation.

The actual forecasts were generated via Monte Carlo simulation. In each repetition of the simulation, an entire integrated multifractal measure is generated to step k = 10, with $b = 2$. The next step is to generate a FBM, and scale it by the sample standard deviation of 10 minute returns, measured over the entire input interval. The i^{th} entry of the FBM would be the price at total clock time elapsed i, while the j^{th} entry of the integrated multifractal indicates the total trading time elapsed.

Compounding according to Equation (2.2) was conducted for each index j by first observing the value of the integrated multifractal measure. This value is then multiplied by b^k. Now compounding can be performed. For instance, if the value was 23.45, the relevant points of the FBM are the 24^{th} and 25^{th} entries. The price for this entry j of the series is a linear interpolation between the values of the FBM for the 24^{th} and 25^{th} entries. The logarithmic return process of this series is the final process $X(t)$.

This process is repeated until it exceeds a length of 10,000 returns, where each return represents a time interval of 10 minutes. The standard deviation of these returns is taken as a forecast of volatility.

c. Testing the GARCH

GARCH estimation was done using the "rugarch" package for the R programming language. The input interval was converted into 10 minute returns, which were then passed to the "ugarchspec" function. The "ugarchfit" and "ugarchsim" functions were used to fit a GARCH model to the data, and simulate 10,000 returns. The standard deviation of these returns is taken as a forecast of volatility.

d. Presentation of Results

For each stock, the root mean-squared error was calculated for the MMAR forecasts and the GARCH models. The differences between the root mean-squared errors for the MMAR and each GARCH model were then computed, and are listed in the table below, along with the t-statistics. The differences were multiplied by 10^6, for presentation.

Differences	**sGARCH**		**eGARCH**	
Stock	Normal	T.Dist	Normal	T.Dist
XOM	0.060	0.070	0.071	0.068
F	-0.222	-0.096	-0.258	-0.081
GE	0.042	-0.004	0.060	0.060
MU	-0.078	-0.068	-0.191	-0.179
PG	-0.028	-0.055	-0.005	0.005
MSFT	0.027	0.026	0.026	0.025
CSCO	-0.055	0.042	0.037	0.043
BAC	-1.086	-0.013	-0.320	-0.167
WMT	-0.012	-0.020	-0.001	0.000
T	0.011	0.011	0.000	0.006
KR	-0.030	-0.030	-0.040	-0.010
VZ	-0.027	-0.037	-0.001	-0.005
GM	-0.062	-0.004	-0.108	-0.031
NKE	0.056	0.045	0.047	0.048
TSN	-0.001	-0.014	-0.009	-0.008
CCL	-0.273	-0.206	-0.112	-0.108
MGM	-0.054	-0.344	-0.185	-0.181
HD	-0.017	-0.012	-0.085	-0.048
BK	-7.226	-0.103	-0.044	-0.091
HES	-0.104	-0.139	-0.166	-0.105
Mean	-0.454	-0.048	-0.064	-0.038
Standard Error	0.361	0.021	0.025	0.018
T Statistic	-1.259	-2.215	-2.594	-2.148

Figure 8.

Two versions of each GARCH variant were tested, one for each type of distribution assigned to the innovations. A negatively significant t-statistic means that the MMAR outperformed the corresponding GARCH model, and vice versa. The MMAR significantly outperformed most of the GARCH models that were tested, but did not significantly outperform all of them. The results suggest that the MMAR provides a promising approach to volatility modelling.[2]

[2] Other GARCH variants were tested, and they did not change the final conclusion.

6. Conclusion

The MMAR introduces a new approach to modelling financial series, by postulating that financial processes can be modelled as a Fractional Brownian Motion compounded by a multifractal time deformation process. Prior work shows that the MMAR can accurately reproduce moment scaling properties found in the data, and that moment scaling exists in daily stock series and exchange rates, as well as in intraday exchange rates.

This paper set out to explore the empirical possibilities of using the MMAR as a forecasting tool. The exercise discovered strong evidence of moment scaling in intraday stock returns, and demonstrated a strong potential for volatility modelling with the MMAR. Future work may be dedicated to improving estimation and simulation procedures for the MMAR, with the end goal of creating a superior forecasting tool.

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Appendix

a. WRDS NBBO Algorithm

The code below has been modified from the original version found on the WRDS website. It has been augmented to have the ability to iterate across several days of data.

```
/*
***************************************************************************
**** */
/* ******************** W R D S   R E S E A R C H   M A C R O S
******************** */
/*
***************************************************************************
**** */
/* WRDS Macro: NBBO
*/
/* Summary   : Simplistic NBBO Derivation using SAS Views
*/
/* Date      : September 7, 2010
*/
/* Author    : Rabih Moussawi, WRDS
*/
/* Variables : - YYYYMMDD is the Date Stamp for the Quote dataset
*/
/*             - OUTSET is the output dataset
*/
/*
***************************************************************************
**** */

*/%MACRO NBBO (YYYYMMDD=19930104,OUTSET=nbbo);

%MACRO NBBO (fdate,ldate,iteration,OUTSET=nbbo);
%let fdated = %sysfunc(InputN(&fdate,yymmdd8.));
%let ldated = %sysfunc(InputN(&ldate,yymmdd8.));
%do date = &fdated %to &ldated;
      %let ndate= %sysfunc(PutN(&date,yymmddn8));
       %let wkday= %sysfunc(weekday(&date));
      %let dcq = taq.cq_&ndate. ;
       %if %index(2 3 4 5 6,&wkday)>0
       %then
       %do;

  options nonotes;
      %put ;
      %put ### START NBBO Calculation for: &NDATE ;
      /* Preliminary: Create an Informat to Convert EX to Numeric */
      /* Keep only Known Exchange Types: as Defined in TAQ Manual */
      proc format;
        invalue ex_to_exn      /* informat to be used in an INPUT function */
            'A'=01   /*AMEX*/
            'N'=02   /*NYSE*/
            'B'=03   /*BOST*/
            'P'=04   /*ARCA*/
            'C'=05   /*NSX -National (Cincinnati) Stock Ex*/
```

```
            'T'=06   /*NASD*/
            'Q'=07   /*NASD*/
            'D'=08   /*NASD-ADF*/
            'X'=09   /*PHIL-NASDAQ OMX PSX*/
            'I'=10   /*ISE */
            'M'=11   /*CHIC*/
            'W'=12   /*CBOE*/
            'Z'=13   /*BATS*/
            'Y'=14   /*BATS Y-Ex*/
            'J'=15   /*DEAX-DirectEdge A*/
            'K'=16   /*DEXX-DirectEdge X*/
             otherwise=17  /* you can drop those quotes in the if statement
*/
            ;
      run;
      /* Additional Exchange Values */
      /* 'E'=17   /*SIP -Market Independent (SIP - Generated)*/
      /* 'S'=18   /*Consolidated Tape System*/

      /* First, Generate Last Prevailing Quote for Each Exchange, every
second */
      data _quotes / view=_quotes;
           set taq.cq_&ndate;
           by symbol date time NOTSORTED ex; length EXN 3.;
            where symbol in ('XOM') and time between '09:30:00't and
'16:00:00't;
      /* Convert EX to EXN for easy array reference */
      EXN=input(ex,ex_to_exn.);
      /* Keep the last prevailing Quote by exchange from consecutive quotes
every second */
      if last.EX and 1 <= EXN <= 17;
      label EXN="Exchange Code (numeric)";
      drop EX MMID;
      run;
/* Second, Derive NBBO from Prevailing Quotes Across Exchange */
data &outset (sortedby= SYMBOL DATE TIME index=(SYMBOL)
              label="WRDS-TAQ NBBO Data");
set _quotes;
by symbol date time;
/* Retain Observations within Each Time Block */
retain nexb1-nexb17 nexo1-nexo17 sexb1-sexb17 sexo1-sexo17;
array nexb nexb:; array nexo nexo:; array sexb sexb:; array sexo sexo:;
/* Step1. Reset NBBO for each new stock and at open and close */
if first.date or (lag(time)<"09:30:00"t <= time) or (lag(time) <= "16:00:00"t
< time) then
do i=1 to 17;
  nexb(i)=.; nexo(i)=.; sexb(i)=.; sexo(i)=.;
end;
/* Step2. Quote Rule: Prevailing Quote Supersedes Previous Quote */
nexb(exn)=bid;nexo(exn)=ofr;sexb(exn)=bidsiz;sexo(exn)=ofrsiz;
/* Step3. Determined if Prevailing Quotes is Eligible for NBBO */
/* See TAQ Manual pp 26 and pp 27 for MODE and NBBO Definitions */
/* Regulatory Trading Halts: MODE in (4,7,9,11,13,14,16,20,21,27,28) */
/* See TickData.com for more information on filters */
if mode not in (1,2,6,10,12,23) or bid >= ofr
then do; nexb(exn)=.; nexo(exn)=.; sexb(exn)=.; sexo(exn)=.; end;
if bid <= 0.01 or bidsiz <= 0 then nexb(exn)=.;
```

```
if ofr <= 0    or ofrsiz <= 0 then nexo(exn)=.;
/* Step4. Calculate NBBO */
BB=max(of nexb:);
BO=min(of nexo:);
/* Step5. Calculate Bid and Ofr Sizes at NBBO */
BBSize=0; BOSize=0;
do i=1 to 17;
 if BB=nexb(i) then BBSize=max(BBSize,sexb(i));
 if BO=nexo(i) then BOSize=max(BOSize,sexo(i));
end;
if missing(BB) then BBSize=.;
if missing(BO) then BOSize=.;
/* Report # of Exchanges with Qualifying Quotes */
length NUMEX 3.;
NUMEX=max(N(of nexb:),N(of nexo:));
/* Keep NBBO Information at the End of Each Second Interval */
if last.time then output;
label BB = 'Best Bid';
label BO = 'Best Offer';
label BBSize='Best Bid Size';
label BOSize='Best Offer Size';
label NUMEX='# of Exchanges with Prevailing Quotes used in the NBBO';
drop EXN MODE BID OFR nexb: nexo: i sexb: sexo: ofrsiz bidsiz;
run;

/* House Cleaning */
proc sql; drop view _quotes; quit;
options notes;
%put ### DONE. NBBO Data Saved as  : &outset ; %put ;
%end;

%if %sysfunc(exist(&outset))%then
 %do;
 proc append base=&iteration data=&outset; run;
 proc sql; drop table &outset; quit;
 %end;
%end;

proc export data=&iteration outfile="XOM_sample_1.csv" DBMS=csv REPLACE; run;
%mend;

/*
****************************************************************************
**** */
/* *************  Material Copyright Wharton Research Data Services
************** */
/* ****************************** All Rights Reserved
****************************** */
/*
****************************************************************************
**** */

%nbbo(fdate=20110104,ldate=20110104,iteration=user_out_set_1,outset=temp.new1
);
```

b. Four Types of Multifractal Measures

The four types of multifractal measures are named according to the distribution, from which the variable *V* is sampled.

$$V = -\log_b M$$

where *M* is the multiplier used to construct the multifractal measure.

For this paper, the quality of spectrum fit is evaluated by the sum of the squared-error at each of the discrete points of the sample spectrum. The best-fit spectrum for each of the four types of measures is calculated in order to find the overall best-fit spectrum. The multifractal component of the sample data is then assumed to be best approximated by the overall best-fit spectrum and the corresponding type of multifractal measure.

Additional information on these multifractal measures can be found in Calvet et al. (1997).

i. Normal Distribution

The multifractal spectrum of the price process takes the form

$$f_P(\alpha) = 1 - \frac{(\alpha - \alpha_0)^2}{4H(\alpha_0 - H)}$$

where α_0 is the value of α corresponding to the peak of the multifractal spectrum. Since *H* has already been estimated from the scaling function, the best-fit version of this spectrum is found by trying many values of α_0.

The variable *V* has a normal distribution with mean $\frac{\alpha_0}{H}$ and variance $\frac{2(\frac{\alpha_0}{H}-1)}{\ln(b)}$.

ii. Binomial Distribution

The multifractal spectrum of the price process takes the form

$$f_P(\alpha) = -\frac{\alpha_{max} - \alpha}{\alpha_{max} - \alpha_{min}} \log_2\left(\frac{\alpha_{max} - \alpha}{\alpha_{max} - \alpha_{min}}\right) - \frac{\alpha - \alpha_{min}}{\alpha_{max} - \alpha_{min}} \log_2\left(\frac{\alpha - \alpha_{min}}{\alpha_{max} - \alpha_{min}}\right)$$

where α_{min} is the smallest value of α and α_{max} is the largest value of α. Both α_{min} and α_{max} have to be fitted by trying many values via grid-search estimation. The variable *V* takes on the value of either α_{min} or α_{max} with equal probability.

iii. Poisson Distribution

The multifractal spectrum of the price process takes the form

$$f_P(\alpha) = 1 - \frac{\alpha_0}{H\ln(b)} + \frac{\alpha}{H}\log_b\left(\frac{\alpha_0 e}{\alpha}\right)$$

where α_0 is the value of α corresponding to the peak of the multifractal spectrum. Since *H* has already been estimated from the scaling function, the best-fit version of this spectrum is found by trying many values of α_0. If *b* (in the context of section 4.d.) is not held constant, it will have to be fitted as well.

The variable V has a Poisson distribution with mean $\frac{\alpha_0}{H}$.

iv. Gamma Distribution

The multifractal spectrum of the price process takes the form

$$f_P(\alpha) = 1 + \gamma \log_b\left(\frac{\alpha}{\alpha_0}\right) + \frac{\gamma(\alpha_0 - \alpha)}{\alpha_0 \ln(b)}$$

where α_0 is the value of α corresponding to the peak of the multifractal spectrum. α_0 and γ have to be fitted by trying many values via grid-search estimation. If b (in the context of section 4.d.) is not held constant, it will have to be fitted as well. The variable V has a Gamma distribution with shape parameter γ and rate parameter $\beta = \frac{\ln(b)}{b^{\frac{1}{\gamma}}-1}$.

c. Stock Tickers

Stocks in Group 1 experienced relatively low trading volume and stocks in Group 3 experienced relatively high trading volume over the time period analyzed. Stocks in Group 2 experienced intermediate levels of volume.

Group 1	Group 2	Group 3
TSN	XOM	AAPL
CCL	WMT	F
MGM	T	GE
HD	KR	MU
BK	VZ	PG
HES	GM	MSFT
	NKE	CSCO