Minimum Variance unbiased estimator [MVUE].

An estimator T of an unknown parameter $\theta \in \Omega$ (= parametric space) or a parametric function $\gamma(\theta)$, it is called the MVUE of θ iff

i) $E_\theta(T) = \theta$, $\forall \theta \in \Omega$ or equivalently, $E_\theta(T) = \gamma(\theta)$, $\theta \in \Omega$

& ii) $V_\theta(T) \le V_\theta(T^*)$, where, T^* represents any other unbiased estimator of θ (or $\gamma(\theta)$), $\forall \theta \in \Omega$.

Eg: Let, $X_1, X_2, \ldots, X_n$ be a random sample of size 'n' from a $N(\mu, \sigma^2)$ distn.

Then, $E(X_i) = \mu$, $\forall i$; $V(X_i) = \sigma^2$, $\forall i$. ~~& $E(\bar{x}) = \frac{1}{n}$~~ E

& $E(\bar{x}) = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu$. $\quad \therefore E(\bar{x}) = \mu$.

Also, $X_1, X_2, \ldots, X_n$ are independent random variables. Then,

$$V(\bar{x}) = V(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}.$$

Thus, $V(\bar{x}) < V(X_i)$, $\forall i$.

Hence, $\bar{x}$ is the ~~best linear unbiased~~ best estimator of μ in this situation.

regarding MVUE of the parameter θ [or the parametric function ...].

Theorem 1: MVUE, if it exists, is unique in nature in the sense that if T_1 & T_2 are both MVUE of an unknown parameter θ, then, $T_1 = T_2$ uniquely.

Proof: Suppose, T_1 & T_2 are both MVUE of an unknown parameter $\theta \in \Omega$.

So, we have, $E(T_1) = \theta = E(T_2)$, $\forall \theta \in \Omega$ & $v(T_1) = v(T_2)$, $\forall \theta \in \Omega$.

Let us now define, $T = \frac{T_1 + T_2}{2}$ as an estimator of θ.

$$\therefore E(T) = \frac{E(T_1) + E(T_2)}{2} = \frac{2\theta}{2} = \theta, \ \forall \theta \in \Omega$$

$\therefore$ T is an unbiased estimator of $\theta \in \Omega$.

Hence, $v(T) \geq v(T_1)$, $\forall \theta \in \Omega \cdots (*)$

Now, $v(T) = v\left(\frac{T_1 + T_2}{2}\right) = \frac{1}{4}[v(T_1) + v(T_2) + 2\,Cov(T_1, T_2)]$.

$$= \frac{1}{4}[v(T_1) + v(T_2) + 2\rho_{T_1,T_2}\sqrt{v(T_1)v(T_2)}], \text{ where, } \rho_{T_1,T_2} = Corr(T_1, T_2).$$

$$= \frac{1}{4}[2v(T_1) + 2\rho_{T_1,T_2}v(T_1)]$$

$\Rightarrow v(T) = \frac{v(T_1)}{2}(1 + \rho_{T_1,T_2}) \geq v(T_1)$, using $(*)$

$\Rightarrow 1 + \rho_{T_1,T_2} \geq 2 \quad \therefore \rho_{T_1,T_2} \geq 1 \Rightarrow \rho_{T_1,T_2} = 1$, $\because -1 \leq \rho_{T_1,T_2} \leq 1$.

As such, T_1 & T_2 are perfectly positively & linearly interrelated, i.e.

we can write $T_2 = a + bT_1$, where, $b > 0$, $\because$, T_1 & T_2 are positively related. 123

Now, $V(T_2) = V(a+bT_1) = b^2 V(T_1)$.

$\Rightarrow b^2 = 1 \Rightarrow b = +1$, since, $b > 0$.

$\Rightarrow T_2 = a + 1.T_1 = a + T_1 \Rightarrow E(T_2) = a + E(T_1)$, $\forall \theta \in \Omega$.

$\Rightarrow \theta = a + \theta$, $\forall \theta \in \Omega \Rightarrow a = 0$ $\forall \theta \in \Omega$. $\therefore$, $T_2 = T_1$, $\forall \theta \in \Omega$.

Hence, MVUE, if exists, is ~~unbiased~~ unique in nature.

Theorem: (2) Let, T_0 be an MVUE of an unknown parameter θ, & T_1 be an unbiased estimator of θ, $\theta \in [\Omega_\theta]$ & $\frac{V_\theta(T_0)}{V_\theta(T_1)} = e_\theta$. Then $\rho_\theta = \sqrt{e_\theta}$, where, $\rho_\theta = Corr_\theta(T_0, T_1)$.

Proof: Let us conventionally consider the following linear combination of T_0 & T_1 as,

$\ldots T + (-\rho_\theta \sqrt{e_\theta} + e_\theta) T_1$.

Theorem: (2) Let, T_0 be an MVUE of an unknown parameter θ & T_1 be an unbiased estimator of θ, $\theta \in \Theta$ $[\equiv \Omega_\theta]$ &

$\frac{V_\theta(T_0)}{V_\theta(T_1)} = e_\theta$. Then $\rho_\theta = \sqrt{e_\theta}$, where, $\rho_\theta = Corr_\theta(T_0, T_1)$.

Proof: Let us conventionally consider the following linear combination of T_0 & T_1 as,

$$(1 - 2\rho_\theta\sqrt{e_\theta} + e_\theta)T = (1 - \rho_\theta\sqrt{e_\theta})T_0 + (-\rho_\theta\sqrt{e_\theta} + e_\theta)T_1.$$

$$\Rightarrow (1 - 2\rho_\theta\sqrt{e_\theta} + e_\theta)E_\theta(T) = (1 - \rho_\theta\sqrt{e_\theta})\theta + (-\rho_\theta\sqrt{e_\theta} + e_\theta)\theta$$
$$= (1 - 2\rho_\theta\sqrt{e_\theta} + e_\theta)\theta.$$

$$\therefore E_\theta(T) = \theta, \quad \forall \theta \in \Theta.$$

Hence ~~As such~~, T belongs to the class of all unbiased estimators of θ, $\forall \theta \in \Theta$. ~~Hence~~ As such, $V_\theta(T) \geq V_\theta(T_0)$, ... (*), $\because T_0$ is MVUE of θ

Now, we can write,

$$(1 - 2\rho_\theta\sqrt{e_\theta} + e_\theta)^2 V_\theta(T) = (1 - \rho_\theta\sqrt{e_\theta})^2 V_\theta(T_0) + (-\rho_\theta\sqrt{e_\theta} + e_\theta)^2 V_\theta(T_1)$$
$$+ 2(1 - \rho_\theta\sqrt{e_\theta})(-\rho_\theta\sqrt{e_\theta} + e_\theta)\,Cov_\theta(T_0, T_1).$$

$$\Rightarrow (1 - 2\rho_\theta\sqrt{e_\theta} + e_\theta)^2 V_\theta(T) = (1 - \rho_\theta\sqrt{e_\theta})^2 V_\theta(T_0) + (\rho_\theta\sqrt{e_\theta} - e_\theta)^2 \frac{V_\theta(T_0)}{e_\theta}$$
$$- 2(1 - \rho_\theta\sqrt{e_\theta})(\rho_\theta\sqrt{e_\theta} - e_\theta)\rho_\theta\sqrt{V_\theta(T_0)\,V_\theta\frac{(T_0)}{e_\theta}},$$

$$\Rightarrow \left(1-2\rho_{\alpha}\sqrt{e_{\alpha}}+e_{\alpha}\right)^2 v_{\alpha}(T) = \left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)^2 v_{\alpha}(T_0) + \left(\rho_{\alpha}\sqrt{e_{\alpha}}-e_{\alpha}\right)^2 \frac{v_{\alpha}(T_0)}{e_{\alpha}}$$

$$-2\left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)\left(\rho_{\alpha}\sqrt{e_{\alpha}}-e_{\alpha}\right)\rho_{\alpha}\sqrt{v_{\alpha}(T_0)\,v_{\alpha}\frac{(T_0)}{e_{\alpha}}},$$

$\because$ ~~$\operatorname{cov}(x,y)$~~. $\quad \because \operatorname{Cov}(X,Y)=\rho_{XY}S_X S_Y$; $\frac{v_{\alpha}(T_0)}{v_{\alpha}(T_1)} = e_{\alpha}$ [Given] $\Rightarrow v_{\alpha}(T_1) = \frac{v_{\alpha}(T_0)}{e_{\alpha}}$.

$$\Rightarrow \left(1-2\rho_{\alpha}\sqrt{e_{\alpha}}+e_{\alpha}\right)^2 v_{\alpha}(T) = \left[\left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)^2 + \left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)^2 - 2\left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)\left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)\rho_{\alpha}\right]$$

$$= \left[\left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)^2 - 2\left(1-\rho_{\alpha}\sqrt{e_{\alpha}}\right)\left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)\rho_{\alpha} + \left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)^2\rho_{\alpha}^2 \quad v_{\alpha}(T_0).\right.$$

$$\left. -\left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)^2\rho_{\alpha}^2 + \left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)^2\right] v_{\alpha}(T_0).$$

~~$= \left[\left(1-\rho_{\alpha}\sqrt{e_{\alpha}} \mp \rho_{\alpha}\sqrt{e_{\alpha}}\right)^2 + \left(1-\rho_{\alpha}^2\right)\left(\rho_{\alpha}-\sqrt{e_{\alpha}}\right)\right] v(T_0).$~~

~~$= 1-\rho_{\alpha}\sqrt{e_{\alpha}} \ldots$~~

$$= [\{1-\rho_\Theta\sqrt{e_\Theta} - (\rho_\Theta-\sqrt{e_\Theta})\rho_\Theta\}^2 + (\rho_\Theta-\sqrt{e_\Theta})^2(1-\rho_\Theta^2)]\, v_\Theta(T_0).$$

$$\Rightarrow [(1-\rho_\Theta^2)^2 + (1-\rho_\Theta^2)(\rho_\Theta-\sqrt{e_\Theta})^2]\, v_\Theta(T_0)$$

$$\Rightarrow (1-\rho_\Theta^2)(1-\rho_\Theta^2+\rho_\Theta^2-2\rho_\Theta\sqrt{e_\Theta}+e_\Theta)\, v_\Theta(T_0)$$

$$= (1-\rho_\Theta^2)(1-2\rho_\Theta\sqrt{e_\Theta}+e_\Theta)\, v(T_0).$$

$$\therefore (1-2\rho_\Theta\sqrt{e_\Theta}+e_\Theta)^2\, v_\Theta(T) = (1-\rho_\Theta^2)(1-2\rho_\Theta\sqrt{e_\Theta}+e_\Theta)\, v_\Theta(T_0).$$

$$\Rightarrow v_\Theta(T) = \frac{(1-\rho_\Theta^2)}{1-2\rho_\Theta\sqrt{e_\Theta}+e_\Theta}\, v_\Theta(T_0) = \cancel{\frac{1-\rho_\Theta^2}{[(\sqrt{e_\Theta})^2-2\rho_\Theta\sqrt{e_\Theta}+\rho_\Theta^2]}}.$$

$$\therefore v_\Theta(T) = \frac{1-\rho_\Theta^2}{\{(\sqrt{e_\Theta})^2-2\rho_\Theta\sqrt{e_\Theta}+\rho_\Theta^2\}+(1-\rho_\Theta^2)}\, v_\Theta(T_0) = \frac{1-\rho_\Theta^2}{(\sqrt{e_\Theta}-\rho_\Theta)^2+(1-\rho_\Theta^2)}\, v_\Theta(T_0)$$

$\because$ $\dfrac{1-\rho_\Theta^2}{(\sqrt{e_\Theta}-\rho_\Theta)^2+(1-\rho_\Theta^2)}$ is a perfect non-negative fraction, $0\le\rho_\Theta^2\le 1$, so, obviously

$$v_\Theta(T) \le v_\Theta(T_0) \quad \ldots\ldots (**), \quad \forall\, \Theta \in \Theta.$$

$\because$ Comparing $(*)$ & $(**)$, we have, $V_\theta(T) = V_\theta(T_\theta)$ $\forall \theta \in \Theta$.

$$\Leftrightarrow \frac{1-\rho_\theta^2}{(\sqrt{e_\theta}-\rho_\theta)^2+(1-\rho_\theta^2)} V_\theta(T_\theta) = V_\theta(T_\theta) \Leftrightarrow \frac{1-\rho_\theta^2}{(\sqrt{e_\theta}-\rho_\theta)^2+(1-\rho_\theta^2)} = 1.$$

$$\Rightarrow 1-\rho_\theta^2 = (\sqrt{e_\theta}-\rho_\theta)^2 + (1-\rho_\theta^2)$$

$$\because (\sqrt{e_\theta}-\rho_\theta)^2 \neq 0 \Leftrightarrow \rho_\theta = \sqrt{e_\theta}, \forall \theta \in \Theta$$ [Proved].

Theorem 2: Let, T_0 be an MVUE of an unknown parameter θ, $\theta \in \Theta$ & T_1 be an unbiased estimator of θ, $\forall \theta \in \Theta$.

If $0 < e_\theta < 1$, where $e_\theta = \dfrac{V_\theta(T_0)}{V_\theta(T_1)}$, then, $\nexists$ any unbiased linear combination of T_0 & T_1 having minimum variance.

Proof: Let us consider $T = a_0 T_0 + a_1 T_1$ as any unbiased linear combination of T_0 & T_1 $(a_0 \neq 0)$ [if $a_0 = 0$ then T cannot be trivially MVUE of θ].

Since, T_0 is an unbiased estimator of θ, so,

$$E(T) = a_0 E(T_0) + a_1 E(T_1)$$

$$\Rightarrow \theta = a_0 \theta + a_1 \theta, \quad \forall \theta \in \Theta.$$

$$\Rightarrow a_0 + a_1 = 1 \ldots\ldots (*), \quad \forall \theta \in \Theta.$$

Now, $V_\theta(T) = a_0^2 V_\theta(T_0) + a_1^2 V_\theta(T_1) + 2a_0a_1 \, Cov_\theta(T_0, T_1)$.

$= a_0^2 V_\theta(T_0) + a_1^2 V_\theta(T_1) + 2a_0a_1 \rho_\theta \sqrt{V_\theta(T_0) V_\theta(T_1)}$

$= a_0^2 V_\theta(T_0) + a_1^2 \dfrac{V_\theta(T_0)}{e_\theta} + 2a_0a_1 \rho_\theta \sqrt{V_\theta(T_0) \dfrac{V(T_0)}{e_\theta}}$, $\because e_\theta = \dfrac{V_\theta(T_0)}{V_\theta(T_1)}$.

$= a_0^2 V_\theta(T_0) + a_1^2 \dfrac{V_\theta(T_0)}{e_\theta} + 2a_0a_1 \rho_\theta \dfrac{V_\theta(T_0)}{\sqrt{e_\theta}}$

$= a_0^2 V_\theta(T_0) + a_1^2 \dfrac{V_\theta(T_0)}{e_\theta} + 2a_0a_1 \rho_\theta \dfrac{V_\theta(T_0)}{\rho_\theta}$, since, we know that, if T_0 be an ~~unbiased~~ MVUE of an unknown parameter θ & if T_1 be an unbiased estimator of θ, $\forall \theta \in \Theta$, $\ni \dfrac{V_\theta(T_0)}{V_\theta(T_1)} = e_\theta$, then, $\rho_\theta = \sqrt{e_\theta}$, where $\rho_\theta = Corr_\theta(T_0, T_1)$.

$\therefore V_\theta(T) = a_0^2 V_\theta(T_0) + a_1^2 \dfrac{V_\theta(T_0)}{e_\theta} + 2a_0a_1 \rho_\theta \dfrac{V(T_0)}{\rho_\theta}$

$= \left(a_0^2 + 2a_0a_1 + \dfrac{a_1^2}{e_\theta}\right) V(T_0)$

$> \left(a_0^2 + 2a_0a_1 + a_1^2\right) V(T_0)$, since $0 < e_\theta < 1$, so, $\dfrac{1}{e_\theta} > 1$.

$= (a_0 + a_1)^2 V(T_0) = V(T_0)$, $\because$ from (*) we have, $a_0 + a_1 = 1$.

$\Rightarrow V_\theta(T) > V_\theta(T_0)$. $\therefore V_\theta(T)$ is not minimum. Hence the proof.

MVUE & Cramer-Rao Inequality.

This well known inequality owing to Cramer, Rao et al., provides simply a lower bound of the variance of an unbiased estimator of an unknown parameter of a distribution.

Let, $X_1, X_2, \ldots, X_n$ be 'n' continuous r.v.'s having the joint p.d.f. $f_\theta(x_1, x_2, \ldots, x_n) = f_\theta(\underset{\sim}{x})$, where, θ is a single valued parameter $\ni$ $\theta \in \Theta$ = known parametric space of θ. Before we go to the desired inequality, we first consider the following important assumptions:

i) Θ is a non-degenerate open interval of the real line, ie. $\Theta \subseteq \mathbb{R}$.

ii) $\frac{\partial}{\partial\theta} f_\theta(\underset{\sim}{x})$ exists finitely.

iii) $\frac{\partial}{\partial\theta} \int_A f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_A \frac{\partial}{\partial\theta} f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}$, where, $A = \{\underset{\sim}{x} : f_\theta(\underset{\sim}{x}) > 0\}$ & A is θ independent.

$\cancel{\text{iii) } \frac{\partial}{\partial\theta} \int_A t(\underset{\sim}{x}) f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_A \frac{\partial}{\partial\theta} t}$

iv) $\frac{\partial}{\partial\theta} \int_A t(\underset{\sim}{x}) f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_A \frac{\partial}{\partial\theta} t(\underset{\sim}{x}) f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}$

[illegible] ^{2} [illegible] $f_\theta(x)$ [illegible] positive for all $\theta \in \Theta$

iv) $\frac{\partial}{\partial\theta}\int_A t(\underline{x})\, f_\theta(\underline{x})\, d\underline{x} = \int_A \frac{\partial}{\partial\theta}\, t(\underline{x})\, f_\theta(\underline{x})\, d\underline{x}$.

v) $E_\theta\left[\frac{\partial}{\partial\theta}\ln f_\theta(\underline{X})\right]^2$ exists finitely, & is positive for all $\theta \in \Theta$.

These conditions or assumptions are called the regularity conditions & the situation, where these regularity conditions hold true is called the regular estimation case.

Cramer - Rao Inequality.

Under the regular estimation case, if 'T' be an unbiased estimator of an estimable & differentiable parametric function $\gamma(\theta)$, then we can write $V_\theta(T) \geq \dfrac{\{\gamma'(\theta)\}^2}{E_\theta\left[\left\{\frac{\partial}{\partial\theta} \ln f_\theta(\underset{\sim}{X})\right\}^2\right]}$, $\forall\, \theta \in \Theta$.

Proof: Since, $X_1, X_2, \cdots, X_n$ are 'n' continuous random variables with joint p.d.f. $f_\theta(\underset{\sim}{x})$, $\theta \in \Theta$,

so $\int_A f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x} = 1$ or, $1 = \int_A f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}$ ~~$\Rightarrow 0 = \frac{\partial}{\partial\theta}\int_A f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}$~~

$$\Rightarrow 0 = \frac{\partial}{\partial\theta}\int_A f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_A \frac{\partial}{\partial\theta} f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}, \text{ by regularity condition } (3).$$

$$= \int_A \left\{\frac{1}{f_\theta(\underset{\sim}{x})} \frac{\partial}{\partial\theta} f_\theta(\underset{\sim}{x})\right\} f_\theta(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$\Rightarrow 0 = \int\limits_A \frac{\partial}{\partial\theta} \ln f_\theta(\underline{x}) \, f_\theta(\underline{x}) \, d\underline{x} \cdots\cdots (1)$

i.e, $0 = E(Y)$; $\cdots (1^*)$ where, $Y = Y(\underline{X}) = \frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})$.

Since, T is an unbiased estimator of $\gamma(\theta)$, $\theta \in \Theta$,

$\therefore$ $E(T) = \gamma(\theta)$, $\forall \theta \in \Theta$, where, $T = T(\underline{X})$.

$\Rightarrow \gamma(\theta) = E(T) = E[T(\underline{X})] = \int\limits_A t(\underline{x}) f_\theta(\underline{x}) \, d\underline{x}$.

$\Rightarrow \gamma'(\theta) = \frac{\partial}{\partial\theta} \int\limits_A t(\underline{x}) f_\theta(\underline{x}) \, d\underline{x} = \int\limits_A t(\underline{x}) . \frac{\partial f_\theta(\underline{x})}{\partial\theta} \, d\underline{x}$, by regularity cond^n (4).

$= \int\limits_A t(\underline{x}) \left\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{x})\right\} . f_\theta(\underline{x}) \, d\underline{x} \cdots\cdots (2)$

Now doing (2) $-$ (1) $\times \gamma(\theta)$, we have,

$\gamma'(\theta) = \int\limits_A \{t(\underline{x}) - \gamma(\theta)\} \frac{\partial}{\partial\theta} \ln f_\theta(\underline{x})\} f_\theta(\underline{x}) \, d\underline{x}$

$= E[\{T(\underline{X}) - \gamma(\theta)\} \frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})]$ $= E[\{T - \gamma(\theta)\}\{Y - E(Y)\}]$,

$\therefore$ by (1^*), $E(Y) = 0$.

$= E(UV)$, where, $U = T - \gamma(\theta)$, $V = Y - E(Y)$.

Now, according to Cauchy Schwartz inequality [C.S. inequality], Now doing ② − ① × $\gamma(\theta)$, we have,

$$\gamma'(\theta) = \int_A \{t(\underline{x}) - \gamma(\theta)\} \frac{\partial}{\partial\theta} \ln f_\theta(\underline{x}) \} f_\theta(\underline{x})\, d\underline{x}$$

$$= E\left[\{T(\underline{X}) - \gamma(\theta)\} \frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})\right] = E\left[\{T - \gamma(\theta)\}\{Y - E(Y)\}\right],$$

$\because$ by (1*), $E(Y) = 0$.

$= E(UV)$, where, $U = T - \gamma(\theta)$, $V = Y - E(Y)$.

Now, ~~assume~~ according to Cauchy Schwartz inequality [C.S. inequality], we have, $E^2(UV) \leq E(U^2)\, E(V^2)$,

ie $\{\gamma'(\theta)\}^2 \leq E[\{T - \gamma(\theta)\}^2]\, E[\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})\}^2]$.

$\Rightarrow V_\theta(T)\, E[\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})\}^2] \geq \{\gamma'(\theta)\}^2$.

$$\therefore V_\theta(T) \geq \frac{\{\gamma'(\theta)\}^2}{E[\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})\}^2]}, \quad \forall \theta \in \Theta.$$

[Note: The statistic for which C.R. inequality hold, is called the minimum variance bound estimator.].

Particular Case:

Let, $\gamma(\theta) = \theta$, $\forall \theta \in \Theta$ & T is an unbiased estimator of $\gamma(\theta) = \theta$, $\forall \theta \in \Theta$

Then, from C.R. inequality we have,

$$V_\theta(T) \geq \frac{1}{E_\theta\left[\left\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{x})\right\}^2\right]}, \quad \forall \theta \in \Theta. \quad [\because \gamma'(\theta) = \frac{d}{d\theta}\theta = 1].$$

Note :

① $E_\theta\left[\left\{\frac{\partial}{\partial\theta}\ln f_\theta(\underset{\sim}{X})\right\}^2\right]$ is called, by Fisher, as the amount of information about θ obtained from the random sample $x_1, x_2, \ldots, x_n$ &

$\dfrac{1}{E_\theta\left[\left\{\frac{\partial}{\partial\theta}\ln f_\theta(\underset{\sim}{X})\right\}^2\right]}$ is called the information limit to $V_\theta(T)$.

It is also known as Fisher's information.

(3) As 'n' increases, ~~$V_\theta(T)$~~ the lower bound for $V_\theta(T)$ gets smaller. Thus, as the Fisher information increases, the lower bound decreases & the 'best' estimator will have smaller variance, consequently more information about θ.

(4) Regularity condition (i) is unnecessarily restrictive. It is only necessary that the conditions (ii) to (v) hold finitely.

In fact cond^n (i) excludes dist^{n}s such as $f_\theta(x) = \frac{1}{\theta}$, $0 < x < \theta$, for which $E\left[\frac{\partial}{\partial\theta} \ln f_\theta(X_i)\right] \neq 0$, i.e. (ii) fails to hold. It also excludes the densities such as, $f_\theta(x) = 1$, $\theta < x < \theta+1$ or, $f_\theta(x) = \frac{2}{\pi}\sin^2(\pi + x)$, $\theta < x < \theta + \pi$ each of which satisfies (ii) so that $E\left[\frac{\partial}{\partial\theta} \ln f_\theta(X_i)\right] = 0$ holds but not (iv)

(5) The C.R. Inequality holds trivially if $E\left[\left\{\frac{\partial}{\partial\theta} \ln f_\theta(\underline{X})\right\}^2\right] \to \infty$, when $\psi'(\theta)$ is finite, or if $V_\theta(T) \to \infty$.

Note on C.R. Inequality

The same derivation of C.R. inequality is also valid when, $X_1, X_2, \ldots, X_n$ ... random variables. The only change performed here

Note on C.R. Inequality

① The same derivation of C.R. inequality is also valid when, $x_1, x_2, \dots, x_n$ are purely discrete random variables. The only change performed here is that the integral sign is replaced by the summation.

V.V.I

130 (2) An unbiased estimator for which CRLB is attained is called the minimum variance bound estimator [MVBE]. Now, in regular estimation case, the variance of an MVUE (if exists) is the least attainable variance of an unbiased estimator of a parameter. Hence, in regular estimation case, $V(MVUE) \geq CRLB$. If the sign of equality holds here, then, it is obvious that $MVUE = MVBE$.

So, MVBE if exists becomes equal to MVUE, but not always it exists. Hence, we can say that, all the MVBE are nothing but MVUE, but not all the MVUE's are the MVBE's.

Moreover, when regularity conditions fail to hold, then, we can have, $CRLB > V(MVUE)$.

Applications:

① Let $X_1, X_2, \dots, X_n \overset{iid}{\sim} P(\lambda)$. Then find an MVUE for λ.

Sol^n: $\because X_i \overset{iid}{\sim} P(\lambda)$, $\forall i = 1(1)n$, so, their common p.m.f. is given by

$$f_\lambda(x) = \begin{cases} \dfrac{e^{-\lambda}\,\lambda^x}{x!}, & x = 0,1,2,\dots \\ & \lambda > 0 \\ 0, & \text{o.w.} \end{cases}$$

$\therefore \ln f_\lambda(x) = -\lambda + x\ln\lambda - \ln x!$

$\therefore \dfrac{\partial \ln f_\lambda(x)}{\partial\lambda} = -1 + \dfrac{x}{\lambda}$; $\dfrac{\partial^2 \ln f_\lambda(x)}{\partial\lambda^2} = -\dfrac{x}{\lambda^2}$.

$\therefore E\left[\dfrac{\partial^2 \ln f_\lambda(x)}{\partial\lambda^2}\right] = E\left[-\dfrac{X}{\lambda^2}\right] = -\dfrac{E(X)}{\lambda^2} = -\dfrac{1}{\lambda}$.

Also, here we choose, $\gamma(\lambda) = \lambda$, $\forall \lambda > 0$. $\therefore \gamma'(\lambda) = 1$, $\forall \lambda > 0$.

$$\therefore \text{CRLB} = \frac{\{\gamma'(\lambda)\}^2}{-n E\left[\frac{\partial^2}{\partial\lambda^2}\ln f_\lambda(X)\right]} = -\frac{1}{n\times\left(-\frac{1}{\lambda}\right)} = \frac{\lambda}{n}, \quad \forall \lambda > 0.$$

Now, we know that $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{n\lambda}{n} = \lambda$, $\forall \lambda > 0$.

$\therefore \bar{X}$ is an u.e. of λ, $\forall \lambda > 0$.

Moreover, $V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\sum_{i=1}^n V(X_i)$, $\because X_i$'s are iid.

$$= \frac{1}{n^2}\times n\lambda = \frac{\lambda}{n} = \text{CRLB}.$$

$\therefore$ For the unbiased estimator $\bar{X}$, CRLB is attained.

Hence, $\bar{X}$ is an MVUE of λ.

$\therefore$ Under the regular estimation case, for the Poisson dist^n, MVUE of λ is $\bar{X}$.

(2) Let, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where, μ is known. Find (if exists) an MVUE of σ^2.

Ans: $\because X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall i = 1(1)n$, so their common p.d.f. is

$$f_{\sigma^2}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, & -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0. \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore \ln f_{\sigma^2}(x) = -\ln\sqrt{2\pi} - \frac{1}{2}\ln\sigma^2 - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}.$$

Now, let, $\sigma^2 = \sigma^*$. Then, we have, $f_{\sigma^*}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^*}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^*}}, & -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma^* > 0 \\ 0, \text{ o.w.} \end{cases}$

$$\therefore \ln f_{\sigma^*}(x) = -\frac{1}{2}\ln\sigma^* - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^*} - \frac{1}{2}\ln 2\pi.$$

$$\therefore \frac{\partial \ln f_{\sigma^*}(x)}{\partial\sigma^*} = -\frac{1}{2\sigma^*} + \frac{(x-\mu)^2}{2\sigma^{*2}}$$

$$\frac{\partial^2 \ln f_{\sigma^*}(x)}{\partial\sigma^{*2}} = \frac{1}{2\sigma^{*2}} - \frac{2(x-\mu)^2}{2\sigma^{*3}} = \frac{1}{2\sigma^{*2}} - \frac{(x-\mu)^2}{\sigma^{*3}}.$$

$$\therefore E\left[\frac{\partial^2}{\partial\sigma^{*2}} f_{\sigma^*}(X)\right] = \frac{1}{2\sigma^{*2}} - \frac{E[(X-\mu)^2]}{\sigma^{*3}} = \frac{1}{2\sigma^{*2}} - \frac{\sigma^*}{\sigma^{*3}}$$

$$= \frac{1}{2\sigma^{*2}} - \frac{1}{\sigma^{*2}} = \frac{1-2}{2\sigma^{*2}} = -\frac{1}{2\sigma^{*2}}.$$

$$= \frac{1}{2\sigma^{*2}} - \frac{1}{\sigma^{*2}} = \frac{1-2}{2\sigma^{*2}} = -\frac{1}{2\sigma^{*2}}.$$

Now, let us choose, $\gamma(\sigma^*) = \sigma^*$.

$\therefore \gamma'(\sigma^*) = 1, \forall \sigma^* > 0.$

$$\therefore CRLB = \frac{\{\gamma'(\sigma^*)\}^2}{-n E\left[\frac{\partial^2}{\partial \sigma^{*2}} \ln f_{\sigma^*}(x)\right]} = -\frac{1}{n\left(-\frac{1}{2\sigma^{*2}}\right)} = \frac{2\sigma^{*2}}{n}$$

Now, we know that, $E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right] = \frac{1}{n} \cdot n E(x_i-\mu)^2 = \frac{n\sigma^2}{n} = \sigma^2 = \sigma^*$

$\therefore \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is an u.e. of σ^*. $\sigma^* > 0$.

Now, we know that, if $x_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall i = 1(1)n$, then $\frac{x_i-\mu}{\sigma} \overset{iid}{\sim} N(0,1)$, $\forall i = 1(1)n$.

$$\therefore \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_n \Rightarrow v\left(\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\right) = v(\chi^2_n) = 2n.$$

$$\Rightarrow \frac{v\left[\sum_{i=1}^{n}(x_i-\mu)^2\right]}{\sigma^{*2}} = 2n \Rightarrow v\left[\sum_{i=1}^{n}(x_i-\mu)^2\right] = 2n\sigma^{*2}.$$

$$\therefore v\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right] = \frac{1}{n^2} v\left[\sum_{i=1}^{n}(x_i-\mu)^2\right] = \frac{2n\sigma^{*2}}{n^2} = \frac{2\sigma^{*2}}{n} = CRLB.$$

Now, since, $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = S_0^2$ attains CRLB, so under the regular estimation case, for Normal dist^n, $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is an MVBE as well as MVUE for σ^2.

(3) Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where σ^2 is known. Find (if exist) an MVUE of μ.

Sol^n: Since, $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall\, i = 1(1)n$, so their common p.d.f. is

$$f_\mu(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2} & , -\infty < x < \infty \\ & -\infty < \mu < \infty;\ \sigma > 0. \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \ln f_\mu(x) = -\ln\sqrt{2\pi} - \ln\sigma - \frac{1}{2\sigma^2}(x-\mu)^2.$$

$$\therefore \frac{\partial \ln f_\mu(x)}{\partial \mu} = +\frac{1}{2\sigma^2}\cdot 2(x-\mu) = \frac{1}{\sigma^2}(x-\mu).$$

$$\frac{\partial^2 \ln f_\mu(x)}{\partial \mu^2} = -\frac{1}{\sigma^2}.$$

$$\therefore E\left[\frac{\partial^2 \ln f_\mu(x)}{\partial \mu^2}\right] = E\left[-\frac{1}{\sigma^2}\right] = \frac{-1}{\sigma^2}.$$

Now, here, we choose $\gamma(\mu) = \mu$ $\quad \therefore \gamma'(\mu) = 1$, $-\infty < \mu < \infty$.

$$\therefore CRLB = \frac{\{\gamma'(\mu)\}^2}{-nE\left[\frac{\partial^2}{\partial\mu^2}\ln f_\mu(x)\right]} = \frac{1}{n/\sigma^2} = \frac{\sigma^2}{n}.$$

Now, $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu.$

$\therefore \bar{X}$ is an u.e. of μ, $\forall$ for $-\infty < \mu < \infty$.

Moreover, $V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i)$, $\because X_i$'s are iid, $\forall\, i = 1(1)n$.

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} = CRLB.$$

$\therefore$ For the u.e. $\bar{X}$ of μ, C.R.L.B. is attained.

Hence, under regular estimation case for $N(\mu, \sigma^2)$ dist^n, $\bar{X}$ is the MVBE as well as the MVUE for μ.

(4) Suppose, $X_1, X_2, \dots, X_n \overset{iid}{\sim} Bin(m,p)$. Then, find an MVUE for 'p'.

Soln: Since, $X_i \overset{iid}{\sim} Bin(m,p)$, $\forall\, i = 1(1)n$, so their ~~joint~~ common p.d.f. is,

$$f_p(x) = \begin{cases} \binom{m}{x} p^x (1-p)^{m-x}, & x = 0(1)m, \; 0<p<1, \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore \ln f_p(x) = \ln\binom{m}{x} + x\ln p + (m-x)\ln(1-p), \quad 0<p<1.$$

$$\therefore \frac{\partial \ln f_p(x)}{\partial p} = \frac{x}{p} - \frac{m-x}{1-p}; \quad \frac{\partial^2 \ln f_p(x)}{\partial p^2} = -\frac{x}{p^2} - \frac{m-x}{(1-p)^2}$$

$$\therefore E\left[\frac{\partial^2 \ln f_p(x)}{\partial p^2}\right] = -\frac{1}{p^2}E(X) - \frac{m-E(X)}{(1-p)^2}$$

$$= -\frac{mp}{p^2} - \frac{m-mp}{(1-p)^2} = -\left[\frac{m}{p} + \frac{m-pm}{(1-p)^2}\right].$$

Now, we choose, $\gamma(p) = p$ $\therefore$ ~~for~~ $\gamma'(p) = 1$, ~~[illegible]~~ . $0 < p < 1$

$$\therefore CRLB = \frac{\{\gamma'(p)\}^2}{-nE\left[\frac{\partial^2}{\partial \mu^2} \ln f_p(x)\right]} = \frac{1}{mn\left[\frac{1}{p} + \frac{1}{1-p}\right]} = \frac{p(1-p)}{mn(1-p+p)} = \frac{p(1-p)}{mn}.$$

Now, $E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{nmp}{n} = mp$ $\therefore E\left(\frac{\bar{x}}{m}\right) = p.$

$\therefore \frac{\bar{x}}{m}$ is an u.e. of p.

Moreover, $V\left(\frac{\bar{x}}{m}\right) = \frac{1}{m^2} V\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{m^2 n^2}\sum_{i=1}^{n} V(x_i) = \frac{nmp(1-p)}{m^2 n^2} = \frac{p(1-p)}{mn} = CRLB.$

$\therefore$, for the u.e. $\frac{\bar{x}}{m}$ of p, CRLB is attained.

Hence, under regular estimation case for $Bin(m,p)$ dist^n, $\frac{\bar{x}}{m}$ is MVBE as well as MVUE for p.